RÉPUBLIQUE ALÉRIENNE DÉMOCRATIQUE ET POPULAIRE
MINISTÈRE DE L'ENSEIGNEMENT SUPÉRIEUR ET DE LA RECHERCHE SCIENTIFIQUE
UNIVERSITÉ MOHAMED BOUDIAF -M'SILA-
FACULTÉ DE MATHÉMATIQUES ET DE L'INFORMATIQUE
DÉPARTEMENT DE MATHÉMATIQUES

جامعة محمد بوضياف - المسيلة
Université Mohamed Boudiaf - M'sila

N° d'ordre: ....

THÈSE

Présentée pour l'obtention du diplôme
de Doctorat en sciences

Spécialité
Mathématiques
Option
Algèbre Appliquée
Par
Soheyb MILLES
Thème

Sur les ensembles ordonnés flous intuitionnistes

Soutenue le 18/12/2017 devant le jury composé de:

Douadi MIHOUBI	**Prof.**	**Université de M'sila**	**Président**
Lemnaouar ZEDAM	**Prof.**	**Université de M'sila**	**Encadreur**
Ewa RAK	**M.C.**	**Université de Rzeszów-Pologne**	**Co-Encadreur**
Abdelaziz AMROUNE	**Prof.**	**Université de M'sila**	**Examinateur**
Lemnaouar NOUI	**Prof.**	**Université de Batna II**	**Examinateur**
Djamel BENTERKI	**Prof.**	**Université de Sétif**	**Examinateur**
Abdelmadjid BOUDAOUD	**Prof.**	**Université de M'sila**	**Invité**

Année Universitaire : 2017/2018

Msila University
Faculty of Mathematics and Informatics

On the intuitionistic fuzzy ordered sets

"Sur les ensembles ordonnés flous intuitionnistes"

Soheyb Milles

Thesis submitted in fulfillment of the requirements for the degree of Doctor of Mathematics
Academic year 2017-2018

Jury president: Prof. dr. Douadi Mihoubi
Faculty of of Mathematics and Informatics
Department of Mathematics
Msila University-Algeria

Supervisor: Prof. dr. Lemnaouar Zedam
Faculty of Mathematics and Informatics
Department of Mathematics
Msila University-Algeria

Co-Supervisor: Dr. Ewa Rak
Faculty of Mathematics and Natural Sciences
University of Rzeszów-Poland

Examination committee: Prof. dr. Abdelaziz Amroune
Faculty of Mathematics and Informatics
Department of Mathematics
Msila University-Algeria

Prof. dr. Lemnaouar Noui
Department of Mathematics
Batna II University-Algeria

Prof. dr. Djamal Benterki
Department of Mathematics
Setif University-Algeria

Soheyb Milles

On the intuitionistic fuzzy ordered sets

Thesis submitted in fulfillment of the requirements for the degree of

Doctor of Mathematics

Academic year 2017-2018

Please refer to this work as follows:
Soheyb Milles (**2017**). *On the intuitionistic fuzzy ordered sets*, Doctoral thesis, Department of Mathematics, Med Boudiaf University of Msila, Algeria.

Acknowledgements

I acknowledge the presence of God who created me and gave me this rare privilege to achieve my dream of attaining the highest qualification.

This thesis is an output of several years of research that was started at the University of Msila under the supervision of Prof. dr. Lemnaouar Zedam with collaboration of dr. Ewa Rak (University of Rzeszów in Poland).

I would like to express my sincere gratitude to my supervisors Prof. dr. L. Zedam and dr. E. Rak for the continuous support of my Ph.D study and research, for their patience, motivation, enthusiasm, and immense knowledge. Their guidance helped me the whole time of research and writing of this thesis. It is a pleasure to convey my gratitude to them all in my humble acknowledgements. I will never forget their guidance and the help they gave me, even during weekends, holidays and all other opportunities.

I sincerely thank the chairman of the jury committee, Prof. dr. D. Mihoubi, and the other jury members, Prof. dr. A. Amroune, Prof. dr. L. Noui and Prof. dr. Dj. Bentorki.

I wish to express my profound appreciation to my colleagues at Msila university, for the friendly atmosphere. I sincerely thank H. Bourmel and N. Ferahtia for their kind help during my work in Msila University.

I am very grateful to my mother and father. Their prayers, passionate encouragements, and generosities have followed me everywhere to give me a lot of power. My deepest gratitude goes to my sisters and brothers. I wish all of you a prosperous life full of happiness and health. My lovely wife and my adorable children 'Maria' and 'Sirine', you were the main supporters of me along my entire Ph.D. thesis. I am deeply grateful for your patience and sacrifices. I hope I can compensate you with all my love for all the moments which I spent far away from you.

Msila, December 2017
Soheyb Milles

Table of contents

Introduction

In 1983, Atanassov [3] introduced a new notion called an intuitionistic fuzzy set (or Atanassov's intuitionistic fuzzy set) as a generalization of Zadeh's fuzzy set [62]. In fuzzy set theory, the non-membership degree of an element x can be viewed as $\nu_A(x) = 1 - \mu_A(x)$ (using the standard strong negation on the real interval $[0, 1]$), which is fixed, while in intuitionistic fuzzy setting, the non-membership degree is a more-or-less independent degree: the only condition is that $\nu_A(x) \leq 1 - \mu_A(x)$. Certainly, fuzzy sets are Atanassov's intuitionistic fuzzy sets by setting $\nu_A(x) = 1 - \mu_A(x)$.

Based on Atanassov's intuitionistic fuzzy set, Burillo and Bustince [15, 16] introduced the concept of intuitionistic fuzzy relation, in particular, they introduced the intuitionistic fuzzy order (or intuitionistic fuzzy ordered set) as a natural generalization of fuzzy order relation previously introduced by Zadeh [63]. Intuitionistic fuzzy relations theory has been applied to many different fields, such as decision making, mathematical modeling, medical diagnosis, control systems, machine learning, market prediction, and so on.

One of the important problems of fuzzy and intuitionistic fuzzy ordered set is to obtain an appropriate concept of the particular elements on such structures, like an upper bound, a maximum, a supremum, maximal elements and their duals, in order to obtain new structures and particular classes of fuzzy and intuitionistic fuzzy ordered sets. Several theoretical and applicational results connected with this problem can be found, e.g. in Bělohlávek [9], Bodenhofer and Klawonn [12], Bustince and Burillo [18, 19], Coppola et al. [21], Tripathy et al. [54] and Zadeh [63]. In particular, Tripathy and his colleagues [54] introduced the concepts of the upper bound, the supremum and their duals of subsets on a universe X with respect to an intuitionistic fuzzy order defined on it. Also, they introduced and studied a concept of lattice with respect to an intuitionistic fuzzy order defined on it. This concept is extensively used and discussed in the fuzzy and intuitionistic fuzzy settings by several authors [9, 36, 50, 56, 68].

Above, there is presented the general context of this thesis. But, in a particular context, this thesis aims to study the following four notions:
(i) the notion of an intuitionistic fuzzy ideal (resp. a filter) on a crisp lattice;
(ii) the notion of an intuitionistic fuzzy ideal (resp. a filter) on an intuitionistic fuzzy ordered lattice;
(iii) the notion of an intuitionistic fuzzy complete lattice;
(iv) the notion of fixed point property of intuitionistic fuzzy lattices.

The notion of an ideal or its dual (a filter) is recognized as one of the most important concepts in the lattices theory and theory of other algebraic structures used in formal fuzzy logic. These notions are mainly used to translate connections between properties on algebraic structures and to define congruence relations and quotient algebras [55, 43]. They are played a central role in Stone representation theorem for Boolean lattice [48] and in the extensive theory of representation of a distributive lattice [22, 27, 45]. In topology and approaches to its analysis, ideals and filters are appeared to provide very general contexts to unify the various notions of sequences convergence and limit in arbitrary topological spaces, and to express completeness and compactness in metric spaces [13, 58]. In fuzzy setting, for the same purposes, several authors introduced and investigated the concepts of some kinds of fuzzy ideals and fuzzy filters in different ways and on different structures. The first approach considered fuzzy ideal and fuzzy filter as fuzzy sets on crisp structures, like on lattices or on residuated lattices [11, 24, 32, 33, 53], on BL-algebras [35] and on ordered ternary semigroups [2, 8, 20]. The second approach proposed similar notions on fuzzy structures, see e.g., Mezzomo et al. [38] for the approache in fuzzy ordered lattices.

Extended approaches based on Atanassov's intuitionistic fuzzy sets were proposed by many authors. One of the approaches considered intuitionistic fuzzy ideals and filters as Atanassov's intuitionistic fuzzy sets on a crisp structure. For example Kim and Jun [31] introduced the notion of intuitionistic fuzzy interior ideals of semigroups. Banerjee and Basnet in [7] studied the notion of intuitionistic fuzzy ideals on a ring, Akram and Dudek [1] introduced and studied a concept of intuitionistic fuzzy Lie ideals. Qin and Liu [34, 44] introduced and investigated the properties of intuitionistic fuzzy filters on a residuated lattice and Thomas and Nair [51, 52] considered intuitionistic fuzzy sublattices, intuitionistic fuzzy ideals and intuitionistic fuzzy filters on a lattice. In [59], Xu introduced the notion of interval valued intuitionistic (T, S)-fuzzy filter on a lattice implication algebra subsequently, some basic properties of it were obtained.

Due to the usefulness of these concepts in different structures, the first aim of this thesis is to investigate the intuitionistic fuzzy ideals and filters on a crisp lattices. The second aim is to extend the results of fuzzy ideals and filters to intuitionistic fuzzy ideals and filters on intuitionistic fuzzy lattices. For both approaches, we present interesting characterizations of these notions in term of lattice operations and in term of their (α, β)-level sets. Moreover, we extend the notion of a prime ideal (resp. prime filter) to intuitionistic fuzzy ideal (resp. intuitionistic fuzzy filter) with respect to the lattice operations and investigate their various characterizations and properties.

As the third purpose of the dissertation, based on the concept of intuitionistic fuzzy lattice previously proposed by Tripathy et al. [54],

we introduce the notion of intuitionistic fuzzy complete lattice and investigate its basic characterizations. This notion is a generalization of the notion of crisp complete lattice. In that point, we extend these characterizations by considering other completeness criteria. The characterizations of intuitionistic fuzzy complete lattices discussed in terms of the existence of the supremum or the infimum of their subsets, in terms of intuitionistic fuzzy chains and maximal chains and in terms of intuitionistic fuzzy ascending (resp. descending) chains are given. Furthermore, we show that an intuitionistic fuzzy lattice X is complete if and only if any intuitionistic fuzzy monotone mapping $f : X \to X$ has a fixed point, i.e., an intuitionistic fuzzification of Tarski-Davis's fixed point theorem [23, 49]. This leads to see clearly that any intuitionistic fuzzy complete lattice has the fixed point property and vice versa. These results show the key role of the fixed point property for establishing a completeness criterion for intuitionistic fuzzy lattices.

Note that many function spaces, in general fuzzy or intuitionistic fuzzy function spaces, can be viewed as intuitionistic fuzzy lattices. This fact allows the obtained results to be used for expressing mathematical problems in fuzzy and intuitionistic fuzzy function spaces. Particularly, the fixed point property for these spaces as intuitionistic fuzzy lattices can be employed to examine the theoretical solvability of linear, integral or differential equations and to develop numerical approaches for their solutions.

This thesis is structured as follows.

- In Chapter 1, we provide generalities on intuitionistic fuzzy sets, intuitionistic fuzzy relations and intuitionistic fuzzy lattices, that we need throughout this thesis.

- In Chapter 2, we investigate the intuitionistic fuzzy ideal and filter concepts on a lattices and their fundamental properties. We present interesting characterizations of these notions in terms of lattice operations and in terms of their (α, β)-level sets. Moreover, we extend the notion of prime ideal (resp. prime filter) to intuitionistic fuzzy ideal (resp. intuitionistic fuzzy filter) with respect to the lattice operations and investigate their various characterizations and properties.

- In Chapter 3, we define the intuitionistic fuzzy ideal and filter on an intuitionistic fuzzy ordered lattice and we present interesting characterizations of these notions in terms of lattice operations and in terms of their (α, β)-level sets. Also, we introduce and investigate two interesting kinds, principal and prime intuitionistic fuzzy ideals and filters.

- In Chapter 4, we introduce the notion of an intuitionistic fuzzy complete lattice and investigate its basic characterizations in terms of the existence of the supremum or the infimum of their subsets or in terms of intuitionistic fuzzy chains and maximal chains. Also, we extend these characterizations by considering others completeness criteria. We provide a characterization of intuitionistic fuzzy complete lattice in terms of intuitionistic fuzzy ascending (resp. descending) chains and maximal chains. In that chapter, we show that an intuitionistic fuzzy complete lattices has the fixed point property and vice versa, i.e., an intuitionistic fuzzification of Tarski-Davis's fixed point theorem [23, 49]. These results show the key role of the fixed point property for establishing a completeness criterion for intuitionistic fuzzy lattices.

- Finally, general conclusions and future research are drawn.

Most parts of results presented in this thesis has already been published or submitted for publication in peer-reviewed international journals. Results included in Chapter 2 has been described in [40] and those included in Chapter 4 has been described in [67].

1 Generalities on intuitionistic fuzzy sets and relations

The purpose of this first chapter is to provide a basic introduction to the binary relations, posets, lattices, t-norm. Next, we recall some basic notions of fuzzy sets, intuitionistic fuzzy sets and intuitionistic fuzzy relations. Many of the properties of these concepts will be used in the next chapters.

1.1. Binary relations, ordered sets and lattices

This section contains the basic definitions and properties of binary relations, posets, lattices.

A binary relation on a set X is a subset of X^2, i.e., it is a set of couples $(x, y) \in X^2$. For a relation $R \subseteq X^2$, we often write xRy instead of $(x, y) \in R$. Two elements x and y of a set X equipped with a relation R are called comparable elements, denoted by $x \nparallel y$, if it holds that xRy or yRx. Otherwise, they are called incomparable elements, denoted by $x \parallel_R y$, or simply $x \parallel y$ when no confusion can occur. We denote by R^c the complement of the relation R on X, i.e., for any $x, y \in X$, xR^cy denotes the fact that $(x, y) \notin R$. We denote by R^t the transpose of the relation R on X, i.e., for any $x, y \in X$, xR^ty denotes the fact that yRx. We denote by R^d the dual of the relation R on X, i.e., for any $x, y \in X$, xR^dy denotes the fact that yR^cx. A relation R on a set X is said to be included in a relation S on the same set X, denoted by $R \subseteq S$, if, for any $x, y \in X$, xRy implies that xSy. The union of two relations R and S on a set X is the relation $R \cup S$ on X defined as $R \cup S = \{(x, y) \in X^2 \mid xRy \vee xSy\}$. Similarly, the intersection of two relations R and S on a set X is the relation $R \cap S$ on X defined as $R \cap S = \{(x, y) \in X^2 \mid xRy \wedge xSy\}$. If $R \cap S = \emptyset$, then R and S are called disjoint relations. The composition of two relations R and S on a set X is the relation $R \circ S$ on X defined as $R \circ S = \{(x, z) \in X^2 \mid (\exists y \in X)(xRy \wedge ySz)\}$. For any $n \in \mathbb{N}^*$, the n-th power relation R^n of R is recursively defined as follows:

$$(R^1 = R) \wedge (\forall n \geqslant 1)(R^{n+1} = R^n \circ R) .$$

A binary relation R on a set X is called:

(i) reflexive, if, for any $x \in X$, it holds that xRx;

(ii) irreflexive, if, for any $x \in X$, it holds hat xR^cx;

(iii) symmetric, if, for any $x, y \in X$, it holds that xRy implies that yRx;

(iv) antisymmetric, if, for any $x, y \in X$, it holds that xRy and yRx imply that $x = y$;

(v) asymmetric, if, for any $x, y \in X$, it holds that xRy implies that $yR^c x$;

(vi) transitive, if, for any $x, y, z \in X$, it holds that xRy and yRz imply that xRz;

(vii) complete, if, for any $x, y \in X$, either xRy or yRx holds.

A binary relation R on a set X is called:

(i) a pseudo-order relation, if it is reflexive and antisymmetric;

(ii) a strict order, if it is irreflexive and transitive;

(iii) an order relation, if it is reflexive, antisymmetric and transitive;

(iv) a total order relation, if it is reflexive, antisymmetric, transitive and complete;

For more details on binary relations, we refer to [22, 45, 47].

A partial order (order, for short) is a binary relation $\leqslant$ over a set X which is reflexive ($a \leqslant a$, for any $a \in X$), antisymmetric ($a \leqslant b$ and $b \leqslant a$ implies $a = b$, for any $a, b \in X$) and transitive ($a \leqslant b$ and $b \leqslant c$ implies $a \leqslant c$, for any $a, b, c \in X$). A set with an order relation is called an ordered set (also called a poset). Further, $\{x, y\}^u$ denotes the set of all upper bounds of x and y, while $\{x, y\}^l$ denotes the set of all lower bounds of x and y, i.e., $\{x, y\}^u = \{z \in X \mid x \leqslant z \ \wedge \ y \leqslant z\}$ and $\{x, y\}^l = \{z \in X \mid z \leqslant x \ \wedge \ z \leqslant y\}$.

A strict order is a binary relation $<$ on a set X that is irreflexive ($a < a$ does not hold for any $a \in X$), asymmetric (if $a < b$, thus $b < a$ does not hold for any $a, b \in X$) and transitive. A given binary relation $\sim$ on a set X is said to be an equivalence relation if it is reflexive, symmetric ($a \sim b$ implies $b \sim a$, for any $a, b \in X$) and transitive. If $\leqslant$ is an order, then the corresponding strict order $<$ is the irreflexive kernel given by:

$$a < b \ \text{ if } \ a \leqslant b \ \text{ and } \ a \neq b.$$

Conversely, if $<$ is a strict order, then the corresponding order $\leqslant$ is the reflexive closure given by

$$a \leqslant b \ \text{ if } \ a < b \ \text{ or } \ a = b.$$

Two elements x and y of X are called comparable if $x \leqslant y$ or $y \leqslant x$; otherwise they are called incomparable, and we write $x \parallel y$. Using the strict order $<$, the relation (x is covered by y) denoted as $x \lll y$, if $x < y$ and there exists no $z \in X$ such that $x < z < y$. A poset can be conveniently represented by a Hasse diagram, displaying the covering relation $<$. Note that $x < y$ if there is a sequence of connected lines upwards from x to y.

For more details about order, strict order and equivalence relations we refer to [45].

Definition 1.1. [22] *Let $(X, \leqslant_1)$ and $(Y, \leqslant_2)$ two posets, then the mapping $f : X \to Y$ is monotone (or order-preserving) if $x \leqslant_1 y \Rightarrow f(x) \leqslant_2 f(y)$, for any $(x, y \in X)$*

Definition 1.2. [22] *Let P be an ordered set. Then P is a chain if for any $x, y \in P$, either $x \leq y$ or $y \leq x$ (that is, if any two elements of P are comparable). Alternative names for a chain are linearly ordered set and totally ordered set.*

Zorn's lemma is a result in set theory that appears in proofs of some non-constructive existence theorems throughout mathematics.

Theorem 1.1. *(Zorn's lemma) Let P be a non-empty ordered set in which every nonempty chain has an upper bound. Then P has a maximal element.*

Many important properties of an order set $(L, \leq)$ are expressed in terms of the existence of certain upper bounds or lower bounds of subsets of X. Particularly, we are interested in two of the most important classes of ordered sets defined in this way. They are a lattice and a complete lattice. We often write $x \vee y$ instead of $\sup\{x, y\}$ when it exists and $x \wedge y$ instead of $\inf\{x, y\}$ when it exists. Similarly, we write $\bigvee S$ (the join of S) and $\bigwedge S$ (the meet of S) instead of $\sup S$ and $\inf S$ when they exist.

Definition 1.3. [22] *Let $(X, \leq)$ be an ordered set.*

(i) If $x \vee y$ exists for any $x, y \in X$, then $(X, \leq)$ is called a $\vee$-semi-lattice.

(ii) If $x \wedge y$ exists for any $x, y \in X$, then $(X, \leq)$ is called a $\wedge$-semi-lattice.

(iii) $(X, \leq)$ is called a lattice if it is both a $\wedge$-semi-lattice and a $\vee$-semi-lattice.

(iv) If $\bigvee S$, $\bigwedge S$ exist for any $S \subseteq X$, then $(X, \leq)$ is called a complete lattice.

A bounded lattice is a lattice that additionally has a greatest element 1 and a smallest element 0, which satisfy $0 \leq x \leq 1$, for any x in X.

A lattice $(L, \leq, \wedge, \vee)$ is distributive if the following additional condition holds

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\,, \text{ for any } x, y, z \in L\,.$$

This means that the meet operation preserves non-empty finite joins. It is known that the above condition is equivalent to its dual

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\,, \text{ for any } x, y, z \in L\,.$$

Definition 1.4. [22] *Let L and L' be two lattices. A mapping $f : L \to L'$ is called an homomorphism if $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$, for any $x, y \in L$. If f is a bijection, then f is called an isomorphism.*

1.2. Ideals and filters on a crisp lattice

Ideals are of fundamental importance in algebra. Filters, the order duals of lattice ideals, have a variety of applications in logic and topology.

Definition 1.5. [22] *A nonempty subset I on a lattice L is called an ideal of L if, for any $x, y \in L$, the following conditions are satisfied:*

1. *if $y \in I$ and $x \leq y$, then $x \in I$,*
2. *if $x, y \in I$ implies $x \vee y \in I$.*

The definition can be more compactly stated by declaring an ideal to be a non-empty down-set closed under join.

A dual ideal is called a filter. Specifically, a non-empty subset of L determined by the following definition.

Definition 1.6. [22] *A nonempty subset F on a lattice L is called a filter if, for any $x, y \in L$, the following conditions are satisfied:*

1. *if $y \in F$ and $y \leq x$, then $x \in F$,*
2. *if $x, y \in F$ implies $x \wedge y \in F$.*

The set of all ideals (resp. filters) of L is denoted by $\mathcal{I}(L)$ (resp. $\mathcal{F}(L)$), and carries the usual inclusion order.

More precisely, an ideal or filter is called proper if it does not coincide with L. More precisely, an ideal I of a lattice with 1 is proper if and only if $1 \notin I$, and dually, a filter F of a lattice with 0 is proper if and only if $0 \notin F$. For any $a \in L$, the set $\downarrow a$ is an ideal (also known as the principal ideal generated by a). Dually, $\uparrow a$ is a principal filter.

Definition 1.7. [22] *An ideal I on a lattice L is called a prime ideal if, $x \wedge y \in L$, then $x \in I$ or $y \in I$, for any $x, y \in L$.*

Definition 1.8. [22] *A filter F on a lattice L is called a prime filter if, $x \vee y \in F$, then $x \in F$ or $y \in F$, for any $x, y \in L$.*

Definition 1.9. [22] *Let L be a lattice. A proper ideal (resp. filter) A is said to be a maximal ideal (resp. maximal filter or more usually known as an ultrafilter) if the only ideal (resp. filter) properly containing A is L.*

Example 1.1. *(i) The following are ideals in $\mathcal{P}(X)$*

   (a) all subsets not containing a fixed element of X,

   (b) all finite subsets (this ideal is non-principal if X is infinite).

(ii) Let $(X, \mathfrak{T})$ be a topological space and let $x \in X$. Then the set $\{V \subseteq X \mid (\exists U \in \mathfrak{T}) x \in U \subseteq V\}$ is a filter in $(X, \mathfrak{T})$.

1.3. t-norms and t-conorms

The history of triangular-norms (*t-norms*) started with Menger [37]. His main idea was to construct metric spaces where probability distributions are used to describe the distance between two elements. Schweizer and Sklar [46] provided the axioms of *t-norms*, as they are used today.

Definition 1.10. [41] *A t-norm T on $[0,1]$ is a function $T : [0,1]^2 \to [0,1]$ satisfies the following four axioms:*

(T1) *Commutativity:* $(\forall x, y \in [0,1])(T(x,y) = T(y,x))$;

(T2) *Associativity:* $(\forall x, y, z \in [0,1])(T(x, T(y,z)) = T(T(x,y), z)))$;

(T3) *Monotonicity:* $(\forall x, y, z \in [0,1)(x \leq y \Rightarrow T(x,z) \leq T(y,z))$;

(T4) *Boundary condition:* $(\forall x \in [0,1])(T(x,1) = x)$.

Conditions (T4) and (T3) imply that for any t-norm T it holds that $T(x,y) \leq x$, $T(x,y) \leq y$, $T(x,y) \leq Min(x,y)$ and $T(x,0) = 0$.

Example 1.2. *The following four operations are the most common t-norms:*

(T5) *Minimum:* $T_M(x,y) = \min\{x,y\}$

(T6) *Product:* $T_P(x,y) = x.y$

(T7) *Lukasiewicz:* $T_L(x,y) = \max\{x+y-1, 0\}$

(T8) *Drastic product:*

$$T_D(x,y) = \begin{cases} x & \textit{if} \;\; y = 1 \\ y & \textit{if} \;\; x = 1 \\ 0 & \textit{if} \;\; x, y < 1. \end{cases}$$

Let T be a t-norm on $[0,1]$.
An element $a \in]0,1[$ is called a zero divisor of T if there exists some $b > 0$ such that $T(a,b) = 0$.
An element $a \in [0,1]$ is called an idempotent element of T if $T(a,a) = a$.
T is called Archimedean if $T(x,x) < x$, for any $x \in [0,1]$.
Each $a \in [a,b]$ is an idempotent element of the Minimum t-norm T_M (Actually T_M is the only t-norm whose set of idempotent is equal $[0,1]$), T_M has no zero divisor.
Each $a \in]0,1[$ is a zero divisor of the Lukasiewicz t-norm T_L as well of the Drastic product t-norm T_D.

For two t-norms T_1 and T_2 on $[0,1]$, we define:

$$T_1 \leq T_2 \Leftrightarrow (\forall x, y \in [0,1])(T_1(x,y) \leq T_2(x,y)) \, .$$

Let be T_1 and T_2 two t-norms. If $T_1 \leq T_2$, then T_1 is called weaker than T_2 (or, equivalently, T_2 is called stronger than T_1). Note that T_D is the weakest t-norm, and T_M is the strongest t-norm, i.e. for any t-norm it holds: (T9) $\mathrm{T_D} \leq \mathrm{T} \leq \mathrm{T_M}$. Since $T_L \leq T_P$, it obviously holds: (T10) $\mathrm{T_D} \leq \mathrm{T_L} \leq \mathrm{T_P} \leq \mathrm{T_M}$.

Triangular conorms (t-conorms) are dual operations of t-norms, we recall the following definition of conorms.

Definition 1.11. [41] *A t-conorm is a function $S : [0,1]^2 \to [0,1]$ that for any $x, y, z \in [0,1]$ satisfies* (T1)-(T3) and the following boundary condition $S(x, 0) = S(0, x) = x$, $S(x, 1) = S(1, x) = 0$

Remark 1.1. *Given a t-norm T, we find the associated dual t-conorm S by $S(x, y) = 1 - T(1 - x, 1 - y)$.*

The dual t-conorms w.r.t. T_M, T_P, T_L and T_D are given by:

(S1) Maximum: $S_M(x, y) = \max\{x, y\}$

(S2) Probabilistic sum: $S_P(x, y) = x + y - s.y$

(T7) Lukasiewicz: $S_L(x, y) = \min\{x + y, 1\}$

(T8) Drastic sum:

$$S_D(x, y) = \begin{cases} 1, & \text{if } (x, y) \in [0, 1)^2 \\ \max\{x, y\}, & \text{otherwise} \end{cases}$$

1.4. Intuitionistic fuzzy sets

This section contains the basic definitions and properties of fuzzy sets, intuitionistic fuzzy sets and several operations on intuitionistic fuzzy sets. The notion of fuzzy set was introduced in 1965 by Lotfi A. Zadeh in the paper [62].

Definition 1.12. [62] *Let X be a nonempty set. A fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ is characterized by a membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A, for $x \in X$.*

For two fuzzy sets A and B on a set X, several operations are defined in the following way (see [62]).

(i) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$, for any $x \in X$;

(ii) $A = B$ if $\mu_A(x) = \mu_B(x)$, for any $x \in X$;

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x)\rangle \mid x \in X\}$;

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x)\rangle \mid x \in X\}$;

(v) $\overline{A} = \{\langle x, 1 - \mu_A(x)\rangle \mid x \in X\}$.

Definition 1.13. [42] *Let A be a fuzzy set on a set X. The support of A is the crisp subset on X given by*

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0\}.$$

Definition 1.14. [42] *Let A be a fuzzy set on a set X. The kernel of A is the crisp subset on X given by*

$$Ker(A) = \{x \in X \mid \mu_A(x) = 1\}.$$

Definition 1.15. [42] *Let A be a fuzzy set on a set X. The height of A is the highest value taken by its membership function given by*

$$H(A) = \sup_{x \in X} \mu_A(x)$$

Example 1.3. *Let $X = [0, 1]$ with $\alpha, \beta \in \mathbb{R}$ and let $a, b \in \mathbb{R}$. We define the fuzzy set A on X by*

$$\mu_A(x) = \begin{cases} 0 \ , \ if \ x < a - \alpha \ or \ b + \beta < x, \\ 1 \ , \quad if \quad a < x < b, \\ 1 + (\frac{x-a}{\alpha}) \ , \quad if \quad a - \alpha < x < a \, , \\ 1 - (\frac{b-x}{\beta}) \ , \quad if \quad b < x < b + \beta. \end{cases}$$

Then $Ker(A) = [0, 1]$, $Supp(A) = [a - \alpha, b + \beta]$ and $H(A) = 1$.

In the sequel, we give the following definition of level sets (which is also often called α-cuts) of a fuzzy set.

Definition 1.16. [42] *Let A be a fuzzy set on a set X. The α-cut of A is a crisp subset*

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\},$$

where $\alpha \in [0, 1]$, $A_0 = X$ and $A_1 = Ker(A)$.

In 1983, Atanassov [3] proposed a generalization of Zadeh membership degree and introduced the notion of the intuitionistic fuzzy set.

Definition 1.17. [3] *Let X be a nonempty set. An intuitionistic fuzzy set (IFS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a membership function $\mu_A : X \to [0, 1]$ and a non-membership function $\nu_A : X \to [0, 1]$ which satisfy the condition:*

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \ for \ any \ x \in X.$$

For any $x \in X$ the number $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the hesitation degree or the intuitionistic index of x to A.

The class of intuitionistic fuzzy sets on X is denoted by $IFS(X)$.

Certainly, fuzzy sets are intuitionistic fuzzy sets by setting $\nu_A(x) = 1 - \mu_A(x)$.

Example 1.4. *Let X be the set of all countries with elective governments. Assume that we know for every country $x \in X$ the percentage of the electorate that have voted for the corresponding government. Denote it by $M(x)$ and let $\mu(x) = \frac{M(x)}{100}$ (degree of membership, validity, etc.). Let $\nu(x) = 1 - \mu(x)$. This number corresponds to the part of electorate who have not voted for the government. Using only the fuzzy set theory, we cannot consider this value in more detail. However, if we define $\nu(x)$ (degree of non-membership, non-validity, etc.) as the number of votes given to parties or persons outside the government, then we can show the part of electorate who have not voted at all or who have given bad voting-paper and the corresponding number will be $\pi(x) = 1 - \mu(x) - \nu(x)$ (degree of indeterminacy, uncertainty, etc.). Thus, we can construct the set $\{\langle x, \mu(x), \nu(x)\rangle \mid x \in X\}$ and obviously,*

$$0 \leq \mu(x) + \nu(x) \leq 1.$$

For two intuitionistic fuzzy sets A and B on a set X, several operations are defined as follows (see, e.g., Atansssov [4, 5, 6], Biswas [10] and Gy [57]). Here we will present only those which are related to the present work.

(i) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for any $x \in X$;

(ii) $A = B$ if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$, for any $x \in X$;

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$;

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$;

(v) $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$;

(vi) $[A] = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$;

(vii) $\langle A\rangle = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \mid x \in X\}$;

(viii) $Ker(A) = \{x \in X \mid \mu_A(x) = 1 \text{ and } \nu_A(x) = 0\}$;

(ix) $Supp(A) = \{x \in X \mid \mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } \nu_A(x) < 1)\}$.

In the sequel, we need the following definition of level sets (which is also often called (α, β)-cuts) of an intuitionistic fuzzy set.

Definition 1.18. [61] *Let A be an intuitionistic fuzzy set on a set X. The (α, β)-cut of A is a crisp subset*

$$A_{\alpha,\beta} = \{x \in X \mid \mu_A(x) \geq \alpha \textit{ and } \nu_A(x) \leq \beta\},$$

where $\alpha, \beta \in [0, 1]$ *with* $\alpha + \beta \leq 1$.

1.5. Intuitionistic fuzzy relations

This section contains the basic definitions and properties of fuzzy relations, intuitionistic fuzzy relations, intuitionistic fuzzy order relations and several operations on intuitionistic fuzzy relations. The notion of fuzzy relations was first introduced by Zadeh [63] as a natural extension to a fuzzy set and plays an important role in the theory of such sets and their applications.

Definition 1.19. [63] *Let* X *and* Y *be two nonempty sets. A binary fuzzy relation from* X *to* Y*, is a fuzzy subset of* $X \times Y$ *characterized by a membership function* μ_R *which associates with each pair* (x, y) *its grade of membership* $\mu_R(x, y)$ *in the interval* $[0, 1]$.

Burillo and Bustince [15, 16] introduced the concept of intuitionistic fuzzy relation as a natural generalization of fuzzy relation.

Definition 1.20. [15, 16] *Let* X *and* Y *be two nonempty sets. An intuitionistic fuzzy binary relation (an intuitionistic fuzzy relation, for short) from* X *to* Y *is an intuitionistic fuzzy subset of* $X \times Y$*, i.e., is an expression* R *given by*

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \mid (x, y) \in X \times Y\},$$

where

$$\mu_R : X \times Y \to [0, 1], \text{ and } \nu_A : X \times Y \to [0, 1]$$

satisfy the condition

$$0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1, \tag{1.1}$$

for any $(x, y) \in X \times Y$. *The value* $\mu_R(x, y)$ *is called the degree of membership of* (x, y) *in* R *and* $\nu_R(x, y)$ *is called the degree of non-membership of* (x, y) *in* R.

Several operations on intuitionistic fuzzy relations are defined (see, e.g., Atansssov [4, 5, 6], Biswas [10] and Gy [57]). Here we will present only those which are related to the present work.

Let R, P be two intuitionistic fuzzy relations. R is said to be contained in P (or we say that P contains R), denoted by $R \subseteq P$, if for any $(x, y) \in X \times Y$ it holds that $\mu_R(x, y) \leq \mu_P(x, y)$ and $\nu_R(x, y) \geq \nu_P(x, y)$. The *transpose* (or the *inverse*) R^t of R is the intuitionistic fuzzy relation from Y to X defined by $R^t = \{\langle (x, y), \mu_{R^t}(x, y), \nu_{R^t}(x, y)\rangle \mid (x, y) \in X \times Y\}$, where $\mu_{R^t}(x, y) = \mu_R(y, x)$ and $\nu_{R^t}(x, y) = \nu_R(y, x)$ for any $(x, y) \in X \times Y$.

The *intersection* of two intuitionistic fuzzy relations R and P is defined as

$$R \cap P = \{\langle (x,y), \min(\mu_R(x,y), \mu_P(x,y)), \max(\nu_R(x,y), \nu_P(x,y))\rangle \mid (x,y) \in X \times Y\}$$

The *union* of two intuitionistic fuzzy relations R and P is defined as

$$R \cup P = \{\langle (x,y), \max(\mu_R(x,y), \mu_P(x,y)), \min(\nu_R(x,y), \nu_P(x,y))\rangle \mid (x,y) \in X \times Y\}.$$

Let R, P and Q be three intuitionistic fuzzy relations from a universe X to a universe Y.

(i) if $R \subseteq P$, then $R^t \subseteq P^t$;

(ii) $(R \cup P)^t = R^t \cup P^t$;

(iii) $(R \cap P)^t = R^t \cap P^t$;

(iv) $(R^t)^t = R$;

(v) $R \cap (P \cup Q) = (R \cap P) \cup (R \cap Q)$ and $R \cup (P \cap Q) = (R \cup P) \cap (R \cup Q)$;

(vi) $R \cup P \supseteq R$, $R \cup P \supseteq P$, $R \cap P \subseteq R$, $R \cap P \subseteq P$;

(vii) if $R \supseteq P$ and $R \supseteq Q$, then $R \supseteq P \cup Q$;

(viii) if $R \subseteq P$ and $R \subseteq Q$, then $R \subseteq P \cap Q$.

Let R be an intuitionistic fuzzy relation (intuitionistic fuzzy relation on X, for short). The following properties are crucial in (see e.g., [15, 16, 54, 65]):

(i) *Reflexivity*: $\mu_R(x,x) = 1$, for any $x \in X$. In this case we note that $\nu_R(x,x) = 0$, for any $x \in X$.

(ii) *Antisymmetry*: for any $x, y \in X$, $x \neq y$ then

$$\begin{cases} \mu_R(x,y) \neq \mu_R(y,x) \\ \nu_R(x,y) \neq \nu_R(y,x) \\ \pi_R(x,y) = \pi_R(y,x) \end{cases}, \tag{1.2}$$

where $\pi_R(x,y) = 1 - \mu_R(x,y) - \nu_R(x,y)$.

(iii) *Perfect antisymmetry*: for any $x, y \in X$ with $x \neq y$ and

$$\begin{cases} \mu_R(x,y) > 0 \\ \quad \text{or} \\ \mu_R(x,y) = 0 \ \text{ and } \nu_R(x,y) < 1, \end{cases}$$

then

$$\begin{cases} \mu_R(y,x) = 0 \\ \text{and} \\ \nu_R(y,x) = 1\,. \end{cases} \tag{1.3}$$

(iv) *Transitivity*:

$$R \supseteq R \circ^{\alpha,\beta}_{\lambda,\rho} R. \tag{1.4}$$

In the above definition, the composition $R \circ^{\alpha,\beta}_{\lambda,\rho} R$ used in the transitivity means that $R \circ^{\alpha,\beta}_{\lambda,\rho} R = \{\langle (x,z),\ \alpha_{y\in X}\{\beta[\mu_R(x,y),\mu_R(y,z)]\},\ \lambda_{y\in X}\{\rho[\nu_R(x,y),\nu_R(y,z)]\}\rangle \mid x,z \in X\}$, where α, β, λ and ρ are t-norms or t-conorms taken under the intuitionistic fuzzy condition

$$0 \le \alpha_{y\in X}\{\beta[\mu_R(x,y),\mu_R(y,z)]\} + \lambda_{y\in X}\{\rho[\nu_R(x,y),\nu_R(y,z)]\} \le 1\,,$$

for any $x,z \in X$. The properties of this composition and the choice of α, β, λ

and ρ, for which this composition fulfills a maximal number of properties, are investigated in [15]-[19], [25]. If no other conditions are imposed, in the following

we will take $\alpha = \sup$, $\beta = \min$, $\lambda = \inf$ and $\rho = \max$.

Note that Bustince and Burillo in [18], mentioned that the definition of intuitionistic fuzzy antisymmetry does not recover the fuzzy antisymmetry for the case in which the considered relation R is fuzzy. However, the definition of intuitionistic fuzzy perfect antisymmetry recovers the definition of fuzzy antisymmetry given by Zadeh [63] when the considered relation is fuzzy. This note justifies the following definition of intuitionistic fuzzy order.

Definition 1.21. [15, 16] *Let X be a nonempty crisp set and R be an intuitionistic fuzzy relation on X. R is called an intuitionistic fuzzy order or a partial intuitionistic fuzzy order if it is reflexive, transitive and perfect antisymmetric.*

A nonempty set X with an intuitionistic fuzzy order R defined on it is called an intuitionistic fuzzy ordered set and is denoted by (X,μ_R,ν_R). It easily follows that each partially ordered set $(X,\le)$ and each fuzzy ordered set (X,R) can be viewed as intuitionistic fuzzy ordered sets.

Example 1.5. *Let $X = \{a,b,c,d,e\}$. Then the intuitionistic fuzzy relation R defined on X by*

$$R = \{\langle (x,y),\mu_R(x,y),\nu_R(x,y)\rangle \mid x,y \in X\},$$

where μ_R and ν_R are given by the following tables:

$\mu_R(.,.)$	a	b	c	d	e
a	1	0	0	0.55	0.40
b	0	1	0	0.35	0.45
c	0	0	1	0	0.70
d	0	0	0	1	0
e	0	0	0	0	1

$\nu_R(.,.)$	a	b	c	d	e
a	0	1	0.40	0.45	0.25
b	0.30	0	0.20	0.35	0.10
c	1	1	0	0.85	0.15
d	1	1	1	0	1
e	1	1	1	0.90	0

is an intuitionistic fuzzy order on X.

Example 1.6. *Let $m, n \in \mathbb{N}$. Then the following intuitionistic fuzzy relation R on $\mathbb{N}$ is an intuitionistic fuzzy order, where*

$$\mu_R(m,n) = \begin{cases} 1\ , \ if\ m = n \\ 1 - \frac{m}{n}\ , \quad if \quad m < n \\ 0\ , \quad if \quad m > n\,, \end{cases} \quad and\ \nu_R(m,n) = \begin{cases} 0\ , \ if \quad m = n \\ \frac{m}{2n}\ , \quad if \quad m < n \\ 1\ , \quad if \quad m > n \end{cases}$$

On the basis of the above definition of perfect antisymmetry we define a complete (or total) intuitionistic fuzzy order as follows

Definition 1.22. [65] *An intuitionistic fuzzy order R on a universe X is called complete (or total) if for any $x, y \in X$ it holds that*

$$[\mu_R(x,y) > 0 \text{ or } (\mu_R(x,y) = 0 \text{ and } \nu_R(x,y) < 1)]$$

or

$$[\mu_R(y,x) > 0 \text{ or } (\mu_R(y,x) = 0 \text{ and } \nu_R(y,x) < 1)].$$

Definition 1.23. [65] *An intuitionistic fuzzy ordered set (X, μ_R, ν_R) in which R is linear is called a linearly intuitionistic fuzzy ordered set or an intuitionistic fuzzy chain.*

1.6. Intuitionistic fuzzy lattices

In this section, we recall the basic definitions and properties of intuitionistic fuzzy lattices and some related notions that will be needed throughout the next chapters. The concept of an intuitionistic fuzzy lattice was introduced by Thomas and Nair

[52] as an intuitionistic fuzzy set on a crisp lattice stable by the supremum and the infimum of the binary operations $\sqcap$ and $\sqcup$.

Definition 1.24. [52] *Let L be a lattice and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in L\}$ be an IFS on L. Then A is called an intuitionistic fuzzy sub-lattice (fuzzy lattice, for short) if for any $x, y \in L$, the following conditions are satisfied:*

(i) $\mu_A(x \sqcup y) \geq \mu_A(x) \wedge \mu_A(y)$;

(ii) $\mu_A(x \sqcap y) \geq \mu_A(x) \wedge \mu_A(y)$;

(iii) $\nu_A(x \sqcup y) \leq \nu_A(x) \vee \nu_A(y)$;

(iv) $\nu_A(x \sqcap y) \leq \nu_A(x) \vee \nu_A(y)$.

Example 1.7. *Figure 1.1 shows the Hasse diagram of a lattice $L = \{0, a, b, 1\}$. The intuitionistic fuzzy set A on L given by $A = \{< 0, 0.5, 0.1 >, < a, 0.4, 0.5 >, < b, 0.4, 0.3 >, < 1, 0.7, 0.3 >\}$ is an intuitionistic fuzzy lattice.*

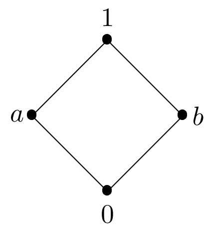

Figure 1.1: Hasse diagram of a lattice $(L, \leqslant, \sqcap, \sqcup)$ with $L = \{0, a, b, 1\}$.

For further details on intuitionistic fuzzy lattices, we refer to [39, 51, 52].

1.7. Intuitionistic fuzzy ideals and filters on a lattice

The notion of intuitionistic fuzzy ideal (resp. filter) on a lattice was first introduced by Thomas and Nair [52].

Definition 1.25. [52] *Let L be a lattice and $I = \{\langle x, \mu_I(x), \nu_I(x)\rangle \mid x \in L\}$ be an IFS on L. Then I is called an intuitionistic fuzzy ideal on L (IF-ideal, for short) if for all $x, y \in L$ the following conditions are satisfied:*

(i) $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$;

(ii) $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$;

(iii) $\nu_I(x \sqcup y) \leq \nu_I(x) \vee \nu_I(y)$;

(iv) $\nu_I(x \sqcap y) \leq \nu_I(x) \wedge \nu_I(y)$.

Example 1.8. *Let L be the lattice given by the Hasse diagram in Figure 2.1. The intuitionistic fuzzy set I on L defined by $I = \{< 0, 0.5, 0.1 >, < a, 0.4, 0.3 >, < b, 0.1, 0.2 >, < 1, 0.1, 0.3 >\}$ is an IF-ideal.*

Definition 1.26. [52] *Let L be a lattice and $F = \{\langle x, \mu_F(x), \nu_F(x)\rangle \mid x \in L\}$ be an IFS on L. Then F is called an intuitionistic fuzzy filter on L (IF-filter, for short) if for any $x, y \in L$, the following conditions are satisfied:*

(i) $\mu_F(x \sqcup y) \geq \mu_F(x) \vee \mu_F(y)$;

(ii) $\mu_F(x \sqcap y) \geq \mu_F(x) \wedge \mu_F(y)$;

(iii) $\nu_F(x \sqcup y) \leq \nu_F(x) \wedge \nu_F(y)$;

(iv) $\nu_F(x \sqcap y) \leq \nu_F(x) \vee \nu_F(y)$.

Example 1.9. *Let L be the lattice given by the Hasse diagram in Figure 2.1. The intuitionistic fuzzy set F on L defined by $F = \{< 0, 0.1, 0.6 >, < a, 0.2, 0.6 >, < b, 0.1, 0.5 >, < 1, 0.4, 0.3 >\}$ is an IF-filter.*

Remark 1.2. *Notice that every IF-ideal on L is an L-IF-lattice, but the converse is not true in general. Indeed, let L be the lattice given by the Hasse diagram in Figure 2.1 and $A \in IFS(L)$ defined by $A = \{< 0, 0.3, 0.1 >, < a, 0.4, 0.5 >, < b, 0.4, 0.3 >, < 1, 0.7, 0.3 >\}$. Then A is an L-IF-lattice, but since $\mu_A(a) = \mu_A(a \sqcap 1) = 0.4 \not\geq \max\{0.4; 0.7\}$, then it holds that A is not an IF-ideal on L. As well since $\mu_A(0) = \mu_A(a \sqcap b) = 0.3 \not\geq \min\{0.4; 0.4\}$, then it holds that A is not an IF-filter on L.*

The following results will be needed throughout this chapter.

Proposition 1.1. *Let L be a lattice, L^d be its order-dual lattice and $A \in IFS(L)$. Then it holds that A is an IF-ideal on L if and only if A is an IF-filter on L^d and conversely.*

Proposition 1.2. [52] *Let L be a lattice, A and B are two intuitionistic fuzzy sets on L. Then it holds that*

(i) if A and B are two IF-ideals on L, then $A \cap B$ is an IF-ideal on L;

(ii) if A and B are two IF-filters on L, then $A \cap B$ is an IF-filter on L.

1.8. Intuitionistic fuzzy ordered lattice

In this section, we recall some basic concepts related to the intuitionistic fuzzy ordered lattices. Further information can be found in [54].

Definition 1.27. [54] *For an intuitionistic fuzzy ordered set* (X, μ_R, ν_R) *and* $x \in X$. *The intuitionistic fuzzy sets* $R_{\geq[x]}$ *and* $R_{\leq[x]}$ *defined in* X *by*

$$R_{\geq[x]} = \{\langle y, \mu_{R_{\geq[x]}}(y), \nu_{R_{\geq[x]}}(y)\rangle \mid y \in X\}, \quad \text{where}$$

$$\mu_{R_{\geq[x]}}(y) = \mu_R(x, y) \text{ and } \nu_{R_{\geq[x]}}(y) = \nu_R(x, y),$$

$$R_{\leq[x]} = \{\langle y, \mu_{R_{\leq[x]}}(y), \nu_{R_{\leq[x]}}(y)\rangle \mid y \in X\}, \text{ where}$$

$$\mu_{R_{\leq[x]}}(y) = \mu_R(y, x) \text{ and } \nu_{R_{\leq[x]}}(y) = \nu_R(y, x).$$

are called, respectively the dominating class of x *and the class dominated by* x.

Remark 1.3. *The notions of the dominating class of* x *and the class dominated by* x *are generalizations of the classical notions* $\uparrow x$ *and* $\downarrow x$ *in a usual poset.*

Next, we recall the definition of the upper bounds, the lower bounds, the supremum and the infimum on intuitionistic fuzzy ordered sets.

Definition 1.28. [54] *Let* (X, μ_R, ν_R) *be an intuitionistic fuzzy ordered set and* A *be a subset of* X.

(i) The set of upper bounds of A *with respect to* R *is the intuitionistic fuzzy subset of* X *defined by*

$$U(R, A)(y) = \bigcap_{x \in A} R_{\geq[x]}(y), \tag{1.5}$$

for $y \in X$

(ii) The set of lower bounds of A *with respect to* R *is the intuitionistic fuzzy subset of* X *defined by*

$$L(R, A)(y) = \bigcap_{x \in A} R_{\leq[x]}(y), \tag{1.6}$$

for $y \in X$.

Definition 1.29. [54] *Let* (X, μ_R, ν_R) *be an intuitionistic fuzzy ordered set and* A *be a subset of* X. *An element* $x \in X$ *is called the least upper bound (or a supremum) of* A *with respect to* R *if:*

(i) $x \in Supp(U(R, A))$ *and*

(ii) for all other $y \in Supp(U(R, A))$, $\mu_R(x, y) > 0$ *or* ($\mu_R(x, y) = 0$ *and* $\nu_R(x, y) < 1$).

An element $x \in X$ *is called the greatest lower bound (or an infimum) of* A *with respect to* R *if:*

(i) $x \in Supp(L(R, A))$ *and*

(ii) for all other $y \in Supp(L(R, A))$, $\mu_R(y, x) > 0$ or ($\mu_R(y, x) = 0$ and $\nu_R(y, x) < 1$).

Remark 1.4. *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and A be a subset of X. If the supremum and the infimum of A with respect to R exist, then from the perfect antisymmetry of R they are unique and denoted by $\sup_R(A)$ and $\inf_R(A)$, respectively.*

Definition 1.30. [54] *An intuitionistic fuzzy ordered set (X, μ_R, ν_R) is called an intuitionistic fuzzy lattice with respect to the intuitionistic fuzzy order R (or simply, intuitionistic fuzzy lattice) if each pair of elements $\{x, y\}$ of X has a supremum and an infimum.*

Example 1.10. *Let R be an intuitionistic fuzzy relation on $X = \{a, b, c, d, e\}$ defined by the following tables:*

$\mu_R(.,.)$	a	b	c	d	e
a	1	0.7	0	0	0.1
b	0	1	0	0	0.1
c	0.5	0.7	1	1	0.8
d	0	0	0	1	0.5
e	0	0	0	0	1

$\nu_R(.,.)$	a	b	c	d	e
a	0	0.2	1	1	0.7
b	0.8	0	1	0.1	0.8
c	0.3	0.2	0	0	0.1
d	0.8	1	1	0	0.4
e	0.7	0.8	0.7	0.6	0

It is easy to verify that R is reflexive, perfect antisymmetric and min-max transitive. This implies that (X, μ_R, ν_R) is an intuitionistic fuzzy ordered set. The following table describes the supremum and the infimum of each subset of two elements $\{x, y\}$ of X.

$\{x, y\}$	$\sup_R\{x, y\}$	$\inf_R\{x, y\}$
$\{a, b\}$	b	a
$\{a, c\}$	a	c
$\{a, d\}$	d	a
$\{a, e\}$	e	a
$\{b, c\}$	b	c
$\{b, d\}$	e	a
$\{b, e\}$	e	b
$\{c, d\}$	d	c
$\{c, e\}$	e	c
$\{d, e\}$	e	d

So, (X, μ_R, ν_R) is an intuitionistic fuzzy lattice.

Definition 1.31. [54] *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set.*

(i) An element $\top \in X$ is called the greatest element (the maximum) of X with

respect to R *or the intuitionistic fuzzy maximum of* X *if*

$$\begin{cases} \mu_R(x, \top) > 0 \\ \text{or} \\ \mu_R(x, \top) = 0 \quad \text{and } \nu_R(x, \top) < 1\,, \end{cases}$$

for any $x \in X$ *.*

(ii) An element $\bot \in X$ *is called the smallest element (the minimum) of* X *with respect to* R *or the intuitionistic fuzzy minimum if*

$$\begin{cases} \mu_R(\bot, x) > 0 \\ \text{or} \\ \mu_R(\bot, x) = 0 \quad \text{and } \nu_R(\bot, x) < 1\,, \end{cases}$$

for any $x \in X$ *.*

2 Characterizations of intuitionistic fuzzy ideals and filters on a lattice

In this chapter, we provide interesting characterizations of intuitionistic fuzzy ideals and filters on a lattice in terms of the lattice operations, and in terms of their (α, β)-level sets. Moreover, we extend the notion of a prime ideal (resp. a prime filter) to an intuitionistic fuzzy ideal (resp. an intuitionistic fuzzy filter) with respect to the lattice operations and investigate their various characterizations and properties.

2.1. Characterizations of intuitionistic fuzzy ideals and filters in terms of lattice operations

In this section, we characterize the notion of IF-ideals and IF-filters on a lattice in terms of the lattice-operations. We start with the key results.

Theorem 2.1. *Let L be a lattice and $A \in IFS(L)$. Then for any $x, y \in L$ the following four statements hold*

(i) $(\mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y))$ *if and only if* $(x \leqslant y \Rightarrow \mu_A(x) \geq \mu_A(y))$;

(ii) $(\mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y))$ *if and only if* $(x \leqslant y \Rightarrow \mu_A(x) \leq \mu_A(y))$;

(iii) $(\nu_A(x \sqcap y) \leq \nu_A(x) \wedge \nu_A(y))$ *if and only if* $(x \leqslant y \Rightarrow \nu_A(x) \leq \nu_A(y))$;

(iv) $(\nu_A(x \sqcup y) \leq \nu_A(x) \wedge \nu_A(y))$ *if and only if* $(x \leqslant y \Rightarrow \nu_A(x) \geq \nu_A(y))$.

Proof. Let $x, y \in L$.

(i) Suppose that $\mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y)$. If $x \leqslant y$ then $x \sqcap y = x$. Since $\mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y)$, it follows that $\mu_A(x) = \mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y)$. Hence, $\mu_A(x) \geq \mu_A(y)$.

Conversely, suppose that $(x \leqslant y \Rightarrow \mu_A(x) \geq \mu_A(y))$. Then it follows that $\mu_A(x \sqcap y) \geq \mu_A(x)$ and $\mu_A(x \sqcap y) \geq \mu_A(y)$. Hence, $\mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y)$.

(ii) Suppose that $\mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y)$. If $x \leqslant y$ then $x \sqcup y = y$. Since $\mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y)$, it follows that $\mu_A(y) = \mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y)$. Hence, $\mu_A(x) \leq \mu_A(y)$.

Conversely, suppose that $(x \leqslant y \Rightarrow \mu_A(x) \leq \mu_A(y))$. Then it follows that $\mu_A(x) \leq \mu_A(x \sqcup y)$ and $\mu_A(y) \leq \mu_A(x \sqcup y)$. Hence, $\mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y)$.

(iii) The proof is similar to (i).

(iv) The proof is similar to (ii).

□

As corollaries, we obtain the following interesting properties of IF-ideals and IF-filters.

Corollary 2.1. *Let L be a lattice and I be an IF-ideal on L. Then for any $x, y \in L$ it holds that*

(i) If $x \leqslant y$, then $\mu_I(x) \geq \mu_I(y)$, (i.e., the map $\mu_I : L \to [0,1]$ is antitone);

(ii) If $x \leqslant y$, then $\nu_I(x) \leq \nu_I(y)$, (i.e., the map $\nu_I : L \to [0,1]$ is monotone).

Remark 2.1. *The converse of the above implications are not necessarily hold. Indeed, let us consider the lattice L given by the Hasse diagram in Figure 2.1 and I the intuitionistic fuzzy ideal on L given by $I = \{< 0, 0.5, 0.1 >, < a, 0.4, 0.3 >, < b, 0.1, 0.2 >, < 1, 0.1, 0.3 >\}$. It is easy to verify that $\mu_I(a) = 0.4 \geq \mu_I(b) = 0.1$, but a, b are incomparable elements.*

Corollary 2.2. *Let L be a lattice and F be an IF-filter on L. Then for any $x, y \in L$ it holds that*

(i) If $x \leqslant y$, then $\mu_F(x) \leq \mu_F(y)$, (i.e., the map $\mu_F : L \to [0,1]$ is monotone);

(ii) If $x \leqslant y$, then $\nu_F(x) \geq \nu_F(y)$, (i.e., the map $\nu_F : L \to [0,1]$ is antitone).

Remark 2.2. *The converse of the above implications are not necessarily hold. Indeed, let us consider the lattice L given by the Hasse diagram in Figure 2.1 and F the intuitionistic fuzzy ideal on L given by $F = \{< 0, 0.1, 0.6 >, < a, 0.2, 0.6 >, < b, 0.1, 0.5 >, < 1, 0.4, 0.3 >\}$. It is easy to verify that $\mu_F(b) = 0.1 \leq \mu_F(a) = 0.2$, but a, b are incomparable elements.*

Corollary 2.3. *Let L be a lattice has smallest element $\bot$ or greatest element $\top$ and I be an IF-ideal on L. Then it holds that*

(i) $\mu_I(\bot) = \max \mu_I(L)$ and $\mu_I(\top) = \min \mu_I(L)$, where $\mu_I(L) = \{\mu_I(x) \mid x \in L\}$;

(ii) $\nu_I(\bot) = \min \nu_I(L)$ and $\nu_I(\top) = \max \nu_I(L)$, where $\nu_I(L) = \{\nu_I(x) \mid x \in L\}$.

Corollary 2.4. *Let L be a lattice has smallest element $\bot$ or greatest element $\top$ and F be an IF-filter on L. Then it holds that*

(i) $\mu_F(\bot) = \min \mu_F(L)$ and $\mu_F(\top) = \max \mu_F(L)$, where $\mu_F(L) = \{\mu_F(x) \mid x \in L\}$;

(ii) $\nu_F(\bot) = \max \nu_F(L)$ and $\nu_F(\top) = \min \nu_F(L)$, where $\nu_F(L) = \{\nu_F(x) \mid x \in L\}$.

In the following theorem we provide a basic characterization of IF-ideals on a lattice.

Theorem 2.2. *Let L be a lattice and $I \in IFS(L)$. Then it holds that I is an IF-ideal on L if and only if the following two conditions are satisfied:*

(i) $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$ *for* $x, y \in L$*;*

(ii) $\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$ *for* $x, y \in L$*.*

Proof. Suppose that I is an IF-ideal on L. Then for any $x, y \in L$ it holds that $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$ and $\nu_I(x \sqcup y) \leq \nu_I(x) \vee \nu_I(y)$. Since $x \leqslant x \sqcup y$ and $y \leqslant x \sqcup y$, from Corollary 2.1 it follows that

$$\mu_I(x) \geq \mu_I(x \sqcup y)$$

and

$$\mu_I(y) \geq \mu_I(x \sqcup y)\,.$$

Hence, $\mu_I(x) \wedge \mu_I(y) \geq \mu_I(x \sqcup y)$. Thus, $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$.

In addition, since $x \leqslant x \sqcup y$ and $y \leqslant x \sqcup y$ we obtain from Corollary 2.1 that

$$\nu_I(x) \leq \nu_I(x \sqcup y)$$

and

$$\nu_I(y) \leq \nu_I(x \sqcup y)\,.$$

Hence, $\nu_I(x) \vee \nu_I(y) \leq \nu_I(x \sqcup y)$. Thus, $\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$.

Conversely, suppose that $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$ and $\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$, for any $x, y \in L$. Then it is easy to see that

$$\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$$

and

$$\nu_I(x \sqcup y) \leq \nu_I(x) \vee \nu_I(y).$$

Next, we show that $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$ and $\nu_I(x \sqcap y) \leq \nu_I(x) \wedge \nu_I(y)$ for $x, y \in L$. Let $x, y \in L$. Since $x \sqcup (x \sqcap y) = x$ and $y \sqcup (x \sqcap y) = y$ then it holds that $\mu_I(x \sqcup (x \sqcap y)) = \mu_I(x)$ and $\mu_I(y \sqcup (x \sqcap y)) = \mu_I(y)$. From Definition 1.25 (hypothesis (i) and (ii)) it follows that

$$\mu_I(x) \wedge \mu_I(x \sqcap y) = \mu_I(x)$$

and

$$\mu_I(y) \wedge \mu_I(x \sqcap y) = \mu_I(y)\,.$$

Hence, $\mu_I(x \sqcap y) \geq \mu_I(x)$ and $\mu_I(x \sqcap y) \geq \mu_I(y)$. Thus, $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$, for any $x, y \in L$. In the same way, we obtain that $\nu_I(x \sqcap y) \leq \nu_I(x) \wedge \nu_I(y)$, for any $x, y \in L$. Therefore, I is an IF-ideal on L. □

In the same manner, the following theorem provides a basic characterization of IF-filters on a lattice.

Theorem 2.3. *Let L be a lattice and $F \in IFS(L)$. Then it holds that F is an intuitionistic fuzzy filter on L if and only if the following conditions are satisfied:*

(i) $\mu_F(x \sqcap y) = \mu_F(x) \wedge \mu_F(y)$ *for* $x, y \in L$;

(ii) $\nu_F(x \sqcap y) = \nu_F(x) \vee \nu_F(y)$ *for* $x, y \in L$.

Proof. The proof is a direct application of Proposition 1.1 and Theorem 2.2. □

2.2. Characterizations of intuitionistic fuzzy ideals and filters in terms of their level sets

In this section, we provide some interesting characterizations and properties of IF-ideals and IF-filters in terms of their level sets.

Proposition 2.1. *Let L be a lattice and $A \in IFS(L)$. The following statements hold*

(i) if A is an IF-ideal, then its support $Supp(A)$ is an ideal on L;

(ii) if A is an IF-filter, then its support $Supp(A)$ is a filter on L.

Proof. Let $A \in IFS(L)$.

(i) Suppose that $A \in IFS(L)$ is an IF-ideal. We show that $Supp(A)$ is an ideal on L.

(a) Let $x \in Supp(A)$ and $y \leqslant x$, then it holds that

$$\mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } \nu_A(x) < 1)\,.$$

There are two cases to consider : ($y \leqslant x$ and $\mu_A(x) > 0$) and ($y \leqslant x$ and ($\mu_A(x) = 0$ and $\nu_A(x) < 1$)).

First case: suppose that $y \leqslant x$ and $\mu_A(x) > 0$. Since $y \leqslant x$, then it holds that $x \sqcup y = x$. This implies that $\mu_A(x) = \mu_A(x \sqcup y) > 0$. From Theorem 2.2 (i), it follows that $\mu_A(x \sqcup y) = \mu_I(x) \wedge \mu_I(y) > 0$. Hence, $\mu_A(y) > 0$. Thus, $y \in Supp(A)$.

Second case: $y \leqslant x$ and ($\mu_A(x) = 0$ and $\nu_A(x) < 1$). Since $y \leqslant x$, then it holds that $x \sqcup y = x$. This implies that $\mu_A(x) = \mu_A(x \sqcup y) = 0$ and $\nu_A(x) = \nu_A(x \sqcup y) < 1$. From Theorem 2.2, it follows that $\mu_A(x \sqcup y) = \mu_I(x) \wedge \mu_I(y) = 0$ and $\nu_A(x \sqcup y) = \nu_I(x) \vee \nu_I(y) < 1$. Hence, $\mu_A(y) > 0$ or ($\mu_A(y) = 0$ and $\nu_A(y) < 1$). Thus, $y \in Supp(A)$.

(b) Let $x, y \in Supp(A)$. We show now that $x \sqcup y \in Supp(A)$. There are four cases to consider.

First case: let $\mu_A(x) > 0$ and $\mu_A(y) > 0$. Since A is an IF-ideal, then from Theorem 2.2 (i) it follows that $\mu_A(x \sqcup y) = \mu_I(x) \wedge \mu_I(y) > 0$. Hence, $x \sqcup y \in Supp(A)$.

Second case: let ($\mu_A(x) = 0$ and $\nu_A(x) < 1$) and ($\mu_A(y) = 0$ and $\nu_A(y) < 1$). From Theorem 2.2 it follows that $\mu_A(x \sqcup y) = 0$ and $\nu_A(x \sqcup y) < 1$. Hence, $x \sqcup y \in Supp(A)$.

Third case: $\mu_A(x) > 0$ and ($\mu_A(y) = 0$ and $\nu_A(y) < 1$). From Theorem 2.2 it follows that $\mu_A(x \sqcup y) = 0$ and $\nu_A(x \sqcup y) < 1$. Hence, $x \sqcup y \in Supp(A)$.

The last case ($\mu_A(x) = 0$ and $\nu_A(x) < 1$) and $\mu_A(y) > 0$ is analogous to the third case. Thus, $Supp(A)$ is an ideal on L.

(ii) Follows from Proposition 1.1 and (i).

□

Remark 2.3. *The converse of the above implications are not necessarily hold. Indeed, let us consider the lattice L given by the Hasse diagram in Figure 2.1 and $A \in IFS(L)$ given by $A = \{< 0, 0.5, 0.1 >, < a, 0.4, 0.5 >, < b, 0.4, 0.3 >, < 1, 0.7, 0.3 >\}$. It is easy to verify that $Supp(A) = L$ is an ideal and a filter on L, but A is neither an IF-ideal nor an IF-filter on L.*

The following theorem provides a characterization of IF-ideals (resp. IF-filters) in terms of their level sets.

Theorem 2.4. *Let L be a lattice and $A \in IFS(L)$. The following statements hold*

(i) A is an IF-ideal if and only if their level sets are ideals on L;

(ii) A is an IF-filter if and only if their level sets are filters on L.

Proof. Let $A \in IFS(L)$ and $A_{\alpha,\beta}$ their level set, where $\alpha, \beta \in [0, 1]$ satisfying $\alpha + \beta \leq 1$.

(i) Suppose that A is an IF-ideal on L. We show that $A_{\alpha,\beta}$ is an ideal on L for $\alpha, \beta \in [0, 1]$ satisfying $\alpha + \beta \leq 1$.

(a) Let $\alpha, \beta \in [0, 1]$ satisfy $\alpha + \beta \leq 1$, $x \in A_{\alpha,\beta}$ and $y \in L$ such that $y \leqslant x$. Since $x \in A_{\alpha,\beta}$, then it holds that $\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$. Since $y \leqslant x$, from Corollary 2.1 it follows that $\mu_A(y) \geq \mu_A(x)$ and $\nu_A(y) \leq \nu_A(x)$. This implies that $\mu_A(y) \geq \alpha$ and $\nu_A(y) \leq \beta$. Hence, $y \in A_{\alpha,\beta}$, for any $\alpha, \beta \in [0, 1]$ satisfying $\alpha + \beta \leq 1$.

(b) Let $\alpha, \beta \in [0, 1]$ satisfy $\alpha + \beta \leq 1$ and $x, y \in A_{\alpha,\beta}$. Then it holds that ($\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$) and ($\mu_A(y) \geq \alpha$ and $\nu_A(y) \leq \beta$).

From Theorem 2.2 it follows that $\mu_A(x \sqcup y) = \mu_A(x) \wedge \mu_A(y) \geq \alpha$ and $\nu_A(x \sqcup y) = \nu_A(x) \vee \nu_A(y) \leq \beta$. Hence, $x \sqcup y \in A_{\alpha,\beta}$ for $\alpha, \beta \in [0,1]$ satisfying $\alpha + \beta \leq 1$. Thus, $A_{\alpha,\beta}$ is an ideal on L for $\alpha, \beta \in [0,1]$ satisfying $\alpha + \beta \leq 1$.

Conversely, suppose that all level sets of A are ideals on L. We show that A is an IF-ideal on L. Let $x, y \in L$, $\alpha = \mu_A(x) \wedge \mu_A(y)$ and $\beta = \nu_A(x) \vee \nu_A(y)$. Then it follows that ($\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$) and ($\mu_A(y) \geq \alpha$ and $\nu_A(y) \leq \beta$). The case $\alpha = 0$ or $\beta = 0$ is obvious. Let $\alpha, \beta \in [0,1]$ satisfying $\alpha + \beta \leq 1$ and $x, y \in A_{\alpha,\beta}$. Since $A_{\alpha,\beta}$ is an ideal on L, then it holds that $x \sqcup y \in A_{\alpha,\beta}$ $\alpha, \beta \in [0,1]$ satisfying $\alpha + \beta \leq 1$. This implies that $\mu_A(x \sqcup y) \geq \alpha$ and $\nu_A(x \sqcup y) \leq \beta$. Hence, $\mu_A(x \sqcup y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x \sqcup y) \leq \nu_A(x) \vee \nu_A(y)$.

On the other hand, let $\alpha = \mu_A(x \sqcup y)$ and $\beta = \nu_A(x \sqcup y)$. The case $\alpha = 0$ or $\beta = 0$ is also obvious. Otherwise $\alpha, \beta \in [0,1]$ satisfy $\alpha + \beta \leq 1$ and $x \sqcup y \in A_{\alpha,\beta}$. Since $A_{\alpha,\beta}$ is an ideal on L, $x \leqslant x \sqcup y$ and $y \leqslant x \sqcup y$, it follows that $x, y \in A_{\alpha,\beta}$, for any $\alpha, \beta \in [0,1]$ satisfy $\alpha + \beta \leq 1$. This implies that ($\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$) and ($\mu_A(y) \geq \alpha$ and $\nu_A(y) \leq \beta$). Hence, $\mu_A(x) \wedge \mu_A(y) \geq \mu_A(x \sqcup y)$ and $\nu_A(x) \vee \nu_A(y) \leq \nu_A(x \sqcup y)$. Thus, $\mu_A(x) \wedge \mu_A(y) = \mu_A(x \sqcup y)$ and $\nu_A(x) \vee \nu_A(y) = \nu_A(x \sqcup y)$. Therefore, Theorem 2.2 guarantees that A is an IF-ideal on L.

(ii) Follows from Proposition 1.1 and (i).

□

2.3. Prime intuitionistic fuzzy ideals (resp. filters) on a lattice

In this section, we introduce and characterize the prime IF-ideals (resp. IF-filters) on a lattice.

Definition 2.1. *An IF-ideal I on a lattice L is called a prime IF-ideal if, for any $x, y \in L$,*

$$\mu_I(x \sqcap y) \leq \mu_I(x) \vee \mu_I(y)$$

and

$$\nu_I(x \sqcap y) \geq \nu_I(x) \wedge \nu_I(y)\,.$$

Definition 2.2. *An IF-filter F on a lattice L is called a prime IF-filter if for $x, y \in L$,*

$$\mu_F(x \sqcup y) \leq \mu_F(x) \vee \mu_F(y)$$

and

$$\nu_F(x \sqcup y) \geq \nu_F(x) \wedge \nu_F(y)\,.$$

A combination of Theorem 2.2 and Definition 1.25 leads to the following characterization of prime IF-ideals.

Proposition 2.2. *Let L be a lattice and $I \in IFS(L)$. Then it holds that I is a prime IF-ideal on L if and only if the following conditions hold:*

(i) $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$*, for any* $x, y \in L$*;*

(ii) $\mu_I(x \sqcap y) = \mu_I(x) \vee \mu_I(y)$*, for any* $x, y \in L$*;*

(iii) $\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$*, for any* $x, y \in L$*;*

(iv) $\nu_I(x \sqcap y) = \nu_I(x) \wedge \nu_I(y)$*, for any* $x, y \in L$*.*

Similarly, Theorem 2.3 and Definition 1.26 lead to the following characterization of prime IF-filters.

Proposition 2.3. *Let L be a lattice and $F \in IFS(L)$. Then it holds that F is a prime IF-filter on L if and only if the following conditions hold:*

(i) $\mu_F(x \sqcup y) = \mu_F(x) \vee \mu_F(y)$*, for any* $x, y \in L$*;*

(ii) $\mu_F(x \sqcap y) = \mu_F(x) \wedge \mu_F(y)$*, for any* $x, y \in L$*;*

(iii) $\nu_F(x \sqcup y) = \nu_F(x) \wedge \nu_F(y)$*, for any* $x, y \in L$*;*

(iv) $\nu_F(x \sqcap y) = \nu_F(x) \vee \nu_F(y)$*, for any* $x, y \in L$*.*

The following proposition shows that the support of a prime IF-ideal (resp. prime IF-filter) on a lattice is a prime ideal (resp. prime filter) on that lattice.

Proposition 2.4. *Let L be a lattice and $A \in IFS(L)$. Then it holds that*

(i) if A is a prime IF-ideal, then its support $Supp(A)$ is a prime ideal on L;

(ii) if A is a prime IF-filter, then its support $Supp(A)$ is a prime filter on L.

Proof. (i) Suppose that A is a prime IF-ideal on a lattice L. From Proposition 2.1, it holds that $Supp(A)$ is an ideal on L. Next, we prove that $Supp(A)$ is prime. Let $x, y \in L$ such that $x \sqcap y \in Supp(A)$. Then $\mu_A(x \sqcap y) > 0$ or ($\mu_A(x \sqcap y) = 0$ and $\nu_A(x \sqcap y) < 1$). We consider the following cases.

(a) If $\mu_A(x \sqcap y) > 0$, then the fact that A is prime IF-ideal on L implies that

$$\mu_A(x) \vee \mu_A(y) = \mu_A(x \sqcap y) > 0\,.$$

This implies that either $\mu_A(x) > 0$ or $\mu_A(y) > 0$. Hence, either $x \in Supp(A)$ or $y \in Supp(A)$.

(b) If ($\mu_A(x \sqcap y) = 0$ and $\nu_A(x \sqcap y) < 1$), then the fact that A is prime IF-ideal on L implies that

$$\mu_A(x) \vee \mu_A(y) = \mu_A(x \sqcap y) = 0$$

and

$$\nu_A(x) \wedge \nu_A(y) = \nu_A(x \sqcap y) < 1\,.$$

These imply that ($\mu_A(x) = 0 \wedge \mu_A(y) = 0$) and ($\nu_A(x) < 1 \vee \nu_A(y) < 1$). Hence, ($\mu_A(x) = 0$ and $\nu_A(x) < 1$) or ($\mu_A(y) = 0$ and $\nu_A(y) < 1$). Thus, either $x \in Supp(A)$ or $y \in Supp(A)$.

Therefore, $Supp(A)$ is a prime ideal on L.

(ii) Follows by using Proposition 1.1 and (i).

□

In the same manner, we get the following theorem which provides a characterization of prime IF-ideals (resp. prime IF-filters) in terms of their level sets.

Theorem 2.5. *Let L be a lattice and $A \in IFS(L)$. Then it holds that*

(i) A is a prime IF-ideal if and only if their level sets are prime ideals;

(ii) A is a prime IF-filter if and only if their level sets are prime filters.

Proof. (i) From Theorem 2.4, A is an IF-ideal on L if and only if $A_{\alpha,\beta}$ is an ideal on L for $\alpha, \beta \in [0,1]$ satisfying $\alpha + \beta \leq 1$. It remains to show the primality. Suppose that A is a prime IF-ideal on L. Let $x, y \in L$ such that $x \sqcap y \in A_{\alpha,\beta}$. Then from Proposition 2.2 it follows that

$$(\mu_A(x \sqcap y) = \mu_A(x) \vee \mu_A(y) \geq \alpha$$

and

$$\nu_A(x \sqcap y) = \nu_A(x) \wedge \nu_A(y) \leq \beta)\,.$$

These imply that either ($\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$) or ($\mu_A(y) \geq \alpha$ and $\nu_A(y) \leq \beta$). Hence, either $x \in A_{\alpha,\beta}$ or $y \in A_{\alpha,\beta}$. Thus, $A_{\alpha,\beta}$ is a prime ideal for any $\alpha, \beta \in [0,1]$ satisfying $\alpha + \beta \leq 1$.

Conversely, suppose that $A_{\alpha,\beta}$ is a prime ideal for any $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$ and A is not a prime IF-ideal on L. Then it holds that there exist $x, y \in L$ such that $\mu_A(x \sqcap y) > \mu_A(x) \vee \mu_A(y)$ and $\nu_A(x \sqcap y) < \nu_A(x) \wedge \nu_A(y)$. These imply that ($\mu_A(x \sqcap y) > \mu_A(x)$ and $\mu_A(x \sqcap y) > \mu_A(y)$) and ($\nu_A(x \sqcap y) < \nu_A(x)$ and $\nu_A(x \sqcap y) < \nu_A(y)$). If we put $\mu_A(x \sqcap y) = \alpha$ and $\nu_A(x \sqcap y) = \beta$, then it follows that ($\mu_A(x) < \alpha$ and $\nu_A(x) > \beta$) and ($\mu_A(y) < \alpha$ and $\nu_A(y) > \beta$). Hence, $x \sqcap y \in A_{\alpha,\beta}$ and $x, y \notin A_{\alpha,\beta}$. That is a contradiction with the fact that $A_{\alpha,\beta}$ is a prime ideal on L, for any $\alpha, \beta \in [0,1]$. Hence, A is a prime IF-ideal on L.

(ii) Follows from Proposition 1.1 and (i).

□

2.4. Interaction of prime intuitionistic fuzzy ideals (resp. filters) with the basic set-operations and a lattices-homomorphism

In this section, we study the interaction of prime IF-ideals (resp. IF-filters) with the basic set-operations and with a lattices-homomorphism.

2.4.1. Interaction of intuitionistic fuzzy ideals (resp. filters) with the basic operations

In this subsection, we discuss the interaction of IF-ideals (resp. IF-filters) with intersection, union, complement and two associated intuitionistic fuzzy sets.

Proposition 2.5. *Let $(A_i)_{i\in I}$ be a family of IF-sets on a lattice L. Then it holds that*

(i) if A_i is a prime IF-ideal on L, for any $i \in I$, then $\underset{i\in I}{\cap} A_i$ is a prime IF-ideal on L;

(ii) if A_i is a prime IF-filter on L, for any $i \in I$, then $\underset{i\in I}{\cap} A_i$ is a prime IF-filter on L.

Proof. (i) Suppose that for any $i \in I$, A_i is a prime IF-ideal on L. From Proposition 1.2, it follows that $\underset{i\in I}{\cap} A_i$ is an IF-ideal on L. It remains to show that $\underset{i\in I}{\cap} A_i$ is prime. Let $x, y \in L$ such that $x \sqcap y \in \underset{i\in I}{\cap} A_i$. Then it follows that $x \sqcap y \in A_i$, for any $i \in I$. Since for $i \in I$, A_i is a prime IF-ideal, we obtain

$$\mu_{A_i}(x \sqcap y) \leq \mu_{A_i}(x) \vee \mu_{A_i}(y)$$

and

$$\nu_{A_i}(x \sqcap y) \geq \nu_{A_i}(x) \wedge \nu_I(y).$$

This implies that

$$\mu_{\underset{i\in I}{\cap} A_i}(x \sqcap y) \leq \mu_{A_i}(x \sqcap y) \leq \mu_{A_i}(x) \vee \mu_{A_i}(y)$$

and

$$\nu_{\bigcap_{i\in I} A_i}(x \sqcap y) \geq \nu_{A_i}(x \sqcap y) \geq \nu_{A_i}(x) \wedge \nu_I(y)\,,$$

for any $i \in I$. Hence,

$$\mu_{\bigcap_{i\in I} A_i}(x \sqcap y) \leq \bigwedge_{i\in I}(\mu_{A_i}(x) \vee \mu_{A_i}(y))$$

and

$$\nu_{\bigcap_{i\in I} A_i}(x \sqcap y) \geq \bigvee_{i\in I}(\nu_{A_i}(x) \wedge \nu_I(y))\,.$$

Thus,

$$\mu_{\bigcap_{i\in I} A_i}(x \sqcap y) \leq \mu_{\bigcap_{i\in I} A_i}(x) \vee \mu_{\bigcap_{i\in I} A_i}(y)$$

and

$$\nu_{\bigcap_{i\in I} A_i}(x \sqcap y) \geq \nu_{\bigcap_{i\in I} A_i}(x) \wedge \nu_{\bigcap_{i\in I} A_i}(y)\,.$$

Therefore, $\bigcap_{i\in I} A_i$ is a prime IF-ideal on L.

(ii) Follows from Proposition 1.1 and (i).

□

Remark 2.4. *The union of two IF-ideals (resp. IF-filers) is not necessarily an IF-ideal (resp. IF-filter). Indeed, let us consider the lattice L given in Example 1.7, A and B be two intuitionistic fuzzy sets on L defined by $A = \{< 0, 0.5, 0.1 >, < a, 0.4, 0.5 >, < b, 0.4, 0.3 >, < 1, 0.1, 0.3 >\}$ and $B = \{< 0, 0.7, 0.2 >, < a, 0.5, 0.5 >, < b, 0.6, 0.3 >, < 1, 0.4, 0.5 >\}$. We can observe that A and B are IF-ideals on L, but $\nu_{A\cup B}(1) = \nu_{A\cup B}(a \sqcup 1) = 0.3 \neq \nu_{A\cup B}(a) \vee \nu_{A\cup B}(1) = 0.5$. Hence, $A \cup B$ is not an IF-ideal.*

The following propositions discuss the relationship between IF-ideal (resp. IF-filter) and its complement.

Proposition 2.6. *Let L be a lattice and $A \in IFS(L)$. Then it holds that*

(i) A is a prime IF-ideal if and only if $\overline{A}$ is a prime IF-filter on L;

(ii) A is a prime IF-filter if and only if $\overline{A}$ is a prime IF-ideal on L.

Proof. (i) Suppose that A is a prime IF-ideal, then for any $x, y \in L$ from Proposition 2.2 it follows that

$$\begin{aligned}\mu_{\overline{A}}(x \sqcup y) &= \nu_A(x \sqcup y)\\ &= \nu_A(x) \vee \nu_A(y)\\ &= \mu_{\overline{A}}(x) \vee \mu_{\overline{A}}(y)\end{aligned}$$

and

$$\begin{aligned}\mu_{\overline{A}}(x \sqcap y) &= \nu_A(x \sqcap y) \\ &= \nu_A(x) \wedge \nu_A(y) \\ &= \mu_{\overline{A}}(x) \wedge \mu_{\overline{A}}(y)\end{aligned}$$

In a similar way, we prove $\nu_{\overline{A}}(x \sqcup y) = \nu_{\overline{A}}(x) \wedge \nu_{\overline{A}}(y)$ and $\nu_{\overline{A}}(x \sqcap y) = \nu_{\overline{A}}(x) \vee \nu_{\overline{A}}(y)$. Applying Proposition 2.3 guarantees that $\overline{A}$ is a prime IF-filter on L.

The sufficient condition follows from Proposition 1.1 and the first implication.

(ii) Follows from the fact that $A = \overline{\overline{A}}$ and (i).

□

Proposition 2.7. *Let L be a lattice and $A \in IFS(L)$. Then it holds that*

(i) A is a prime IF-ideal if and only if $[A]$ is a prime IF-ideal on L;

(ii) A is a prime IF-filter if and only if $[A]$ is a prime IF-filter on L.

Proof. (i) Suppose that A is a prime IF-ideal on a lattice L. Since $[A] = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$, then it easily to see that $[A]$ is an IF-ideal on L. Moreover, we have that

$$\begin{aligned}\mu_{[A]}(x \sqcap y) &= \mu_A(x \sqcap y) \\ &= \mu_A(x) \vee \mu_A(y) \\ &= \mu_{[A]}(x) \vee \mu_{[A]}(y)\end{aligned}$$

and

$$\begin{aligned}\nu_{[A]}(x \sqcap y) &= 1 - \mu_A(x \sqcap y) \\ &= 1 - (\mu_A(x) \vee \mu_A(y)) \\ &= (1 - \mu_A(x)) \wedge (1 - \mu_A(y)) \\ &= \nu_{[A]}(x) \wedge \nu_{[A]}(y).\end{aligned}$$

Thus, we can conclude that $[A]$ is a prime IF-ideal on L.

Conversely, suppose that $[A]$ is a prime IF-ideal. By using the same method as above we get that A is a prime IF-ideal on L.

(ii) Follows from Proposition 1.1 and (i).

□

Proposition 2.8. *Let L be a lattice and $A \in IFS(L)$. Then it holds that*

(i) A is a prime IF-ideal if and only if $\langle A \rangle$ is a prime IF-ideal on L;

(ii) A is a prime IF-filter if and only if $\langle A \rangle$ is a prime IF-filter on L.

Proof. The proof is similar as in Proposition 2.7 □

2.4.2. Interaction of intuitionistic fuzzy ideals (resp. filters) with a lattices homomorphism

In this subsection, we study the interaction of IF-ideals (resp. IF-filters) with a lattices-homomorphism.

Definition 2.3. [30] *Let $f : L \to L'$ be a mapping from a lattice L to another lattice L' and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in L\}$ be an IFS on L. The image of A by f is defined by $f(A) = \{\langle y, f(\mu_A)(y), f(\nu_A)(y)\rangle \mid y \in L'\}$, where*

$$f(\mu_A)(y) = \begin{cases} sup\{\mu_A(x) \mid x \in f^{-1}(y)\}, \ \textit{if } f^{-1}(y) \neq \phi \\ 0 \ , \ \textit{if } f^{-1}(y) = \phi, \end{cases}$$

and

$$f(\nu_A)(y) = \begin{cases} inf\{\nu_A(x) \mid x \in f^{-1}(y)\}, \ \textit{if } f^{-1}(y) \neq \phi \\ 1 \ , \ \textit{if } f^{-1}(y) = \phi. \end{cases}$$

Similarly, if $A' = \{\langle y, \mu_{A'}(y), \nu_{A'}(y)\rangle \mid y \in L'\}$ is an IFS on L', then

$$f^{-1}(A') = \{\langle x, f^{-1}(\mu_{A'})(x), f^{-1}(\nu_{A'})(x)\rangle \mid x \in L\},$$

where $f^{-1}(\mu_{A'})(x) = \mu_{A'}(f(x))$ and $f^{-1}(\nu_{A'})(x) = \nu_{A'}(f(x))$.

In the following theorem, we will show that the image of an IF-ideal (resp. IF-filter) is an IF-ideal (resp. IF-filter).

Theorem 2.6. *Let L, L' be two lattices and $f : L \to L'$ be a lattices-homomorphism. Then it holds that*

(i) if A is an IF-ideal on L, then $f(A)$ is an IF-ideal on L';

(ii) if A is an IF-filter on L, then $f(A)$ is an IF-filter on L'.

Proof. (i) Let A be an IF-ideal on L. For $y, z \in L'$ we have

$$\begin{aligned}
f(\mu_A)(y \sqcup z) &= \sup\{\mu_A(x) \mid x \in f^{-1}(y \sqcup z)\} \\
&= \sup\{\mu_A(u \vee v) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\
&= \sup\{(\mu_A(u) \wedge \mu_A(v)) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\
&= \sup\{\mu_A(u) \mid u \in f^{-1}(y)\} \wedge \sup\{\mu_A(v) \mid v \in f^{-1}(z)\} \\
&= f(\mu_A)(y) \wedge f(\mu_A)(z).
\end{aligned}$$

Similarly, for $y, z \in L'$

$$\begin{aligned}
f(\nu_A)(y \sqcup z) &= \inf\{\nu_A(x) \mid x \in f^{-1}(y \sqcup z)\} \\
&= \inf\{\nu_A(u \vee v) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\
&= \inf\{(\mu_A(u) \vee \mu_A(v)) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\
&= \inf\{\nu_A(u) \mid u \in f^{-1}(y)\} \vee \inf\{\nu_A(v) \mid v \in f^{-1}(z)\} \\
&= f(\nu_A)(y) \vee f(\nu_A)(z).
\end{aligned}$$

Thus, we can conclude that $f(A)$ is an IF-ideal on L'.

(ii) Follows from Proposition 1.1 and (i).

□

In the following theorem, we will show that the inverse image of an IF-ideal (resp. IF-filter) is an IF-ideal (resp. IF-filter).

Theorem 2.7. *Let L, L' be two lattices and $f : L \to L'$ be a lattices-homomorphism. Then it holds that*

(i) if A' is an IF-ideal on L', then $f^{-1}(A')$ is an IF-ideal on L;

(ii) if A' is an IF-filter on L', then $f^{-1}(A')$ is an IF-filter on L.

Proof. (i) Let A' be an IF-ideal on L'. For any $x, y \in L$ it holds that

$$\begin{aligned}
f^{-1}(\mu_{A'})(x \sqcup y) &= \mu_{A'}(f(x \sqcup y)) \\
&= \mu_{A'}(f(x)) \wedge \mu_{A'}(f(y)) \\
&= f^{-1}(\mu_{A'})(x) \wedge f^{-1}(\mu_{A'})(y)
\end{aligned}$$

and

$$\begin{aligned} f^{-1}(\nu_{A'})(x \sqcup y) &= \nu_{A'}(f(x \sqcup y)) \\ &= \nu_{A'}(f(x)) \vee \nu_{A'}(f(y)) \\ &= f^{-1}(\nu_{A'})(x) \vee f^{-1}(\nu_{A'})(y). \end{aligned}$$

Therefore, $f^{-1}(A')$ is an IF-ideal on L.

(ii) Follows from Proposition 1.1 and (i).

□

The next results show that the above Theorems 2.6 and 2.7 of the image and the inverse image of an IF-ideal (resp. IF-filter) are also relevant to a prime IF-ideal (resp. prime IF-filter).

Theorem 2.8. *Let L, L' be two lattices and $f : L \to L'$ be a lattices-homomorphism. Then it holds that*

(i) if A is a prime IF-ideal on L, then $f(A)$ is a prime IF-ideal on L';

(ii) if A is a prime IF-filter on L, then $f(A)$ is a prime IF-filter on L'.

Proof. (i) Let A be a prime IF-ideal on L. Theorem 2.6 guarantees that $f(A)$ is an IF-ideal on L'. Next, we show that $f(A)$ is prime. The fact that A is a prime IF-ideal implies that for $y, z \in L$

$$\begin{aligned} f(\mu_A)(y \sqcap z) &= \sup\{\mu_A(x) \mid x \in f^{-1}(y \sqcap z)\} \\ &= \sup\{\mu_A(u \wedge v) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{(\mu_A(u) \vee \mu_A(v)) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{\mu_A(u) \mid u \in f^{-1}(y)\} \vee \sup\{\mu_A(v) \mid v \in f^{-1}(z)\} \\ &= f(\mu_A)(y) \vee f(\mu_A)(z). \end{aligned}$$

Similarly, for $y, z \in L'$ it holds that

$$\begin{aligned} f(\nu_A)(y \sqcap z) &= \inf\{\nu_A(x) \mid x \in f^{-1}(y \sqcap z)\} \\ &= \inf\{\nu_A(u \wedge v) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \inf\{(\mu_A(u) \wedge \mu_A(v)) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \inf\{\nu_A(u) \mid u \in f^{-1}(y)\} \wedge \inf\{\nu_A(v) \mid v \in f^{-1}(z)\} \\ &= f(\nu_A)(y) \wedge f(\nu_A)(z). \end{aligned}$$

We conclude that $f(A)$ is a prime IF-ideal on L'.

(ii) Follows from Proposition 1.1 and (i).

□

Theorem 2.9. *Let L, L' be two lattices and $f : L \to L'$ be a lattices-homomorphism. Then it holds that*

(i) if A' is a prime IF-ideal on L', then $f^{-1}(A')$ is a prime IF-ideal on L;

(ii) if A' is a prime IF-filter on L', then $f^{-1}(A')$ is a prime IF-filter on L.

Proof. (i) Let A' be a prime IF-ideal on L'. Theorem 2.7 guarantees that $f^{-1}(A')$ is an IF-ideal on L. We only need to show that $f^{-1}(A')$ is prime. Since A' is a prime IF-ideal, it follows that for any $x, y \in L$,

$$\begin{aligned} f^{-1}(\mu_{A'})(x \sqcap y) &= \mu_{A'}(f(x \sqcap y)) \\ &= \mu_{A'}(f(x)) \vee \mu_{A'}(f(y)) \\ &= f^{-1}(\mu_{A'})(x) \vee f^{-1}(\mu_{A'})(y) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\nu_{A'})(x \sqcap y) &= \nu_{A'}(f(x \sqcap y)) \\ &= \nu_{A'}(f(x)) \wedge \nu_{A'}(f(y)) \\ &= f^{-1}(\nu_{A'})(x) \wedge f^{-1}(\nu_{A'})(y). \end{aligned}$$

Therefore, we conclude that $f^{-1}(A')$ is a prime IF-ideal on L.

(ii) Follows from Proposition 1.1 and (i).

□

3 Characterizations of intuitionistic fuzzy ideals and filters on an intuitionistic fuzzy ordered lattice

In this chapter, we introduce the notions of intuitionistic fuzzy ideal and filter on an intuitionistic fuzzy ordered lattice and we present interesting characterizations of these notions in terms of the intuitionistic fuzzy ordered lattice operations and in terms of their (α, β)-level sets. Moreover, we introduce two interesting kinds, principal and prime intuitionistic fuzzy ideals and filters, and then we investigate their different properties.

3.1. Intuitionistic fuzzy ideals and filters on an intuitionistic fuzzy ordered lattice

In this section, we generalize the notion of fuzzy ideals and filters given by Mezzomo, et al.[38] to the intuitionistic fuzzy case and providing their basic characterizations.

Definition 3.1. *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and A be an intuitionistic fuzzy subset on X. Then A is called an intuitionistic fuzzy ideal on X if for any $x, y \in X$ the following conditions are satisfied:*

(i) $\mu_A(x \sqcup y) \geq \mu_A(x) \wedge \mu_A(y)$;

(ii) $\nu_A(x \sqcup y) \leq \nu_A(x) \vee \nu_A(y)$;

(iii) $\mu_A(x) \geq \mu_A(y) \wedge \mu_R(x, y)$;

(iv) $\nu_A(x) \leq \nu_A(y) \vee \nu_R(x, y)$.

Definition 3.2. *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and A be a subset on X. Then A is called an intuitionistic fuzzy filter on X if for any $x, y \in X$ the following conditions are satisfied:*

(i) $\mu_A(x \sqcap y) \geq \mu_A(x) \wedge \mu_A(y)$;

(ii) $\nu_A(x \sqcap y) \leq \nu_A(x) \vee \nu_A(y)$;

(iii) $\mu_A(x) \geq \mu_A(y) \wedge \mu_R(y, x)$;

(iv) $\nu_A(x) \leq \nu_A(y) \vee \nu_R(y, x)$.

Example 3.1. *Consider the intuitionistic fuzzy lattice of Example 1.7, A and B are two intuitionistic fuzzy sets on X*

(i) $A = \{< a, 0.5, 0.1 >, < b, 0.4, 0.3 >, < c, 0.1, 0.2 >, < d, 0.1, 0.3 >, \}$ *is an IF-ideal on X;*

(ii) $B = \{< a, 0.1, 0.6 >, < b, 0.2, 0.6 >, < c, 0.1, 0.5 >, < d, 0.4, 0.3 >, \}$ *is an IF-filter on X.*

We state the following result based on Proposition 3.1.

Proposition 3.1. *Let* $(X, \mu_R, \nu_R, \sqcap, \sqcup)$ *be an intuitionistic fuzzy lattice,* $(X, \mu_{R^t}, \nu_{R^t}, \sqcup, \sqcap)$ *be its order-dual lattice and* $A \in IFS(X)$*. Then it holds that A is an IF-ideal on* $(X, \mu_R, \nu_R, \sqcap, \sqcup)$ *if and only if A is an IF-filter on* $(X, \mu_{R^t}, \nu_{R^t}, \sqcup, \sqcap)$ *and vice versa.*

Combining Definition 3.1 and Definition 3.2 we easily get also the following one.

Proposition 3.2. *Let* (X, μ_R, ν_R) *be an intuitionistic fuzzy ordered set, A and B are two intuitionistic fuzzy sets on X. Then it holds that*

(i) if A and B are two IF-ideals on X, then $A \cap B$ *is an IF-ideal on X;*

(ii) if A and B are two IF-filters on X, then $A \cap B$ *is an IF-filter on X.*

Proof. (i) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$

and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in X\}$ be two intuitionistic fuzzy sets. Then $A \cap B = \{\langle x, \mu_{A\cap B}(x), \nu_{A\cap B}(x)\rangle \mid x \in X\}$, where $\mu_{A\cap B}(x) = \{\mu_A(x) \wedge \mu_B(x)\}$ and $\nu_{A\cap B}(x) = \{\nu_A(x) \vee \nu_B(x)\}$.

$$\begin{aligned}
\mu_{A\cap B}(x \sqcup y) &= \{\mu_A(x \sqcup y) \wedge \mu_B(x \sqcup y)\} \\
&\geq \{\{\mu_A(x) \wedge \mu_A(y)\} \wedge \{\mu_B(x) \wedge \mu_B(y)\}\} \\
&\geq \{\{\mu_A(x) \wedge \mu_B(x)\} \wedge \{\mu_A(y) \wedge \mu_B(y)\}\} \\
&\geq \{\mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y)\}.
\end{aligned}$$

Similarly, we get that $\nu_{A\cap B}(x \sqcup y) \leq \{\nu_{A\cap B}(x) \vee \nu_{A\cap B}(y)\}$.

$$\begin{aligned}
\mu_{A\cap B}(x) &= \{\mu_A(x) \wedge \mu_B(x)\} \\
&\geq \{\{\mu_A(y) \wedge \mu_R(x, y)\} \wedge \{\mu_B(y) \wedge \mu_R(x, y)\}\} \\
&\geq \{\{\mu_A(y) \wedge \mu_B(y)\} \wedge \mu_R(x, y)\} \\
&\geq \{\mu_{A\cap B}(y) \wedge \mu_R(x, y)\}.
\end{aligned}$$

Similarly, we get that $\nu_{A\cap B}(x) \leq \{\nu_{A\cap B}(y) \vee \nu_R(x, y)\}$.

Hence, $A \cap B$ is an IF-ideal on X.

(ii) Follows from Proposition 3.1 and (i).

□

Remark 3.1. *The union of two intuitionistic fuzzy ideals (resp. filers) does not necessarily be an intuitionistic fuzzy ideal (resp. filter). Indeed, let us consider the lattice X given in Example 1.7, A and B are two intuitionistic fuzzy sets on X defined by $A = \{< a, 0.5, 0.1 >, < b, 0.4, 0.5 >, < c, 0.4, 0.3 >, < d, 0.1, 0.3 >\}$ and $B = \{< a, 0.7, 0.2 >, < b, 0.5, 0.5 >, < c, 0.6, 0.3 >, < d, 0.4, 0.5 >\}$. We can see that A and B are two intuitionistic fuzzy ideals on X. We have,*

$A \cup B = \{< a, 0.7, 0.1 >, < b, 0.5, 0.5 >, < c, 0.6, 0.3 >, < d, 0.4, 0.3 >\}$.

$\nu_{A\cup B}(b) = 0.5 \nleq \nu_{A\cup B}(d) \vee \nu_R(b, d) = 0.3$. *which prove that $A \cup B$ is not an IF-ideal on X.*

3.2. Characterizations of intuitionistic fuzzy ideals and filters in terms of the intuitionistic fuzzy ordered lattice operations

In this section, we characterize the notion of IF-ideals and IF-filters on IF-lattice in terms of the IF-ordered lattice operations. First, we need the following key results.

Theorem 3.1. *Let X be an IF-lattice and $A \in IFS(X)$. Then for any $x, y \in X$, the following four statements hold*

(i) $(\mu_A(x) \geq \mu_A(y) \wedge \mu_R(x, y))$ *if and only if* $(\mu_R(x, y) > 0 \Rightarrow \mu_A(x) \geq \mu_A(y))$;

(ii) $(\mu_A(x) \geq \mu_A(y) \wedge \mu_R(y, x))$ *if and only if* $(\mu_R(x, y) > 0 \Rightarrow \mu_A(x) \leq \mu_A(y))$;

(iii) $(\nu_A(x) \leq \nu_A(y) \vee \nu_R(x, y))$ *if and only if* $(\nu_R(x, y) < 1 \Rightarrow \nu_A(x) \leq \nu_A(y))$;

(iv) $(\nu_A(x) \leq \nu_A(y) \vee \nu_R(y, x))$ *if and only if* $(\nu_R(x, y) < 1 \Rightarrow \nu_A(x) \geq \nu_A(y))$.

Proof. Let $x, y \in X$.

(i) Suppose that $\mu_A(x) \geq \mu_A(y) \wedge \mu_R(x, y)$. Then $\mu_A(x) \geq \mu_A(y)$ or $\mu_A(x) \geq \mu_R(x, y)$. If $\mu_R(x, y) > 0$, then $\mu_A(x) \geq \mu_R(x, y)$ does not hold in general. Hence, $\mu_A(x) \geq \mu_A(y)$.

Conversely, suppose that $(\mu_R(x, y) > 0 \Rightarrow \mu_A(x) \geq \mu_A(y))$. We show that $\mu_A(x) \geq \mu_A(y) \wedge \mu_R(x, y)$. There are three cases to consider $\mu_A(y) = 0$ or $\mu_A(y) \geq \mu_R(x, y)$ or $\mu_A(y) \leq \mu_R(x, y)$.

First case: suppose that $\mu_A(y) = 0$. This implies $\mu_A(y) \wedge \mu_R(x, y) = \mu_A(y)$. Since $\mu_A(x) \geq \mu_A(y)$, then we get $\mu_A(x) \geq \mu_A(y) \wedge \mu_R(x, y)$.

Second case: suppose that $\mu_A(y) \geq \mu_R(x, y)$. Since $\mu_A(x) \geq \mu_A(y)$, then it holds that $\mu_A(x) \geq \mu_R(x, y)$. Hence, $\mu_A(x) \geq \mu_A(y) \wedge \mu_R(x, y)$.

Third case: Supposing $\mu_A(y) \leq \mu_R(x, y)$, we have $\mu_A(y) \wedge \mu_R(x, y) = \mu_A(y)$. Since $\mu_A(x) \geq \mu_A(y)$, then it holds that $\mu_A(x) \geq \mu_A(y) \wedge \mu_R(x, y)$.

Thus, $\mu_A(x) \geq \mu_A(y) \wedge \mu_R(x, y)$, for any $x, y \in X$.

(ii) The proof is similar as in (i).

(iii) Suppose that $\nu_A(x) \leq \nu_A(y) \vee \nu_R(x, y)$. Then $\nu_A(x) \leq \nu_A(y)$ or $\nu_A(x) \leq \nu_R(x, y)$. If $\nu_R(x, y) < 1$, then $\nu_A(x) \leq \nu_R(x, y)$ does not hold in general. Hence, $\nu_A(x) \leq \nu_A(y)$.

Conversely, suppose that $(\nu_R(x, y) < 1 \Rightarrow \nu_A(x) \leq \nu_A(y))$. We show that $\nu_A(x) \leq \nu_A(y) \vee \nu_R(x, y)$. There are three cases to consider $\nu_A(y) = 1$ or $\nu_A(y) \leq \nu_R(x, y)$ or $\nu_A(y) \geq \nu_R(x, y)$.

First case: suppose that $\nu_A(y) = 1$. This implies that $\nu_A(y) \vee \nu_R(x, y) = \nu_A(y)$. Since $\nu_A(x) \leq \nu_A(y)$, then it holds that $\nu_A(x) \leq \nu_A(y) \vee \mu_R(x, y)$.

Second case: suppose that $\nu_A(y) \leq \nu_R(x, y)$. Since $\nu_A(x) \leq \nu_A(y)$, then we get $\nu_A(x) \leq \nu_R(x, y)$. Hence, $\nu_A(x) \leq \nu_A(y) \vee \nu_R(x, y)$.

Third case: Supposing $\nu_A(y) \geq \nu_R(x, y)$, we have $\nu_A(y) \vee \nu_R(x, y) = \nu_A(y)$. Since $\nu_A(x) \leq \nu_A(y)$, then it holds that $\nu_A(x) \leq \nu_A(y) \vee \nu_R(x, y)$.

Thus, $\nu_A(x) \leq \nu_A(y) \vee \nu_R(x, y)$, for any $x, y \in X$.

(iv) The proof is similar as in (iii).

□

As corollaries of Theorem 3.1, we obtain the following interesting properties of IF-ideals and IF-filters. The proofs are straightforward.

Corollary 3.1. *Let (X, μ_R, ν_R) be an IF-lattice and I be an IF-ideal on X. Then for $x, y \in X$ it holds that*

(i) if $\mu_R(x, y) > 0$, then $\mu_I(x) \geq \mu_I(y)$, (i.e., the map $\mu_I : X \to [0, 1]$ is antitone);

(ii) if $\mu_R(x, y) > 0$, then $\nu_I(x) \leq \nu_I(y)$, (i.e., the map $\nu_I : X \to [0, 1]$ is monotone).

Corollary 3.2. *Let (X, μ_R, ν_R) be an IF-lattice and F be an IF-filter on X. Then for any $x, y \in X$ it holds that* *(i) if* $\mu_R(x,y) > 0$*, then* $\mu_F(x) \leq \mu_F(y)$*, (i.e., the map* $\mu_F : X \to [0,1]$ *is monotone);*

(ii) if $\mu_R(x,y) > 0$*, then* $\nu_F(x) \geq \nu_F(y)$*, (i.e., the map* $\nu_F : X \to [0,1]$ *is antitone).*

Corollary 3.3. *Let* (X, μ_R, ν_R) *an IF-lattice has smallest element* $\perp$ *or greatest element* $\top$ *and* I *be an IF-ideal on* X*. Then it holds that*

(i) $\mu_I(\perp) = \max \mu_I(X)$ *and* $\mu_I(\top) = \min \mu_I(X)$*, where*

$\mu_I(X) = \{\mu_I(x) \mid x \in X\}$,

(ii) $\nu_I(\perp) = \min \nu_I(X)$ *and* $\nu_I(\top) = \max \nu_I(X)$*, where*

$\nu_I(X) = \{\nu_I(x) \mid x \in X\}$.

Corollary 3.4. *Let* (X, μ_R, ν_R) *be an IF-lattice has smallest element* $\perp$ *or greatest element* $\top$ *and* F *be an IF-filter on* X*. Then it holds that*

(i) $\mu_F(\perp) = \min \mu_I(X)$ *and* $\mu_F(\top) = \max \mu_F(X)$*, where*

$\mu_F(X) = \{\mu_F(x) \mid x \in X\}$,

(ii) $\nu_F(\perp) = \max \nu_I(X)$ *and* $\nu_F(\top) = \min \nu_F(X)$*, where*

$\nu_F(X) = \{\nu_F(x) \mid x \in X\}$.

In the following theorem we provide a basic characterization of IF-ideals on IF-lattice.

Theorem 3.2. *Let* X *be an IF-lattice and* $I \in IFS(X)$*. Then it holds that* I *is an IF-ideal on* X *if and only if the following two conditions are satisfied:*

(i) $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$*, for any* $x, y \in X$*;*

(ii) $\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$*, for any*$x, y \in X$*.*

Proof. Suppose that I is an IF-ideal on X. Then for $x, y \in X$ it holds that $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$ and $\nu_I(x \sqcup y) \leq \nu_I(x) \vee \nu_I(y)$. Moreover, since $\mu_R(x, x \sqcup y) > 0$ and $\mu_R(y, x \sqcup y) > 0$, from Corollary 3.1 it follows that

$$\mu_I(x) \geq \mu_I(x \sqcup y)$$

and

$$\mu_I(y) \geq \mu_I(x \sqcup y)\,.$$

Hence, $\mu_I(x) \wedge \mu_I(y) \geq \mu_I(x \sqcup y)$ and lastly $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$.

Also, since $\mu_R(x, x \sqcup y) > 0$ and $\mu_R(y, x \sqcup y) > 0$ from Corollary 3.1 we obtain that

$$\nu_I(x) \leq \nu_I(x \sqcup y)$$

and

$$\nu_I(y) \leq \nu_I(x \sqcup y)\,.$$

Hence, $\nu_I(x) \vee \nu_I(y) \leq \nu_I(x \sqcup y)$. Thus, $\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$.

Conversely, suppose that $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$ and $\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$, for $x, y \in X$. Then it is obvious that

$$\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$$

and

$$\nu_I(x \sqcup y) \leq \nu_I(x) \vee \nu_I(y)$$

for $x, y \in X$. Next, we show that $\mu_I(x) \geq \mu_I(y) \wedge \mu_R(x, y)$ and $\nu_I(x) \leq \nu_I(y) \vee \nu_R(x, y)$, for $x, y \in X$.

Let $x, y \in X$ and $\mu_R(x, y) > 0$. Then $x \sqcup y = y$. Since $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$, it follows that $\mu_I(y) = \mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$. Hence, $\mu_I(x) \geq \mu_I(y)$. Using Theorem 3.1 (i) we get $\mu_I(x) \geq \mu_I(y) \wedge \mu_R(x, y)$. In the same way, we obtain that $\nu_I(x) \leq \nu_I(y) \vee \nu_R(x, y)$, for $x, y \in X$. Therefore, I is an IF-ideal on X. □

In the same manner, the following theorem provides a basic characterization of IF-filters on a IF-lattice.

Theorem 3.3. *Let X be an IF-lattice and $F \in IFS(X)$. Then it holds that F is an IF-filter on X if and only if the following conditions are satisfied:*

(i) $\mu_F(x \sqcap y) = \mu_F(x) \wedge \mu_F(y)$, *for any* $x, y \in X$;

(ii) $\nu_F(x \sqcap y) = \nu_F(x) \vee \nu_F(y)$, *for any* $x, y \in X$.

Proof. The proof is a direct application of Proposition 3.1 and Theorem 3.2. □

3.3. Characterizations of intuitionistic fuzzy ideals and filters in terms of their level sets

The present section contains some interesting characterizations and properties about IF-ideals and IF-filters in terms of their level sets.

Proposition 3.3. *Let (X, μ_R, ν_R) be an IF-lattice and $A \in IFS(X)$. The following statements hold*

(i) if A is an IF-ideal, then its support Supp(A) is an ideal on X;

(ii) if A is an IF-filter, then its support Supp(A) is a filter on X.

Proof. (i) Let $A \in IFS(X)$ be an IF-ideal. We show that $Supp(A)$ is an ideal on X.

(a) Let $x \in Supp(A)$ and $\mu_R(y,x) > 0$. Then it holds that

$$\mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } \nu_A(x) < 1)$$

There are two cases to consider ($\mu_R(y,x) > 0$ and $\mu_A(x) > 0$) or ($\mu_R(y,x) > 0$ and ($\mu_A(x) = 0$ and $\nu_A(x) < 1$)).

First case: suppose that $\mu_R(y,x) > 0$ and $\mu_A(x) > 0$. From Corollary 3.1 it follows that $\mu_A(y) \geq \mu_A(x)$, which implies $\mu_A(y) > 0$. Thus, $y \in Supp(A)$.

Second case: $\mu_R(y,x) > 0$ and ($\mu_A(x) = 0$ and $\nu_A(x) < 1$). Since $\mu_R(y,x) > 0$, then it holds that $x \sqcup y = x$. This implies that $\mu_A(x \sqcup y) = \mu_A(x) = 0$ and $\nu_A(x \sqcup y) = \nu_A(x) < 1$. From Theorem 3.2, it follows that $\mu_A(x \sqcup y) = \mu_I(x) \wedge \mu_I(y) = 0$ and $\nu_A(x \sqcup y) = \nu_I(x) \vee \nu_I(y) < 1$. Hence, $\mu_A(y) > 0$ or ($\mu_A(y) = 0$ and $\nu_A(y) < 1$).

Thus, $y \in Supp(A)$.

(b) Let $x, y \in Supp(A)$. We need to show that $x \sqcup y \in Supp(A)$. There are four cases to consider.

First case: $\mu_A(x) > 0$ and $\mu_A(y) > 0$. Since A is an IF-ideal, then from Theorem 3.2 (i) it follows that $\mu_A(x \sqcup y) = \mu_I(x) \wedge \mu_I(y) > 0$. Hence, $x \sqcup y \in Supp(A)$.

Second case: ($\mu_A(x) = 0$ and $\nu_A(x) < 1$) and ($\mu_A(y) = 0$ and $\nu_A(y) < 1$). Using Theorem 3.2 we have that $\mu_A(x \sqcup y) = 0$ and $\nu_A(x \sqcup y) < 1$. Hence, $x \sqcup y \in Supp(A)$.

Third case: $\mu_A(x) > 0$ and ($\mu_A(y) = 0$ and $\nu_A(y) < 1$). From Theorem 3.2 we get $\mu_A(x \sqcup y) = 0$ and $\nu_A(x \sqcup y) < 1$. Hence, $x \sqcup y \in Supp(A)$.

The last case ($\mu_A(x) = 0$ and $\nu_A(x) < 1$) and $\mu_A(y) > 0$ is analogous to the third case. Thus, $Supp(A)$ is an ideal on X.

(ii) Follows from Proposition 3.1 and (i).

□

In the following theorem we show a characterization of the IF-ideals (resp. filters) in terms of their level sets.

Theorem 3.4. *Let (X, μ_R, ν_R) be an IF-lattice and $A \in IFS(X)$. The following statements hold*

(i) A is an IF-ideal if and only if its level sets are crisp ideals on X;

(ii) A is an IF-filter if and only if its level sets are crisp filters on X.

Proof. Let $A \in IFS(L)$ and $A_{\alpha,\beta}$ their level set, where $\alpha, \beta \in [0,1]$ satisfy $\alpha+\beta \leq 1$.

(i) Suppose that A is an IF-ideal on X. We show that $A_{\alpha,\beta}$ is an ideal on X, for any $\alpha, \beta \in [0,1]$ satisfying $\alpha + \beta \leq 1$.

(a) Let $\alpha, \beta \in [0,1]$ satisfy $\alpha + \beta \leq 1$, $x \in A_{\alpha,\beta}$ and $y \in X$ such that $\mu_R(y,x) > 0$. Since $x \in A_{\alpha,\beta}$, then it holds that $\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$. Since $\mu_R(y,x) > 0$, from Corollary 3.1 it follows that $\mu_A(y) \geq \mu_A(x)$ and $\nu_A(y) \leq \nu_A(x)$. This implies that $\mu_A(y) \geq \alpha$ and $\nu_A(y) \leq \beta$. Hence, $y \in A_{\alpha,\beta}$, for any $\alpha, \beta \in [0,1]$, which satisfy $\alpha + \beta \leq 1$.

(b) Let $\alpha, \beta \in [0,1]$ satisfy $\alpha + \beta \leq 1$ and $x, y \in A_{\alpha,\beta}$. Then it holds that $(\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta)$ and $(\mu_A(y) \geq \alpha$ and $\nu_A(y) \leq \beta)$. From Theorem 3.2 it follows that $\mu_A(x \sqcup y) = \mu_A(x) \wedge \mu_A(y) \geq \alpha$ and $\nu_A(x \sqcup y) = \nu_A(x) \vee \nu_A(y) \leq \beta$. Hence, $x \sqcup y \in A_{\alpha,\beta}$, for any $\alpha, \beta \in [0,1]$ satisfying $\alpha + \beta \leq 1$. Thus, $A_{\alpha,\beta}$ is an ideal on X, for any $\alpha, \beta \in [0,1]$, such that $\alpha + \beta \leq 1$.

Conversely, suppose that all level sets of A are ideals on X. Now we show that A is an IF-ideal on X. Let $x, y \in X$, $\alpha = \mu_A(x) \wedge \mu_A(y)$ and $\beta = \nu_A(x) \vee \nu_A(y)$. Then it follows that $(\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta)$ and $(\mu_A(y) \geq \alpha$ and $\nu_A(y) \leq \beta)$. The case $\alpha = 0$ or $\beta = 0$ is obvious. Otherwise, let $\alpha, \beta \in [0,1]$ satisfy $\alpha + \beta \leq 1$ and $x, y \in A_{\alpha,\beta}$. Since $A_{\alpha,\beta}$ is an ideal on X, then it holds that $x \sqcup y \in A_{\alpha,\beta}$ for any $\alpha, \beta \in [0,1]$ satisfying $\alpha + \beta \leq 1$. This implies that $\mu_A(x \sqcup y) \geq \alpha$ and $\nu_A(x \sqcup y) \leq \beta$. Hence, $\mu_A(x \sqcup y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x \sqcup y) \leq \nu_A(x) \vee \nu_A(y)$.

On the other hand, let $\alpha = \mu_A(x \sqcup y)$ and $\beta = \nu_A(x \sqcup y)$. The case $\alpha = 0$ or $\beta = 0$ is also obvious. Otherwise, let $\alpha, \beta \in [0,1]$ satisfy $\alpha + \beta \leq 1$ and $x \sqcup y \in A_{\alpha,\beta}$. Since $A_{\alpha,\beta}$ is an ideal on X, $\mu_R(x, x \sqcup y) > 0$ and $\mu_R(y, x \sqcup y) > 0$, it follows that $x, y \in A_{\alpha,\beta}$, for any $\alpha, \beta \in [0,1]$ satisfying $\alpha + \beta \leq 1$. This implies that $(\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta)$ and $(\mu_A(y) \geq \alpha$ and $\nu_A(y) \leq \beta)$. Hence, $\mu_A(x) \wedge \mu_A(y) \geq \mu_A(x \sqcup y)$ and $\nu_A(x) \vee \nu_A(y) \leq \nu_A(x \sqcup y)$. Thus, $\mu_A(x) \wedge \mu_A(y) = \mu_A(x \sqcup y)$ and $\nu_A(x) \vee \nu_A(y) = \nu_A(x \sqcup y)$. Therefore, Theorem 3.2 guarantees that A is an IF-ideal on X.

(ii) Follows from Proposition 3.1 and (i).

□

3.4. Principal intuitionistic fuzzy ideals and filters

In this section, at first we recall the notion of down-set (resp. up-set) in the crisp case and secondly we define the down-set (resp. up-set) of an intutionistic fuzzy ordered set. Then we present their basic properties.

The down-set $\downarrow S$ (resp. the up-set $\uparrow S$) of S in a crisp lattice L is defined as

$$\downarrow S = \{x \in L : x \leq y, \text{ for some } y \in S\}$$

$$(\text{resp. } \uparrow S = \{x \in L : y \leq x, \text{ for some } y \in S\}).$$

Similarly, for a given element x of a set X, the down-set $\downarrow x$ (resp. the up-set $\uparrow x$) is defined as

$$\downarrow x = \{y \in L \mid y \leq x\} \text{ (resp. } \uparrow x = \{y \in L \mid x \leq y\}).$$

Note that both $\downarrow S$ and $\uparrow S$ coincide with S, when S is an ideal or a filter respectively.

In the following definition, we define analogously to down-set and up-set on a crisp lattice L, two intuitionistic fuzzy sets $\Downarrow S$ and $\Uparrow S$ on intuitionistic fuzzy lattice X as follows:

Definition 3.3. *Let (X, μ_R, ν_R) be an IF-lattice and $A \in IFS(X)$.*

(i) The IF-down set of S denoted by $\Downarrow S$ is defined by

$$\mu_{\Downarrow S}(x) = \sup_{y \in X}(\mu_S(y) \wedge \mu_R(x, y)),$$

$$\nu_{\Downarrow S}(x) = \inf_{y \in X}(\nu_S(y) \vee \nu_R(x, y)).$$

(ii) The IF-up set of S denoted by $\Uparrow S$ is defined by

$$\mu_{\Uparrow S}(x) = \sup_{y \in X}(\mu_S(y) \wedge \mu_R(y, x)),$$

$$\nu_{\Uparrow S}(x) = \inf_{y \in X}(\nu_S(y) \vee \nu_R(y, x)).$$

Remark 3.2. *For any crisp set S on a lattice L, it holds that*

(i) $\Downarrow S = \downarrow S$;

(ii) $\Uparrow S = \uparrow S$.

Now, we present the following result.

Proposition 3.4. *Let (X, μ_R, ν_R) be an IF-lattice and S, T be an intuitionistic fuzzy subsets on X. Then we have*

(i) $S \subseteq \Downarrow S$;

(ii) If $S \subseteq T$, *then* $\Downarrow S \subseteq \Downarrow T$;

(iii) $\Downarrow (\Downarrow S) = \Downarrow S$;

(iv) $\Downarrow (S \cup T) = \Downarrow S \cup \Downarrow T$;

(v) $\Downarrow (S \cap T) \subseteq \Downarrow S \cap \Downarrow T$.

Proof. (i) Let S be an IFS on X and $x \in X$. Then it holds that $\mu_S(x) = \mu_S(x) \wedge \mu_R(x,x) \leq \sup\limits_{y\in X}(\mu_S(y) \wedge \mu_R(x,y))$. Similarly, we get that $\nu_S(x) = \nu_S(x) \vee \nu_R(x,x) \geq \inf\limits_{y\in X}(\nu_S(y) \vee \nu_R(x,y))$. Hence, $S \subseteq \Downarrow S$.

(ii) On the one hand, $\mu_{\Downarrow S}(x) = \sup\limits_{y\in X}(\mu_S(y) \wedge \mu_R(x,y))$. Since $\mu_S(y) \leq \mu_T(y)$, it follows that $\mu_{\Downarrow S}(x) \leq \sup\limits_{y\in X}(\mu_T(y) \wedge \mu_R(x,y)) = \mu_{\Downarrow T}(x)$. On the other hand, $\nu_{\Downarrow S}(x) = \inf\limits_{y\in X}(\nu_S(y) \vee \nu_R(x,y))$. Since $\nu_S(y) \geq \nu_T(y)$, it follows that $\nu_{\Downarrow S}(x) \geq \inf\limits_{y\in X}(\nu_T(y) \vee \nu_R(x,y)) = \nu_{\Downarrow T}(x)$. Therefore, $\Downarrow S \subseteq \Downarrow T$.

(iii) On the one hand,

$$\begin{aligned}
\mu_{\Downarrow(\Downarrow S)}(x) &= \sup_{y\in X}(\mu_{\Downarrow S}(y) \wedge \mu_R(x,y)) \\
&= \sup_{y\in X}(\sup_{z\in X}((\mu_S(z) \wedge \mu_R(y,z))) \wedge \mu_R(x,y)) \\
&= \sup_{y\in X}(\sup_{z\in X}(\wedge(\mu_S(z), \mu_R(y,z), \mu_R(x,y)))) \\
&= \sup_{y,z\in X}(\wedge(\mu_S(z), \mu_R(y,z), \mu_R(x,y))) \\
&= \sup_{z\in X}((\mu_S(z) \wedge \sup_{y\in X}(\mu_R(x,y) \wedge \mu_R(y,z))) \\
&= \sup_{z\in X}(\mu_S(z), \mu_R(x,z)) \\
&= \mu_{\Downarrow S}(x).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\nu_{\Downarrow(\Downarrow S)}(x) &= \inf_{y\in X}(\nu_{\Downarrow S}(y) \vee \nu_R(x,y)) \\
&= \inf_{y\in X}(\inf_{z\in X}((\nu_S(z) \vee \nu_R(y,z))) \vee \nu_R(x,y)) \\
&= \inf_{y\in X}(\inf_{z\in X}(\vee(\nu_S(z), \nu_R(y,z), \nu_R(x,y)))) \\
&= \inf_{y,z\in X}(\vee(\nu_S(z), \nu_R(y,z), \nu_R(x,y))) \\
&= \inf_{z\in X}((\nu_S(z) \vee \inf_{y\in X}(\nu_R(x,y) \vee \nu_R(y,z))) \\
&= \inf_{z\in X}(\nu_S(z), \nu_R(x,z)) \\
&= \nu_{\Downarrow S}(x).
\end{aligned}$$

Therefore, $\Downarrow(\Downarrow S) = \Downarrow S$.

(iv) On the one hand,

$$\begin{aligned}
\mu_{\Downarrow(S\cup T)}(x) &= \sup_{y\in X}(\mu_{S\cup T}(y)\wedge\mu_R(x,y))\\
&= \sup_{y\in X}((\mu_S(y)\vee\mu_T(y))\wedge\mu_R(x,y))\\
&= \sup_{y\in X}((\mu_S(y)\wedge\mu_R(x,y))\vee(\mu_T(y)\wedge\mu_R(x,y)))\\
&= \sup_{y\in X}(\mu_S(y)\wedge\mu_R(x,y))\vee\sup_{y\in X}(\mu_T(y)\wedge\mu_R(x,y))\\
&= \mu_{\Downarrow S}(x)\vee\mu_{\Downarrow T}(x).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\nu_{\Downarrow(S\cup T)}(x) &= \inf_{y\in X}(\nu_{S\cup T}(y)\vee\nu_R(x,y))\\
&= \inf_{y\in X}((\nu_S(y)\wedge\nu_T(y))\vee\nu_R(x,y))\\
&= \inf_{y\in X}((\nu_S(y)\vee\nu_R(x,y))\wedge(\nu_T(y)\vee\nu_R(x,y)))\\
&= \inf_{y\in X}(\nu_S(y)\vee\nu_R(x,y))\wedge\inf_{y\in X}(\nu_T(y)\vee\nu_R(x,y))\\
&= \mu_{\Downarrow S}(x)\wedge\mu_{\Downarrow T}(x).
\end{aligned}$$

Hence, $\Downarrow(S\cup R) = \Downarrow S \cup \Downarrow R$.

(v) On the one hand,

$$\begin{aligned}
\mu_{\Downarrow(S\cap T)}(x) &= \sup_{y\in X}(\mu_{S\cap T}(y)\wedge\mu_R(x,y))\\
&= \sup_{y\in X}((\mu_S(y)\wedge\mu_T(y))\wedge\mu_R(x,y))\\
&\leq \sup_{y\in X}(\mu_S(y)\wedge\mu_R(x,y))\wedge\sup_{y\in X}(\mu_T(y)\wedge\mu_R(x,y))\\
&= \mu_{\Downarrow S}(x)\wedge\mu_{\Downarrow T}(x).
\end{aligned}$$

On the other hand,

$$\begin{aligned} \nu_{\Downarrow(S\cap T)}(x) &= \inf_{y\in X}(\nu_{S\cap T}(y)\vee\nu_R(x,y)) \\ &= \inf_{y\in X}((\nu_S(y)\vee\nu_T(y))\vee\nu_R(x,y)) \\ &\geq \inf_{y\in X}(\nu_S(y)\vee\nu_R(x,y))\vee\inf_{y\in X}(\nu_T(y)\vee\nu_R(x,y)) \\ &= \nu_{\Downarrow S}(x)\vee\nu_{\Downarrow T}(x). \end{aligned}$$

Hence, $\Downarrow(S\cap R)\subseteq\Downarrow S\cap\Downarrow R$.

□

In the same manner, we obtain a dual version of Proposition 3.4 for the intuitionistic fuzzy down-set.

Proposition 3.5. *Let (X,μ_R,ν_R) be an IF-lattice and S,T be an intuitionistic fuzzy subsets on X. Then it holds that*

(i) $S\subseteq\Uparrow S$;

(ii) If $S\subseteq T$*, then* $\Uparrow S\subseteq\Uparrow T$;

(iii) $\Uparrow(\Uparrow S)=\Uparrow S$;

(iv) $\Uparrow(S\cup T)=\Uparrow S\cup\Uparrow T$;

(v) $\Uparrow(S\cap T)\subseteq\Uparrow S\cap\Uparrow T$.

Proof. The proof is a direct application of Proposition 3.1 and Theorem 3.2. □

The following corollary follows immediately from Proposition 3.4 and Proposition 3.5.

Corollary 3.5. *Let (X,μ_R,ν_R) be an IF-lattice and S be an intuitionistic fuzzy subset on X. Then $\Downarrow S$ and $\Uparrow S$ define a topological closure on intuitionistic fuzzy sets on X.*

Combining Proposition 3.4, Proposition 3.5, Corollary 3.1 and Corollary 3.2 we obtain the following theorem.

Theorem 3.5. *Let (X,μ_R,ν_R) be an IF-lattice and S be an intuitionistic fuzzy subset on X. Then it holds that*

(i) If S is an IF-ideal, then $\Downarrow S=S$;

(ii) If S is an IF-filter, then $\Uparrow S=S$.

Proof. (i) Suppose that S an IF-ideal on X. On the one hand, Proposition 3.4 guaranties that $S \subseteq \Downarrow S$. On the other hand, let $x, y \in X$ such that $\mu_R(x, y) > 0$.

As $\mu_{\Downarrow S}(x) = \sup_{y \in X}(\mu_S(y) \wedge \mu_R(x, y))$, from Corollary 3.1 it follows that $\mu_{\Downarrow S}(x) \leq \sup_{y \in X}(\mu_S(x) \wedge \mu_R(x, y))$. Since $\mu_R(x, x) = 1$, then

$$\begin{aligned} \mu_{\Downarrow S}(x) &\leq \mu_S(x) \wedge \mu_R(x, x) \\ &\leq \mu_S(x). \end{aligned}$$

Hence, $\mu_{\Downarrow S}(x) \leq \mu_S(x),$ for any $x \in X$.

In the same way, $\nu_{\Downarrow S}(x) = \inf_{y \in X}(\nu_S(y) \vee \nu_R(x, y))$ and from Corollary 3.1 it follows that $\nu_{\Downarrow S}(x) \geq \inf_{y \in X}(\nu_S(x) \vee \nu_R(x, y))$. Since $\nu_R(x, x) = 0$, then

$$\begin{aligned} \nu_{\Downarrow S}(x) &\geq \nu_S(x) \vee \nu_R(x, x) \\ &\geq \nu_S(x) \end{aligned}$$

Hence, $\nu_{\Downarrow S}(x) \geq \nu_S(x),$ for any $x \in X$. Thus, $\Downarrow S = S$.

(ii) Follows from Proposition 3.1 and (i).

□

Remark 3.3. *The converse of the above theorem does not necessarily hold. Indeed, let us consider X as the IF-lattice of Example 1.7 and $S \in IFS(X)$ given by $S = \{< a, 0.7, 0.1 >, < b, 0.4, 0.2 >, < c, 0.3, 0.1 >, < d, 0.1, 0.3 >\}$*

$$\begin{aligned} \mu_{\Downarrow S}(a) &= \sup_{y \in X}(\mu_S(y) \wedge \mu_R(a, y)) = 0.7 \\ \textit{and } \nu_{\Downarrow S}(a) &= \inf_{y \in X}(\nu_S(y) \vee \nu_R(a, y)) = 0.1 \\ \mu_{\Downarrow S}(b) &= \sup_{y \in X}(\mu_S(y) \wedge \mu_R(b, y)) = 0.4 \\ \textit{and } \nu_{\Downarrow S}(b) &= \inf_{y \in X}(\nu_S(y) \vee \nu_R(b, y)) = 0.2 \\ \mu_{\Downarrow S}(c) &= \sup_{y \in X}(\mu_S(y) \wedge \mu_R(c, y)) = 0.3 \\ \textit{and } \nu_{\Downarrow S}(c) &= \inf_{y \in X}(\nu_S(y) \vee \nu_R(c, y)) = 0.1 \\ \mu_{\Downarrow S}(d) &= \sup_{y \in X}(\mu_S(y) \wedge \mu_R(d, y)) = 0.1 \\ \textit{and } \nu_{\Downarrow S}(d) &= \inf_{y \in X}(\nu_S(y) \vee \nu_R(d, y)) = 0.3. \end{aligned}$$

Therefore, $\Downarrow S = \{< a, 0.7, 0.1 >, < b, 0.4, 0.2 >, < c, 0.3, 0.1 >, < d, 0.1, 0.3 >\}$.

Thus, $\Downarrow S = S$ *but* S *is not an IF-ideal on* X.

From the above we get the following corollary

Corollary 3.6. *Let* (X, μ_R, ν_R) *be an IF-lattice and* S *be an intuitionistic fuzzy subset on* X. *Then it holds that*

(i) if S *is an IF-ideal, then* $\Downarrow S$ *is an IF-ideal;*

(ii) if S *is an IF-filter, then* $\Uparrow S$ *is an IF- filter.*

The following result follows immediately from Proposition 3.4 and Theorem 3.5.

Corollary 3.7. *Let* (X, μ_R, ν_R) *be an IF-lattice and* S *be an intuitionistic fuzzy subset on* X. *Then it holds that*

(i) If S *is an IF-ideal, then* $\Downarrow S$ *is the lowest ideal containing* S*;*

(ii) If S *is an IF-filter, then* $\Uparrow S$ *is the lowest filter containing* S.

Definition 3.4. *Let* (X, μ_R, ν_R) *be an IF-lattice. For any* $x \in X$*, we define the intuitionistic fuzzy singleton* $\tilde{x} : X \to [0, 1]$ *by*

$$\mu_{\tilde{x}}(t) = \begin{cases} 1 \ , \ if \ x = t \\ \alpha(t) \ , \quad otherwise, \end{cases}$$

and

$$\nu_{\tilde{x}}(t) = \begin{cases} 0 \ , \ if \ x = t \\ \beta(t) \ , \quad otherwise, \end{cases}$$

where $\alpha(t)$ *(resp.* $\beta(t)$*) is an monotone map (resp. antitone map) with the condition:*

$$0 \leq \alpha(t) + \beta(t) \leq 1, \ for \ any \ t \in X.$$

In the following Proposition we show that the intuitionistic fuzzy down-set (resp. up-set) of any intuitionistic fuzzy singleton is an intuitionistic fuzzy ideal (resp. filter).

Proposition 3.6. *Let* (X, μ_R, ν_R) *be an IF-lattice. Then it holds that*

(i) $\Downarrow \tilde{x}$ *is an IF-ideal for any* $x \in X$*;*

(ii) $\Uparrow \tilde{x}$ *is an IF-filter for any* $x \in X$.

Proof. Let $x \in X$. We show that $\mu_{\Downarrow \tilde{x}}(a \sqcup b) = \mu_{\Downarrow \tilde{x}}(a) \wedge \mu_{\Downarrow \tilde{x}}(b)$ and $\nu_{\Downarrow \tilde{x}}(a \sqcup b) = \nu_{\Downarrow \tilde{x}}(a) \vee \nu_{\Downarrow \tilde{x}}(b)$, for any $a, b \in X$.

(i) On the one hand, we have

$$\begin{aligned}\mu_{\Downarrow\tilde{x}}(a \sqcup b) &= \sup_{y\in X}(\mu_{\tilde{x}}(y) \wedge \mu_R(a \sqcup b, y)) \\ &= \sup_{y\in X}(\mu_{\tilde{x}}(y) \wedge (\mu_R(a, y) \wedge \mu_R(b, y))) \\ &= \sup_{y\in X}(\mu_{\tilde{x}}(y) \wedge \mu_R(a, y)) \wedge \sup_{y\in X}(\mu_{\tilde{x}}(y) \wedge \mu_R(b, y)) \\ &= \mu_{\Downarrow\tilde{x}}(a) \wedge \mu_{\Downarrow\tilde{x}}(b).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\nu_{\Downarrow\tilde{x}}(a \sqcup b) &= \inf_{y\in X}(\nu_{\tilde{x}}(y) \vee \nu_R(a \sqcup b, y)) \\ &= \inf_{y\in X}(\nu_{\tilde{x}}(y) \vee (\nu_R(a, y) \vee \nu_R(b, y))) \\ &= \inf_{y\in X}(\nu_{\tilde{x}}(y) \vee \nu_R(a, y)) \vee \inf_{y\in X}(\nu_{\tilde{x}}(y) \vee \nu_R(b, y)) \\ &= \nu_{\Downarrow\tilde{x}}(a) \vee \nu_{\Downarrow\tilde{x}}(b).\end{aligned}$$

Therefore, Theorem 3.2 guarantees that $\Downarrow \tilde{x}$ is an IF-ideal on X.

(ii) Follows from Proposition 3.1 and (i).

□

Definition 3.5. *Let (X, μ_R, ν_R) be an IF-lattice and x be an element of X. Then*

(i) $\Downarrow \tilde{x}$ is called a principal ideal on X generated by x;

(ii) $\Uparrow \tilde{x}$ is called a principal filter on X generated by x.

3.5. Prime intuitionistic fuzzy ideals and filters

In this section, we treat with the concept of a prime IF-ideal and IF-filter and their properties. First we introduce the following definitions.

Definition 3.6. *Let (X, μ_R, ν_R) be an IF-lattice. A nonempty IF-set A on X is called a proper IF-set, if there exists $x \in X$ such that $\mu_A(x) = 0$ and $\nu_A(x) > 0$.*

Definition 3.7. *Let (X, μ_R, ν_R) be an IF-lattice. A proper IF-ideal I on X is called a prime IF-ideal if for any $x, y \in X$*

$$\mu_I(x \wedge y) \leq \mu_I(x) \vee \mu_I(y),$$

and

$$\nu_I(x \wedge y) \geq \nu_I(x) \wedge \nu_I(y).$$

Definition 3.8. *Let (X, μ_R, ν_R) be an IF-lattice. A proper IF-filter F on X is called a prime IF-filter if for any $x, y \in X$*

$$\mu_F(x \vee y) \leq \mu_F(x) \vee \mu_F(y),$$

and

$$\nu_F(x \vee y) \geq \nu_F(x) \wedge \nu_F(y).$$

Combining Theorem 3.2 and Definition 3.1 we get the following characterization of prime IF-ideals.

Proposition 3.7. *Let (X, μ_R, ν_R) be an IF-lattice and $I \in IFS(X)$. Then I is a prime IF-ideal on X if and only if the following conditions hold:*

(i) $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$*, for any* $x, y \in X$;

(i) $\mu_I(x \sqcap y) = \mu_I(x) \vee \mu_I(y)$*, for any* $x, y \in X$;

(iii) $\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$*, for any* $x, y \in X$;

(iv) $\nu_I(x \sqcap y) = \nu_I(x) \wedge \nu_I(y)$*, for any* $x, y \in X$.

Also, Theorem 3.2 and Definition 3.2 we get the following characterization of prime IF-filters.

Proposition 3.8. *Let (X, μ_R, ν_R) be an IF-lattice and $F \in IFS(X)$. Then F is a prime IF-filter on X if and only if the following conditions hold:*

(i) $\mu_F(x \sqcup y) = \mu_F(x) \vee \mu_F(y)$*, for any* $x, y \in X$;

(i) $\mu_F(x \sqcap y) = \mu_F(x) \wedge \mu_F(y)$*, for any* $x, y \in X$;

(iii) $\nu_F(x \sqcup y) = \nu_F(x) \wedge \nu_F(y)$*, for any* $x, y \in X$;

(iv) $\nu_F(x \sqcap y) = \nu_F(x) \vee \nu_F(y)$*, for any* $x, y \in X$.

3.5.1. Basic properties of prime intuitionistic fuzzy ideals and filters

Now we proceed with some basic properties about prime IF-ideals and IF-filters including intersection, union, complement and two associated intuitionistic fuzzy sets.

Proposition 3.9. *Let X be an IF-lattice and $A \in IFS(X)$. Then*

(i) A is a prime IF-ideal if and only if $\overline{A}$ is a prime IF-filter on X;

(ii) A is a prime IF-filter if and only if $\overline{A}$ is a prime IF-ideal on X.

Proof. (i) Suppose that A is a prime IF-ideal. Then for any $x, y \in X$ from Proposition 3.7 it follows that

$$\begin{aligned}\mu_{\overline{A}}(x \sqcup y) &= \nu_A(x \sqcup y)\\ &= \nu_A(x) \vee \nu_A(y)\\ &= \mu_{\overline{A}}(x) \vee \mu_{\overline{A}}(y)\end{aligned}$$

$$\begin{aligned}\mu_{\overline{A}}(x \sqcap y) &= \nu_A(x \sqcap y)\\ &= \nu_A(x) \wedge \nu_A(y)\\ &= \mu_{\overline{A}}(x) \wedge \mu_{\overline{A}}(y).\end{aligned}$$

In similar way, we can prove that $\nu_{\overline{A}}(x \sqcup y) = \nu_{\overline{A}}(x) \wedge \nu_{\overline{A}}(y)$ and $\nu_{\overline{A}}(x \sqcap y) = \nu_{\overline{A}}(x) \vee \nu_{\overline{A}}(y)$. Applying Proposition 3.8 it guarantees that $\overline{A}$ is a prime IF-filter on X.

The sufficient condition follows from Proposition 3.1 and the first implication.

(ii) Follows from the fact that $A = \overline{\overline{A}}$ and (i).

□

Proposition 3.10. *Let (X, μ_R, ν_R) be an IF-lattice and $(A_i)_{i \in I}$ be a family of intuitionistic fuzzy sets on X. Then*

(i) if $(A_i)_{i\in I}$ is a family of prime IF-ideals on X, then $\bigcap\limits_{i\in I} A_i$ is a prime IF-ideal on X;

(ii) if $(A_i)_{i\in I}$ is a family of prime IF-filters on X, then $\bigcap\limits_{i\in I} A_i$ is a prime IF-filter on X.

Proof. (i) Suppose that for any $i \in I$, A_i is a prime IF-ideal on X. From Proposition 3.2 it follows that $\bigcap\limits_{i\in I} A_i$ is an IF-ideal on X. It remains to show that $\bigcap\limits_{i\in I} A_i$ is prime. Let $x, y \in L$ such that $x \sqcap y \in \bigcap\limits_{i\in I} A_i$. Then we have $x \sqcap y \in A_i$, for any $i \in I$. Since for any $i \in I$, A_i is a prime IF-ideal, it follows that

$$\mu_{A_i}(x \sqcap y) \leq \mu_{A_i}(x) \vee \mu_{A_i}(y)$$

and

$$\nu_{A_i}(x \sqcap y) \geq \nu_{A_i}(x) \wedge \nu_I(y)\,,$$

This implies that

$$\mu_{\underset{i\in I}{\cap} A_i}(x \sqcap y) \leq \mu_{A_i}(x \sqcap y) \leq \mu_{A_i}(x) \vee \mu_{A_i}(y)$$

and

$$\nu_{\underset{i\in I}{\cap} A_i}(x \sqcap y) \geq \nu_{A_i}(x \sqcap y) \geq \nu_{A_i}(x) \wedge \nu_I(y)\,,$$

for any $i \in I$. Hence,

$$\mu_{\underset{i\in I}{\cap} A_i}(x \sqcap y) \leq \bigwedge_{i\in I}(\mu_{A_i}(x) \vee \mu_{A_i}(y))$$

and

$$\nu_{\underset{i\in I}{\cap} A_i}(x \sqcap y) \geq \bigvee_{i\in I}(\nu_{A_i}(x) \wedge \nu_I(y))\,.$$

Thus,

$$\mu_{\underset{i\in I}{\cap} A_i}(x \sqcap y) \leq \mu_{\underset{i\in I}{\cap} A_i}(x) \vee \mu_{\underset{i\in I}{\cap} A_i}(y)$$

and

$$\nu_{\underset{i\in I}{\cap} A_i}(x \sqcap y) \geq \nu_{\underset{i\in I}{\cap} A_i}(x) \wedge \nu_{\underset{i\in I}{\cap} A_i}(y)\,.$$

Therefore, $\underset{i\in I}{\cap} A_i$ is a prime IF-ideal on X.

(ii) Follows from Proposition 3.1 and (i).

□

Remark 3.4. *The union of two prime intuitionistic fuzzy ideals (resp. filters) does not necessarily be a prime intuitionistic fuzzy ideal (resp. filter). Indeed, from Remark 3.1 the union of two intuitionistic fuzzy ideals (resp. filters) does not necessarily be an intuitionistic fuzzy ideal (resp. filter).*

Proposition 3.11. *Let (X, μ_R, ν_R) be an IF-lattice and $A \in IFS(X)$. The following statements hold*

(i) A is a prime IF-ideal if and only if $[A]$ is a prime IF-ideal on X;

(ii) A is a prime IF-filter if and only if $[A]$ is a prime IF-filter on X.

Proof. (i) Suppose that A is a prime IF-ideal on IF-lattice X. Since $[A] = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$, then $[A]$ is an IF-ideal on X. We need to show that $[A]$ is prime, so we have that

$$\begin{aligned} \mu_{[A]}(x \sqcap y) &= \mu_A(x \sqcap y) \\ &= \mu_A(x) \vee \mu_A(y) \\ &= \mu_{[A]}(x) \vee \mu_{[A]}(y) \end{aligned}$$

and

$$\begin{aligned}\nu_{[A]}(x \sqcap y) &= 1 - \mu_A(x \sqcap y) \\ &= 1 - (\mu_A(x) \vee \mu_A(y)) \\ &= (1 - \mu_A(x)) \wedge (1 - \mu_A(y)) \\ &= \nu_{[A]}(x) \wedge \nu_{[A]}(y).\end{aligned}$$

Hence, we can conclude that $[A]$ is a prime IF-ideal on X.

Conversely, suppose that $[A]$ is a prime IF-ideal. In the same way, it follows that A is a prime IF-ideal on X.

(ii) Follows from (i).

□

Proposition 3.12. *Let (X, μ_R, ν_R) be an IF-lattice and $A \in IFS(X)$. The following statements hold*

(i) A is a prime IF-ideal if and only if $\langle A \rangle$ is a prime IF-ideal on X;

(ii) A is a prime IF-filter if and only if $\langle A \rangle$ is a prime IF-filter on X.

Proof. The proof is analogous to Proposition 3.11. □

3.5.2. Characterizations of prime intuitionistic fuzzy ideals and filters in terms of their level sets

In this subsection, we provide some interesting characterizations and properties about prime IF-ideals and IF-filters in terms of their level sets.

Proposition 3.13. *Let X be an IF-lattice and $A \in IFS(X)$. Then*

(i) if A is a prime IF-ideal, then its support $Supp(A)$ is a prime ideal on X;

(ii) if A is a prime IF-filter, then its support $Supp(A)$ is a prime filter on X.

Proof. (i) Suppose that A is a prime IF-ideal on a IF-lattice X. From Proposition 3.3 it holds that $Supp(A)$ is an ideal on X. Next, we prove that $Supp(A)$ is prime. Let $x, y \in X$ such that $x \sqcap y \in Supp(A)$. Then it holds that $\mu_A(x \sqcap y) > 0$ or $(\mu_A(x \sqcap y) = 0$ and $\nu_A(x \sqcap y) < 1)$. We consider the following cases:

(a) If $\mu_A(x \sqcap y) > 0$, then the fact that A is prime IF-ideal on X implies that

$$\mu_A(x) \vee \mu_A(y) = \mu_A(x \sqcap y) > 0\,.$$

This implies that either $\mu_A(x) > 0$ or $\mu_A(y) > 0$. Hence, either $x \in Supp(A)$ or $y \in Supp(A)$.

(b) If $(\mu_A(x \sqcap y) = 0$ and $\nu_A(x \sqcap y) < 1)$, then the fact that A is prime IF-ideal on X implies that

$$\mu_A(x) \vee \mu_A(y) = \mu_A(x \sqcap y) = 0$$

and

$$\nu_A(x) \wedge \nu_A(y) = \nu_A(x \sqcap y) < 1\,.$$

These imply that $(\mu_A(x) = 0 \wedge \mu_A(y) = 0)$ and $(\nu_A(x) < 1 \vee \nu_A(y) < 1)$. Hence, $(\mu_A(x) = 0$ and $\nu_A(x) < 1)$ or $(\mu_A(y) = 0$ and $\nu_A(y) < 1)$. Thus, either $x \in Supp(A)$ or $y \in Supp(A)$.

Therefore, $Supp(A)$ is a prime ideal on X.

(ii) Follows from Proposition 3.1 and (i).

□

In the same manner, the following theorem provides a characterization of prime IF-ideals (resp. prime IF-filters) in terms of their level sets.

Theorem 3.6. *Let X be an IF-lattice and $A \in IFS(X)$. Then*

(i) A is a prime IF-ideal if and only if their level sets are prime ideals;

(ii) A is a prime IF-filter if and only if their level sets are prime filters.

Proof. (i) From Theorem 3.4, A is an IF-ideal on X if and only if $A_{\alpha,\beta}$ is an ideal on L, for any $\alpha, \beta \in [0,1]$, such that $\alpha + \beta \leq 1$. It remains to show the primality. Let A be a prime IF-ideal on X. Let $x, y \in X$ such that $x \sqcap y \in A_{\alpha,\beta}$, then from Proposition 3.7 it follows that

$$(\mu_A(x \sqcap y) = \mu_A(x) \vee \mu_A(y) \geq \alpha$$

and

$$\nu_A(x \sqcap y) = \nu_A(x) \wedge \nu_A(y) \leq \beta)\,.$$

These imply that either $(\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta)$ or $(\mu_A(y) \geq \alpha$ and $\nu_A(y) \leq \beta)$. Hence, either $x \in A_{\alpha,\beta}$ or $y \in A_{\alpha,\beta}$. Thus, $A_{\alpha,\beta}$ is a prime ideal for any $\alpha, \beta \in [0,1]$ satisfying $\alpha + \beta \leq 1$.

Conversely, suppose that $A_{\alpha,\beta}$ is a prime ideal for any $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$ and A is not a prime IF-ideal on X. Then it holds that there exist $x, y \in X$ such that $\mu_A(x \sqcap y) > \mu_A(x) \vee \mu_A(y)$ and $\nu_A(x \sqcap y) < \nu_A(x) \wedge \nu_A(y)$. These imply that $(\mu_A(x \sqcap y) > \mu_A(x)$ and $\mu_A(x \sqcap y) > \mu_A(y))$ and $(\nu_A(x \sqcap y) < \nu_A(x)$ and $\nu_A(x \sqcap y) < \nu_A(y))$. Putting $\mu_A(x \sqcap y) = \alpha$

and $\nu_A(x \sqcap y) = \beta$ we get $(\mu_A(x) < \alpha$ and $\nu_A(x) > \beta)$ and $(\mu_A(y) < \alpha$ and $\nu_A(y) > \beta)$. Hence, $x \sqcap y \in A_{\alpha,\beta}$ and $x, y \notin A_{\alpha,\beta}$. It is a contradiction with the fact that $A_{\alpha,\beta}$ is a prime ideal on X, for any $\alpha, \beta \in [0,1]$. Thus, A is a prime IF-ideal on X.

(ii) Follows from Proposition 3.1 and (i).

□

4 Characterizations of intuitionistic fuzzy ordered complete lattices

In this chapter, we introduce the notion of an intuitioniostic fuzzy complete lattice (recently published in [39]) and then we proceed with some basic characterizations in terms of the existence of the supremum and the infimum of its subsets, and in terms of its intuitionistic fuzzy chains and maximal intuitionistic fuzzy chains. These basic characterizations are used as auxiliary results to show that any intuitionistic fuzzy complete lattice has the fixed point property and vice versa.

We start with the definitions of the intuitionistic fuzzy complete lattice, the intuitionistic fuzzy monotone mapping, the fixed point property and some related notions that will be needed throughout this chapter.

Definition 4.1. *An intuitionistic fuzzy ordered set* (X, μ_R, ν_R) *is called an intuitionistic fuzzy complete lattice if* $\sup_R(A)$ *and* $\inf_R(A)$ *exist for every nonempty subset* $A \subseteq X$.

Remark 4.1. *Every intuitionistic fuzzy complete lattice must have the greatest element (or a maximum) and the smallest element (or a minimum). The greatest element will be denoted by* $\top_X$ *and the smallest element* $\bot_X$. *It obviously holds that*

$$\top_X = \sup_R(X) = \inf_R(\emptyset) \text{ and } \bot_X = \inf_R(X) = \sup_R(\emptyset).$$

Next, we need the following lemma.

Lemma 4.1. *Let* (X, μ_R, ν_R) *be an intuitionistic fuzzy ordered set,* A *be a subset of* X *and* $x \in X$. *Then it holds that*

(i) $x = \sup_R(A)$ *with respect to* R *if and only if* $x = \inf_{R^t}(A)$ *with respect to* R^t;

(ii) $x = \inf_R(A)$ *with respect to* R *if and only if* $x = \sup_{R^t}(A)$ *with respect to* R^t;

(iii) (X, μ_R, ν_R) *is an intuitionistic fuzzy complete lattice if and only if* $(X, \mu_{R^t}, \nu_{R^t})$ *is intuitionistic fuzzy complete lattice.*

Definition 4.2. *Let* (X, μ_R, ν_R) *be an intuitionistic fuzzy ordered set. A mapping* $f : X \to X$ *is called monotone mapping with respect to the intuitionistic fuzzy order* R *or (intuitionistic fuzzy monotone mapping, for short) if* $\mu_R(f(x), f(y)) \geq \mu_R(x, y)$ *and* $\nu_R(f(x), f(y)) \leq \nu_R(x, y)$, *for any* $x, y \in X$.

Definition 4.3. *Let* (X, μ_R, ν_R) *be an intuitionistic fuzzy ordered set. Then*

(i) an element $x \in X$ *is called a fixed point of a mapping* $f : X \to X$ *if* $f(x) = x$. *The set of all fixed points of* f *will be denoted by* $Fix(f)$;

(ii) *X is said to have the fixed point property with respect to the intuitionistic fuzzy order R or (the fixed point property, for short) if every intuitionistic fuzzy monotone mapping f of (X, μ_R, ν_R) into itself has a fixed point.*

4.1. Basic characterizations of intuitionistic fuzzy complete lattices

In this section, we provide some basic characterizations of intuitionistic fuzzy complete lattices.

The following theorem characterize the intuitionistic fuzzy complete lattices in terms of the existence of the supremum or the existence of the infimum of their subsets.

Theorem 4.1. *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set. The following statements hold*

(i) *(X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice if and only if $\sup_R(A)$ exists for any $A \subseteq X$;*

(ii) *(X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice if and only if $\inf_R(A)$ exists for any $A \subseteq X$.*

Proof. Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and $A \subseteq X$.

(i) If (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice, then it obviously holds that $\sup_R(A)$ exists for any $A \subseteq X$.

Conversely, suppose that $\sup_R(A)$ exists for any $A \subseteq X$. We show that every nonempty subset $A \subseteq X$ has an infimum. Let $L(R, A)$ be the intuitionistic fuzzy set of lower bounds of A with respect to R. Then it holds that $\sup_R(Supp(L(R, A)))$ exists. We set $m = \sup_R(Supp(L(R, A)))$ and we show that $m = \inf_R(A)$.

First, since $m \in Supp(U(R, Supp(L(R, A))))$, then it holds that

$$\mu_{U(R,Supp(L(R,A)))}(x) > 0 \text{ or } [\mu_{U(R,Supp(L(R,A)))}(x) = 0 \text{ and }$$

$$\nu_{U(R,Supp(L(R,A)))}(x) < 1].$$

By (1.5) it follows that $U(R, Supp(L(R, A)))(y) = \bigcap_{x \in Supp(L(R,A))} R_{\geq [x]}(y)$.

Since

$$R_{\geq [x]} = \{\langle y, \mu_{R_{\geq [x]}}(y), \nu_{R_{\geq [x]}}(y)\rangle \ / \ y \in X\},$$

where

$$\mu_{R_{\geq[x]}}(y) = \mu_R(x,y) \text{ and} \nu_{R_{\geq[x]}}(y) = \nu_R(x,y),$$

and by the fact that $R_{\geq[x]} = R^t_{\leq[x]}$ and $U(R,A) = L(R^t,A)$, it follows that

$$U(R, Supp(L(R,A))) = L(R^t, Supp(L(R,A))) = L(R,A).$$

Hence, $m \in Supp(L(R,A))$.

In the same way, for any $y \in Supp(L(R,A))$, it holds that $\mu_R(y,m) > 0$ or ($\mu_R(y,m) = 0$ and $\nu_R(y,m) < 1$). Thus $m = \inf_R(A)$, which implies that $\inf_R(A)$ exists. Therefore, (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice.

(ii) Follows from Lemma 4.1 and (i).

□

Remark 4.2. *In the above Theorem 4.1 the existence of* $\inf_R(\emptyset)$ *guarantees the existence of the greatest element of* (X, μ_R, ν_R)*, and in similar way, the existence of* $\sup_R(\emptyset)$ *guarantees the smallest element of* (X, μ_R, ν_R)*. So, an equivalent formulation of Theorem 4.1 is the following.*

(i) (X, μ_R, ν_R) *is an intuitionistic fuzzy complete lattice if and only if it has the smallest element and* $\sup_R(A)$ *exists for any nonempty* $A \subseteq X$*;*

(ii) (X, μ_R, ν_R) *is an intuitionistic fuzzy complete lattice if and only if it has the greatest element and* $\inf_R(A)$ *exists for any nonempty* $A \subseteq X$*.*

In the following theorem, we characterize the intuitionistic fuzzy complete lattices in terms of their intuitionistic fuzzy chains and maximal chains.

Theorem 4.2. *Let* (X, μ_R, ν_R) *be an intuitionistic fuzzy lattice. Then the following statements are equivalent:*

(i) (X, μ_R, ν_R) *is intuitionistic fuzzy complete lattice;*

(ii) (X, μ_R, ν_R) *is intuitionistic fuzzy chain-complete (i.e., every nonempty intuitionistic fuzzy chain in* (X, μ_R, ν_R) *has a supremum and an infimum);*

(iii) every maximal intuitionistic fuzzy chain of X *is an intuitionistic fuzzy complete lattice.*

Proof. (i)⇒ (ii) is obvious.

To prove (ii)⇒ (iii), let C be a maximal intuitionistic fuzzy chain (with respect to the set inclusion) of X. First, we will show that C has an intuitionistic fuzzy maximum and an intuitionistic fuzzy minimum. Since C is an intuitionistic fuzzy chain in (X, μ_R, ν_R), it holds from (ii) that C has a supremum and an infimum.

By using the fact that C is maximal (with respect to the set inclusion) we obtain that $c_1 = \sup_R(C)$ is the maximum and $c_2 = \inf_R(C)$ is the minimum.

Second, let $A \subseteq C$. Then it holds that A is an intuitionistic fuzzy chain. From (ii), it follows that $\sup_R(A)$ exists in (X, μ_R, ν_R) and we denote it by m. Now, it suffices to show that $m \in C$.

Suppose that $m \notin C$. We consider three cases.

(a) If $[\mu_R(x, m) > 0$ or $(\mu_R(x, m) = 0$ and $\nu_R(x, m) < 1)]$ or

$[\mu_R(m, x) > 0$ or $(\mu_R(m, x) = 0$ and $\nu_R(m, x) < 1)]$, for any $x \in C$, then $C \cup \{m\}$ is an intuitionistic fuzzy chain in (X, μ_R, ν_R). This is a contradiction with the fact that C is maximal.

(b) If there exists $x \in C$ such that $[\mu_R(x, m) = 0$ and $\nu_R(x, m) = 1]$, then it holds from the transitivity of R that

$$\mu_R(x, c_1) \wedge \mu_R(c_1, m) \leq \mu_R(x, m)$$

and

$$\nu_R(x, c_1) \vee \nu_R(c_1, m) \geq \nu_R(x, m).$$

Since $[\mu_R(x, m) = 0$ and $\nu_R(x, m) = 1]$, then it holds that $\mu_R(c_1, m) = 0$ and $\nu_R(c_1, m) = 1$. Hence, $\sup_R\{c_1, m\} \notin C$. Thus, $C \bigcup \{\sup_R\{c_1, m\}\}$ is an intuitionistic fuzzy chain, which is a contradiction with the maximality of C.

(c) If there exists $x \in C$ such that $[\mu_R(m, x) = 0$ and $\nu_R(m, x) = 1]$, then it follows similarly as (b).

As a consequence of the above cases we get $m \in C$. Thus, A has a supremum in C. Therefore, directly from Theorem 4.1 C is an intuitionistic fuzzy complete lattice.

(iii)⇒ (i) Suppose that every maximal intuitionistic fuzzy chain of X is an intuitionistic fuzzy complete lattice. We show that (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice.

Let $A \subseteq X$. $\mathcal{BFC}(Supp(U(R, A)))$ denote the set of all intuitionistic fuzzy chains $C \subseteq Supp(U(R, A))$, ordered in the classical way by $C_1 \sqsubseteq C_2$ if and only if C_1 is an intuitionistic fuzzy filter of C_2,i.e., $C_1 \subseteq C_2$ or [if $x \in C_1$ and $y \in C_2$ with $\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1)$ then $y \in C_1$].

Next, let $\{C_i : i \in I \subseteq N\}$ be a chain of $\mathcal{BFC}(Supp(U(R, A)))$ under the crisp order defined above. On the one hand, since C_i is an intuitionistic fuzzy chain of $Supp(U(R, A))$ and $C_i \subseteq C_{i+1}$, for any $i \in I$, then $\bigcup_{i \in I} C_i$ is an intuitionistic fuzzy chain of $Supp(U(R, A))$. Hence, $\bigcup_{i \in I} C_i \in \mathcal{BFC}(Supp(U(R, A)))$. On the other hand, $\bigcup_{i \in I} C_i$ is an upper bound of $\{C_i\}_{i \in I}$.

By Zorn's Lemma, we know that $\mathcal{BFC}(Supp(U(R, A)))$ has a maximal element denoted by C_m with respect to the above crisp order $\sqsubseteq$.

Let K be a maximal intuitionistic fuzzy chain such that $C_m \subseteq K$. By hypothesis, K is an intuitionistic fuzzy complete lattice, which implies that C_m has an infimum denoted by c in (K, μ_R, ν_R).

Now, we will show that $c = \sup_R(A)$. Let $x \in A$. Since $C_m \subseteq Supp(U(R, A))$, then it holds that

$$\mu_R(x, y) > 0 \text{ or } (\mu_R(x, y) = 0 \text{ and } \nu_R(x, y) < 1)\,,$$

for any $y \in C_m$. Hence,

$$\mu_R(x, c) > 0 \text{ or } (\mu_R(x, c) = 0 \text{ and } \nu_R(x, c) < 1)\,.$$

Thus, $c \in Supp(U(R, A))$. For all other $y \in Supp(U(R, A))$, it holds that

$$\mu_R(c, y) > 0 \text{ or } (\mu_R(c, y) = 0 \text{ and } \nu_R(c, y) < 1)\,.$$

Otherwise, we get a contradiction with the maximality of C_m. Thus, $c = \sup_R(A)$. Directly from Theorem 4.1 (i) we get that (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice, which end the proof. □

4.2. Characterizations of intuitionistic fuzzy complete lattice in terms of fixed point property

In this section, we show that any intuitionistic fuzzy complete lattice has the fixed point property and vice versa, i.e., an intuitionistic fuzzification of Tarski-Davis's fixed point theorem.

4.2.1. An intuitionistic fuzzification of Tarski's fixed point theorem

In this subsection, we show that the intuitionistic fuzzy complete lattices has the fixed point property, i.e., Tarski's fixed point theorem for intuitionistic fuzzy complete lattice. Moreover, we show that the set of fixed points of an intuitionistic fuzzy monotone mapping of an intuitionistic fuzzy complete lattice into itself is also an intuitionistic fuzzy complete lattice.

Theorem 4.3. *Any intuitionistic fuzzy complete lattice has the fixed point property.*

Proof. Let (X, μ_R, ν_R) be an intuitionistic fuzzy complete lattice and

$$A = \{x \in X \mid \mu_R(x, f(x)) > 0 \text{ or } (\mu_R(x, f(x)) = 0 \text{ and } \nu_R(x, f(x)) < 1)\}.$$

Since $\perp \in A$, then $A \neq \emptyset$. As (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice, then it holds that A has a supremum, denoted by m. We will show that m is a fixed point of f. By the intuitionistic fuzzy monotonicity of f, we obtain for any $x \in A$ that

$$\mu_R(f(x), f(f(x))) \geq \mu_R(x, f(x)) > 0$$

or

$$(\mu_R(f(x), f(f(x))) = 0 \text{ and } \nu_R(f(x), f(f(x))) \leq \nu_R(x, f(x)) < 1).$$

This implies that

$$f(A) \subseteq A\,. \tag{4.1}$$

We also get that

$$\mu_R(f(x), f(m)) \geq \mu_R(x, m) > 0 \tag{4.2}$$

or

$$\mu_R(f(x), f(m)) = 0 \text{ and } \nu_R(f(x), f(m)) \leq \nu_R(x, m) < 1. \tag{4.3}$$

By (4.2) and (4.3), it holds that $f(m) \in Supp(U(R, A))$. Now, since $m = \sup_R(A)$ and $f(m) \in Supp(U(R, A))$, it follows that

$$\mu_R(m, f(m)) > 0 \tag{4.4}$$

or

$$\mu_R(m, f(m)) = 0 \text{ and } \nu_R(m, f(m)) \leq \nu_R(x, m) < 1. \tag{4.5}$$

By (4.4) and (4.5) we get that $m \in A$, which implies from (4.1) that $f(m) \in A$. Since $m = \sup_R(A)$ and $f(m) \in A$ then it holds that

$$\mu_R(f(m), m) > 0 \tag{4.6}$$

or

$$\mu_R(f(m), m) = 0 \text{ and } \nu_R(f(m), m) < 1. \tag{4.7}$$

Therefore, $m = f(m)$ follows from (4.4),(4.5),(4.6),(4.7) and the perfect antisymmetry of R. □

Now, we present the result showing that the set of fixed points of an intuitionistic fuzzy monotone mapping of an intuitionistic fuzzy complete lattice into itself is also an intuitionistic fuzzy complete lattice. First, we need to show the following lemma.

Lemma 4.2. *Let (X, μ_R, ν_R) be an intuitionistic fuzzy complete lattice, $f : X \to X$ be an intuitionistic fuzzy monotone mapping and $A \subseteq Fix(f)$. Then it holds that*

(i) if $B = \{x \in X \mid ((\mu_R(y, f(x)) > 0 \text{ or } (\mu_R(y, f(x)) = 0 \text{ and } \nu_R(y, f(x)) < 1)) \text{ and } (\mu_R(f(x), x) > 0 \text{ or } (\mu_R(f(x), x) = 0 \text{ and } \nu_R(f(x), x) < 1)), \forall y \in A)\}$, then $\inf_R(B)$ is a fixed point of f in B;

(ii) if $C = \{x \in X \mid ((\mu_R(f(x), y) > 0 \text{ or } (\mu_R(f(x), y) = 0 \text{ and } \nu_R(f(x), y) < 1)) \text{ and } (\mu_R(x, f(x)) > 0 \text{ or } (\mu_R(x, f(x)) = 0 \text{ and } \nu_R(x, f(x)) < 1)), \forall y \in A)\}$ *, then* $\sup_R(C)$ *is a fixed point of* f *in* C*.*

Proof. Let (X, μ_R, ν_R) be an intuitionistic fuzzy complete lattice, $f : X \to X$ be an intuitionistic fuzzy monotone mapping and $A \subseteq Fix(f)$.

(i) Let $m = \inf_R(B)$, i.e.,

(a) $m \in Supp(L(R, B))$ and

(b) for all other $y \in Supp(L(R, B))$, $\mu_R(y, m) > 0$ or $(\mu_R(y, m) = 0$ and $\nu_R(y, m) < 1)$.

Hence,

$$\mu_R(m, x) > 0 \text{ or } (\mu_R(m, x) = 0 \text{ and } \nu_R(m, x) < 1),$$

for any $x \in B$. The monotonicity of f implies that

$$\mu_R(f(m), f(x)) > 0 \text{ or } (\mu_R(f(m), f(x)) = 0 \text{ and } \nu_R(f(m), f(x)) < 1),$$

for any $x \in B$. Since $\mu_R(f(x), x) > 0$ or $(\mu_R(f(x), x) = 0$ and $\nu_R(f(x), x) < 1)$ for any $x \in B$, from the transitivity of R it follows that

$$\mu_R(f(m), x) > 0 \text{ or } (\mu_R(f(m), x) = 0 \text{ and } \nu_R(f(m), x) < 1),$$

This implies that $f(m) \in Supp(L(R, B))$. Hence,

$$\mu_R(f(m), m) > 0 \text{ or } (\mu_R(f(m), m) = 0 \text{ and } \nu_R(f(m), m) < 1). \quad (4.8)$$

In the other hand, since

$$(\mu_R(y, f(x)) > 0 \text{ or } (\mu_R(y, f(x)) = 0 \text{ and } \nu_R(y, f(x)) < 1))$$

and

$$\mu_R(f(x), x) > 0 \text{ or } (\mu_R(f(x), x) = 0 \text{ and } \nu_R(f(x), x) < 1),$$

for any $y \in A$ and $x \in B$, from the transitivity of R it follows that

$$\mu_R(y, x) > 0 \text{ or } (\mu_R(y, x) = 0 \text{ and } \nu_R(y, x) < 1),$$

for any $y \in A$. Hence,

$$\mu_R(y, m) > 0 \text{ or } (\mu_R(y, m) = 0 \text{ and } \nu_R(y, m) < 1),$$

for any $y \in A$. The monotonicity of f implies that $\mu_R(f(f(y)), f(f(m))) > 0$ or $(\mu_R(f(f(y)), f(f(m))) = 0$ and $\nu_R(f(f(y)), f(f(m))) < 1)$, for any $y \in$

A. Since $A \subseteq Fix(f)$ it holds that

$$\mu_R(y, f(f(m))) > 0 \text{ or } (\mu_R(y, f(f(m))) = 0 \text{ and } \nu_R(y, f(f(m))) < 1)\,. \tag{4.9}$$

Now, from condition (4.8) and the monotonicity of f we obtain

$$\mu_R(f(f(m)), f(m)) > 0 \text{ or } (\mu_R(f(f(m)), f(m)) = 0 \text{ and } \nu_R(f(f(m)), f(m)) < 1)\,. \tag{4.10}$$

Eqs. (4.9) and (4.10) imply that $f(m) \in B$. Thus,

$$\mu_R(m, f(m)) > 0 \text{ or } (\mu_R(m, f(m)) = 0 \text{ and } \nu_R(m, f(m)) < 1)\,. \tag{4.11}$$

On account of (4.8),(4.11) and the perfect antisymmetry of R we have $m = f(m)$. Therefore, $m = \inf_R(B)$ is a fixed point of f in B.

(ii) Follows from Lemma 4.1 and (i).

□

Theorem 4.4. *Let (X, μ_R, ν_R) be an intuitionistic fuzzy complete lattice and $f : X \to X$ be an intuitionistic fuzzy monotone mapping. Then the set $Fix(f)$ is an intuitionistic fuzzy complete lattice.*

Proof. Suppose that (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice and let $f : X \to X$ be an intuitionistic fuzzy monotone mapping. Theorem 4.3 guarantees that $Fix(f)$ is a nonempty set. Now, let A be a subset of $Fix(f)$. We show that A has a supremum with respect to R in $Fix(f)$, and then Theorem 4.1(i) implies that $(Fix(f), \mu_R, \nu_R)$ is an intuitionistic fuzzy complete lattice.

Let

$$B = \{x \in X \mid (\forall y \in A)(\mu_R(y, f(x)) > 0 \text{ or } (\mu_R(y, f(x)) = 0 \text{ and } \nu_R(y, f(x)) < 1))$$
$$\text{and } (\mu_R(f(x), x) > 0 \text{ or } (\mu_R(f(x), x) = 0 \text{ and } \nu_R(f(x), x) < 1))\}\,.$$

Since (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice, then it holds that B has an infimum, denoted by m. Lemma 4.2 guarantees that m is a fixed point of f in B, i.e., $m = f(m)$ and $m \in B$. Next, we will show that m is the supremum of A with respect to R.

Since $m = f(m) \in B$ and $(\mu_R(y, x) > 0$ or $(\mu_R(y, x) = 0$ and $\nu_R(y, x) < 1))$, for any $y \in A$ and $x \in B$, it follows that

$$\mu_R(y, m) > 0 \text{ or } (\mu_R(y, m) = 0 \text{ and } \nu_R(y, m) < 1)\,,$$

, for any $y \in A$. Hence,

$$m \in Supp(U(R, A))\,. \tag{4.12}$$

Now, let s be an other fixed point of f and $s \in Supp(U(R,A))$. Then it holds that

$$\mu_R(y,s) > 0 \text{ or } (\mu_R(y,s) = 0 \text{ and } \nu_R(y,s) < 1)\,,$$

for any $y \in A$.

Since $A \subseteq Fix(f)$, it follows from the monotonicity of f that

$$\mu_R(y,f(s)) > 0 \text{ or } (\mu_R(y,f(s)) = 0 \text{ and } \nu_R(y,f(s)) < 1)\,,$$

for any $y \in A$. Now, because $f(s) = s$ and by the reflexivity of R we have $\mu_R(f(s),s) = 1 > 0$. Hence, $s \in B$ and

$$\mu_R(m,s) > 0 \text{ or } (\mu_R(m,s) = 0 \text{ and } \nu_R(m,s) < 1)\,, \tag{4.13}$$

for any $s \in Supp(U(R,A)) \cap Fix(f)$. Therefore, conditions (4.13) and (4.12) imply that m is the supremum of A with respect to R in $Fix(f)$. □

Combining Theorem 4.3 and Theorem 4.4, we obtain the following characterizations of intuitionistic fuzzy complete lattice in terms of the least and the greatest fixed points of its intuitionistic fuzzy monotone maps.

Corollary 4.1. *Let (X, μ_R, ν_R) be an intuitionistic fuzzy lattice. Then the following statements are equivalent:*

(i) (X, μ_R, ν_R) *is an intuitionistic fuzzy complete lattice;*

(ii) every intuitionistic fuzzy monotone mapping $f : X \to X$ has a least fixed point;

(iii) every intuitionistic fuzzy monotone mapping $f : X \to X$ has a greatest fixed point.

4.2.2. An intuitionistic fuzzification of Davis's fixed point theorem

In this subsection, we introduce the notion of intuitionistic fuzzy ascending (resp. descending) chain as necessary to establish a characterization theorem of non-complete intuitionistic fuzzy lattice. Based on this characterization, we show that every intuitionistic fuzzy lattice has the fixed point property is complete, i.e., an intuitionistic fuzzification of Davis's fixed point theorem.

Definition 4.4. *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and $\{a_i\}_{i \in I \subseteq N}$ be a subset of elements of X. Then*

(i) $\{a_i\}_{i \in I \subseteq N}$ is called an intuitionistic fuzzy ascending chain (or an ascending chain with respect to R) if $\mu_R(a_i, a_{i+1}) > 0$ or $(\mu_R(a_i, a_{i+1}) = 0$ and

$\nu_R(a_i, a_{i+1}) < 1$), for any $i \in I$. Descending intuitionistic fuzzy chains (or an descending chain with respect to R) are defined dually.

(ii) *(X, μ_R, ν_R) is said to satisfy the intuitionistic fuzzy ascending chain condition (or the ACC_R, for short) if every intuitionistic fuzzy ascending chain $\{a_i\}_{i\in I\subseteq N}$ of elements of X is eventually stationary (i.e. there exist a positive integer $n \in I$ such that $a_m = a_n$, for any $m > n$). In other words, (X, R) contains no infinite intuitionistic fuzzy ascending chain.*

(iii) *Similarly, (X, μ_R, ν_R) is said to satisfy the intuitionistic fuzzy descending chain condition (or the DCC_R, for short) if every intuitionistic fuzzy descending chain $\{a_i\}_{i\in I\subseteq N}$ of elements of X is eventually stationary.*

The following result establish a characterization that an intuitionistic fuzzy lattice is not complete in terms of its intuitionistic fuzzy ascending (resp. descending) chains.

Theorem 4.5. *Let (X, μ_R, ν_R) be an intuitionistic fuzzy lattice. (X, μ_R, ν_R) is not an intuitionistic fuzzy complete lattice if and only if there exists an intuitionistic fuzzy chain C satisfying the ACC_R which does not have an infimum and there exists an intuitionistic fuzzy chain D satisfying the DCC_R which does not have a supremum, such that*

(i) *$\mu_R(d, c) > 0$ or ($\mu_R(d, c) = 0$ and $\nu_R(d, c) < 1$), for any $d \in D$ and $c \in C$;*

(ii) *for any $x \in X$, either there exists $c \in C$ with ($\mu_R(x, c) = 0$ and $\nu_R(x, c) = 1$) or there exists $d \in D$ with ($\mu_R(d, x) = 0$ and $\nu_R(d, x) = 1$), i.e., there does not exist an element $x \in X$ such that*

$$x \in Supp(L(R, C)) \bigcap Supp(U(R, D)) \,.$$

Proof. Let (X, μ_R, ν_R) be an intuitionistic fuzzy lattice. If on (X, μ_R, ν_R) exist an intuitionistic fuzzy chain C satisfying the ACC_R without an infimum or there exists an intuitionistic fuzzy chain D satisfying the DCC_R and having no supremum , then it is not an intuitionistic fuzzy complete lattice. Conversely, assume that (X, μ_R, ν_R) is not an intuitionistic fuzzy complete lattice. Then either (X, μ_R, ν_R) has no the greatest element $\top_X$ or there exists an intuitionistic fuzzy chain $C \subseteq X$ that satisfies the ACC_R without an infimum. Suppose that (X, r) has the greatest element $\top_X$ and that every intuitionistic fuzzy chain C in X satisfying the ACC_R has an infimum. We will show that every subset A of X has a supremum, which proves by applying Theorem 4.1 that (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice.

Indeed, let A be a subset of X and $U(R, A)$ the set of upper bounds of A with respect to R. Since $U(R, A)(\top_X) = 1$ (i.e., $\top_X \in Supp(U(R, A))$), it follows that

$$Supp(U(R, A)) \neq \emptyset.$$

Now, let $IFC_{ACC}(U(R,A))$ denote the set of all intuitionistic fuzzy chains $C \subseteq Supp(U(R,A))$ satisfying the ACC_R, ordered in the classical way by $C_1 \sqsubseteq C_2$ if and only if C_1 is an intuitionistic fuzzy filter of C_2 with respect to R, i.e. $C_1 \subseteq C_2$ and $(\forall x \in C_1, \forall y \in C_2)$ $(\mu_R(x,y) > 0$ or $(\mu_R(x,y) = 0$ and $\nu_R(x,y) < 1) \Rightarrow y \in C_1)$.

Next, let $\{C_i : i \in I \subseteq N\}$ be a chain of $IFC_{ACC}(U(R,A))$ under the order defined above. On the one hand, since C_i is an intuitionistic fuzzy chain of $Supp(U(R,A))$ satisfying the ACC_R and $C_i \subseteq C_{i+1}$, for any $i \in I$, then it follows that $\bigcup_{i\in I} C_i$ is an intuitionistic fuzzy chain of $Supp(U(R,A))$ satisfying the ACC_R. Hence,

$$\bigcup_{i\in I} C_i \in IFC_{ACC}(U(R,A)) .$$

Now, we show that $C_j \sqsubseteq \bigcup_{i\in I} C_i$, for any $j \in I$. Indeed, let $x \in C_j$ and $y \in \bigcup_{i\in I} C_i$. Since $y \in \bigcup_{i\in I} C_i$, it follows that there exists $k \in I$ such that $y \in C_k$. We consider two cases:

(i) if $j \leq k$, then since $C_j \sqsubseteq C_k$, it follows that $y \in C_j$. Hence, C_j is an intuitionistic fuzzy filter of $\bigcup_{i\in I} C_i$ with respect to R.

(ii) if $j > k$, then it trivially holds that $y \in C_j$. Hence, C_j is an intuitionistic fuzzy filter of $\bigcup_{i\in I} C_i$ with respect to R.

Thus, $C_j \sqsubseteq \bigcup_{i\in I} C_i$, for any $j \in I$. Therefore, $\bigcup_{i\in I} C_i$ is an upper bound of $\{C_j\}_{j\in I}$. By using Zorn's Lemma, it follows that $IFC_{ACC}(U(R,A))$ has a maximal element denoted by C_m. By hypothesis, C_m has an infimum in (X, μ_R, ν_R). Setting $c = \inf_R(C_m)$, and we will prove that $c = \sup_R(A)$. Indeed, since $C_m \subseteq Supp(U(R,A))$, it follows that $\mu_{U(R,A)}(s) > 0$ or $(\mu_{U(R,A)}(s) = 0$ and $\nu_{U(R,A)}(s) < 1)$, for any $s \in C_m$. Since $U(R,A)(s) = \bigcap\limits_{x\in A} R_{\geq[x]}(s)$, it follows that

$$\mu_R(x,s) > 0 \text{ or } (\mu_R(x,s) = 0 \text{ and } \nu_R(x,s) < 1) ,$$

for any $x \in A$ and $s \in C_m$. This implies that

$$\mu_R(x,c) > 0 \text{ or } (\mu_R(x,c) = 0 \text{ and } \nu_R(x,c) < 1) ,$$

for any $x \in A$. Hence,

$$U(R,A)(c) = \bigcap_{x\in A} R_{\geq[x]}(c) > 0 ,$$

i.e.

$$\mu_{U(R,A)}(c) > 0 \text{ or } (\mu_{U(R,A)}(c) = 0 \text{ and } \nu_{U(R,A)}(c) < 1) .$$

Thus, $c \in Supp(U(R,A))$.

On the other hand, suppose that $\mu_R(c,y) = 0$ and $\nu_R(c,y) = 1$, for some $y \in$

$Supp(U(R,A))$. Since $\inf_R\{c,y\} \in Supp(U(R,A))$ and

$$[\mu_R(\inf_R\{c,y\},x) > 0 \text{ or } (\mu_R(\inf_R\{c,y\},x) = 0 \text{ and } \nu_R(\inf_R\{c,y\},x) < 1)],$$

for any $x \in C_m$, it follows that $C_m \cup \inf_R\{c,y\}$ is an intuitionistic fuzzy chain of $Supp(U(R,A))$. Moreover, $C_m \cup \inf_R\{c,y\}$ satisfies the ACC_R. Hence, $C_m \cup \inf_R\{c,y\} \in IFC_{ACC}(U(R,A))$, a contradiction with the maximality of C_m. Thus, for all other $y \in Supp(U(R,A))$,

$$\mu_R(c,y) > 0 \text{ or } (\mu_R(c,y) = 0 \text{ and } \nu_R(c,y) < 1).$$

Therefore, $c = \sup_R(A)$. We conclude that if (X, μ_R, ν_R) is not an intuitionistic fuzzy complete lattice, then either (X, μ_R, ν_R) has no the greatest element or there exists an intuitionistic fuzzy chain $C \subseteq X$ that satisfies the ACC_R without an infimum. Also, if (X, μ_R, ν_R) has no the greatest element, then we take $C = C_m$ an intuitionistic fuzzy chain satisfying the ACC_R but having no infimum; otherwise we take $C = \emptyset$ (because $\emptyset$ can be considered as an intuitionistic fuzzy chain satisfying the ACC_R and since (X, μ_R, ν_R) has no greatest element, it follows that $\emptyset$ does not have an infimum).

Dually, let $IFC_{DCC}(L(R,C))$ denote the set of all intuitionistic fuzzy chains $D \subseteq Supp(L(R,C))$ satisfying the DCC_R, ordered in the classical way by $D_1 \sqsubseteq D_2$ if and only if D_1 is an intuitionistic fuzzy ideal of D_2 with respect to R, i.e. $D_1 \subseteq D_2$ and $\forall x \in D_1$ and $y \in D_2$

$$(\mu_R(y,x) > 0 \text{ or } (\mu_R(y,x) = 0 \text{ and } \nu_R(y,x) < 1) \Rightarrow y \in D_1).$$

If $L(R,C) \neq \emptyset$, then $IFC_{DCC}(L(R,C)) \neq \emptyset$. By using Lemma 4.1 and the same steps as above, we obtain that $IFC_{DCC}(L(R,C))$ has a maximal element denoted by D_m. We take $D = D_m$ if $IFC_{DCC}(L(R,A)) \neq \emptyset$ and $D = \emptyset$ otherwise.

Moreover,

(i) Since $D \subseteq Supp(L(R,C))$, it follows that $\mu_R(d,c) > 0$ or $(\mu_R(d,c) = 0$ and $\nu_R(d,c) < 1)$, for any $d \in D$ and $c \in C$.

(ii) Let $x \in X$. Suppose that $\mu_R(x,c) > 0$ or $(\mu_R(x,c) = 0$ and $\nu_R(x,c) < 1)$, for any $c \in C$ and $\mu_R(d,x) > 0$ or $(\mu_R(d,x) = 0$ and $\nu_R(d,x) < 1)$, for any $d \in D$. Since $x \in Supp(L(R,C))$, it follows that $Supp(L(R,C)) \neq \emptyset$. Hence, $FC_{DCC}(L(R,C)) \neq \emptyset$. Thus, $D = D_m$. On the other hand, since $x \in Supp(L(R,C))$ and $\inf_R(C)$ does not exist, it follows that there exists $y \in Supp(L(R,C))$ such that $\mu_R(y,x) = 0$ and $\nu_R(y,x) = 1$. Since

$$\mu_R(d,x) > 0 \text{ or } (\mu_R(d,x) = 0 \text{ and } \nu_R(d,x) < 1),$$

for any $d \in D$ and

$$\mu_R(x, \sup_R\{x,y\}) > 0 \text{ or } (\mu_R(x, \sup_R\{x,y\}) = 0 \text{ and } \nu_R(x, \sup_R\{x,y\}) < 1)\,,$$

it follows from the transitivity of R that

$$\mu_R(d, \sup_R\{x,y\}) > 0 \text{ or } (\mu_R(d, \sup_R\{x,y\}) = 0 \text{ and } \nu_R(d, \sup_R\{x,y\}) < 1)\,,$$

for any $d \in D$. Hence, $D \cup \{\sup_R\{x,y\}\}$ is an intuitionistic fuzzy chain of $Supp(L(R,C))$. Also, since D satisfy the DCC_R, then also $D \cup \{\sup_R\{x,y\}\}$ satisfy the DCC_R. Hence, $D \cup \{\sup_R\{x,y\}\} \in IFC_{DCC}(L(R,C))$. This is a contradiction with the maximality of D. Therefore, for any $x \in X$, either there exists $c \in C$ with

$$(\mu_R(x,c) = 0 \text{ and } \nu_R(x,c) = 1)$$

or there exists $d \in D$ with

$$(\mu_R(d,x) = 0 \text{ and } \nu_R(d,x) = 1)\,,$$

i.e. there does not exist an element $x \in X$ such that

$$x \in Supp(L(R,C)) \bigcap Supp(U(R,D))\,.$$

□

The following theorem shows that any intuitionistic fuzzy lattice has the fixed point property is complete.

Theorem 4.6. *Every intuitionistic fuzzy lattice has the fixed point property is complete.*

Proof. Assume that (X, μ_R, ν_R) is not an intuitionistic fuzzy complete lattice. We show that there exist an intuitionistic fuzzy monotone mapping $f : X \to X$ which does not have a fixed point. From Theorem 4.5 we know that there exists an intuitionistic fuzzy chain C satisfying the ACC_R without an infimum and an intuitionistic fuzzy chain D satisfying the DCC_R without an supremum. For any $x \in X$ we have that

$$C_x = \{c \in C \quad : \quad \mu_R(x,c) = 0 \text{ and } \nu_R(x,c) = 1\}$$

and

$$D_x = \{d \in D \quad : \quad \mu_R(d,x) = 0 \text{ and } \nu_R(d,x) = 1\}.$$

From Theorem 4.5 (ii) it is easy to see that for any $x \in X$, only one of the above two subsets is nonempty. Suppose that $C_x \neq \emptyset$. Since $C_x \subseteq C$ and C satisfies the ACC_R, then it follows that C_x has the greatest element.

Similarly, if we suppose that $D_x \neq \emptyset$, then from the fact that $D_x \subseteq D$ and D satisfies the DCC_R we get that D_x has the smallest element. Now, we define the mapping $f : X \to X$ as: $f(x)$ is the greatest element of C_x if $C_x \neq \emptyset$ or $f(x)$ is the smallest element of D_x, if $D_x \neq \emptyset$. Obviously, f is an intuitionistic fuzzy monotone mapping with respect to R. Next, we show that f does not have a fixed point. Indeed, let $x \in X$. Since $f(x) \in C_x$ or $f(x) \in D_x$, it follows that

$$\mu_R(x, f(x)) = 0 \text{ and } \nu_R(x, f(x)) = 1$$

or

$$\mu_R(f(x), x) = 0 \text{ and } \nu_R(f(x), x) = 1.$$

Thus, $x \neq f(x)$. Therefore, f does not have a fixed point, which ends the proof. $\square$

General conclusions and future research

In this thesis, we characterized the notions of an intuitionistic fuzzy ideal (resp. filter) on a lattice in terms of the lattice operations and in terms of their associated crisp sets. As interesting kinds, we introduced the notions of a prime intuitionistic fuzzy ideal and filter and investigated their various characterizations and properties.

In this work, we introduced the notions of an intuitionistic fuzzy ideal and filter on an intuitionistic fuzzy lattice based on an intuitionistic fuzzy order relation as a generalization of the notions of a fuzzy ideal and filter given by Mezzomo, et al., and investigated their most interesting properties. We gave a characterization of the intuitionistic fuzzy ideal (resp. filter) in terms of the lattice operations and their level sets. As interesting kinds, we introduced the notions of principal and prime intuitionistic fuzzy ideals and filters and investigated their different characterizations and properties.

Also, we introduced the notion of an intuitionistic fuzzy complete lattice and some characterizations have been expressed in terms of the supremum and the infimum of its subsets, chains and maximal chains. In the main contribution, we have shown that an intuitionistic fuzzy lattice is complete if and only if it satisfies the fixed point property what leads to the fact that the fixed point problem has a complete solution when we restrict to the class of intuitionistic fuzzy lattices.

Future work is anticipated in multiple directions. We think it makes sense to study the notions of an intuitionistic fuzzy ideal and an intuitionistic fuzzy filter for other types of lattices. Moreover, we intend to extend this work to other kinds of intuitionistic fuzzy ideals and filters. Also, we will investigate other classes of intuitionistic fuzzy ordered sets satisfying the fixed point property, in order to combine the properties of these classes with the properties of its intuitionistic fuzzy monotone maps.

Bibliography

[1] M. Akram, W. Dudek, Interval-valued intuitionistic fuzzy Lie ideals of Lie algebras, World Applied Sciences Journal, 7 (7) (2009) 812-819.

[2] T. Anitha, (λ, μ)-Fuzzy ideals in ordered ternary semigroups, Malaya Journal of Matematik, 2 (2015) 326-335.

[3] K. Atanassov, Intuitionistic fuzzy sets, VII ITKRs Scientific Session, Sofia, (1983).

[4] K. Atanassov, intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1986) 87-96.

[5] K. Atanassov, Intuitionistic Fuzzy Sets, Springer-Verlag, Heidelberg / New York, (1999).

[6] K. Atanassov, On Intuitionistic Fuzzy Sets, Springer, Berlin, (2012).

[7] B. Banerjee, D. Kr. Basnet, Intuitionstic fuzzy sub rings and ideals, The Journal of Fuzzy Mathematics, 11 (1) (2003) 139-155.

[8] S. Bashir, X. Du, On weakly regular fuzzy ordered ternary semigroups, Applied Mathematics and Information Sciences, 10 (2016) 2247-2254.

[9] R. Bělohlávek, Concept lattices and order in fuzzy logic, Annals of Pure and Applied Logic, 128 (2004) 277-298.

[10] R. Biswas, On fuzzy sets and intuitionistic fuzzy sets, NIFS, 3 (1997) 3-11.

[11] Y. Bo, W. Wangming, Fuzzy Ideals on a distributive lattice, Fuzzy Sets and Systems, 35 (1990) 231-240.

[12] U. Bodenhofer, F. Klawonn, A Formal Study of Linearity Axioms for Fuzzy Orderings, Fuzzy Sets and Systems, 145 (2004) 323-354.

[13] N. Bourbaki, Topologie générale, Springer-Verlag, Berlin Heidelberg, (2007).

[14] P. Burillo and H. Bustince, Estructuras algebraicas en conjuntos IFS, II Congreso nacional de lógica y tecnologia fuzzy, Boadilla del Monte, Madrid, Spain, (1992) 135-147.

[15] P. Burillo, H. Bustince, Intuitionistic fuzzy relations (Part I), Mathware and Soft Computing Magasine, 2 (1995) 5-38.

[16] P. Burillo, H. Bustince, Intuitionistic fuzzy relations (Part II), Effect of Atanassov's operators on the properties of the intuitionistic fuzzy relations, Mathware and Soft Computing Magasine, 2 (1995) 117-148.

[17] H. Bustince, P. Burillo, Antisymmetrical intuitionistic fuzzy relation, Order on the referential set induced by an bi-fuzzy relation, Fuzzy Sets and Systems, 62 (1995) 17-22.

[18] H. Bustince, P. Burillo, Structures on intuitionistic fuzzy relations, Fuzzy Sets and Systems, 3 (1996) 293–303.

[19] H. Bustince, Construction of intuitionistic fuzzy relations with predetermined properties, Fuzzy Sets and Systems, 3 (2003) 79-403.

[20] R. Chinram, S. Sompob, Fuzzy ideals and fuzzy filters of ordered ternary semigroups, Journal of Mathematics Research, 2 (2010) 93-101.

[21] C. Coppola, G. Gerla, T. Pacelli, Convergence and fixed points by fuzzy orders, Fuzzy Sets and Systems, 159 (2008) 1178-1190.

[22] B. A. Davey, H. A. Priestley, Introduction to Lattices and Order, Second Edition, Cambridge University Press, Cambridge, (2002).

[23] A. C. Davis, A characterization of complete lattices, Pacific Journal of Mathematics, 5 (1955) 311-319.

[24] B. Davvaz, O. Kazanci, A new kind of fuzzy Sublattice (Ideal, Filter) of a lattice, International Journal of Fuzzy Systems, 13 (1) (2011) 55-63.

[25] G. Deschrijver, E. E. Kerre, On the composition of intuitionistic fuzzy relations, Fuzzy Sets and Systems, 136 (2003) 333-361.

[26] T. Gerstenkorn, J. Mańko, Intuitionistic fuzzy probabilistic sets, Fuzzy Sets and Systems, 71 (1995) 207-214.

[27] G. Grätzer, General Lattice Theory, Academic Press, New York, (1978).

[28] P. Grzegorzewski, E. Mrówka, Some notes on (Atanassov's) intuitionistic fuzzy sets, Fuzzy Sets and Systems, 156 (2005) 492-495.

[29] K. Hur, S. Y. Jang, H. W. Kang, Intuitionistic fuzzy subgroupoids, International Journal of Fuzzy Logic and Intelligent Systems, 3 (1) (2003), 72-77.

[30] K. Hur, S. Youn Jang, The lattice of Intuitionistic fuzzy congruences, International Mathematical Forum, 1 (5) (2006) 211-236.

[31] K. H. Kim, Y. B. Jun, Intuitionistic fuzzy interior ideals of semigroups, International Journal of Mathematics and Mathematical Sciences, 27 (5) (2001) 261-267.

[32] Y. Liu, K. Qin, Y. Xu, Fuzzy prime filters of lattice implication algebras, Fuzzy Information and Engineering, 3 (2011) 235-246.

[33] Y. Liu, M. Zheng, Characterizations of fuzzy ideals in coresiduated lattices, Advances in Mathematical Physics, (2016), doi: 10. 1155/2016/6423735.

[34] Y. Liu, K. Qin, Y. Xu, Interval-valued intuitionistic (T, S)-fuzzy filters theory on residuated lattices, International Journal of Machine Learning and Cybernetics, 5 (2014) 683-696.

[35] X. Ma, J. Zhan, W. A. Dudekb, Some kinds of View the MathML source$(\mp \in, \mp \in \vee \mp q)$-fuzzy filters of BL-algebras, Computers and Mathematics with Applications, 58 (2009) 248-256.

[36] P. Martinek, Completely lattice L-ordered sets with and without L-equality, Fuzzy Sets and Systems, 166 (2011) 44-55.

[37] K. Menger, Statistical metrics, Proceedings of the National Academy of Sciences, USA, (1942).

[38] I. Mezzomo, B. C. Bedregal, R. H. N. Santiago, Types of fuzzy ideals in fuzzy lattices, Journal of Intelligent and Fuzzy Systems, 28 (2015) 929-945.

[39] S. Milles, E. Rak, L. Zedam, Intuitionistic fuzzy complete lattices, Novel Developments in Uncertainty Representation and Processing Advances in Intelligent Systems and Computing, Springer International Publishing Switzerland, 401 (2016) 149-160.

[40] S. Milles, L. Zedam, E. Rak, Characterizations of intuitionistic fuzzy ideals and filters based on lattice operations, Journal of Fuzzy Set Valued Analysis, 2 (2017) 143-159.

[41] W. Näther, Copulas and t-norms, Mathematical tools for combinig probabilistic and fuzzy information, with application to error propagation and interaction, Structural Safety, 32 (2010) 366-371.

[42] W. Pedrycz, F. Gomide, An introduction to fuzzy sets, Analysis and Deeign, A Bradford Book, Cambbridge, London, England, (1998).

[43] H. Prodinger, Congruences defined by languages and filters, Information and Control, 44 (1980) 36-46.

[44] Y. Qin, Y. Liu, Intuitionistic (T, S)-fuzzy filters on residuated lattices, General Mathematics Notes, 16 (2013) 55-65.

[45] B. S. Schröder, Ordered Sets, Birkhauser, Boston, (2002).

[46] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific Journal of Mathematics, 1 (1960) 313-334.

[47] H. L. Skala, Trellis theory, Algebra Universalis, 1 (1971) 218-233.

[48] M. H. Stone, The theory of representations of Boolean algebras, Transactions of the American Mathematical Society, 40 (1936) 37-111.

[49] A. Tarski, A lattice-theoretical fixpoint theorem and its applications, Pacific Journal of Mathematics, 5 (1955) 285-309.

[50] A. Tepavčevic, G. Trajkovski, L-fuzzy lattices: an introduction, Fuzzy Sets and Systems, 123 (2001) 209-216.

[51] K. V. Thomas, L. S. Nair, Quotient of ideals of an intuitionistic fuzzy lattices, Advances in Fuzzy System, (2010), doi: 10, 1155/2010/78167 2.

[52] K. V. Thomas, L. S. Nair, Intuitionistic fuzzy sublattices and ideals, Fuzzy Information and Engineering, 3 (2011) 321-331.

[53] M. Tonga, Maximality on fuzzy filters of lattices, Afrika Matematika, 22 (2011) 105-114.

[54] B. K. Tripathy, M. K. Satapathy, P. K. Choudhury, Intuitionistic fuzzy lattices and intuitionistic fuzzy Boolean algebras, International Journal of Engineering and Technology, 5 (2013) 2352–2361.

[55] B. Van Gasse, G. Deschrijver, C. Cornelis, E. E. Kerre, Filters of residuated lattices and triangle algebras, Information Sciences, 480 (2010) 3006-3020.

[56] K. Wang, B. Zhao, Join-completions of L-ordered sets, Fuzzy Sets and Systems, 199 (2012) 92-107.

[57] G. J. Wang, Y. Yutte, Intuitionistic fuzzy sets and L-fuzzy sets, Fuzzy Sets and Systems, 110 (2000) 271-274.

[58] S. Willard, General Topology, Addison-Wesley Publishing Company, Reading, Massachusetts, (1970).

[59] J. Xu, Some properties of intuitionistic (T, S)-fuzzy filters on lattice implication algebras, Theoretical Mathematics and Applications, 3 (2013) 79-89.

[60] Z. S. Xu, Intuitionistic fuzzy preference relations and their application in group decision making, Information Sciences, 177 (2007) 2363-2379.

[61] R. R. Yager, Level sets and the representation theorem for intuitionistic fuzzy sets, Soft Computing, 14 (2010) 1-7.

[62] L. A. Zadeh, Fuzzy sets, Information and Control, 8 (1965) 331-352.

[63] L. A. Zadeh, Similarity relations and fuzzy orderings, Information Sciences, 3 (1971) 177-200.

[64] L. Zedam, A. Amroune, On the representation of L-M algebra by intuitionistic fuzzy subsets, Revue Africaine de la Recherche en Informatique et Mathématiques Appliquées, 4 (2006) 72–85.

[65] L. Zedam, A. Amroune, B. Davvaz, Szpilrajn theorem for intuitionistic fuzzy orderings, Annals of Fuzzy Mathematics and Informatics, 9 (2015) 703–718.

[66] L. Zedam, E. Rak, S. Milles, On intuitionistic fuzzy lattices, Proceeding of The 2015 IEEE International Conference on fuzzy systems (FUZZ-IEEE2015), WordPress, (2015) 1-4.

[67] L. Zedam, S. Milles, E. Rak, The fixed point property for intuitionistic fuzzy lattices, Fuzzy Information and Engineering, 9 (2017) 359-381.

[68] Q.Y. Zhang, W. Xie, L. Fan, Fuzzy complete lattices, Fuzzy Sets and Systems, 160 (2009) 2275-2291.

ملخّـص

في هذه الأطروحة، قمنا أولا بدراسة المثاليات الضبابية الحدسية والمُرَشّحات داخل شبكة كلاسيكية. ثانيا قمنا بتوسعة بعض المفاهيم الخاصة بالمثاليات الضبابية والمرشحات إلى الحالة الضبابية الحدسية داخل شبكة ضبابية حدسية, في كلا المقاربتين قدمنا خصائص مهمة لهذه المفاهيم بواسطة عمليات الشبكة وبواسطة المقاطع -(α,β), علاوة على ذلك, قمنا بتوسعة مفهوم المثالي الأولي (على التوالي المرشح الأولي) إلى المثالي الأولي الضبابي الحدسي (على التوالي المرشح الأولي الضبابي الحدسي) بالنسبة إلى عمليات الشبكة وكذا دراسة مختلف مميزاتها وخواصها. بالنسبة للنقطة الثالثة في هذا العمل فترتكز على مفهوم الشبكة الضبابية الحدسية المقدم من طرف Tripathy وزملائه، قمنا بتقديم مفهوم الشبكة التامة الضبابية الحدسية ودراسة مميزاتها الأساسية. في هذه النقطة قمنا بتوسيع خواصها باستخدام معايير أخرى لعملية التتميم. مميزات الشبكات الضبابية الحدسية المعبر عنها بوجود الحد الأعلى والحد الأدنى لمجموعاتها الجزئية و المعبر عنها بالسلاسل الضبابية الحدسية وكذا السلاسل الكبرى وأيضا المعبر عنها بالسلاسل الصاعدة (على التوالي النازلة) الضبابية الحدسية كلها تم إثباتها. أخيرا قمنا بتعميم نظرية النقطة الصامدة الخاصة بـــ Tarski-Davis إلى الحالة الضبابية الحدسية.

Abstract

In this thesis. First, we investigate the intuitionistic fuzzy ideals and filters on a crisp lattices. Second, we extend the results of fuzzy ideals and filters to intuitionistic fuzzy ideals and filters on intuitionistic fuzzy lattices. For the two approaches, we present interesting characterizations of these notions in terms of lattice operations and in terms of their (α,β)-level sets. Moreover, we extend the notion of prime ideal (resp. prime filter) to prime intuitionistic fuzzy ideal (resp. prime intuitionistic fuzzy filter) with respect to the lattice operations and investigate their various characterizations and properties. For the third point of this work, based on the concept of intuitionistic fuzzy lattice previously proposed by Tripathy et al., we introduce the notion of intuitionistic fuzzy complete lattice and investigate its basic characterizations. In that point, we extend these characterizations by considering others completeness criterions. The characterizations of intuitionistic fuzzy complete lattices expressed in terms of the existence of the supremum or the infimum of their subsets, in terms of intuitionistic fuzzy chains and maximal chains and in terms of intuitionistic fuzzy ascending (resp. descending) chains are given. Furthermore, we will show an intuitionistic fuzzification of Tarski-Davis's fixed point theorem.

Résumé

Dans cette thèse. Tout d'abord, nous étudions les idéaux et les filtres flou intuitionnistes dans un treillis classique. Deuxièmement, nous étendons des résultats des idéaux et des filtres flous aux idéaux et filtres flous intuitionnistes dans un treillis flou intuitionniste. Pour les deux approches, nous présentons des caractérisations intéressantes de ces notions en termes des opérations du treillis et en termes de ses (α,β)-coupes. En outre, nous étendons la notion d'idéal premier (respectivement filtre premier) à l'idéal premier flou intuitionniste (respectivement filtre premier flou intuitionniste) par rapport aux opérations du treillis et étudions leurs différentes caractérisations et propriétés. Pour le troisième point de ce travail, basé sur le concept de treillis flou intuitionniste précédemment proposé par Tripathy et al., on introduit la notion du treillis complet flou intuitionniste et étudie ses caractérisations de base. Dans ce point, nous étendons ces caractérisations en considérant d'autres critères de complétude. Les caractérisations des treillis flous intuitionnistes exprimés en termes de l'existence de la borne supérieure et de la borne inférieure de ses sous-ensembles, en termes des chaînes floues intuitionnistes et des chaînes maximales et en termes des chaînes ascendantes (respectivement descendantes) floues intuitionnistes sont données. En outre, nous allons montrer une intuitionniste fuzzification du théorème du point fixe de Tarski-Davis.