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**Existence and regularity of the solution of some
elliptic problems**

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Dedication

I dedicate this modest work :

-To my parents,

-To my brothers and sisters,

-To my aunts,

-To all my family,

-To all friends and all my department family,

-To all my adorable ones that i have known during all my life ...

Berrabeh Souad

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Notation

We introduce the necessary notations and definition which are used in the sequel.

\mathcal{H}	Hilbert space.
$\Omega \subset \mathbb{R}^N$	open set in \mathbb{R}^N
X'	topological dual of X .
A^c	complement of the set A .
(\cdot, \cdot)	scalar product.
\mathbb{R}^N	Euclidean space of dimension N , where N is a nonzero natural number.
x	vector in \mathbb{R}^N , $x = (x_1, x_2, \dots, x_N)$, $x_i \in \mathbb{R}$, $1 \leq i \leq N$
dx	Lebesgue measure in N -dimensional space.
$\partial\Omega = \Gamma$	boundary of Ω .
$ E $	or mes (E) measure of the set E .
$B(x_0, r)$	open ball of radius r centered at x_0 .
B_E	$= \{x \in E; \ x\ \leq 1\}$.
$ \cdot $	Hilbert norm.
χ_E	characteristic function of set E .
Ω	open subset of \mathbb{R}^N .
$\langle \cdot, \cdot \rangle$	duality bracket between X and its dual space.
$\int_{\Omega} f(x) dx$	integral of f in Ω with respect to the Lebesgue measure.
$\text{supp } u$	Support of the function u .
$\frac{\partial u}{\partial n}$	outward normal derivative.
$\nabla u = Du$	$= (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_N})$ gradient of the function u .
$\text{div } u$	divergence of the vector u , $\text{div } u = \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + \dots + \frac{\partial u}{\partial x_N}$
$\Delta u = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$	Laplacian of u .
$f_n \rightarrow f$	denotes that the sequence $\{f_n\}$ converge to f .
$f_n \rightharpoonup f$	denotes that the sequence $\{f_n\}$ converge weakly to f .
$L^p(\Omega)$	$= \{u : \Omega \rightarrow \mathbb{R} \text{ measurable and } (\int_{\Omega} u(x) ^p dx < +\infty \text{ such that } 1 \leq p < \infty)\}$.

- $|u|_p = \left[\int_{\Omega} |u(x)|^p dx \right]^{1/p} = |u|_{L^p}.$
 $L^{\infty}(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \text{ measurable, } \exists M > 0 \mid |u(x)| \leq M \text{ a.e.}\}.$
 $\|u\|_{L^{\infty}} = \inf\{C : |u(x)| \leq C \text{ a.e. on } \Omega\}.$
 q Hölder conjugate of $p : q = \frac{p}{p-1}$ if $p > 1$ and $q = \infty$ if $p = 1$
 $L^q(\Omega) \subset L^p(\Omega), \quad \forall 1 \leq p \leq q < \infty$
 $\|u\|_{C_c(\Omega)} = \sup_{x \in \Omega} |u(x)| = \max_{x \in \Omega} |u(x)|$
 $C_c(\Omega)$ space of continuous functions with compact support.
 $\mathcal{D}(\Omega)$ space of infinitely differentiable functions on Ω with compact support in Ω .
 $W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \mid \nabla u \in (L^p(\Omega))^N \right\}.$
 $W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega), \text{ with } u = 0 \text{ on } \partial\Omega \right\}.$
 $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ for $1 \leq p < \infty$ and $a, b \geq 0$.
a.e. almost everywhere.

General Introduction

Partial differential equations, which will be abbreviated as "PDE" in the following, constitute an important branch of applied mathematics and this field is becoming increasingly important in modern times. PDEs have great utility in modeling of many phenomena of different natures such as physics, chemistry, biology and other sciences. In other words, PDEs intervene in many other fields: in chemistry to model reactions, in economics to study market behavior, in finance to study derivative products and in image processing to restore damage. These problems boil down to mathematical models usually written in the form

$$L(u) = f, \quad (1)$$

where L is a definite operator from a function space E to a function space F , u is the unknown function and f a given function. PDEs probably first appeared during the 17th century. In the field of PDEs has been enriched as the sciences have developed and especially physics. However, the systematic study of PDEs is much more recent and it is only during of the 20th century that mathematicians began to develop and indeed this theory has recently experienced a great theoretical and practical advancement. The mathematical analysis of these partial differential equations requires an appropriate choice of functional spaces and a clear definition of the notion of solution (the existence and sometimes uniqueness). One of the things to keep in mind about PDEs is to ask the question: can we get solutions explicitly?. So what mathematics can do on the other hand, is to say if one or more solutions exist, and to describe sometimes very precisely- ment some properties of these solutions. So usually an exact solution of problem (1) is not easy to find and sometimes we cannot

even find the explicit solution. This leads to the introduction of the notion of the weak solution which appeared in 1934 in the work of Jean Leray. therefore, in most cases it is very difficult, if not impossible, to exhibit the solutions of a PDE. In some cases we manage to try to show that the problem admits a unique solution (it is said to be well-posed). But sometimes we can calculate numerical approximations of the solutions.

Moreover, the work presented in this dissertation concerns a particular case of the equations to partial derivatives of the elliptic type involving the divergent operator

$$Au = -\operatorname{div}(a(x, \nabla u))$$

where $a = |\nabla u|^{p-2} \nabla u$ a Carathéodory function, verifies certain hypotheses.

We are interested in this thesis to study the following problem:

$$(P) \quad \begin{cases} Au = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Ω is a bounded open set of \mathbb{R}^N with $N > 2$, we assume that $f \in L^1(\Omega)$ and $f \in W^{-1,p'}(\Omega)$.

We first illustrate the main difficulties that may arise when studying of the problem (P)

1) The non-linearity of the differential operator $A(u) = -\operatorname{div}(|\nabla u|^{p-2} \nabla u)$

2) The irregularity in the case where $f \in L^1(\Omega)$; i.e. the second member of (P) does not belong to space $W^{-1,p'}(\Omega)$.

Technique

We are going to prove the existence and regularity of the weak solutions of the problem (P).

To do this, we will approximate the problem (P) by a sequence of approximate problems (P_n) given in L^∞ whose existence of the solution approximate is guaranteed by Theorem 2.2. Then we will prove some estimates uniform on the sequence of solutions of these problems (P_n) and their partial derivatives. Once this is done, the linearity of the operator with respect to the gradient as well as the boundedness and the continuity of a will make it possible to pass to the limit, thus finding the solution.

Thesis plan

This thesis is divided into three chapters as follows: The first chapter is dedicated to giving some basic definitions and results with functional analysis tools essential to the achievement of the objectives for the study of the (P) problem. For example we recall functional spaces (Lebesgue, Sobolev) and we give brief reminders of weak and weak star convergence.

In the second chapter we recall some definitions on the operators (Bounded, hime-continuous, Monotone and coercive). We also present the method of monotonic (pseudo-monotonic) operators in the general framework to prove the existence of a solution for the equation

$$Au = f.$$

According to hypotheses on the functions $a = |\nabla u|^{p-2}\nabla u$ and f ($f \in W^{-1,p'}(\Omega)$) and for to prove the existence of the solution we use the Theorem 2.1.

In the second chapter, we study the weak solution for a class of nonlinear elliptic problem with L^1 data $Au = f$ and $u = 0$ with $f \in L^1(\Omega)$. We limit ourselves to the case where $p > 2 - \frac{1}{N}$. The weak solution u of this problem be in the space $W_0^{1,q}(\Omega)$, for all $q \in [1, \frac{N}{N-1}(p-1)[$. As a prototype example, we consider the model problem $Au = \delta$ where δ is the Dirac measure.

Preliminaries and basic tools

The purpose of this first chapter is to present a number of analytical tools from the theory of Sobolev spaces that will be used throughout this thesis. We will also take the opportunity to introduce the main notations.

1.1 Reminders and some definitions

Throughout this chapter Ω will denote a bounded domain in \mathbb{R}^N , $N \geq 1$, i.e. a connected and bounded open subset of \mathbb{R}^N . Its boundary will be denoted by Γ or $\partial\Omega$ and its closure by $\bar{\Omega}$.

Let $u = u(x_1, \dots, x_N)$ be a function defined in $\Omega \subset \mathbb{R}^N$. Assuming that exists, the gradient of u at point x is defined as the vector:

$$\nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), \dots, \frac{\partial u}{\partial x_N}(x) \right)$$

The Euclidean norm of ∇u is denoted by $|\nabla u|$:

$$|\nabla u| = \left[\left| \frac{\partial u}{\partial x_1} \right|^2 + \dots + \left| \frac{\partial u}{\partial x_N} \right|^2 \right]^{1/2}$$

Definition 1.1. (Space of test functions): The space $\mathcal{D}(\Omega)$ or $C_c^\infty(\Omega)$ is the set of infinitely differentiable functions $\varphi : \Omega \rightarrow \mathbb{R}$ (of class $C^\infty(\Omega)$) with compact support in Ω .

$\mathcal{D}(\Omega)$ is a vector space, and each element of this space is called a test function.

- $\text{supp } \varphi = \overline{\{x \in \Omega, \varphi(x) \neq 0\}}$,
- $\mathcal{D}(\Omega) = C^\infty(\Omega) \cap C_c(\Omega)$.

Definition 1.2 (Separable Spaces). A Banach space E is said to be separable if there exists a countable set that is dense in E .

Definition 1.3 (Dual Space). . The topological dual of a vector space E over the field \mathbb{K} , (where $\mathbb{K} = \mathbb{R}$ or \mathbb{C}) is defined as the set of continuous linear forms on E , i.e. continuous linear maps from E to \mathbb{K} . We denote by $E' = \mathbb{L}(E, \mathbb{K})$.

E' equipped with the norm $\|\cdot\|$ defined by :

$$\|u\|_{E'} = \sup_{x \in E, \|x\|=1} |u(x)| = \sup_{x \in E, \|x\| \leq 1} |u(x)| = \sup_{x \in E, \|x\| \neq 0} \frac{|u(x)|}{\|x\|}$$

Proposition 1.1. 1. $E_1 \subset E_2$ so $E_2' \subset E_1'$,

2. $\langle u, x \rangle_{E'E} \leq \|u\|_{E'} \|x\|_E$.

1.2 Functional spaces

1.2.1 L^p Spaces

Let Ω be an open subset of \mathbb{R}^N and let p be a real number greater than or equal to 1. The Lebesgue Space $L^p(\Omega)$ is defined as:

$$L^p(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ is measurable and } \int_{\Omega} |u(x)|^p < \infty \right\}.$$

We equip it with the following norm

$$\|u\|_{L^p(\Omega)} = \|u\|_p = \left(\int_{\Omega} |u(x)|^p \right)^{\frac{1}{p}}, \quad p \geq 1.$$

By $L^\infty(\Omega)$ we denote the set

$$L^\infty(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \text{ measurable, } \exists M > 0 \mid |u(x)| \leq M \text{ a.e.}\}$$

It is equipped with the following norm:

$$\|u\|_{\infty, \Omega} = \inf \{M > 0 : |u| \leq M \text{ a.e. in } \Omega\}$$

The space $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{2, \Omega} = \int_{\Omega} uv dx.$$

Remark 1.1. Let $1 \leq p \leq \infty$, be denoted by p' the conjugate exponent of p

$$\text{i.e. } \frac{1}{p} + \frac{1}{p'} = 1$$

Let $1 \leq p < \infty$. Then the dual space of $L^p(\Omega)$ is $L^{p'}(\Omega)$.

1.2.2 Main inequalities

Let $1 \leq p \leq q \leq \infty$ be two real numbers; and let p' be the conjugate exponent of p .

Lemma 1.1 (Hölder's inequality). If $f \in L^p(\Omega)$, and $g \in L^{p'}(\Omega)$, then $f \cdot g \in L^1(\Omega)$ and

$$\|f \cdot g\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}$$

If more $|\Omega| < \infty$ and $f \in L^q(\Omega)$, so $f \in L^p(\Omega)$ and

$$\|f\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(\Omega)}$$

In particular,

$$L^q(\Omega) \subset L^p(\Omega), \quad \forall 1 \leq p \leq q < \infty$$

Lemma 1.2. Let $p_i \in [1, +\infty]$ be exponents with $1 \leq i \leq k$ such that:

$1/p = 1/p_1 + \dots + 1/p_k \leq 1$. Then, for all functions $f_i \in L^{p_i}(\Omega)$, We have $f = f_1 \dots f_k \in L^p(\Omega)$ and generalized Hölder inequality

$$\|f\|_p \leq \|f\|_{p_1} \dots \|f\|_{p_k}.$$

Lemma 1.3 (Young's inequality). For all non-negative real numbers a, b and every $1 < p < \infty$, then we have:

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad p' = \frac{p}{p-1}. \quad (1.1)$$

Corollary 1.1. Generalized Young's inequality: for $1 < p < \infty$, $p' = \frac{p}{p-1}$ and any positive ϵ we have:

$$ab \leq \epsilon^p \frac{a^p}{p} + \frac{1}{\epsilon^{p'}} \frac{b^{p'}}{p'}.$$

Theorem 1.1 (Fisher-Riesz). L^p is a Banach space for all $1 \leq p \leq \infty$.

Lemma 1.4 (Divergence and Green's formulas). Let Ω be a domain in \mathbb{R}^N , and $n(x)$ its exterior normal. Let u and v be two regular functions, and w be a vector field defined on Ω . Then,

$$\int_{\Omega} \operatorname{div} w \, dx = \int_{\partial\Omega} w \cdot n \, d\sigma \quad (\text{divergence formula})$$

$$\int_{\Omega} (\Delta u)v \, dx = - \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, d\sigma \quad (\text{green formula})$$

Theorem 1.2 (Rellich-Kondrachov[3]). Let Ω be a bounded open set of \mathbb{R}^N and $p \geq 1$. If $p < N$ then for any q such that $1 \leq q \leq p^*$ (with $1/p^* = 1/p - 1/N$), the injection of $W_0^{1,p}(\Omega)$ in $L^q(\Omega)$ is continuous and for all q such that $1 \leq q < p^*$ the injection is compact, that is, the bounded of $W_0^{1,p}(\Omega)$ are relatively compact in $L^q(\Omega)$.

1.2.3 Convergence theorems

In this section, we present some definitions and results on the convergence sequences of measurable functions

Definition 1.4. Let (u_n) be a sequence of measurable functions on Ω and u a measurable function on Ω .

- The sequence (u_n) converges almost everywhere on Ω to u if and only if

$$\operatorname{meas}\{x \in \Omega : u_n(x) \text{ does not converge to } u(x)\} = 0,$$

- The sequence (u_n) is said to be Cauchy in measure if for every $\varepsilon > 0$ and every $\eta > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$, so

$$\operatorname{meas}\{x \in \Omega : |u_n(x) - u_m(x)| > \eta\} \leq \varepsilon$$

Lemma 1.5. [7] Let (u_n) be a sequence of measurable functions of in Ω and \mathbb{R}^N . If (u_n) of Cauchy in measure then there exists a sub-sequence of (u_n) converge almost everywhere

Lemma 1.6. [8] *Let f be a strictly positive measurable function. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every measurable $A \subset \Omega$, we have*

$$\int_A f dx \leq \delta \Rightarrow \text{meas}(A) \leq \varepsilon. \quad (1.2)$$

Lemma 1.7 ([2]). *Let (u_n) the sequence in $L^p(\Omega)$ with $1 < p < \infty$.*

Suppose that

- (u_n) is bounded in $L^p(\Omega)$;
- $u_n \rightarrow u$ a.e. on Ω .

Then, $u_n \rightarrow u$ in $L^q(\Omega)$, for all $1 \leq q < p$ and weakly in $L^p(\Omega)$, i.e.,

$$\lim_{n \rightarrow +\infty} \int_{\Omega} u_n v dx = \int_{\Omega} u v dx, \text{ for all } v \in L^{p'}(\Omega)$$

Theorem 1.3 (Giuseppe Vitali convergence theorem [4]). *Let $p \in [1, \infty[$ and (f_n) be a sequence of functions in $L^p(\Omega)$ such that*

- $f_n \rightarrow f$ a.e. in Ω
- (f_n) equal-integration in Ω , i.e. $\forall \varepsilon > 0, \exists \delta > 0, \forall E \subset \Omega$ with $|E| \leq \delta$, such that

$$\int_E |f_n(x)|^p dx \leq \varepsilon, \quad \forall n \in \mathbb{N}$$

Or

$$\limsup_{|E| \rightarrow 0} \int_E |f_n(x)|^p dx = 0, \quad \forall n \in \mathbb{N}$$

Then

$$f \in L^p(\Omega) \text{ and } f_n \rightarrow f \text{ in } L^p(\Omega) \text{ strongly.}$$

Theorem 1.4 (Lebesgue's Dominated Convergence Theorem[3]). *Let $p \in [1, \infty)$, and (u_n) be a sequence of functions in $L^p(\Omega)$ such that*

- $u_n \rightarrow u$ a.e in Ω

- there is a function $v \in L^p(\Omega)$ satisfying

$$|u_n(x)| \leq v(x) \quad \text{a.e in } \Omega \quad \forall n \in \mathbb{N}$$

Then $u_n \rightarrow u$ in $L^p(\Omega)$, that is

$$u \in L^p(\Omega), \quad \text{and} \quad \|u_n - u\|_{L^p(\Omega)} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

1.3 Sobolev Spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 1.5. The Sobolev space $W^{1,p}(\Omega)$, is defined by

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega), \nabla u \in (L^p(\Omega))^N \right\}$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_p + \|\nabla u\|_p \tag{1.3}$$

and for $p = \infty$

$$\|u\|_{W^{1,\infty}} = \max(\|u\|_{L^\infty}, \|\nabla u\|_{L^\infty}).$$

Definition 1.6. the space $D(\Omega)$ denotes the set of functions in $C^\infty(\Omega)$ with compact support in Ω , we define $1 \leq p < \infty$

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega), \quad \text{with} \quad u = 0 \quad \text{on} \quad \partial\Omega \right\}.$$

Remark 1.2. Since, for $1 \leq p < \infty$, the space $D(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ by definition, we can identify the dual of $W_0^{1,p}(\Omega)$ with a subspace of the space of distributions $D'(\Omega)$. we denote this as: $W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))'$

In the particular case where $p = 2$, $W^{1,2}(\Omega)$, $W_0^{1,2}(\Omega)$, $W^{-1,2}(\Omega)$, are respectively denoted $H^1(\Omega)$, $H_0^1(\Omega)$, $H^{-1}(\Omega)$

The space H^1 is a Hilbert¹ space equipped with the scalar product.

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}$$

The associated norm

$$\|u\|_{H^1(\Omega)} = (\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2)^{1/2}$$

is a norm equivalent to that of $W^{1,2}(\Omega)$

The following proposition represents the fundamental properties of the spaces $W^{1,p}$, $W_0^{1,p}$. For the proof we refer the reader to reference [3].

Proposition 1.2. • The space $W^{1,p}$ is a Banach space for $1 \leq p \leq \infty$.

- The space $W^{1,p}$ is a separable space for $1 \leq p < \infty$.
- The space $W^{1,p}$ is a reflexive space for $1 < p < \infty$.
- The space $W_0^{1,p}$ is a separable space . Also reflexive for $1 < p < \infty$.
- The space H_0^1 is a separable Hilbert space.

1.4 Sobolev injections

Sobolev injections are widely used. they provide inequalities between the norms of sobolev spaces and L^p norms. This compactness result is a powerful tool in the study of PDEs, allowing us to transition from a sobolev space to a space of Lebesgue.

Theorem 1.5 (compactness theorem). Let Ω a bounded open subset of \mathbb{R}^N with $N \geq 1$ and $1 \leq p < \infty$, any bounded subset of $W_0^{1,p}(\Omega)$ is relatively compact in $L^p(\Omega)$, this implies that for any bounded sequence in $W_0^{1,p}(\Omega)$, we can extract a sub-sequence that converges in $L^p(\Omega)$.

¹David Hilbert (1862-1943) is a German mathematician. He is often considered one of the greatest mathematicians of the 20th century.

Theorem 1.6 (continuous Injection). *Let Ω be a bounded open subset of \mathbb{R}^N with $N \geq 1$ and $1 \leq p < \infty$.*

- *If $1 \leq p < N$ then $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ with $p^* = \frac{Np}{N-p} > p$ and the injection is continuous, i.e $\exists C \geq 0$ (depending only on p, N) such that*

$$\|u\|_{L^{p^*}(\Omega)} \leq C \|u\|_{W_0^{1,p}(\Omega)}$$

- *If $p > N$ then $W_0^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{N}{p}}$. where for $\alpha > 0$, $C^{0,\alpha}$ is the set of Holder continuous functions with exponent α , defined as*

$$C^{0,\alpha} = \{u \in C(\Omega, \mathbb{R}) / \exists K \geq 0, |u(x) - u(y)| \leq K \|x - y\|^\alpha, \forall (x, y) \in \Omega^2\}$$

- *If $p = N$ then $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q \geq 1$*

Lemma 1.8 (Poincare's inequality). *Suppose that $1 \leq p < \infty$, and Ω is a bounded open set. Then there exists a constant C (depending on Ω and p) such that*

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}. \quad (1.4)$$

Moreover, by the density of $\mathcal{D}(\Omega)$ in $W_0^{1,p}(\Omega)$, the inequality (1.4) stay true for any function $u \in W_0^{1,p}(\Omega)$.

1.5 Weak convergence and (weak-*) convergence

1.5.1 Weak convergence

Definition 1.7. *Let E be a Banach space, E' its dual and $\langle \cdot, \cdot \rangle$ the duality bracket EE' .*

We say that the sequence (x_n) in E weakly converges to $x \in E$ if and only if:

$$\langle f, x_n \rangle \xrightarrow{n \rightarrow +\infty} \langle f, x \rangle, \quad \forall f \in E'.$$

we denote this as

$$x_n \rightharpoonup x \quad \text{in } E.$$

Proposition 1.3. • if $x_n \rightarrow x$ strongly then $x_n \rightharpoonup x$ in E .

• if $x_n \rightharpoonup x$ in E then $\|x_n\|_E$ is bounded.

•

$$\begin{cases} \text{If } x_n \rightharpoonup x, \text{ weakly in } E \\ \text{And } f_n \rightarrow f \text{ strongly in } E' \end{cases}$$

then $\langle f_n, x_n \rangle \rightarrow \langle f, x \rangle$ as $n \rightarrow +\infty$.

Theorem 1.7. Let E be a reflexive Banach space. Then, for any bounded sequence $(x_n)_n \subset E$, there exists a sub-sequence $(x_{n_k})_{n_k}$ that converges weakly in E .

1.5.2 (weak*) convergence

Definition 1.8. We say that the sequence (f_n) in E' weakly* converges to $f \in E'$ if and only if:

$$\langle f_n, g \rangle \xrightarrow{n \rightarrow +\infty} \langle f, g \rangle, \quad \forall g \in E$$

we denote this as

$$f_n \rightharpoonup^* f \quad \text{in } E'.$$

Proposition 1.4.

$$f_n \rightarrow f \text{ strongly} \implies f_n \rightharpoonup^* f \text{ in } E'$$

For more details, you can refer to the book[3].

Elliptic Problem with regular data

2.1 Introduction

in this chapter we study the Dirichlet problem for the Nonlinear elliptic equation in the Sobolev space of the following type

$$(P) \quad \begin{cases} Au = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded open set of \mathbb{R}^N ($N \geq 2$) with a Lipschitz boundary denoted by $\partial\Omega$, $f \in W^{-1,p'}(\Omega)$ and A is the operator given by

$$Av = -\Delta_p(v) = -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial x_i} \right), \quad v \in W_0^{1,p}(\Omega)$$

2.2 Some notions and results on operators

2.3 Monotone Operators

V denotes a real Banach space, V' denotes its topological dual of V .

Definition 2.1. *An operator $A : V \rightarrow V'$ is said to be:*

- *Monotone if* $\langle Au - Av, u - v \rangle \geq 0, \forall u, v \in V$
- *strictly monotone if* $\langle Au - Av, u - v \rangle > 0, \forall u, v \in V, u \neq v$

Example 2.1. *The operator $A : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ defined by $Au = -\Delta u$ is monotone, where $H_0^1(\Omega)$ is equipped with the norm of the gradient. Indeed, for $u, v \in H_0^1(\Omega)$,*

we have:

$$\begin{aligned} \langle Au - Av, u - v \rangle &= \int_{\Omega} \nabla(u - v) \cdot \nabla(u - v) \\ &= \|u - v\|_{H_0^1(\Omega)}^2 \\ &\geq 0. \end{aligned}$$

2.4 Bounded operator

Definition 2.2. Let V and W be two Banach spaces and let $A : V \rightarrow W$ be an operator.

We say that A is bounded if it maps every bounded set in V to a bounded set in W ; i.e

$$\forall \rho > 0, \quad \exists C_\rho > 0 : \quad A(B_V(0, \rho)) \subset B_W(0, C_\rho)$$

2.5 Hemicontnuous Operators

Definition 2.3. An operator $A : V \rightarrow W$ is said to be hemicontinuous at the point u_∞ in V if, for any sequence $\{u_n\}$ converging to u_∞ , the sequence $\{Au_n\}$ converges weakly to Au_∞ in W , in other words :

$$\forall v \in V, \quad \forall \{\lambda_n\} \subset \mathbb{R}, \quad \lambda_n \rightarrow 0, \quad A(u_\infty + \lambda_n v) \rightharpoonup Au_\infty.$$

If A is hemicontnuous at every point in V , we say that it is hemicontinuous on V . In reflexive spaces and when $W = V'$, and passing from sequential to continuous, we can define hemicontinuity on V by requiring that:

$$\forall u, v, w \in V \quad \text{the map} \quad \mathbb{R} \ni \lambda \mapsto \langle A(u + \lambda v), w \rangle \in \mathbb{R}$$

is continuous .

Example 2.2. The operator $A : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ defined by

$Au = -\Delta u = -\text{div}(\nabla u)$ is hemicontinuous. Indeed, let $u, v, w \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$, we

have :

$$\begin{aligned}\langle A(u + \lambda v), w \rangle &= \int_{\Omega} \nabla(u + \lambda v) \cdot \nabla w \\ &= \int_{\Omega} \nabla u \cdot \nabla w + \lambda \int_{\Omega} \nabla v \cdot \nabla w \\ &= a + b\lambda.\end{aligned}$$

This shows that $\lambda \rightarrow \langle A(u + \lambda v), w \rangle$ is continuous.

2.6 The p-Laplacian operator

The p-Laplacian operator is a quasi-linear second-order elliptic partial differential operator defined by

$$\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2} \nabla u), \quad 1 < p < \infty$$

this operator takes the form of a divergence when $p \neq 2$ and for $p = 2$, the p-Laplacian coincides with the usual laplacian Δ

2.7 Properties of the p-Laplacian operator

Let Ω be a bounded open subset of \mathbb{R}^N , and let the operator

$$A : V = W_0^{1,p}(\Omega) \rightarrow V' = W^{-1,p'}(\Omega)$$

defined by

$$Au = -\operatorname{div}(|\nabla u|^{p-2} \nabla u), \quad 1 < p < \infty$$

According to the green's formula, we have

$$\forall \varphi \in W_0^{1,p}(\Omega) : \langle Au, \varphi \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi$$

Proposition 2.1. *The operator $Au = -\Delta_p$ is bounded from $W_0^{1,p}(\Omega)$ in $W^{-1,p'}(\Omega)$*

Proof. From the expression of the norm in a dual space, let $\rho > 0$, for $u \in B_V(0, \rho)$, we can write:

$$\| Au \|_{V'} = \sup_{\substack{\varphi \in V \\ \|\varphi\| \leq 1}} |\langle Au, \varphi \rangle| = \sup_{\substack{\varphi \in V \\ \|\varphi\| \leq 1}} \left| \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \right|.$$

But

$$\begin{aligned} \left| \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \right| &\leq \int_{\Omega} |\nabla u|^{p-1} |\nabla \varphi| \\ &\leq \left(\int_{\Omega} |\nabla u|^p \right)^{\frac{1}{p}} \left(\int_{\Omega} |\nabla \varphi|^p \right)^{\frac{1}{p}} \\ &= \| u \|_V^{p-1} \| \varphi \|_V \\ &\leq \rho^{p-1} \end{aligned}$$

Hence $\| Au \|_{V'} \leq \rho^{p-1}$. this shows that $A(B_V(0, \rho)) \subset B_{V'}(0, \rho^{p-1})$

□

Proposition 2.2. *The operator A is hemicontinuous from V to V' .*

Proof. Let $v \in V$, and $\{\lambda_n\}$ be a real sequence converging to 0.

$$\forall g \in V, \quad \langle A(u_{\infty} + \lambda_n v), g \rangle \rightarrow \langle Au_{\infty}, g \rangle$$

we have

$$\langle Au_{\infty}, g \rangle = \int_{\Omega} |\nabla u_{\infty}|^{p-2} \nabla u_{\infty} \cdot \nabla g$$

then

$$\begin{aligned} \langle A(u_{\infty} + \lambda_n v), g \rangle &= \int_{\Omega} |\nabla(u_{\infty} + \lambda_n v)|^{p-2} \nabla(u_{\infty} + \lambda_n v) \cdot \nabla g \\ &= \int_{\Omega} |(\nabla u_{\infty} + \lambda_n \nabla v)|^{p-2} (\nabla u_{\infty} + \lambda_n \nabla v) \cdot \nabla g \end{aligned}$$

$$1. \quad |\nabla(u_{\infty} + \lambda_n \nabla v)|^{p-2} (\nabla u_{\infty} + \lambda_n \nabla v) \cdot \nabla g \rightarrow^{p,p} |\nabla u_{\infty}|^{p-2} (\nabla u_{\infty}) \cdot \nabla g$$

we can assume $|\lambda_n| \leq 1$, then

$$\begin{aligned}
 \left| |\nabla(u_\infty + \lambda_n \nabla v)|^{p-2} (\nabla u_\infty + \lambda_n \nabla v) \nabla g \right| &= |\nabla(u_\infty + \lambda_n \nabla v)|^{p-1} |\nabla g| \\
 &\leq \left[|\nabla u_\infty| + |\lambda_n| |\nabla v| \right]^{p-1} |\nabla g| \\
 &\leq \left[|\nabla u_\infty| + |\nabla v| \right]^{p-1} |\nabla g|
 \end{aligned}$$

we recall that

$$\left(\sum_{i=1}^N a_i \right)^\alpha \leq \max\{1, N^{\alpha-1}\} \sum_{i=1}^N a_i^\alpha, \quad \forall a_i \geq 0, \alpha > 0 \quad (2.1)$$

Using Young's inequality (1.3) and (2.1), we can write

$$\begin{aligned}
 \left| |\nabla(u_\infty + \lambda_n \nabla v)|^{p-2} (\nabla u_\infty + \lambda_n \nabla v) \nabla g \right| &\leq \frac{|\nabla g|^p}{p} + \frac{\left[|\nabla u_\infty| + |\nabla v| \right]^p}{p'} \\
 &\leq \frac{1}{p} |\nabla g|^p + \frac{\max(1, 2^{p-1})}{p'} \left(|\nabla u_\infty|^p + |\nabla v|^p \right).
 \end{aligned}$$

the fact that $g, v, u_\infty \in W_0^{1,p}(\Omega)$, implies that

$$\frac{1}{p} |\nabla g|^p + \frac{2^{p-1}}{p'} \left(|\nabla u_\infty|^p + |\nabla v|^p \right) \in L^1(\Omega)$$

According to Lebesgue's dominated convergence theorem

$$\lim_{n \rightarrow +\infty} \langle A(u_\infty + \lambda_n v), g \rangle = \langle A(u_\infty), g \rangle$$

Which shows the hemicontinuity of A . □

Proposition 2.3. *the operator A is coercive from V to V' .*

Proof.

$$\begin{aligned}
 \lim_{\|v\|_V \rightarrow +\infty} \frac{\langle Av, v \rangle}{\|v\|_V} &= \lim_{\|v\|_V \rightarrow +\infty} \frac{\int_\Omega |\nabla v|^p}{\|v\|_V} \\
 &= \lim_{\|v\|_V \rightarrow +\infty} \frac{\|v\|_V^p}{\|v\|_V} \\
 &= \lim_{\|v\|_V \rightarrow +\infty} \|v\|_V^{p-1} = +\infty \quad \text{car } 1 < p < +\infty.
 \end{aligned}$$

□

Proposition 2.4. *The operator A is monotone from V to V' .*

Proof. we recall that

$$\forall x, y \in \mathbb{R}^N : (|x|^{p-2}x - |y|^{p-2}y)(x - y) \geq \alpha|x - y|^p, \alpha > 0.$$

So, for all $\forall u, v \in V$

$$\begin{aligned} \langle Au - Av, u - v \rangle &= \int_{\Omega} \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \nabla(u - v) \\ &\geq \int_{\Omega} \alpha |\nabla u - \nabla v|^p \\ &\geq 0 \end{aligned}$$

□

2.8 Pseudo-monotonic Operators

Definition 2.4. *An operator $A : V \rightarrow V'$*

i) We say that A is pseudo-monotonic (in sense 1) if it is

- *for all $u_n \rightharpoonup u$ in V with $\lim_{n \rightarrow \infty} \sup \langle Au_n, u_n - u \rangle \leq 0$*

we have

$$\liminf_{n \rightarrow \infty} \langle Au_n, u_n - v \rangle \geq \langle Au_n, u - v \rangle, \quad \forall v \in V$$

ii) We say that A is pseudo-monotonic (in sense 2) if it is

- *for all $u_n \rightharpoonup u$ in V with $A(u_n) \rightharpoonup \xi$ in V' and*

$$\limsup_{n \rightarrow \infty} \langle Au_n, u_n \rangle \leq \langle \xi, u \rangle$$

We have

$$\xi = A(u) \quad \text{and} \quad \langle Au_n, u_n \rangle \rightarrow \langle Au, u \rangle$$

Proposition 2.5. *If A is bounded , then A is pseudo-monotonic in sense 1 if only if it is pseudo-monotonic in sense 2.*

Proof. Let us first assume that A is pseudo-monotonic in sens 1 (but not necessarily bounded).

Let u_n be a sequence such that $u_n \rightharpoonup u$, $A(u_n) \rightharpoonup \xi$ and

$$\limsup \langle A(u_n), u_n \rangle \leq \langle \xi, u \rangle$$

We deduce that

$$\limsup \langle A(u_n), u_n - u \rangle = \limsup (\langle A(u_n), u_n \rangle - \langle A(u_n) - u \rangle) \leq 0.$$

So, by pseudo-monotony in sense 1, it follows that for all $v \in V$,

$$\liminf \langle A(u_n), u_n - v \rangle \geq \langle A(u), u - v \rangle,$$

hence by developing the left hook,

$$\liminf \langle A(u_n), u_n \rangle - \langle \xi, v \rangle \geq \langle A(u), u - v \rangle.$$

Taking $v = u$, we deduce that $\liminf \langle A(u_n), u_n \rangle \geq \langle \xi, v \rangle$, from where

$$\langle A(u_n), u_n \rangle \rightarrow \langle \xi, u \rangle$$

and by reporting in the inequality above

$$\langle \xi, u - v \rangle \geq \langle A(u), u - v \rangle.$$

Taking $v = u + w$, we deduce that $\xi = A(u)$ and that A is pseudo-monotonic in sense 2.

Now suppose that A is pseudo-monotonic in sense 2 and bounded. Let u_n be a sequence such that $u_n \rightharpoonup u$ and $\limsup \langle A(u_n), u_n - u \rangle \leq 0$. Let $v \in V$. As A is bounded, we can extract a sequence such that

$$A(u_{n'}) \rightharpoonup \xi \quad \text{and} \quad \langle A(u_{n'}), u_{n'} - v \rangle \rightarrow \liminf \langle A(u_n), u_n - v \rangle.$$

As $\langle A(u_{n'}), u \rangle \rightarrow \langle \xi, u \rangle$, we first have $\limsup \langle A(u_{n'}), u'_n \rangle \leq \langle \xi, u \rangle$. So, by pseudo-monotonicity in sense 2, it follows that $\xi = A(u)$ and $\langle A(u'_n), u'_n \rangle \rightarrow \langle A(u), u \rangle$. We therefore see that

$$\langle A(u'_n), u'_n - v \rangle \rightarrow \langle A(u), u - v \rangle$$

So A is pseudo-monotonic in sense 1. □

Proposition 2.6. *If $A : V \rightarrow V'$ is bounded, hemicontinuous and monotonic, then A is pseudo-monotonic (in sense 1).*

Proof. Let $\{u_j\}$ be a sequence that converges weakly to u in V . Suppose that

$$\limsup_{j \rightarrow \infty} \langle Au_j, u_j - u \rangle \leq 0$$

If A is monotonic, we have

$$\lim_{j \rightarrow \infty} \langle Au_j, u_j - u \rangle \rightarrow 0 \tag{2.2}$$

Indeed, the monotonicity of A and the weak convergence of $\{u_j\}$ to u implies that

$$\langle Au_j, u_j - u \rangle \geq \langle Au, u_j - u \rangle \rightarrow 0 \quad \text{for } j \rightarrow \infty$$

And so

$$0 \geq \limsup_{j \rightarrow \infty} \langle Au_j, u_j - u \rangle \geq \liminf_{j \rightarrow \infty} \langle Au_j, u_j - u \rangle \geq \limsup_{j \rightarrow \infty} \langle Au, u_j - u \rangle = 0$$

Hence (2.2)

b) for $v \in V$ and $t \in]0, 1[$, let $w = (1-t)u + tv$. we have $\langle Au_j - Aw, u_j - w \rangle$ of fate that:

$$t \langle Au_j, u - v \rangle \geq -\langle Au_j, u_j - u \rangle + \langle Aw, u_j - u \rangle - t \langle Aw, v - u \rangle.$$

Hence, thanks to (2.2):

$$t \liminf_j \langle Au_j, u - v \rangle \geq -t \langle Aw, v - u \rangle,$$

hence , dividing by t and taking into account (2.2):

$$\liminf_j \langle Au_j, u_j - v \rangle \geq \langle Aw, u - v \rangle \quad (2.3)$$

$$w = (1 - t)u + tv \quad \forall t \in]0, 1[$$

By finishing tend t to 0 in (2.3), and using hemicontinuity, we deduce

$$\liminf_{j \rightarrow \infty} \langle Au_j, u_j - v \rangle \geq \langle Au, u - v \rangle, \quad \forall v \in V$$

Which means that A is pseudo-monotonic in sense 1 . □

2.9 Monotonic operator theorems

2.9.1 General result

Let V be a reflexive and separable Banach space, and $A : V \rightarrow V'$ be an operator.

Theorem 2.1 (Existence theorem). *Suppose that $A : V \rightarrow V'$ is an operator :*

- *bounded*
- *hemicontinuous*
- *coercive, meaning that*

$$\lim_{\|v\|_V \rightarrow +\infty} \frac{\langle Av, v \rangle}{\|v\|_V} = +\infty$$

- *monotone*

Let $f \in V'$. Then, there exists a function $u \in V$ and only one such that:

$$Au = f$$

For the proof this theorem, we refer the reader to the book [5]

Theorem 2.2. *Let $f \in W^{-1,p'}(\Omega)$. Then the problem (P) defined by*

$$(P) \quad \begin{cases} -\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = f & \text{on } \Omega; \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

has a unique solution $u \in W_0^{1,p}(\Omega)$, for all $1 < p \leq +\infty$.

Proof. According to the previous propositions, we have the principal operator

$$Au = -\operatorname{div}(|\nabla u|^{p-2} \nabla u)$$

which is coercive and bounded, hemicontinuous, monotone, therefore, according to theorem

2.1. The problem (P) has a unique solution $u \in W_0^{1,p}(\Omega)$, for all $1 < p \leq +\infty$. \square

Elliptic Problem with irregular data

In this chapter, we prove existence and regularity of weak solutions for a class of non-linear boundary value problems of elliptic type associated with p-Laplacian operator but with a non-regular second member. i.e, not belonging to a dual space. we consider second members in L^1 .

3.1 The problem (P)

We are interested in problems of the type

$$(P) \quad \begin{cases} -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = f & \text{on } \Omega; \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

where Ω is a bounded open subset of \mathbb{R}^N ($N > 2$) with boundary denoted by $\partial\Omega$ and p is a real number such that $p \geq 1$, $f \in L^1(\Omega)$.

3.1.1 Weak solutions

Definition 3.1. says that u is a weak solution to problem (P) if: $u \in W_0^{1,1}(\Omega)$ and

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx, \quad \forall \varphi \in W_0^{1,p}(\Omega). \quad (3.1)$$

The result proved in this chapter is the following theorem

Theorem 3.1. Let $2 - \frac{1}{N} < p \leq N$ and $f \in L^1(\Omega)$. Then The problem (P) has at least one weak solution u which is in $W_0^{1,q}(\Omega)$, for all

$$q \in \left[1, q_c = \frac{N}{N-1}(p-1) \right].$$

3.1.2 Model Example

Let $\Omega = B(0, 1)$, the open Euclidean unit ball centered at zero with radius 1. find $u : \bar{\Omega} \rightarrow \mathbb{R}$ as a solution of

$$(P_0) \quad \begin{cases} -\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \delta & \text{in } D'(\Omega); \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

where $p > 1$ is a real number and δ is the Dirac distribution at the origin.

the problem (P_0) has a solution (model) u_{mod} is given by (see [1]):

$$u_{mod} = \begin{cases} C(|x|^{\frac{p-N}{p-1}} - 1) & \text{if } p \neq N; \\ C \log |x| & \text{if } p = N; \end{cases}$$

3.1.3 Why the condition $p > 2 - \frac{1}{N}$

In which space -or set- should we look for a solution of (P) ?

In the case $p \neq N$, we have

$$|\nabla u_{mod}| = C_1 |x|^{\frac{p-N}{p-1}-1} = C_1 |x|^{\frac{1-N}{p-1}}$$

then

$$\begin{aligned} \int_{\Omega} |\nabla u_{mod}| dx &= C_1 \int_{B(0,1)} |x|^{\frac{1-N}{p-1}} dx \\ &= C_2 \int_0^1 r^{\frac{1-N}{p-1}} r^{N-1} dr \\ &= C_2 \int_0^1 r^{\frac{1-N}{p-1} + N-1} dr \end{aligned}$$

This integral is finite if $\frac{1-N}{p-1} + N - 1 > -1$. Thus, if $p > 2 - \frac{1}{N}$, this calculation shows that:

$$u_{mod} \text{ is in } W_0^{1,1}(B(0,1)) \text{ only if } p > 2 - \frac{1}{N}.$$

Suppose that $p > 2 - \frac{1}{N}$ and we are looking for critical value of the exponent $q \geq 1$ so that

$u_{mod} \in W_0^{1,q}(B(0,1))$ we have :

$$\int_{\Omega} |\nabla u_{mod}|^q dx = C_1 \int_0^1 r^{\frac{q(1-N)}{p-1} + N-1} dr dx.$$

Our integral will be finite if $\frac{q(1-N)}{p-1} + N - 1 > -1$, which gives

$$u_{mod} \in W_0^{1,q}(B(0,1)) \quad \text{for all } q \text{ such that } 1 \leq q < \frac{N}{N-1}(p-1).$$

Now, the proof of the Theorem 3.1 needs several steps: First, we approximate the problem (P) with sequence of problems (P_n) having smooth solutions (u_n) . Then, after deriving uniform estimates on u_n , we pass to the limit.

3.2 Approximation of (P)

Consider the following approximated problem :

$$(P_n) \quad \begin{cases} -\operatorname{div}(|\nabla u_n|^{p-2} \nabla u_n) = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega; \end{cases}$$

i.e., satisfying

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi dx = \int_{\Omega} f_n \varphi dx, \quad (3.2)$$

for every $\varphi \in W_0^{1,p}(\Omega)$, with f_n be a sequence of bounded functions defined in Ω defined by

$$f_n(x) = \frac{f(x)}{1 + \frac{|f(x)|}{n}}. \quad (3.3)$$

such that

$$\begin{cases} \|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}, \\ f_n \rightarrow f \text{ strongly in } L^1(\Omega) \end{cases} \quad (3.4)$$

The existence of a solution u_n in $W_0^{1,p}(\Omega)$ of the problem (P_n) is classical, since $f_n \in W^{-1,p'}(\Omega)$ and the fact that the operator $Au_n = -\operatorname{div}(|\nabla u_n|^{p-2} \nabla u_n)$ is coercive and is a monotone (see Theorem 2.1), i.e., there exists a function $u_n \in W_0^{1,p}(\Omega)$ solution of problem (P_n) .

3.3 Uniform Estimates

Since we only have $\|f_n\|_{L^1(\Omega)} \leq C$, $\forall n \geq 1$ in order to bound the term $\int_{\Omega} f_n \varphi$ uniformly with respect to n , we need to choose a test function

$$\varphi_n = \text{a function of } u_n \text{ such that } \|\varphi_n\|_{L^\infty(\Omega)} \leq C, \forall n \geq 1 \quad \text{with } \varphi_n \in W_0^{1,p}(\Omega)$$

To choose the test functions, we use the following result due to Stampacchia :

Lemma 3.1 (Stampacchia). *Let $T : \mathbb{R} \rightarrow \mathbb{R}$ be a globally lipschitz function, i.e.*

$$\exists C > 0 \quad \text{such that} \quad |T(s) - T(t)| \leq C|s - t|, \quad \forall s, t \in \mathbb{R},$$

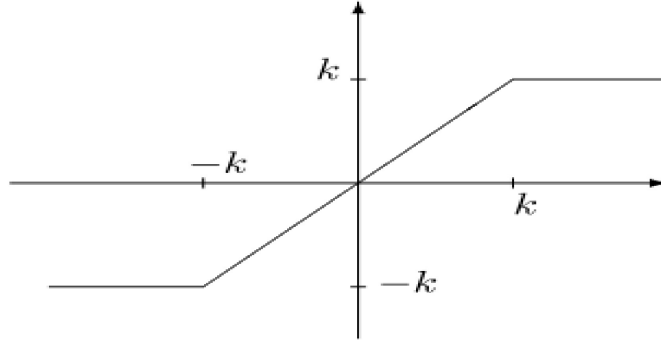
such that $T(0) = 0$. then, $\forall v \in W_0^{1,p}(\Omega)$ with $1 \leq p \leq \infty$ we have :

$T(v) \in W_0^{1,p}(\Omega)$ and $\nabla T(v) = T'(v)\nabla v$ in $\mathcal{D}'(\Omega)$ and almost everywhere in Ω

Example 3.1. *Let $k > 0$. The truncation at levels $-k$ and k is defined by the function T_k of \mathbb{R} from \mathbb{R} given by*

$$T_k(r) = \begin{cases} k, & \text{if } r \geq k, \\ r, & \text{if } |r| < k, \\ -k, & \text{if } r \leq -k. \end{cases}$$

It can be verified that the function T_k is a globally Lipschitz function, The graph of T_k is:



We can verify that the function T_k is a globally Lipschitz function satisfying $|T_k(r)| \leq k$ and $|T_k(r)| \leq |r|$.

Lemma 3.2. *Let $\delta > 0$, there exists a constant C_δ independent of n such that*

$$\int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{1+\delta}} dx \leq C_\delta, \quad \forall n \geq 1.$$

Proof. Let $\delta > 0$. the function ψ_δ given by

$$\psi_\delta(t) = \int_0^t \frac{d\sigma}{(1 + |\sigma|)^{1+\delta}} = \frac{1}{\delta} \left(1 - \frac{1}{(1 + |t|)^\delta} \right) \text{sign}(t), \quad t \in \mathbb{R}$$

is globally Lipschitz $\psi_\delta(0) = 0$ and $|\psi_\delta(t)| \leq \frac{1}{\delta}$, $\forall t \in \mathbb{R}$. Taking $\psi_\delta(u_n)$ as a test function in (P_n) , we obtain:

$$\langle Au_n, \psi_\delta(u_n) \rangle = \langle \mu_n, \psi_\delta(u_n) \rangle, \quad Au_n = -\text{div}(|\nabla u_n|^{p-2} \nabla u_n)$$

so that

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla \psi_\delta(u_n) dx = \int_{\Omega} f_n \psi_\delta(u_n) dx$$

Using the fact that $\nabla \psi_\delta(u_n) = \nabla u_n \psi'_\delta(u_n)$ with $\psi'_\delta(u_n) = \frac{1}{(1 + |u_n|)^{1+\delta}}$, we have

$$\begin{aligned} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla \psi_\delta(u_n) dx &= \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla u_n \psi'_\delta(u_n) dx \\ &= \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \frac{\nabla u_n}{(1 + |u_n|)^{1+\delta}} dx, \end{aligned}$$

Therefore,

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \frac{\nabla u_n}{(1 + |u_n|)^{1+\delta}} dx = \int_{\Omega} f_n \psi_\delta(u_n) dx \leq \frac{1}{\delta} \|f_n\|_{L^1(\Omega)},$$

then, we obtain

$$\begin{aligned} \int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{1+\delta}} dx &\leq \frac{1}{\delta} \|f_n\|_{L^1(\Omega)}, \quad \forall n \geq 1 \\ &\leq C \end{aligned}$$

□

Lemma 3.3. *Let p and q be such that*

$$2 - \frac{1}{N} < p \leq N \quad \text{and} \quad q \in [1, \frac{N}{N-1}(p-1)].$$

Then,

- the sequence (∇u_n) is bounded in $L^q(\Omega)$. i.e

there exists a constant $C = C(q) > 0$ such that :

$$\int_{\Omega} |\nabla u_n|^q dx \leq C$$

Proof. Note that

$$\frac{N}{N-1}(p-1) - p = \frac{p-N}{N-1} \leq 0 \implies 1 \leq q < \frac{N}{N-1}(p-1) \leq p \leq N.$$

Let q^* be the sobolev exponent associated with q , i.e., $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{N} \Leftrightarrow q^* = \frac{Nq}{N-q}$. For

$$\delta = q^* \frac{p-q}{q} - 1 \Leftrightarrow q^* = (1+\delta) \frac{q}{p-q}$$

Lemma 3.2 implies that there exists a constant $C = C(\delta) > 0$ such that :

$$\int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{1+\delta}} dx \leq C, \quad \forall n \geq 1$$

we can write :

$$y_n = \int_{\Omega} |\nabla u_n|^q dx = \int_{\Omega} \frac{|\nabla u_n|^q}{(1 + |u_n|)^{(1+\delta)q/p}} (1 + |u_n|)^{(1+\delta)q/p} dx.$$

Using Hölder's inequality, we see that:

$$\begin{aligned} y_n = \int_{\Omega} |\nabla u_n|^q dx &\leq \left(\int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{(1+\delta)}} \right)^{q/p} \cdot \left(\int_{\Omega} (1 + |u_n|)^{(1+\delta)\frac{q}{p-q}} \right)^{1-q/p} \\ &\leq C^{q/p} \left(\int_{\Omega} (1 + |u_n|)^{q^*} \right)^{1-q/p}. \end{aligned} \quad (3.5)$$

Using the inequality

$$(a + b)^p \leq 2^{p-1}(a^p + b^p) \quad \text{for } 1 \leq p < \infty \quad \text{and } a, b \geq 0, \quad (3.6)$$

we can write

$$\begin{aligned} y_n = \int_{\Omega} |\nabla u_n|^q dx &\leq C^{q/p} (2^{q^*-1})^{1-q/p} \left(\int_{\Omega} 1 dx + \int_{\Omega} |u_n|^{q^*} \right)^{1-q/p} \\ &\leq C^{q/p} (2^{q^*-1})^{1-q/p} (2^{-q/p}) \left\{ |\Omega|^{1-q/p} + \left(\int_{\Omega} |u_n|^{q^*} \right)^{1-q/p} \right\}. \end{aligned}$$

But, by the sobolev Poincare inequality, we have:

$$\left(\int_{\Omega} |u_n|^{q^*} \right)^{1/q^*} \leq C_{q,N} \|\nabla u_n\|_{L^q(\Omega)} = C_{q,N} y_n^{1/q}.$$

Therefore,

$$y_n \leq a + b y_n^{\frac{q^*}{q}(1-\frac{q}{p})} \quad (3.7)$$

where a and b are constants depending on q, N and $|\Omega|$. Since, in the case $p < N$, we have:

$$\begin{aligned} \frac{q^*}{q} \left(1 - \frac{q}{p}\right) - 1 &= \frac{Nq}{N-q} \frac{p-q}{qp} - 1 \\ &= \frac{N(p-q)}{(N-q)p} - 1 \\ &= \frac{qp - N}{p(N-q)} < 0. \end{aligned}$$

that is to say $\frac{q^*}{q} \left(1 - \frac{q}{p}\right) < 1$, then the previous inequality (3.7), satisfied by y_n , shows that this sequence is bounded.

In the case $p = N$, we repeat the calculation, but choosing $\delta = \alpha \frac{N-q}{q} - 1$ with $\max\{1, \frac{q}{N-q}\} < \alpha < q^*$. We use the continuity of the injection from $W_0^{1,q}(\Omega)$ in $L^\alpha(\Omega)$ and we obtain

$$y_n \leq a + b y_n^{\frac{\alpha}{q}(1-\frac{q}{N})}, \quad \forall n \geq 1 \quad \text{with} \quad \frac{\alpha}{q} \left(1 - \frac{q}{N}\right) < \frac{q^*}{q} \left(1 - \frac{q}{p}\right) = 1$$

This implies that y_n is also bounded in \mathbb{R} in this case. \square

3.4 passage to the limit

Proposition 3.1. *For all $q \in [1, q_c = \frac{N}{N-1}(p-1)[$ and since $(u_n)_n$ is bounded in the space $W_0^{1,q}(\Omega)$, according to Rellich-Kondrachov theorem, we can extract from the sequence (u_n) a sub-sequence, also denote by (u_n) such that*

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,q}(\Omega), \quad (3.8)$$

$$u_n \longrightarrow u \quad \text{strongly in } L^q(\Omega). \quad (3.9)$$

Therefore

$$u_n \longrightarrow u \quad \text{a.e in } \Omega, \quad (3.10)$$

Proof. For the proof of this proposition, we can refer the reader to the book [3] □

Now, we have the operator $-\Delta_p$ depends on the gradient ∇u , so we need more convergence on the gradients ∇u_n .

Lemma 3.4. *The sequence (∇u_n) converges almost everywhere to ∇u (possibly up to a sub-sequence).*

Proof. We show that (∇u_n) is Cauchy in measure, which implies that $\nabla u_n \longrightarrow \nabla u$ almost everywhere, for a sub-sequence. This consists of proving that

$$\forall \delta > 0, \forall \varepsilon > 0, \exists n_0 \text{ such that } \forall p, q \geq n_0 \quad \text{meas}\{x \in \Omega \mid |(\nabla u_p - \nabla u_q)(x)| \geq \delta\} \leq \varepsilon$$

To do this, let us fix $\delta > 0$ and $\varepsilon > 0$, and note that for $\lambda > 0$ and $\eta > 0$, we have

$$\{x \in \Omega \mid |(\nabla u_p - \nabla u_q)(x)| \geq \delta\} \subset E_1 \cup E_2 \cup E_3 \cup E_4$$

where

$$E_1 = \{x \in \Omega \mid |\nabla u_p| \geq \lambda\}, \quad E_2 = \{x \in \Omega \mid |\nabla u_q| \geq \lambda\}$$

$$E_3 = \{x \in \Omega \mid |u_p - u_q| \geq \eta\}$$

and

$$E_4 = \{|\nabla u_p - \nabla u_q| \geq \delta, |\nabla u_p| \leq \lambda, |\nabla u_q| \leq \lambda, |u_p - u_q| \leq \eta\}.$$

And since the sequence (∇u_n) remains in a bounded of $L^q(\Omega)$ for all $q \in [1, p - \frac{N}{N+1}]$ [See lemma 3.3], by choosing λ large, we can make $\text{meas}(E_1)$ and $\text{meas}(E_2)$ arbitrarily small.

For example

$$\text{meas}(E_1) = \int_{E_1} 1 dx = \frac{1}{\lambda} \int_{E_1} \lambda dx \leq \frac{1}{\lambda} \int_{E_1} |\nabla u_p| dx \leq \frac{1}{\lambda} \int_{\Omega} |\nabla u_p| dx \leq \frac{C}{\lambda} \leq \varepsilon.$$

Then,

$$\text{meas}(E_1) \rightarrow 0 \quad \text{if} \quad \lambda \rightarrow +\infty,$$

hence

$$\text{meas}(E_1) \leq \varepsilon.$$

for $\text{meas}(E_3)$, we have

$$\text{meas}(E_3) \leq \frac{1}{\eta} \int_{E_3} |u_p - u_q| dx \leq \frac{1}{\eta} \int_{\Omega} |u_p - u_q| dx$$

Since (u_n) is a Cauchy sequence in $L^1(\Omega)$ [see (3.9)], then for fixed $\eta > 0$, we see that

$$\text{meas}(E_3) \rightarrow 0 \quad \text{if} \quad p, q \rightarrow +\infty$$

hence

$$\text{meas}(E_3) \leq \varepsilon.$$

It remains to control $\text{meas}(E_4)$. According to the monotonicity of the function

$$a : (x, \nabla u) \mapsto a(x, \nabla u) = |\nabla u|^{p-2} \nabla u \quad (\text{see chapter 2}),$$

we have $\left(a(x, \xi_1) - a(x, \xi_2)\right)(\xi_1 - \xi_2) > 0$ and the set

$$\{(\xi_1, \xi_2) \mid |\xi_1| \leq \lambda, |\xi_2| \leq \lambda, |\xi_1 - \xi_2| \leq \delta\}$$

is a compact set and a is continuous in ξ , then $\left(a(x, \xi_1) - a(x, \xi_1)\right)(\xi_1 - \xi_2)$ attains its minimum on this compact set, which we will denote as $\mu(x)$ such that

$$(a(x, \xi_1) - a(x, \xi_2))(\xi_1 - \xi_2) \geq \mu(x) > 0.$$

Therefore, by (1.2) for any $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that

$$\int_{E_4} \mu(x) dx < \varepsilon' \Rightarrow \text{meas}(E_4) < \varepsilon. \quad (3.11)$$

To obtain $\text{meas}(E_4) < \varepsilon$, it suffices to show that $\int_{E_4} \mu(x) dx < \varepsilon'$. Using the definition of $\mu(x)$ and E_4 , we can write

$$\int_{E_4} \mu(x) dx \leq \int_{E_4} \left(|\nabla u_p|^{p-2} \nabla u_p - |\nabla u_q|^{p-2} \nabla u_q \right) \nabla(u_p - u_q) \mathbf{1}_{\{|u_p - u_q| \leq \eta\}} dx.$$

Moreover, the integral term is positive and $\nabla T_\eta(u_p - u_q) = \nabla(u_p - u_q) \mathbf{1}_{\{|u_p - u_q| \leq \eta\}}$, so we have

$$\int_{E_4} \mu(x) dx \leq \int_{E_4} \left(|\nabla u_p|^{p-2} \nabla u_p - |\nabla u_q|^{p-2} \nabla u_q \right) \nabla T_\eta(u_p - u_q) dx. \quad (3.12)$$

Where T_η is the truncation function at levels η and T'_η is defined by

$$T'_\eta(\sigma) = \begin{cases} 1, & |\sigma| \leq \eta; \\ 0, & |\sigma| > \eta. \end{cases}$$

By taking $T_\eta(u_p - u_q)$ as a test function in (P_n) for u_p and u_q , we have

$$\int_{\Omega} |\nabla u_p|^{p-2} \nabla u_p \nabla T_\eta(u_p - u_q) dx = \int_{\Omega} f_n T_\eta(u_p - u_q) dx \quad (3.13)$$

and

$$\int_{\Omega} |\nabla u_q|^{p-2} \nabla u_q \nabla T_\eta(u_p - u_q) dx = \int_{\Omega} f_n T_\eta(u_p - u_q) dx. \quad (3.14)$$

Then, subtracting the inequality resulting from (3.13) and (3.14), we obtain

$$\int_{\Omega} (|\nabla u_p|^{p-2} \nabla u_p - |\nabla u_q|^{p-2} \nabla u_q) \nabla T_\eta(u_p - u_q) dx = 0.$$

So, we can write that

$$\begin{aligned} & \int_{\Omega} (|\nabla u_p|^{p-2} \nabla u_p - |\nabla u_q|^{p-2} \nabla u_q) \nabla T_\eta(u_p - u_q) dx \\ & \leq \eta \rightarrow^{\eta \rightarrow 0} 0 \text{ (uniformly in } p \text{ and } q). \end{aligned} \quad (3.15)$$

For η sufficiently small, (3.12) and (3.15) imply

$$\int_{E_4} \mu(x) dx < \varepsilon',$$

and also by (3.11) we have $meas(E_4) \leq \varepsilon$. Therefore for all $p, q \geq n_0$ we have

$$meas\{x \in \Omega \mid |(\nabla u_p - \nabla u_q)(x)| \geq \delta\} \leq 4\varepsilon.$$

Thus, we have the convergence of ∇u_n to ∇u in measure, as well as the lemma 1.5 (after extraction of a sequence)

$$\nabla u_n \rightarrow \nabla u \quad \text{almost everywhere in } \Omega.$$

□

Now let us fix $\varphi \in W_0^{1,p}(\Omega)$. We have

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi = \int_{\Omega} f_n \varphi.$$

1) Limit Passage in $\int_{\Omega} f_n \varphi$. We have :

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f_n \varphi = \langle f, \varphi \rangle$$

2) Limit Passage in $\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi dx$.

As $\nabla u_n \rightarrow \nabla u$ almost everywhere in Ω , we have:

$$|\nabla u_n|^{p-2} \nabla u_n \rightarrow |\nabla u|^{p-2} \nabla u \quad \text{almost everywhere in } \Omega. \quad (3.16)$$

Since

$$\frac{N}{N-1}(p-1) - (p-1) = (p-1) \left(\frac{N}{N-1} - 1 \right) = \frac{p-1}{N-1} > 0$$

So that

$$\frac{N}{N-1}(p-1) > p-1$$

Now, for any measurable subset for $E \subset \Omega$ we have :

$$\begin{aligned} \int_E || \nabla u |^{p-2} \nabla u \nabla \varphi| dx &\leq \| \nabla \varphi \|_{L^\infty(E)} \int_E |\nabla u_n|^{p-1} dx \\ &\leq \| \nabla \varphi \|_{L^\infty(\Omega)} \int_E |\nabla u_n|^{p-1} dx \\ &\leq C \int_E |\nabla u_n|^{p-1} dx \end{aligned}$$

We can choose $q > p - 1$ and according to Hölder inequality, we have :

$$\begin{aligned} \int_E || \nabla u_n |^{p-2} \nabla u_n \nabla \varphi| dx &\leq C \left(\int_E |\nabla u_n|^q dx \right)^{(p-1)/q} \cdot \left(\int_E 1 dx \right)^{1-(p-1)/q} \\ &\leq C \left(\int_E |\nabla u_n|^q dx \right)^{(p-1)/q} \cdot |E|^{1-(p-1)/q} \end{aligned}$$

Using lemma 3.3 we have (∇u_n) is bounded in $L^q(\Omega)$, we have :

$$\int_E || \nabla u_n |^{p-2} \nabla u_n \nabla \varphi| dx \leq C |E|^{1-(p-1)/q} = C'$$

This proves the equi-integrability of the sequence $(| \nabla u_n |^{p-2} \nabla u_n \nabla \varphi)_n$. It follows from Vitali's convergence theorem 1.3 that

$$\lim_{n \rightarrow +\infty} \int_\Omega | \nabla u_n |^{p-2} \nabla u_n \nabla \varphi dx = \int_\Omega | \nabla u |^{p-2} \nabla u \nabla \varphi dx$$

It has therefore been proven that

$$\int_\Omega | \nabla u |^{p-2} \nabla u \nabla \varphi = \langle \mu, \varphi \rangle, \quad \forall \varphi \in D(\Omega)$$

Then u is a solution of the problem

$$\begin{cases} u \in W_0^{1,q}(\Omega), & q \in [1, \frac{N}{N-1}(p-1)[\\ Au = f & \text{ on } \Omega; \end{cases}$$

We therefore have to prove that u is a solution to problem (P). This finishes the proof of Theorem 3.1.

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Abstract

in this work, we prove the existence and regularity of a weak solutions of elliptic problem (P) defined by

$$(P) \quad \begin{cases} Au = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

the operator $Au = -\operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $1 < p < \infty$ is a pseudo-monotone operator, $f \in L^1(\Omega)$. The method of solving our problem consist of obtaining local estimates for suitable approximate problems and then passing to the limit.

keywords: Sobolev spaces, pseudo-monotone, operator nonlinear, elliptic equation, weak solution.

Résumé

Dans ce travail, nous prouvons l'existence et la régularité d'une solution faible d'un problème elliptique non régulière définie par

$$(P) \quad \begin{cases} Au = f & \text{dans } \Omega; \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

L'opérateur $Au = -\operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $1 < p < \infty$ est un opérateur pseudo-monotone, $f \in L^1(\Omega)$. Les étapes principales de la preuve consister à approcher par une suite de problèmes à donnée dans L^∞ , ensuite obtenir des estimations uniformes et locales pour la suite des solutions approchées u_n et ∇u_n , puis le passage à la limite.

mots-clés: Espace de Sobolev, pseudo-monotone, opérateur non linéaire, équation elliptique, solution faible.

ملخص

في هذه المذكرة نسعى الى اثبات وجود وصقالة الحلول الضعيفة و للمعادلة التناقصية وغير الخطية المعرفة كما يلي: $Au = f$ تحت الشرط التالي $u = 0$ على الحافة حيث ان الطرف الايمن f ينتمي الى الفضاء $L^1(\Omega)$ المؤثر $1 < p < \infty$ $Au = -\text{div}(|\nabla u|^{p-2} \nabla u)$ معرف بشكل جيد بين الفضاء $W_0^{1,p}$ وفضاءه الثنوي، الخطوات الرئيسية للاثبات تتمحور فيمايلي:

نقرب الجملة (P) بمتتالية جمل (P_n) ثم نبرهن ان الجملة المقربة تتمتع بحل ضعيف محدود u_n ثم الحصول على التقديرات لمتتالية الحلول التقريبية u_n و ∇u_n . وفي الاخير نمر بالنهاية في المسألة التقريبية للحصول على حل للجملة (P) كلمات مفتاحية: معادلات تناقصية غير خطية ، مؤثر غير خطي، فضاء سوبولاف، حل ضعيف.