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Study of polynomials of Fibonacci, Lucas and Jacobsthall

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Study of polynomials of Fibonacci, Lucas and Jacobsthall

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Thank you

Contents

Introduction

In the mathematical literature, there are many famous families of polynomials. Among these families, by way of examples but not limited to them, we find the polynomials of Tchebyshev, Hermite, Legender,... .

Often, these polynomials are defined by recursive relations and they have important applications in mathematics and other sciences; the thing that immortalized them in the names of the mathematicians who discovered them.

The polynomials considered in this memory are those of Fibonacci, Lucas and Jacobsthal. These polynomials are defined by recursive relations and are intimately related to the Fibonacci and Lucas numbers. In this dissertation we are interested in the study of the properties of these polynomials and their applications; especially the rediscovery of some identities concerning the Fibonacci and Lucas numbers.

In this memory, the first chapter is devoted to fundamental concepts and tools. The study of Fibonacci and Lucas numbers is the subject of the second chapter. In the third chapter we deal with Fibonacci and Lucas polynomials. We end with the fourth chapter which is devoted to Jacobsthal polynomials.

Chapter 1

SOME BASIC TOOLS AND PRINCIPLES

1.1 PRINCIPLE OF MATHEMATICAL INDUCTION (PMI)

Let $(P(n))_{n\geq 0}$ be a sequence of proposition. The method of mathematical induction assists in proving that $P(n)$ is true for all $n \geq n_0$, where n_0 is a given nonnegative integer.

1.1.1 MATHEMATICAL INDUCTION (weak form)

Suppose that

1. $P(n_0)$ is true ;

2. For all $k \geq n_0$, $P(k)$ is true implies $P(k+1)$ is true.

Then $P(n)$ is true for all $n \geq n_0$.

1.1.2 MATHEMATICAL INDUCTION (strong form)

Suppose that

1. $P(n_0)$ is true ;

2. For all $k \geq n_0$, $P(m)$ is true for all m with $n_0 \leq m \leq k$ implies $P(k+1)$ is true.

Then $P(n)$ is true for all $n \geq n_0$.
This principle is amply used in mathematics.

1.2 PASCAL'S TRIANGLE

It is remarakble that Fibonacci numbers have a stong connection with the elements of Pascal's triangle.

1.2.1 BINOMIAL COEFFICIENTS

Definition 1.1. *Let n and k be nonnegative integers. and $0 \leq k \leq n$,the binomial coefficients is defined by*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

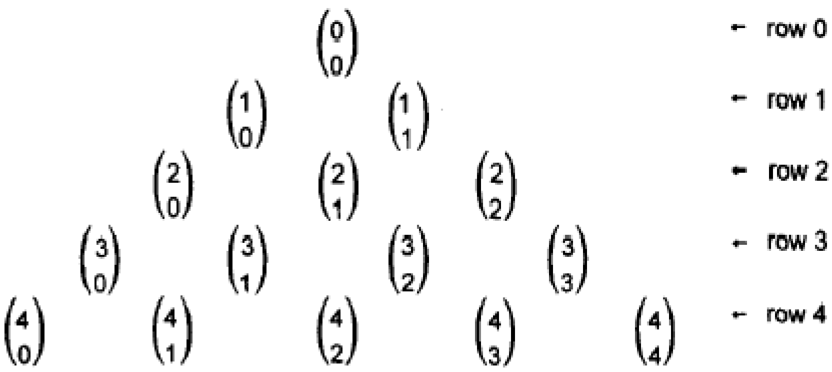
where $0! = 1$ *and* $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$.

Example 1.1.

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{24}{4} = 6.$$

1.2.2 PASCAL'S TRIANGLE

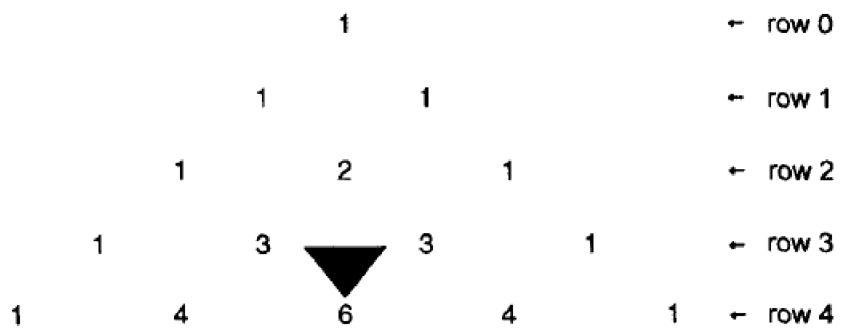
The various binomial coefficients $\binom{n}{k}$, where $0 \leq k \leq n$, can be arranged in the form of a triangle, called Pascal's triangle as shown in follwing Figure



PASCAL'S TRIANGLE

Theorem 1.1 (The Binomial of Newton)**.** *Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. Then*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$



PASCAL'S TRIANGLE

Example 1.2.

$$\begin{aligned}(x+y)^3 &= \sum_{k=0}^{3}\binom{3}{k}x^{3-k}y^k \\ &= \binom{3}{0}x^{3-0}y^0+\binom{3}{1}x^{3-1}y^1+\binom{3}{2}x^{3-2}y^2+\binom{3}{3}x^{3-3}y^3 \\ &= x^3+3x^2y+3xy^2+y^3.\end{aligned}$$

1.2.3 RELATION BETWEEN PASCAL'S TRIANGLE AND FIBONACCI NUMBRES

We show that the Fibonacci numbers related to Pascal's triangle.
Now add the numbers along the diagonals of the northeast .The sums are $1, 1, 2, 3, 5, 8, 13, \cdots$, and they appear to be the Fibonacci numbers as in the following figure

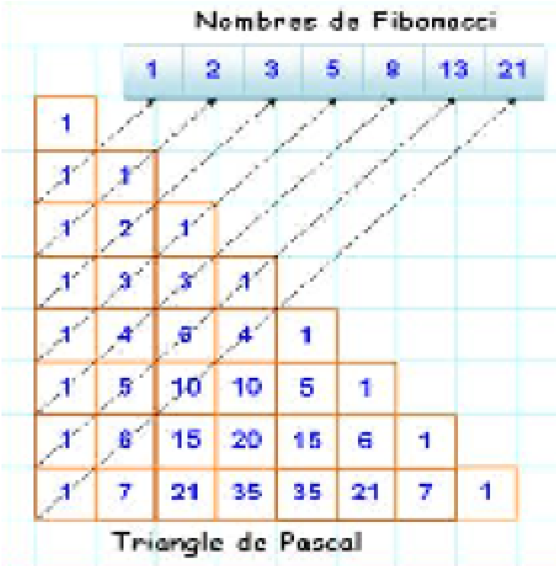
1.3 LINEAR RECURRENCE SEQUENCE

A linear recurrence relation is an equation that relates a term in a sequence to previous terms using recursion. The use of the word linear refers to the fact that previous terms are arranged as a 1^{st} degree polynomial in the recurrence relation.

Definition 1.2. *A linear recurrence relation is an equation that defines the n^{th} term in a sequence in terms of the k previous terms in the sequence. The recurrence relation is in the form*

$$x_n = c_1x_{n-1} + c_2x_{n-2} + \cdots + c_kx_{n-k}.$$

Where each c_i is a constant coefficient.



PASCAL'S TRIANGLE

Definition 1.3. *A solution to a recurrence relation is the value of x_n in terms of n, and does not requires the value of any previous terms.*

Example 1.3. *Let the recurrence relation : $x_1 = 3$, $x_n = 3x_{n-1}$*
Each term in the sequence can be calculated with a previous term. The first term, $x_1 = 3$, is given. The next term can be calculated using the relation $x_n = 3x_{n-1}$ thas

$$x_2 = 3x_1 = 3 \cdot 3 = 9.$$

This process is repeated for the other terms.

$$x_3 = 3x_2 = 3 \cdot 9 = 27 \ , \ x_4 = 3x_3 = 3 \cdot 27 = 81 = 3^4, \cdots .$$

These are powers of 3.

Let us say we have a recurrence

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k}$$

and a solution $x_n = ar^n$.
Remplacing in the last equation, we get

$$ar^n = c_1 ar^{n-1} + c_2 ar^{n-2} + \cdots + c_k ar^{n-k}.$$

Since a, $r \neq 0$, we can divide by ar^{n-k}

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k.$$

Moving the terms over, we get

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

which is a polynomial in r, so the solution satisfies the recurrence only if r is a root of this polynomial. This polynomial is called the characteristic polynomial of the recurrence.
Also, note that if two geometric series satisfy a recurrence, the sum of them also satisfies the recurrence. Then, we can find the following method for solving recurrences

(i) Find the characteristic polynomial;

(ii) Find the roots $r_1, r_2, \cdots, r_k$ of the characteristic polynomial;

(iii) Assuming no multiple roots, the form expression is

$$x_n = a_1(r_1)^n + a_2(r_2)^n + \cdots + a_k(r_k)^n$$

for some constant $a\prime s$.

(iv) Use the initial values to find the values of the $a\prime s$.

Let us try an example

Example 1.4. *Let the sequence x_n be defined by*

$$x_n = 5x_{n-1} + 6x_{n-2} \quad , \quad n \geq 2$$

$$x_0 = 1 \quad , \quad x_1 = 3.$$

Then the characteristic polynomial is

$$x^2 = 5x + 6.$$

The roots are -1 and 6. Thus, the form expression looks like

$$x_n = a_1(-1)^n + a_2(6)^n.$$

For in $n = 0$

$$1 = a_1 + a_2.$$

For in $n = 1$

$$3 = -a_1 + 6a_2.$$

Solving, we get $a_1 = \frac{3}{7}$ *and* $a_2 = \frac{4}{7}$*, and the closed-form expression is*

$$x_n = \frac{3}{7}(-1)^n + \frac{4}{7}(6)^n.$$

This can be verified by remplacing into the recurrence.

1.4 GENERATING FUNCTIONS

The generating functions help us to solve linear homogeneous recurrence relations.

Definition 1.4. *The generating functions for the sequence* $a_0, a_1, a_2, \cdots, a_k, \cdots$ *of real numbers series is difined by*

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

Example 1.5. *The generating function for the sequence* $1, 1, 1, 1, \cdots$ *is*

$$G(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

Example 1.6. *The generating functions for the sequence* $1, a, a^2, a^3, \cdots$ *is*

$$G(x) = 1 + ax + a^2x^2 + \cdots = \sum_{k=0}^{\infty} a^k x^k = \frac{1}{1-ax}.$$

1.4.1 Using Generating Functions to solov Recurrence Relation

Example 1.7. *Find the generating function for the sequence* $\{a_n\}$ *defined by* $a_0 = 1$ *and*

$$a_n = 4a_{n-1} + 5^{n-1} \tag{1.1}$$

for $n \geq 1$*. then find explicit formula for* a_n*.*

Solution.

We multiply both sides of the recurrence relation (1.1) by x^n *to obtain*

$$a_n x^n = 4a_{n-1} x^n + 5^{n-1} x^n. \tag{1.2}$$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ *be the generating function of the sequence* $a_0, a_1, a_2, \cdots$. *We sum both sides of the equation (1.2), starting with* $n = 1$, *to find that*

$$\begin{aligned}
G(x) - 1 &= \sum_{n=1}^{\infty} a_n x^n \\
&= \sum_{n=1}^{\infty} (4a_{n-1}x^n + 5^{n-1}x^n) \\
&= 4\sum_{n=1}^{\infty} a_{n-1}x^n + x\sum_{n=1}^{\infty} 5^{n-1}x^{n-1} \\
&= 4x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 5^{n-1}x^{n-1} \\
&= 4x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 5^n x^n \\
&= 4xG(x) + \frac{x}{1-5x}
\end{aligned}$$

where we have use example(1.6) to evaluate the second summation. Therefore , we have

$$G(x) - 1 = 4xG(x) + \frac{x}{1-5x}.$$

Soloving for $G(x)$ *we obtain*

$$\begin{aligned}
G(x) - 4xG(x) &= 1 + \frac{x}{1-5x} \\
G(x)(1-4x) &= 1 + \frac{x}{1-5x} \\
G(x)(1-4x) &= \frac{1-5x+x}{1-5x} \\
G(x)(1-4x) &= \frac{1-4x}{1-5x}.
\end{aligned}$$

Therefore

$$G(x) = \frac{1}{1-5x}. \tag{1.3}$$

Since $G(x) = \sum_{n=0}^{\infty} 5^n x^n$ *and* $G(x) = \sum_{n=0}^{\infty} a^n x^n$, *then* $a_n = 5^n$ *for* $n \geqslant 0$.

Verification

$$\begin{aligned}
a_n &= 4a_{n-1} + 5^{n-1} \\
5^n &= 4 \cdot 5^{n-1} + 5^{n-1} \\
&= (4+1)5^{n-1} \\
&= 5 \cdot 5^{n-1} = 5^n
\end{aligned}$$

is true.

Chapter 2

FIBONACCI AND LUCAS SUITE

2.1 FIBONACCI SEQUENCE

2.1.1 FIBONACCI SEQUENCE

Definition 2.1. *The numbers* $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$ *is called the Sequence of Fibonacci numbers each term greater than or equal to 2 is obtained by adding the two previous terms i.e.*

$2 = 1 + 1,\ 3 = 1 + 2,\ 5 = 2 + 3 \cdots$.

Thus we have the following recursive definition of the n^{th} *Fibonacci number* F_n

$$\begin{aligned} F_1 &= F_2 = 1 \\ F_n &= F_{n-1} + F_{n-2} \quad n \geqslant 3. \end{aligned}$$

We deducted that $F_0 = 0$. *The characterisic polynomail of the linear recurrence relation defining Fibonacci numbers is*

$$x^2 - x - 1 = 0.$$

The roots are $\alpha = \dfrac{1+\sqrt{5}}{2}$, $\beta = \dfrac{1-\sqrt{5}}{2}$

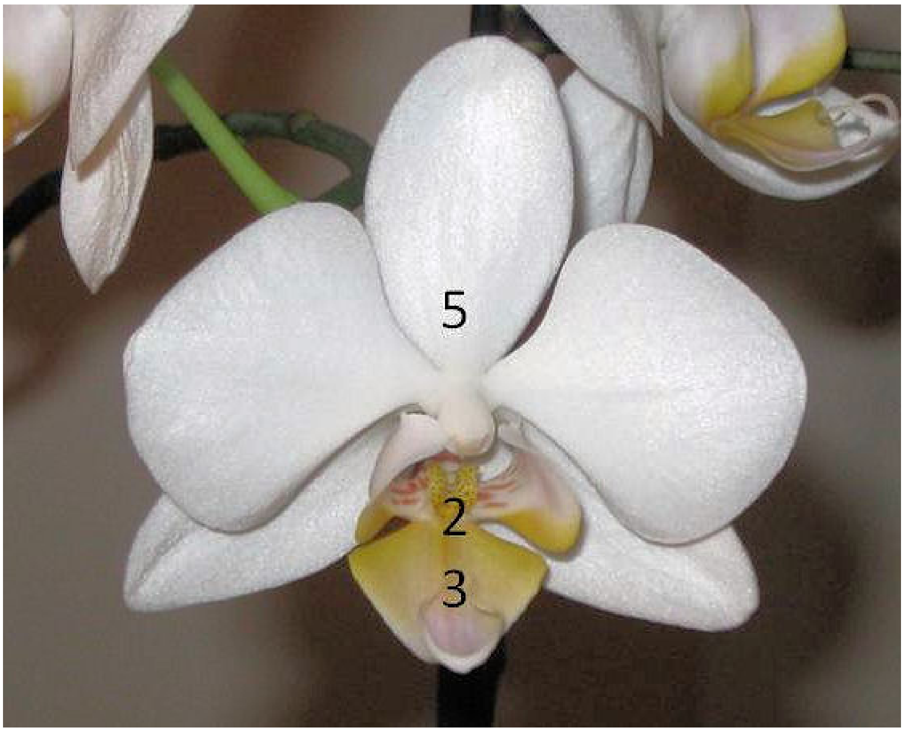
and in this case

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad , \ n \geqslant 3.$$

Example 2.1.

The following example shows the occurence of fibonacci numbers in petals of plants

Plant	Number of Petals
Enchanter's nightshade	2
Iris, lilly	3
Buttercup, columbine, delphinium, larkspur, wall lettuce	5
Celandine, delphinium, field senecio, squalid senecio	8



2.1.2 FIBONACCI IDENTITIES

Both Fibonacci and Lucas numbers satisfy numerous identities that have been discovered over the centuries. In the rest of this chapter we explore several of these fundamental identities.

Theorem 2.1. *(LUCAS,1876)*

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1. \tag{2.1}$$

Proof. Using the Fibonacci recurrence relation, we have

$$F_1 = F_3 - F_2$$

$$F_2 = F_4 - F_3$$

$$F_3 = F_5 - F_4$$

$$\vdots$$

$$F_{n-1} = F_{n+1} - F_n$$

$$F_n = F_{n+2} - F_{n-1}.$$

Adding these equations, we get

$$\sum_{1}^{n} F_i = F_{n+2} - F_2 = F_{n+2} - 1.$$

□

2.1.3 AN ALTERNATE METHOD

An alternate method of proving identity (2.1) is to apply the principle of mathematical induction (weak from). Since $F_1 = F_3 - 1$, the formula works for $n = 1$.

Now assume it is true for an arbitrary positive integer $k \geq 1$ i.e.

$$\sum_{1}^{k} F_i = F_{k+2} - 1.$$

Then

$$\begin{aligned}
\sum_{1}^{k+1} F_i &= \sum_{1}^{k} F_i + F_{k+1} \\
&= (F_{k+2} - 1) + F_{k+1} \qquad \text{by the inductive hypothesis} \\
&= (F_{k+1} + F_{k+2}) - 1 \\
&= F_{k+3} - 1.
\end{aligned}$$

Thus, by PMI, the formula is true for every positive integer n.

Example 2.2. $\sum_{1}^{20} F_i = F_{22} - 1 = 17711 - 1 = 17710.$

Example 2.3. *Prove that* $\sum_{j=0}^{k} F_{i+j} + F_{i+1} = F_{i+k+2}$.

Solution

$$\begin{aligned}\sum_{j=0}^{k} F_{i+j} + F_{i+1} &= \sum_{1}^{i+k} F_i - \sum_{1}^{i-1} F_i + F_{i+1} \\ &= (F_{i+k+2} - 1) - (F_{i+1} - 1) + F_{i+1} \\ &= F_{i+k+2}.\end{aligned}$$

This example justify the validity of the puzzle of W. H. Huff : Add up any finite number of consecutive Fibonacci numbers. Now add the second term to this sum. The resulting sum is a Fibonacci number.

Theorem 2.2 (Lucas)**.**

$$\sum_{1}^{n} F_{2i-1} = F_{2n}.$$

Proof. By Fibonacci recurrence relation, we have

$$F_1 = F_2 - F_0$$

$$F_3 = F_4 - F_2$$

$$F_5 = F_6 - F_4$$

$$\vdots$$

$$F_{2n-3} = F_{2n-2} - F_{2n-4}$$

$$F_{2n-1} = F_{2n} - F_{2n-2}.$$

Adding these equations, we obtai

$$\sum_{1}^{n} F_{2i-1} = F_{2n} - F_0 = F_{2n}.$$

□

Example 2.4.

$$\sum_{1}^{10} F_{2i-1} = F_{20} = 6765.$$

Corollary 2.1.

$$\sum_{1}^{n} F_{2i} = F_{2n+1} - 1.$$

Proof.

$$\begin{aligned}\sum_{1}^{n} F_{2i} &= \sum_{1}^{2n} F_i - \sum_{1}^{n} F_{2i-1} \\ &= (F_{2n+2}-1) - F_{2n} \quad \text{by Theorems (2.1) and (2.2)} \\ &= (F_{2n+2} - F_{2n}) - 1 \\ &= F_{2n+1} - 1 \quad \text{by the Fibonacci recurrence relation.}\end{aligned}$$

□

Theorem 2.3.

$$\sum_{1}^{n} F_i^2 = F_n F_{n+1}.$$

Proof. When n = 1, the left-hand side

$$\sum_{1}^{1} F_i^2 = F_i^2 = 1 = 1.1 = F_1 F_2.$$

Which is the right-hand side. So the result is true when $n = 1$.

Assume the formula for a positive integer k

$$\sum_{1}^{k} F_i^2 = F_k F_{k+1}.$$

Then

$$\begin{aligned}\sum_{1}^{k+1} F_i^2 &= \sum_{1}^{k} F_i^2 + F_{k+1}^2 \\ &= F_k F_{k+1} + F_{k+1}^2 \\ &= F_{k+1}(F_k + F_{k+1}) \\ &= F_{k+1} F_{k+2} \quad \text{by the Fibonacci recurrence relation.}\end{aligned}$$

So the statement is true when n = k + 1. Thus,by (PMI), it is true for every positive integer n.

□

Also we recall here the following importanet theorem

Theorem 2.4 (Binet's formula)**.** *Let α be the positive root of the quadratic equation* $x^2 - x - 1 = 0$ *and β its negative root. Then*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad , \quad L_n = \alpha^n + \beta^n \quad for\ n \geqslant 0.$$

2.1.4 HOW DO THE FIBONACCI NUMBERS ENLARGE ?

Theorem 2.5. *For all* $n \geqslant 3$ *we have*

$$\alpha^{n-1} \geqslant F_n \geqslant \alpha^{n-2}$$

2.2 LUCAS SEQUENCE

2.2.1 LUCAS SEQUENCE

Definition 2.2. *The numbers* $L = 1, 3, 4, 7, 11, \cdots$ *is called the Sequence of Lucas numbers. Each term greater than or equal to 2 is obtained by adding the two previous terms*
$4 = 1 + 3,\ 7 = 4 + 3,\ 11 = 4 + 7 \cdots.$
Thus we have the following recursive definition of the n^{th} *Lucas number* L_n

$$\begin{aligned} L_n &= L_{n-1} + L_{n-2} \quad , \quad n \geq 3 \\ L_1 &= 1 \textit{ and } L_2 = 3. \end{aligned}$$

We deducted that $L_0 = 2$.

2.2.2 LUCAS IDENTITIES

Like the case of Fibonacci Numbres, there are analogous results for Lucas number

$$\sum_1^n L_i = L_{n+2} - 3 \tag{2.2}$$

$$\sum_1^n L_{2i-1} = L_{2n} - 2 \tag{2.3}$$

$$\sum_1^n L_{2i} = L_{2n+1} - 1 \tag{2.4}$$

$$L_{n-1}L_{n+1} - L_n^2 = 5(-1)^{-1} \tag{2.5}$$

$$\sum_1^n L_i^2 = L_n L_{n+1} - 2. \tag{2.6}$$

These identities can also be established by using PMI.

2.3 MEXED FIBONNCI AND LUCAS IDENTITIES

Theorem 2.6.

$$L_n = F_{n-1} + F_{n+1} \quad , \quad n \geqslant 0. \tag{2.7}$$

Proof. By Binet's formula $L_n = \alpha^n + \beta^n$ *and* $F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$.
Then (2.7) is equivalent to

$$\alpha^n + \beta^n - \frac{1}{\alpha - \beta}(\alpha^{n-1} - \beta^{n-}) - \frac{1}{\alpha - \beta}(\alpha^{n+1} - \beta^{n+1}) = 0$$

Or to

$$(\alpha - \beta)(\alpha^n + \beta^n) - \alpha^{n-1} + \beta^{n-1} - \alpha^{n+1} + \beta^{n+1} = 0. \tag{2.8}$$

The right hand side of (2.8)is equal to $-\alpha^{n-1}(1 + \alpha\beta) + \beta^{n-1}(1 + \alpha\beta)$ which equal to 0, since $1 + \alpha\beta = 0$. □

The two Binet formulas can be used together to derive an array of identities.

Corollary 2.2.

$$F_{2n} = F_n L_n \tag{2.9}$$
$$F_{n-1} + F_{n+1} = L_n \tag{2.10}$$
$$F_{n+2} - F_{n-2} = L_n \tag{2.11}$$
$$F_{n-1} + F_{n+1} = 5F_n. \tag{2.12}$$

Example 2.5.

$$\begin{aligned} F_{20} = 6765 &= 55 \times 123 = F_{10} L_{10} \\ F_{11} + F_{13} &= 89 + 233 = 322 = L_{12} \\ F_{11} - F_7 &= 89 - 13 = 76 = L_9. \end{aligned}$$

And

$$L_{10} + L_{12} = 123 + 322 = 445 = 5 \times 89 = 5F_{11}.$$

Corollary 2.3.

$$F_{2^m} = L_1 L_2 L_4 L_8 \cdots L_{2^{m-2}} L_{2^{m-1}}.$$

Proof. We have

$$\begin{aligned} F_{2^m} &= L_{2^{m-1}} F_{2^{m-1}} \\ &= L_{2^{m-1}} (L_{2^{m-2}} F_{2^{m-1}}) = L_{2^{m-1}} L_{2^{m-2}} F_{2^{m-2}} \\ &= L_{2^{m-1}} L_{2^{m-2}} (L_{2^{m-3}} F_{2^{m-3}}) = L_{2^{m-1}} L_{2^{m-2}} F_{2^{m-3}}. \end{aligned}$$

Continuing like this we get

$$F_{2^m} = L_{2^{m-1}} L_{2^{m-2}} \cdots L_8 L_4 L_2 L_1.$$

$\square$

Chapter 3

FIBONACCI AND LUCAS POLYNOMIALS

3.1 FIBONACCI POLYNOMIALS

The Fibonacci polynomails were studied by the Belgian mathematician Eugene Charles Catalan(1814-1894) and the German mathematician E. Jacobsthal.
Eugene Charles Catalan studied The polynomials $f_n(x)$ defined by

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x) \tag{3.1}$$

where $f_1(x) = 1,\ f_2(x) = x$, and $n \geq 3$.

3.1.1 CATALAN'S FIBONACCI POLYNOMIALS

The first six membres of this Fibonacci family are :

$$\begin{aligned} f_1(x) &= 1 \\ f_2(x) &= x \\ f_3(x) &= x^2+1 \\ f_4(x) &= x^3+2x \\ f_5(x) &= x^4+3x^2+1 \\ f_6(x) &= x^5+4x^3+3x \end{aligned}$$

We note that $f_i(1) = F_i$ for $1 \leq i \leq 10$. In fact, $f_n(1) = F_n$ for all n ; this follows dircctly from the recurrence relation (3.1). Also we note that $f_1(2) = 1,\ f_2(2) = 2$,
and $f_n(2) = 2f_{n-1}(2) + f_{n-2}(2)$, where $n \geq 3$. So $f_n(2) = P_n$, where $(P_n)_{n\geqslant 1}$ is the Pell numbres

$1, 2, 5, 12, 29, ...$.

The Fibonacci polynomials to negative subscripts is defined as follwing

$$f_{-n}(x) = (-)^{n+1} f_n(x)$$

and $f_0(x) = 0$.

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	
1	1									
2	0	1								
3	1	0	1							
4	0	2	0	1						
5	1	0	3	0	1					
6	0	3	0	4	0	1				
7	1	0	6	0	5	0	1			
8	0	4	0	10	0	6	0	1		
9	1	0	10	0	15	0	7	0	1	
10	0	5	0	20	0	21	0	8	0	1

A Pascal row

Diagonal sum = 2^4

← Row sum = F_7

Table 2

Table 2 shows the vurious coefficients of the first ten Fibonacci polynomials, when arranged in increasing exponents. We have Three obscrvations

(i) The elements on every rising diagonal beginning on row $2n$ are zero ;

(ii) The alternate rising diagonals form the various Pscal rows ;

(iii) The sum of the elements on the rith rising diagonal is $2^{(n-1)/2} = 2 \cdot 2^{(n-3)/2}$, where n is old. For example, the sum of the numbres on row 5 is $1 + 2 + 1 = 4$.

n	Expansion of $(x+1)^n$	
0	1	
1	$x+1$	
2	x^2+2x+1	$f_5(x)$
3	x^3+3x^2+3x+1	$f_7(x)$
4	$x^4+4x^3+6x^2+4x+1$	
5	$x^5+5x^4+10x^3+10x^2+5x+1$	
6	$x^6+6x^5+15x^4+20x^3+15x^2+6x+1$	

Table 3

The sums of the elements along the rising diagonals in Table 3 yield the various Fibonacci polynomials. For instance, the sum of the elements along the diagonal beginning at row 5 is x^5+4x^3+3x, which is $f_6(x)$; similarly, the diagonal beginning at row 7 yields $f_8(x)$.

3.1.2 EXPLICIT FORMULA FOR $f_n(x)$

More generally, the sum of the elements along the diagonal beginning at row n is $f_{n+1}(x)$. Thus we have the following

Theorem 3.1.

$$f_{n+1}(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} x^{n-2j}, \ n \geq 0$$

$$f_n(x) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-j-1}{j} x^{n-2j-1}, \ n \geq 0. \tag{3.2}$$

Proof. Notice that

$$g_1(x) = \sum_{j=0}^{0} \binom{-j}{j} x^{-2j} = 1 \quad and \quad g_2(x) = \sum_{j=0}^{0} \binom{1-j}{j} x^{1-2j} = x.$$

Besides,

$$xg_{n-1}(x) + g_{n-2}(x) = \sum_{j=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-j-2}{j} x^{n-2j-1} + \sum_{j=0}^{\lfloor (n-3)/2 \rfloor} \binom{n-j-1}{j} x^{n-2j-3}.$$

When n is even,$n = 2m$

$$\begin{aligned}
RHS &= \sum_{j=0}^{m-1} \binom{2m-j-2}{j} x^{2m-2j-1} + \sum_{j=0}^{m-2} \binom{2m-j-3}{j} x^{2m-2j-3} \\
&= \sum_{j=0}^{m-1} \binom{2m-j-2}{j} x^{2m-2j-1} + \sum_{j=0}^{m-1} \binom{2m-j-2}{j} x^{2m-2j-1} \\
&= \sum_{j=0}^{m-1} \binom{2m-j-2}{j-1} x^{2m-2j-1} + \sum_{j=0}^{m-1} \binom{2m-j-2}{j-1} x^{2m-2j-1} \\
&= \sum_{j=0}^{m-1} [\binom{2m-j-2}{j-1} + \binom{2m-j-2}{j-1}] x^{2m-2j-1} \\
&= \sum_{j=0}^{m-1} \binom{2m-j-1}{j-1} x^{2m-2j-1} = g_{2m}(x).
\end{aligned}$$

Similarly, when $n = 2m+1, RHS = g_{2m+1}(x)$.Thus, in both case, $g_n(x)$ satisfies the recurrence relation (3.1) $g_n(x) = f_n(x)$

□

Example 3.1.

$$\begin{aligned}
f_6(x) &= \sum_{j=0}^{\lfloor (6-1)/2 \rfloor} \binom{5-j}{j} x^{5-2j} \\
&= \binom{5}{0} x^5 + \binom{4}{1} x^3 + \binom{3}{2} x^1 \\
&= x^5 + 4x^3 + 3x = F_6.
\end{aligned}$$

Thus is consistent with wate we have meutioned eralier .

3.1.3 OTHER EXPLICIT FORMULA FOR explicit FOR $f_n(x)$

$$f_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \tag{3.3}$$

where $\alpha(x) = \frac{x+\sqrt{x^2+4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2+4}}{2}$

Example 3.2. *Let us compute $f_5(x)$. It is easy to verify that*

$$(x + \sqrt{x^2+4})^5 - (x - \sqrt{x^2+4})^5 = 32(x^4 + 3x^2 + 1)\sqrt{x^2+4}.$$

So $f_5(x) = x^4 + 3x^2 + 1$.

Next we establish a few properties of Fibonacci polynomials, which are generalizations of some formulas that we encountered in chapter 2.

Theorem 3.2.

$$x\sum_{1}^{n} f_i(x) = f_{n+1}(x) + f_n(x) - 1.$$

Proof. By using the recurrence relation 3.1,

$$\sum_{1}^{n} f_{i+1}(x) = x\sum_{1}^{n} f_i(x) + \sum_{1}^{n} f_{i-1}(x).$$

That is,

$$f_n(x) + f_{n+1}(x) = x\sum_{1}^{n} f_i(x) + f_0(x) + f_1(x).$$

Since $f_0(x) = 0$, it follows that

$$x\sum_{1}^{n} f_i(x) = f_{n+1}(x) + f_n(x) - 1.$$

□

Example 3.3.

$$\begin{aligned} x\sum_{1}^{5} f_i(x) &= x[1 + x + (x^2+1) + (x^3+2x) + (x^4+3x^2+)] \\ &= x^5 + x^4 + 4x^3 + 3x^2 + 3x. \end{aligned}$$

On the other hand

$$\begin{aligned} f_6(x) + f_5(x) - 1 &= x^5 4x^3 + 3x + (x^4 + 3x^2 + 1) - 1 \\ &= x^5 + x^4 + 4x^3 + 3x^2 + 3x \\ &= x\sum_{1}^{5} f_i(x). \end{aligned}$$

As expected

Corollary 3.1.

$$\sum_{1}^{n} F_i = F_{n+2} - 1$$

This corollary from the theorem,since $f_i(1) = F_i$.

3.1.4 A GENERATING FUNCTION FOR $f_n(x)$

Next, we will find the genarating function for $f_n(x)$. To this end, we let

$$g(t) = \sum_0^\infty f_n(x)t^n$$

$$xtg(t) = \sum_0^\infty xf_n(x)t^{n+1}$$

$$t^2g(t) = \sum_0^\infty f_n(x)t^{n+1}.$$

Then

$$(1 - xt - t^2)g(t) = f_0(x) + tf_1(x) - xtf_0(x) = t.$$

Thus

$$g(t) = \frac{t}{1 - xt - t^2}$$

generates $f_n(x)$.

3.1.5 AS AN APPLICATION OF THE GENERATING FUNCTIONS

we have

Theorem 3.3.

$$f_{m+n+1}(x) = f_{m+1}(x)f_{n+1}(x) + f_m(x)f_n(x).$$

Proof.

$$\frac{y}{1 - xy - y^2} = \sum_0^\infty f_n(x)y^n.$$

Then

$$\frac{yf_m(x)}{1 - xy - y^2} = \sum_0^\infty f_m(x)f_n(x)y^n. \tag{3.4}$$

And

$$\frac{f_{m+1}(x)}{1 - xy - y^2} = \sum_0^\infty f_{m+1}(x)f_{n+1}(x)y^n. \tag{3.5}$$

Then

$$\frac{f_{m+1}(x)f_n(x)y}{1 - xy - y^2} = \sum_0^\infty (f_{m+1}(x)f_{n+1}(x) + f_m(x)f_n(x))y^n.$$

On the other hand, by Zeitlin

$$\frac{f_{m+1}(x) + f_m(x)y}{1 - xy - y^2} = \sum_0^\infty f_{m+n+1}(x)y^n. \tag{3.6}$$

Henc

$$f_{m+n+1}(x) = f_{m+1}(x)f_{n+1}(x) + f_m(x)f_n(x).$$

□

This theorem illustrates an alternate method for constructing new membres of the family of Fibonacci polynomials

Example 3.4. *let* $m = 2$ *and* $n = 3$. *Then*

$$\begin{aligned} f_{m+1}(x)f_{n+1}(x) + f_m(x)f_n(x) &= f_3(x)f_4(x) + f_2(x)f_3(x) \\ &= (x^2+1)(x^3+2x) + (x)(x^2+) \\ &= x^5 + 4x^3 + 3x \\ &= f_6(x) = f_{m+n+1(x)}. \end{aligned}$$

Also theorem (3.3) yields the following Fibonacci identity .

Corollary 3.2. $F_{m+n} = F_{m+1}F_n + F_mF_{n-1}$

Theorem 3.4.

$$f_n(x) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-j-1}{j} x^{n-2j-1}, \quad n \geqslant 0$$

Proof. We have

$$\frac{y}{1-xy-y^2} = \sum_{0}^{\infty} f_n(x)y^n. \tag{3.7}$$

But

$$\frac{1}{1-2tz+z^2} = \sum_{0}^{\infty}[\sum_{j=0}^{\lfloor n/2 \rfloor}(-1)^j \binom{n-j}{j}]z^n \tag{3.8}$$

$$U_n(t) = \sum_{j=0}^{\lfloor n/2 \rfloor}(-1)^j \binom{n-j}{j} (2t)^{n-2j}$$

is the Chepyshev polynomial of the second kind. Let $z = iy$ and $t = x/2i$, where $i^2 = -1$. Then Eq(3.8) yields

$$\frac{1}{1-xy-y^2} = \sum_{n=0}^{\infty} i^n U_n(x/2i)y^n$$

$$\frac{y}{1-xy-y^2} = \sum_{n=0}^{\infty} i^n U_n(x/2i)y^{n+1}.$$

From Eqs (3.7) and (3.8), it follows that

$$\begin{aligned} f_{n+1}(x) &= i^n U_n(x/2i) y^n && (3.9)\\ &= i^n \sum_{j=0}^{[n/2]} (-1)^j \begin{pmatrix} n-j \\ j \end{pmatrix} (x/i)^{n-2j} && (3.10)\\ &= \sum_{j=0}^{[n/2]} \begin{pmatrix} n-j \\ j \end{pmatrix} (x)^{n-2j} && (3.11)\\ f_n(x) &= \sum_{j=0}^{\lfloor n/2 \rfloor} \begin{pmatrix} n-j-1 \\ j \end{pmatrix} (x)^{n-2j-1}. && (3.12) \end{aligned}$$

□

The next result, which we encountered in Pascal triangle, follows from this formula.

Corollary 3.3 (lucas, 1876)**.**

$$F_n = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \begin{pmatrix} n-j-1 \\ j \end{pmatrix}$$

Theorem 3.5.

$$f_n'(x) = \sum_{i=1}^{n-1} f_i(x) f_{n-1}(x)$$

where $f_n'(x)$ denotes the derivative of $f_n(x)$ with respect to x and $n \geq 1$.

Proof. Defferentiating Eq 3.7 with respect to x,

$$\begin{aligned} \sum_{n=0}^{\infty} f_n'(x) y^n &= (\frac{y}{1-xy-y^2})^2 \\ &= [\sum_{0}^{\infty} f_n(x) y^n]^2 \\ &= \sum_{0}^{\infty} [\sum_{i=0}^{n} f_i(x) f_{n-i}(x)] y^n. \end{aligned}$$

Comparing the coefficients of y^n, we get

$$f_n'(x) = \sum_{i=0}^{n} f_i(x) f_{n-1}(x)$$

$$f_n'(x) = \sum_{i=1}^{n-1} f_i(x) f_{n-1}(x).$$

since $f_0(x) = 0$. □

Example 3.5. *We have*

$$f_5(x) = x^4 + 3x^2$$

$$f_5'(x) = 4x^3 + 6x$$

$$\begin{aligned}\sum_1^5 f_i(x)f_{5-i}(x) &= f_1(x)f_4(x) + f_2(x)f_3(x) + f_3(x)f_2(x) + f_4(x)f_1(x) \\ &= x^3 + 2x + x^3 + x + x^3 + x + x^3 + 2x \\ &= 4x^3 + 6x \\ &= f_5'(x).\end{aligned}$$

3.2 LUCAS POLYNOMIALS

Lucas polynomials $l_n(x)$, originally studied in 1970 by Bicknell, are defined by

$$l_n = xl_{n-1}(x) + l_{n-2}(x)$$

where $l_0(x) = 2$, $l_1(x) = x$, and $n \geq 2$.

The first seven Lucas polynomials are

$$\begin{aligned}l_1(x) &= x \\ l_2(x) &= x^2 + 2 \\ l_3(x) &= x^3 + 3x \\ l_4(x) &= x^4 + 4x^2 + 2 \\ l_5(x) &= x^5 + 5x^3 + 5x \\ l_6(x) &= x^6 + 6x^4 + 9x^2 + 2 \\ l_7(x) &= x^7 + 7x^5 + 9x^3 + 7x.\end{aligned}$$

It follows from the recursive definition that $l_n(1) = L_n$ for $n \geq 0$; that is, the sum of the coefficient of $l_n(x)$ is L_n. We can verify this by computing $l_n(1)$ for these Lucas polynomials.

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}
1	0	1									
2	2	0	1								
3	0	3	0	1							
4	2	0	4	0	1						
5	0	5	0	5	0	1					
6	2	0	9	0	6	0	1				
7	0	7	0	14	0	7	0	1			
8	2	0	16	0	20	0	8	0	1		
9	0	9	0	30	0	27	0	9	0	1	
10	2	0	25	0	50	0	35	0	10	0	1

Sum $= 48 = 3 \cdot 2^4$

$\leftarrow$ Sum $= L_7 = 29$

Table 4

We can prove tha the Lucas polynomials satisfy three additional properites

- $l_n = f_{n+1}(x) + f_{n-1}(x) = xf_n(x) + 2f_{n-1}(x)$;
- $xl_n(x) = f_{n+2}(x) - f_{n-2}(x)m$
- $l_{-n}(x) = (-1)^n l_n(x)$.

In addition $l_n(2) = f_{n+1}(2) + f_{n-1}(2) = P_{n+1} + P_{n-1}$ where P_n denotes the nth Pell number.

Example 3.6. $l_5(2) = 82 = 70 + 12 = P_6 + P_4$.

Arranging the coefficients of the various polynomials in ascending order of exponents, we get the array in Table 4. The sum of the element along the nth rising diagonal is $3 \cdot 2^{(n-2)/2}$, where $n \geq 2$ is even.

3.2.1 BINET'S FOFMULAS FOR $f_n(x)$ AND $l_n(x)$

Next we find Binet's formulas for both Fibonacci and Lucas polynomials.
Let $\alpha(x)$ and $\beta(x)$ be the solutions of the quadratic equation $t^2 - xt - t = 0$

$$\alpha(x) = \frac{x+\sqrt{x^2+4}}{2} \text{ and } \beta(x) = \frac{x-\sqrt{x^2+4}}{2}.$$

Notice that $\alpha(1) = \alpha$ and $\beta(1) = \beta$; $\alpha(2) = 1 + \sqrt{2}$ and $\beta(2) = 1 - \sqrt{2}$ are the eharacteristic roots of the fell recurrence realation $x^2 - 2x - 1 = 0$. We have

Theorem 3.6. $f_n(x) = \frac{\alpha^n(x)-\beta^n(x)}{\alpha(x)-\beta(x)}$ *and* $l_n = \alpha^n(x) - \beta^n(x)$.

Proof. Let

$$\alpha = \alpha(x) \ \ and \ \ \beta = \beta(x)$$

Then $l_1(x) = \alpha + \beta = x$; $l_2(x) = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = x^2 - 2(-1) = x^2 + 2$.

And

$$\begin{aligned}
xl_{n-1}(x) + l_{n-2}(x) &= x(\alpha^{n-1} + \beta^{n-1}) + (\alpha^{n-2} + \beta^{n-2}) \\
&= \alpha^{n-2}(x\alpha + 1) + \beta^{n-2}(x\beta + 1) \\
&= \alpha^{n-2}\alpha^2 + \beta^{n-2}\beta^2 \\
&= \alpha^n + \beta^n \\
&= l_n(x).
\end{aligned}$$

Is the Lucas polynomial. □

3.2.2 OTHER FORMULA FOR POLYNOMIAL FIBONACCI AND LUCAS

Ther is polynomials $g_n(x)$ *and* $h_n(x)$ corespond for the Fibonacci and Lucas polynomials defined by

$$\begin{aligned}
g_0(x) &= 0 \quad g_1(x) = 1 \\
g_n(x) &= xg_{n-1}(x) - g_{n-2}(x) \quad n \geqslant 2 \\
h_0(x) &= 0 \quad h_1(x) = 1 \\
h_n(x) &= xh_{n-1}(x) - h_{n-2}(x) \quad n \geqslant 2.
\end{aligned}$$

Let$\gamma(x)$ *and* $\delta(x)$ the roots of quadratic equation $t^2 - xt + 1 = 0$.

Then

$$\gamma(x) = \frac{x + \sqrt{x^2 - 4}}{2} \ \ and \ \ \delta(x) = \frac{x - \sqrt{x^2 - 4}}{2}.$$

We have

Theorem 3.7. *(Hoggatt, Jr, Bicknell, and King,1972).*

$$f_{nk}(x) = \begin{cases} f_k(x).f_n(L_k(x)) & if \ \ k \ is \ odd; \\ f_k(x).g_n(L_k(x)) & otherwise. \end{cases}$$

Proof. The polynomials $f_n(x)$, $l_n(x)$, $g_n(x)$ and $h_n(x)$ yield interesting divisibility dividends. To extract some of them, recall that

$$\alpha(x) = \frac{x+\sqrt{x^2+4}}{2} \text{ and } \beta(x) = \frac{x-\sqrt{x^2+4}}{2}$$

Then

$$\begin{aligned} \alpha(L_{2m+1}(x)) &= \frac{L_{2m+1}}{(}x) + \sqrt{x^2+4}F_{2m+1}2 \\ &= \alpha^{2m+1}(x) \end{aligned}$$

by EX15. Similary.

$$\beta(L_{2m+1}(x)) = \beta^{2m+1}(x)$$

$$\begin{aligned} f_n(L_{2m+1}(x)) &= \frac{\alpha^n(L_{2m+1}(x)) - \beta^n(L_{2m+1}(x))}{\alpha(L_{2m+1}(x)) - \beta(L_{2m+1}(x))} \\ &= \frac{\alpha^{(2m+1)n}(x) - \beta^{(2m+1)n}(x)}{\alpha^{(2m+1)}(x) - \beta^{(2m+1)}(x)} = \frac{f_{(2m+1)n}(x)}{f_{(2m+1)}(x)} \end{aligned}$$

Similarly, using the identity $l_{2m}^2(x) - 4 = (x^2+4)f_{2m}^2(x)$, we can show that :

$$\gamma(l_{2m}(x)) = \frac{l_{2m}(x) + \sqrt{x^2+4}f_{2m}(x)}{2} = \alpha^{2m}(x)$$

$$\delta(l_{2m}(x)) = \frac{l_{2m}(x) - \sqrt{x^2+4}f_{2m}(x)}{2} = \beta^{2m}(x)$$

$$\begin{aligned} g_n(L_{2m}(x)) &= \frac{\gamma^n(L_{2m}(x)) - \delta^n(L_{2m}(x))}{\alpha(L_{2m}(x)) - \gamma(L_{2m}(x))} \\ &= \frac{\gamma^{2mn}(x) - \delta^{2mn}(x)}{\gamma^{2m}(x) - \delta^{2m}(x)} = \frac{f_{2mn}(x)}{f_{2m}(x)} \end{aligned}$$

□

We deducted for the theorem (3.7)

Corollary 3.4. $f_k(x)|f_{nk}(x)$

Example 3.7.

$$\begin{aligned} f_4(x) &= x^3 + 2x \\ &= (x)(x^2+2) \\ &= f_2(x) \cdot l_2(x) = f_2(x) \cdot f_2(l_2(x)) \end{aligned}$$

Corollary 3.5. $F_k|F_{nk}$

This follows from corollary (3.4), since $f_n(1) = F_k$; note that we already knew this from

Corollary 3.6.

$$f_{nk}(x) = f_k(x) \sum_{j=0}^{(n-1)/2} \binom{n-j-1}{j} (-1)^{(k+1)j} l_k^{n-2j-1}(x)$$

Example 3.8.

$$\begin{aligned} f_4(x) &= f_2(x) \cdot \sum_{0}^{0} \begin{pmatrix} 2-i \\ i \end{pmatrix} (-1)^i l_2^{2-2i}(x) \\ &= x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} l_2^1(x) = x[(x^2+2)] \\ &= x^3 + 2x \end{aligned}$$

as we found in the preceding chapter

We can also employ the polynomials $l_n(x)$ and $h_n(x)$ to derive similar divisibility properties. To this end, recall that

$$\alpha(l_{2m+1}(x)) = \alpha^{2m+1}(x) \text{ and } \beta(l_{2m+1}(x)) = \beta^{2m+1}(x)$$

$$l_n(l_{2m+1}(x)) = \alpha^{(2m+1)n}(x) + \beta^{(2m+1)n}(x) = l_{(2m+1)n}(x)$$

Since $\gamma(l_{2m+1}(x)) = \alpha^{2m}$ and $\delta(l_{2m}(x)) = \beta^{(2m)n}(x)$, it follows that

$$h_n(l_{2m}(x)) = \alpha^{2mn}(x) + \beta^{2mn}(x) = l_{2mn}(x)$$

Thus, we have the following theorem.

Theorem 3.8. $l_n(l_{2m-1}(x)) = l_{(2m-1)n}(x)$ *and* $h_n(l_{2m}(x)) = l_{2mn}(x)$

Corollary 3.7. $l_n(x)|l_{(2m-1)n}(x)$

$$\begin{aligned} l_{10}(x) = l_{5\cdot 2}(x) &= x^{10} + 10x^8 + 35x^6 + 50x^4 + 25x^2 + 2 \\ &= (x^2+2)(x^8 + 8x^6 + 20x^4 + 16x^2 + 1) \\ &= l_2(x) \cdot [l_8(x) - 1] \end{aligned}$$

So $l_2(x)|l_{5\cdot 2}(x)$

Since $l_n(1) = L_n$, this corollary yields the following result

Corollary 3.8. $L_n|L_{(2m-1)n}$

Chapter 4

JACOBSTHAL POLYNOMIALS

Jacobsthal polynomials, $J_n(x)$, are related to Fibonacci polynomials. They are defined by

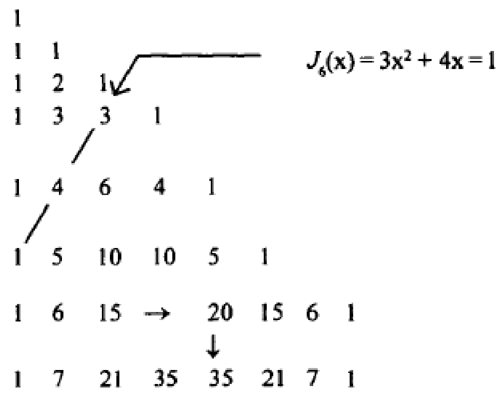
$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x) \tag{4.1}$$

where $J_1(x) = 1 = J_2(x)$. Clearly, $J_n(1) = F_n$, The first 8 Jacobsthal polynomials are

$$\begin{aligned}
J_1(x) &= 1 \\
J_2(x) &= 1 \\
J_3(x) &= x + 1 \\
J_4(x) &= 2x + 1 \\
J_5(x) &= x^2 + 3x + 1 \\
J_6(x) &= 3x^2 + 4x + 1 \\
J_7(x) &= x^3 + 6x^2 + 5x + 1 \\
J_8(x) &= 4x^3 + 10x^2 + 6x + 1
\end{aligned}$$

We have the following observation

- The Jacobsthal polynomials $J_{2n-1}(x)$ and $J_{2n}(x)$ have the same degree.
- The degree of $J_n(x)$ is $\lfloor (n-1)/2 \rfloor$.
- The leading coefficient of $J_{2n-1}(x)$ is one, whereas that of $J_{2n}(x)$ is n.
- The coefficients of $J_{2n-1}(x)$ are the same of those of $f_n(x)$, but in the reverse order.
- The coefficients of the Jacobsthal polynomials lie on the rising diagonals of the left-justified Pascal's triangle, in the reverse order, as shown below :



4.1 EXPLICIT FORMULA FOR $J_n(x)$

It is easy to prove that

$$J_n(x) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{\lfloor n/2 \rfloor + j}{(\lfloor n-1 \rfloor/2) - j} x^{\lfloor (n-1)/2 \rfloor - j}. \tag{4.2}$$

Example 4.1.

$$\begin{aligned} J_5(x) &= \sum_0^2 \binom{2+j}{2-j} x^{2-j} \\ &= \binom{2}{2} x^2 + \binom{3}{1} x^1 + \binom{4}{0} x^0 \\ &= x^2 + 3x + 1. \end{aligned}$$

Similarly

$$J_7(x) = x^3 + 6x^2 + 5x + 1.$$

4.2 BINET'S FORMULA FOR $J_n(x)$

Now, we find Binet's formula for $J_n(x)$. To this end, let r and s be the solutions of the characteristic equation $t^2 - t - x = 0$ of the recurrence relation Eq (4.1). Then

$$r = \tfrac{1+\sqrt{1+4x}}{2} \quad s = \tfrac{1-\sqrt{1+4x}}{2}$$
$$r + s = 1 \ rs = -x \text{ and } r - s = \sqrt{1+4x}.$$

Thus implies that

$$J_n(x) = \frac{r^n - s^n}{\sqrt{1+4x}}, \qquad n \geq 1 \tag{4.3}$$

(4.3)is the Binet's formula for$J_n(x)$

4.3 ANOTHER FAMILY OF POLYNOMIALS $K_n(x)$

Next, we introduce yet another family of polynomials $K_n(x)$, which are closely related to Jocobsthal polynomials.

We define the polynomials $K_n(x)$ by

$$K_n(x) = K_{n-1}(x) + xK_{n-2}(x) \tag{4.4}$$

where $K_1(x) = 1$ and $K_2(x) = x$. The first 6 membres of this family are :

$$\begin{aligned} K_1(x) &= 1 \\ K_2(x) &= x \\ K_3(x) &= 2x \\ K_4(x) &= x^2 + 2x \\ K_5(x) &= 3x^2 + 2x \\ K_6(x) &= x^3 + 5x^2 + 2x. \end{aligned}$$

The use of plynomais $J_n(x)$ and $K_n(x)$ allows us to find recover formulas concering the Fibonacci and lucas numbers.But we can't do all the detail becaus this will make the memory even bigger.

Conclusion

In this memory, we study the Fibonacci and Lucas series and determine their properties and the relationship between them. We also study the Fibonacci and Lucas polynomials and Jacobstahl and their importance. Work remains to study the properties of the Fibonacci polynomials, Lucas and Jacobstahl.

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ملخص

في هذه المذكرة، قمنا بالتذكير أولا بأرقام فيبوناشي ولوكاس والخصائص المرتبطة بها. بعد ذلك ، ندرس كثيرات حدود فيبوناتشي ولوكاس وجاكوب ستال وبعض تطبيقاتها ؛ ولا سيما إعادة اكتشاف بعض الخصائص المتعلقة بأرقام فيبوناتشي ولوكاس.

كلمات مفتاحية

كثير حدود، فيبوناشي، لوكاس، جاكوب ستال، اعداد فيبوناشي، اعداد لوكاس.

Abstract

In this memory, we recalled, first of all, Fibonacci and Lucas numbers and the identities linked to them. Next, we study Fibonacci, Lucas and Jacobsthal polynomials and some of their applications ; in particular the rediscovery of certain identities concerning Fibonacci and Lucas numbers.

Key words

Polynomail, Fibonacci, Lucas, Jacobsthal, Fibonacci numbers, Lucas numbers.

Résumé

Dans ce mémoire on a donné, tout d'abord, un rappel sur les nombres de Fibonacci, Lucas et les identités qui leur sont liées. Ensuite, nous avions fait une étude sur les polynômes de Fibonacci, Lucas et Jacobsthal et certaines de leurs applications ; en particulier la redécouverte de certaines identités concernant les nombres de Fibonacci et Lucas.

Mot-clés

Polynôme, Fibonacci, Lucas, Jacobsthal, nombres des Fibonacci, nombres des Lucas.