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Thème

BOUNDEDESS OF SOME SUBLINEAR OPERATORS ON VARIABLE WEAK HERZ-TYPE HARDY SPACES

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♡

❖ *Abdelhalim*

Contents

Notation

- $\mathbb{R}^n$ we denote the n-dimensional real Euclidean space.
- $\mathbb{N}$ we denote the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- $\mathbb{Z}$ we denote the set of all integer numbers.
- For $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| = \alpha_1 + ... + \alpha_n$.
- The Euclidean scalar product of $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ is given by $xy = x_1y_1 + ... + x_ny_n$.
- The expression $f \lesssim g$ means that $f \leqslant cg$ for some independent constant c (and non-negative functions f and g).
- $f \approx g$ means $f \lesssim g \lesssim f$.
- As usual for any $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to x.
- supp$f$ is the support of the function f , i.e., the closure of its non-zero set.
- If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E.
- χ_E denotes its characteristic function.
- $\mathcal{S}(\mathbb{R}^n)$ is used in place of set of all Schwartz functions on $\mathbb{R}^n$.
- $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$.

- Given a function $f \in L^1_{loc}(\mathbb{R}^n)$; the Hardy-Littlewood maximal operator is defined by
$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy, \forall x \in \mathbb{R}^n.$$

Introduction

The history of Herz space goes back to the authors Beurling and Herz in the sixty's of the last century [14]. In 1968, Herz and Beurling presented some fundamental form of Herz spaces to study some convolution algebras. Later, Herz generalized these space to convergence for Hardy space. In 1989 Chen and Lau [5] introduced Hardy space associated with Beurling algebras.

Recently, as a generalization of Lebesgue space with variable exponent, Herz space with variable exponents are introduced. In fact, in 2010 Izuki proved the boundedness of sublinear operators on Herz space with variable exponents $\dot{K}^{\alpha}_{p(\cdot),q}(\mathbb{R}^n)$ and $K^{\alpha}_{p(\cdot),q}(\mathbb{R}^n)$ in [16]. In 2012, Almeida and Drihem obtained boundedness results for a wide class of classical operators on Herz space $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$ and $K^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$ in [2]. In 2016, Drihem and Seghiri gave a new norm equivalents of the variable Herz spaces $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, and they proved the atomic decomposition for Herz-type Hardy spaces of variable smoothness and integrability.

Our work is divided in to three chapters.

In the first one, we collects fundamental notation and concepts. We also give some key lemmas needed in the proofs of main statements.

In the second chapter we define Herz type-Hardy space with variable smoothness and integrability and present a few aspects of their properties.

In the last chapter, we presented the atomic decomposition of weakly Herz type Hardy space.

All the results of this thesis are taken from [4].

Chapter 1

Variable Herz spaces

In this chapter, we present some fundamental proprieties of our work. We also give some key technical lemmas needed in the proofs of the main statements.

1.1 Modular space

In this section we start by recalling about the semi-modular functional space. In vector space defined on a over of $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

1.1.1 Definition and basic properties

Definition 1.1 *Let X be a real vector space. A function $\varrho : X \longrightarrow [0, +\infty]$ is called a semi-modular on X if it satisfies the following condition:*

1. $\varrho(0) = 0$.
2. $\varrho(\lambda x) = \varrho(x)$ *for all* $x \in X$, *and for all scalar* λ *with* $|\lambda| = 1$.
3. ϱ *is quasi-convex.*
4. $\varrho(\lambda x) = 0$ *for all* $\lambda > 0$ *implies* $x = 0$.
5. ϱ *is left-continuous on* $[0, +\infty)$ *for every* $x \in X$.

   A semi-modular ϱ is called a modular if

6. $\varrho(x) = 0$ *implies* $x = 0$.

A semi-modular ϱ *is called continuous if*

7. *the mapping* $\lambda \longrightarrow \varrho(\lambda x)$ *is continuous on* $[0, +\infty)$ *for every* $x \in X$.

Example 1.2 *Let* Ω *is a Lebesgue measurable subset of* $\mathbb{R}^n$.

(i)- *If* $1 \leq p < \infty$, *then*

$$\varrho_p(f) = \int_\Omega |f(x)|^p dx$$

defines a continuous modular on the space of all measurable functions on Ω.

(ii)- *If* $1 \leq p < \infty$, *then*

$$\varrho_p((x_j)) = \sum_{j=0}^{\infty} |x_j|^p$$

defines a continuous modular on $\mathbb{R}^n$.

Definition 1.3 *If* ϱ *be a semimodular or modular on* X, *then*

$$X_\varrho = \{x \in X : \lim_{\lambda \longrightarrow 0} \varrho(\lambda x) = 0\}$$

is called a semimodular space or modular space, respectively.

Proposition 1.4 *If* ϱ *is a semimodular or modular on* X, *then*

$$X_\varrho = \{x \in X, \exists \lambda > 0 : \varrho(\lambda x) < \infty\}$$

is called a semimodular space or modular space, respectively.

Theorem 1.5 *Let* ϱ *be a semimodular on* X. *Then* X_ϱ *is a quasi-normed R-vector space. The quasi-norm, called the Luxemburg quasi-norm, is defined by*

$$\|x\|_\varrho := \inf \left\{ \lambda > 0 : \varrho(\frac{x}{\lambda}) \leqslant 1 \right\}.$$

Lemma 1.6 *(Norm-modular unit ball property). Let ϱ be semi-modular on X. Then*

$$\|x\|_\varrho \leq 1 \Leftrightarrow \varrho(x) \leq 1.$$

If ϱ is continuous, then also

$$\|x\|_\varrho < 1 \Leftrightarrow \varrho(x) < 1, \text{ and : } \|x\|_\varrho = 1 \Leftrightarrow \varrho(x) = 1.$$

Corollary 1.7 *Let ϱ be a semi-modular on X and $x \in X_\varrho$.*

(a) *If $\|x\|_\varrho \leq 1$, then $\varrho(x) \leq \|x\|_\varrho$.*

(b) *If $1 < \|x\|_\varrho$, then $\|x\|_\varrho \leq \varrho(x)$.*

(c) $\|x\|_\varrho \leq \varrho(x) + 1$.

Remark 1.8 *The proof of the above results can be found in [7].*

1.2 Variable Lebesgue spaces

In this section we recall and present some properties of variable Lebesgue spaces.

1.2.1 Basic properties of variable exponents

Give an open set $\Omega \subset \mathbb{R}^n$. We put

$$\mathcal{P}_0(\Omega) := \{p \text{ measurable} : p(\cdot) : \Omega \longrightarrow [c, \infty[: \text{for some } c > 0\}.$$

The elements of $\mathcal{P}_0(\Omega)$ are called exponent functions. In order to distinguish between variable and constant exponents, we will always denote exponent functions by $p(\cdot)$.

Notation 1.9 *We denote by*

$$\mathcal{P}(\Omega) := \{p \text{ measurable} : p(\cdot) : \Omega \subset \mathbb{R}^n \longrightarrow [1, \infty[\}.$$

Given $p \in \mathcal{P}_0(\Omega)$ *and a set* $E \subseteq \Omega$, *let*

$$p^-(E) = \operatorname*{ess\,inf}_{x\in E} p(x), \qquad p^+(E) = \operatorname*{ess\,sup}_{x\in E} p(x).$$

If the domain $E = \Omega = \mathbb{R}^n$, *we will simply write*

$$p^- = p^-(\mathbb{R}^n), \qquad p^+ = p^+(\mathbb{R}^n).$$

Definition 1.10 *Given* Ω *and* $p \in \mathcal{P}_0(\Omega)$. *The variable Lebesgue space* $L^{p(\cdot)}(\Omega)$ *is defined by*

$$L^{p(\cdot)}(\Omega) := \left\{ f \ \textit{measurable} : \exists \lambda > 0 : \lim_{\lambda \longrightarrow 0} \varrho_{L^{p(\cdot)}(\Omega)}(\lambda f) = 0 \right\},$$

equipped with the following quasi-norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\{\lambda > 0 : \varrho_{L^{p(\cdot)}(\Omega)}(f/\lambda) \leqslant 1\}.$$

Definition 1.11 *Given* Ω *and* $p \in P_0(\Omega)$, *define* $L^{p(\cdot)}_{loc}(\Omega)$ *by*

$$L^{p(\cdot)}_{loc}(\Omega) := \{f \ \textit{measurable} : f \in L^{p(\cdot)}(K), \ \textit{for every compact set} : K \subset \Omega\}.$$

Definition 1.12 *The weak Lebesgue space with variable exponent* $WL^{p(\cdot)}$ *consists of all Lebesgue measurable function* f *satisfying*

$$\|f\|_{WL^{p(\cdot)}} = \sup_{\lambda > 0} \lambda \|\chi_{\{x:|f(x)>\lambda\}}\|_{L^{p(\cdot)}} < \infty.$$

1.2.2 Logarithmic Hölder continuity

Definition 1.13 *We say that a function* $g : \mathbb{R}^n \to \mathbb{R}$ *is* locally log-Hölder continuous, *if there exists a constant* $c_{\log} > 0$ *such that*

$$|g(x) - g(y)| \leq \frac{c_{\log}}{\ln(e + 1/|x-y|)}$$

for all $x, y \in \mathbb{R}^n$. *If*

$$|g(x) - g(0)| \leq \frac{c_{\log}}{\ln(e + 1/|x|)}$$

for all $x \in \mathbb{R}^n$*, then we say that* g *is* log-Hö lder continuous at the origin *(or has a* log decay at the origin *). If, for some* $g_\infty \in \mathbb{R}$ *and* $c_{\log} > 0$*, there holds*

$$|g(x) - g_\infty| \leq \frac{c_{\log}}{\ln(e + |x|)}$$

for all $x \in \mathbb{R}^n$*, then we say that* g *is* log-Hö lder continuous at infinity *(or has a* log *decay at infinity).*

1.2.3 The mixed Lebesgue-sequence space

Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$ -functions by the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) := \sum_v \inf \left\{ \lambda_v > 0 : \varrho_{p(\cdot)} \left(\frac{f_v}{\lambda^{1/q(\cdot)}} \right) \leqslant 1 \right\}.$$

The (quasi)-norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf \left\{ \mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left(\frac{1}{\mu} (f_v)_v \right) \leqslant 1 \right\}.$$

1. If $q^+ < \infty$, then

$$\inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left(\frac{f}{\lambda^{1/q(\cdot)}} \right) \leqslant 1 \right\} = \||f|^{q(\cdot)}\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

2. If p and q are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^p(L^q)$.

Theorem 1.14 *Let* $p, q \in \mathcal{P}_0(\mathbb{R}^n)$*.* $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ *is a quasi-norm on the mixed Lebesgue-sequence space* $\ell^{q(\cdot)}(L^{p(\cdot)})$*.*
Let $p, q \in \mathcal{P}(\mathbb{R}^n)$*. If* $p(x) \geqslant 1$ *is constant almost everywhere (a.e.) on* $\mathbb{R}^n$ *and* $q \geqslant 1$*, or if* $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} \leqslant 1$ *a.e. on* $\mathbb{R}^n$*, or if* $1 \leqslant q(\cdot) \leqslant p(\cdot) < \infty$ *a.e. on* $\mathbb{R}^n$*, then* $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ *is a norm on the mixed Lebesgue-sequence space* $\ell^{q(\cdot)}(L^{p(\cdot)})$*.*

1.2.4 Key Lemmas

By $\mathcal{P}_0^{\ln}(\mathbb{R}^n)$ and $\mathcal{P}_\infty^{\ln}(\mathbb{R}^n)$ we denote the class of all exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which have a log decay at the origin and at infinity, respectively. The notation $\mathcal{P}^{\ln}(\mathbb{R}^n)$ is used for all those exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which are locally log-Hölder continuous and have a log decay at infinity, with $p_\infty := \lim_{|x|\to\infty} p(x)$. Obviously we have $\mathcal{P}^{\ln}(\mathbb{R}^n) \subset \mathcal{P}_0^{\ln}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\ln}(\mathbb{R}^n)$. Note that $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ if and only if $p' \in \mathcal{P}^{\ln}(\mathbb{R}^n)$, and since $(p')_\infty = (p_\infty)'$ we write only p'_∞ for any of these quantities.

Lemma 1.15 *Let $p \in \mathcal{P}^{\log}$. For any cubes (balls) P and Q, such that $P \subset Q$, we have*

$$C\left(\frac{|Q|}{|P|}\right)^{1/p^+} \leq \frac{\|\chi_Q\|_{p(\cdot)}}{\|\chi_P\|_{p(\cdot)}} \leq c\left(\frac{|Q|}{|P|}\right)^{1/p^-}$$

with $c, C > 0$ are independent of $|Q|$ and $|P|$.

For the proof see [7]. The next lemma is a Hardy-type inequality which is easy to prove.

Lemma 1.16 *Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of positive real numbers, such that*

$$\left\|\{\varepsilon_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} = I < \infty.$$

Then the sequences $\left\{\delta_k : \delta_k = \sum_{j\leq k} a^{k-j}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ and $\left\{\eta_k : \eta_k = \sum_{j\geq k} a^{j-k}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ belong to ℓ^q, and

$$\left\|\{\delta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} + \left\|\{\eta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} \leq c\, I,$$

with $c > 0$ only depending on a and q.

The proof of the following results are given in [2].

Lemma 1.17 *Let $\alpha \in L^\infty(\mathbb{R}^n)$ and $r_1 > 0$. If α is* log-*Hölder continuous both at the origin and at infinity, then*

$$r_1^{\alpha(x)} \lesssim r_2^{\alpha(y)} \times \begin{cases} \left(\frac{r_1}{r_2}\right)^{\alpha^+} & \text{if} \quad 0 < r_2 \leq \frac{r_1}{2} \\ 1 & \text{if} \quad \frac{r_1}{2} < r_2 \leq 2r_1 \\ \left(\frac{r_1}{r_2}\right)^{\alpha^-} & \text{if} \quad r_2 > 2r_1 \end{cases}$$

for any $x \in B(0,r_1) \setminus B(0,\frac{r_1}{2})$ and $y \in B(0,r_2) \setminus B(0,\frac{r_2}{2})$, with the implicit constant not depending on x, y, r_1 and r_2.

Lemma 1.18 *Let $p \in \mathcal{P}_\infty^{\ln}(\mathbb{R}^n)$ and let $R = B(0,r) \setminus B(0,\frac{r}{2})$. If $|R| \geq 2^{-n}$, then*

$$\|\chi_R\|_{p(\cdot)} \approx |R|^{\frac{1}{p(x)}} \approx |R|^{\frac{1}{p_\infty}}$$

with the implicit constants independent of r and $x \in R$.
The left-hand side equivalence remains true for every $|R| > 0$ if we assume, additionally, $p \in \mathcal{P}_0^{\ln}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\ln}(\mathbb{R}^n)$.

1.3 Variable Herz-type space

In this section, we give the definition of Herz spaces with variable exponent. Also, we present basic properties.

1.3.1 Variable Herz space

We begin by the definition of Herz space.

Definition 1.19 *For convenience, we set*

$$B_k := B(0, 2^k)\,, \quad R_k := B_k \setminus B_{k-1} \quad \text{and} \quad \chi_k = \chi_{R_k}\,, \quad k \in \mathbb{Z}.$$

Definition 1.20 *Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The* inhomogeneous Herz space $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *consists of all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ such that*

$$\|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|f\,\chi_{B_0}\|_{p(\cdot)} + \left\| \left(2^{k\alpha(\cdot)} f\,\chi_k\right)_{k\geq 1} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty. \tag{1.1}$$

Similarly, the homogeneous Herz space $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all $f \in L^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$ such that*

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left\| \left(2^{k\alpha(\cdot)} f\,\chi_k\right)_{k\in\mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty. \tag{1.2}$$

If p, q and α are constant, then $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = K_{p,q}^{\alpha}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = \dot{K}_{p,q}^{\alpha}(\mathbb{R}^n)$ are the classical Herz spaces.

Let us denote

$$\|\{g_k\}\|_{\ell^q_>(L^{p(\cdot)})} := \left(\sum_{k=0}^{\infty} \|g_k\|_{p(\cdot)}^q\right)^{1/q}$$

and

$$\|\{g_k\}\|_{\ell^q_<(L^{p(\cdot)})} := \left(\sum_{k=-\infty}^{-1} \|g_k\|_{p(\cdot)}^q\right)^{1/q}$$

for sequences $\{g_k\}_{k\in\mathbb{Z}}$ of measurable functions (with the usual modification if $q = \infty$).

Proposition 1.21 *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. If α and q are* log-*Hö lder continuous at infinity, then*

$$K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = K_{p(\cdot),q_\infty}^{\alpha_\infty}(\mathbb{R}^n).$$

Additionally, if α and q have a log *decay at the origin, then*

$$\|f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \approx \|\{2^{k\alpha(0)} f\,\chi_k\}\|_{\ell^{q(0)}_<(L^{p(\cdot)})} + \|\{2^{k\alpha_\infty} f\,\chi_k\}\|_{\ell^{q_\infty}_>(L^{p(\cdot)})}. \tag{1.3}$$

Proof. *Step 1.* We will prove that

$$K_{p(\cdot),q_\infty}^{\alpha_\infty}(\mathbb{R}^n) \hookrightarrow K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n),$$

which is equivalent to

$$\|f\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \lesssim \|f\|_{K_{p(\cdot),q_\infty}^{\alpha_\infty}}$$

for any $f \in K_{p(\cdot),q_\infty}^{\alpha_\infty}(\mathbb{R}^n)$. By the scaling argument, we see that it suffices to consider the case $\|f\|_{K_{p(\cdot),q_\infty}^{\alpha_\infty}} = 1$ and show that the modular of f on the left-hand side is bounded. In particular, we will show that

$$\sum_{k=1}^{\infty} \left\| \left| c\, 2^{k\alpha(\cdot)} f\,\chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1 \tag{1.4}$$

for some constant $c > 0$. Since α has logarithmic decay at infinity, then for $k \geq 1$ and $x \in R_k$ we have

$$k|\alpha(x) - \alpha_\infty| \lesssim \frac{k}{\ln(e + |x|)} \lesssim 1.$$

Therefore, $2^{k\alpha(x)} \approx 2^{k\alpha_\infty}$ with constants independent of k and x, and hence

$$\sum_{k=1}^{\infty} \left\| \left| c\, 2^{k\alpha(\cdot)} f\, \chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \approx \sum_{k=1}^{\infty} \left\| \left| c\, 2^{k\alpha_\infty} f\, \chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Our estimate (1.4), clearly follows from the inequality

$$\left\| \left| c\, 2^{k\alpha_\infty} f\, \chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq \left\| 2^{k\alpha_\infty} f\, \chi_k \right\|_{p(\cdot)}^{q_\infty} + 2^{-k} = \delta. \tag{1.5}$$

This claim can be reformulated as

$$\left\| \delta^{-1} \left| c\, 2^{k\alpha_\infty} f\, \chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1,$$

which is equivalent to

$$\left\| c\, \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha_\infty} f\, \chi_k \right\|_{p(\cdot)} \leq 1.$$

For any $x \in R_k$, we have

$$\delta^{-\frac{1}{q(x)}} = (2^k \delta)^{\frac{1}{q_\infty} - \frac{1}{q(x)}} 2^{k(\frac{1}{q(x)} - \frac{1}{q_\infty})} \delta^{-\frac{1}{q_\infty}}.$$

Since q has logarithmic decay at infinity, then for $k \geq 1$ and $x \in R_k$ we have

$$\frac{k|q(x) - q_\infty|}{q_\infty q(x)} \leq \frac{k|q(x) - q_\infty|}{q_\infty q^-} \lesssim \frac{k}{\ln(e + |x|)} \lesssim 1.$$

Therefore, $2^{k(\frac{1}{q_\infty} - \frac{1}{q(x)})} \approx 1$ with constants independent of k and x. Also, since $1 < 2^k \delta < 2^{k+1}$,

$$(2^k \delta)^{\frac{1}{q_\infty} - \frac{1}{q(x)}} \leq (2^{k+1})^{\left|\frac{1}{q_\infty} - \frac{1}{q(x)}\right|} \lesssim 1.$$

Hence, with an appropriate choice of $c > 0$

$$\left\| c\, \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha_\infty} f\, \chi_k \right\|_{p(\cdot)} \leq \left\| \delta^{-\frac{1}{q_\infty}} 2^{k\alpha_\infty} f\, \chi_k \right\|_{p(\cdot)} \leq 1,$$

since of

$$\left\| 2^{k\alpha_\infty} f\, \chi_k \right\|_{p(\cdot)} \leq \delta^{\frac{1}{q_\infty}}.$$

Step 2. We will prove that

$$K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) \hookrightarrow K_{p(\cdot),q_\infty}^{\alpha_\infty}(\mathbb{R}^n),$$

which is equivalent to

$$\|f\|_{K^{\alpha_\infty}_{p(\cdot),q_\infty}} \lesssim \|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$$

for any $f \in K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. By the scaling argument, we see that it suffices to consider the case $\|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = 1$ and show that

$$\sum_{k=1}^{\infty} \left\| c\ 2^{k\alpha_\infty} f\,\chi_k \right\|_{p(\cdot)}^{q_\infty} \lesssim 1 \tag{1.6}$$

for some constant $c > 0$. As before, we have for $k \geq 1$

$$\left\| 2^{k\alpha_\infty} f\,\chi_k \right\|_{p(\cdot)}^{q_\infty} \lesssim \left\| 2^{k\alpha(\cdot)} f\,\chi_k \right\|_{p(\cdot)}^{q_\infty}.$$

Now, our estimate (1.6), clearly follows from the inequality

$$\left\| c\ 2^{k\alpha(\cdot)} f\,\chi_k \right\|_{p(\cdot)}^{q_\infty} \leq \left\| \left| 2^{k\alpha(\cdot)} f\,\chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^{-k} = \delta. \tag{1.7}$$

This claim can be reformulated as

$$\left\| c\ \delta^{-\frac{1}{q_\infty}} 2^{k\alpha(\cdot)} f\,\chi_k \right\|_{p(\cdot)} \leq 1.$$

From above, $\delta^{-\frac{1}{q_\infty}} \lesssim \delta^{-\frac{1}{q(x)}}$, then with an appropriate choice of $c > 0$

$$\left\| c\ \delta^{-\frac{1}{q_\infty}} 2^{k\alpha(\cdot)} f\,\chi_k \right\|_{p(\cdot)} \leq \left\| \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha(\cdot)} f\,\chi_k \right\|_{p(\cdot)}.$$

The left-hand side is less than or equal to 1 if and only if

$$\left\| \left| \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha(\cdot)} f\,\chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1.$$

We see that the right-hand side can be rewritten as

$$\delta^{-1} \left\| \left| 2^{k\alpha(\cdot)} f\,\chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1$$

which follows immediately from the definition of δ.

Step 3. Let us prove that

$$\|\{2^{k\alpha(0)} f\,\chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} + \|\{2^{k\alpha_\infty} f\,\chi_k\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})} \lesssim \|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

We suppose that $\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq 1$. If, in addition, α has a log decay at the origin, then we also have $2^{k\alpha(x)} \approx 2^{k\alpha(0)}$ for $k<0$ and $x \in R_k$. Thus

$$\|\{2^{k\alpha(0)} f\,\chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} \approx \|\{2^{k\alpha(\cdot)} f\,\chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}.$$

As in Step 2 we can prove that

$$\left\|c\; 2^{k\alpha(\cdot)} f\,\chi_k\right\|^{q(0)}_{p(\cdot)} \leq \left\|\left|2^{k\alpha(\cdot)} f\,\chi_k\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^k$$

for any $k<0$ and for some constant $c>0$. Then $\|\{2^{k\alpha(0)} f\,\chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} \lesssim 1$. Using the estimate (1.7) we obtain

$$\|\{2^{k\alpha_\infty} f\,\chi_k\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})} \lesssim 1.$$

Therefore,

$$\|\{2^{k\alpha(0)} f\,\chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} + \|\{2^{k\alpha_\infty} f\,\chi_k\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})} \lesssim 1.$$

The desired estimate can be obtained by the scaling argument.

Finally

$$\|\{2^{k\alpha(0)} f\,\chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} \leq 1,$$

and

$$\|\{2^{k\alpha_\infty} f\,\chi_k\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})} \leq 1.$$

As in Step 1 we have for any $k<0$ and for some constant $c>0$

$$\left\|\left|c\; 2^{k\alpha(\cdot)} f\,\chi_k\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq \left\|2^{k\alpha(0)} f\,\chi_k\right\|^{q(0)}_{p(\cdot)} + 2^k$$

by using (1.5) we obtain

$$\sum_{k=-\infty}^{\infty} \left\|\left|2^{k\alpha(\cdot)} f\,\chi_k\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \lesssim 1.$$

Therefore,

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim 1$$

and the result follows by the scaling argument. ■

Chapter 2

Boundedness of singular integral operators on variable weak Herz space

In this chapter we present the boundedness of class of operators on variable weak Herz type space, study is properties and show some application.

2.1 Variable weak Herz space

For $k \in \mathbb{Z}$ and $\lambda > 0$, let

$$A_k(\lambda, f) := \{x \in R_k : |f(x)| > \lambda\}.$$

For $k \in \mathbb{N}_0$ and $\lambda > 0$, let

$$A_k(\lambda, f) := \{x \in R_k : |f(x)| > \lambda\}$$

and

$$A_0(\lambda, f) := \{x \in B(0, 1) : |f(x)| > \lambda\}.$$

Now we will give the definition of the weak Herz space.

Definition 2.1 *Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The* inhomogeneous weak Herz space $WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *consists of all measurable functions f such that*

$$\|f\|_{WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \sup_{\lambda>0} \lambda \left\| \left(2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right)_{k\geq 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Similarly, the homogeneous weak Herz space $W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all measurable functions f such that*

$$\|f\|_{W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \sup_{\lambda>0} \lambda \left\| \left(2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right)_{k\in\mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Now we present the main result of this section.

Proposition 2.2 *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. If α and q are* log-*H ölder continuous at infinity, then*

$$WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = WK^{\alpha_\infty}_{p(\cdot),q_\infty}(\mathbb{R}^n).$$

Additionally, if α and q have a log *decay at the origin, then*

$$\|f\|_{W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \sup_{\lambda>0}(\lambda\|\{2^{k\alpha(0)}\chi_{A_k(\lambda,f)}\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} + \lambda\|\{2^{k\alpha_\infty}\chi_{A_k(\lambda,f)}\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}). \tag{2.1}$$

Proof. *Step 1.* We will prove that

$$WK^{\alpha_\infty}_{p(\cdot),q_\infty}(\mathbb{R}^n) \hookrightarrow WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n),$$

which is equivalent to

$$\|f\|_{WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|_{WK^{\alpha_\infty}_{p(\cdot),q_\infty}}$$

for any $f \in WK^{\alpha_\infty}_{p(\cdot),q_\infty}(\mathbb{R}^n)$. By the scaling argument, we see that it suffices to consider the case $\|f\|_{WK^{\alpha_\infty}_{p(\cdot),q_\infty}} = 1$ and

$$\sum_{k=0}^{\infty} \left\| \left|c\ \lambda\ 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1 \tag{2.2}$$

for any $\lambda > 0$ and some constant $c > 0$ independent of λ. Since α has logarithmic decay at infinity, then for $k \geq 1$ and $x \in R_k$ we have

$$k|\alpha(x) - \alpha_\infty| \lesssim \frac{k}{\ln(e + |x|)} \lesssim 1.$$

Therefore, $2^{k\alpha(x)} \approx 2^{k\alpha_\infty}$ with constants independent of k and x, and hence

$$\sum_{k=1}^{\infty} \left\| \left| c\ \lambda\ 2^{k\alpha(\cdot)} \chi_{A_k(\lambda,f)} \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \approx \sum_{k=1}^{\infty} \left\| \left| c\ \lambda\ 2^{k\alpha_\infty} \chi_{A_k(\lambda,f)} \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

In addition

$$\left\| \lambda \chi_{A_0(\lambda,f)} \right\|_{p(\cdot)} \leq 1$$

for any $\lambda > 0$ which yields that

$$\left\| \left| \lambda \chi_{A_0(\lambda,f)} \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1$$

for any $\lambda > 0$. Our estimate (2.2), clearly follows from the inequality

$$\left\| \left| c\ \lambda\ 2^{k\alpha_\infty} \chi_{A_k(\lambda,f)} \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq \left\| \lambda\ 2^{k\alpha_\infty} \chi_{A_k(\lambda,f)} \right\|_{p(\cdot)}^{q_\infty} + 2^{-k} = \delta, \quad k \geq 1. \tag{2.3}$$

This claim can be reformulated as showing that

$$\left\| \delta^{-1} \left| c\ \lambda\ 2^{k\alpha_\infty} \chi_{A_k(\lambda,f)} \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1,$$

which is equivalent to

$$\left\| c\ \lambda\ \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha_\infty} \chi_{A_k(\lambda,f)} \right\|_{p(\cdot)} \leq 1.$$

For any $x \in R_k$, we have

$$\delta^{-\frac{1}{q(x)}} = (2^k \delta)^{\frac{1}{q_\infty} - \frac{1}{q(x)}} 2^{k(\frac{1}{q(x)} - \frac{1}{q_\infty})} \delta^{-\frac{1}{q_\infty}}.$$

Since q has logarithmic decay at infinity, then for $k \geq 1$ and $x \in R_k$ we have

$$\frac{k|q(x) - q_\infty|}{q_\infty q(x)} \leq \frac{k|q(x) - q_\infty|}{q_\infty q^-} \lesssim \frac{k}{\ln(e + |x|)} \lesssim 1.$$

Therefore, $2^{k(\frac{1}{q_\infty} - \frac{1}{q(x)})} \approx 1$ with constants independent of k and x. Also, since $1 < 2^k\delta < 2^{k+1}$, we have

$$(2^k \delta)^{\frac{1}{q_\infty} - \frac{1}{q(x)}} \leq (2^{k+1})^{|\frac{1}{q_\infty} - \frac{1}{q(x)}|} \lesssim 1.$$

Hence, with an appropriate choice of $c > 0$

$$\left\| c\ \lambda\ \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha_\infty} \chi_{A_k(\lambda,f)} \right\|_{p(\cdot)} \leq \left\| \lambda\ \delta^{-\frac{1}{q_\infty}} 2^{k\alpha_\infty} \chi_{A_k(\lambda,f)} \right\|_{p(\cdot)} \leq 1,$$

since $\left\|\lambda\ 2^{k\alpha_\infty}\chi_{A_k(\lambda,f)}\right\|_{p(\cdot)} \leq \delta^{\frac{1}{q_\infty}}$.

Step 2. We will prove that

$$WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow WK^{\alpha_\infty}_{p(\cdot),q_\infty}(\mathbb{R}^n),$$

which is equivalent to

$$\|f\|_{WK^{\alpha_\infty}_{p(\cdot),q_\infty}} \lesssim \|f\|_{WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$$

for any $f \in WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. By the scaling argument, we see that it suffices to consider the case $\|f\|_{WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = 1$ and show that

$$\sum_{k=0}^{\infty}\left\|c\ \lambda\ 2^{k\alpha_\infty}\chi_{A_k(\lambda,f)}\right\|^{q_\infty}_{p(\cdot)} \lesssim 1 \tag{2.4}$$

for some constant $c > 0$. As before, we have for $k \geq 1$

$$\left\|\lambda\ 2^{k\alpha_\infty}\chi_{A_k(\lambda,f)}\right\|^{q_\infty}_{p(\cdot)} \lesssim \left\|\lambda\ 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right\|^{q_\infty}_{p(\cdot)}.$$

Now, our estimate (2.4), clearly follows from the inequality

$$\left\|c\ \lambda 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right\|^{q_\infty}_{p(\cdot)} \leq \left\|\left|\lambda\ 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^{-k} = \delta. \tag{2.5}$$

This claim can be reformulated as showing that

$$\left\|c\ \lambda\ \delta^{-\frac{1}{q_\infty}} 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right\|_{p(\cdot)} \leq 1.$$

From above, $\delta^{-\frac{1}{q_\infty}} \lesssim \delta^{-\frac{1}{q(x)}}$, then with an appropriate choice of $c > 0$

$$\left\|c\ \lambda\ \delta^{-\frac{1}{q_\infty}} 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right\|_{p(\cdot)} \leq \left\|\lambda\ \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right\|_{p(\cdot)}.$$

The left-hand side is less than or equal to 1 if and only if

$$\left\|\left|\lambda\ \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1.$$

We see that the right-hand side can be rewritten us

$$\delta^{-1}\left\|\left|\lambda\ 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1$$

which follows immediately from the definition of δ.

Step 3. Let us prove that

$$\|\{\lambda\ 2^{k\alpha(0)}\chi_{A_k(\lambda,f)}\}\|_{\ell^{q(0)}_<(L^{p(\cdot)})}+\|\{\lambda\ 2^{k\alpha_\infty}\chi_{A_k(\lambda,f)}\}\|_{\ell^{q_\infty}_>(L^{p(\cdot)})}\lesssim\|f\|_{W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

for any $\lambda>0$. We suppose that $\|f\|_{W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}\le 1$. If, in addition, α has a log decay at the origin, then we also have $2^{k\alpha(x)}\approx 2^{k\alpha(0)}$ for $k<0$ and $x\in R_k$. Thus

$$\|\{\lambda\ 2^{k\alpha(0)}\chi_{A_k(\lambda,f)}\}\|_{\ell^{q(0)}_<(L^{p(\cdot)})}\approx\|\{\lambda\ 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\}\|_{\ell^{q(0)}_<(L^{p(\cdot)})}.$$

As in Step 2 we can prove that

$$\left\|c\ \lambda\ 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right\|_{p(\cdot)}^{q(0)}\le\left\|\left|\lambda\ 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}+2^k$$

for any $k<0$ and for some constant $c>0$. Then

$$\|\{\lambda\ 2^{k\alpha(0)}\chi_{A_k(\lambda,f)}\}\|_{\ell^{q(0)}_<(L^{p(\cdot)})}\lesssim 1.$$

Using the estimate (2.5) we obtain

$$\|\{\lambda\ 2^{k\alpha_\infty}\chi_{A_k(\lambda,f)}\}\|_{\ell^{q_\infty}_>(L^{p(\cdot)})}\lesssim 1.$$

Therefore,

$$\|\{\lambda\ 2^{k\alpha(0)}\chi_{A_k(\lambda,f)}\}\|_{\ell^{q(0)}_<(L^{p(\cdot)})}+\|\{\lambda\ 2^{k\alpha_\infty}\chi_{A_k(\lambda,f)}\}\|_{\ell^{q_\infty}_>(L^{p(\cdot)})}\lesssim 1.$$

The desired estimate can be obtained by the scaling argument.

Now let

$$\|\{\lambda\ 2^{k\alpha(0)}\chi_{A_k(\lambda,f)}\}\|_{\ell^{q(0)}_<(L^{p(\cdot)})}\le 1$$

and

$$\|\{\lambda\ 2^{k\alpha_\infty}\chi_{A_k(\lambda,f)}\}\|_{\ell^{q_\infty}_>(L^{p(\cdot)})}\le 1.$$

As in Step 1 we have for any $k<0$ and for some constant $c>0$

$$\left\|\left|c\ \lambda\ 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\le\left\|\lambda\ 2^{k\alpha(0)}\chi_{A_k(\lambda,f)}\right\|_{p(\cdot)}^{q(0)}+2^k$$

and using (2.3) we obtain

$$\sum_{k=-\infty}^{\infty} \left\| \left| \lambda \ 2^{k\alpha(\cdot)} \chi_{A_k(\lambda,f)} \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \lesssim 1.$$

Therefore, $\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim 1$ and hence the result follows by the scaling argument. ■

2.2 Main result and its proof

We consider the sublinear operators satisfying the size condition

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\, dy, \quad x \notin \operatorname{supp} f, \tag{2.6}$$

for integrable and compactly supported functions f. Condition (2.6) was first considered in [28] and it is satisfied by several classical operators in Harmonic Analysis, such as Calderó n-Zygmund operators, the Carleson maximal operator and the Hardy-Littlewood maximal operator (see [28], [20]).

We have the following results:

Theorem 2.3 *Let* $0 < q^+ \leq \infty$, $p \in \mathcal{P}^{\ln}_{\infty}(\mathbb{R}^n)$ *with* $1 < p^- \leq p^+ < \infty$, *and let* $\alpha \in L^\infty(\mathbb{R}^n)$ *be* log-*Hölder continuous at infinity with*

$$-\frac{n}{p_\infty} < \alpha_\infty < n\left(1 - \frac{1}{p_\infty}\right). \tag{2.7}$$

Suppose that T *is a sublinear operator satisfying estimate* (2.6) *. If* T *is bounded from* $L^{p(\cdot)}(\mathbb{R}^n)$ *to* $L^{p(\cdot),\infty}(\mathbb{R}^n)$, *then* T *is bounded from* $K^{\alpha(\cdot)}_{p(\cdot),q\ (\cdot)}(\mathbb{R}^n)$ *to* $WK^{\alpha(\cdot)}_{p(\cdot),q\ (\cdot)}(\mathbb{R}^n)$.

For homogeneous spaces we have the following statement:

Theorem 2.4 *Let* $q \in \mathcal{P}_0(\mathbb{R}^n)$, $p \in \mathcal{P}^{\ln}_0(\mathbb{R}^n) \cap \mathcal{P}^{\ln}_{\infty}(\mathbb{R}^n)$ *with* $1 < p^- \leq p^+ < \infty$ *and* $0 < q^- \leq q^+ < 1$. *Let* $\alpha \in L^\infty(\mathbb{R}^n)$ *be* log-*Hölder continuous, both at the origin and at infinity, such that*

$$-\frac{n}{p^+} < \alpha^- \leq \alpha^+ < n\left(1 - \frac{1}{p^-}\right). \tag{2.8}$$

Then every sublinear operator T satisfying (2.6) which is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot),\infty}(\mathbb{R}^n)$ is also bounded from $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Proof. In view of Proposition 2.2 we use the equivalent quasinorm (2.1). We split the operator into

$$|Tf| \leq \left|T\big(f\chi_{B_{k-2}}\big)\right| + \left|T\big(f\chi_{\widetilde{R}_k}\big)\right| + \left|T\big(f\chi_{\mathbb{R}^n\setminus B_{k+2}}\big)\right|,$$

with $k \in \mathbb{Z}$, where $\widetilde{R}_k := \left\{x \in \mathbb{R}^n : 2^{k-2} \leq |x| < 2^{k+2}\right\}$.

Estimate of $T\big(f\chi_{B_{k-2}}\big)$. Let $\lambda > 0$,

$$I_1 = \lambda\left\|\{2^{k\alpha(0)}\chi_{A_k(\lambda,T(f\chi_{B_{k-2}}))}\}\right\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}$$

and

$$I_2 = \lambda\left\|\{2^{k\alpha_\infty}\chi_{A_k(\lambda,T(f\chi_{B_{k-2}}))}\}\right\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}.$$

Given $k \leq -1$ and $x \in R_k$ we write

$$\begin{aligned}\left|T\big(f\chi_{B_{k-2}}\big)(x)\right| &\lesssim \int_{B_{k-2}} |x-y|^{-n}|f(y)|\,dy \\ &= \sum_{j=-\infty}^{k-2}\int_{R_j} |x-y|^{-n}|f(y)|\,dy.\end{aligned}$$

Noting that $|x-y| > 2^{k-2}$ for $x \in R_k$ and $y \in R_j$, we use successively Hölder's inequality to obtain the estimate

$$2^{k\alpha(0)}\left|T\big(f\chi_{B_{k-2}}\big)(x)\right| \lesssim 2^{-kn}\sum_{j=-\infty}^{k-2} 2^{(k-j)\alpha^+}\left\|2^{j\alpha(\cdot)}f\chi_j\right\|_{p(\cdot)}\left\|\chi_j\right\|_{p'(\cdot)}.$$

Therefore

$$\lambda 2^{k\alpha(0)}\left\|\chi_{A_k(\lambda,T(f\chi_{B_{k-2}}))}\right\|_{p(\cdot)}$$

can be estimated by

$$\begin{aligned}&2^{-kn}\sum_{j=-\infty}^{k-2} 2^{(k-j)\alpha^+}\left\|2^{j\alpha(\cdot)}f\chi_j\right\|_{p(\cdot)}\left\|\chi_j\right\|_{p'(\cdot)}\left\|\chi_k\right\|_{p(\cdot)} \\ \lesssim\ &2^{-kn}\sum_{j=-\infty}^{k-2} 2^{(k-j)\alpha^+}\left\|2^{j\ \alpha(\cdot)}f\chi_j\right\|_{p(\cdot)} 2^{j\frac{n}{p'(y)}}\,2^{k\frac{n}{p(x)}},\end{aligned}$$

where the positive constant c is independent of k, $x \in R_k$ and $y \in R_j$. Using to Lemma 1.17, we estimate the last term by

$$c \sum_{j=-\infty}^{k-2} 2^{(k-j)(\alpha^+ - n + \frac{n}{p^-})} \left\| 2^{j\ \alpha(\cdot)} f \chi_j \right\|_{p(\cdot)}$$

Since $\alpha^+ - n + \frac{n}{p^-} < 0$, we apply Lemma 1.16 we get

$$I_1 \lesssim \left\| f \right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},$$

by Proposition 1.21. Now we estimate I_2. In view of the estimation of I_1, we have

$$\lambda 2^{k\alpha_\infty} \left\| \chi_{A_k(\lambda, T(f\chi_{B_{k-2}}))} \right\|_{p(\cdot)}$$

can be estimated by

$$c \sum_{j=-\infty}^{k-2} 2^{(k-j)(\alpha^+ - n + \frac{n}{p^-})} \left\| 2^{j\ \alpha(\cdot)} f \chi_j \right\|_{p(\cdot)}, \quad k \geq 0.$$

Since $\alpha^+ - n + \frac{n}{p^-} < 0$, we apply Lemma 1.16 we get

$$I_2 \lesssim \left\| f \right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},$$

by Proposition 1.21.

Estimate of $T\big(f\chi_{\widetilde{R}_k}\big)$. Let $\lambda > 0$,

$$V_1 = \lambda \left\| \{ 2^{k\alpha(0)} \chi_{A_k(\lambda, T(f\chi_{\widetilde{R}_k}))} \} \right\|_{\ell_<^{q(0)}(L^{p(\cdot)})}$$

and

$$V_2 = \lambda \left\| \| \{ 2^{k\alpha_\infty} \chi_{A_k(\lambda, T(f\chi_{\widetilde{R}_k}))} \} \right\|_{\ell_>^{q_\infty}(L^{\ p(\cdot)})}.$$

By similarity we estimate only V_1. Since T is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot),\infty}(\mathbb{R}^n)$, it follows that

$$\lambda \left\| \chi_{A_k(\lambda, T\left(f\chi_{\widetilde{R}_k}\right))} \right\|_{p(\cdot)} \lesssim \left\| f\chi_{\widetilde{R}_k} \right\|_{p(\cdot)},$$

where the implicit constant is independent of λ and k. Consequently

$$V_1 \lesssim \left\| \{ 2^{k\alpha(0)} f\chi_{\widetilde{R}_k} \} \right\|_{\ell_<^{q(0)}(L^{p(\cdot)})} \lesssim \left\| f \right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},$$

where we use Proposition 1.21.

Estimate of $T\big(f\chi_{\mathbb{R}^n\setminus B_{k+2}}\big)$. Let $\lambda>0$

$$J_1=\lambda\big\|\{2^{k\alpha(0)}\chi_{R_k(\lambda,T(f\chi_{\ \mathbb{R}^n\setminus B_{k+2}}))}\}\big\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}.$$

and

$$J_2=\lambda\big\|\{2^{k\alpha_\infty}\chi_{A_k(\lambda,T(f\chi_{\ \mathbb{R}^n\setminus B_{k+2}}))}\}\big\|_{\ell^{q_\infty}_{>}(L^{\ p(\cdot)})}.$$

By similarity we estimate only J_1. Given $k\le -1$ and $x\in R_k$ we write

$$\begin{aligned}\big|T\big(f\chi_{\mathbb{R}^n\setminus B_{k+2}}\big)(x)\big| &\lesssim \int_{\mathbb{R}^n\setminus B_{k+2}}|x-y|^{-n}|f(y)|\,dy\\ &= \sum_{j=k+3}^{\infty}\int_{R_j}|x-y|^{-n}|f(y)|\,dy.\end{aligned}$$

Noting that $|x-y|>2^j$ for $x\in R_k$ and $y\in R_j$, we use successively Hölder's inequality to obtain the estimate

$$2^{k\alpha(0)}\big|T\big(f\chi_{B_{k-2}}\big)(x)\big|\lesssim\sum_{j=k+3}^{\infty}2^{-jn}2^{(k-j)\alpha^-}\big\|2^{j\alpha(\cdot)}f\chi_j\big\|_{p(\cdot)}\big\|\chi_j\big\|_{p'(\cdot)}.$$

Therefore

$$\lambda 2^{k\alpha(0)}\big\|\chi_{A_k(\lambda,T(f\chi_{B_{k-2}}))}\big\|_{p(\cdot)}$$

can be estimated by

$$\begin{aligned}& c\sum_{j=k+3}^{\infty}2^{-jn}2^{(k-j)\alpha^-}\big\|2^{j\alpha(\cdot)}f\chi_j\big\|_{p(\cdot)}\big\|\chi_j\big\|_{p'(\cdot)}\big\|\chi_k\big\|_{p(\cdot)}\\ \lesssim\ & \sum_{j=k+3}^{\infty}2^{-jn}2^{(k-j)\alpha^-}\big\|2^{j\ \alpha(\cdot)}f\chi_j\big\|_{p(\cdot)}\,2^{j\frac{n}{p'(y)}}\,2^{k\frac{n}{p(x)}},\end{aligned}$$

where the positive constant c is independent of k, $x\in R_k$ and $y\in R_j$. Using to Lemma 1.17, we estimate the last term by

$$c\sum_{j=k+3}^{\infty}2^{(k-j)(\alpha^-+\frac{n}{p^+})}\big\|2^{j\ \alpha(\cdot)}f\chi_j\big\|_{p(\cdot)}$$

Since $\alpha^- + \frac{n}{p^+} > 0$, we apply Lemma 1.16 we get

$$J_1 \lesssim \|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},$$

by Proposition 1.21. The proof is complete. $\blacksquare$

Chapter 3

Boundedness singular integral operators on variable Herz-type Hardy space

In this chapter we study the boundedness of class of operators on variable weak Herz-type Hardy space dy using the atomic decomposition. these spaces.

3.1 Variable Herz-type Hardy space

In recent years, it turned out that atomic decomposition of some function spaces are extremely useful in many aspects. This concerns, for instance, the investigation of (compact) embeddings between function spaces. But this applies equally to questions of mapping properties of some operators, such as Calderón-Zygmund operators, the commutator of Calderón-Zygmund operator with a BMO function and to trace problems, where arguments can be equivalently transferred to the sequence space, which is often more convenient to handle. The main goal of this section is to prove an atomic decomposition result for $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. First we introduce the basic notation.

Let $G_N f$ be the grand maximal function of f defined by

$$G_N f(x) := \sup_{\varphi \in \mathcal{A}_N} |\varphi_N^*(f)(x)|,$$

where

$$\mathcal{A}_N = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha| \leq N, |\beta| \leq N} |x^\alpha \partial^\beta \varphi(x)| \leq 1\}$$

and

$$\varphi_N^*(f)(x) = \sup_{t>0} |\varphi_t * f(x)|,$$

with $\varphi_t = t^{-n}\varphi(\frac{\cdot}{t})$.

Definition 3.1 *Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$ and $N > n+1$. The* inhomogeneous Herz-type Hardy space $HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ *consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $G_N f \in K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and we define*

$$\|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} := \|G_N f\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.$$

Similarly, the homogeneous Herz-type Hardy space $H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $G_N f \in \dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and we define*

$$\|f\|_{H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} := \|G_N f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.$$

We have G_N satisfies the size condition (2.6). Let $q \in \mathcal{P}_0(\mathbb{R}^n)$, $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$, and α and q are log-H ölder continuous at infinity, with $\alpha \in L^\infty(\mathbb{R}^n)$ and

$$-\frac{n}{p_\infty} < \alpha_\infty < \Big(1 - \frac{1}{p_\infty}\Big).$$

Then

$$HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) \cap L^{p(\cdot)}(\mathbb{R}^n) = K_{p(\cdot),q\ (\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n).$$

Let $q \in \mathcal{P}_0(\mathbb{R}^n)$, $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$, and let α and q are log-Hölder continuous, both at the origin and at infinity, such that $\alpha \in L^\infty(\mathbb{R}^n)$ and

$$-\frac{n}{p^+} < \alpha^- \leq \alpha^+ < n\Big(1 - \frac{1}{p^-}\Big).$$

Then

$$H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}\left(\mathbb{R}^n\right) \cap L^{p(\cdot)}\left(\mathbb{R}^n \setminus \{0\}\right) = \dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}\left(\mathbb{R}^n\right).$$

Definition 3.2 *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p \in \mathcal{P}(\mathbb{R}^n)$, $q \in \mathcal{P}_0(\mathbb{R}^n)$ and $s \in \mathbb{N}_0$. A function a is said to be a central $(\alpha(\cdot), p(\cdot))$-atom, if*

(i) $\operatorname{supp} a \subset \overline{B(0,r)} = \{x \in \mathbb{R}^n : |x| \leq r\}, r > 0.$

(ii) $\|a\|_{p(\cdot)} \leq |\overline{B(0,r)}|^{-\alpha(0)/n}, \qquad 0 < r < 1.$

(iii) $\|a\|_{p(\cdot)} \leq |\overline{B(0,r)}|^{-\alpha_\infty/n}, \qquad r \geq 1.$

(iv) $\int_{\mathbb{R}^n} x^\beta a(x)dx = 0, \quad |\beta| \leq s.$

A function a on $\mathbb{R}^n$ is said to be a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type, if it satisfies the conditions (iii), (vi) above and $\operatorname{supp} a \subset B(0,r), r \geq 1$.

Now we come to the atomic decomposition theorems.

Theorem 3.3 *Let α and q be* log*-Hölder continuous at infinity and $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. For any $f \in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k, \tag{3.1}$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$,each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type with $\operatorname{supp} a \subset B_k$ *and*

$$\Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\Big)^{1/q_\infty} \leq c\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Conversely,if $\alpha_\infty \geq n(1 - \frac{1}{p_\infty})$ and $s \geq [\alpha_\infty + n(\frac{1}{p_\infty} - 1)]$, and if (3.1) holds, then $f \in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, and

$$\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \inf\Big\{\Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\Big)^{1/q_\infty}\Big\},$$

where the infimum is taken over all the decompositions of f as above.

Theorem 3.4 *Let α and q are be* log-*Hölder continuous, both at the origin and at infinity and $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$. For any $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, \tag{3.2}$$

where the series converges in the sense of distributions, $\lambda_k \ge 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom with supp$a \subset B_k$ *and*

$$\Big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\Big)^{1/q(0)} + \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\Big)^{1/q_\infty} \le c\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Conversely,if $\alpha(\cdot) \ge n(1 - \frac{1}{p^-})$ and $s \ge [\alpha^+ + n(\frac{1}{p^-} - 1)]$, and if (3.2) *holds, then $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, and*

$$\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \inf\Big\{\Big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\Big)^{1/q(0)} + \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\Big)^{1/q_\infty}\Big\},$$

where the infimum is taken over all the decompositions of f as above.

For the proof see [8], Theorem 4.

Remark 3.5 *Corresponding statements to Theorems 3.3 and 3.4 were proved by Liu and Wang [21], with α and q constants, under the assumption that the maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (both in the homogeneous and the inhomogeneous situation). Here we are requiring the log-Hölder continuity at two points only (zero and infinity).*

Next we will give the definition of the weak Herz-type Hardy space with variable exponents.

Definition 3.6 *Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$ and $N > n+1$. The* inhomogeneous weak Herz-type Hardy space $WHK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $G_N f \in WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and we define*

$$\|f\|_{WHK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|G_N f\|_{WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Similarly, the homogeneous weak Herz-type Hardy space $WH\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that* $G_N f \in W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *and we define*

$$\|f\|_{WH\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|G_N f\|_{W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

3.2 Boundedness of some singular integral operators

First we establish the boundedness for a class of operators on Herz type spaces with variable exponent. Let T be an operator defined on a linear space of all almost everywhere finite measurable functions on $\mathbb{R}^n$. Then T is called a sublinear operator if

$$|T(f_1 + f_2)(x)| \leq |Tf_1(x)| + |Tf_2(x)| \text{ and } |T(\lambda f)| = |\lambda| |Tf|, \quad \lambda \in \mathbb{C}$$

Now we give the corresponding result about the sublinear operator T on Herz type spaces with variable exponent. Let K be a measurable function defined on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{x = y\}$ that satisfies the size condition

$$|K(x,y) - K(x,0)| \leq \frac{|y|^\delta}{|x|^{n+\delta}}, \tag{3.3}$$

if $2|y| < |x|$ for some $\delta \in (0,1]$. Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a linear operator such that for any $f \in L^2(\mathbb{R}^n)$ with compact support and almost everywhere $x \notin$supp$f$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy, \tag{3.4}$$

which are more general than the standard Calderón-Zygmund operator.

This section is devoted to the behavior of T on variable Herz type Hardy space.

Theorem 3.7 *Let* α *and* q log-*Hölder continuous at infinity and* $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ *with* $1 < p^- \leq p^+ < \infty$ *and* $0 < q^- \leq q^+ < \infty$. *With some assumption on* α, *every linear operator* T *satisfying* (3.4) *which is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$ *is also bounded from* $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *to* $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

The counterpart for homogeneous Herz spaces runs as follows:

Theorem 3.8 *Let α and q be* log-*Hölder continuous, both at the origin and at infinity and $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < \infty$. With some assumption on α, every linear operator T satisfying* (3.4) *which is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ is also bounded from $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.*

Theorem 3.9 *Let α and q be* log-*Hölder continuous at infinity and $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < 1$. With some assumption on α, every linear operator T satisfying* (3.4) *which is bounded $L^{p(\cdot)}(\mathbb{R}^n)$ is also bounded from $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.*

For homogeneous spaces we have the following statement:

Theorem 3.10 *Let α and q be* log-*Hölder continuous, both at the origin and at infinity and $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < 1$. With some assumption on α, every linear operator T satisfying* (3.4) *which is bounded $L^{p(\cdot)}(\mathbb{R}^n)$ is also bounded from $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.*

If the operator T in Theorems 3.9 and 3.10 is of convolution type, we then can obtain the following result.

Theorem 3.11 *Let α and q are be* log-*Hölder continuous at infinity and , $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < 1$. Let T defined by*

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} K(x-y) f(y)\, dy,$$

where the kernel K satisfies

$$|K(x-y) - K(x)| \leq \frac{|y|^{\delta}}{|x|^{n+\delta}},$$

if $2|y| < |x|$ for some $\delta \in (0,1]$. Assume that T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. With some assumption on α, T is also bounded from $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $WHK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

The counterpart for homogeneous Herz spaces runs as follows:

Theorem 3.12 *Let α and q be* log-*Hölder continuous, both at the origin and at infinity and $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < 1$. Let T be as in Theorem 3.12. Assume tha T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. With some assumption on α, T is also bounded from $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $WH\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.*

Remark 3.13 *The proof of the results of this chapter are given in [4]. Corresponding statements of the results of this thesis were proved in [29] , with α and q constants, under the assumption that the maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (both in the homogeneous and the inhomogeneous situation). Here we are requiring the log-Hölder continuity at two points only (zero and infinity).*

3.3 Commutator of Calderón–Zygmund singular integral operator

For $0 < \beta \leq 1$, the Lipschitz spase $Lip_\beta(\mathbb{R}^n)$ is defined as

$$Lip_\beta(\mathbb{R}^n) := \left\{ f : \|f\|_{Lip_\beta} = \sup_{y,t\in\mathbb{R}^n; x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty \right\}.$$

Let $b \in Lip_\beta(\mathbb{R}^n)$, and let T be a Calderón–Zygmund singular integral operator that is,

$$Tf(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy,$$

$p.v.$ or (principal value integrals) where $\Omega \in C^2(S^{n-1})$ is homogeneous of degree zero and has mean value zero on the unit sphere. The commutator $[b,T]$ generated by b and T is defined by

$$[b,T](f)(x) := b(x)\,Tf(x) - T(bf)(x).$$

Let $0 < \beta < n$, The fractional integral operator I_β is defined by

$$I_\beta(f)(x) := \frac{1}{\gamma(\beta)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}} dy,$$

where $0<\beta<n$ and

$$\gamma(\beta):=\frac{\pi^{n/2}2^{\beta}\Gamma(\beta/2)}{\Gamma((n-\beta)/2)}.$$

We have the following results.

Theorem 3.14 *Let $0<\beta\leq 1, q\in\mathcal{P}_0(\mathbb{R}^n)$, and $p_1\in\mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1<p_1^-\leq p_1^+<\frac{n}{\lambda}$. Let α and q be log-Hölder continuous at infinity, such that $\alpha\in L^{\infty}(\mathbb{R}^n)$ and*

$$\beta-\frac{n}{(p_1)_{\infty}}<\alpha_{\infty}<n\Big(1-\frac{1}{(p_1)_{\infty}}\Big).$$

If $b\in Lip_{\beta}(\mathbb{R}^n)$, then $[b,T]$ is bounded from $K^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$ into $K^{\alpha(\cdot)}_{p_2(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Theorem 3.15 *Let $0<\beta\leq 1$, $p_1\in\mathcal{P}^{\ln}_0(\mathbb{R}^n)\cap\mathcal{P}^{\ln}_{\infty}(\mathbb{R}^n)$, $q\in\mathcal{P}_0(\mathbb{R}^n)$ with $1<p_1^-\leq p_1^+<\frac{n}{\beta}$, $\frac{1}{p_1(\cdot)}-\frac{1}{p_2(\cdot)}=\frac{\beta}{n}$ and let α and q are be log-Hölder continuous, both at the origin and at infinity such that $\alpha\in L^{\infty}(\mathbb{R}^n)$ and*

$$\beta-\frac{n}{p_1^+}<\alpha^-\leq\alpha^+<n\Big(1-\frac{1}{p_1^-}\Big).$$

If $b\in Lip_{\beta}(\mathbb{R}^n)$, then $[b,T]$ is bounded from $\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}$ into $\dot{K}^{\alpha(\cdot)}_{p_2(\cdot),q(\cdot)}$.

Theorem 3.16 *Let $0<\beta\leq 1, q\in\mathcal{P}_0(\mathbb{R}^n)$, and $p_1\in\mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1<p_1^-\leq p_1^+<\frac{n}{\lambda}$. Let α and q be log-Hölder continuous at infinity, such that $\alpha\in L^{\infty}(\mathbb{R}^n)$ and*

$$n\Big(1-\frac{1}{(p_1)_{\infty}}\Big)\leq\alpha_{\infty}<\beta+n\Big(1-\frac{1}{(p_1)_{\infty}}\Big).$$

If $b\in Lip_{\beta}(\mathbb{R}^n)$, then $[b,T]$ is bounded from $HK^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$ into $K^{\alpha(\cdot)}_{p_2(\cdot),q(\cdot)}(\mathbb{R}^n)$.

3.4 Commutator of the fractional integral operator

Let $b\in Lip_{\beta}(\mathbb{R}^n), 0<\lambda<n$ and let I_{λ} denote the fractional integral operator. The commutator of fractional integral operator

$$[b,I_{\lambda}]f(x):=\int_{\mathbb{R}^n}\frac{(b(x)-b(y))}{|x-y|^{n-\lambda}}f(y)\,dy.$$

Theorem 3.17 *Let* $0 < \beta < 1, 0 < \lambda < n - \beta, p_1 \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ *and* $q \in \mathcal{P}_0(\mathbb{R}^n)$ *with* $1 < p_1^- \leq p_1^+ < \frac{n}{\beta+\lambda}$, $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta+\lambda}{n}$, *and let* α *and* q *are be log-Hölder continuous, at infinity such that* $\alpha \in L^\infty(\mathbb{R}^n)$ *and*

$$n\Big(1 - \frac{1}{(p_1)_\infty}\Big) \leq \alpha_\infty < \beta + n\Big(1 - \frac{1}{(p_1)_\infty}\Big).$$

If $b \in Lip_\beta(\mathbb{R}^n)$, *then* $[b, I_\lambda]$ *is bounded from* $HK^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}$ *into* $K^{\alpha(\cdot)}_{p_2(\cdot),q(\cdot)}$.

Theorem 3.18 *Let* $0 < \beta < 1, 0 < \lambda < n - \beta, p_1 \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ *and* $q \in \mathcal{P}_0(\mathbb{R}^n)$ *with* $1 < p_1^- \leq p_1^+ < \frac{n}{\beta+\lambda}$, $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta+\lambda}{n}$, *and let* α *and* q *are be log-Hölder continuous, both at the origin and at infinity such that* $\alpha \in L^\infty(\mathbb{R}^n)$, $\alpha(\cdot) \geq n(1 - \frac{1}{p_1^-})$ *and*

$$n\Big(1 - \frac{1}{p_1^-}\Big) \leq \alpha^- \leq \alpha^+ < \beta + n\Big(1 - \frac{1}{p_1^-}\Big).$$

If $b \in Lip_\beta(\mathbb{R}^n)$, *then* $[b, I_\lambda]$ *is bounded from* $H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}$ *into* $\dot{K}^{\alpha(\cdot)}_{p_2(\cdot),q(\cdot)}$.

Suppose that S^{n-1} denotes the unit sphere of $\mathbb{R}^n \quad (n \geq 2)$ equipped with normalized Lebesgue measure. Let $\Omega \in Lip_\nu(S^{n-1})$ for $0 < \nu \leq 1$ be a homogeneous function of degree zero and

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x'),$$

where $x' = x/|x|$ for any $x \neq 0$.The Marcinkiewicz integral operator μ is defined by

$$\mu(f)(x) := \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}},$$

where

$$F_t(f)(x) := \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} dy.$$

Let $b \in Lip_\beta(\mathbb{R}^n)$, the commutator $[b, \mu]$ generated by the Marcinkiewicz integral operator μ and b is defined by

$$[b, \mu](f)(x) := \left(\int_0^\infty \Big| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)]\, f(y)\, dy\Big|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}.$$

Theorem 3.19 *Suppose that* $\Omega \in Lip_\nu\left(S^{n-1}\right) \quad (0<\nu\leq 1)$ *and* $b \in Lip_\beta(\mathbb{R}^n), \quad 0<\beta\leq \frac{\nu}{2}$. *Let* $p_1 \in \mathcal{P}^{\ln}\left(\mathbb{R}^n\right)$ *and* $q \in \mathcal{P}_0\left(\mathbb{R}^n\right)$ *with* $1<p_1^- \leq p_1^+ < \frac{n}{\lambda}$, *and let* α *and* q *be log-Hölder continuous at infinity, such that* $\alpha \in L^\infty\left(\mathbb{R}^n\right)$ *and*

$$n\Big(1-\frac{1}{(p_1)_\infty}\Big) \leq \alpha_\infty < \beta + n\Big(1-\frac{1}{(p_1)_\infty}\Big),$$

then $[b,\mu]$ *is bounded from* $HK^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}\left(\mathbb{R}^n\right)$ *into* $K^{\alpha(\cdot)}_{p_2(\cdot),q(\cdot)}\left(\mathbb{R}^n\right)$.

Theorem 3.20 *Suppose that* $\Omega \in Lip_\nu\left(S^{n-1}\right) \quad (0<\nu\leq 1)$ *and* $b \in Lip_\beta(\mathbb{R}^n), \quad 0<\beta\leq \frac{\nu}{2}$. *Let* $p_1 \in \mathcal{P}^{\ln}\left(\mathbb{R}^n\right)$ *and* $q \in \mathcal{P}_0\left(\mathbb{R}^n\right)$ *with* $1<p_1^- \leq p_1^+ < \frac{n}{\beta}$, $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta}{n}$ *and let* α *and* q *are be log-Hölder continuous, both at the origin and at infinity such that* $\alpha \in L^\infty\left(\mathbb{R}^n\right)$, $\alpha\left(\cdot\right) \geq n(1-\frac{1}{p_1^-})$ *and*

$$n\Big(1-\frac{1}{p_1^-}\Big) \leq \alpha^- \leq \alpha^+ < \beta + n\Big(1-\frac{1}{p_1^-}\Big),$$

then $[b,\mu]$ *is bounded from* $H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}$ *into* $\dot{K}^{\alpha(\cdot)}_{p_2(\cdot),q(\cdot)}$.

Remark 3.21 *The proof of the results of Sections 3.3 and 3.4 are given in [10].*

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الملخص:

في هذا العمل قمنا بدراسة بعض الخصائص في فضاءات "هارز" و فضاءات "هارز" الضعيفة ذات الأدلة المتغيرة، ثم قمنا بدراسة محدودية المؤثرات تحت الخطية في هذه الفضاءات " و في الاخير درسنا محدودية المؤثرات تحت الخطية في فضاءات "هارز-هاردي" و "هارز-هاردي" الضعيفة ذات الأدلة المتغيرة باستعمال التفكيك الذري لهذه الفضاءات.

الكلمات المفتاحية:

فضاءات "هارز"، فضاءات "هارز-هاردي"، مؤثرات تحت الخطية، أدلة متغيرة، الدالة الأعظمية، الذرة.

Résumé:

Dans ce travail, nous avons étudié certaines propriétés des espaces de Herz et de Herz faible avec des exposants variables, finalement, nous avons étudié la continuité des opérateurs sous linéaires dans les espaces Herz-Hardy et Herz-Hardy faible avec des exposants variables en utilisant la décomposition atomique de ces espaces.

Mots clés : Espaces de Herz, Espaces de Herz-Hardy, Opérateurs sous linéaire, Exposants variables, La fonction maximale, Atome.

Abstract:

In this memory we have studied some properties of Herz space with variable exponent, then we studied the boundedness of some sublinair operators in Herz space and in weak Herz space with variable exponent, finally we studied the boundedness of sublinear operators in Herz-Hardy spaces and in weak Herz-Hardy spaces with variables exponents by using atomic decomposition of these spaces.

Key words: Herz spaces, Herz-type Hardy spaces, Sublinear operators, Variable exponent, Maximal function, Atom.