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Non-linear isometries in Lipschitz free spaces

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إهـــداء:

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نـــســـريـــن.

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Introduction

Ancient Greeks like the great engineer Archimedes (c.287–c.212.BC) were able to calculate the measurements of geometrical shapes using rectangles, triangles and squares. Later by the development of symbolism, the intuitive continuity's notion was formulated by R.Lipschitz (1832-1903) who used symbols to represent the same approximative quadrature inequalities and using:
- Numeric discretization : Newton (1642-1727), Leibnitz (1646-1716).
- Symbolic discretization : Cauchy (1789-1857), Weierstrass (1815-1897).
- Constructions of numbers sets : Dedekind (1831-1916).
- Constructions of transfinite numbers[1]: Cantor (1845-1918).
- Constructions of topological spaces: Hilbert (1862-1943), Hausdroff (1868-1942), Fréchet (1878-1973), and Banach (1892-1945).

The main subject of the functional analysis is to study functional spaces and sequences, where we distinguish two different classes: The first class (type theory) is the finite non singular dimension case which is the subject of the real and complex analysis (resolving the open equivalent problems related to the existence of an "atomic" quasi-Banach space, which contains no infinite-dimensional proper closed subspace), the second class (category theory) is the infinite singular dimension case which is the subject of linear algebra and algebraic topology (relative to the singular homology, consisting of finding exact sequences of morphisms between objects, like groups, rings, modules, such that the image of one morphism equals the kernel of the next).

We give in this thesis the modern definition of the Lipschitz functional space, constructed using the same constructive axiomatic (symbolic) system:

• In the first chapter, we introduce preliminary notions about the Lipschitz space $LIP(X,Y)$ and the properties of the Lipschitz functions.

• Then, in the second chapter we present some fundamental results related to the Lipschitzian operators on the free Lipschitz space $\mathcal{F}(M)$.

• Finally, in the last chapter we give the main important results about the non-linear isometries in $\mathcal{F}(M)$, explaining the universal shifting isometric property applied to the open problem of the existing of an infinite linear shifted space isometric to a strict part of a countable separated convex space.

‘

[1]The mathematicians: Kronecker (1823–1891), Newton, Leibnitz, Cauchy, Hilbert, Hausdroff, Dedekind were skepticals about transfinite set theory unlike the mathematicians: Cantor, Lipschitz, Weierstrass, Banach and Fréchet, because of Banach-Taski paradox (1924) related to the axiom of choice (paradoxical equivalent of the zorn's conjecture).

Chapter 1

Structure of general Lipschitz spaces $LIP(X, Y)$

In 1906, Maurice Fréchet submitted his outstanding thesis, « Quelques Points du Calcul Fonctionnel » , introducing the notion of metric space (with a systematic study of functional operations). In 1914, Felix Hausdorff published his famous: « Grundzüge der Mengenlehre », establishing the theory of metric spaces (metrische Räume).

1.1 Normed vector spaces

We study : normed spaces, completeness, linear operators on Banach spaces, duality, operators on Hilbert spaces and the essential classical theorems in the linear functional analysis (For Complete and detailed results see [Mezrag.20]).

1.1.1 Metric spaces

Definition 1.1.1.

Let X be an arbitrary non empty set. A metric, or distance, on X is an application (a map) $d : X \times X \to \mathbb{R}^+$, such that for all x, y, z in X

(i) $\mathrm{d}(x, y) = 0 \Leftrightarrow x = y$ (separation),

(ii) $\mathrm{d}(x, y) = \mathrm{d}(y, x)$ (symmetry),

(iii) $\mathrm{d}(x, z) \leq \mathrm{d}(x, y) + d(y, z)$ (triangular inequality).

A metric space (X, d) is a set X equipped with a metric d.

Example 1.1.1.

If we let $\mathrm{d}(x,y) = |x-y|$, then $(\mathbb{R},\ \mathrm{d})$ is a metric space.

Example 1.1.2.

Let $X = \mathcal{C}([a,b],r)$ be the set of all continuous functions $f : [a,b] \to \mathbb{R}$.

There are many ways to measure the distance between two functions in X.

$$\mathrm{d}_1(f,g) = \int_a^b |f(x) - g(x)|dx$$

is a metric on X. The same for

$$\mathrm{d}_2(f,g) = \left(\int_a^b |f(x) - g(x)|^2 dx\right)^{\frac{1}{2}}$$

and

$$\mathrm{d}_\infty(f,g) = \sup\{|f(x) - g(x)|,\ x \in [a,b]\}$$

Definition 1.1.2.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements in X.

- We say that $(x_n)_{n\in\mathbb{N}}$ converges to an element $x \in X$, and we denote $\lim\limits_{n\to\infty} x_n = x \in X$ if and only if $\lim\limits_{n\to\infty} \mathrm{d}(x_n, x) = 0$.
- The sequence $(x_n)_{n\in\mathbb{N}}$ is said to be a Cauchy sequence if and only if

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N}\ :\ \forall m,\ n > n_\varepsilon \Rightarrow \mathrm{d}(x_n, x_m) < \varepsilon.$$

Definition 1.1.3.

A metric space X is called complete iff every Cauchy sequence in X converges to a limit point in X.

1.1.1.1 The open and closed sets in metrics spaces.

Let (X,d) be a metric space.

Definition 1.1.4.

The open ball of radius $r > 0$ centered at $x_0 \in X$ is denoted as

$$\mathcal{B}(x_0,r) = \{x \in X\ :\ d(x_0,x) < r\}.$$

The closed ball of radius $r > 0$ centered at $x_0 \in X$ is denoted as

$$\bar{\mathcal{B}}(x_0, r) = \{x \in X \ : \ d(x_0, x) \leq r\}.$$

Definition 1.1.5.

Let S be a subset in X.

- $x_0 \in \text{Int}(S) = \mathring{S}$ if and only if there exists $r > 0$ such that $\mathcal{B}(x_0, r) \subset S$.
- $x_0 \in \text{Ext}(S)$ if and only if there exists $r > 0$ such that $\mathcal{B}(x_0, r) \cap S = \emptyset$.
- $\bar{S} = \text{Ext}(S)^C, Fr(S) = \bar{S} - \mathring{S}$

Definition 1.1.6.

- A subset $S \subset X$ is said to be open if and only if:

  for every $x_0 \in S$, there exists $r > 0$ such that $\mathcal{B}(x_0, r) \subset S$.

- A subset S of a metric space X is closed if and only if its complement S^c is open.

Proposition 1.1.1.

All open balls $\mathcal{B}(x_0, r)$ are open sets, while all closed balls $\bar{\mathcal{B}}(x_0, r)$ are closed sets.

Theorem 1.1.1.

Let (X, d) be a metric space.

1. *The sets $\emptyset$, X are opened.*
2. *Any union of opened sets is opened.*
3. *Finite intersection of opened sets is closed.*

Theorem 1.1.2. *(Complementary)*

1. *The sets $\emptyset$, X are closed.*
2. *Any intersection of closed sets is closed.*
3. *Finite union of closed sets is closed.*

1.1.1.2 Continuous functions.

Definition 1.1.7.

A function $f : X \to Y$ between two metrics spaces is called continuous iff it is continuous at all points x_0 in X.

-Topological definition of continuity:

$$\forall x_0 \in X, [\forall \varepsilon > 0,\ \exists \delta > 0 :\ \mathrm{d}_X(x, x_0) < \delta \Rightarrow \mathrm{d}_Y(f(x), f(x_0)) < \varepsilon]$$

which is equivalent to

$$\forall x_0 \in X, [\forall \varepsilon > 0,\ \exists \delta > 0 : \mathcal{B}(x_0, \delta) \subset f^{-1}(\mathcal{B}(f(x_0), \varepsilon))]$$

-Sequential definition of continuity:

$$\forall x_0 \in X, \left[\forall (x_n)_{n\in\mathbb{N}} \subset X\ :\ \lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = f(x_0)\right]$$

1.1.2 Normed and Banach spaces

Definition 1.1.8.

A set X is a vector space or linear space over a field $\mathbb{F}$ =($\mathbb{R}$ or $\mathbb{C}$) if

1. for all x, $y \in X$ and $\lambda \in \mathbb{F}$, we have $x + y \in X$ and $\lambda x \in X$,
2. the set X is an Abelian group with respect to the operation + of addition,
3. the identities

$$\begin{aligned} \lambda(x + y) &= \lambda x + \lambda y \\ (\lambda + \mu)x &= \lambda x + \mu x \\ (\lambda\mu)x &= \lambda(\mu x) \\ 1x &= x \end{aligned}$$

are satisfied for all x, $y \in X$ and λ, $\mu \in \mathbb{F}$.

Definition 1.1.9.

A norm $\|\cdot\|$ on a real or complex vector space X is a function,

$$\|\cdot\| : X \to \mathbb{R}^+$$

such that for all x, y in X and $\lambda \in \mathbb{F}$, we have

(i) $\|x\| = 0 \Leftrightarrow x = 0$ (non degenerate),

(ii) $(\|\lambda x\| = |\lambda|\|x\|$ (one-homogeneity),

(iii) $\|x + y\| \leq \|x\| + \|y\|$ (triangular inequality).

A normed space X, $\|.\|$ is a vector space X equipped with a norm $\|\cdot\|$

Remark 1.1.1.

A normed space is a metric space with the associate metric

$$\mathrm{d}(x, y) = \|x - y\|.$$

Definition 1.1.10.

The open ball of radius $r > 0$ centered at $x_0 \in X$ is denoted as

$$\mathcal{B}(x_0, r) = \{x \in X \ : \ \|x_0 - x\| < r\}.$$

The closed ball of radius $r > 0$ centered at $x_0 \in X$ is denoted as

$$\bar{\mathcal{B}}(x_0, r) = \{x \in X \ : \ \|x_0 - x\| \leq r\}.$$

The closed bull $\bar{B}(x_0, r)$ is the closed $\mathcal{B}(\bar{x_0}, r)$ of the open bull $\mathcal{B}(x_0, r)$.

Definition 1.1.11.

A Banach space is a complete normed space.

Remark 1.1.2.

The complete normed spaces are called Banach spaces after the Polish mathematician Stephan Banach. He is considered to be one of the founders of functional analysis.

Example 1.1.3.

The spaces are Banach ones:

- The spaces $L_p(K)$, $l_p(K)$, $\mathcal{C}(K)$ for K compact and $1 \leq p \leq \infty$
- The space $l^\infty(\mathbb{R})$ is a complete Banach space.
- The space $(\mathcal{C}([a,b\,]\,,\,\|\cdot\|_\infty)$

Remark 1.1.3.

The interval $[a,b]$ can be replaced by any compact subset K of any topological space and $\mathbb{R}$ can be replaced by any complete metric space.

1.1.2.1 Properties of finite-dimensional spaces

For any Banach or normed space X, we denote by $\mathcal{B}_X$, the closed unit ball in X. The following theorem, due to F. Riesz characterizes finite-dimensional spaces among normed linear spaces.

Theorem 1.1.3. *(F. Rieszs theorem)*

For a normed linear space X, the following statements are equivalent.

(a) *The closed unit ball $\mathcal{B}_X$ is compact.*

(b) *Every closed bounded set in X is compact.*

(c) *The space X is finite-dimensional.*

Proposition 1.1.2.

A subspace E of a Banach space X is complete if and only if, E is closed.

1.1.2.2 Equivalent norms.

Two equivalent norms on the same vector space X are two norms for which the topologies induced on X are identical. This equivalence of norms on X translates the equivalence of the associated distances.

Definition 1.1.12.

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space X are said to be equivalent iff there exist positive real numbers a, b such that for all $x \in X$ the inequalities 1.1 holds.

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \tag{1.1}$$

Proposition 1.1.3.

Let X be a vector space. Then, the following properties are equivalent.

(i) *The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent in X,*

(ii) *The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ generate the same topology on X.*

(iii) *Let (x_n) be in X, then $x_n \xrightarrow{\|\cdot\|_1} x$ if and only if, $x_n \xrightarrow{\|\cdot\|_2} x$.*

Theorem 1.1.4.

Let X be a finite dimensional vector space. Then all norms on X are equivalent.

Example 1.1.4.

We have for all $(x)_{1\leq i\leq n}$ in $\mathbb{R}^n$ and $1 \leq p \leq q \leq \infty$

$$\left(\sum_{i=1}^{n} |x_i|^q\right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \leq n^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{i=1}^{n} |x_i|^q\right)^{\frac{1}{q}}$$

Remark 1.1.4.

- The norms $\|\cdot\|_p$ are equivalent for all n and for all $1 \leq p \leq \infty$.
- If n is infinite the norms $\|\cdot\|_p$ are not equivalent.

  Take the sequence $x_n = \dfrac{1}{n}$ in l_1 and l_2.
- If X is not of finite dimensional, the precedent theorem is false as shown by this example.

Example 1.1.5.

Let $\mathcal{C}([0,1])$ denote the set of continuous real-valued functions $f : [0,1] \to \mathbb{R}$, then the norms $\|f\|_\infty = \sup_{x\in[0,1]} \|f(x)\|$ and $\|f\|_1 = \int_0^1 |f(x)|dx$ are not equivalent on $\mathcal{C}([0,1])$, (the first one is a Banach space, but not the second one).

Corollary 1.1.1.

Every finite dimensional normed vector space is a Banach space.

1.1.3 Series in normed spaces

Definition 1.1.13.

Let $(X, \|\cdot\|)$ be a normed vector space. Set $(x_n)_{n\geq 1}$ a sequence in X .

- we say the formal series $\sum_{n=1}^{\infty} x_n$ is convergent iff the sequence $(s_N)_{N\geq 1}$ of partial sums $s_N = \sum_{n=1}^{N} x_n$ converges to a point $x \in X$ i.e.,

$$\|x - \sum_{n=1}^{N} x_n\| \to 0 \quad \text{as} \quad n \to \infty$$

  In this case we write $x = \sum_{n=1}^{\infty} x_n$.

- If $\sum_{n=1}^{\infty} \|x_n\|$ converges in $\mathbb{R}$ then we say that $\sum_{n=1}^{\infty} x_n$ is absolutely convergent.

Theorem 1.1.5.

Every absolutely convergent series in a Banach space is also convergent.

$$\|\sum_{n=1}^{\infty} x_n\| \leq \sum_{n=1}^{\infty} \|x_n\|.$$

Theorem 1.1.6.

A normed space X is a Banach space if and only if the absolute convergence of all series in X implies the usual convergence.

1.1.4 Hilbert spaces

Let H be a complex vector space.

Definition 1.1.14. (Inner Product)

A Hermitian inner product on H is a function, $\langle \cdot, \cdot \rangle : H \times H \longrightarrow \mathbb{C}$, such that for all x, y, z in H and all a, b in $\mathbb{C}$, we have

1. $\langle ax + y,\ z\rangle = a\langle x,\ z\rangle + \langle y,\ z\rangle$ $(i.e.,\ x \longmapsto \langle x, z\rangle$ is linear)

2. $\langle x,\ y\rangle = \overline{\langle y, x\rangle}$ (hermitian symmetric)

3. $\langle x,\ x\rangle \geq 0$ (with equality $\langle x,\ x\rangle = 0$ if and only if x = 0 positive definite)

Definition 1.1.15.

We call a linear space with a hermitian inner product a pre-Hilbert space.

Definition 1.1.16.

Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space,

we say x, $y \in H$ are orthogonal and write $x \perp y$ iff, $\langle x, y\rangle = 0$.

Notation 1.1.1.

- More generally if, $A \subset H$ is a set, $x \in H$ is orthogonal to A and write $x \perp A$ if, and only if, $\langle x, y\rangle = 0$ for all $y \in A$.
- Let $A^{\perp} = \{x \in H : x \perp A\}$ be the set of vectors orthogonal to A.
- We also say that a set $S \subset H$ is orthogonal if, $x \perp y$ for all $x \neq y \in S$
- If, S further satises, $\|x\| = 1$, for all $x \in S$, then S is said to be orthonormal.
- Note also that if S is orthogonal, then $\{x/\|x\| \ :\ x \in S\backslash\{\mathrm{O}\}\}$ is orthonormal.

Definition 1.1.17. (Hilbert space)

A Hilbert space is a complete pre-Hilbert space

Example 1.1.6.

Let (X, M, μ) be a measure space then $\mathcal{H} := L^2(X, M, \mu)$ with inner product

$$\langle f,\ g\rangle = \int_X f(x)\mathrm{g}(x)d\mu$$

is a Hilbert space.

We now state the most important theorem in Hilbert space theory.

Theorem 1.1.7. *(Projection Theorem)*

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{K}$ a closed subspace of $\mathcal{H}$.

(i) *For each $x \in \mathcal{H}$, there is a unique $y_0 \in \mathcal{K}$ such that*

$$\|x - y_0\| = \mathrm{d}(x, \mathcal{K}) = \min_{y \in \mathcal{K}} \|x - y\|$$

(ii) *$y_0 \in \mathcal{K}$ and $\|x - y_0\| = \min\{\|x - y\| \ : \ y \in \mathcal{K}\}$ if and only if, $y_0 \in \mathcal{K}$ and $(x - y_0) \in \mathcal{K}^{\perp}$ where $\mathcal{K}^{\perp}$is the orthogonal complement to $\mathcal{K}$ in $\mathcal{H}$.*

The element y_0 is called the orthogonal projection of x onto $\mathcal{K}$.

Corollary 1.1.2.

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{K}$ be a closed subspace of $\mathcal{H}$. For all $x \in \mathcal{H}$, denote by $p_{\mathcal{K}}(x) \in \mathcal{K}$ the unique element of $\mathcal{K}$ such that:

$$\|x - p_{\mathcal{K}}(x)\| = \min\{\|x - y\| \ : \ y \in K\}.$$

- *The operator $p_{\mathcal{K}} : \mathcal{H} \to \mathcal{K}$ is linear, continuous, of norm 1 and $p_{\mathcal{K}}|\mathcal{K} = \mathrm{Id}_{\mathcal{K}}$.*
- *We have $x \in \mathcal{K}$ if and only if, $p_{\mathcal{K}}(x) = x$.*
- *We have $\mathcal{K} \oplus \mathcal{K}^{\perp} = \mathcal{H}$.*
- *For all $x = x_1 + x_2 \in \mathcal{H}$ with $x_1 \in \mathcal{K}$ and $x_2 \in \mathcal{K}^{\perp}$, one have $p(x) = x_1$.*
- *We have $x \in \mathcal{K}^{\perp} if$ and only if $p_{\mathcal{K}}(x) = 0$*
- *$\mathcal{K}_1 \subseteq \mathcal{K}_2$ if and only if $P_{\mathcal{K}_2}(P_{\mathcal{K}_1}(x)) = P_{\mathcal{K}_2}$.*
- *$\|x\|^2 = \|P_{\mathcal{K}}(x)\|^2 + \|(\mathrm{Id} - P_{\mathcal{K}})x\|^2$*
- *$P_{\mathcal{K}}^* = P_{\mathcal{K}}$, ($P_{\mathcal{K}}$ is self-adjoint).*
- *$\langle P_{\mathcal{K}}^*(x),\ y\rangle = \langle x,\ P_{\mathcal{K}}^*(y)\rangle = \langle x,\ y\rangle$.*
- *Ran $(P_{\mathcal{K}}) = \mathcal{K}$ and $\ker(P_{\mathcal{K}}) = \mathcal{K}^{\perp}$.*
- *The operator $P_{\mathcal{K}}$ is called orthogonal projection.*

Remark 1.1.5.

Let E be a Banach space non isomorphic to a Hilbert space, then there is always a closed subspace without closed supplementary space.

1.1.5 Continuous linear operators on normed spaces

We will interest to a new normed linear space, namely the mappings linear space.

Definition 1.1.18.

A linear map or linear operator T between real (or complex) linear spaces X, Y is a function $T : X \to Y$ such that

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for all α, $\beta \in \mathbb{R}$ (or $\mathbb{C}$) and x, $y \in X$.

If X, Y are normed spaces, then we can denote the notion of a bounded linear operator or continuous linear operator. As we will see in the sequel, the boundedness of a linear operator is equivalent to its continuity.

Definition 1.1.19.

Let X and Y be two normed linear spaces. We denote both the X and Y norms by $\|\cdot\|$. A linear map $T : X \to Y$ is bounded iff there is a constant C such that

$$\forall x \in X, \quad \|T(x)\| \leq C\|x\|.$$

Notation 1.1.2.

- If $T : X \to Y$ is a bounded linear map, then we denote the operator norm $\|T\|$ by

$$\|T\| = \inf\{C : \|T(x)\| \leq C\|x\|, x \in X\} = \sup\{\frac{\|T(x)\|}{\|x\|},\ x \in X\}.$$

- We denote the set of all linear maps $T : X \to Y$ by $L(x, y)$, and the set of all bounded linear maps T: $X \to Y$ by $\mathcal{B}(x, y)$ or $\mathcal{L}(x, y)$.

- If $Y = \mathbb{R}$ or $\mathbb{C}$ then $\mathcal{B}(X, r) = X^*$. We will say that X^* is the dual of X.

Remark 1.1.6.

Equivalent expressions for $\|T\|$ are

$$\|T\| = \sup\left\{\frac{\|T(x)\|}{\|x\|},\ x \in X\right\} = \sup\left\{\frac{\|T(x)\|}{\|x\|},\ \|x\| \leq 1\right\} = \sup\{\|T(x)\|,\ \|x\| = 1\}.$$

Example 1.1.7. (Norm atteined)

Let $X = l_1$ and $T : X \to \mathbb{C}$ denoted for $x = (X)$ by

$$T(x) = \sum_{n=1}^{\infty} \frac{x_n}{n}.$$

Then T is linear and $\|T\| = 1$ (we have $T(e_1) = 1$).

Example 1.1.8. (Norm not atteined)

Let $X = c_0$ equipped with the norm $\|\cdot\|_1$. Let $T : X \to \mathbb{C}$ denoted by

$$T(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

Then T is linear and we have, $|T(x)| \leq \|x\|_\infty$. Thus, $T \in \mathcal{B}(X, \mathbb{C})$ and $\|T\| \leq 1$.

Proposition 1.1.4.

Let $T : X \to Y$ be a linear operator between the normed vector spaces X and Y.

Then, the following assertions are equivalent.

(a) *T is continuous,*

(b) *T is continuous at* 0,

(c) *T is bounded*

Theorem 1.1.8.

Let X be a normed vector space, Y a Banach space then $\mathcal{L}(x, y)$ with the operator norm

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|$$

is a Banach space.

Remark 1.1.7.

The converse is true. If Y is a Banach space then so is $\mathcal{B}(x, y)$. (We do not need to assume that X is a Banach space):

Definition 1.1.20. (Normed Dual space X^*)

Let X be a normed space. Then the set of all bounded linear functionals on X constitutes a Banach space, with the norm

$$\|f\| = \sup_{x\neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)| = \sup_{\|x\|\leq 1} |f(x)| = \sup_{\|x\|<1} |f(x)|.$$

This is the normed (or topological) dual of X. In general we denote f by x^* and we write $x^*(x) = \langle x^*, x\rangle$ (brackets of duality).

Definition 1.1.21.

Let T be in $\mathcal{B}(x, y)$.

1. We say that T is

   (a) an isometry iff $\|T(x)\| = \|x\|$, for all $x \in X$,

   (b) an isometric isomorphism iff T is a linear isometry which is also a surjection,

   (c) an isomorphic isomorphism iff T and T^{-1} are bounded.

2. We say that X is embeddable in Y iff it is isomorphic with a subspace of Y.

Definition 1.1.22.

The norm dual of X^* is called the second dual (bidual or the double dual) of X and is denoted by X. The normed space X can be embedded isometrically in X^{**}

$$\begin{array}{rccc} i: & X & \longrightarrow & X^{**} \\ & x & \longmapsto & i(x) \end{array}$$

via the formula $i(x)(x^*) = x^*(x) = \langle x^*,\ x\rangle$ for each $x^* \in X^*$

Lemma 1.1.1.

Let X be a Banach space. For each $x \in X$, we have $\|i(x)\| = \max_{\|x^*\|=1} |x^*(x)| = \|x\|$

Remark 1.1.8.

- Let X be a Banach space. The mapping $x \longmapsto i(x)$ from a Banach X into X^{**} is a linear isometry, so X can be identified with a subspace $i(X)$ of X^{**}.

- When the linear isometry $x \longmapsto i(x)$ from a Banach X into X^{**} is surjective, the Banach space X is called reflexive. That is, we have the following definition.

Definition 1.1.23.

A Banach space X is called reflexive iff $X \overset{i}{\equiv} i(X) = X^{**}$, i.e., i is surjective.

Theorem 1.1.9. *(Finite dimension)*

Let X, Y be normed spaces, $\dim X < \infty$*, then any linear operator $T : X \to Y$ is bounded*

1.1.5.1 Adjoint operator.

The word adjoint has a number of related meanings. In linear analysis, it refers to the conjugate transpose and is most commonly denoted T^*.

Definition 1.1.24.

Let X, Y be two Banach spaces and T be a bounded operator from X into Y. The adjoint (or dual or transpose) of T, noted T^*, is the bounded operator from Y^* to X^* such that

$$T^*(y^*)(x) = \langle T^*(y^*),\ x\rangle = \langle y^*,\ T(x)\rangle\,.$$

Theorem 1.1.10.

Let X, Y be two Banach spaces. The map from $\mathcal{B}(x, y)$ into $\mathcal{B}(Y^*,\ X^*)$ which associates to T its adjoint T^* is an isometry (i.e., $\|T\| = \|T^*\|$ for all $T \in \mathcal{B}(x, y)$).

Proposition 1.1.5.

Let X, Y, Z be three Banach, T, S in $\mathcal{B}(X, Y)$ and R in $\mathcal{B}(Z, X)$. Then

(a) $(T + S)^* = T^* + S^*$ *and* $(T \mathrm{o} R)^* = R^* \mathrm{o} T^*$.

b) *If $T : X \to Y$ is a topological isomorphism, then $T^* : Y^* \to X^*$ is also a topological isomorphism, and in fact* $(T^*)^{-1} = (T^{-1})^*$.

(c) *If $T : X \to Y$ is an isometric isomorphism, then $T^* : Y^* \to X^*$ is also an isometric isomorphism.*

1.1.5.2 Quotient space.

Proposition 1.1.6.

1. *Let X be a Banach space and M be a closed linear subspace of X.*

   Then M is a Banach space.

2. *Let X be a normed linear space and let M be a linear subspace of X.*

   Then $\bar{M}$ is also a linear subspace of X.

Notation 1.1.3.

- Let X be any vector space and let M be any linear subspace of X. Then, the quotient space X/M is denoted to be the linear space of cosets

$$\bar{\mathrm{x}} = x + M = \{x + m \ : \ m \in M\}$$

  under the algebraic operations

$$\bar{\mathrm{x}} + \bar{\mathrm{y}} \equiv (x + M) + (y + M) = (x + y) + M = x \bar{+} y \quad for \quad \bar{\mathrm{x}}, \bar{\mathrm{y}} \in X/M$$

  and

$$\lambda\bar{\mathrm{x}} \equiv \lambda(x + M) = \lambda x + M = \bar{\lambda} x \ for \ \bar{\mathrm{x}} \in X/M \ and \ \lambda \in \mathbb{R} \ or \ \mathbb{C}$$

- The coset $\bar{0}$ is the zero vector of X/M and is simply denoted 0. The axioms of the vector space are satisfied equipped with the following quotient space norm:

$$\|\mathrm{x}\|_{X/M} = \mathrm{d}(0, x + M) = \inf_{y \in x+M} \|y\| = \inf_{m \in M} \|x + m\|.$$

- We call $\| \cdot \|_{X/M}$ the quotient norm on the space X/M

Theorem 1.1.11.

1. *The space $(X/M, \| \cdot \|)$ is a normed space iff, M is a closed linear subspace of X.*

2. *Let X be a Banach space, M a closed linear subspace, then X/M is a Banach space*

3. *The quotient map*

$$\begin{array}{rccc} q: & X & \longrightarrow & X/M \\ & x & \longmapsto & q(x) = \bar{x} \end{array}$$

   is a bounded linear operator ($\|q\| \leq 1$) , surjective and open.

1.1.6 Operators on Hilbert spaces

Definition 1.1.25.

Let $\mathcal{H}$ be a Hilbert space. For all x in $\mathcal{H}$, one denote a linear form $\varphi_x : \mathcal{H} \to \mathbb{K}$ by $\varphi_x(y) = \langle y,\ x\rangle$ for any y in $\mathcal{H}$. We define H^* the dual of H, by all such linear forms φ,

$$H^* = \{\varphi_x : \mathcal{H} \to \mathbb{K},\ \ [\forall y \in \mathcal{H},\ \varphi_x(y) = \langle y,\ x\rangle]\,,\ x \in \mathcal{H}\}$$

Proposition 1.1.7.

Consider φ_x as the definition 1.1.25 above, then it is verifying the following assertions:

(i) *The linear form φ_x is continuous and $\|\varphi_x\| = \|x\|$.*

(ii) *The application*

$$\begin{array}{rccc} \varphi : & \mathcal{H} & \longrightarrow & \mathcal{H}^* \\ & x & \longmapsto & \varphi_x \end{array}$$

is an anti-linear isometric (linear isometric iff $\mathbb{K} = \mathbb{R}$).

Theorem 1.1.12. *(Rieszs Theorem)*

Let $\mathcal{H}$ be a Hilbert space. Then, the anti-linear isometric φ is bijective.

1.2 Mappings in Lipschitz space

Definition 1.2.1. [Deza.16]

Given a metric space $(X,\ d)$, a mapping $f : X \to X$ verifying the condition

$$d(f(x), f(y)) = r \cdot d(x, y) \tag{1.2}$$

for some $r > 0$ and any $x \in X$, is called **r-**dilation.

Definition 1.2.2.

A homeomorphism $f : X \to Y$, is called a quasi**-** conformal mapping (or, specifically, **C-**quasi **-**conformal mapping), iff there exists a constant $C > 0$, such that the inequality (1.3), holds for each $x \in X$.

$$\limsup_{r\to 0}\left(\frac{\max\left\{\ d_Y(f(x), fy)),\quad \text{such that}\quad d_X(x,y) \le r\ \right\}}{\min\left\{\ d_Y(f(x), fy)),\quad \text{such that}\quad d_X(x,y) \ge r\ \right\}}\right) \quad \le \quad C. \tag{1.3}$$

The smallest such constant C is called the conformal dilation.

Remark 1.2.1.

- The conformal dimension of a metric space (X, d) is the infimum of the Hausdorff dimension over all quasi-conformal mappings of $(X,\ d)$ into some metric space.
- For the middle-third Cantor set on $[0, 1]$, it is 0, but, for any of its quasi-conformal images, it is strictly positive.

1.2.1 Hölder mappings

Definition 1.2.3.

Let $c, \alpha \geq 0$, be constants. Given metric spaces $(X,\ d_X)$ and $(Y,\ d_Y)$.
A function $f : X \to Y$ is called α-Hölder mapping , (the constant α should be mentioned) if for all $x, y \in X$, the inequality (1.4), holds.

$$d_Y(f(x), f(y)) \ \leq c \cdot (d_X(x, y))^{\alpha}. \tag{1.4}$$

Remark 1.2.2.

- The number α is called the exponent of the Hölder condition.
- The **0**-Hölder mapping means that the metric d_Y is bounded.
- The **1**-Hölder mapping is a Lipschitz mapping
- If $0 < \alpha < 1$, the condition (1.4), implies that the function f is continuous.
- If $\alpha > 1$, then any function satisfying the condition (1.4), is constant.
- If $X = Y = \mathbb{R}$, then we get a chain of inclusions over any compact real subset

$$\begin{array}{ccc} \text{Continuously Differentiable} & \Rightarrow & \text{Lipschitz Continuous} \\ \Downarrow & & \Downarrow \\ \text{Uniformly Continuous} & \Leftarrow & \alpha\text{-Hölder Continuous} \\ \Downarrow & & \\ \text{Continuous} & & \end{array}$$

1.2.2 Lipschitz mappings

Definition 1.2.4.

Let c be a positive constant (the constant c should be mentioned). Given metric spaces $(X,\ d_X)$ and $(Y,\ d_Y)$, then a function $f : X \to Y$ is called a Lipschitz continuous mapping (or **c-**Lipschitz mapping) iff for all $x, y \in X$ the inequality (1.5), holds.

$$d_Y(f(x), fy)) \ \leq c \cdot d_X(x, y). \tag{1.5}$$

Remark 1.2.3.

- If $c = 1$, then the **c-**Lipschitz mapping is called a metric mapping.
- If $0 < c < 1$, then it is called a contraction.

1.2.2.1 bi-Lipschitz mapping

Definition 1.2.5.

Given metric spaces $(X,\ d_X), (Y,\ d_Y)$ and a constant $c > 1$, then a function $f : X \to Y$ is called a c**-bi-**Lipschitz mapping, (or **c-**Lipschitz embedding) iff there exists a number $r > 0$ such that for any $x, y \in X$, the inequality (1.6), holds.

$$r \cdot d_X(x, y) \ \leq \ d_Y(f(x), fy)) \quad \leq \quad c \cdot r \cdot d_X(x, y). \tag{1.6}$$

Remark 1.2.4.

- Any **bi-**Lipschitz mapping is a quasi**-**symmetric mapping.
- We call distortion of f, the smallest c for which f is a **c-bi-**Lipschitz mapping.
- Any **k-**point metric space, c**-**embeds into an Euclidean space with distortion $O(\ln k)$.

Definition 1.2.6.

Two metrics d_1 and d_2 on X are called **bi-**Lipschitz equivalent metrics iff there are positive constants c and C such that for all $x, y \in X$, the two following inequalities (1.7) holds

$$c \cdot d_1(x, y) \leq \ d_2(x, y) \leq C \cdot d_1(x, y) \tag{1.7}$$

i.e., the identity mapping is a **bi-**Lipschitz mapping from $(X,\ d_1)$ into $(X,\ d_2)$.

Remark 1.2.5.

- The **bi-**Lipschitz equivalent metrics are equivalent, i.e., generate the same topology.
- A **bi-**Lipschitz mapping $f : X \to Y$ is a c-isomorphism $f : X \to f(X)$.

Example 1.2.1.

The L_1-metric and L_2-metric on $\mathbb{R}$ are distinct topologies (having different distances), so they are not **bi-**Lipschitz equivalent.

1.2.2.2 c-isomorphism of metric spaces

Definition 1.2.7.

Given two metric spaces $(X,\ d_X)$ and $(Y,\ d_Y)$, the Lipschitz norm $\|\cdot\|_{Lip}$ on the set of all injective mappings $f : X \to Y$ is defined by

$$L(f) = \|f\|_{Lip} = \sup_{x,y\in X, x\neq y} \frac{d_Y(f(x), fy))}{d_X(x, y)}. \tag{1.8}$$

Definition 1.2.8.

Two metric spaces X and Y are called c-isomorphic iff there exists an injective mapping $f : X \to Y$ such that,

$$\|f\|_{Lip} \times \left\|f^{-1}\right\|_{Lip} \quad \leq \quad c. \tag{1.9}$$

1.2.2.3 Uniform metric mappings

Definition 1.2.9.

Given metric spaces $(X,\ d_X)$ and $(Y,\ d_Y)$, a function $f : X \to Y$ is called a uniform metric mapping iff there are two nondecreasing functions g_1 and g_2 from $\mathbb{R}^+$ to itself with $\lim_{r\to\infty} g_i(r) = 1$ for $i = 1, 2$, such that the inequality (1.10), holds for all $x, y \in X$.

$$g_1(d_X(x, y)) \ \leq d_Y(f(x), fy)) \ \leq g_2(d_X(x, y)) \tag{1.10}$$

Remark 1.2.6.

Any **bi-**Lipschitz mapping is a uniform metric mapping with linear functions.

1.2.2.4 Metric compression

Definition 1.2.10.

Given metric spaces $(X,\ d_X)$ (which could be unbounded) and $(Y,\ d_Y)$,

then we say that a function $f : X \to Y$ is a large scale Lipschitz mapping iff,

for some $c > 0, D \geq 0$ and for all $x, y \in X$, the inequality (1.11), holds

$$d_Y(f(x), fy)) \ \leq c \cdot d_X(x, y) + D. \tag{1.11}$$

Remark 1.2.7.

- The compression of such a mapping f is

$$\rho_f(r) = \inf_{d_X(x,y) \geq r} d_Y(f(x), fy)). \tag{1.12}$$

- The metric compression of $(X,\ d_X)$ in $(Y,\ d_Y)$ is defined by

$$R(x, y) = \sup_f \left\{ \underline{\lim}_{r \to \infty} \frac{\log\left(\max\{\rho_f(r), 1\}\right)}{\log r} \right\}, \tag{1.13}$$

where the supremum is over all large scale Lipschitz mappings f.

Definition 1.2.11.

Given $\alpha \geq 0$ and two metric spaces $(X,\ d_X)$, $(Y,\ d_Y)$, the α-Hölder norm $\|\cdot\|_{\ Hol}$ on the set of all injective functions $f : X \to Y$ is defined by (1.14).

$$\|f\|_{Hol} = \sup_{x,y \in X, x \neq y} \left(\frac{d_Y(f(x), fy))}{d_X(x, y)^\alpha} \right). \tag{1.14}$$

The Lipschitz norm $\|\cdot\|_{\ Lip}$ is the case $\alpha = 1$ of $\|\cdot\|_{\ Hol}$.

Definition 1.2.12.

The Lipschitz distance between metric spaces $(X,\ d_X)$ and $(Y,\ d_Y)$ is defined by (1.15), where the infimum is taken over all bijective functions $f : X \to Y$.

$$d_{Lip}(X, Y) = \ln \left(\inf_f \left(\|f\|_{Lip} \times \left\|f^{-1}\right\|_{Lip} \right) \right), \tag{1.15}$$

Remark 1.2.8.

- Equivalently, it is the infinitum of numbers $\ln(a)$ such that there exists a bijective **bi-**Lipschitz mapping between $(X,\ d_X)$ and $(Y,\ d_Y)$ with constants $\exp(-a), \exp(a)$.
- It becomes a metric (Lipschitz metric) on the set of all isometry classes of compact metric spaces (Hausdorff**-**Lipschitz distance).
- This distance is analogue to the Banach**-**Mazur distance.
- The real Banach spaces could coincides with it in the case of finite **-**dimensional .
- It also coincides with the Hilbert projective metric on nonnegative projective spaces, obtained by starting with $\mathbb{R}^n_+$ and identifying any point x with $c \cdot x,\ (c > 0)$

Definition 1.2.13.

Given a compact metric space $(X,\ d)$, the Lipschitz seminorm $lip(\cdot) = \|\cdot\|_{\ Lip}$ on the set of all functions $f\colon X \to \mathbb{R}$ is defined by

$$lip(f) = \|f\|_{Lip} = \sup_{x,y\in X, x\neq y} \frac{|f(x)-f(y)|}{d(x,y)}. \tag{1.16}$$

The Lipschitz distance between measures μ and v on X is defined by

$$d_{Lip}(\mu, v) = \sup_{\|f\|_{Lip}\ \leq\ 1} \int f d(\mu - v). \tag{1.17}$$

If μ, v are probability measures, then (1.17) is the transportation distance.

1.3 Lipschitz functions properties

Proposition 1.3.1. *[Wea.99]*

Let X, Y, and Z be metric spaces and let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be Lipschitz functions. Then $g \circ f : X \longrightarrow Z$ is also Lipschitz function, with

$$L(g \circ f) \leq L(f).L(g).$$

Proof.

For any $p, q \in X$, we put: $(p', q') = (f(p), f(q)) \in Y \times Y$, since g is a Lipschitz function, then $d(g(p', q')) \leq L(g) \cdot \ d(p', q')$. We have also f a Lipschitz function, it follows:

$$\begin{aligned} d(p', q') &= d(f(p), f(q)) && (1.18) \\ &\leq L(f) \cdot d(p, q). && (1.19) \end{aligned}$$

We put $h = g \circ f$, then since all norms are positive and by the inequality (1.18) we deduce

$$\begin{aligned} d(h(p), h(q)) &= d(g \circ f(p), g \circ f(q)) \\ &= d(g(f(p)), g(f(q))) \\ &= d(g(p', q')) \\ &\leq L(g) \cdot d(p', q') \leq L(g) \cdot L(f) \cdot d(p, q) \end{aligned}$$

then

$$\frac{d(h(p), h(q))}{d(p, q)} \leq L(g) \cdot L(f) = C_{f,g},$$

hence

$$L(g \circ f) = L(h) \leq C_{f,g} = L(g) \cdot L(f)$$

□

Proposition 1.3.2. *[Ram-San.20]*

Set (X, d) a metric space, and $f, g : (X, d) \longrightarrow \mathbb{R}$ two Lipschitz functions and $a \in \mathbb{R}$, then

1. $lip(f + g) \leq lip(f) + lip(g)$
2. $lip(\lambda f) \leq |\lambda| \ lip(f)$
3. $lip(f \vee g) \leq \max\{lip(f), lip(g)\} and\ lip(f \wedge g) \leq \max\{lip(f), lip(g)\}$
4. $lip(f \cdot g) \leq lip(f) g_\infty + lip(g) \|f\|_\infty$

Proof. 1.

$$\begin{aligned} lip(f+g) &= sup_{x\neq y}\frac{|f(x)+g(x)\ -(f(y)+g(y)|)}{d(x,y)} \\ &= sup_{x\neq y}\frac{|f(x)-f(y)+g(x)-g(y)|}{d(x,y)} \\ &= sup_{x\neq y}(\frac{|(f(x)-f(y)|}{d(x,y)}+\frac{|g(x)-g(y)|}{d(x,y)}) \\ &\leq sup_{x\neq y}\frac{|(f(x)-f(y)|}{d(x,y)}+sup_{x\neq y}\frac{|g(x)-g(y)|}{d(x,y)} \\ &\leq lip(f)\cdot lip(g) \end{aligned}$$

2. Taking suprema on both sides of the following, then (2) yields.

$$\begin{aligned} lip(\lambda f) &= sup_{x\neq y}\frac{|\lambda f(x)\ -\lambda f(y)|}{d(x,y)} \\ &= sup_{x\neq y}\frac{|\lambda(f(x)\ -f(y))|}{d(x,y)} \\ &= |\lambda|\, sup_{x\neq y}\frac{|f(x)\ -f(y)|}{d(x,y)} \\ &= \ |\lambda|\, lip(f) \end{aligned}$$

3. To prove (3), let $h = f \vee g$ and suppose $h(x) \geq h(y)$ without loss of generality.

   If $h(x) = f(x)$ then

$$\begin{aligned} \frac{h(x)-h(y)}{d(x,y)} &= \frac{f(x)-h(y)}{d(x,y)} \\ &\leq \frac{f(x)-f(y)}{d(x,y)} \\ &\leq \|f\|_L\,, \end{aligned}$$

   and if $h(x) = g(x)$ then we have the same with g in place of f.

   In any case,taking suprema; then (3) yields

$$\|h\|_L \leq max\{\|f\|_L\,, \|g\|_L\}.$$

   The corresponding inequality for

$$h = f \wedge g = -((-f) \vee (-g))$$

   follows immediately.

4. If $diam(x) < \infty$, then the product of any two complex-valued Lipschitz functions on X is again a Lipschitz function.

$$\begin{aligned}\frac{|(f\cdot g)(x) - (f\cdot g)(y)|}{d(x,y)} &\leq \frac{|f(x)\cdot g(x) - f(x)\cdot g(y)|}{d(x,y)} + \frac{|f(x)\cdot g(x) - f(x)\cdot g(y)|}{d(x,y)} \\ &= f(x)\frac{|g(x) - g(y)|}{d(x,y)} + |g(x)|\,\frac{|f(x) - f(y)|}{d(x,y)} \\ &\leq \|f\|_\infty \|g\|_L + \|g\|_\infty \|f\|_L .\end{aligned}$$

□

Proposition 1.3.3. *[Wea.99]*

Let X be a metric space and let $f, g : X \longrightarrow \mathbb{R}$ be Lipschitz maps, then

1. $lip(f\cdot g) \leq \|f\|_\infty \cdot lip(g) + \|g\|_\infty \cdot lip(f)$
2. $lip(\frac{1}{f}) \leq \frac{ip(f)}{\epsilon^2}$, *if* $|f(x)| \geq \epsilon > 0$, *for all* $x \in X$.
3. *If* $diam(X) < \infty$, *then the product of any two scalar valued Lipschitz functions is a Lipschitz function.*

Proof. 1.

$$\begin{aligned}|f\cdot g(p) - f\cdot g(q)| &\leq |f(p)\cdot g(p) - f(p)\cdot g(q)| + |f(p)\cdot g(q) - f(q)\cdot g(q)| \\ &= |f(p)|\cdot|g(p) - g(q|) + |g(q)|\cdot|f(p) - f(q)| \\ &\leq (\|f\|_\infty \cdot L(g) + \|g\|_\infty \cdot L(f))d(p,q)\end{aligned}$$

2. and the second follows from the estimate

$$\left|\frac{1}{f(p)} - \frac{1}{f(q)}\right| = \frac{|f(q) - f(p)|}{|f(p)\cdot f(q)|} \leq \frac{L(f)\cdot d(p,q)}{\epsilon^2}.$$

3. The final assertion is a consequence by the fact that every complex-valued Lipschitz function on a finite diameter metric space is bounded

□

Chapter 2: Operators in Lipschitz free spaces $\mathcal{F}(M)$

In this chapter, (X, e, d) is a pointed metric space (quasi-metric space) with a distinguished fixed point (selected or chosen) called base point (origin or neutral element, which is taken to be the zero element if X is a normed space). In most of our examples M is a subset of a Banach space X and the origin is the origin of the Banach space. The assumption of an origin is a convenience to avoid considering spaces of Lipschitz functions modulo constants, and the particular choice of origin does not affect the theory substantially (convenient to normalize Lipschitz functions).

Lipschitz free spaces $\mathcal{F}(M)$ are canonical linearisation of arbitrary complete metric spaces M. More specifically, $\mathcal{F}(M)$ is the unique Banach space that contains an isometric copy of M that is linearly dense, and such that any Lipschitz mapping from M into some Banach space X extends to a bounded linear operator from $\mathcal{F}(M)$ into X. Those spaces are very powerful tool for studies of the non-linear geometry of Banach spaces, as they allow the application of well-known classical linear techniques to non-linear problems.

2.1 Basic notions

In 1931, Nikodym proved that $\mathrm{L}_0([0,1])$ has trivial dual. Day's result that $\mathrm{L}_{p'}$, $0 < \mathrm{p} < 1$, has trivial dual appeared in 1940 and the clual of $\mathrm{L}_\mathrm{p}(\mu)$ is isometric to $l_\infty(A)$ where A is th collection of atoms of μ.. The notion of Banach envelope in the 1959 paper of Duren−Romberg-Shields. In the early 1970's, Stiles studied the basic structure of $\ell_{p'}$, $0 < \mathrm{p} < 1$. Although mathematicians did not know whether an atomic space exists, but they tried to give constructions of such spaces, the most famous one is the **F**-space whose only endomorphisms are scalar multiples of the identity and they found that:

an infinite dimensional quasi-Banach space X is atomic iff every proper closed subspace of X is finite-dimensional.

Definition 2.1.1.

Given a measurable space (X, Σ) and a measure $\mu : \Sigma \longrightarrow \mathbb{R}$.

A set $A \subset X$ is called an atom iff

$$\mu(A) > 0 \text{ and } \forall B \subset A,\ \mu(B) < \mu(A) \Rightarrow \mu(B) = 0$$

If A is an atom, all the subsets in the μ-equivalence class $[A]$ of A are atoms, and $[A]$ is called an atomic class.

If μ is a σ-finite measure, then there are countably many atomic classes.

Example 2.1.1.

• Consider the set $X = \{1, 2, ..., 9, 10\}$ and let the sigma-algebra be the power set of X. Define the measure of a set to be its cardinality, that is, the number of elements in the set. Then, each of the singletons $\{i\}$, for $i = 1, 2, ..., 9, 10$ is an atom.

• Consider the Lebesgue measure on the real line. This measure has no atoms.

Definition 2.1.2.

A metric space X is atomic iff every proper closed subspace of X is finite dimensional. Of course an atomic space is separable.

Definition 2.1.3.

A measurable function $a : \mathbb{R}^n \to \mathbb{R}$ is said to be a $(k,\ \lambda,\ q)$-atom iff for some cube $Q \in K(\mathbb{R}^n)$ we have:

(a) supp a$\subset Q$;

(b) $\|a\|_{L_q(Q)} \leq |Q|^{-\lambda/n}$;

(c) $\int_Q a(x)x^\alpha dx = 0$ for all $|\alpha| < k$.

We say that a function $m : \mathbb{R}^n \to \mathbb{R}^n$ is a$(k,\ \lambda,\ q)$-molecula iff m is a linear combination of $(k,\ \lambda,\ q)$-atoms whose supports are pairwise disjoint.

Let $m = \sum_i c_i a_i$ be such presentation of the molecula; then for $1 \leq p \leq \infty$ we set

$$[m]_p := \{\sum_i |c_i|^p\}^{\frac{1}{p}}$$

Now we define a space $U_p^{k,\lambda}(L_q)$ to be the linear space of functions f presented as infinite sums of $(k,\ \lambda,\ q)$-moleculas convergent in $L_q(\mathbb{R}^n)$; it is equipped by the norm

$$\|f\|_{\mathcal{U}_p^{k,\lambda}(L_q)} := \inf\{\sum_i [m_i]_p\}$$

Theorem 2.1.1.

Let X be any locally bound metric complete space. Then either

(a) X contains an infinite dimensional atomic subspace

(b) X has a finite-dimension al subspace F so that X/F is not minimal.

Remark 2.1.1. The space $L^\infty(\Omega)$ is not separable except when Ω consists of a finite number of atoms. In order to prove this fact it is convenient to use the following property. When Ω is a general measure space we split Ω into its atomic part Ω_a and its non-atomic (=diffuse) part Ω_d; then we distinguish two cases:

(i) Ω_d is not a null set.

(ii) Ω_d is a null set.

For example, The space $L^1(\Omega)$ is never reflexive except in the trivial case where Ω consists of a finite number of atoms–and then $L^1(\Omega)$ is finite-dimensional. Indeed suppose, by contradiction, that $L^1(\Omega)$ is reflexive and consider two cases:

(i) $\forall \varepsilon > 0 \exists \omega \subset \Omega$ measurable with $0 < \mu(\omega) < \varepsilon$.

(ii) $\exists \varepsilon > 0$ such that $\mu(\omega) \geq \varepsilon$ for every measurable set $\omega \subset \Omega$ with $\mu(\omega) > 0$

From the both two cases, we get absurde contradictions.

The following table summarizes the main properties of the space $L^p(\Omega)$ when Ω is a measurable subset of $\mathbb{R}^n$

	Reflexive	Separable	Dual space
L^p with $1 < p < \infty$	Yes	Yes	$L^{p'}$
L^1	No	Yes	L^∞
L^∞	No	No	Strictly bigger than L^1

Example 2.1.2.

Consider $\Omega = \mathbb{R}^N$ with the Lebesgue measure, then the measure space Ω is diffuse (i.e., Ω has no atoms).

Example 2.1.3.

Given a measure space $(X,\ A,\ \nu)$ such that $\{x\} \in \mathcal{A}$ for every $x \in X$, we put

$$\mathbf{At}(\nu) = \{x \in X : \nu(x) > 0\}$$

The elements of the set $\mathbf{At}(\nu))$ are called atoms of ν. We say that a measure space $(X,\ \mathcal{A},\ \nu)$ is discrete (or that $\nu : \mathcal{A} \to \bar{\mathbb{R}}_+$ is a discrete measure on X) iff $\mathcal{A} = 2^X$, card $(\mathbf{At}(\nu)) \leq \aleph_0$, $\nu(X \backslash \mathbf{At}(\nu)) = 0$ and $\nu(x) < \infty$ for all $x \in X$. Clearly, such ν is σ-finite, $\mathbf{At}(\nu) = \{x \in X : 0 < \nu(x) < \infty\}$, $\nu(\triangle) = \nu(\triangle \cap \mathbf{At}(\nu))$ for all $\triangle \in 2^X$ and

$$\text{for every } \triangle \in 2^X, \nu(\triangle) = 0 \text{ if and only if } \triangle \subseteq X \backslash \mathbf{At}(\nu)$$

Example 2.1.4.

We say that a measure $\tau : \mathcal{C} \to \bar{\mathbb{R}}+$ is atomic iff $\mathcal{C}$ is an atomic σ-algebra of subsets of a nonempty set Z, the measure τ is σ-finite and $\tau(\triangle) > 0$ for every $\triangle \in \text{At}(\mathcal{C})$. If this is the case, then the set At $(\mathcal{C})$ is countable.

Notation 2.1.1.

- B_X is the closed unit ball of a Banach space X and $\overset{0}{B}_X$ is its open unit ball.
- M will denote a complete pointed metric space with metric d and base point 0,

- $B(p,\ r)$ will stand for the closed ball of radius r around $p \in M$, and we will use the notation $d(p,\ A) = \inf\{d(x,p)\ :\ x \in A\}$, called the distance from p to $A \subset M$

- rad $(A) = \sup\{d(x,\ 0)\ :\ x \in A\}$, the radius of A.

- Lip (M) will be the space of all real-valued Lipschitz functions on M, and $\mathrm{Lip}_0(\mathrm{M})$ will consist of all $f \in \mathrm{Lip}(M)$ such that $f(\mathrm{O}) = 0$. For $f \in \mathrm{Lip}(M)$

- We will denote its Lipschitz constant by $\|f\|_L$. Recall that $\|f\|_L$ is a norm on $\mathrm{Lip}_0(\mathrm{M})$ that turns it into a dual Banach space.

- For any $x \in M$, we will use the notation $\delta(x)$ for the evaluation functional $f \mapsto f(x)$, Note that δ is a (non-linear) isometric embedding of M into $\mathrm{Lip}_0(\mathrm{M})^*$, and in fact $\mathcal{F}(M) = \overline{\mathrm{span}}\ \delta(M)$ is the canonical isometric predual of $\mathrm{Lip}_0(\mathrm{M})$, which we will call the Lipschitz free space over M.

- The weak$*$ topology induced by $\mathcal{F}(M)$ on $\mathrm{Lip}_0(\mathrm{M})$ coincides with the topology of pointwise convergence on norm-bounded subsets of $\mathrm{Lip}_0(\mathrm{M})$. In what follows the weak$*$ topology will always be denoted w^*.

- We will say that $f \in \mathrm{Lip}(M)$ is positive if $f \geq 0$, i.e. if $f(x) \geq 0$ for all $x \in M$.

- Recall that the pointwise order is a partial order in Lip(M), and that a functional $\phi \in \mathcal{F}(M)$ (or $\mathrm{Lip}_0(M)^*$) is positive if $\phi(f) \geq 0$ for any positive $f \in \mathrm{Lip}_0(\mathrm{M})$. In that case, we will write $\phi \geq 0$;

- more generally, we will write $\phi \geq \psi$ whenever $\phi(f) \geq \psi(f)$ for all positive f.

- Given a subset K of M, we will also consider the subspace $\mathcal{F}_M(K) = \overline{\mathrm{span}}\ \delta(K)$ of $\mathcal{F}(M)$ and the subspace $\mathcal{I}_M(K)$ of $\mathrm{Lip}_0(\mathrm{M})$ defined by

$$\mathcal{I}_M(K) = \{f \in \mathrm{Lip}_0(M) : f(x) = 0 \quad \text{for all} \quad x \in K\}.$$

- In the above, if $M \neq \emptyset$, we adopt the the convention that span $\emptyset = \{0\}$. Thus $\mathcal{F}_M(K) = \mathcal{F}_M(K \cup \{0\})$ and $\mathcal{I}_M(K) = \mathcal{I}_M(K \cup \{0\})$ for all $K \subset M$,

- $\mathcal{F}_M(K)$ can be identified with the Lipschitz free space $\mathcal{F}(K\mathrm{U}\{0\})$.
- Let us also recall that $\mathcal{F}_M(K)^\perp = \mathcal{I}_M(K)$ and $\mathcal{I}_M(K)_\perp = \mathcal{F}_M(K)$.

2.2 Arens-Eells space Æ(X)

Definition 2.2.1. [Ram-San.20]

A molecule on X is a real valued function m on X with finite support set (i.e., the set where m has non-zero values) and satisfies

$$\sum_{x\in\ supp(m)} m(x) = 0$$

We denote by $M(X)$ the real linear space of molecules m on X, such that:

$$m = \sum_{x\in supp(m)} m(x)\cdot 1_{\{x\}} = \sum_{i=1}^{n} m(x_i)\cdot 1_{\{x_i\}}.$$

Where $supp(m) = \{x_1, ..., x_n\}$ and $1_{\{x\}}$ denotes the characteristic function of the set $\{x\}$. For $x_1,\ x_2 \in X$ we define the molecule $m_{x_1x_2} = 1_{\{x_1\}} - 1_{\{x_2\}}$ ($\{x_1\}, \{x_2\}$ are called atoms). It is easy to see that every molecule m can be written as a (non unique) finite linear combination of basic molecule (the condition $\sum_{i=1}^{n} m(x_i) = 0$ insures that such representations of m exist):

Indeed, we have: $m = \sum_{j=1}^{l} \lambda_j \cdot (1_{\{x_j\}} - 1_{\{x'_j\}}) = \sum_{j=1}^{l} \lambda_j \cdot m_{x_jx'j}$.

Put now $\|m\|_{M(x)} = \inf\left\{\sum_{j=1}^{l} |\lambda_j|\, d_x(x_j, x'_j)\right\}$,

over all representation of $m = \sum_{j=1}^{l} \lambda_j \cdot (1_{\{x_j\}} - 1_{\{x'_j\}})$.

Example 2.2.1. [Mezrag.20]

Consider $m : \mathbb{R} \longrightarrow \mathbb{R}$, such that

$$\begin{cases} m(0) = -4 \\ m(1) = 1 \\ m(2) = 3 \\ 0 \qquad \text{othewise.}, (m \notin \{0,1,2\}) \end{cases}$$

Then

$$\begin{aligned} m &= -4 \cdot 1_{\{0\}} + 1 \cdot 1_{\{1\}} + 3 \cdot 1_{\{2\}} \\ &= -3 \cdot 1_{\{0\}} - 1 \cdot 1_{\{0\}} + 1 \cdot 1_{\{1\}} + 3 \cdot 1_{\{2\}} \\ &= 1 \cdot (1_{\{1\}} - 1_{\{0\}}) + 3 \cdot (1_{\{2\}} - 1_{\{0\}}). \end{aligned}$$

It follows that $\|.\|_{M(x)}$ is a norm on the vector space $M(X)$. We denote by Æ(X, d_x) the completion of the normed space $(M(X), \|.\|_{M(x)})$. This space was first introduced by Arens and Eells in 1956. Originally, the basic idea goes back to Kantorovich. The terminology of Arens-Eells space Æ(X, d) is due to Weaver [[Wea.99]]. A different notation and appellation was used by Godefroy and Kalton. It is the Lipschitz free space denoted by $\mathcal{F}(X, d)$. Similar space was introduced by Pestov under the name of free Banach space.

Every molecule m is uniquely expressible in the form

$$m = \sum_{j=1}^{l} \lambda_j \cdot (1_{\{x_j\}} - 1_{\{e\}}).$$

Where the points x_j are all distinct and none equals to e.

Indeed, suppose that there is two representations

$$\sum_{j=1}^{l} \lambda_j \cdot (1_{\{x_j\}} - 1_{\{e\}}) = \sum_{j=1}^{l} \alpha_j \cdot (1_{\{y_j\}} - 1_{\{e\}})$$

where $x_i \neq x_j \neq e$ and $y_i \neq y_j \neq e$. We have

$$\begin{array}{ccccccc} \lambda_1 & = & \alpha_{i1} & x_1 & = & y_{j1} \\ \lambda_2 & = & \alpha_{i2} & x_2 & = & y_{j2} \\ \vdots & = & \vdots & \vdots & = & \vdots \\ \lambda_l & = & \alpha_{il} & x_l & = & y_{jl} \end{array}$$

Proposition 2.2.1.

Let (X, e, d) be a pointed metric space, then:

$Lip_0(X)$ is a dual space, for every metric space X of class M_0.

Remark 2.2.1.

On bounded sets the weak topology agrees with the topology of point wise convergence.

We now prove that $(Æ(X))^* \cong Lip_0(X)$. (isometrically)

Proposition 2.2.2. *[Mezrag.20]*

Let (X, e, d) be a pointed metric space, then:

$(Æ(X))^*$ is isometrically isomorphic to $Lip_0(X)$.

Proof.

Define

$$S : Æ^*(X, d) \longrightarrow lip_0(x).$$

By

$$(S\ \varphi)(x) = \varphi((1_{\{x\}} - 1_{\{e\}})).$$

Since $\left\|1_{\{x\}} - 1_{\{x'\}}\right\|_{Æ(X,d)} = d(x, x')$ or all $x, x' \in X$,

we have

$$\begin{aligned} |(S\ \varphi)(x) - (S\ \varphi)(x')| &= \left|\varphi((1_{\{x\}} - 1_{\{e\}})) - \varphi((1_{\{x'\}} - 1_{\{e\}}))\right| \\ &= \left|\varphi((1_{\{x\}} - 1_{\{x'\}}))\right| \\ &\leq \|\varphi\|\, d(x, x'). \end{aligned}$$

Also $(S\ \varphi)(e) = \varphi(0)$, so indeed $S\ \varphi \in Lip_0(X)$.

It follows that S is a non expansive linear mapping from $Æ^*(X, d)$ to $Lip_0(X)$.

Define now $R : Lip_0(X) \longrightarrow Æ^*(X, d)$ by

$$(Rf)(m) = \sum_x m(x)f(x) \quad ,$$

for $f \in lip_0(X)$ and m a molecule.

If $m = \sum_{j=1}^{l} \lambda_j \cdot (1_{\{x_j\}} - 1_{\{x'_j\}})$, we have

$$\begin{aligned} |(R\ f)(m)| &= \left|\sum_x m(x)f(x)\right| \\ &\leq \sum_{j=1}^{l} |\lambda_j| \left|f(x_j) - f(x'_j)\right| \\ &\leq lip(f) \sum_{j=1}^{l} |\lambda_j|\, d(x_j, x'_j) \quad . \end{aligned}$$

Hence $|(R\ f)(m)| \leq \text{Lip(f)}\ \|m\|_{M(x)}$, which uniquely extends to a continuous linear functional on the completion Æ(X,d) of $M(X)$, denoted by the same symbol $R\ f$.

Thus $R\ f \in$ Æ*(X, d) and $\|R\ f\| \leq Lip(f)$.

Straightforward calculations show that R and S are inverses.

Indeed, for all $x \in X$

$$\begin{aligned}(S \circ R)(f)(x) &= S(R(f))(x) \\ &= R(f)(1_{\{x\}} - 1_{\{e\}}) \\ &= f(x)\ ,\end{aligned}$$

and for all $m \in M(X)$

$$\begin{aligned}(R \circ S)(\varphi)(m) &= R(S(\varphi))(m) \\ &= \sum_x m(x)S(\varphi)(x) \\ &= \sum_{j=1}^{l} \lambda_j \varphi(1_{\{x_j\}} - 1_{\{x'_j\}}) \\ &= \varphi(m)\ .\end{aligned}$$

The operators R, S are non expansive and $R \circ S = S \circ R = Id$, so S is isometric

$$\begin{aligned}\|x\| &= \|(R \circ S)(x)\| \\ &\leq \|R\| \cdot \|S(x)\| \\ &\leq \|S(x)\|\ .\end{aligned}$$

Hence $Lip_0(X)$ is isometrically isomorphic to Æ$^*(X, d_x)$ □

Properties of Arens-Eells space

Proposition 2.2.3. *[Mezrag.20]*

Let (X, e, d) be a pointed metric space.

1. *For any molecule m we have*

$$\|m\|_{Æ(x,d_x)} = \sup\left\{|\langle m, f\rangle| = \left|\sum_{x\in X} m(x)\cdot f(x)\right| : f \in \beta_{x\neq}\right\}\ ,$$

and there exists $f \in B_{x\neq}$ such that $\langle m, f\rangle = \|m\|_{Æ(x,d_x)}$.

2. $\|m\|_{Æ(x,d_x)}$ *is a norm on* $M(X)$ *satisfying the followig condition:*

$$\forall x, y \in X, \quad \left\|1_{\{x\}} - 1_{\{y\}}\right\|_{Æ} = d(x,y), \tag{2.20}$$

3. $\|m\|_{Æ(x,d_x)}$ *is the largest seminorm on* $M(X)$ *which satisfies* (2.20)

Proof.

1. This follows from the identification of $Lip_0(X,d)$ with $Æ(X,d)^*$ and the Hahn-Banach theorem.

2. The inequality $\left\|1_{\{x\}} - 1_{\{y\}}\right\| \leq d(x,y)$ follows from the definition. Conversely, fix x in X and define $f_x(y) = d(x,y) - d(x,e)$.

   We have $f_x \in B_{Lip_0(X,d)}$, because $f_x(e) = 0$ and $Lip(f_x) = 1$.

   Indeed,

$$\begin{aligned} Lip(f_x) &= sup_{y_1 \neq y_2} \frac{|f_x(y_1) - f_x(y_2)|}{d(y_1,y_2)} \\ &\geq sup_{X \neq y} \frac{|f_x(y) - f_x(x)|}{d(x,y)} \\ &\geq \frac{d(x,y)}{d(x,y)} = 1. \end{aligned}$$

   and

$$\begin{aligned} Lip(f_x) &= sup_{y_1 \neq y_2} \frac{|f_x(y_1) - f_x(y_2)|}{d(y_1,y_2)} \\ &\leq sup_{y_1 \neq y_2} \frac{|d(x,y_1) - d(x,y_2)|}{d(y_1,y_2)} \\ &\leq \frac{d(y_1,y_2)}{d(y_1,y_2)} = 1. \end{aligned}$$

   By part (1), we have

$$\begin{aligned} \left\|1_{\{x\}} - 1_{\{y\}}\right\|_{Æ} &\geq |\langle m_{xy}, f_x \rangle| \\ &\geq |m(x)f_x(x) + m(y)f_x(y)| \\ &\geq |-m(x)d(x,e) + m(y)d(x,y) + m(y)d(x,e)| \\ &\geq |m(y)d(x,y)| \\ &\geq d(x,y) \end{aligned}$$

3. Let $\|\cdot\|_0$ be any semi norm such that for all $x, y \in X$

$$\left\|1_{\{x\}} - 1_{\{y\}}\right\|_0 \leq d(x, y).$$

Let $m = \sum_{i=1}^{n} \alpha_i m_{x_i y_i}$ be a molecule. We have

$$\begin{aligned} \|m\|_0 &= \left\|\sum_{i=1}^{n} \alpha_i m_{x_i y_i}\right\|_0 \\ &\leq \sum_{i=1}^{n} |\alpha_i| \, \|m_{x_i y_i}\|_0 \\ &\leq \sum_{i=1}^{n} |\alpha_i| \, d(x_i, y_i). \end{aligned}$$

Taking the infimum of all such representation of m , then (3) yields

$$\|m\|_0 \leq \|m\|_{Æ} \quad .$$

The application $i_x : X \longrightarrow Æ(X, d_x)$ defined by

$$i_X(x) = 1_{\{x\}} - 1_{\{e\}} = m_{xe},$$

is an isometric embedding of X into $Æ(X, dX)$.

□

Theorem 2.2.1. *[Wea.99] (The linearisation of Lipschitz operators)*
Let (X, ρ_x, e) be a pointed metric space, E a Banach space and $T : X \longrightarrow E$ A Lipschitz map which preserves base point (i.e., $T(e) = 0$), then there is a unique bounded linear operator (noted T_L),

$$T_L : Æ(X, \rho_x) \longrightarrow E,$$

Such that $T = T_L \circ i$ and $\|T_L\| = Lip(T)$ $\qquad (i_X : X \longrightarrow Æ(X, \rho_x))$.

2.3 Banach free space

The following theorem was independently proved by Weaver [Wea.99], p. 41.

Theorem 2.3.1. *[Wea.99]*

Let (X, d, e) be a pointed metric space, then there exists a unique, up to an isometric isomorphism, Banach space $B(X)$ over the field $\mathbb{F} =(\mathbb{R}$ or $\mathbb{C})$ and an isometric embedding

$$i_x : X \longrightarrow B(X)$$

such that

1. *The linear span of $i_x(X)$ is dense in $B(X)$.*

2. *Every map T in $Lip_0(X, E)$ can be extended to a continuous linear operator*

$$T^L : B(X) \longrightarrow E$$

such that $T^L = Lip(T)$ for any arbitrary normed space.

Proposition 2.3.1. *(Kernel complex functions.)*

Set Ω a domain (connected, open set*) of Bounded point evaluations in $\mathbb{C}^N$, and $H \neq \{\mathrm{O}\}$ a non-triviality Hilbert space of holomorphic functions on Ω, there is for each $z \in \Omega$, a unique kernel function $k_z \in H$, (called the reproducing kernel) with*

$$f(z) = \langle f,\ k_z \rangle\,, \qquad (f \in H)\ .$$

Theorem 2.3.2.

Let X be a compact metric space with metric d_X and that Y is a compact and connected metric space with metric, then for any commutative C^*-algebra with unit A, there exists a compact Hausdorff space K such that A is isometrically $*$-isomorphic to $C(K)$, where $C(K)$ is the commutative C^*-algebra of $\mathbb{C}$-valued continuous functions on K.

Definition 2.3.1.

Let $\mathrm{M}_{\mathrm{Lip}(X,C(K))}$ be the maximal ideal space of $\mathrm{Lip}(X, C(K))$. and Let $(x,\ \xi) \in X \times K$. We define $\tau_{(x,\xi)} \in \mathrm{M}_{\mathrm{Lip}(X,C(K))}$ the evaluation functiofnal at $(x,\ \xi)$ by

$$\tau_{(x,\xi)}(f) = f(x,\ \xi) \qquad (f \in \mathrm{Lip}(X,\ C(K)))\ .$$

Example 2.3.1.

Choose a distinguished "base point $e \in M$

Let $(M,\ d)$ be a complete pointed metric space with the base point $e \in M$. Set $\mathrm{Lip}_0(M)$, the space of Lipschitz functions defined as

$$\mathrm{Lip}_0(M) = \{f : M \to \mathbb{R} \quad \text{Lipschitz}\ , f(\mathrm{e}) = 0\}$$

with the norm $\|\mathrm{f}\| = \mathrm{Lip(f)}$, it is a Banach space. For $x \in M$ consider the evaluation functional $\delta(x) \in \mathrm{Lip}_0(M)^*$ defined by

$$\delta(x)\,(f) = \langle f, \delta(x)\rangle = f(x) \qquad f \in \mathrm{Lip}_0(M).$$

Then the Dirac map $\delta :\ M \to \mathrm{Lip}_0(M)^*$ is an isometric embedding.

and denote by $\mathrm{Lip}_0(M)$ the space of all real-valued Lipschitz functions on M

that map $0 \in M$ to $0 \in \mathbb{R}$. It becomes a Banach space if we define the norm of f to be its minimal Lipschitz constant. $\delta(0)$ is just the 0 element of $Lip_0(M)^*$because every $f \in Lip_0(M)$ vanishes at the base point. The space $\mathcal{F}(M)$ is defined to be the closed linear span of $\{\delta(x); x \in M\}$ with the dual space

Proposition 2.3.2.

Let $(X,\ d,\ \theta)$ be a pointed metric space. For every pair x, y of distinct points in X there exists a Lipschitz function $f \in Lip(X)$ with $L(f) = 1$ such that $f(x) - f(y) = d(x,\ y)$. In particular, for every $x \neq \theta$ in X there exists a Lipschitz function $f \in \mathrm{Lip}_0(\mathrm{X})$ with $L(f) = 1$ such that $f(x) = d(\theta,\ x)$.

Proof.

Suppose first that $x \neq y, x, y \in X\backslash\{\theta\}$. Let

$$g(x) = d(\theta,\ x), g(y) = d(\theta,\ x) - d(x,\ y), \quad \text{and}\ \ g(\theta) = 0.$$

Then

$$\begin{aligned} |g(x) - g(\theta)| &= g(x) = d(\theta,\ x)\ , \\ |g(y) - g(\theta)| &= |d(\theta,\ x) - d(x,\ y)| \leq d(\theta,\ y)\ \text{ and} \\ g(x) - g(y) &= d(x,\ y)\ . \end{aligned}$$

It follows that $L(g) = 1$. Extending g to a function $f \in \mathrm{Lip}_0(\mathrm{X})$ with $L(f) = L(g)$ one obtains the desired function.

The second assertion follows from the first one taking $y = \theta$.

For $x \in X$ define $e_x : \mathrm{Lip}_0(X) \longrightarrow \mathbb{R}$ by

$$e_x(f) = f(x) = \langle e_x, f \rangle , \qquad f \in \mathrm{Lip}_0(X) \tag{2.21}$$

The functionals e_x are called evaluation functionals. □

Proposition 2.3.3.

The functional e_x is linear and continuous with $\|e_x\| = d(\theta,\ x)$ for all $x \in X$. The mapping $i_X(x) = e_x, x \in X$, is an isometric embedding of X into $(\mathrm{Lip}_0(X))^*$, i.e.,

$$\|e_x - e_y\| = d(x,\ y) \tag{2.22}$$

for all $x, y \in X$.

If X is a Banach space, then i_X is nowhere Gâteaux differentiable.

Proof.

The linearity of e_x is obvious. Suppose $x \neq \theta$. The inequality

$$|e_x(f)| = |f(x)| \leq L(f)d(\theta,\ x)$$

is valid for all $f \in Lip_0(X)$ yields the continuity of e_x and the inequality $\|e_x\| \leq d(x,\ \theta)$. By Proposition 2.3.2, there exists $f \in \mathrm{Lip}_0(\mathrm{X})$ with $L(f) = 1$ such that $f(x) = d(\theta,\ x)$, implying

$$\|e_x\| \geq |e_x(f)| = d(x,\ \theta)$$

and so $\|e_x\| = d(\theta,\ x)$.

Let now $x, y \in X$. Observe first that

$$|e_x(f) - e_y(f)| = |f(x) - f(y)| \leq L(f)d(x,\ y) ,$$

for all $f \in \mathrm{Lip}_0(X)$, implying $\|e_x - e_y\| \leq d(x,\ y)$. Apply again Proposition 2.3.2 to obtain a function $f \in \mathrm{Lip}_0(\mathrm{X})$ such that $L(f) = 1$ and $f(x) - f(y) = d(x,\ y)$. Then

$$\|e_x - e_{\mathrm{y}}\| \geq |(e_x - e_{\mathrm{y}})(f)| = |f(x) - f(y)| = d(x,\ y)\ ,$$

so that $\|e_x - e_y\| = d(x,\ y)$.

Suppose that X is Banach and that, for some $x, h \in X$ with $h \neq 0$, the limit

$$\lim_{t\to 0} \frac{i_X(x+th) - i_X(x)}{t} = \psi(x,\ h)$$

exists. Taking into account (2.22), one obtains the contradiction

$$\lim_{t\to 0} \frac{|t|}{t}\|h\| = \lim_{t\to 0} \frac{\|e_{x+th} - e_x\|}{t} = \lim_{t\to 0} \frac{\|i_X(x+th) - i_X(x)\|}{t} = \|\psi(x,\ h)\|$$

Let

$$X_e = \{e_x : x \in X\} \subseteq (Lip(X))^* \quad textand \quad \mathcal{F}(X) = \overline{\mathrm{span}}(X_e), \tag{2.23}$$

where the closure is the norm-closure in $(Lip(X))^*$. By Proposition 2.3.3, the set X_e is an isometric copy of X. □

2.4 Lipschitz free spaces

Corollary 2.4.1.

The evaluation functionals are linearly independent.

Proof.

Indeed, given any finite set $E \subset M\backslash\{\mathrm{O}\}$ and any $x \in E$, the function that takes x to 1 and all other points of E to 0 is Lipschitz on E (any function on a finite metric space is clearly Lipschitz). By McShane's theorem, it can be extended to $f \in Lip_0(M)$ such that $\langle f, \delta(x)\rangle = 1$ and $\{f, \delta(y)\rangle = 0$ for $y \in E, y \neq x$. Hence $\delta(E)$ is a linearly independent set. □

Proposition 2.4.1.

δ is an isometric embedding of M into $Lip_0(M)^*$ that is continuous with respect to the weak topology of $Lip_0(M)^*$.*

Proof.

Let $x \neq y \in M$. Then

$$\begin{aligned} \|\delta(x) - \delta(y)\| &= \sup\{\{f,\ \delta(x) - \delta(y)\rangle : f \in B_{\mathrm{Lip}_0(M)}\} \\ &= \sup\{f(x) - f(y) : f \in B_{\mathrm{Lip}_0(M)}\}. \end{aligned}$$

On one hand, $f(x) - f(y) \leq d(x,\ y)$ for any $f \in B_{\mathrm{Lip}_0(M)}$. On the other hand, the function $f : z \mapsto d(z,\ y) - d(\mathrm{O},\ y)$ belongs to $B_{\mathrm{Lip}_0(M)}$ and $f(x) - f(y) = d(x,\ y)$. Thus the supremum in the expression above is exactly $d(x,\ y)$, and δ is indeed an

isometry. It is also clear that if $x_i \to x$ in M then

$$\{f,\ \delta(x_i)\rangle = f(x_i) \to f(x) = \{f,\ \delta(x)\rangle$$

for any $f \in \mathrm{Lip}_0(M)$ because such an f is continuous, thus $\delta(x_i) \xrightarrow{w^*} \delta(x)$. That is, δ is a weak$*$ continuous mapping. □

Definition 2.4.1.

. The Lipschitz free space over M is the subspace of $Lip_0(M)^*$, given by

$$\mathcal{F}(M) = \mathrm{sp\bar{a}n}^{\|\cdot\|}\delta(M) = \mathrm{sp\bar{a}n}\{\delta(\mathrm{x}) \ :\ x \in M\} \subseteq \mathrm{Lip}_0(M)^*$$

The norm in $\mathcal{F}(M)$ will be simply the restriction of the norm inherited from $Lip_0(M)^*$.

Theorem 2.4.1.

$\mathcal{F}(M)^*$ is linearly isometric to $Lip_0(M)$ under the mapping $\Lambda \mapsto \Lambda \circ \delta$, and the corresponding weak* topology on $Lip_0(M)$ coincides on bounded sets with the topology of pointwise convergence

Theorem 2.4.2. *[Wea.99]*

Let (X, d_x, e) be a pointed metric space and let E be a Banach space. Let $T : X \longrightarrow E$ be a Lipschitz map such that $T(e) = 0$. Then, there is a unique linear map

$$u : \mathcal{F}(X, d_x) \longrightarrow E,$$

such that $u \circ \delta_x = T$ and $\|u\| = Lip(T)$.

Chapter 3

Affine isometries in Lipschitz free spaces $\mathcal{F}(M)$

S. Mazur and S. Ulam, proved in 1932 that every isometry between two real normed spaces is affine. In 1968, T. Figiel proved more general results for metric spaces, while few years later J.A. Baker proved in 1971 that an isometry mapping from a real normed linear space into a strictly convex one is affine. More recently In 2003, J. Vaisala gave a very simple proof using modern techniques.

3.1 Affine isometries in vectorial normed spaces

In this section we denote by $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$, two vectorial normed spaces.

Definition 3.1.1.

An affine space is a set A together with a vector space E verifying three axioms:

i) $\forall \alpha \in A,\ \alpha + 0_E = \alpha,$ (Right identity)

ii) $\forall u, v \in E,\ \forall \alpha \in A,\ (\alpha + u) \oplus v = \alpha + (u \oplus v),$ (Associativity)

iii) $\forall \alpha \in A$, the mapping $\left\| \begin{array}{l} A \times E \longrightarrow E \\ \\ (\alpha, u) \longmapsto \alpha + u \end{array} \right.$ is a bijection.(free and transitive)

Remark 3.1.1.

The elements of the affine space A are called points, and those of the associated vector

space are translations, or sometimes free vectors. Explicitly, the definition above means that the action is a correspondence mapping, generally denoted as an addition.

Definition 3.1.2.

- A map $f : E \longrightarrow F$ is an isometry if and only if

$$\forall u, u' \in E, \quad \|f(u) - f(u')\|_F = \|u - u'\|_E . \tag{3.1}$$

- A function $f : E \longrightarrow F$, is affine if and only if,

$$\forall u, u' \in E,\ \forall \lambda \in [0,1], \quad f((1-\lambda)\cdot u + \lambda \cdot u') = (1-\lambda)\cdot f(u) + \lambda \cdot f(u') . \tag{3.2}$$

Definition 3.1.3.

- $X \subset E$ is convex if and only if it is closed under convex combinations.

$$\forall \lambda_i \geq 0, \forall x_i \in X, \quad \sum_{i=1}^{n\in\mathbb{N}^*} \lambda_i = 1 \Rightarrow \sum_{i=1}^{n} \lambda_i x_i \in X$$

- $X \subset E$ is called absolutely convex if and only if it is convex and balanced,

$$\forall \lambda \in [-1,1], \quad \lambda X = \{\lambda \cdot x, \quad x \in X\} \subseteq X$$

Example 3.1.1.

Let E be a bounded part of the real line $\mathbb{R}$ and $F = \mathbb{R} \times \mathbb{R}$ be the plane with the norm:

$$\|v\|_F = \|(v_1, v_2)\|_F = \max(|v_1|,\ |v_2|).$$

Let φ be a Lipschitz function and setting $v = f(u) = (x,\ \varphi(x))$, we get a map verifying the isometry's condition (3.1), but for $\varphi(x) = \sin x$, the affine's condition (3.2) doesn't match, so in general an isometery is not usually affine.

Lemma 3.1.1.

If the map $T : \left\| \begin{array}{l} E \longrightarrow F \\ \\ u \longmapsto T(u) = f(u) - f(0_E) \end{array} \right.$ *, is linear then f is affine.*

Proof.

If T is linear, then for every $u, u' \in E$, and any $\lambda \in [0,1]$:

$$\begin{aligned} f((1-\lambda)u + \lambda u') &= T((1-\lambda)u + \lambda u') + f(0_E) \\ &= (1-\lambda)T(u) + \lambda T(u') + [(1-\lambda) + \lambda] f(0_E) \\ &= (1-\lambda)(T(u) + f(0_E)) + \lambda(T(u') + f(0_E)) = (1-\lambda) f(u) + \lambda f(u') . \end{aligned}$$

□

Proposition 3.1.1.

A continuous map $f : E \to F$ is affine if and only if

$$\forall u,\ u' \in\ E,\ f\left(\frac{u+u'}{2}\right) = \frac{f(u)+f(u')}{2} \tag{3.3}$$

Proof. 1. It's obvious that if f is affine, by taking $\lambda = \frac{1}{2}$, then we get (3.3).

2. Reciprocally, assuming that (3.3 is verified, and if $f(0_E) = 0_F$, then we get:

$f\left(\frac{u+0_E}{2}\right) = \frac{f(u)}{2} = \frac{f(u)+f(0_E)}{2}$,

$\frac{f(-u)+f(u)}{2} = f\left(\frac{-u+u}{2}\right) = 0_F$,

$f(u+u') = f(\frac{1}{2}(2u+2u')) = f(\frac{2u}{2}) + f(\frac{2u'}{2}) = f(u) + f(u')$.

By induction, for all $n \in \mathbb{N}$, we get: $f(\frac{k}{2^n}u) = \frac{k}{2^n}f(u)$ and $f(\frac{k}{2^n}u') = \frac{k}{2^n}f(u')$.

Setting $D = \{\frac{k}{2^n},\ n \in \mathbb{N},\ k \in \{0, 1,\ 2^n\}\} \subset [0,1]$, then D is dense in $[0,1]$.

Let $\lambda \in [0,1]$ and $\varepsilon > 0$, then there exit $\lambda_1 = k/2^n$ and $\lambda_2 = k'/2^{n'}$, such that:

$|\lambda - \lambda_1| < \varepsilon$ and $|(1-\lambda) - \lambda_2| < \varepsilon$, hence :

$$\begin{aligned} f(\lambda_1 u + \lambda_2 u') &= f(\lambda_1 u) + f(\lambda_2 u') = f(\frac{k}{2^n}u) + f(\frac{k'}{2^{n'}}u') = \frac{k}{2^n}f(u) + \frac{k'}{2^{n'}}f(u') \\ &= \lambda_1 f(u) + \lambda_2 f(u'). \end{aligned}$$

Since f is continuous, and tending ε to 0, $f(\lambda u + (1-\lambda)u') = \lambda f(u) + (1-\lambda)f(u')$

Moreover $f(0_E) = 0_F$, thus f is linear.

If $f(0_E) = a \neq 0_F$, setting $g(u) = f(u) - a$, then f and g are both continuous, verifying the midpoint condition (3.3). Moreover, $g(0_E) = 0_F$, so g is linear, we deduce that f is affine. □

Remark 3.1.2.

There are two important general cases when every isometry is affine:

(i) The metric space F is strictly convex, that is, no sphere contains a line segment: Thus (3.3) follows from the fact that the spheres with centres at $f(u)$ and $f(u')$, with radius $\|u-u'\|/2$ meet only at $(f(u)+f(u'))/2$. (See theorem 3.1.3).

(ii) The map's function f is bijective (equivalently surjective): This result was proved by Mazur-Ulam, and later by J.Vaisala, using the points' reflections.

3.1.1 Mazur-Ulam isometries

Theorem 3.1.1. *(Mazur-Ulam Theorem)*

Every isometry of a real normed linear space onto a real normed linear space is a linear mapping up to translation (Geometrically).

Proof.

f is an isometry map $\Rightarrow$

$$\forall u, u' \in E, \forall \lambda \in [0,1],\ r := \|u - u'\|_E = \|f(u) - f(u')\|_F$$

For strictly convex spaces we have

$$\bar{B}(u,\ (1-\lambda)\cdot r) \cap B(u',\ \lambda \cdot r) = \{\lambda \cdot u + (1-\lambda)\cdot u'\}$$

f is an injective map $\Rightarrow$

$$\begin{aligned} f\left(\bar{B}(u,\ (1-\lambda)\cdot r) \cap \bar{B}(u',\ \lambda\cdot r)\right) &= f\left(\{\lambda\cdot u + (1-\lambda)\cdot u'\}\right) \\ &= \{f\left(\lambda\cdot u + (1-\lambda)\cdot u'\right)\} \end{aligned}$$

In an other hand we have:

$$\begin{aligned} f\left(\bar{B}(u,\ (1-\lambda)\cdot r) \cap \bar{B}u',\ \lambda\cdot r\right) &= f\left(\bar{B}(u,\ \lambda\cdot r)\right) \cap f\left(\bar{B}(u',\ \lambda\cdot r)\right) \\ &= \bar{B}\left(f\left(u,\ \lambda\cdot r\right)\right) \cap \bar{B}\left(f(u',\ \lambda\cdot r)\right) \\ &= \{\lambda f\left(u\right) + (1-\lambda) f\left(u'\right)\} \end{aligned}$$

We deduce that

$$f\left(\lambda\cdot u + (1-\lambda)\cdot u'\right) = \lambda f\left(u\right) + (1-\lambda) f\left(u'\right)$$

Therefore f is an affine map. This argument fails in the general case, because in a normed space which is not strictly convex: Two tangent balls may meet in some flat convex region of their boundary, not just a single point. □

Corollary 3.1.1. *[Baker.71]*

Let E and F be real normed linear spaces and suppose that F is strictly convex.

If $f : E \to F$ is an isometry, then f is affine.

Proof.

For the complete proof, see [Baker.71] □

Corollary 3.1.2. *[vaisala.03]*

Every bijective isometry $f : E \to F$ between two normed spaces is affine.

Proof.

Let E be a real normed space, and fix two points $u_1, u_2 \in E$. We check that:

$f\left(\frac{u_1+u_2}{2}\right) = \frac{f(u_1)+f(u_2)}{2}$, for every bijective isometry f defined on u.

For any such f, we let $\mathrm{def}(f) = \left\| f\left(\frac{u_1+u_2}{2}\right) - \frac{f(u_1)+f(u_2)}{2} \right\|$ denotes the affine defect. One observe that we have a uniform bound on the defect:

$$\mathrm{def}(f) \leq \frac{1}{2}\left\| f\left(\frac{u_1+u_2}{2}\right) - f(u_1) \right\| + \frac{1}{2}\left\| f\left(\frac{u_1+u_2}{2}\right) - f(u_2) \right\| = \frac{\|u_1-u_2\|}{2}.$$

Define again another bijective isometry f' on E whose defect is twice the defect of f.

Indeed, the bijectivity of f allows us to consider ρ the reflection of $\frac{f(u_1)+f(u_2)}{2}$, such that $f' = f^{-1}\rho f$, $\rho\colon z \mapsto f(u_1)+f(u_2)-z$, and $f'(u_1) = u_2$, $f'(u_2) = u_1$.

As f^{-1} is an isometry, we have:

$$\begin{aligned} \mathrm{def}\,(f') &= \left\| f^{-1}\left(f(u_1)+f(u_2) - f\left(\frac{u_1+u_2}{2}\right)\right) - \frac{u_1+u_2}{2} \right\| \\ &= \left\| f(u_1)+f(u_2) - f\left(\frac{u_1+u_2}{2}\right) - f\left(\frac{u_1+u_2}{2}\right) \right\| = 2\mathrm{def}(f). \end{aligned}$$

Since def satisfies the uniform bound property we conclude that $\mathrm{def}(f) = 0$.

Once we know that a bijective isometry f defined on u satisfies the following:

$$\forall u_1, u_2 \in E, \quad f\left(\frac{u_1+u_2}{2}\right) = \frac{f(u_1)+f(u_2)}{2},$$

and by the continuity of f, we deduce its affinity. □

3.1.2 Figiel isometries

Definition 3.1.4.

Set E a vectorial normed space. A point x of E is called smooth if and only if $\|.\|_E$ is gâteaux-differentiable i.e there exitsts a linear continuous mapping $T : E \to \mathbb{R}$ such that $\forall h \in E, \|x + th\| = \|x\| + tT(h) + o(|t|)$ when $t \longrightarrow 0$

Theorem 3.1.2.

Any convex continuous function on an open set of separable Banach space with real values is gâteaux-differentiable on a $\mathcal{G}_\delta$-dense of its domain.

Lemma 3.1.2.

Set E a vectorial normed space and a continuous, convex function $f : E \to \mathbb{R}$.

Then f is gâteaux-differentiable on $x_0 \in E$ if and only if

$$\forall u \in E,\ \lim_{t\to 0} \frac{f(x_0 + tu) + f(x_0 - tu) - 2f(x_0)}{t} = 0_F$$

Proposition 3.1.2.

Set U an open set of E and $f : U \to \mathbb{R}$ convex and continuous on $x_0 \in E$.

Then f is locally Lipschitzian on x_0 :

$$\exists k > 0, \exists \delta > 0 : B(x_0,\ \delta) \subset U, \quad |f(x) - f(y)| \leq k\|x - y\|, \forall x, y \in B(x_0,\ \delta)$$

Theorem 3.1.3. *(T.Figiel)*

Set E and F two vectorial normed spaces and $f : E \to F$ an isometry such that $f(0_E) = 0_F$, then there exists a unique mapping Q from $\overline{vect}\ f(E)$ to E , linear and continuous such that

$$\|Q\| \leq 1 \quad \text{and} \quad \forall x \in E, \qquad Q \circ f(x) = x.$$

Proof. For the complete proof, see [Figiel.68]. □

Proposition 3.1.3.

Set E and F two V.N.spaces and $f : E \to F$ an isometriy such that $f(0_E) = 0_F$, then

$$\forall n \in \mathbb{N}, \forall \lambda_1, \ldots, \lambda_n \in \mathbb{R}, \forall x_1, \ldots, x_n \in E, \quad \|\sum_{i=1}^{n} \lambda_i x_i\|_E \leq \|\sum_{i=1}^{n} \lambda_i f(x_i)\|_F$$

3.2 isometries in separable metric spaces

- In this section the separable metric spaces' elements are considered as points.
- The main results of this section will not be proven (taken as relatively true).

Definition 3.2.1.

Set $(M,\ d)$ a metric space. We define the space $Lip_0(M)$ of Lipschitzian functions f defined from M, to $\mathbb{R}$, such that $f(0) = 0$. equipped with the following norm:

$$\forall f \in Lip_0(M),\ \|f\|_{lip} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$

The value of $\|f\|_{lip}$ corresponds to the Lipschits function's constant.

Proposition 3.2.1.

The space $(Lip_0(M),\ \|.\|_{lip})$ *is a Banach space.*

Definition 3.2.2.

Set M be a metric space, we define $\mathcal{F}(M)$ the free Lipschitz space on M as:

$$\mathcal{F}(M) = \overline{vect}\ \{\delta_x,\ x \in M\} \subset Lip_0(M)^*$$

Proposition 3.2.2.

The dual of $\mathcal{F}(M)$ *is isometric to* $Lip_0(M)$.

Example 3.2.1. We can identify for the real line the same points

$$(isometrically)\ \ \mathcal{F}(\mathbb{R}) \equiv L_1(\mathbb{R})$$

Remark 3.2.1.

Any Lipschitzian function is derivable almost everywhere.

3.2.1 Isometric universal property

Lemma 3.2.1.

Set X *and* Y *two Banach spaces . If* $T : Y^* \to X^*$ *is a* w^* *continuous operator then there exists an operator* $S : X \to Y$ *such that* $T = S^*$.

Proposition 3.2.3.

Set $(M,\ d)$ and $(N,\ d')$ two metric spaces. Set $L : M \to N$ a Lipschitzian mapping, such that $L(0) = 0$. Then there exists only one single linear mapping $\hat{L} : \mathcal{F}(M) \to \mathcal{F}(N)$ such that $\hat{L}\delta_M = \delta_N L$. Moreover, $||\hat{L}|| = \|L\|_{lip}$.

Definition 3.2.3.

Set X a Banach space. Let β_X be the mapping defined on vect $\{\delta_x,\ x \in X\}$ by :

$$\sum_{i=1}^{n} \alpha_i \delta_{x_i} \in vect\ \{\delta_x,\ x \in X\} \Rightarrow\ \beta_X(\sum_{i=1}^{n} \alpha_i \delta_{x_i}) = \sum_{i=1}^{n} \alpha_i x_i$$

such that $\beta : \mathcal{F}(X) \to X$.

Lemma 3.2.2.

β_X is the quotient mapping from $\mathcal{F}(X)$ to X and its right inverse is δ_X :

$$\beta_X \delta_X = Id_X$$

Moreover β_X is linear, continuous and injective.

Proposition 3.2.4.

Set X and Y two Banach spaces and $L : X \to Y$ a Lipschitzian mapping such that $L(0) = 0$. Then there exists a unique linear mapping $\bar{L} : \mathcal{F}(X) \to Y$ such that

$$\bar{L}\delta_X = L \qquad and \qquad ||\bar{L}|| = \|L\|_{lip}.$$

Definition 3.2.4.

• The sequence $0 \to Z \xrightarrow{R} Y \xrightarrow{S} X \longrightarrow 0$ is exact when R is injective, S is surjective and $Im(R) = Ker(S)$.

• An exact sequence is called a divided sequence iff there exists a continuous linear mapping $V : X \to Y$, such that $SV = Id_X$.

• An exact sequence is called Lipschitz-divided if there exists a Lipschitzian mapping (not necessarily linear) $L : X \to Y$ such that $SL = Id_X$.

Definition 3.2.5.

A Banach space X verifies the isometric universal's property, iff there exitsts a continuous linear mapping (of norm 1) $T : X \to \mathcal{F}(X)$ such that $\beta_X T = Id_X$.

Proposition 3.2.5.

Set X, a Banach space. Then X verifies the Universal's property if and only if any exact sequence $0 \to Z \to Y \to X \to 0$ which is Lipschitz-divided, is also linearly-divided.

Corollary 3.2.1.

Set X and Y two Banach spaces such that X is verifying the universal's property, set $Q : Y \to X$ a quotient. mapping .

Iff there exitsts an isometry (not necessarily linear) $L : X \to Y$ such that $QL = Id_X$,

Then there exitsts a linear isometry $V : X \to Y$ such that $QV = Id_X$.

Lemma 3.2.3.

$\mathcal{F}(X)$ verifies the isometric Universal's property, for any Banach space X.

Theorem 3.2.1.

Set X be a Banach space. Then the following assertions are equivalents:

i. *X verifies the isometric Universal's property.*

ii. *If Y is a Banach space such that there exitsts an isometry $L : X \to Y$ (not necessarily linear) with $L(0) = 0$ and $Y = \overline{vect}\ L(X)$, then Y contains a complemented subspace linearly isometric to X.*

3.2.2 Godefroy-Kalton isometries

Definition 3.2.6.

A function $f : I = [a,\ b] \to \mathbb{C}$ is absolutely continuous on I iff for any $\varepsilon > 0$, there exitsts $\delta > 0$ such that for any $n \in \mathbb{N}$ and for any disjoint intervals of $I,]\ \alpha_1, \beta_1[,$ ... $]\alpha_i, \beta_i[\ldots]\alpha_n, \beta_n[$, such that for $\sum_{i=1}^{n}(\beta_i - \alpha_i) < \delta$, we have $\sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| < \varepsilon$.

Remark 3.2.2.

If f is absolutely continuous and $\sum_{i=1}^{+\infty}(\beta_i - \alpha_i) < \delta$, then for any $n \in \mathbb{N}$, $\sum_{i=1}^{n}(\beta_i - \alpha_i) < \delta$.

Thus, for any $n \in \mathbb{N}, \sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| < \varepsilon$, and hence $\sum_{i=1}^{+\infty} |f(\beta_i) - f(\alpha_i)| < \varepsilon$.

Definition 3.2.7.

Let $\mathcal{M}$ be a tribe, and μ be a positive meausure on $\mathcal{M}$ and λbe an arbitrary measure on $\mathcal{M}$. If $\lambda(E) = 0$, for any $E \in \mathcal{M}$ such that $\mu(E) = 0$, we say that λ is absolutely $\mu-$ continuous ($\lambda << \mu$).

Theorem 3.2.2. *[Radon-Nikodym]*

Set $\mathcal{M}$, a tribe on X, and μ a σ-finite positive measure on $\mathcal{M}$ and λ a complex measure on $\mathcal{M}$. Then:

a) *there exists a unique couple $(\lambda_a,\ \lambda_b)$ with a complex measure on $\mathcal{M}$ such that*

$$\begin{aligned} \lambda &= \lambda_a + \lambda_b, \qquad \lambda_a << \mu \qquad \text{and} \quad \lambda_b \perp \mu \\ 0 &< \lambda < \infty \Rightarrow 0 < \lambda_a, \lambda_b < \infty \ \text{ and } \lambda_a \perp \lambda_b \end{aligned}$$

b) *there exists a unique mapping $h \in L^1(\mu)$ such that $\forall E \in \mathcal{M}, \lambda_a(E) = \int_E h d\mu$*

Theorem 3.2.3.

Set $I = [a,\ b]$ an interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$, an increasing absolutely continuous function on I. Then f is almost everywhere differentiable on I.

Proposition 3.2.6.

Set a Lipschitzian function $f : \mathbb{R} \to \mathbb{R}$, then f is absolutely continuous.

Proposition 3.2.7.

Set a Lipschitzian function $f : \mathbb{R} \to \mathbb{R}$, then f is derivable almost everywhere.

Theorem 3.2.4. *(Rademacher)*

Set $n \in \mathbb{N}$ and a Lipschitzian. function $f : \mathbb{R}^n \to \mathbb{R}$,

then f is Fréchet-differentiable almost everywhere.

Lemma 3.2.4.

If $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitzian and gâteaux-differentiable at $x_0 \in \mathbb{R}^n$,

then f is Fréchet-differentiable at x_0.

Lemma 3.2.5.

Set X and Y two Banach spaces and a Lipschitzian mapping $f : X \to Y$.

Set G an additive dense sub-group of X . If there exists $x_0 \in X$ such that

$$\forall u \in G, \quad \lim_{t\to 0} \frac{f(x_0 + tu) - f(x_0)}{t},$$

does exist and is additive to a function of u, then f is Gâteaux-differentiable at x_0.

Theorem 3.2.5. *(Godefroy-Kalton)*

Any separable Banach space X verifies the isometric universal's property.

3.2.3 Main theorems

Theorem 3.2.6. *[Dalet.15]*

Set X and Y two Banach spaces such that X is separable .

Set $Q : Y \to X$ a linear quotient mapping .

If Q admits a Lipschitzian right inverse at L, then Q admits a linear right inverse V such that $\|V\| \leq \|L\|_{lip}$.

Theorem 3.2.7. *[Dalet.15]*

Set X a separable Banach space.

If there exitsts an isometric injection of X in a Banach space Y, then Y contains a vectorial subspace isometric to X.

Example 3.2.2.

We Consider the mapping f: $\left\| \begin{array}{ccc} \mathbb{R} & \to & (\ \mathbb{R}^2,\ \|.\|_\infty) \\ t & \mapsto & (t,\ \sin t) \end{array} \right.$

Then, for all $t, t' \in \mathbb{R}$,

$$\begin{aligned} \|f(t) - f(t')\|_\infty &= \|(t,\ \sin t) - (t',\ \sin t')\|_\infty \\ &= \|(t - t',\ \sin t - \sin t')\|_\infty \\ &= \max\{|t - t'|,\ |\sin t - \sin t'|\} \\ &= |t - t'| \end{aligned}$$

hence f is an isometry. Moreover, $f\,(0) = (0, \sin 0) = (0,\ 0)$

Example 3.2.3.

From the Theorem 3.1.3 of Figiel,

$$Q : \left\| \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x \end{array} \right.$$

is the unique mapping such that $Qf = Id_{\mathbb{R}}$ and $\|Q\| \leq 1$

$$\forall (x,y) \in \mathbb{R}^2, \qquad |Q(x,y)| = |x| \leq \|(x,y)\|_\infty).$$

Moreover Q is a quotient. mapping From the Corollairy (3.2.1) there exitsts a linear isometry $S : \mathbb{R} \to \mathbb{R}^2$ such that

$$QS = Id_{\mathbb{R}}$$

Moreover, $\mathbb{R}$ is complemented in $\mathbb{R}^2$, and we can write

$$f(x) = S(x) + (f(x) - S(x))$$

It means that f can be decomposed as sum of a linear isometry and a non linear part which the norm doesn't change.

Any isometry on separable Banach space can be decomposed with the same form: Set a separable Banach space X and a Banach space Y. Set $j : X \to j(X) \subset Y$ an isometry. We suppose that $j(0) = 0$. By the theorem 3.1.3 of Figiel, there exists a unique continue and linear mapping $Q : \overline{vect}\ \ j(X) \to X$ such that $\|Q\| \leq 1$ and $Q \circ j = Id_X$. In particular, Q is surjective. According to the Corollary 3.2.1, there exists a linear isometry $S : X \to j(X) \subset Y$ such that $Q \circ S = Id_X$. Moreover, according to the theorem 3.2.6 $S(X)$ is complemented in $\overline{vect}\ \ j(X)$. Finally, we can write

$$\boxed{\forall x \in X, \quad j(x) = S(x) + (j(x) - S(x)) \in S(X) \oplus \ker Q} \tag{3.4}$$

3.3 Conclusion

- Thanks to the Theorem 3.1.1 of Mazur-Ulam: any surjective isometry f between two normed vectorial spaces is affine. If Moreover $f(0) = 0$, it is obviously linear.

- Moreover, by mixing the theorem 3.1.3 of Figiel (Any isometry between two vectorial normed spaces verifying $f(0) = 0$ admits a unique left, linear, continue inverse with a norm less than 1, thanks also to theorem 3.2.5 of Godefroy-Kalton (Any separable Banach space verifies the isometric shifting property)

- If X and Y are two Banach spaces such that X is separable and $\varphi : X \to Y$ is an isometry, then Y contains a vectorial closed sub-space isometrically linear to X.

- Introducing the Lipschitz free spaces, the main linear decomposition formula 3.4 of non separable general spaces can be investigated for future research (open problem).

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الملخص

- في الفصل الأول ، نقدم مفاهيم أولية حول فضاء ليبشينز، وخصائص الدوال الليبشينزية.
- في الفصل الثاني نشرح بعض النتائج الأساسية للمؤثرات الليبشيتزية في فضاء ليبشينز الحر $\mathcal{F}(M)$.
- في الفصل الثالث نبين أهمية فضاء ليبشستز الحر في تحويل الفضاء اللاخطي إلى فضاء خطي باستعمال تساويات القياس

الكلمات المفتاحية

فضاء ليبشينز الحر، فضاء متري قابل للفصل، تحويل تساوي القياس، تطبيق تآلفي، الخاصية الأساسية المميزة للإزاحة

Résumé

- Dans le premier chapitre , on introduit des notions préliminaires sur l'espace de Lipschitz, et les propriétés des fonctions de Lipschitz.
- Ensuite au second chapitre on présente quelques résultats fondamentaux relatifs aux opérateurs de Lipschitz dans l'espace libre de Lipschitz $\mathcal{F}(M)$.
- Enfin au dernier chapitre on donne quelques résultats sur les isométries non linéaires expliquant la propriété isométrique du relèvement.

Mots clés

Espace de Lipschitz libre, espace métrique séparable, sous-espace convexe, isométrie, application affine, caractéréstique principale de la propriété de relèvement.

Abstract

- In the first chapter, we introduce preliminary notions about Lipschitz space, and the properties of Lipschitz functions.
- Then in the second chapter we present some fundamental results related to the Lipschitzian operators on the free lipschitz space $\mathcal{F}(M)$.
- Finally in the last chapter we give some results about the non-linear isometries, and the isometric universal shifting property (infinite shift space).

Key words

Free Lipschitz spaces, Metric separable space, convex sub-space, isometry, affine mapping, universal characteristic of the shifting property.