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Etude de quelques propriétés classiques dans le cas non-linéaire

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ملخص
في هذه الأطروحة قدمنا وعرفنا مفهوم إستمرار المؤثرات متعددة الخطية المعرف على فضاءات نظيمية لامتماثلة.
ومنها توصلنا إلى خصائص الفضاءات الناتجة لهذه التطبيقات
كتطبيق لهذا المفهوم قمنا بإثبات كل من نظرية بناخ شتينهاوس ونظرية البيان المغلق بالنسبة للتطبيقات متعددة الخطية المستمرة المعرفة على فضاءات نظيمية لامتماثلة.
الهدف الثاني لهذه الأطروحة هو عرض مفهوم خطية للمؤثرات متعددة الخطية المستمرة المعرفة على فضاءات نظيمية لامتماثلة عبر فضاء الجداء الموتري لعدة فضاءات نظيمية لامتماثلة .

الكلمات المفتاحية:
نظيم لا متماثل، مؤثر متعدد الخطية المستمرة، نظرية بناخ ستينهاوس، نظرية البيان المغلق، نظيم لامتماثل موتري.

Abstract
In this thesis we introduce and characterize the continuity of multilinear mappings between asymmetric normed spaces. In particular, we study the completeness properties of the asymmetric normed semi-vector space of these mappings. As an application, we prove multilinear versions of the Banach-Steinhaus and closed graph theorems in the framwork of asymmetric normed spaces. The second purpose of the thesis is to present the concept of linearization of continuous multilinear operators, on asymmetric normed spaces, through the tensor product of asymmetric normed spaces.

Ky words: Asymmetric norm, continuous multilinear operator, Banach-Steinhaus theorem, closed graph theorem, tensor asymmetric norm.

Résumé
Dans cette thèse, nous introduisons et caractérisons la continuité des opérateurs multilinéaires entre espaces asymétriques normés. Nous étudions en particulier les propriétés de complétude de semi-espace vectoriel asymétrique normé de ces opérateurs. Comme application, nous prouvons des versions multilinéaires des théorèmes de Banach-Steinhaus et de graphe fermé dans le cadre des espaces asymétriques normés. Le deuxième objectif de la thèse est de présenter le concept de linéarisation d'opérateurs multilinéaires continus sur des espaces asymétriques normés, à travers le produit tensoriel des espaces asymétriques normés.

Mots clés: Asymétrique norme, opérateur multilinéaire continu, théorème de Banach-Steinhaus, théorème de graphe fermé, asymétrique norme tensorielle.

Contents

0.1 Introduction

The main topic treated in this thesis is the study of the continuity of multilinear mappings between asymmetric normed spaces, following the classical scheme of linear operators. It is difficult to localize the first moment when asymmetric norms were used, but it goes back as early as 1968 in a paper by Duffin and Karlovitz (1968) [16], who proposed the term asymmetric norm. Talking about Asymmetric Topology, there are two basic references: the book of Murdeshwar and Naimpally [36] and the book of Fletcher and Lindgren [19], the survey paper [8] may be viewed as a skeleton of this book.

A systematic study of the properties of asymmetric normed spaces started with some papers ([38], [39], [40], [41]) of S. Romaguera from the polytechnic university of Valencia and his collaborators from other universities in spain: Alegre, Ferrer, Garcia-Raffi, Sánchez Pérez, Sánchez Alvarez, Sanchis, Valero.

One of the recent advances in Computer Science was due to the possibility of establishing a Mathematical model that account the distance between algorithms and programs when they are analyzed in terms of their computational complexity (complexity distance), where computational complexity is interpreted in terms of running time, for example. Several authors have done a big effort in obtaining a robust Mathematical theory, which was a useful tool that played, in this context, a similar role that normed vector spaces have played in different scientific areas. In the context of Computational Complexity, it is shown that asymmetric normed vector spaces constitute a very satisfactory model (see [20]).

An asymmetric metric (also known as a quasi metric) is a mapping satisfying all the axioms of a metric except the symmetry condition. An asymmetric norm on a real vector space X is a positive function satisfies the subadditivity, the homogeneity and that $p(x) = p(-x) = 0$ implies $x = 0$.

Concerning the continuity of linear operators between asymmetric normed spaces, in spite of the existing differences, some results from the symmetric case have their counterparts in the asymmetric one, a study that was initiated in [23].

The characterization of the compactness and precompactness of subsets on asymmetric normed vector spaces can be found in [21] and [3]. More information on the structure of these spaces and compact sets on them (with respect to different topologies) can be found in [11].

Our main motivation is to show that the results that worked for the case of linear operators between asymmetric normed spaces could be extended to the multilinear operators, so the main goal of this thesis is to introduce and study the continuity of multilinear mappings on asymmetric normed spaces. As far as we know that is a first attempt in this regard. Let $X_1, ..., X_m$ and Y be vector spaces and T a mapping from $X_1 \times ... \times X_m$ into Y. We may fix $m-1$ coordinates and so obtain a mapping from an X_i into Y. If such a mapping is linear for each X_i, then T is said to be m-linear (multilinear). We give some characterizations, for the class of continuous multilinear operators, by asymmetric norm inequalities similar to linear case and using the notion of N-asymmetric norm. Also we prove some fundamental theorems concerning this mappings in the framwork of asymmetric normed spaces.

A second purpose of the thesis is to present the concept of linearization of continuous multilinear operators, on asymmetric normed spaces, through the m-fold tensor product $X_1 \otimes ... \otimes X_m$ of N-asymmetric normed spaces $X_1, ..., X_m$. We show that the continuous multilinear operators T, on asymmetric normed spaces, associated with a unique continuous linear mapping T_L defined on $X_1 \otimes ... \otimes X_m$, endowed with a projective N-asymmetric norm. The results related to the symmetric case for order $m > 2$ have been treated in some classical works. For example, in the symmetric case, we can found some details about these concepts in [42] and for $m = 2$ in [44] or [14].

The thesis consists of four chapters. In Chapter 1 we establish the notation of the thesis. We introduce some important results concerning asymmetric norm and continuous linear mappings between asymmetric normes spaces and we recall the main definitions and properties of the continuous multilinear mappings on normed spaces.

In Chapter 2 of this thesis we give a result that gives the characterization of the continuous multilinear mappings between asymmetric normes spaces. We study the completeness properties of the asymmetric normed semi-vector space of these mappings.

As an application, in Chapter 3, using the asymmetric version of the Banach-Steinhaus theorem for linear operators we prove that the separately continuity of a multilinear operator implies the continuity of this mapping between asymmetric normed spaces. Also we present a multilinear versions of the Banach-Steinhaus and closed graph theorems in asymmetric normed spaces.

In the last chapter (Chapter 4) we recall the most important results for the algebraic theory of tensor product of vector spaces. We introduce an asymmetric norm on tensor product of vector spaces to realized a linearization of the continuous multilinear mappings defined in this tensor product.

Chapter 1

Asymmetric normed spaces.

Contents of the chapter:

1) Quasi-metric spaces.

2) Asymmetric normed spaces.

3) Multilinear linear operators between normed spaces.

In this chapter we present the concepts and results used throughout the thesis on asymmetric normed spaces and multilinear linear operators between normed spaces.

1.1 Quasi-metric spaces

In the following, let X be a non-empty set.

Definition 1.1.1 *A quasi-metric on X is a function $d : X \times X \longrightarrow \mathbb{R}^+$ with the following properties*

1. *for all $x, y \in X$ we have $d(x, y) = d(y, x) = 0$ if and only if $x = y$.*

2. *for all $x, y, z \in X$ we have $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inquality).*

The pair (X, d) is called quasi-metric space.

If $d(x, y) = d(y, x) = 0$ does not imply $x = y$ for some $x, y \in X$, the function d is called a quasi-semimetric, and the pair (X, d) is called a quasi-semimetric space.

Remark 1.1.2 *If d is a quasi-metric on X, then the function $\overline{d}$ defined on $X \times X$ by*

$$\overline{d}(x, y) = d(y, x), \quad x, y \in X,$$

is a quasi-metric on X called the conjugate of d.

And the function d^s defined on $X \times X$ by

$$d^s(x, y) = \max\{d(x, y), \overline{d}(x, y)\}, \quad x, y \in X,$$

is a metric on X.

Definition 1.1.3 (Balls) *Let (X, d) be a quasi-metric space, $x_0 \in X$ and $r > 0$. The open ball, of radius r centered at x_0, is the set defined by*

$$B_d(x_0, r) = \{x \in X : d(x_0, x) < r\}.$$

The closed ball of radius r centered at x_0 is defined by

$$B_d[x_0, r] = \{x \in X : d(x_0, x) \leq r\}.$$

Let $x \in X$, we say that a set $V \subset X$ is a neighborhood of the point x if and only if

$$\exists r > 0 : B_d(x, r) \subset V.$$

The collection neighborhoods of the point x is denoted by $V_d(x)$.

The topology τ_d of a quasi-metric space (X, d) can be defined starting from the family $V_d(x)$.

Definition 1.1.4 *The convergence of a sequence $(x_n)_n$ to x with respect to τ_d, called d-convergence, in symbols $x_n \xrightarrow{d} x$, can be characterized in the following way:*

$$x_n \xrightarrow{d} x \Longleftrightarrow d(x, x_n) \longrightarrow 0.$$

Proposition 1.1.5 *[7, Page 4] Let $(x_n)_n$ be a sequence in a quasi-metric space (X, d).*

1. If $(x_n)_n$ is d-convergent to x and $\overline{d}$-convergent to y, then $d(x, y) = 0$.

2. If $(x_n)_n$ is d-convergent to x and $d(y, x) = 0$, then $(x_n)_n$ is also d-convergent to y.

1.2 Asymmetric normed spaces

For the general theory of asymmetric normed spaces we refer the reader to the monograph [7]. In the following, let X be a real vector space.

Definition 1.2.1 *A function $p : X \longrightarrow \mathbb{R}^+$ is an asymmetric norm on the the real vector space X if for every $x, y \in X$ and $\alpha \in \mathbb{R}^+$ we have*

1. *$p(x) = p(-x) = 0$ if and only if $x = 0$*

2. *$p(\alpha x) = \alpha p(x)$*

3. $p(x+y) \leq p(x)+p(y)$.

We say that the pair (X,p) is an asymmetric normed space.

If $p(x)=p(-x)=0$ does not imply $x=0$ for some $x \in X$ (i.e. satisfies only the conditions 2) and 3)), the function p is called a asymmetric seminorm, and the pair (X,p) is called an asymmetric seminormed vector space.

Asymmetric norm conjugate and symmetrization

The asymmetric norm conjugate to p is the function $\overline{p} : X \longrightarrow \mathbb{R}^+$ defined by $\overline{p}(x) = p(-x)$. As a consequence, the asymmetric norm p induces a norm p^s defined on X by the formula $p^s(x) = \max\{p(x), p(-x)\}$, this norm is referred to as the symmetrization of the asymmetric norm p.

Topology of asymmetric norm

Every asymmetric norm p, on a vector space X, induces a quasi-metric d_p on $X \times X$ defined by

$$d_p(x,y) = p(y-x), \quad x,y \in X$$

If p is an asymmetric norm on X, then the topology τ_{d_p} will be simply denoted by τ_p and we will say that τ_p is the topology induced by p.

τ_p is a T_0 topology on X, that is for any pair x,y of distinct points in X, at least one of them has a neighborhood not containing the other. The topology τ_p is generated by the asymmetric open balls $B_p(x,\varepsilon) = \{y \in X : p(y-x) < \varepsilon\}$, where $\varepsilon > 0$.

Moreover the collection $\{B_p(x,\varepsilon) : \varepsilon > 0\}$ forms a fundamental system of neighborhoods for the topology τ_p. However, in general this topology is not Hausdorff (see [24]).

Proposition 1.2.2 *[7] If (X,p) is an asymmetric space, then any ball $B_p(x_0,r)$ is open in the topology τ_p and any ball $B_p[x_0,r]$ is closed in the topology $\tau_{\overline{p}}$.*

Also, the following inclusions hold:

$$B_{p^s}(x_0, r) \subset B_p(x_0, r) \text{ and } B_{p^s}(x_0, r) \subset B_{\overline{p}}(x_0, r),$$

with similar inclusions for the closed balls.

Convergence in asymmetric normed space

A sequence $(x_n)_n$ in an asymmetric normed space (X, p) is convergent to $x \in X$ with respect to τ_p if and only if $\lim_{n \longrightarrow +\infty} p(x_n - x) = 0$. From this we obtain the following result (see [34, Remark 1.1]).

Proposition 1.2.3 *Let Z be a linear subspace of X. Then Z is closed in (X, p) if and only if it is closed in $(X, \overline{p})$.*

Completeness in asymmetric normed spaces

There are several notions of Cauchy sequence and more notions of completeness in asymmetric normed spaces (see [7] and [22]). We present only the following notions:

- A sequence $(x_n)_n$ of elements of X is said to be left (right) K-Cauchy if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $p(x_n - x_k) < \varepsilon$ (resp. $p(x_k - x_n) < \varepsilon$) whenever $n \geq k \geq n_0$.

- (X, p) is called left (right) K-complete if every left (right) K-Cauchy sequence in X is convergent whith respect to τ_p.

- (X, p) is called bicomplete (or bi-Banach) if the normed space (X, p^s) is complete.

Example 1.2.4 *As an important example, let u the asymmetric norm on the real vector space $\mathbb{R}$ defined by*

$$u(x) := x^+ = \max\{x, 0\}. \tag{1.1}$$

In this case $\overline{u}(x) = \max\{-x, 0\} = x^-$ *and* $u^s(x) = \max\{-x, x\} = |x|$. *Obviously* $(\mathbb{R}, u)$ *is a bi-Banach space.*

The asymmetric norm u is called *usual asymmetric norm.*

1.3 Continuous linear operators.

Let (X, p) and (Y, q) two asymmetric normed spaces. We denote by $LC(X, Y)$ the set of all continuous linear mappings from (X, p) into (Y, q) and by $LC^s(X, Y)$ the set of all continuous linear mappings from normed space (X, p^s) to normed space (Y, q^s).

Definition 1.3.1 *A linear map* $T : (X, p) \longrightarrow (Y, q)$ *is called bounded if there exist a positive constant* K *such that*

$$q(T(x)) \leq Kp(x),$$

for all $x \in X$.

The next result and its consequences can be found in [18] or [23] and will be used in the sequel.

Theorem 1.3.2 $T \in LC(X, Y)$ *if and only if* T *is bounded.*

The following example show that the set $LC(X, Y)$ is not necessarily a vector space but it is a cone (or normed semi-vector space). That is, $T+S \in LC(X, Y)$ and $\alpha T \in LC(X, Y)$ for all $T, S \in LC(X, Y)$ and $\alpha \geq 0$.

Example 1.3.3 *Let id be the identity function from* $(\mathbb{R}, u)$ *into itself. Clearly id is a continuous linear function but* $-id$ *is not continuous, because if* $x < 0, u(-x) = -x$, *so*

$$\sup\{u(-x) : u(x) \leq 1\} = \infty$$

Thus we conclude that $LC(X, Y)$ *is not a vector space in general.*

Following [23, Theorem 1], we can consider an asymmetric norm on the cone $LC(X,Y)$ of all linear continuous mappings T from (X,p) into (Y,q) defined by the formula

$$p_q^*(T) := \sup\{q(T(x)) : p(x) \leq 1\}$$

and also

$$p_q^*(T) = \inf\{K > 0 : q(T(x)) \leq Kp(x)\}.$$

Proposition 1.3.4 *If the linear map* $T : (X,p) \longrightarrow (Y,q)$ *is continuous, then* $T : (X,\overline{p}) \longrightarrow (Y,\overline{q})$ *is continuous. Hence* $LC(X,Y) \subseteq LC^s(X,Y)$.

Proof. Let $T \in LC(X,Y)$. Then there is $K > 0$ such that

$$\overline{q}(T(x)) = q\left(T(-x)\right) \leq K\overline{p}(x).$$

Therefore T is continuous from $(X,\overline{p})$ to $(X,\overline{q})$, hence

$$q^s(T(x)) \leq \max\{Kp(x), K\overline{p}(x)\} = Kp^s(x)$$

We conclude that $LC(X,Y) \subseteq LC^s(X,Y)$. ■

Dual of an asymmetric normed space

A relevant special case of continuous linear operator between asymmetric normed spaces arises when we take $(Y,q) = (\mathbb{R},u)$. In this case p_u^* will be simply denoted by p^* and we put

$$X^* = \{T : (X,p) \longrightarrow (\mathbb{R},u), T \text{ is linear and continuous}\}.$$

This cone is referred to as the dual space of (X,p). Observe that (X^*,p^*) is a bi-Banach cone where $p^* := p_u^*$ (see [23, Theorem 1]).

1.4 Example of a bi-Banach space

In what follows, we study and detail an important example given in [20].

For $1 \leq p < \infty$, we will denote by ℓ_p the set of infinite sequences $\mathbf{x} := (x_n)_n \subset \mathbb{R}$ such that

$$\ell_p = \left\{ (x_n)_n \subset \mathbb{R} : \sum_{n=0}^{\infty} |x_n|^p < \infty \right\}.$$

It is well known that $(\ell_p, \|\cdot\|_p)$ is a Banach space, where $\|\cdot\|_p$ is the norm on ℓ_p defined by

$$\|x\|_p = \left(\sum_{n=0}^{\infty} |x_n|^p\right)^{\frac{1}{p}}, \quad x = (x_n)_n \in \ell_p.$$

Fix $p \geq 1$, for each $x = (x_n)_n \in \ell_p$ define

$$x^+ := (x_n^+)_n \quad \text{and} \quad \|x\|_{+p} = \|x^+\|_p := \left(\sum_{n=0}^{\infty} (x_n^+)^p\right)^{\frac{1}{p}}.$$

We will show that $\|\cdot\|_{+p}$ is an asymmetric norm on ℓ_p, such that the norm $\left(\|\cdot\|_{+p}\right)^s$ is equivalent to $\|\cdot\|_p$. We need the following lemma.

Lemma 1.4.1 *For $x = (x_n)_n \in \ell_p$, $y = (y_n)_n \in \ell_p$ and $a \in \mathbb{R}^+$ the following statements hold*

(a) $x = x^+ - (-x)^+$.

(b) $(ax)^+ = ax^+$.

(c) $(x_n + y_n)^+ \leq x_n^+ + y_n^+$ *for all* $n \in \mathbb{N}$.

Proposition 1.4.2 *For each $p \geq 1$, $\| \cdot \|_{+p}$ is an asymmetric norm on ℓ_p.*

Proof. Let $x = (x_n)_x \in \ell_p$ such that $\|x\|_{+p} = \| - x\|_{+p} = 0$. Then $x^+ = (-x)^+$ and by (a) in the previous lemma, $x = 0$. On the other hand, it is clear that $\|\mathbf{0}\|_{+p} = \mathbf{0}$.

Now let $a \in \mathbb{R}^+, x = (x_n)_x \in \ell_p$ and $y = (y_n)_n \in \ell_p$, then $\|(ax)\|_{+p} = \|(ax)^+\|_p = a\|x\|_{+p}$, by (b) in the previous lemma.

Finally by (c) in the previous lemma we have

$$\|x+y\|_{+p} = \|(x+y)^+\|_p \leq (\sum_{n=0}^{\infty}(x_n^+ + y_n^+)^p)^{1/p}$$

and so

$$\|x+y\|_{+p} \leq \|x^+ + y^+\|_p \leq \|x^+\|_p + \|y^+\|_p = \|x\|_{+p} + \|y\|_{+p}.$$

■

Corollary 1.4.3 $(\ell_p, \|\cdot\|_{+p})$ *is an asymmetric normed linear space, for each* $p \geq 1$.

Proposition 1.4.4 *For each* $p \geq 1$ *we have*

$$(\|x\|_{+p})^s \leq \|x\|_p \leq \|x\|_{+p} + \|-x\|_{+p},$$

whenever $x \in \ell_p$.

Proof. Let $x = (x_n)_x \in \ell_p$. Then, it is clear that

$$(\|x\|_{+p})^s = max\{\|x\|_{+p}, \|-x\|_{+p}\} \leq \|x\|_p.$$

Finally, by (a) in the previous lemma, we obtain

$$\|x\|_p = \|x^+ - (-x)^+\|_p \leq \|x^+\|_p + \|(-x)^+\|_p = \|x\|_{+p} + \|-x\|_{+p}.$$

■

Corollary 1.4.5 *For each* $p \geq 1$,

$$(\|\cdot\|_{+p})^s \leq \|\cdot\|_p \leq 2(\|\cdot\|_{+p})^s.$$

Therefore $(\|\cdot\|_{+p})^s$ *and* $\|\cdot\|_p$ *are equeivalent norms in* ℓ_p.

Corollary 1.4.6 *For each* $p \geq 1$, $(\ell_p, \|\cdot\|_{+p})$ *is a bi-Banach space.*

1.5 Multilinear linear operators between normed spaces.

Let $m \in \mathbb{N}$ and consider $X_j (j = 1, ..., m)$, Y the normed spaces over $\mathbb{K}$, (either $\mathbb{R}$ or $\mathbb{C}$). A mapping $T : X_1 \times ... \times X_m \longrightarrow Y$ is called multilinear (or m-linear) if the mappings

$$\begin{array}{rcl} T_j : \; X_j & \longrightarrow & Y \\ x^j & \longmapsto & T(x^1, ..., x^j, ..., x^m), \end{array}$$

are linear for each set of fixed $x^k \in X_k, k \neq j$, i.e.

$$T(x^1, ..., \lambda x^j + y^j, ..., x^m) = \lambda T(x^1, ..., x^j, ..., x^m) + T(x^1, ..., y^j, ..., x^m),$$

for all $\lambda \in K$ and $x^j, y^j \in X_j (j = 1, ..., m)$.
The vector space of such mappings is denoted by $L(X_1, ..., X_m; Y)$. If $Y = \mathbb{K}$, we write $L(X_1, ..., X_m)$.

Remark 1.5.1 *The set $\mathcal{S}$ of all vectors in Y of the form $T(x^1, ..., x^m)$, $x^j \in X_j (j = 1, ..., m)$ is not in general a vector subspace of Y. In order to see this, let $X_1 = X_2$ and Y be vector spaces such that $\dim(X_1) = \dim(X_2) = 2$ and $\dim(Y) = 4$. Select a basis $\{a_1, a_2\}$ in X_1 and a basis $(e_i)_{i=1}^4$ in Y and define the bilinear mapping T by*

$$T(x^1, x^2) = \xi_1 \eta_1 e_1 + \xi_1 \eta_2 e_2 + \xi_2 \eta_1 e_3 + \xi_2 \eta_2 e_4$$

where $x^1 = \xi_1 a_1 + \xi_2 a_2$ and $x^2 = \eta_1 a_1 + \eta_2 a_2$. It is easy to see that

$$\mathcal{S} = \left\{ z = \sum_{i=1}^{4} \lambda_i e_i \in Y : \lambda_1 \lambda_4 - \lambda_2 \lambda_3 = 0 \right\}$$

Let $z_1 = 2e_1 + 2e_2 + e_3 + e_4$ and $z_2 = e_1 + e_3$. It is clear that $z_1, z_2 \in \mathcal{S}$ but $z_1 - z_2 \notin \mathcal{S}$, it follows that $\mathcal{S}$ is not a subspace of Y. (see [26, section 1.1]).

Let us consider the space $X_1 \times ... \times X_m$ endowed with the norms $\|\cdot\|_\infty$ and s defined by

$$\|x\|_\infty = \max_{1\leq j\leq m} \|x^j\| \quad \text{and} \quad s(x) = \sum_{j=1}^m \|x^j\|,$$

for all $x = (x^1, ..., x^m) \in X_1 \times ... \times X_m$.

Definition 1.5.2 *An m-linear mapping $T : X_1 \times ... \times X_m \longrightarrow Y$ is continuous if it is continuous as a function between two normed spaces.*

As a consequence of this definition, similar to the linear case, we have a result that gives the characterization of the continuous m-linear mapping.

Theorem 1.5.3 *Let $X_1, ..., X_m, Y$ be normed spaces. For $T \in L(X_1, ..., X_m; Y)$ the following assertions are equivalent.*

(1) T is continuous.

(2) T is continuous in $(0, ..., 0)$.

(3) There is a constant $K \geq 0$ with

$$\left\|T(x^1, ..., x^m)\right\| \leq K \left\|x^1\right\| ... \|x^m\|, \tag{1.2}$$

for all $x^j \in X_j (j = 1, ..., m)$.

(4) $\|T\| := \sup_{\|x^j\|\leq 1, j=1,...,m} \|T(x^1, ..., x^m)\| < \infty$.

We will write $\mathcal{L}(X_1, ..., X_m; Y)$ for the vector space of all continuous m-linear mappings. If $Y = \mathbb{K}$, we write $\mathcal{L}(X_1, ..., X_m)$.
It is easy to see that

$$\|T\| = \inf \{K \geq 0, \text{ verifying the inequality } (1.2)\},$$

defines a norm on $\mathcal{L}(X_1, ..., X_m; Y)$ which is complete norm when $\|.\|_Y$ is complete. For the general theory of multilinear mappings we refer to [35] or [15].

Chapter 2

Continuous multilinear operators between asymmetric normed spaces

Contents of the chapter:

1) Characterization of continuous multilinear mappings.
2) Completeness properties.
3) Adjoint of multililinear mapping.

The results obtained in this chapter have been published in the Journal of Colloquium Mathematicum. In this chapter we introduce and characterize the continuity of multilinear mappings between asymmetric normed spaces. In particular, we study the completeness properties of the asymmetric normed semi-vector space of these mappings. Also we present the definition of adjoint of an m-linear mapping but the adjoint operator obtained in this way is additive, positively homogeneous defined from a bi-Banach cone to another bi-Banach cone.

2.1 Characterization of continuous multilinear mappings between asymmetric normed spaces.

We give a characterization of continuous multilinear mappings in a way analogous to that used to characterize linear mappings between asymmetric normed spaces. For studying the continuity of multilinear mappings between asymmetric normed spaces, we use the N-asymmetric norms instead of asymmetric norms. We will see why this in the Remark 2.1.5 after characterize the continuity by means of an inequality.

Definition 2.1.1 *An N-asymmetric norm is an asymmetric norm p on the real vector space X, for which $p(x) = 0$ implies $x = 0$. We say that the pair (X, p) is a N-asymmetric normed space.*

Example 2.1.2 *It is easy to see that the function*

$$p : \mathbb{R} \longrightarrow \mathbb{R}^+, \quad p(x) = |x| + \max\{x, 0\} \tag{2.1}$$

is an N-asymmetric (callad the usual N-asymmetric norm). More generally, we can define an N-asymmetric norm p on the Banach lattice X by the formula

$$p(x) = \|x\| + \|\max\{x, 0\}\|.$$

Throughout this chapter, (X_1, p_1)..., (X_m, p_m) will be real N-asymmetric normed spaces and (Y, q) be an asymmetric normed space. Let us consider the space $X_1 \times ... \times X_m$ endowed with the N-asymmetric norms p_∞ and s defined by

$$p_\infty(x) = \max_{1 \le j \le m} p_j(x^j) \quad \text{and} \quad s(x) = \textstyle\sum_{j=1}^m p_j(x^j),$$

for all $x = (x^1, ..., x^m) \in X_1 \times ... \times X_m$. We know that s and p_∞ are equivalent asymmetric norms on $X_1 \times ... \times X_m$ whose induced topology coincides with the product topology (see [1, Lemma 6]).

Definition 2.1.3 *An m-linear mapping $T : X_1 \times ... \times X_m \longrightarrow Y$ is continuous if it is continuous as a function between two asymmetric normed spaces.*

By $LC_{(p_1,...,p_m;q)}(X_1,...,X_m;Y)$ we denote the set of all continuous multilinear mappings between the N-asymmetric normed space $X_1 \times ... \times X_m$ and the asymmetric normed space Y and by $LC^s_{(p_1^s,...,p_m^s;q^s)}(X_1,...,X_m;Y)$ the normed vector space of all continuous multilinear operators between the normed vector spaces $(X_1, p_1^s), \times ... \times (X_m, p_m^s)$ and (Y, q^s).

Theorem 2.1.4 *Let $(X_1, p_1), ..., (X_m, p_m)$ be N-asymmetric normed spaces, (Y, q) be an asymmetric normed space and $T : X_1 \times, ..., \times X_m \longrightarrow Y$ be a multilinear mapping. The following statements are equivalent:*

(i) T is continuous.

(ii) T is continuous in $(0, ..., 0)$.

(iii) There is a constant $M \geq 0$ such that

$$q(T(x^1, ..., x^m)) \leq M p_1(x^1)...p_m(x^m), \tag{2.2}$$

for every $x^j \in X_j, j = 1, ..., m$.

(iv)

$$\|T|_{(p_1,...,p_m;q)} := \sup_{p_j(x^j)\leq 1, j=1,...,m} q(T(x^1, ..., x^m)) < \infty. \tag{2.3}$$

Proof. (i) implies (ii) is obvious.

$(ii) \Rightarrow (iii)$. Assume that T is continuous at $(0, ..., 0)$. Then we can choose $r > 0$ such that $T(B_{p_\infty}(0, r)) \subset B_q(0, 1)$. Since

$$p_\infty\left(\frac{rx^1}{2p_1(x^1)}, ..., \frac{rx^m}{2p_m(x^m)}\right) = \frac{r}{2} < r,$$

for all $(x^1,...,x^m) \in X_1\times,...,\times X_m$ with $x^j \neq 0$, $j=1,...,m$, one has

$$q\left(T(\frac{rx^1}{2p_1(x^1)},...,\frac{rx^m}{2p_m(x^m)})\right) < 1.$$

By multilinearity of T we obtain (2.2) with $M=\frac{2^m}{r^m}$. If $x^j=0$ for some $j=1,...,m$ we have $T(x^1,...,x^m)=0$ and the inequality (2.2) remains valid.

$(iii)\Rightarrow(i)$. Let us consider the space $X_1\times...\times X_m$ endowed with the asymmetric norm s. Let us fix $a=(a^1,...,a^m)\in X_1\times...\times X_m$ and we prove that $T(B_s(a,r))\subset B_q(T(a),\epsilon)$, for every $\epsilon>0$, where $r<\min\left\{1,\frac{\epsilon}{kM}\right\}$ and $k=\max_{1\leq j\leq m}\{p_\infty(a)^{j-1}(1+p_\infty(a))^{m-j}\}$. Let $y=T(z)\in T(B_s(a,r))$ with $z=(z^1,...,z^m)$. Then, using the inequality (2.2) and taking into account that

$$T(z)-T(a)=\sum_{j=1}^{m}T(a^1,...,a^{j-1},z^j-a^j,z^{j+1},...,z^m),$$

we obtain

$$\begin{aligned} q(y-T(a)) &\leq \textstyle\sum_{j=1}^m Mp_1(a^1)...p_{j-1}(a^{j-1})p_j(z^j-a^j)p_{j+1}(z^{j+1})...p_m(z^m)\\ &\leq \textstyle\sum_{j=1}^m Mp_j(z^j-a^j)p_\infty(a)^{j-1}p_\infty(z)^{m-j}. \end{aligned}$$

On the other hand, since $p_\infty\leq s$, we get that

$$p_\infty(z)\leq s(z-a)+p_\infty(a)<r+p_\infty(a)<1+p_\infty(a)$$

which yields

$$\begin{aligned} q(y-T(a)) &< \sum_{j=1}^{m}Mp_j(z^j-a^j)\left(1+p_\infty(a)\right)^{m-j}p_\infty(a)^{j-1}\\ &< \sum_{j=1}^{m}Mp_j(z^j-a^j)k\\ &= kMs(z-a)\\ &< kMr<kM\frac{\epsilon}{kM}=\epsilon. \end{aligned}$$

$(iii) \Rightarrow (iv)$. Starting from (2.2) we get

$$\|T|_{(p_1,\ldots,p_m;q)} \; : \; = \sup_{p_j(x^j)\leq 1, j=1,\ldots,m} q(T(x^1,...,x^m)) \leq \sup_{p_j(x^j)\leq 1, j=1,\ldots,m} Mp_1(x^1)...p_m(x^m) = M < \infty$$

$(iv) \Rightarrow (iii)$. If $p_j(x^j) = 0$ for some $j = 1, ..., m$, the the inequality (2.2) is evident. Suppose that $\|T|_{(p_1,\ldots,p_m;q)} < \infty$, then there exists a constant $M \geq 0$ such that

$$q(T(\frac{x^1}{p_1(x^1)}, ..., \frac{x^m}{p_m(x^m)})) \leq M,$$

for all $x^j \in X_j$ with $p_j(x^j) \neq 0, j = 1, ..., m$ and we obtain (2.2). ■

Remark 2.1.5 *For many asymmetric norms, there are no continuous multilinear mappings. Those are the ones that has may non-zero elements $x^1 \in X_1$ such that $p_1(x^1) = 0$ (i.e. p_1 is not an N-asymmetric norm). Indeed, if the m-linear mapping T is continuous, then $T(x^1, ..., x^m) = 0$ for all $x^j \in X_j$ with $j > 1$. This is obvious, since if $T(x^1, ..., x^m) \neq 0$ we have*

$$q\left(T(x^1, x^2, ..., x^m)\right) > 0 \quad or \quad q\left(T(x^1, -x^2, ..., x^m)\right) > 0$$

what contradicts that

$$0 < q\left(T(x^1, x^2, ..., x^m)\right) \leq Mp_1(x^1)p_2(x^2)...p_m(x^m) = 0$$

or

$$0 < q\left(T(x^1, -x^2, ..., x^m)\right) \leq Mp_1(x^1)p_2(-x^2)...p_m(x^m) = 0.$$

Now we give an easy example of a continuous bilinear mapping.

Example 2.1.6 *We can define the bilinear map $T : (\mathbb{R}, p) \times (\mathbb{R}, p) \longrightarrow (\mathbb{R}, u)$ by $T(x, y) = xy$, where u is the usual asymmetric norm on $\mathbb{R}$ defined by (1.1) and p is the usual N-asymmetric norm defined by (2.1). It is easy to see that*

$$u\left(T(x, y)\right) = \max\left\{xy, 0\right\} \leq |x|\,|y| \leq p(x)p(y),$$

for all $x, y \in \mathbb{R}$.

Proposition 2.1.7 *Let* $T \in LC_{(p_1,\dots,p_m;q)}(X_1, ..., X_m; Y)$. *Then,*

$$q(T(x^1, ..., x^m)) \leq \|T|_{(p_1,\dots,p_m;q)}\, p_1(x^1)...p_m(x^m), \tag{2.4}$$

for all $x^j \in X_j, j = 1, ..., m$. *Moreover* $\|T|_{(p_1,\dots,p_m;q)}$ *can be calculated also by the formula*

$$\|T|_{(p_1,\dots,p_m;q)} = \inf\left\{M \geq 0 :\ M \text{ satisfies } (2.2)\right\}. \tag{2.5}$$

Proof. For every $x^j \in X_j$ such that $x^j \neq 0$, $j = 1, ..., m$ we get, from (2.3),

$$q(T(\frac{x^1}{p_1(x^1)}, ..., \frac{x^m}{p_m(x^m)})) \leq \|T|_{(p_1,\dots,p_m;q)}$$

and we obtain the inequality (2.4). If $x^j = 0$ for some $j = 1, ..., m$, the inequality is obvious. On the other hand, if λ is the right side member of the equality (2.5) then it is clear that $\lambda \leq \|T|_{(p_1,\dots,p_m;q)}$. For the reverse inequality, if $M \geq 0$ satisfies (2.2), follows that

$$\|T|_{(p_1,\dots,p_m;q)} = \sup_{p_j(\ x^j) \leq 1, j=1,\dots,m} q(T(x^1, ..., x^m)) \leq M$$

and so $\|T|_{(p_1,\dots,p_m;q)} \leq \lambda$. ■

An immediate consequence of the Theorem 2.1.4 is the following corollary.

Corollary 2.1.8 *Let* $m \in \mathbb{N}$. *The following are equivalent for the multilinear mapping* $T : X_1 \times ... \times X_{2m+1} \longrightarrow Y$.

(i) T *is continuous from* $(X_1, p_1) \times ... \times (X_{2m+1}, p_{2m+1})$ *to* (Y, q).

(ii) T *is continuous from* $(X_1, \overline{p_1}) \times ... \times (X_{2m+1}, \overline{p}_{2m+1})$ *to* $(Y, \overline{q})$.

Consequently,

$$LC_{(p_1,\dots,p_{2m+1};q)}(X_1, ..., X_{2m+1}; Y) \subset LC^s_{(p^s_1,\dots,p^s_{2m+1};q^s)}(X_1, ..., X_{2m+1}; Y),$$

and

$$\|T\| := \|T\|_{(p_1^s,\dots,p_{2m+1}^s;q^s)} \le \|T|_{(p_1,\dots,p_{2m+1};q)} ,$$

for all $T \in LC_{(p_1,\dots,p_{2m+1};q)}(X_1,\dots,X_{2m+1};Y)$.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ it follow from

$$\begin{aligned}\sup_{\substack{\overline{p}_j(x^j)\le 1\\ j=1,\dots,2m+1}} \overline{q}(T(x^1,\dots,x^{2m+1})) &= \sup_{\substack{p_j(-x^j)\le 1\\ j=1,\dots,2m+1}} q(T(-x^1,\dots,-x^{2m+1}))\\ &= \sup_{\substack{p_j(x^j)\le 1\\ j=1,\dots,2m+1}} q(T(x^1,\dots,x^{2m+1})).\end{aligned}$$

With this we have $\|T|_{(p_1,\dots,p_{2m+1};q)} = \|T|_{(\overline{p}_1,\dots,\overline{p}_{2m+1};\overline{q})}$.
Now let $T \in LC_{(p_1,\dots,p_{2m+1};q)}(X_1,\dots,X_{2m+1};Y)$. For all $x^j \in X_j$ such that $p_j^s(x^j) \le 1$, $j = 1,\dots,2m+1$, we have

$$q(T(x^1,\dots,x^{2m+1})) \le \|T|_{(p_1,\dots,p_{2m+1};q)}$$

and

$$\overline{q}(T(x^1,\dots,x^{2m+1})) \le \|T|_{(\overline{p}_1,\dots,\overline{p}_{2m+1};\overline{q})} .$$

This implies that

$$\|T\| = \sup_{p_j^s(x^j)\le 1, j=1,\dots 2m+1} q^s(T(x^1,\dots,x^{2m+1})) \le \|T|_{(p_1,\dots,p_{2m+1};q)} < \infty$$

and the proof follows. ■

Remark 2.1.9 *As in the linear case, the set* $LC_{(p_1,\dots,p_m;q)}(X_1,\dots,X_m;Y)$ *is a cone (or normed semi-vector space), that is* $T+S, \alpha T \in LC_{(p_1,\dots,p_m;q)}(X_1,\dots,X_m;Y)$ *for all* T,S *belongs to* $LC_{(p_1,\dots,p_m;q)}(X_1,\dots,X_m;Y)$ *and all* $\alpha > 0$. *We have no proof for the fact that this set is a vector space or not.*

Now we introduce an asymmetric semi-norm on the cone $LC_{(p_1,...,p_m;q)}(X_1,...,X_m;Y)$. In the following proposition, the properties of the asymmetric semi-norm are easy to verify.

Proposition 2.1.10 *The number $\|\cdot|_{(p_1,...,p_m;q)}$ is an asymmetric semi-norm on the cone of all continuous multilinear mappings between N-asymmetric normed spaces $X_1 \times ... \times X_m$ and Y.*

As the linear case, we consider an extended asymmetric norm on the space $LC^s_{(p_1^s,...,p_m^s;q^s)}(X_1,...,X_m;Y)$ by the same formula

$$\|T|_{(p_1,...,p_m;q)} = \sup_{p_j(x^j)\leq 1} q\left(T(x^1,...,x^m)\right),$$

with the possibility that $\|T|_{(p_1,...,p_m;q)} = +\infty$.

With the asymmetric norm $\|\cdot|_{(p_1,...,p_m;q)}$ we associates an extended norm on $LC^s_{(p_1^s,...,p_m^s;q^s)}(X_1,...,X_m;Y)$ defined by

$$\|T|^s_{(p_1,...,p_m;q)} = \max\left\{\|T|_{(p_1,...,p_m;q)}, \|-T|_{(p_1,...,p_m;q)}\right\}.$$

Corollary 2.1.11 *For all $T \in LC^s_{(p_1^s,...,p_m^s;q^s)}(X_1,...,X_m;Y)$ we have*

$$\|T\| \leq \|T|^s_{(p_1,...,p_m;q)}. \tag{2.6}$$

Proof. In order to establish (2.6), we may suppose $\|T|^s_{(p_1,...,p_m;q)} < \infty$. Then we obtain $\|T|_{(p_1,...,p_m;q)} < \infty$ and $\|-T|_{(p_1,...,p_m;q)} < \infty$. For every $x^j \in X_j, j = 1,...,m$, we get

$$\begin{aligned} q(T(x^1,...,x^m)) &\leq \|T|_{(p_1,...,p_m;q)} p_1(x^1)...p_m(x^m) \\ &\leq \|T|^s_{(p_1,...,p_m;q)} p_1^s(x^1)...p_m^s(x^m) \end{aligned}$$

and

$$\begin{aligned} \overline{q}(T(x^1,...,x^m)) &\leq \|-T|_{(p_1,...,p_m;q)} p_1(x^1)...p_m(x^m) \\ &\leq \|T|^s_{(p_1,...,p_m;q)} p_1^s(x^1)...p_m^s(x^m). \end{aligned}$$

Consequently,

$$q^s(T(x^1,...,x^m)) \leq \|T|^s_{(p_1,...,p_m;q)} p_1^s(x^1)...p_m^s(x^m)$$

and thus $\|T\| \leq \|T|^s_{(p_1,...,p_m;q)}$. ■

2.2 Completeness properties

For studing the completeness properties of the cone $LC_{(p_1,...,p_m;q)}(X_1,...,X_m;Y)$ we need the following

Lemma 2.2.1 *The extended norm $\|\cdot|^s_{(p_1,...,p_m;q)}$ can be calculated by the following formula*

$$\|T|^s_{(p_1,...,p_m;q)} = \sup_{p_j(x^j)\leq 1, j=1,...,m} q^s(T(x^1,...,x^m)), \tag{2.7}$$

for all $T \in LC^s_{(p_1^s,...,p_m^s;q^s)}(X_1,...,X_m;Y)$.

Proof. Let α is the right side member of the equality (2.7). Then

$$\begin{aligned} \|-T|_{(p_1,...,p_m;q)} &= \sup_{p_j(\ x^j)\leq 1, j=1,...,m} q(-T(x^1,...,x^m)) \\ &\leq \sup_{p_j(x^j)\leq 1, j=1,...,m} q^s\ (T(x^1,...,x^m)) = \alpha \end{aligned}$$

and

$$\begin{aligned} \|T|_{(p_1,...,p_m;q)} &= \sup_{p_j(\ x^j)\leq 1, j=1,...,m} q(T(x^1,...,x^m)) \\ &\leq \sup_{p_j(x^j)\leq 1, j=1,...,m} q^s\ (T(x^1,...,x^m)) = \alpha. \end{aligned}$$

This implies $\|T|^s_{(p_1,...,p_m;q)} \leq \alpha$. Also, for all $x^j \in X_j$ such that $p_j(x^j) \leq 1, j = 1,...,m$, we have

$$q(T(x^1,...,x^m)) \leq \|T|_{(p_1,...,p_m;q)} \leq \|T|^s_{(p_1,...,p_m;q)}$$

and

$$q(-T(x^1,...,x^m)) \leq \|-T|_{(p_1,...,p_m;q)} \leq \|T|^s_{(p_1,...,p_m;q)}.$$

Then $\alpha \leq \|T|^s_{(p_1,...,p_m;q)}$. ■

Proposition 2.2.2 *Let m an odd natural number. The set $LC_{(p_1,...,p_m;q)}(X_1,...,X_m;Y)$ is closed in the space*

$$\left(LC^s_{(p_1^s,...,p_m^s;q^s)}(X_1,...,X_m;Y),\ \|\cdot|^s_{(p_1,...,p_m;q)}\right).$$

Proof. Let $(T_n)_n$ be a sequence in $LC_{(p_1,...,p_m;q)}(X_1,...,X_m;Y)$ that converges to $T \in LC^s_{(p_1^s,...,p_m^s;q^s)}(X_1,...,X_m;Y)$ with respect to the extended norm $\|\cdot|^s_{(p_1,...,p_m;q)}$. We will show that $T \in LC_{(p_1,...,p_m;q)}(X_1,...,X_m;Y)$. Indeed, there exist $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \|T|_{(p_1,...,p_m;q)} &\leq \|T|^s_{(p_1,...,p_m;q)} \\ &\leq \|T_{n_0} - T|^s_{(p_1,...,p_m;q)} + \|T_{n_0}|^s_{(p_1,...,p_m;q)} \\ &\leq 1 + \|T_{n_0}|^s_{(p_1,...,p_m;q)} < \infty. \end{aligned}$$

■

In the following theorem we show that the cone $LC_{(p_1,...,p_m;q)}(X_1,...,X_m;Y)$ is bi-Banach if Y is a bi-Banach asymmetric space.

Theorem 2.2.3 *Let $(X_1,p_1),...,(X_m,p_m)$ be N-asymmetric normed spaces, (Y,q) be an asymmetric normed space and assume that (Y,q) is bi-Banach. Then $LC^s_{(p_1^s,...,p_m^s;q^s)}(X_1,...,X_m;Y)$ is a bi-Banach space with respect to the asymmetric norm $\|\cdot|_{(p_1,...,p_m;q)}$. Consequently, if m is odd, $\left(LC_{(p_1,...,p_m;q)}(X_1,...,X_m;Y), \|\cdot|_{(p_1,...,p_m;q)}\right)$ is a bi-Banach cone.*

Proof. Consider a Cauchy sequence $(T_n)_n \subset LC^s_{(p_1^s,...,p_m^s;q^s)}(X_1,...,X_m;Y)$ with respect to the extended norm $\|\cdot|^s_{(p_1,...,p_m;q)}$. Hence for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|T_n - T_k\| \leq \|T_n - T_k|^s_{(p_1,...,p_m;q)} < \varepsilon,$$

for all $n, k \geq n_0$. Which means that $(T_n)_n$ is a Cauchy sequence in the Banach space

$$\left(LC^s_{(p_1^s,...,p_m^s;q^s)}(X_1, ..., X_m; Y), \|\cdot\|_{(p_1^s,...,p_m^s;q^s)}\right).$$

Thus, there exists $T \in LC^s_{(p_1^s,...,p_m^s;q^s)}(X_1, ..., X_m; Y)$ such that $\|T_n - T\| \longrightarrow 0$. As $T_n - T$ is continuous we get that

$$q^s\left(T_n(x^1, ..., x^m) - T(x^1, ..., x^m)\right) \leq \|T_n - T\| \, p_1^s(x^1), ..., p_m^s(x^m)$$

and $(T_n(x^1, ..., x^m))_n$ is convergent to $T(x^1, ..., x^m)$ in the Banach space (Y, q^s) for all $x^j \in X_j$, $j = 1, ..., m$. Then there is $k \geq n_0$ such that $q^s\left((T - T_k)(x^1, ..., x^m)\right) < \varepsilon$. By using the formula (2.7), for every $n \geq n_0$ and $x^j \in X_j$ with $p_j(x^j) \leq 1$, $j = 1, ..., m$, we have

$$\begin{aligned} q^s\left((T - T_n)(x^1, ..., x^m)\right) &= q^s\left(T(x^1, ..., x^m) - T_n(x^1, ..., x^m)\right) \\ &\leq \begin{array}{l} q^s\left(T(x^1, ..., x^m) - T_k(x^1, ..., x^m)\right) \\ +q^s\left(T_k(x^1, ..., x^m) - T_n(x^1, ..., x^m)\right) \end{array} \\ &< \varepsilon + \|T_n - T_k|^s_{(p_1,...,p_m;q)} \\ &< 2\varepsilon. \end{aligned}$$

By taking the supremum over all $x^j \in X_j$ with $p_j(x^j) \leq 1$, $j = 1, ..., m$, we obtain

$$\|T - T_n|^s_{(p_1,...,p_m;q)} < 2\varepsilon, \text{ for every } n \geq n_0.$$

The second claim is a consequence of the first one and of the above proposition. ■

2.3 Adjoint of multililinear mapping

The definition of adjoint of an m-linear mapping between normed spaces is due to Ramanujan and Schock [37]. We now present a similar definition for the multilinear mappings between asymmetric normed spaces but the adjoint operator obtained in this way is additive, positively homogeneous defined from a bi-Banach cone to another bi-Banach cone.

Definition 2.3.1 *Let* $(X_1, p_1), ..., (X_m, p_m)$ *and* (Y, q) *be* N*-asymmetric normed spaces. If* $T \in LC_{(p_1,...,p_m;q)}(X_1, ..., X_m; Y)$ *we define the adjoint of* T *by*

$$T^* : Y^* \longrightarrow LC_{(p_1,...,p_m;u)}(X_1, ..., X_m; \mathbb{R}), \quad f \longmapsto T^*(f) : X_1 \times ... \times X_m \longrightarrow \mathbb{R},$$

with

$$T^*(f)(x^1, ..., x^m) = f(T(x^1, ..., x^m)).$$

For the proof of the main theorem of this section we need the following consequence of Hahn-Banach theorem for asymmetric normed spaces (see [7, Theorem 2.2.2]).

Proposition 2.3.2 *Let* (X, p) *be a space with asymmetric seminorm. If* $x_0 \in X$ *with* $p(x_0) > 0$*, then there exists a continuous linear functional* $\varphi : X \longrightarrow \mathbb{R}$ *such that*

$$\|\varphi| = 1 \quad and \quad \varphi(x_0) = p(x_0). \tag{2.8}$$

Theorem 2.3.3 *The mapping* T^* *is additive, positively homogeneous, bounded and* $\|T^*| = \|T|_{(p_1,...,p_m;q)}$*, where* $\|T^*|$ *is the smallest bounded constant for* T^**.*

Proof. For all $f \in Y^*$ we can write

$$\begin{aligned} \|T^*(f)|_{(p_1,...,p_m;u)} &= \sup_{\substack{p_j(x^j)\leq 1 \\ j=1,...m}} u\left(f(T(x^1, ..., x^m))\right) \\ &\leq q^*(f) \sup_{\substack{p_j(x^j)\leq 1 \\ j=1,...m}} q\left(T(x^1, ..., x^m)\right) \\ &= q^*(f)\, \|T|_{(p_1,...,p_m;q)}, \end{aligned}$$

which means that T^* is bounded and $\|T^*| \leq \|T|_{(p_1,...,p_m;q)}$. In order to establish the reverse inequality, let $x^j \in X_j$ such that $p_j(x^j) \leq 1$ for all $j = 1, ..., m$ and $T(x^1, ..., x^m) \neq 0$.

By the above proposition there exist $f \in Y^*$ such that $q^*(f) = 1$ and $f\left(T(x^1, ..., x^m)\right) = q\left(T(x^1, ..., x^m)\right)$. Hence

$$\|T^*(f)|_{(p_1,\ldots,p_m;u)} \geq u\left(f(T(x^1, ..., x^m))\right) = u\left(q(T(x^1, ..., x^m))\right) = q\left(T(x^1, ..., x^m)\right),$$

from which it follows that

$$\|T^*| \geq \|T^*(f)|_{(p_1,\ldots,p_m;u)} \geq \sup_{p_j(x^j)\leq 1, j=1,\ldots m} q\left(T(x^1, ..., x^m)\right) = \|T|_{(p_1,\ldots,p_m;q)}.$$

■

Chapter 3

Fundamental theorems

Contents of the chapter:

1) Separately continuity of continuous multilinear mappings.

2) Multilinear asymmetric Banach-Steinhaus theorem.

3) Closed graph theorem.

The results obtained in this chapter have been published in the Journal of Colloquium Mathematicum. In this chapter we prove multilinear versions of the Banach-Steinhaus and closed graph theorems in the framwork of asymmetric normed spaces.

3.1 Separately continuity of multilinear mappings

Definition 3.1.1 *[27]The multilinear mapping $T : X_1 \times ... \times X_m \longrightarrow Y$ between normed spaces is separately continuous if it is continuous with respect to each variable while the other variable is fixed.*

Remark 3.1.2 *Obviously, the continuity implies the separately continuity, but the reverse implication is true if $X_1, ..., X_m$ are Banach spaces (see [27, Page 4] or [14, Page 8]).*

This result is true in the asymmetric framework under some requirements. For the proof we need some linear preliminaries.

Definition 3.1.3 *[34]An asymmetric normed space (X, p) is said to be of the half second category if the condition $X = \cup_{n\geq 1} E_n$ implies $int_p\left(cl_{\overline{p}}(E_m)\right) \neq \emptyset$ for some $m \in \mathbb{N}$, where $int_p(A)$ is the interior of the set A in the topological space (X, τ_p) and $cl_{\overline{p}}(A)$ is the closure of A in the topological space $(X, \tau_{\overline{p}})$.*

Note that if p is a norm on X, the notion of space of the half second category coincides with the classical notion of space of the second category (see [2] or [34]).

The next result, an asymmetric version of the Banach-Steinhaus theorem for linear operators, can be found in [2] and will be used in the sequel. For the proof of this theorem we need the following lemma.

Lemma 3.1.4 *If (X, d) is a quasi-metric space of the half second category and $\mathcal{F}$ is a family of real valued lower semicontinuous functions on $(X, \overline{d})$ such that for each $x \in X$ there exists $b_x > 0$ such that $f(x) \leq b_x$ for all $f \in \mathcal{F}$, then there exist a nonempty open set U in (X, d) and $b > 0$ such that $f(x) \leq b$ for all $f \in \mathcal{F}$ and $x \in U$.*

Proof. For each $n \in \mathbb{N}$ let

$$E_n = \bigcap_{f \in \mathcal{F}} f^{-1}(]-\infty, n]).$$

Then each E_n is closed in $(X, \overline{d})$. Moreover $X = \cup_{n=1}^{+\infty} E_n$. Indeed, by our hypothesis, given $x \in X$ there exists $n_x \in \mathbb{N}$ such that $f(x) \leq n_x$ for all $f \in \mathcal{F}$, so $x \in E_{n_x}$.

Hence, there exists $m \in \mathbb{N}$ such that $U \neq \emptyset$, where

$$U = int_d\left(cl_{\overline{d}}(E_m)\right) = int_d\left(E_m\right).$$

Then, for each $x \in U$ and $f \in \mathcal{F}$ we obtain $f(x) \leq m$. This completes the proof. ■

Theorem 3.1.5 *[2, Theorem 2.6] Let (X, p) and (Y, q) be two asymmetric normed spaces. Suppose that (X, p) is of the half second category. If $\mathcal{F}$ is a family of continuous linear operators such that $\sup_{T \in \mathcal{F}} q(T(x)) < \infty$ for every $x \in X$, then*

$$\sup_{T \in \mathcal{F}} \sup \{q(T(x)) : p(x) \leq 1\} < \infty.$$

Proof. For each $x \in X$ there exists $b_x > 0$ with

$$q(T(x) \leq b_x, \quad \text{for all } T \in \mathcal{F}.$$

For each $T \in \mathcal{F}$ define a function $h_T : X \longrightarrow \mathbb{R}_+$ by $h_T(x) = q(T(x))$. We first show that h_T is lower semicontinuous on X with respect to the quasi-metric $d_{\overline{p}}$. Indeed, let $x \in X$ and $(x_n)_n$ be a sequence in X such that $d_{\overline{p}}(x, x_n) \longrightarrow 0$. Then $p(x - x_n) \longrightarrow 0$. By the linear continuity of T we deduce that $q(T(x) - T(x_n)) \longrightarrow 0$. From the fact that

$$q(T(x)) - q(T(x_n)) \leq q(T(x) - T(x_n)), \quad \text{for all } n \in \mathbb{N},$$

it follows that for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $h_T(x) - h_T(x_n) < \varepsilon$ for all $n \geq n_0$. Since x is arbitrary, we conclude that h_T is lower semicontinuous on $(X, d_{\overline{p}})$.

Put the family

$$\mathcal{H} = \{h_T : T \in \mathcal{F}\}.$$

Since for each $x \in X$, $h_T(x) = q(T(x)) \leq b_x$ for all $T \in \mathcal{F}$, we can apply Lemma 3.1.4 and thus there exists a nonempty open subset U on the asymmetric normed space (X,p) and a $\delta > 0$ such that $h_T(x) \leq \delta$ for all $T \in \mathcal{F}$ and $x \in U$.

Fix $z \in U$. Then, there exists $r > 0$ such that $B_p(z,r) \subset U$. Take an $\varepsilon \in \left]0,r\right[$. Then, $B_p[z,\varepsilon] \subset U$, where $B_p[z,\varepsilon] = \{y \in X : p(y-z) \leq \varepsilon\}$. Put $b = (\delta + b_{-z})/\varepsilon$ and let $x \in X$ such that $p(x) \leq 1$. We will prove that $q(T(x)) \leq b$ for all $T \in \mathcal{F}$. Indeed, first note that $p(\varepsilon x + z - z) = \varepsilon p(x) \leq \varepsilon$, so $\varepsilon x + z \in U$.

Now take $T \in \mathcal{F}$. Then

$$\begin{aligned} q(T(x)) &= \frac{1}{\varepsilon} q(T(\varepsilon x)) = \frac{1}{\varepsilon} q(T(\varepsilon x + z - z)) = \frac{1}{\varepsilon} q(T(\varepsilon x + z) + T(-z)) \\ &\leq \frac{1}{\varepsilon}\left[q(T(\varepsilon x + z)) + q(T(-z))\right] = \frac{1}{\varepsilon}\left[h_T(\varepsilon x + z)) + h_T(-z)\right] \\ &\leq \frac{1}{\varepsilon}(\delta + b_{-z}) = b. \end{aligned}$$

Thus

$$\sup_{T \in \mathcal{F}} \sup \{q(T(x)) : p(x) \leq 1\} \leq b < \infty.$$

■

Now we give the main result of this section.

The main theorem

Theorem 3.1.6 *Let $(X_1,p_1),...,(X_m,p_m)$ be N-asymmetric normed spaces and (Y,q) be an asymmetric normed space. Suppose that (X_j,p_j) is of the half second category for all $j = 1,...,m$. The m-linear mapping $T : X_1,...,X_m \longrightarrow Y$ is separately continuous if and only if T is continuous.*

Proof. The part "if" is obvious. We prove the "only if" part by induction on $m \in \mathbb{N}$. For $m = 1$ there is nothing to prove. Suppose that the statement is true for $m-1$. Given $(x^1, ..., x^m) \in X_1 \times ... \times X_m$, define the mappings $T_{x^1,...,x^{m-1}} : (X_m, p_m) \longrightarrow (Y, q)$ and $T_{x^m} : (X_1, p_1) \times ... \times (X_{m-1}, p_{m-1}) \longrightarrow (Y, q)$ by

$$T_{x^m}(x^1, ..., x^{m-1}) = T_{x^1,...,x^{m-1}}(x^m) := T(x^1, ..., x^m).$$

By the assumption it is clear that the linear mapping $T_{x^1,...,x^{m-1}}$ is continuous and the $(m-1)$-linear mapping T_{x^m} is separately continuous. Then T_{x^m} is continuous by the inductive hypothesis. Now consider the family

$$\mathcal{F} = \left\{ T_{x^1,...,x^{m-1}} : p_j(x^j) \leq 1, j = 1, ..., m-1 \right\}.$$

The continuity of T_{x^m} implies that

$$\sup_{T_{x^1,...,x^{m-1}} \in \mathcal{F}} q(T_{x^1,...,x^{m-1}}(x^m)) = \sup_{p_j(x^j) \leq 1, j=1,...,m-1} q(T_{x^m}(x^1, ..., x^{m-1})) < \infty,$$

for every $x^m \in X_m$. So by the asymmetric version of the Banach-Steinhaus theorem (Theorem 3.1.5) we have

$$\sup_{T_{x^1,...,x^{m-1}} \in \mathcal{F}} \left(\sup_{p_m(x^m) \leq 1} q(T_{x^1,...,x^{m-1}}(x^m)) \right) < \infty.$$

It follows that

$$\begin{aligned}
\|T|_{(p_1,...,p_m;q)} &= \sup_{p_j(x^j) \leq 1, j=1,...,m} q\left(T(x^1, ..., x^m)\right) \\
&= \sup_{p_j(x^j) \leq 1, j=1,...,m-1} \left(\sup_{p_m(x^m) \leq 1} q(T_{x^1,...,x^{m-1}}(x^m)) \right) \\
&= \sup_{T_{x^1,...,x^{m-1}} \in \mathcal{F}} \left(\sup_{p_m(x^m) \leq 1} q(T_{x^1,...,x^{m-1}}(x^m)) \right) < \infty,
\end{aligned}$$

which proves that T is continuous from $(X_1, p_1) \times ... \times (X_m, p_m)$ to (Y, q). ■

3.2 Multilinear asymmetric Banach-Steinhaus theorem

To prove the main theorem, we need some linear preliminaries.

Definition 3.2.1 *Let be (X,p), (Y,q) asymmetric normed space. A family $\mathcal{F}\subset LC(X,Y)$ is called pointwisely bounded if*

$$\sup_{T\in\mathcal{F}} q(T(x)) < \infty, \quad \textit{for every } x\in X. \tag{3.1}$$

In this case the condition (3.1) is equivalent to

$$\sup_{T\in\mathcal{F}} \overline{q}(T(x)) < \infty, \quad \text{for every } x\in X. \tag{3.2}$$

The following version of the uniform boundedness principle was proved in [34].

Theorem 3.2.2 *Let (X,p) be a right p-K-complete asymmetric normed space, (Y,q) an asymmetric normed space and $\mathcal{T}\subset LC(X,Y)$. Suppose that the family $\mathcal{T}$ is pointwisely bounded. Then*

$$\sup_{T\in\mathcal{F}}\sup_{\overline{p}(x)\le 1} q(T(x)) < \infty \quad \textit{and} \quad \sup_{T\in\mathcal{F}}\sup_{p(x)\le 1} \overline{q}(T(x)).$$

As it is shown by Example 2.2 in [2] if the asymmetric normed space (X,p) is biBanach, then the Banach-Steinhaus theorem could not hold, even for linear functionals.

Example 3.2.3 *[2, Example 2.2] We consider the asymmetric normed space (ℓ_1,p), where*

$$\ell_1=\left\{(x_n)_n\subset\mathbb{R} : \|(x_n)_n\|_1 := \sum_{n=1}^{+\infty}|x_n|<\infty\right\},$$

and

$$p(x)=\sum_{n=1}^{+\infty}x_n^+, \quad x=(x_n)_n\in\ell_1. \tag{3.3}$$

Let $X_k = \{(x_n)_n \in \ell_1 : 2kx_{2k-1} + x_{2k} = 0\}$. *Since* X_k *is a closed vector subspace of* ℓ_1 *for every* $k \in \mathbb{N}$, *the subspace*

$$X = \bigcap_{k=1}^{+\infty} X_k$$

is a closed vector subspace of ℓ_1.

Since the norm p^s *is equivalent to the norm* $\|\cdot\|_1$ *on* ℓ_1, *we have that* X *is a closed vector subspace of the Banach space* (ℓ_1, p^s) *and then* (X, p^s), *is a Banach space. Therefore* (X, p) *is a biBanach space.*

Let $f_n : X \longrightarrow \mathbb{R}$ *be given by* $f_n(x) = (2n+1)x_{2n-1}$, *for every* $n \in \mathbb{N}$. *Let* $x = (x_n)_n \in X$. *Since*

$$f_n(x) \leq (2n+1)x^+_{2n-1} \leq (2n+1)p(x),$$

we have that f_n *is a continuous linear map from* (X, p) *to* $(\mathbb{R}, u)$, *for every* $n \in \mathbb{N}$.

$x = (x_n)_n \in X$, *then* $\|x\|_1 = \sum_{n=1}^{+\infty}(2n+1)\,|x_{2n-1}|$, *so that*

$$(2n+1)x_{2n-1} \leq (2n+1)\,|x_{2n-1}| \leq \|x\|_1\,.$$

Therefore, $f_n(x) \leq \|x\|_1$ *for every* $n \in \mathbb{N}$.

Now, we will prove that $\sup\{f_n(x) : p(x) \leq 1\} = 2n+1$, *for every* $n \in \mathbb{N}$. *Indeed, if* $p(x) \leq 1$, *then* $f_n(x) = 2n+1$. *If we consider* $t = (x^n_i)_n$ *such that*

$$x^n_i = \begin{cases} 1, & \text{if } i = 2n-1 \\ -2n, & \text{if } i = 2n \\ 0, & \text{if } i \in \mathbb{N} - \{2n-1, 2n\} \end{cases},$$

then $t \in X$, $p(t) = 1$ *and* $f_n(t) = 2n+1$. *Hence,*

$$\sup\{f_n(x) : p(x) \leq 1\} = 2n+1,$$

Consequently,

$$\sup\sup\{f_n(x) : p(x) \leq 1\} = +\infty.$$

Let $(X_1, p_1), ..., (X_m, p_m)$ be N-asymmetric normed spaces and (Y, q) be an asymmetric normed space. As the linear case, a family $\mathcal{F} \subset LC_{(p_1,...,p_m;q)}(X_1, ..., X_m; Y)$ is called pointwisely bounded if

$$\sup_{T\in\mathcal{F}} q(T(x^1, ..., x^m)) < \infty, \tag{3.4}$$

for every $x^j \in X_j, j = 1, ..., m$. In this case the condition (3.4) is equivalent to

$$\sup_{T\in\mathcal{F}} \overline{q}(T(x^1, ..., x^m)) < \infty,$$

for every $x^j \in X_j, j = 1, ..., m$.

We also need the following lemma.

Lemma 3.2.4 *If (X, p) is a right K-complete asymmetric normed space such then $(X, \overline{p})$ is too.*

Proof. Let $(x_n)_n$ a right K-Cauchy sequence in $(X, \overline{p})$. Then for each $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\overline{p}(x_s - x_r) < \varepsilon, \quad \text{for } r \geq s \geq n_0.$$

And then

$$p(-x_s + x_r) = \overline{p}(x_s - x_r) < \varepsilon, \quad \text{for } r \geq s \geq n_0.$$

This proves that is a right K-Cauchy sequence in (X, p). Thus, by hypothesis, ther exists $y \in X$ such that $(-x_n)_n$ converges to y in (X, p).

Therefore, for each $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$p(-x_n - y) < \varepsilon, \quad \text{for } n \geq n_0,$$

and thus we have that

$$\overline{p}(x_n + y) = p(-x_n - y) < \varepsilon, \quad \text{for } n \geq n_0.$$

Hence the sequence $(x_n)_n$ converges to $-y$ in $(X, \overline{p})$. ■

Mimicking the proof of [43, Theorem 1] we present a Banach-Steinhaus theorem for multilinear mappings between asymmetric normed spaces.

Theorem 3.2.5 *Let $(X_1, p_1), ..., (X_m, p_m)$ be right K-complete N-asymmetric normed spaces, (Y, q) an asymmetric normed space and $\mathcal{F}$ be a pointwisely bounded family in $LC_{(p_1,...,p_m;q)}(X_1, ..., X_m; Y)$. Then*

$$\sup_{T\in\mathcal{F}} \|T|_{(p_1,...,p_m;q)} < \infty.$$

Proof. First, note that the product space $X_1 \times ... \times X_m$, endowed with the N-asymmetric norm $p_\infty := \max\{p_1, ..., p_m\}$, is right K-complete (see [1, Lemma 6]). It follows that $X_1 \times ... \times X_m$ is also right K-complete with $\overline{p}_\infty$. For each integer $n \in \mathbb{N}$ cosider F_n the subset of $X_1 \times ... \times X_m$ of all $(x_1, ..., x_m)$ such that

$$\sup_{T\in\mathcal{F}} q(T(x^1, ..., x^m)) \le n \quad \text{and} \quad \sup_{T\in\mathcal{F}} \overline{q}(T(x^1, ..., x^m)) \le n.$$

Each F_n is closed. Perhaps the simplest way to see this is to note that the maps $\sup_{T\in\mathcal{F}} q \circ T$ and $\sup_{T\in\mathcal{F}} \overline{q} \circ T$ are lower semi-continuous on $X_1 \times ... \times X_m$ with respect to the N-asymmetric norm p_∞, since $q, \overline{q} : Y \longrightarrow \mathbb{R}$ are semi-continuous with respect to q and $\overline{q}$ respectively (see [7, Proposition 1.1.8]) and all $T \in \mathcal{F}$ are continuous. In addition,

$$X_1 \times ... \times X_m = \cup_{n\ge1} F_n,$$

because if $(x_1, ..., x_m) \in X_1 \times ... \times X_m$, there exists $n_1, n_2 \in \mathbb{N}$ so that

$$\sup_{T\in\mathcal{F}} q(T(x^1, ..., x^m)) \le n_1 \quad \text{and} \quad \sup_{T\in\mathcal{F}} \overline{q}(T(x^1, ..., x^m)) \le n_2.$$

Then $(x^1, ..., x^m) \in F_n$ with $n = \max\{n_1, n_2\}$. By the asymmetric Baire category theorem (see [34, Theorem 1.11]), there must be some $n_0 \in \mathbb{N}$ such that F_{n_0} contains an open ball

$B_{p_\infty}(a,r)$ with $(a^1,...,a^m) \in X_1 \times ... \times X_m$. It is clear that, for every $(t^1,...,t^m) \in B_{p_\infty}(0,r)$ we have $(0,t^2,t^3,...,t^m) \in B_{p_\infty}(0,r)$. We can compute, for $T \in \mathcal{F}$,

$$\begin{aligned}
&q(T(t^1, a^2+t^2, ..., a^m+t^m)) \\
&\leq q(T(a^1+t^1, a^2+t^2, ..., a^m+t^m)) + \overline{q}(T(a^1, a^2+t^2, ..., a^m+t^m)) \\
&\leq 2n_0.
\end{aligned}$$

Using the same argument and taking account that $(t^1,0,t^3,...,t^m)$ and $(0,0,t^3,...,t^m)$ belongs to $B_{p_\infty}(0,r)$, we get $\overline{q}(T(t^1,a^2,a^3+t^3,...,a^m+t^m)) \leq 2n_0$ and then

$$\begin{aligned}
&q(T(t^1, t^2, a^3+t^3, ..., a^m+t^m)) \\
&\leq q(T(t^1, a^2+t^2, ..., a^m+t^m)) + \overline{q}(T(t^1, a^2, a^3+t^3, ..., a^m+t^m)) \\
&\leq 4n_0.
\end{aligned}$$

By repeating this argument m times, we obtain

$$q(T(t^1,...,t^m)) \leq 2^m n_0,$$

for every $(t^1,...,t^m) \in B_{p_\infty}(0,r)$. On the other hand,

$$\begin{aligned}
\|T|_{(p_1,...,p_m;q)} &= \sup\{q(T(x^1,...,x^m)) : p_j(x^j) \leq 1, j=1,...,m\} \\
&= \sup\left\{q(T(\frac{t^1}{r},...,\frac{t^m}{r})) : p_j(t^j) \leq r, j=1,...,m\right\} \\
&\leq \frac{2^m n_0}{r^m}.
\end{aligned}$$

Since this holds for every $T \in \mathcal{F}$ the result is proved. ■

3.3 Closed graph theorem

The graph Γ_f of a mapping $f : X \longrightarrow Y$ is the subset of $X \times Y$ given by

$$\Gamma_f = \{(x,y) \in X \times Y : y = f(x)\}.$$

The closed graph theorem for the continuous multilinear operators between asymmetric normed spaces can easily be derived from the asymmetric closed graph theorem in the linear case [34, Theorem 4.2] and Theorem 3.1.6.

For two asymmetric normed spaces (X,p) and (Y,q) consider $X\times Y$ endowed with the asymmetric norm

$$r(x,y)=p(x)+q(y),\quad (x,y)\in X\times Y.$$

Theorem 3.3.1 ***(linear case).*** *Let (X,p) and (Y,q) be asymmetric normed spaces. Suppose that (X,p) is right p-K-complete and of half second category, and (Y,q) is right q-K-complete. If $T:X\longrightarrow Y$ is a linear operator with closed graph, then T is continuous.*

Let $T: X_1\times ...\times X_m \longrightarrow Y$ be a multilinear mapping between asymmetric spaces $(X_1,p_1),...,(X_m,p_m)$ and (Y,q). The graph of T, in symbols $G(T)$, is the set of elements $((x^1,...,x^m),y)\in (X_1\times ...\times X_m)\times Y$ such that $y=T(x^1,...,x^m)$. Consider the space $(X_1\times ...\times X_m)\times Y$ endowed with the asymmetric norm r defined by

$$r(z)=q(y)+\sum_{j=1}^{m}p_j(x^j),$$

for all $z=((x^1,...,x^m),y)\in (X_1\times ...\times X_m)\times Y$.

Theorem 3.3.2 *Assume that $(X_j,p_j), j=1,...,m,$ is Hausdorff N-asymmetric normed space, right K-complete and of the half second category, (Y,q) is right K-complete and the graph of T is closed in $(X_1,...,X_m)\times Y$. Then T is continuous.*

Proof. For each set of fixed x^k, $k\neq j$, $j=1,...,m$, define the linear mapping

$$T_j: X_j\longrightarrow Y,\quad T_j(x):=T(x^1,...,x^{j-1},x,x^{j+1},...,x^m)$$

and the set

$$Z_j=G(T)\cap\left(\left\{x^1\right\}\times ...\times\left\{x^{j-1}\right\}\times X_j\times\left\{x^{j+1}\right\}\times ...\times\{x^m\}\right)\times Y.$$

It is easy to check that Z_j is closed in $(X_1 \times ... \times X_m) \times Y$ and $G(T_j) = \psi(Z_j)$, for every $j = 1, ..., m$, where ψ is the homeomorphism

$$\psi : \left(\left\{x^1\right\} \times ... \times \left\{x^{j-1}\right\} \times X_j \times \left\{x^{j+1}\right\} \times ... \times \{x^m\}\right) \times Y \longrightarrow X_j \times Y$$

defined by

$$\psi((x^1, ..., x^{j-1}, x, x^{j+1}, ..., x^m), y) := (x, y).$$

Then $G(T_j)$ is closed in $X_j \times Y$ and by the closed graph theorem (see [34, Theorem 4.2]) each T_j is continuous. It follows that T is separately continuous. Therefore, by Theorem 3.1.6, T is continuous. ■

Chapter 4

Linearization of Continuous Multilinear Mappings.

Contents of the chapter:

This chapter has a twofold purpose: firsty, we offer a depth of theory of tensor product of bi-Banach spaces for the order $m > 2$ especially the projective N-asymmetric norm. Secondly we present the concept of linearization of continuous multilinear operators.

4.1 The algebraic theory of tensor products.

4.1.1 Tensor products of vector spaces.

The basic question answered by tensor product constructions is the following

Is there a vector space V such that $L(V, Y)$ coincides with (is isomorphic to) $L(X_1, ..., X_m; Y)$? i.e. can we, in some way, linearize multilinear mappings? The object is we construct the tensor product $X_1 \otimes ... \otimes X_m$, of $X_1, ..., X_m$.

The tensor product $X_1 \otimes ... \otimes X_m$ of the vector spaces $X_1, ..., X_m$ can be constructed from the elements of the space $L(X_1, ..., X_m)^*$. For $x^j \in X_j (j = 1, ..., m)$ we define the linear mapping

$$x^1 \otimes ... \otimes x^m : L(X_1, ..., X_m) \longrightarrow \mathbb{R},$$

by

$$x^1 \otimes ... \otimes x^m(\phi) := \phi(x^1, ..., x^m),$$

for each m-linear form ϕ on $X_1 \times ... \times X_m$. The functional $x^1 \otimes ... \otimes x^m$ is called an elementary tensor.

Definition 4.1.1 *The subspace of $L(X_1, ..., X_m)^*$ spanned by the collection of elementary tensors*

$$\left\{x^1 \otimes ... \otimes x^m, x^j \in X_j (j = 1, ..., m)\right\},$$

is called the tensor product of $X_1, ..., X_m$ and will be denoted by $X_1 \otimes ... \otimes X_m$;

$$X_1 \otimes ... \otimes X_m = spam\left(\left\{x^1 \otimes ... \otimes x^m, x^j \in X_j (j = 1, ..., m)\right\}\right).$$

The elements of this space are called tensors.

So a typical tensor $u \in X_1 \otimes ... \otimes X_m$ has the form

$$u = \sum_{i=1}^{n} \lambda_i x_i^1 \otimes ... \otimes x_i^m, \tag{4.1}$$

where $(\lambda_i)_{i=1}^n \subset \mathbb{R}, (x_i^j)_{i=1}^n \subset X_j (j = 1, ..., m)$ and $n \in \mathbb{N}$ is arbitrary.

Here are some basic algebraic properties of elementary tensors.

Proposition 4.1.2 *Let $X_1, ..., X_m$ be vector spaces. Then for all $x^j, y^j \in X_j (j = 1, ..., m)$ and $\lambda \in \mathbb{K}$ we have*

1) $x^1 \otimes ... \otimes (x^j + y^j) \otimes ... \otimes x^m = x^1 \otimes ... \otimes x^j \otimes ... \otimes x^m + x^1 \otimes ... \otimes y^j \otimes ... \otimes x^m.$

2) $\lambda(x^1 \otimes ... \otimes x^j \otimes ... \otimes x^m) = (x^1 \otimes ... \otimes (\lambda x^j) \otimes ... \otimes x^m)$

3) If $x^j = 0$ for some $j = 1, ..., m$ then $x^1 \otimes ... \otimes x^m = 0$.

Note that the typical description of an element $u \in X_1 \otimes ... \otimes X_m$ given in (4.1) can always be rewritten, using 3) above, in the form

$$u = \sum_{i=1}^n x_i^1 \otimes ... \otimes x_i^m$$

In the next proposition we show a criterion useful to identify when a tensor u is null.

Definition 4.1.3 *A subset $\mathcal{S}$ of the algebric dual space X^* is said to be separating if for any $x, y \in X$, $x \neq y$ there exists $\varphi \in \mathcal{S}$ such that $\varphi(x) \neq \varphi(y)$, equivalently, if $\varphi(x) = 0$ for every $\varphi \in \mathcal{S}$, then $x = 0$.*

Proposition 4.1.4 *Let $u = \sum_{i=1}^n x_i^1 \otimes ... \otimes x_i^m \in X_1 \otimes ... \otimes X_m$. The following are equivalent*

(i) $u = 0$.

(ii) $\sum_{i=1}^n \phi^1(x_i^1)...\phi^m(x_i^m) = 0$ for all $\varphi^j \in \mathcal{S}_j$ where $\mathcal{S}_j$ is separating subset of $X_j^* (j = 1, ..., m)$.

(iii) $\sum_{i=1}^n \varphi^1(x_i^1)...\varphi^m(x_i^m) = 0$ for all $\varphi^j \in X_j^* (j = 1, ..., m)$.

Proof. (i)$\Longrightarrow$(ii) Suppose that $u = \sum_{i=1}^n x_i^1 \otimes ... \otimes x_i^m = 0$. Let $\varphi^j \in \mathcal{S}_j$, separating subsets of $X_j^* (j = 1, ..., m)$, consider the m-linear form $T \in L(X_1, ..., X_m)$ defined by

$$T(x^1, ..., x^m) = \varphi^1(x^1)...\varphi^m(x^m),$$

so that we have

$$0 = u(T) = \sum_{i=1}^{n} T(x_i^1, ..., x_i^m) = \sum_{i=1}^{n} \varphi^1(x_i^1)...\varphi^m(x_i^m).$$

(ii)$\Longrightarrow$(iii) Let $\mathcal{S}_j \subset X_j^*$ be separating subsets of X_j^* such that

$$\sum_{i=1}^{n} \phi^1(x_i^1)...\phi^m(x_i^m) = 0,$$

for all $\phi^j \in \mathcal{S}_j (j = 1, ..., m)$. We have

$$\phi^1 \left(\sum_{i=1}^{n} x_i^1 \phi^2(x_i^2)...\phi^m(x_i^m) \right) = 0.$$

Since $\mathcal{S}_1$ is separating subsets of X_1^*,

$$\sum_{i=1}^{n} x_i^1 \phi^2(x_i^2)...\phi^m(x_i^m) = 0,$$

for all $\phi^j \in \mathcal{S}_j (j = 2, ..., m)$. Therefore

$$\sum_{i=1}^{n} \varphi^1(x_i^1)\phi^2(x_i^2)...\phi^m(x_i^m) = \varphi^1 \left(\sum_{i=1}^{n} x_i^1 \phi^2(x_i^2)...\phi^m(x_i^m) \right) = 0,$$

for all $\varphi^1 \in X_1^*$ and all $\phi^j \in \mathcal{S}_j (j = 2, ..., m)$. Then

$$\phi^2 \left(\sum_{i=1}^{n} \varphi^1(x_i^1) x_i^2 \phi^3(x_i^3)...\phi^m(x_i^m) \right) = 0.$$

Since $\mathcal{S}_2$ is separating subsets of X_2^* we have

$$\sum_{i=1}^{n} \varphi^1(x_i^1) x_i^2 \phi^3(x_i^3)...\phi^m(x_i^m) = 0,$$

for all $\varphi^1 \in X_1^*$ and all $\phi^j \in \mathcal{S}_j (j = 3, ..., m)$. Therefore

$$\sum_{i=1}^{n} \varphi^1(x_i^1)\varphi^2(x_i^2)\phi^3(x_i^3)...\phi^m(x_i^m) = \varphi^2 \left(\sum_{i=1}^{n} \varphi^1(x_i^1) x_i^2 \phi^3(x_i^3)...\phi^m(x_i^m) \right) = 0,$$

for all $\varphi^j \in X_j^*(j=1,2)$ and all $\phi^j \in \mathcal{S}_j(j=3,...,m)$. By repeating this process a finite number of times we obtain (iii).

(iii)$\Longrightarrow$(i), let us suppose that $\sum\limits_{i=1}^{n}\varphi^1(x_i^1)...\varphi^m(x_i^m)=0$ for all $\varphi^j \in X_j^*(j=1,...,m)$ and show that $u(T)=0$ for all $T \in L(X_1,...,X_m)$.

Let $T \in L(X_1,...,X_m)$ and define $Z_j = span\left\{x_1^j,...x_n^j\right\}, j=1,...,m$. Let S denote the restriction of T to $Z_1 \times ... \times Z_m$. For $l=1,...,m-1$ consider

$$\mathcal{B}_l = \left\{e_1^l,...e_{n_l}^l\right\}, n_l \leq n,$$

a base for the finite dimensional space Z_l. Given $(x^1,...,x^m) \in Z_1 \times ... \times Z_m$ then for $l=1,...,m-1$ we can write, uniquely,

$$x^l = \sum_{i_l=1}^{n_l}\lambda_{i_l}^l e_{i_l}^l,$$

where $\left(\lambda_{i_l}^l\right)_{i_l=1}^{n_l} \subset \mathbb{R}$. Thus

$$\begin{aligned} S(x^1,...,x^m) &= \sum_{i_1=1}^{n_1}...\sum_{i_{m-1}=1}^{n_{m-1}}\lambda_{i_1}^1...\lambda_{i_{m-1}}^{m-1}S\left(e_{i_1}^1,...,e_{i_{m-1}}^{m-1},x^m\right) \\ &= \sum_{i_1=1}^{n_1}...\sum_{i_{m-1}=1}^{n_{m-1}}\theta_{i_1}^1(x^1)...\theta_{i_{m-1}}^{m-1}(x^{m-1})\theta_{i_1,...,i_{m-1}}^m(x^m), \end{aligned}$$

where $\theta_{i_l}^l \in Z_l^*$ and $\theta_{i_1,...,i_{m-1}}^m \in Z_m^*$ with $1 \leq l \leq m-1$, $1 \leq i_l \leq n_l$ are defined by

$$\theta_{i_l}^l(x^l) = \theta_{i_l}^l\left(\sum_{i_l=1}^{n_l}\lambda_{i_l}^l e_{i_l}^l\right) = \lambda_{i_l}^l \quad \text{and} \quad \theta_{i_1,...,i_{m-1}}^m(x^m) = S\left(e_{i_1}^1,...,e_{i_{m-1}}^{m-1},x^m\right),$$

finally we obtain

$$S(x^1,...,x^m) = \sum_{l=1}^{k}\phi_l^1(x^1)...\phi_l^m(x^m),$$

where $k = n_1...n_{m-1}$ and $\left(\phi_l^j\right)_{l=1}^{k} \subset Z_j^*(j=1,...,m)$.

Now choose algebraic complements, H_j for Z_j so that

$$X_j = Z_j \oplus H_j (j = 1, ..., m),$$

then if $x^j = x'^j + x''^j \in X_j$ with $x'^j \in Z_j$ and $x''^j \in H_j$ define $\widetilde{\phi_l^j}(x^j) = \phi_l^j(x'^j)$. Then we obtain $\widetilde{\phi_l^j} \in X_j^*$ extension of ϕ_l^j for $j = 1, ..., m$ and $l = 1, ..., k$. Now T and S may be different m-linear forms on $X_1 \times ... \times X_m$ but they coincides on $Z_1 \times ... \times Z_m$. Thus we have

$$\begin{aligned} u(T) &= \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m(T) = \sum_{i=1}^{n} T(x_i^1, ..., x_i^m) \\ &= \sum_{i=1}^{n} S(x_i^1, ..., x_i^m) = \sum_{i=1}^{n}\sum_{l=1}^{k} \phi_l^1(x_i^1)...\phi_l^m(x_i^m) \\ &= \sum_{l=1}^{k}\sum_{i=1}^{n} \widetilde{\phi_l^1}(x_i^1)...\widetilde{\phi_l^m}(x_i^m) \\ &= \sum_{l=1}^{k} 0 = 0. \end{aligned}$$

■

4.1.2 Linearization of multilinear mappings.

The purpose of this is to show in what sense the tensor product is the vector space through witch multilinear mappings can be linearized. To achieve this we need to introduce a special multilinear mapping that will be very useful.

For the vector spaces $X_1, ..., X_m$ we consider the canonical mapping

$$\sigma_m : X_1 \times ... \times X_m \longrightarrow X_1 \otimes ... \otimes X_m.$$

Define by

$$\sigma_m(x^1, ..., x^m) = x^1 \otimes ... \otimes x^m.$$

It is clear that the mapping σ_m is multilinear.

Theorem 4.1.5 *Let $X_1, ..., X_m$ and Y be vector spaces. For every m-linear mapping $T : X_1 \times ... \times X_m \longrightarrow Y$ there exists a unique linear mapping $T_L : X_1 \otimes ... \otimes X_m \longrightarrow Y$ given by*

$$T_L\left(\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m\right) = \sum_{i=1}^{n} T\left(x_i^1, ..., x_i^m\right),$$

such that $T = T_L \circ \sigma_m$ i.e. the following diagram commutes

$$T = T_L \circ \sigma_m : X_1 \times ... \times X_m \overset{\sigma_m}{\longrightarrow} X_1 \otimes ... \otimes X_m \overset{T_L}{\longrightarrow} Y$$

The correspondence $T \longleftrightarrow T_L$ establishes an isomorphism between the vector spaces $L(X_1, ..., X_m; Y)$ and $L(X_1 \otimes ... \otimes X_m, Y)$. The linear operator T_L is called linearization of the m-linear mapping T.

Proof. To show that the linear mapping T_L is well defined. First note that if $\varphi \in Y^*$ then $\varphi \circ T$ is a multilinear form in $X_1 \times ... \times X_m$. Now, if $u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m = 0$, then for all functional $\varphi \in Y^*$ we have

$$\begin{aligned} \varphi\left(\sum_{i=1}^{n} T\left(x_i^1, ..., x_i^m\right)\right) &= \sum_{i=1}^{n} \varphi \circ T\left(x_i^1, ..., x_i^m\right) \\ &= \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m\left(\varphi \circ T\right) = u(\varphi \circ T) = 0. \end{aligned}$$

As the only element of a vector space that negates all linear functionals is the zero of this space, we have

$$0 = \sum_{i=1}^{n} T\left(x_i^1, ..., x_i^m\right) = T_L(u),$$

then $T_L(u) = 0$. Therefore T_L is well defined.

Now, for all $(x^1, ..., x^m) \in X_1 \times ... \times X_m$ we have

$$T_L \circ \sigma_m\left(x^1, ..., x^m\right) = T_L(x^1 \otimes ... \otimes x^m) = T\left(x^1, ..., x^m\right).$$

It follow that $T = T_L \circ \sigma_m$.

For the uniqueness of the linearization, suppose $s \in L(X_1 \otimes ... \otimes X_m, Y)$ such that $T = s \circ \sigma_m$. Thus, for any $u = \sum\limits_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \in X_1 \otimes ... \otimes X_m$,

$$\begin{aligned} s(u) &= \sum_{i=1}^{n} s(x_i^1 \otimes ... \otimes x_i^m) = \sum_{i=1}^{n} s\left(\sigma_m(x_i^1, ..., x_i^m)\right) \\ &= \sum_{i=1}^{n} s \circ \sigma_m(x_i^1, ..., x_i^m) = \sum_{i=1}^{n} T_L \circ \sigma_m(x_i^1, ..., x_i^m) \\ &= \sum_{i=1}^{n} T_L(x_i^1 \otimes ... \otimes x_i^m) \\ &= T_L(u), \end{aligned}$$

which is proves that $s = T_L$.

Finally, let us show that the mapping

$$\Phi : L(X_1, ..., X_m; Y) \longrightarrow L(X_1 \otimes ... \otimes X_m, Y), T \longmapsto T_L,$$

is an isomorphism.

Clearly, if $S, T \in L(X_1, ..., X_m; Y)$ and $\lambda \in \mathbb{K}$ we have

$$(S + \lambda T)_L = S_L + \lambda T_L,$$

and then Φ is linear.

Now let $T \in \ker \Phi$. In this case,

$$T_L \left(\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \right) = 0,$$

for all $\sum\limits_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \in X_1 \otimes ... \otimes X_m$. In particular

$$0 = T_L(x^1 \otimes ... \otimes x^m) = T(x^1, ..., x^m),$$

for any $x^j \in X_j (j = 1, ..., m)$. It follows that $T = 0$ and then Φ is injective.
To prove that the map Φ is also surjective, consider the linear mapping $\phi \in L(X_1 \otimes ... \otimes X_m, Y)$ and define the mapping $S : X_1 \times ... \times X_m \longrightarrow Y$ by

$$S(x^1, ..., x^m) = \phi(x^1 \otimes ... \otimes x^m),$$

the linearity of u and Proposition 4.1.2 assert that S is m-linear. From we have demonstrated in this proof, there exists a unique linear mapping $S_L \in L(X_1 \otimes ... \otimes X_m, Y)$ such that

$$S = S_L \circ \sigma_m,$$

on the other hand, for all $x^j \in X_j (j = 1, ..., m)$ we have

$$\phi \circ \sigma_m(x^1, ..., x^m) = \phi(x^1 \otimes ... \otimes x^m) = S(x^1, ..., x^m),$$

thus $S = \phi \circ \sigma_m$. Now, since S_L is the sole mapping checks $S = S_L \circ \sigma_m$, it follows that

$$\phi = S_L = \Phi(S).$$

Where the surjectivity of Φ. ■

Remark 4.1.6 *By the isomorphism stated in the previous theorem and if we take $Y = \mathbb{R}$, we can write the identification of the algebric dual of tensor product;*

$$(X_1 \otimes ... \otimes X_m)^* = L(X_1, ..., X_m).$$

The property of the tensor product described in Theorem 4.1.5 is called *universal property of tensor products.* A part from enjoying this property, the tensor product is the unique vector space to the extent of being able to linearize m-linear maps:

Proposition 4.1.7 ***(Uniqueness of the tensor product)***

Let $m \in \mathbb{N}$ and $X_1, ..., X_m$ be vector spaces. Suppose there exists a vector space W and a multilinear mapping $S : X_1 \times ... \times X_m \longrightarrow W$ with the property that, for every vector space Y and every multilinear mapping T from $X_1 \times ... \times X_m$ into Y, there is a unique linear mapping $u : W \longrightarrow Y$ such that $T = u \circ S$. Then there is an isomorphism $\Phi : X_1 \otimes ... \otimes X_m \longrightarrow W$ such that

$$\Phi(x^1 \otimes ... \otimes x^m) = S(x^1, ..., x^m),$$

for all $x^j \in X_j (j = 1, ..., m)$.

Proof. We start by showing that the vector space W is spanned by

$$S(X_1 \times ... \times X_m) = \left\{ S(x^1, ..., x^m), x^j \in X_j (j = 1, ..., m) \right\}.$$

Suppose that

$$W \neq span \left\{ S(X_1 \times ... \times X_m) \right\}.$$

Then

$$W = span \left\{ S(X_1 \times ... \times X_m) \right\} \oplus Z.$$

Where $Z \neq \{0\}$ being some proper subspace of W.

So, a mapping u would not be uniquely determined by the requirement $T = u \circ S$ because such a relation would be preserved under modifications of u in the subspace Z. Now, applying the given property of W and S to the canonical m-linear mapping

$$\sigma_m : X_1 \times ... \times X_m \longrightarrow X_1 \otimes ... \otimes X_m,$$

we obtain a linear mapping $u : W \longrightarrow X_1 \otimes ... \otimes X_m$ such that $\sigma_m = u \circ S$, i.e.

$$u(S(x^1, ..., x^m)) = x^1 \otimes ... \otimes x^m,$$

for all $x^j \in X_j (j = 1, ..., m)$.

On the other hand, if $S_L : X_1 \otimes ... \otimes X_m \longrightarrow W$ is the unique linearization of the m-linear mapping S we have

$$S_L(x^1 \otimes ... \otimes x^m) = S(x^1, ..., x^m).$$

Thus

$$S_L \circ u(S(x^1, ..., x^m)) = S(x^1, ..., x^m),$$

and

$$u \circ S_L(x^1 \otimes ... \otimes x^m) = x^1 \otimes ... \otimes x^m,$$

for every $x^j \in X_j (j = 1, ..., m)$.

Since $X_1 \otimes ... \otimes X_m$ and W are spanned by the elements $x^1 \otimes ... \otimes x^m$ and $S(x^1, ..., x^m)$ respectively, it follows that $u \circ S_L$ and $S_L \circ u$ are the application identity of $X_1 \otimes ... \otimes X_m$ and W respectively. We conclude that $\Phi = S_L$ is the required isomorphism. ■

Now we will use Proposition 4.1.4 and Theorem 4.1.5 to prove another interesting property of tensor products, namely the *commutativity*. Consider a permutation

$$\eta : \{1, ..., m\} \to \{1, ..., m\}.$$

For each $u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \in X_1 \otimes ... \otimes X_m$, it is obvious that

$$u^\eta = \sum_{i=1}^{n} x_i^{\eta(1)} \otimes ... \otimes x_i^{\eta(m)} \in X_{\eta(1)} \otimes ... \otimes X_{\eta(m)}.$$

Proposition 4.1.8 *If $\eta : \{1, ..., m\} \to \{1, ..., m\}$ is any permutation. Then the vector space $X_1 \otimes ... \otimes X_m$ is isomorphic to $X_{\eta(1)} \otimes ... \otimes X_{\eta(m)}$.*

Proof. Fixed a permutation $\eta : \{1, ..., m\} \to \{1, ..., m\}$ and consider the mapping

$$T : X_1 \times ... \times X_m \to X_{\eta(1)} \otimes ... \otimes X_{\eta(m)},$$

defined by

$$T(x^1, ..., x^m) = x^{\eta(1)} \otimes ... \otimes x^{\eta(m)}.$$

The Proposition 4.1.2 asserts that T is m-linear. By Theorem 4.1.5 there exists a unique linear mapping T_L

$$T_L \in L(X_1 \otimes ... \otimes X_m, X_{\eta(1)} \otimes ... \otimes X_{\eta(m)}),$$

such that

$$T_L(\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m) = \sum_{i=1}^{n} T(x_i^1, ..., x_i^m) = \sum_{i=1}^{n} x_i^{\eta(1)} \otimes ... \otimes x_i^{\eta(m)}.$$

Let us verify that the linear mapping T_L is isomorphism. If $\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \in \ker(T_L)$ we have

$$\sum_{i=1}^{n} x_i^{\eta(1)} \otimes ... \otimes x_i^{\eta(m)} = 0.$$

Applying Proposition 4.1.4 we have

$$\sum_{i=1}^{n} \varphi_{\eta(1)} \left(x_i^{\eta(1)}\right) ... \varphi_{\eta(m)} \left(x_i^{\eta(m)}\right) = 0,$$

for all $\varphi_{\eta(j)} \in X_{\eta(j)}^*(j = 1, ..., m)$. Rearranging the terms of the last sum we get

$$\sum_{i=1}^{n} \varphi_1(x^1) ... \varphi_m(x^m) = 0,$$

for all $\varphi_j \in X_j^*(j = 1, ..., m)$. Again, applying Proposition 4.1.4 we obtain

$$\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m = 0,$$

and therefore T_L is injective. It remains to show the surjectivity, given

$$v = \sum_{i=1}^{n} x_i^{\eta(1)} \otimes ... \otimes x_i^{\eta(m)} \in X_{\eta(1)} \otimes ... \otimes X_{\eta(m)}.$$

If we take $u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m$ we obtain directly $T_L(u) = v$. Then T_L is surjective and hence it is an isomorphism. ■

Now we introduce the tensor product of linear operators.

Proposition 4.1.9 *Let $s_j : X_j \longrightarrow Y_j (j = 1, ..., m)$ be linear mappings between vector spaces. Then there is a unique linear mapping*

$$s : X_1 \otimes ... \otimes X_m \longrightarrow Y_1 \otimes ... \otimes Y_m,$$

such that

$$s(x^1 \otimes ... \otimes x^m) = s_1(x^1) \otimes ... \otimes s_m(x^m).$$

We write

$$s = s_1 \otimes ... \otimes s_m.$$

Proof. Define an m-linear mapping

$$T \in L(X_1 \times ... \times X_m; Y_1 \otimes ... \otimes Y_m),$$

by

$$T(x^1, ..., x^m) = s_1(x^1) \otimes ... \otimes s_m(x^m).$$

The linearization of T gives the unique linear mapping

$$T_L \in L(X_1 \otimes ... \otimes X_m, Y_1 \otimes ... \otimes Y_m),$$

such that

$$T_L(x^1 \otimes ... \otimes x^m) = s_1(x^1) \otimes ... \otimes s_m(x^m).$$

Then taking $s_1 \otimes ... \otimes s_m = T_L$. ■

4.1.3 Tensor as multilinear forms.

The tensor product $X_1 \otimes ... \otimes X_m$ is defined as a space of linear functionals on the space $L(X_1, ..., X_m)$. There are other, equally natural approaches, some of which we describe in the following. We show that tensors can be viewed as multilinear forms.

Proposition 4.1.10 *For all vector spaces $X_1, ..., X_m$ we have a canonical embedding*

$$X_1 \otimes ... \otimes X_m \subset L(X_1^*, ..., X_m^*).$$

Proof. With each $(x^1, ..., x^m) \in X_1 \times ... \times X_m$ we can associate a multilinear form $T_{x^1,...,x^m} \in L(X_1^*, ..., X_m^*)$ where

$$T_{x^1,...,x^m}(\varphi^1, ..., \varphi^m) = \varphi^1(x^1)...\varphi^m(x^m),$$

for all $\varphi^j \in X_j^*(j = 1, ..., m)$. By the linearity of all φ^j we can prove that the mapping

$$\Psi : (x^1, ..., x^m) \longmapsto T_{x^1,...,x^m},$$

is multilinear. Let $\Psi_L : X_1 \otimes ... \otimes X_m \longrightarrow L(X_1^*, ..., X_m^*)$ the unique linearization of Ψ, to see that this mapping is injective. Suppose that $\Psi_L\left(\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m\right) = 0$, it follows that $\sum_{i=1}^{n} T_{x_i^1,...,x_i^m} = 0$. Then

$$\sum_{i=1}^{n} \varphi^1(x_i^1)...\varphi^m(x_i^m) = 0,$$

for all $\varphi^j \in X_j^*(j = 1, ..., m)$ and so, by Proposition 4.1.4 this becomes

$$\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m = 0.$$

Thus, we have the canonical embedding

$$X_1 \otimes ... \otimes X_m \subset L(X_1^*, ..., X_m^*).$$

■

Remark 4.1.11 *By the last canonical embedding, the elementary tensors* $x^1 \otimes ... \otimes x^m \in X_1 \otimes ... \otimes X_m$ *is identified with the multilinear form that maps* $(\varphi^1, ..., \varphi^m) \in X_1^* \otimes ... \otimes X_m^*$ *to* $\langle \varphi^1, x^1 \rangle ... \langle \varphi^m, x^m \rangle$*. In other word we have*

$$x^1 \otimes ... \otimes x^m(\varphi^1, ..., \phi^m) = \varphi^1(x^1)...\varphi^m(x^m),$$

for all $\varphi^j \in X_j^*(j = 1, ..., m)$.

4.2 The asymmetric projective tensor product.

The results related to the projective norm for of order $m > 2$ have been treated in some document, for example, we found some details about these concepts in [42].

4.2.1 The projective N-asymmetric norm.

Let $(X_1, p_1), ..., (X_m, p_m)$ be N-asymmetric normed vector spaces. The aim of this subsection is to introduce an N-asymmetric norm on $X_1 \otimes ... \otimes X_m$ to realized linearization of the continuous multilinear mappings defined in $X_1 \times ... \times X_m$. More precisely, we want an N-asymmetric norm on $X_1 \otimes ... \otimes X_m$ such that for every asymmetric normed space (Y, q), an m-linear mapping $T : X_1 \times ... \times X_m \longrightarrow Y$ is continuous if and only if

$$T_L : X_1 \otimes ... \otimes X_m \longrightarrow Y,$$

is continuous linear mapping relative to this N-asymmetric norm. Consider first the elementary tensor $x^1 \otimes ... \otimes x^m, x^j \in X_j(j = 1, ..., m)$, since $T = T_L \circ \sigma_m$ where

$$\sigma_m(x^1, ..., x^m) = x^1 \otimes ... \otimes x^m,$$

it is natural to require that the mapping

$$\sigma_m : (X_1, p_1) \times ... \times (X_m, p_m) \longrightarrow (X_1 \otimes ... \otimes X_m, p)$$

is continuous. It follows that there is a constant $C > 0$ such that

$$p\left(x^1 \otimes ... \otimes x^m\right) \le Cp_1(x^1)...p_m(x^m),$$

for all $x^j \in X_j (j = 1, ..., m)$. Incorporating the constant $\frac{1}{C}$ in the N-asymmetric norm to be defined in the tensor product, it must then be true that

$$p\left(x^1 \otimes ... \otimes x^m\right) \le p_1(x^1)...p_m(x^m),$$

for any elementary tensor $x^1 \otimes ... \otimes x^m$.

Now let u be arbitrary tensor in $X_1 \otimes ... \otimes X_m$, with the representation

$$u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m$$

then it follow from the triangle inequality that the N-asymmetric norm must satisfy

$$p(u) \le \sum_{i=1}^{n} p_1(x_i^1)...p_m(x_i^m).$$

Since this holds for every representation of u, it follows that

$$p(u) \le \inf \sum_{i=1}^{n} p_1(x_i^1)...p_m(x_i^m),$$

the infimum being taken over all representations of u.

The quantity that is mentioned to the right of last inequality is the biggest possible candidate for a "natural" N-asymmetric norm on $X_1 \otimes ... \otimes X_m$.

Definition 4.2.1 *Let $X_1, ..., X_m$ be N-asymmetric normed vector spaces over $\mathbb{K}$. For each tensor $u \in X_1 \otimes ... \otimes X_m$, we put*

$$\pi_a(u) = \inf \sum_{i=1}^{n} p_1(x_i^1)...p_m(x_i^m), \tag{4.2}$$

where the infimum is taken over all possible representations of u.

To prove that π_a is an N-asymmetric norm on the tensor product, we need the following lemma.

Lemma 4.2.2 *Let X be an N-asymmetric normed vector space. Then the topological dual space X^* is separating subset of the algebraic dual.*

Proof. Let $x \in X$ and suppose that

$$\varphi(x) = 0, \quad \text{for every } \varphi \in X^*.$$

Then we have two cases; $p(x) > 0$ or $p(x) = 0$.
If $p(x) > 0$, by the consequences of the asymmetric Hahn-Banach (Proposition 2.3.2) we have

$$p(x) = \sup\{\varphi(x), \varphi \in X^*, \|\varphi| \leq 1\} = 0,$$

which is a contradiction. Consequently we have $p(x) = 0$ and then $x = 0$ because p is an N-asymmetric norm. ■

Proposition 4.2.3 *Let $(X_1, p_1), ..., (X_m, p_m)$ be N-asymmetric normed vector spaces over $\mathbb{R}$. Then π_a, as defined in (4.2), is an N-asymmetric norm on $X_1 \otimes ... \otimes X_m$. Furthermore*

$$\pi_a(x^1 \otimes ... \otimes x^m) = p_1(x^1)...p_m(x^m), \quad \textit{for all } x^j \in X_j (j = 1, ..., m).$$

The N-asymmetric norm π_a is known as the projective N-asymmetric norm and we shall denote by $X_1 \otimes_{\pi_a} ... \otimes_{\pi_a} X_m$ the tensor product $X_1 \otimes ... \otimes X_m$ endowed with the projective N-asymmetric norm π_a.
Proof. First note that, if $\varphi^j \in X_j^* (j = 1, ..., m)$ then the value of the sum $\sum_{i=1}^{n} \varphi^1(x_i^1)...\varphi^m(x_i^m)$ is independent of the representation of $u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m$. In order to see this, let $u \in X_1 \otimes ... \otimes X_m$ admits two representations

$$u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m = \sum_{i=1}^{k} y_i^1 \otimes ... \otimes y_i^m,$$

consider the multilinear form $T \in L(X_1, ..., X_m)$ defined by

$$T(x^1, ..., x^m) = \varphi^1(x^1)...\varphi^m(x^m).$$

By definition we have

$$\begin{aligned} \sum_{i=1}^{n} \varphi^1(x_i^1)...\varphi^m(x_i^m) &= \left(\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \right)(T) \\ &= \left(\sum_{i=1}^{k} y_i^1 \otimes ... \otimes y_i^m \right)(T) \\ &= \sum_{i=1}^{k} \varphi^1(y_i^1)...\varphi^m(y_i^m). \end{aligned}$$

It is clear that $\pi_a(u) \geq 0$ for any $u \in X_1 \otimes ... \otimes X_m$. Suppose that $\pi_a(u) = 0$. Then, for every $\varepsilon > 0$ there is a representation of u of the form $\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m$ such that

$$\sum_{i=1}^{n} p_1(x_i^1)...p_m(x_i^m) \leq \varepsilon.$$

Hence, for every $\varphi^j \in X_j^*(j = 1, ..., m)$ we have

$$\left| \sum_{i=1}^{n} \varphi^1(x_i^1)...\varphi^m(x_i^m) \right| \leq \varepsilon \left\| \varphi^1 \right| ... \left\| \varphi^m \right| . \tag{4.3}$$

Since the value of the $\sum_{i=1}^{n} \varphi^1(x_i^1)...\varphi^m(x_i^m)$ is independent of the representation of u it follows that, by passage to the limit in (4.3), for $\varepsilon \longrightarrow 0$ we obtain

$$\sum_{i=1}^{n} \varphi^1(x_i^1)...\varphi^m(x_i^m) = 0.$$

But, by the previous lemma, the dual spaces $X_j^*(j = 1, ..., m)$ are separating subsets of the respective algebraic duals and so, by Proposition 4.1.4, it follows that $u = 0$.

Now, we prove that $\pi_a(\lambda u) = \lambda \pi_a(u)$ for all $\lambda \geq 0$. Is clear that this equality is satisfied for $\lambda = 0$. Suppose that $\lambda \neq 0$. If $\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m$ is a representation of u then

$$\lambda u = \sum_{i=1}^{n} (\lambda x_i^1) \otimes ... \otimes x_i^m,$$

and so we have

$$\pi_a(\lambda u) \leq \sum_{i=1}^{n} p_1(\lambda x_i^1)...p_m(x_i^m) = \lambda \sum_{i=1}^{n} p_1(x_i^1)...p_m(x_i^m).$$

Since this holds for every representation of u, it follows that $\pi_a(\lambda u) \leq \lambda \pi_a(u)$. On the other hand, using this last inequality for the scalar λ^{-1}, we have

$$\pi_a(u) = \pi_a(\lambda^{-1}\lambda u) \leq \lambda^{-1}\pi_a(\lambda u),$$

this gives $\lambda\pi_a(u) \leq \pi_a(\lambda u)$. We conclude that $\pi_a(\lambda u) = \lambda\pi_a(u)$ for all $\lambda \geq 0$ and all $u \in X_1 \otimes ... \otimes X_m$.

Now to prove that π_a satisfies the triangle inequality. Given $u, v \in X_1 \otimes \cdots \otimes X_m$, for any $\varepsilon > 0$, we can find a representation of u and v of the form

$$u = \sum_{i=1}^{n} x_i^1 \otimes \cdots \otimes x_i^m, \quad v = \sum_{i=1}^{k} y_i^1 \otimes \cdots \otimes y_i^m,$$

such that

$$\sum_{i=1}^{n} p_1(x_i^1)...p_m(x_i^m) \leq \pi_a(u) + \frac{\varepsilon}{2} \quad \text{and} \quad \sum_{i=1}^{k} p_1(y_i^1)...p_m(y_i^m) \leq \pi_a(v) + \frac{\varepsilon}{2}.$$

Let us pose $u + v = \sum\limits_{i=1}^{n+k} z_i^1 \otimes \cdots \otimes z_i^m$ with

$\begin{cases} z_i^j = x_i^j \text{ if } 1 \leq i \leq n \\ z_i^j = y_{i-n}^j \text{ if } n+1 \leq i \leq n+k \end{cases}$ and $j = 1, ..., m$. Then we can write

$$\begin{aligned} & \sum_{i=1}^{n} p_1(x_i^1)...p_m(x_i^m) + \sum_{i=1}^{k} p_1(y_i^1)...p_m(y_i^m) \\ = & \sum_{i=1}^{n} p_1(x_i^1)...p_m(x_i^m) + \sum_{i=n+1}^{n+k} p_1(y_{i-n}^1)...p_m(y_{i-n}^m) \\ = & \sum_{i=1}^{n+k} p_1(z_i^1)...p_m(z_i^m). \end{aligned}$$

It follows that

$$\pi_a(u+v) \leq \sum_{i=1}^{n+k} p_1(z_i^1)...p_m(z_i^m) \leq \pi_a(u) + \pi_a(v) + \varepsilon,$$

for all $\varepsilon > 0$. This proves that $\pi_a(u+v) \leq \pi_a(u) + \pi_a(v)$.

Now, let us show that $\pi_a(x^1 \otimes \cdots \otimes x^m) = p_1(x^1)...p_m(x^m)$ for any $x^j \in X_j (j = 1, ..., m)$. It is clear that

$$\pi_a(x^1 \otimes \cdots \otimes x^m) \leq p_1(x^1)...p_m(x^m).$$

By the consequences of the asymmetric Hahn-Banach theorem, (see Proposition 2.3.2) for any $x^j \in X_j (j = 1, ..., m)$ there exists $\phi^j \in B_{X_j^*}$ such that

$$\left\|\phi^j\right| = 1 \quad \text{and} \quad \phi^j(x^j) = p_j(x^j), \ (j = 1, ..., m) \text{ with } p_j(x^j) > 0.$$

If $p_j(x^j) = 0$ for $1 \leq j \leq m$ then $x^j = 0$ because p_j is an N-asymmetric, and the equality required is satisfied.

Consider the m-linear form $T \in L(X_1, ..., X_m)$ (of finite type) defined by

$$T(x^1, \cdots, x^m) = \phi^1(x^1)...\phi^m(x^m).$$

By Theorem 4.1.5, the linearization of T is a linear functional

$$T_L : X_1 \otimes \cdots \otimes X_m \longrightarrow \mathbb{R},$$

such that for any $u = \sum\limits_{i=1}^{n} x_i^1 \otimes \cdots \otimes x_i^m \in X_1 \otimes \cdots \otimes X_m$ we have

$$\begin{aligned} |T_L(u)| &\leq \sum_{i=1}^{n} |T_L(x_i^1 \otimes \cdots \otimes x_i^m)| \\ &= \sum_{i=1}^{n} |T(x_i^1, \cdots, x_i^m)| \\ &= \sum_{i=1}^{n} \left|\phi^1(x_i^1)\right| ... \left|\phi^m(x_i^m)\right| \\ &= \sum_{i=1}^{n} p_1(x_i^1)...p_m(x_i^m). \end{aligned}$$

Taking the infimum over all representations of u we conclude that

$$|T_L(u)| \leq \pi_a(u),$$

for all $u \in X_1 \otimes \cdots \otimes X_m$. In particular, for the elementary tensor we have

$$p_1(x^1)...p_m(x^m) = \left|\phi^1(x^1)...\phi^m(x^m)\right| = \left|T_L(x^1 \otimes \cdots \otimes x^m)\right| \leq \pi_a(x^1 \otimes \cdots \otimes x^m),$$

it follows that $\pi_a(x^1 \otimes \cdots \otimes x^m) = p_1(x^1)...p_m(x^m)$ for all $x^j \in X_j (j = 1, ..., m)$. ■

4.3 Linearization of continuous multilinear mappings.

Before addressing the linearization of continuous multilinear mapping, we consider the projective tensor product of continuous linear operators.

Proposition 4.3.1 *Let* $s_j \in LC(X_j, Y_j)$, $(j = 1, ..., m)$ *be continuous linear operators between asymmetric normed vector spaces* (X_j, p_j) *and* (Y_j, q_j). *Then there is a unique continuous linear operator* $s_1 \otimes ... \otimes s_m : X_1 \otimes_{\pi_a} ... \otimes_{\pi_a} X_m \longrightarrow Y_1 \otimes_{\pi_a} ... \otimes_{\pi_a} Y_m$ *such that*

$$s_1 \otimes ... \otimes s_m(x^1 \otimes ... \otimes x^m) = s_1(x^1) \otimes ... \otimes s_m(x^m),$$

for every $x^j \in X_j, (j = 1, ..., m)$. *Moreover*

$$\|s_1 \otimes ... \otimes s_m| = \prod_{j=1}^{m} \|s_j| \,.$$

Proof. By Proposition 4.1.9 there is a unique linear operator

$$s_1 \otimes ... \otimes s_m : (X_1 \otimes ... \otimes X_m) \longrightarrow (Y_1 \otimes ... \otimes Y_m)\,,$$

such that $s_1 \otimes ... \otimes s_m(x^1 \otimes ... \otimes x^m) = s_1(x^1) \otimes ... \otimes s_m(x^m)$ for every $x^j \in X_j, j = 1, ..., m$. We may suppose $s_j \neq 0$, $j = 1, ..., m$. Let $u \in X_1 \otimes ... \otimes X_m$ and let $\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m$ be a representation of u. Then

$$\begin{aligned}
\pi_a\left(s_1 \otimes ... \otimes s_m(u)\right) &= \pi_a\left(\sum_{i=1}^{n} s_1(x_i^1) \otimes ... \otimes s_m(x_i^m)\right) \\
&\leq \sum_{i=1}^{n} \pi_a\left(s_1(x_i^1) \otimes ... \otimes s_m(x_i^m)\right) \\
&= \sum_{i=1}^{n} q_1(s_1(x_i^1)) ... q_m(s_m(x_i^m)) \\
&\leq \prod_{j=1}^{m} \|s_j| \sum_{i=1}^{n} p_1(x_i^1) ... p_m(x_i^m),
\end{aligned}$$

from which it follows that

$$\pi_a\left(s_1 \otimes ... \otimes s_m(u)\right) \leq \prod_{j=1}^{m} \|s_j| \, \pi_a(u),$$

so that the linear operator $s_1 \otimes ... \otimes s_m$ is continuous for the projective asymmetric norm on $X_1 \otimes ... \otimes X_m$ and $Y_1 \otimes ... \otimes Y_m$. Moreover

$$\|s_1 \otimes ... \otimes s_m| \leq \prod_{j=1}^{m} \|s_j| \, .$$

On the other hand,

$$\begin{aligned}
\|s_1| ... \|s_m| &= \sup_{p_1(x^1) \leq 1} q_1(s_1(x^1)) ... \sup_{p_m(x^m) \leq 1} q_m(s_m(x^m)) \\
&= \sup_{p_j(x^j) \leq 1 (j=1,...,m)} q_1(s_1(x^1)) ... q_m(s_m(x^m)) \\
&= \sup_{p_j(x^j) \leq 1 (j=1,...,m)} \pi_a\left(s_1(x^1) \otimes ... \otimes s_m(x^m)\right) \\
&= \sup_{p_j(x^j) \leq 1 (j=1,...,m)} \pi_a\left(s_1 \otimes ... \otimes s_m(x^1 \otimes ... \otimes x^m)\right) \\
&\leq \sup\left\{\pi_a\left(s_1 \otimes ... \otimes s_m(u)\right), u \in X_1 \otimes_\pi ... \otimes_\pi X_m \text{ with } \pi_a(u) \leq 1\right\} \\
&= \|s_1 \otimes ... \otimes s_m| \, .
\end{aligned}$$

Thus we have

$$\|s_1 \otimes ... \otimes s_m| = \|s_1| ... \|s_m| .$$

■

Theorem 4.3.2 *Let $(X_1, p_1), ..., (X_m, p_m)$ be real N-asymmetric normed vector spaces and let (Y, q) be a real asymmetric normed vector space. For every continuous m-linear mapping $T : X_1 \times ... \times X_m \longrightarrow Y$ there exists a unique continuous linear operator T_L : $X_1 \otimes_{\pi_a} ... \otimes_{\pi_a} X_m \longrightarrow Y$ satisfying*

$$T_L\left(x_i^1 \otimes ... \otimes x_i^m\right) = T\left(x_i^1, ..., x_i^m\right),$$

for every $x^j \in X_j (j = 1, ..., m)$. Moreover,

$$\|T_L| = \|T|_{(p_1,...,p_m;q)} .$$

Proof. Let $T_L : X_1 \otimes ... \otimes X_m \longrightarrow Y$ the unique linearization of T. Let us show that T_L is continuous for the projective asymmetric norm on $X_1 \otimes ... \otimes X_m$. For $u = \sum\limits_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \in X_1 \otimes ... \otimes X_m$ we have

$$q(T_L(u)) \leq \sum_{i=1}^{n} q(T(x_i^1, ..., x_i^m)) \leq \|T|_{(p_1,...,p_m;q)} \sum_{i=1}^{n} p_1(x_i^1)...p_m(x_i^m).$$

Taking the infimum over all representations of u it follows that

$$q(T_L(u)) \leq \|T|_{(p_1,...,p_m;q)} \pi_a(u),$$

showing that

$$T_L \in LC(X_1 \otimes_{\pi_a} ... \otimes_{\pi_a} X_m, Y) \quad \text{and} \quad \|T_L| \leq \|T|_{(p_1,...,p_m;q)} .$$

On the other hand for any $x^j \in X_j (j = 1, ..., m)$,

$$q(T(x^1, ..., x^m)) = q(T_L(x^1 \otimes ... \otimes x^m)) \leq \|T_L| \, p_1(x^1)...p_m(x^m),$$

this becomes $\|T|_{(p_1,...,p_m;q)} \leq \|T_L|$. Thus, we conclude that $\|T_L| = \|T|_{(p_1,...,p_m;q)}$. ■

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