DEMOCRATIC AND POPULAR REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Mohamed Boudiaf University of Msila
Faculty of Mathematics and Computer Science
Department of Mathematics

Master's Dissertation

Field : Mathematics and Computer Science

Branch : Mathematics

Option : Numerical and Mathemtical Analysis

Theme

A Study of an Inverse Problem for a Parabolic Equation of Fractional Order with Non Local Boundary Condition

Presented by :

Dahmani Mouna

Publicly defended on June 11, 2026 in front of the jury composed of :

Noui Djaidja	MCA,	University of Msila	**President,**
Toufik Heraiz	MCA,	University of Msila	**Supervisor,**
Amina Khirani	MCA,	University of Msila	**Examiner.**

University year 2025/2026

Acknowlegments

First and foremost, I wish to express my gratitude to Allah for giving me the courage stength and patience to complete this work. I would like to deeply thank my research supervisor, Dr. **Toufik Heraiz** for the honor of supervising my work and for guiding me throughout my research. I am immensely grateful for his valuable remarks, guidance, continuos support, and encouragement in this year. I warmely thank Dr. **Noui Djaidja** for agreeing to preside over the commitee of this dissertation, I am also highly grateful to Dr. **Amina Khirani** for accepting to praticipate in this committee and examine my work. My special thanks go to my parents and my entire family for their endless motivation and for supporting me both physically and spiritually throughout my life. Finally, I would like to thank all the teachers who have taught me throughout my life, as well as my friends and everyone who supported me during my academic journey

Table of contents

Acknowledgments **i**

Introduction **1**

1 Basic Properties **3**

1.1 Special functions 3

1.1.1 Gamma function 3

1.1.2 Beta function 5

1.1.3 Relation between Gamma and Beta 6

1.1.4 The function of Mittag-Leffler 7

1.2 Fractional derivative 7

1.2.1 Fractional derivative within the meaning of Rieman-Liouville 7

1.2.2 Fractional derivative in the sense of Caputo 9

1.2.3 Some other Types of fractional derivatives 11

1.2.4 Laplace Transform of the Caputo Derivation 12

1.3 Function Spaces 13

1.3.1 The Space of Continuous Functions $C(\Omega_T)$ 13

1.3.2 Space of Functions with Spatial Derivatives $C_x^2(0,1)$ 13

1.3.3 The Space W(0,T) and Sobolev spaces 14

1.3.4 Fractional Sobolev Spaces 14

2 Spectral Theory and Sturm–Liouville Problem **16**

2.1 Classical Sturm–Liouville Theory 17

2.1.1 Regular Sturm–Liouville problem 17
2.1.2 Lagrange identity and self-adjointness 18
2.2 Nonlocal Sturm–Liouville Problem 20
2.2.1 Adjoint problem and biorthogonal system 22
2.2.2 Construction of a Riesz basis 24
2.2.3 Expansion and spectral representation 25

3 Inverse Problem of Parabolic Equation of Fractional Order 28
3.1 Ordinary Differential Equation of Fractional order 28
3.2 Analysis of a time-dependent inverse coefficient problem 29
3.2.1 Direct Problem, Existence and Uniquness 29
3.3 Transformation to Homogeneous Boundary Data 30
3.4 Solution of the Direct Problem via Eigenfunction Expansion 31
3.4.1 Derivation of the Modal Equations 33
3.4.2 Solving the Fractional ODEs 33
3.4.3 Series Solution and Convergence 34
3.5 The Input–Output Mapping 35
3.5.1 Formal Derivation of the Observation Functional 35
3.5.2 Definition and Properties of the Operator 35
3.5.3 Stability Estimate for the Time-Dependent Component 36
3.6 Distinguishability and Uniqueness 37
3.6.1 Main Distinguishability Lemma 38
3.6.2 Uniqueness Theorem for the Inverse Source Problem 39
3.6.3 Injectivity of the Input–Output Map 39
3.6.4 Identification of the Fractional Order 40
3.7 Numerical Method 41
3.7.1 Finite Difference Discretization 41
3.7.2 Example 43

Conclusion 46

Bibliography 46

Notations

$\Gamma(.)$	The Gamma function
$\beta(.,.)$	The Beta function
$E_\alpha(.)$	The function of Mittag-Leffler
$D^\alpha f$	The fractional derivative of the function f
${}^{RL}D^\alpha f$	The fractional derivative in the sense of Riemann-Liouville of the function f of order α
${}^{C}D^\alpha f$	The fractional derivative in the Caputo sense of the function f of order α
${}_aD_t^\alpha f(t)$	TheLeft-sided Riemann-Liouville derivatives
${}_a^CD_t^\alpha f(t)$	The left-sided Caputon fractional derivative
${}_tD_b^\alpha f(t)$	The right-sided Riemann-Liouville derivatives
${}_t^CD_b^\alpha f(t)$	The right-sided Caputon fractional derivative
${}_aR_b^\alpha f$	The fractional Riesz derivative of the function f
${}^{RL}D^\alpha f$	Fractional derivative in Riemann-Liouville type of order α
$C(\Omega_T)$	The Space of Continuous Functions
$W^{s,p}(\Omega)$	Fractional Sobolev Spaces
$H^s(\Omega)$	Sobolev Spaces
$C_x^2(0,1)$	Space of Functions with Spatial Derivatives

Introduction

The mathematical framework of fractional calculus, which extends traditional differentiation and integration to non-integer orders, has its conceptual roots in the 19th-century works of mathematicians such as Riemann and Liouville. In recent years, fractional partial differential equations (FPDEs) have become a vital tool in applied mathematics and physics, particularly for modeling phenomena that exhibit memory effects and complex hereditary properties, including anomalous subdiffusion. Within this field, inverse problems specifically, reconstructing unknown system parameters such as time-dependent sources from measured output data present significant challenges. When fractional parabolic equations are coupled with nonlocal boundary conditions, the mathematical analysis becomes highly complex. Classical self-adjoint spectral theory is often insufficient because the standard orthogonality of eigenfunctions may be lost, necessitating the adoption of biorthogonal systems and specialized basis constructions. While inverse problems for the classical diffusion equation have been extensively studied (see, e.g., [9, 11, 21, 31]), the study of inverse problems for fractional differential equations has emerged much more recently than the analysis of forward fractional problems. Most existing work focuses on fractional-order generalizations of classical inverse problems for parabolic equations, typically involving a subdiffusion model where the standard time derivative $\frac{\partial U}{\partial t}$ is replaced by $\partial_t^\alpha U$ with $0 < \alpha \leq 1$, using the Djrbashianâ-Caputo fractional derivative to naturally accommodate the initial condition $u(x, 0)$. Comprehensive discussions appear in the subdiffusion chapter of Kaltenbacher and Rundell (see [23, Chapter 10]), with additional references found in the tutorial paper [22]. Furthermore, Ashyralyev et al. [4, 5, 6] developed finite difference methods for fractional parabolic and hyperbolic equations under various conditions. This thesis addresses a rigorous treatment of an inverse problem for a one-dimensional time-fractional parabolic equation

with nonlocal boundary conditions as states in [28] in which the authors focusing on recovering an unknown time-dependent source parameter from boundary observations through a rigorous spectral analysis for fractional parabolic operators with nonlocal Samarskii Ionkin-type boundary conditions, a proof of distinguishability and uniqueness for the inverse source problem and a convergent finite difference scheme for numerical approximation. The dissertation is organized as follows. **Chapter 1** establishes the fundamental mathematical tools of fractional calculus, including the Gamma, Beta, and Mittag-Leffler functions, the Riemann-Liouville and Caputo derivatives, and the continuous and fractional Sobolev spaces required for rigorous solution definitions. **Chapter 2** develops the spectral theory of linear differential operators, contrasting classical regular Sturm-Liouville problems with nonlocal boundary value problems. Since nonlocal problems are generally non-self-adjoint, this chapter demonstrates the construction of an adjoint problem, the establishment of a biorthogonal system, and the modification of eigenfunctions to form a valid Riesz basis for spatial expansion. **Chapter 3** applies this theoretical framework to the inverse problem, transforming the system to feature homogeneous boundary conditions, solving the direct problem via eigenfunction expansion, constructing the input-output mapping, and proving distinguishability and uniqueness. A numerical finite difference method is also provided.

Chapter 1

Basic Properties

1.1 Special functions

1.1.1 Gamma function

The Gamma function is one of the basic functions in fractional calculus, it is also called Euler's Gamma function

Definition 1.1.1 *The Gamma function* $\Gamma(z)$ *is defined by the integral* $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$

Example 1.1.1 *Let's calculate* $\Gamma(2)$

$$\begin{aligned}\Gamma(2) &= \int_0^\infty e^{-t}tdt \\ &= \left[-te^{-t} - e^{-t}\right]_0^\infty \\ &= 1\end{aligned}$$

Proposition 1.1.1 *[33]*

1) $\Gamma(z+1) = z\Gamma(z)$

2) For every natural number n *:* $\Gamma(n+1) = n!$

3) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Proof. *1)Representations* $\Gamma(z+1)$ *by Euler's integral and let's integrate in parts*

$$\Gamma(z+1) = \int_0^{\infty} t^z e^{-t} dt \tag{1.1.1}$$

$$= [-t^z e^{-t}]_0^{\infty} + z\int_0^{\infty} t^z e^{z-1} dt \tag{1.1.2}$$

$$= z\Gamma(z) \tag{1.1.3}$$

2)It's sufficient to apply property 1 for $z = n$

$$\Gamma(n+1) = \int_0^{+\infty} t^n e^{-t} dt$$

Let's integrate by part n *times, we get*

$$\Gamma(n+1) = n(n-1)(n-2).....1$$

$$\Gamma(n+1) = n!$$

3)

$$\Gamma(\frac{1}{2}) = \int_0^{+\infty} t^{-\frac{1}{2}} e^{-t} dt$$

We put $t = x^2, dt = 2xdx$, *So*

$$\Gamma(\frac{1}{2}) = 2\int_0^{+\infty} \frac{1}{x} e^{-x^2} x dx$$

$$= 2\int_0^{+\infty} e^{-x^2} dx$$

$$[\Gamma(\frac{1}{2})]^2 = 4\int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)} dxdy$$

$$= 4\int_0^{\frac{\pi}{2}}\int_0^{+\infty} e^{-r^2} drd\theta = \pi$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

■

Notation 1.1.1 *The Gamma function does not have simple poles for point* $z = 0; -1; -2; -3; ...$

$\Gamma(0_+) = +\infty, \ \ \Gamma(1) = 1$

$\Gamma(n)$ *Decreasing function for* $0 < n \leq 1$

1.1.2 Beta function

for all $(u, \upsilon) \in \mathbb{C} \times \mathbb{C}$, Where $\mathrm{Re}(u) > 0; \mathrm{Re}(\upsilon) > 0$, We defined the Beta function as follows[17]

$$\beta(u, \upsilon) = \int_0^1 t^{u-1}(1-t)^{\upsilon-1} dt \tag{1.1.4}$$

$$\begin{aligned} \beta(1,2) &= \int_0^1 (1-t)dt && (1.1.5)\\ &= t]_0^1 - \frac{1}{2}t^2]_0^1 && (1.1.6)\\ &= 1 - \frac{1}{2} && (1.1.7)\\ \beta(1,2) &= \frac{1}{2} && (1.1.8) \end{aligned}$$

For each $(u, \upsilon) \in \mathbb{C} \times \mathbb{C}$, Where $\mathrm{Re}(u) > 0; \mathrm{Re}(\upsilon) > 0$

a) $\beta(u, \upsilon) = \beta(\upsilon, u)$

b) $\beta(u, \upsilon + 1) = \dfrac{\upsilon}{u}\beta(u + 1, \upsilon)$

c) $\beta(u + 1, \upsilon) = \dfrac{u}{u + \upsilon}\beta(u, \upsilon)$

d) $\beta(u, n + 1) = \dfrac{n}{u(u + 1)(u + 2)...(u + n)}, \ \ n \in \mathbb{N}^*$

e) Integral can be found using the Beta and Gamma function

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^{2u-1}\theta \cos^{2\upsilon-1}\theta d\theta &= \frac{1}{2}\beta(u, \upsilon) && (1.1.9)\\ \int_0^{\frac{\pi}{2}} \frac{t^{p-1}}{1+t} dt &= \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi} \quad 0 < p < 1 && (1.1.10) \end{aligned}$$

1.1.3 Relation between Gamma and Beta

Theorem 1.1.1 *The relationship between the Gamma and Beta function is given by*

$$\beta(u,\upsilon)=\frac{\Gamma(u)\Gamma(\upsilon)}{\Gamma(u+\upsilon)},\quad \forall u,\upsilon>0 \tag{1.1.11}$$

Proof. Let $H=[0,+\infty[\times[0,+\infty[$, We have

$$\begin{aligned}\Gamma(u)\Gamma(\upsilon) &= \int_0^{\infty}e^{-x}x^{u-1}dx\int_0^{\infty}e^{-y}y^{\upsilon-1}dy,\\ &= \int_0^{\infty}\int_0^{\infty}e^{-(x+y)}x^{u-1}y^{\upsilon-1}dxdy;\end{aligned}$$

Let's put; $y=p-x; dy=dp$

$$\begin{aligned}\Gamma(u)\Gamma(\upsilon) &= \int_0^{\infty}\int_0^{p}e^{-p}x^{u-1}(p-x)^{\upsilon-1}dxdp;\\ \Gamma(u)\Gamma(\upsilon) &= \int_0^{\infty}e^{-p}\int_0^{p}x^{u-1}(p-x)^{\upsilon-1}dxdp;\end{aligned}$$

We put; $x=tp; dx=pdt$,

$$\begin{aligned}\Gamma(u)\Gamma(\upsilon) &= \int_0^{\infty}e^{-p}\int_0^{1}t^{u-1}p^{u-1}(1-t)^{\upsilon-1}p^{\upsilon}dtdp;\\ &= \int_0^{\infty}e^{-p}p^{u+\upsilon-1}dp\int_0^{\infty}t^{u-1}(1-t)^{\upsilon-1}dt,\\ \Gamma(u)\Gamma(\upsilon) &= \Gamma(u+\upsilon)\beta(u,\upsilon)\end{aligned}$$

Therefore

$$\beta(u,\upsilon)=\frac{\Gamma(u)\Gamma(\upsilon)}{\Gamma(u+\upsilon)}$$

■

Example 1.1.2 *We calculate* $\beta(2,1)$

$$\beta(2,1)=\frac{\Gamma(2)\Gamma(1)}{\Gamma(2+1)}=\frac{0!1!}{2!}=\frac{1}{2} \tag{1.1.12}$$

1.1.4 The function of Mittag-Leffler

The Mittag-Leffler function plays an important role in the theory of integer differential equations. It is also widely used in the search for solutions for fractional order differenatial equations, this functions was introduced by G.M. Mittag-Leffler[27]

Definition 1.1.2 *[30]for $z \in \mathbb{C}$, the function of Mittag-Leffler $E_\alpha(z)$ is defined by*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0 \tag{1.1.13}$$

The two-parameter Mittag-Leffler function also plays a very important role in fractional computing theory. The latter was introduced by Wiman [36] and is defined by the following serial developpement

$$E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0$$

$E_0(z) = \frac{1}{1-z}, \quad |z| < 1.$

$E_1(z) = E_{1,1}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{+\infty} \frac{z^k}{k!} = e^z.$

$E_2(z) = E_{2,1}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(2k+1)} = \sum_{k=0}^{+\infty} \frac{(\sqrt{z})^{2k}}{(2k)!} = \cosh\sqrt{z}.$

$E_{2,2}(z) = \sum_{k=0}^{+\infty} \frac{z^{2k}}{\Gamma(2k+2)} = \frac{1}{z} \sum_{k=0}^{+\infty} \frac{z^{2k}}{(2k+1)!} = \frac{1}{z} \sinh z.$

1.2 Fractional derivative

1.2.1 Fractional derivative within the meaning of Rieman-Liouville

Definition 1.2.1 *On the interval $[a, b]$, $T > 0$, The function $f \in L^1([0,1]), T > 0$, the fractional derivative in the sense of Riemann-Liouville of the function f of order $\alpha \in \mathbb{C}$ $(\mathrm{Re}(\alpha) > 0)$, denoted ${}^{RL}D^\alpha f$ is defined by*

$$\begin{aligned} {}^{RL}D^\alpha f(x) &= D^n I^{n-\alpha} f(x) && (1.2.1) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t) dt && (1.2.2) \end{aligned}$$

Proposition 1.2.1 *Let $\alpha, \beta > 0$ such that $n-1 < \alpha \leq n$,$m-1 < \beta \leq m$ such that $n, m \in \mathbb{N}$*

If $\alpha > \beta > 0$, Then for $f \in L^1([0,T])T\rangle 0$, The relation

$$D^{\beta}(I^{\alpha} f(x)) = I^{\alpha-\beta} f(x) \tag{1.2.3}$$

is true almost everywhere on $t \in [0,T]$

If $\beta > \alpha > 0$ and the fractional derivative $D^{\beta-\alpha} f$ exists, Then we have

$$D^{\beta}(I^{\alpha} f(x)) = D^{\beta-\alpha} f(x)$$

For $\alpha > 0, k \in \mathbb{N}^*$, If the fractional derivatives $D^{\alpha} f$ and $D^{k+\alpha}$ f exist,Then

$$D^{k}(D^{\alpha} f(x)) = D^{k+\alpha} f(x) \tag{1.2.4}$$

Example 1.2.1 *Let $f(x) = x^2$, We calculate the derivative of order $\frac{1}{2}$ of f*
We have the definition

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t) dt$$

For $\alpha = \frac{1}{2}$ therefore $n = 1$ we find

$$D^{\frac{1}{2}} f(x) = \frac{1}{\Gamma(\frac{1}{2})} \frac{d^1}{dx^1} \int_0^x (x-t)^{-\frac{1}{2}} t^2 dt$$

We set $I = \int_0^x (x-t)^{-\frac{1}{2}} t^2 dt$, *and according to the integral by parts twice, We obtain*

$$\begin{aligned} I &= \left[(-2)t^2(x-t)^{\frac{1}{2}}\right]_0^x + \int_0^x 4t(x-t)^{\frac{1}{2}}dt \\ &= 4\int_0^x t(x-t)^{\frac{1}{2}}dt \\ &= \left[4.(-\frac{2}{3}).t(x-t)^{\frac{3}{2}}\right]_0^x + \frac{8}{3}\int_0^x (x-t)^{\frac{3}{2}}dt \\ &= \frac{8}{3}\int_0^x (x-t)^{\frac{3}{2}}dt \\ &= \left[\frac{8}{3}.(-\frac{2}{5})\,(x-t)^{\frac{5}{2}}\right]_0^x \\ &= \frac{16}{15}x^{\frac{5}{2}} \end{aligned}$$

Therefore

$$D^{\frac{1}{2}} f(x) = \frac{1}{\Gamma(\frac{1}{2})}\frac{d^1}{dx^1}(\frac{16}{15}x^{\frac{5}{2}}) \tag{1.2.5}$$

$$= \frac{8}{3\sqrt{\pi}}x^{\frac{3}{2}} \tag{1.2.6}$$

1.2.2 Fractional derivative in the sense of Caputo

Definition 1.2.2 *The fractional derivative in the Caputo sense of a function* f *given on the interval* $[a,b]$ *and* $n \in \mathbb{N}$ *is defined by the following relation*

$${}^cD_t^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}dt,\ [\alpha] = n-1 \tag{1.2.7}$$

Notation 1.2.1 *The main advantage of Caputo 's approach is that the initial conditions of thz fractional derivative in the Caputo sense of fractional differential equations take the same form as in the case of integer order differential equations*

Lemma 1.2.1 *The fractional derivative of Caputo for the constant function is equal to zero i.e*

${}^cD^\alpha c = 0$, *such as* $c = const$

Proof. *Let* $\alpha > 0$ *and* $\alpha \in]n-1; n[\,, n \in \mathbb{N}, i.e\ n \geq 1$ *apply the deffinition of the Caputo derivative*

$$ {}^cD^\alpha f(x) = I^{n-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, s > 0 \tag{1.2.8} $$

and since the nth *derivative* $c^{(n)}$ *is equal to* 0 *with* $n \in \mathbb{N}$ *and* $n \geq 1$, *it follows*

$$ {}^cD_c^\alpha = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{c^{(n)}(t)}{(t-s)^{\alpha+1-n}} ds = 0 \tag{1.2.9} $$

■

Example 1.2.2 *We calculate the derivative of order* $\frac{1}{2}$ *of* g *in the Caput sense, such that* $g(x) = x^2$ *we have the definitio*

$$ {}^cD_t^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}} dt $$

,*For* $\alpha = \frac{1}{2}$ *So* $n = 1$ *we find*

$$ {}^cD^{\frac{1}{2}} f(x) = \frac{1}{\Gamma(\frac{1}{2})} \int_a^x \frac{2}{(x-t)^{\frac{1}{2}}} dt $$

According to the integral per part, we gey

$$
\begin{aligned}
{}^{c}D^{\frac{1}{2}}(x^2) &= \frac{1}{\Gamma(\frac{1}{2})} 2 \int_a^x (x-t)^{-\frac{1}{2}} dt && (1.2.10)\\
&= \frac{1}{\Gamma(\frac{1}{2})} \left[\left[2x(x-t)^{\frac{1}{2}} - \frac{3}{2}(x-t)(x-t)^{\frac{1}{2}} \right]_0 \right] && (1.2.11)\\
&= \frac{2}{\sqrt{\pi}} \frac{4}{3} x^{\frac{3}{2}} && (1.2.12)\\
&= \frac{8}{3\sqrt{\pi}} x^{\frac{3}{2}} && (1.2.13)
\end{aligned}
$$

Proposition 1.2.2 *Let $f\colon [0;b] \mapsto \mathbb{R}$ and $\alpha > 0$, we have*

The relationship between The Caputo derivative and that of Rieman-Liouville is given by

$$D^{\alpha} f(x) = {}^{RL}\ D^{\alpha} f(x) - \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} x^{k-\alpha}$$

$f\ f(x) = (x-a)^p$, With $p > n-1$, Then

$$D^{\alpha} f(x) = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} (x-a)^{p-\alpha}, 0 \leq n-1 \leq \alpha \leq n, \qquad (1.2.14)$$

1.2.3 Some other Types of fractional derivatives

Left-sided Riemann-Liouville derivatives are defined as follows [30]

$${}_{a}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau) d\tau}{(t-\tau)^{\alpha+1-n}}, (n-1 < \alpha < n) \qquad (1.2.15)$$

Another definition of the left-sided Caputon fractional derivative was introduced by [12]

$${}_{a}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(\tau) d\tau}{(t-\tau)^{\alpha+1-n}}, (n-1 < \alpha < n) \qquad (1.2.16)$$

Relationship between the left-sided Riemann-Liouville and the left-sided Caputo fractional derivatives has discussed by [30].

The Right-sided Riemann-Liouville derivative is defined as follows [30]

$${}_{t}D_b^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b \frac{f(\tau) d\tau}{(t-\tau)^{\alpha+1-n}}, (n-1 < \alpha < n) \qquad (1.2.17)$$

Inspite of the remark made by [30] , the definition of the right-sided Caputo derivative did not appear in the literature until very recent works in [2, 3]

The right-sided Caputo derivative is defined by analogy with the right-sided Riemann-Liouville

$$ {}_t^C D_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \frac{f^n(\tau)d\tau}{(\tau - t)^{\alpha+1-n}}, (n-1 < \alpha < n) \tag{1.2.18} $$

The Riesz potential [30, 32](sometimes it is written with a factor of $[2\cos(\pi\alpha/2)]^{-1}$ in the right hand side

$$ {}_aR_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^b f(\tau) \left|\tau - t\right|^{\alpha-1} d\tau \tag{1.2.19} $$

is the sum of the left-sided and the right-sided Riemann-Liouville fractional integrals

$$ {}_aR_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau) \left|t - \tau\right|^{\alpha-1} d\tau + \frac{1}{\Gamma(\alpha)} \int_t^b f(\tau) \left|\tau - t\right|^{\alpha-1} d\tau \tag{1.2.20} $$

Based on this, we can consider the fractional Riesz derivative

$$ {}_aR_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^b \frac{f(\tau)d\tau}{\left|t - \tau\right|^{\alpha+1-n}}, (n-1 < \alpha < n) \tag{1.2.21} $$

and we can also introduce the fractional Riesz-Caputo derivative

$$ {}_a^C R_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^b \frac{f^{(n)}(\tau)d\tau}{\left|t - \tau\right|^{\alpha+1-n}}, (n-1 < \alpha < n) \tag{1.2.22} $$

1.2.4 Laplace Transform of the Caputo Derivation

Lemma 1.2.2 *Let $\alpha \in (0,1)$, $n = 1$, $f \in C^1(\mathbb{R}_+)$, and suppose that f and f' have Laplace transforms. Then*

$$ \mathcal{L}\{\partial_t^\alpha f\}(z) = z^\alpha \hat{f}(z) - z^{\alpha-1} f(0), \quad \mathrm{Re}(z) > 0. \tag{1.2.23} $$

1.3 Function Spaces

Function spaces are the cornerstone of mathematical analysis and theoretical physics. A rigorous analysis of boundary value problems for fractional partial differential equation requires the precise specification of the functionl framework with in which solutions are defined . The choice of funnction space directly affectes the formulation of initial and boundary conditions ,the proof of existence of solutions , and the analysis of stability with respect to data.

1.3.1 The Space of Continuous Functions $C(\Omega_T)$

Lemma 1.3.1 *The space of continuous functions $C(\Omega)$ over a closed domain Ω is equipped with the supremum norm $\| f \|_{C(\Omega)} = \max_{x\in\Omega} | f(x) |$ is a Banach space ,*

1.3.2 Space of Functions with Spatial Derivatives $C_x^2(0,1)$

The space $C^m(\Omega)$ consists of all functions prosseing continuons derivatives up to order m on Ω ,equipped with :

$$\| f \| C^m(\Omega) = \sum_{|K|\leq m} \| D^K f \|_{C(\Omega)} . \tag{1.3.1}$$

The space $C^2(0,1)$ is employed in the formulation of boundary value problems for the fractional diffusion equation $\partial_t^\alpha u - \partial_x^2 u = f(0,1)\times(0,T)$, where twice differentiability in x is required.

Remark 1.3.1 *It requires the function and its first and second derivatives with respect to x to be continuous*

It is the natural space for seeking "Classical Solutions" for diffusion or wave equations in one dimension. For example, in the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, u must belong to this space to allow for the calculation of the second spatial derivative.

1.3.3 The Space W(0,T) and Sobolev spaces

The Sobolev Spaces $H^s(\Omega) = W^{s,2}(\Omega)$ are defined with the inner product :

$$(f,g)_{H^s(\Omega)} = \sum_{|k|\leq s} (D^k f, D^k g)_{L^2(\Omega).} \tag{1.3.2}$$

In the context of fractional diffusion equations , the relevant solution space is:

$$W(0,T) = \{u \in L^2(0,T;H_0^1(\Omega)) :_0^C D_t^\alpha u \in L^2(0,T;H^{-1}(\Omega))\}; \tag{1.3.3}$$

This is the natural function space for weak solutions of the fractional diffusion equation $\partial_t^\alpha u - \Delta u = f$, and it provides the analytical infrastructure for proving the existence of weak solutions and establisshing their continuous dependence on data a prerequisite for any rigorous treatment of the associated inverse probems.

1.3.4 Fractional Sobolev Spaces

We also need to look at the case of fractional derivatives, that is, taking k to have noninteger values, in which case the space is often named Sobolev-Slobodeckij space. We do this here only briefly, and we suggest the references [26, 18, 24] for further reading.

We define, for $s = \{k + \sigma, k \in \mathbb{N}, \sigma \in [0,1]\}$

$$W^{s,p}(\Omega) = \left\{u \in W^{s,p}(\Omega) \mid\mid u \mid_{W^{\sigma,p}(\Omega)} < \infty\right\} \tag{1.3.4}$$

where

$$\|u\|_{W^{s,p}(\Omega)} := (\|u\|^p_{W^{k,p}(\Omega)} + \sum_{|\gamma|=k} \int_\Omega \int_\Omega \frac{\mid D^\gamma u(x) - D^\gamma u(y) \mid^p}{\mid x - y \mid^{d+\sigma p}} dxdy)^{\frac{1}{p}} \tag{1.3.5}$$

For any $\beta \geq 0$, we denote $H^\beta(0,1)$ to be the Sobolev space of order β on the unit interval $D = (0,1)$, and we denote $\widetilde{H}^\beta(D)$ to be the set of functions in $H^\beta(0,1)$ whose extensions by zero to $\mathbb{R}$ are in $H^\beta(\mathbb{R})$.

Remark 1.3.2 *For $\beta \in (0,1)$, the space $\widetilde{H}^\beta(D)$ coincides with the interpolation space $[L^2(D), H_0^1(D)]_\beta$.*

It is important for further study to note that $\phi \in \widetilde{H}^{\beta}(D), \beta > \frac{3}{2}$, satisfies the boundary conditions

$$\phi(0) = \phi'(0) = 0 \text{ and } \phi(1) = \phi'(1) = 0 \tag{1.3.6}$$

Left and Right Spaces

Analogously, we define $\widetilde{H}_L^{\beta}(D)$ (resp, $\widetilde{H}_R^{\beta}(D)$) to be the set of functions u whose extension by zero u is in $H^{\beta}(-\infty, 1)$(resp, $\widetilde{H}^{\beta}(0, \infty)$).

For $u \in \widetilde{H}_L^{\beta}(D)$, we set:

$$\|u\|_{\widetilde{H}_L^{\beta}(D)} := \|u\|_{H^{\beta}(-\infty,1)} \tag{1.3.7}$$

with the analogous definition for the norm in $\widetilde{H}_R^{\beta}(D)$.

Remark 1.3.3 *Let $\widetilde{C}_L^{\infty}[0,1]$ (resp,$\widetilde{C}_R^{\infty}[0,1]$) denote the set of functions $v \in C^{\infty}[0,1]$ satisfying $v(0) = v'(0) = ... = v^{(k)}(0) = 0$respectively, $v(1) = v'(1) = ... = v^{(k)}(1) = 0$. For any nonnegative integer $k < \beta - \frac{1}{2}$, the spaces $\widetilde{C}_L^{\infty}[0,1]$ and $\widetilde{C}_R^{\infty}[0,1]$ are dense in the spaces $\widetilde{H}_L^{\beta}(D)$ and $\widetilde{H}_R^{\beta}(D)$, respectively. Moreover, these are Banach spaces as well.*

Chapter 2

Spectral Theory and Sturm–Liouville Problem

Spectral theory provides the natural language for studying linear differential operators through their eigenvalues, eigenfunctions, and expansion formulas. In the theory of boundary value problems, Sturm–Liouville operators occupy a central position because they arise from separation of variables in the heat, wave, Laplace, and Helmholtz equations. The classical self-adjoint theory gives real eigenvalues, orthogonality of eigenfunctions, and completeness in suitable Hilbert spaces. However, once the boundary conditions become nonlocal, the situation changes substantially: orthogonality may be lost, the operator may fail to be self-adjoint, and the system of eigenfunctions may be complete and minimal without being a basis. This makes the construction of a biorthogonal or biorthonormal system essential.

In this section we first recall the classical regular Sturm–Liouville theory, then analyze a model nonlocal Sturm–Liouville problem. The nonlocal problem illustrates an important phenomenon in non-self-adjoint spectral theory: although the eigenfunctions are complete, they do not automatically form a Riesz basis. A suitable modification of the system restores good expansion properties and allows one to use generalized Fourier methods.

2.1 Classical Sturm–Liouville Theory

2.1.1 Regular Sturm–Liouville problem

Definition 2.1.1 *Let $a < b$ and consider the differential equation*

$$-\frac{d}{dx}\left(p(x)\frac{dX}{dx}\right) + q(x)X(x) = \lambda w(x)X(x), \qquad a < x < b, \tag{2.1.1}$$

together with the separated homogeneous boundary conditions

$$\alpha_1 X(a) + \beta_1 X'(a) = 0, \qquad \alpha_2 X(b) + \beta_2 X'(b) = 0. \tag{2.1.2}$$

We assume that

$$p, p', q, w \in C[a,b], \qquad p(x) > 0, \qquad w(x) > 0 \quad \text{on } [a,b].$$

Under these assumptions, (2.1.1)–(2.1.2) *is called a* regular Sturm–Liouville problem.

It is convenient to introduce the weighted inner product

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}\, w(x)\, dx, \tag{2.1.3}$$

with corresponding norm

$$\|f\|_w^2 = \langle f, f\rangle_w.$$

The natural state space is therefore the Hilbert space $L^2_w(a,b)$.

Lemma 2.1.1 *The Sturm–Liouville differential expression may be written as*

$$\mathcal{L}X := -\frac{d}{dx}(p(x)X'(x)) + q(x)X(x),$$

so that the eigenvalue equation is simply

$$\mathcal{L}X = \lambda w X.$$

When the boundary conditions are chosen so that the associated operator is self-adjoint in $L^2_w(a,b)$, the spectral theory becomes especially clean.

2.1.2 Lagrange identity and self-adjointness

A basic tool in the theory is the Lagrange identity. For sufficiently smooth functions u, v, one has

$$u\,\mathcal{L}v - v\,\mathcal{L}u = -\frac{d}{dx}\Big(p(x)\big(u(x)v'(x) - v(x)u'(x)\big)\Big). \tag{2.1.4}$$

Integrating over $[a, b]$ yields Green's formula:

$$\int_a^b \big(u\,\mathcal{L}v - v\,\mathcal{L}u\big)\,dx = -\Big[p(x)\big(u(x)v'(x) - v(x)u'(x)\big)\Big]_{x=a}^{x=b}. \tag{2.1.5}$$

If u and v satisfy the same admissible separated boundary conditions, then the boundary term vanishes, and therefore

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}v\rangle.$$

This is the formal basis of self-adjointness.

Theorem 2.1.1 *For a regular self-adjoint Sturm–Liouville problem (2.1.1)–(2.1.2), all eigenvalues are real.*

Proof. Let X_n be an eigenfunction corresponding to the eigenvalue λ_n:

$$\mathcal{L}X_n = \lambda_n w X_n.$$

Taking the complex conjugate gives

$$\mathcal{L}\overline{X_n} = \overline{\lambda_n} w \overline{X_n}.$$

Multiply the first equation by $\overline{X_n}$, the second by X_n, and subtract:

$$\overline{X_n}\,\mathcal{L}X_n - X_n\,\mathcal{L}\overline{X_n} = (\lambda_n - \overline{\lambda_n}) w |X_n|^2.$$

Integrating over $[a, b]$ and using Green's formula, the left-hand side reduces to a boundary term which vanishes because the boundary conditions are self-adjoint. Hence

$$(\lambda_n - \overline{\lambda_n}) \int_a^b w(x)|X_n(x)|^2\,dx = 0.$$

Since $w(x) > 0$ and $X_n \not\equiv 0$, the integral is strictly positive. Therefore $\lambda_n = \overline{\lambda_n}$, which means that λ_n is real. ∎

Theorem 2.1.2 *Eigenfunctions corresponding to distinct eigenvalues are orthogonal in $L^2_w(a,b)$.*

Proof. Let X_n and X_m satisfy

$$\mathcal{L}X_n = \lambda_n w X_n, \qquad \mathcal{L}X_m = \lambda_m w X_m, \qquad \lambda_n \neq \lambda_m.$$

Multiply the first equation by X_m, the second by X_n, and subtract:

$$X_m \mathcal{L}X_n - X_n \mathcal{L}X_m = (\lambda_n - \lambda_m) w X_n X_m.$$

Integrate over $[a,b]$ and apply Green's formula. The boundary term again vanishes, so

$$(\lambda_n - \lambda_m) \int_a^b X_n(x) X_m(x) w(x)\, dx = 0.$$

Because $\lambda_n \neq \lambda_m$, we obtain

$$\int_a^b X_n(x) X_m(x) w(x)\, dx = 0.$$

Hence $X_n \perp X_m$ in $L^2_w(a,b)$. ■

Spectral properties

Proposition 2.1.1 *The classical regular Sturm–Liouville problem possesses a discrete spectrum*

$$\lambda_1 < \lambda_2 < \cdots, \qquad \lambda_n \to +\infty,$$

and each eigenvalue admits an eigenfunction X_n having exactly $n-1$ zeros in (a,b). Moreover, the family $\{X_n\}_{n=1}^{\infty}$ is complete in $L^2_w(a,b)$, so every sufficiently regular function can be expanded into a generalized Fourier series

$$f(x) \sim \sum_{n=1}^{\infty} c_n X_n(x), \qquad c_n = \frac{\langle f, X_n \rangle_w}{\langle X_n, X_n \rangle_w}. \tag{2.1.6}$$

Thus the self-adjoint theory combines spectral discreteness, orthogonality, and an expansion principle in a single framework.

Definition 2.1.2 *A further useful quantity is the Rayleigh quotient*

$$R[y] = \frac{-p(x)y(x)y'(x)\Big|_a^b + \displaystyle\int_a^b \left(p(x)|y'^2 + q(x)|y(x)|^2\right)\, dx}{\displaystyle\int_a^b w(x)|y(x)|^2\, dx}, \tag{2.1.7}$$

defined for admissible nonzero functions y. For eigenfunctions one has $R[X_n] = \lambda_n$. In particular, the first eigenvalue admits the variational characterization

$$\lambda_1 = \min_{y \neq 0} R[y],$$

which is extremely useful both analytically and numerically.

Why the classical theory is not enough

Remark 2.1.1 *The classical theory depends heavily on self-adjointness. When the boundary conditions are nonlocal, for example when values at one endpoint are linked to values at the other endpoint, the Green boundary form need not vanish for pairs of functions in the same domain. In that case orthogonality with respect to the usual inner product may fail. One then replaces orthogonality by* biorthogonality*: one studies eigenfunctions of the original operator together with eigenfunctions of the adjoint operator. This is the idea that leads to the spectral analysis of nonlocal Sturm–Liouville problems.*

2.2 Nonlocal Sturm–Liouville Problem

Problem of Samarskii–Ionkin type

Proposition 2.2.1 *The nonlocal spectral problem*

$$-X''(x) = \lambda X(x), \qquad 0 < x < \pi, \tag{2.2.1}$$

subject to the boundary conditions

$$X(0) = 0, \qquad X'(0) + X'(\pi) + \alpha X(\pi) = 0, \qquad \alpha > 0. \tag{2.2.2}$$

This is a model nonlocal Sturm–Liouville problem of Samarskii–Ionkin type. The condition at $x = \pi$ is nonlocal because it couples the derivative at $x = \pi$ with the derivative at $x = 0$ and the value $X(\pi)$.

The differential equation (2.2.1) is elementary, but the spectral structure is not. The boundary conditions (2.2.2) are regular, yet not strengthened regular, and this is precisely why unusual basis phenomena appear. In particular, the resulting eigenfunction system is complete and minimal, but it does not form a basis in $L^2(0, \pi)$.

Eigenvalues and eigenfunctions

Let $\lambda \neq 0$, and write $\rho = \sqrt{\lambda}$. Then a nontrivial solution of (2.2.1) satisfying $X(0) = 0$ has the form

$$X(x) = C\sin(\rho x).$$

Substituting into the second boundary condition gives

$$\rho + \rho\cos(\rho\pi) + \alpha\sin(\rho\pi) = 0.$$

Using the trigonometric identities

$$1 + \cos(\rho\pi) = 2\cos^2\left(\frac{\rho\pi}{2}\right), \qquad \sin(\rho\pi) = 2\sin\left(\frac{\rho\pi}{2}\right)\cos\left(\frac{\rho\pi}{2}\right),$$

we obtain the factorization

$$2\cos\left(\frac{\rho\pi}{2}\right)\left(\rho\cos\left(\frac{\rho\pi}{2}\right) + \alpha\sin\left(\frac{\rho\pi}{2}\right)\right) = 0.$$

Hence the spectrum splits into two branches.

First branch.

If

$$\cos\left(\frac{\rho\pi}{2}\right) = 0,$$

then

$$\frac{\rho\pi}{2} = \frac{(2k+1)\pi}{2}, \qquad k = 0, 1, 2, \ldots,$$

so that

$$\lambda_k^{(1)} = (2k+1)^2, \qquad X_{k,1}(x) = \sin\big((2k+1)x\big). \tag{2.2.3}$$

Second branch.

If

$$\rho\cos\left(\frac{\rho\pi}{2}\right) + \alpha\sin\left(\frac{\rho\pi}{2}\right) = 0,$$

and if we put $\rho = 2\beta$, then

$$\cot(\beta\pi) = -\frac{\alpha}{2\beta}. \tag{2.2.4}$$

Let β_k be the roots of (2.2.4). Then the corresponding eigenvalues and eigenfunctions are

$$\lambda_k^{(2)} = (2\beta_k)^2, \qquad X_{k,2}(x) = \sin(2\beta_k x), \qquad k = 0, 1, 2, \ldots. \tag{2.2.5}$$

Moreover, the roots satisfy the localization

$$2k+1 < 2\beta_k < 2k+2, \qquad k = 0, 1, 2, \dots. \tag{2.2.6}$$

It is convenient to introduce

$$\delta_k = \beta_k - k - \frac{1}{2}.$$

Then $\delta_k \to 0$ as $k \to \infty$, and the second branch approaches the first branch asymptotically.

Thus the spectrum consists of two interlacing families:

$$\{\lambda_k^{(1)}\}_{k\geq 0} \cup \{\lambda_k^{(2)}\}_{k\geq 0}.$$

The associated eigenfunctions are

$$\{X_{k,1}, X_{k,2}\}_{k\geq 0}.$$

2.2.1 Adjoint problem and biorthogonal system

Remark 2.2.1 *Since the original problem is not self-adjoint, orthogonality of the system* $\{X_{k,1}, X_{k,2}\}$ *is not expected. The correct replacement is obtained through the adjoint problem*

$$-Y''(x) = \lambda Y(x), \qquad 0 < x < \pi, \tag{2.2.7}$$

with boundary conditions

$$Y(0) + Y(\pi) = 0, \qquad Y'(\pi) + \alpha Y(\pi) = 0. \tag{2.2.8}$$

The reason for these boundary conditions becomes clear from Green's identity.

Proof. Indeed,

$$\int_0^\pi \big((-X'')Y - X(-Y'')\big)\,dx = \Big[-X'(x)Y(x) + X(x)Y'(x)\Big]_0^\pi.$$

If X satisfies (2.2.2) and Y satisfies (2.2.8), then

$$\Big[-X'Y + XY'\Big]_0^\pi = 0,$$

so the operator pair is adjoint in $L^2(0,\pi)$. The eigenfunctions of the adjoint problem corresponding to the first branch may be chosen in the form

$$Y_{k,1}(x) = \frac{2}{\pi}\left[\sin\big((2k+1)x\big) - \frac{2k+1}{\alpha}\cos\big((2k+1)x\big)\right], \qquad k \geq 0. \tag{2.2.9}$$

For the second branch one may choose

$$Y_{k,2}(x) = C_{k,2}\left[\sin(2\beta_k x) - \frac{2\beta_k}{\alpha}\cos(2\beta_k x)\right], \qquad k \geq 0, \tag{2.2.10}$$

where the constants $C_{k,2}$ are fixed by normalization. ■

Proposition 2.2.2 *The systems* $\{X_{k,1}, X_{k,2}\}_{k\geq 0}$ *and* $\{Y_{k,1}, Y_{k,2}\}_{k\geq 0}$ *can be normalized so that*

$$\langle X_{k,i}, Y_{m,j}\rangle = \int_0^\pi X_{k,i}(x)Y_{m,j}(x)\,dx = \delta_{km}\delta_{ij}, \tag{2.2.11}$$

where δ_{km} *is the Kronecker delta.*

Proof. Assume

$$-X'' = \lambda X, \qquad -Y'' = \mu Y,$$

with X satisfying (2.2.2) and Y satisfying (2.2.8). Multiply the first equation by Y, the second by X, and subtract:

$$(-X'')Y - X(-Y'') = (\lambda - \mu)XY.$$

Integrating from 0 to π, we get

$$(\lambda - \mu)\int_0^\pi X(x)Y(x)\,dx = \Big[-X'(x)Y(x) + X(x)Y'(x)\Big]_0^\pi.$$

The boundary term vanishes by the choice of the adjoint boundary conditions, hence

$$(\lambda - \mu)\int_0^\pi X(x)Y(x)\,dx = 0.$$

Therefore, if $\lambda \neq \mu$, then

$$\int_0^\pi X(x)Y(x)\,dx = 0.$$

This proves biorthogonality for distinct eigenvalues. For equal eigenvalues, the normalization constants are chosen so that the diagonal products become equal to 1. Hence (2.2.11) follows. ■

The family $\{Y_{k,1}, Y_{k,2}\}$ is therefore the biorthogonal, and after normalization the biorthonormal, system associated with $\{X_{k,1}, X_{k,2}\}$.

Remark 2.2.2 *Although the eigenfunctions of the nonlocal problem are complete and minimal, they do not form a basis in $L^2(0,\pi)$. The obstruction is visible in the asymptotic geometry of the two branches. Since $\delta_k \to 0$, the functions $X_{k,1}$ and $X_{k,2}$ become almost linearly dependent for large k.*

In fact,

$$\lim_{k\to\infty} \left\langle \frac{X_{k,1}}{\|X_{k,1}\|}, \frac{X_{k,2}}{\|X_{k,2}\|} \right\rangle = 1. \tag{2.2.12}$$

Thus the angle between the normalized vectors tends to zero. A system with such behavior cannot satisfy the uniform minimality estimates required for a basis. Equivalently, the norms of the biorthogonal functionals blow up, which prevents the original eigenfunction family from being a Schauder basis, and in particular from being a Riesz basis.

This phenomenon is typical for non-self-adjoint problems with regular but not strengthened regular boundary conditions. The spectral data still carry enough information to be useful, but the eigenfunctions themselves must be reorganized.

2.2.2 Construction of a Riesz basis

To restore the basis property, one introduces a modified system built from nondegenerate linear combinations of the original eigenfunctions. Define

$$\varphi_{2k}(x) = X_{k,1}(x) = \sin\big((2k+1)x\big), \tag{2.2.13}$$

and

$$\varphi_{2k+1}(x) = \frac{X_{k,2}(x) - X_{k,1}(x)}{2\delta_k}, \qquad k = 0, 1, 2, \dots. \tag{2.2.14}$$

The corresponding dual system is chosen as

$$\psi_{2k}(x) = Y_{k,2}(x) + Y_{k,1}(x), \qquad \psi_{2k+1}(x) = 2\delta_k\, Y_{k,2}(x). \tag{2.2.15}$$

A direct computation using the relations

$$\langle X_{k,i}, Y_{m,j} \rangle = \delta_{km}\delta_{ij}$$

shows that

$$\langle \varphi_n, \psi_m \rangle = \delta_{nm}, \qquad n, m \geq 0.$$

Hence $\{\psi_n\}$ is biorthogonal to $\{\varphi_n\}$.

The importance of the system $\{\varphi_n\}$ is that it is quadratically close to a known Riesz basis. Consider the auxiliary system

$$w_{2k}(x) = \sin\big((2k+1)x\big), \qquad w_{2k+1}(x) = x\cos\big((2k+1)x\big). \tag{2.2.16}$$

The system $\{w_n\}_{n\geq 0}$ is a Riesz basis in $L^2(0,\pi)$, and one checks that

$$\varphi_{2k} = w_{2k},$$

while for odd indices

$$\varphi_{2k+1}(x) = \frac{\sin(2\beta_k x) - \sin((2k+1)x)}{2\delta_k} = \frac{\sin(\delta_k x)}{\delta_k x}\, x\cos((2k+1+\delta_k)x).$$

Since $\delta_k \to 0$, the factor $\sin(\delta_k x)/(\delta_k x)$ tends to 1, and therefore

$$\varphi_{2k+1}(x) \approx x\cos((2k+1)x) = w_{2k+1}(x) \quad \text{for large } k.$$

More precisely, one proves

$$\sum_{n=0}^{\infty} \|\varphi_n - w_n\|^2_{L^2(0,\pi)} < \infty.$$

Quadratic closeness to a Riesz basis, together with completeness and minimality, implies that $\{\varphi_n\}$ itself is a Riesz basis in $L^2(0,\pi)$.

Theorem 2.2.1 *The modified system $\{\varphi_n\}_{n\geq 0}$ defined by (2.2.13)–(2.2.14) forms a Riesz basis in $L^2(0,\pi)$, and $\{\psi_n\}_{n\geq 0}$ given by (2.2.15) is its unique biorthogonal dual system.*

Proof. The system $\{\varphi_n\}$ is obtained from a complete minimal family by nondegenerate linear transformations on each two-dimensional spectral block, so it remains complete and minimal. The quadratic closeness to the Riesz basis $\{w_n\}$ then yields the Riesz basis property by a standard perturbation argument. Since a complete minimal system has a unique biorthogonal dual, the system $\{\psi_n\}$ is uniquely determined. ∎

2.2.3 Expansion and spectral representation

Proposition 2.2.3 *Once a Riesz basis is available, every $f \in L^2(0,\pi)$ admits a unique expansion*

$$f(x) = \sum_{n=0}^{\infty} \langle f, \psi_n\rangle \varphi_n(x), \tag{2.2.17}$$

where the convergence holds in $L^2(0,\pi)$*. This formula plays the role that the usual orthogonal Fourier expansion plays in the self-adjoint theory. The difference is that the coefficients are computed with the dual system* $\{\psi_n\}$*, not with the same system* $\{\varphi_n\}$*.*

For the original eigenfunctions $\{X_{k,1}, X_{k,2}\}$, one should therefore distinguish between three levels of structure:

1. the system is complete and minimal, so it detects all functions in the space;
2. it possesses a biorthogonal system coming from the adjoint problem;
3. after a suitable block transformation, one obtains a genuine Riesz basis.

This is the correct spectral framework for nonlocal Sturm–Liouville problems of the present type.

The model problem (2.2.1)–(2.2.2) shows clearly how the spectral theory of nonlocal operators differs from the classical self-adjoint case. In the classical theory, the same family of eigenfunctions provides orthogonality, expansion coefficients, and basis properties. In the nonlocal case, these roles split apart. The eigenfunctions remain meaningful and complete, but orthogonality is replaced by biorthogonality, and the basis property is recovered only after a suitable renormalization and recombination of the spectral blocks.

This distinction is not merely technical. In applications to partial differential equations with nonlocal boundary conditions, one must expand the solution in a system having stable reconstruction properties. A Riesz basis is precisely the right substitute for an orthonormal basis in a non-self-adjoint setting. It guarantees uniqueness of expansion, norm equivalence of coefficients, and robustness under perturbation.

Therefore, the nonlocal Sturm–Liouville problem provides an instructive example of the transition from orthogonal spectral theory to biorthogonal spectral theory. The adjoint problem supplies the dual system, and the modified family $\{\varphi_n\}$ supplies the basis required for analytic and computational applications.

For clarity, we collect the principal conclusions

1. In the classical regular self-adjoint Sturm–Liouville problem, the eigenvalues are real, simple in the separated case, and form a discrete sequence tending to $+\infty$; eigen-

functions corresponding to different eigenvalues are orthogonal in the weighted space $L^2_w(a,b)$.

2. For the nonlocal problem

$$-X'' = \lambda X, \qquad X(0) = 0, \qquad X'(0) + X'(\pi) + \alpha X(\pi) = 0,$$

the eigenvalues split into two branches:

$$\lambda_k^{(1)} = (2k+1)^2, \qquad \lambda_k^{(2)} = (2\beta_k)^2,$$

where β_k solves

$$\cot(\beta_k \pi) = -\frac{\alpha}{2\beta_k}.$$

3. The corresponding eigenfunctions are

$$X_{k,1}(x) = \sin((2k+1)x), \qquad X_{k,2}(x) = \sin(2\beta_k x).$$

4. The adjoint problem is

$$-Y'' = \lambda Y, \qquad Y(0) + Y(\pi) = 0, \qquad Y'(\pi) + \alpha Y(\pi) = 0,$$

and its eigenfunctions generate a biorthonormal system $\{Y_{k,1}, Y_{k,2}\}$ satisfying

$$\langle X_{k,i}, Y_{m,j} \rangle = \delta_{km}\delta_{ij}.$$

5. The original eigenfunction family is complete and minimal but not a basis in $L^2(0,\pi)$.

6. The modified system

$$\varphi_{2k} = X_{k,1}, \qquad \varphi_{2k+1} = \frac{X_{k,2} - X_{k,1}}{2\delta_k},$$

with dual system

$$\psi_{2k} = Y_{k,2} + Y_{k,1}, \qquad \psi_{2k+1} = 2\delta_k Y_{k,2},$$

forms a Riesz basis in $L^2(0,\pi)$.

Chapter 3

Inverse Problem of Parabolic Equation of Fractional Order

This chapter develops the complete mathematical analysis of the inverse problem for a one-dimensional time-fractional diffusion equation of subdiffusion type. The goal is to recover an unknown time-dependent source parameter $p(t)$ from a boundary observation $h(t) = u(1,t)$, where the state $u(x,t)$ satisfies a Caputo-type fractional partial differential equation on $(0,1) \times (0,T]$. The analysis follows four connected stages: reduction to homogeneous boundary conditions, eigenfunction expansion, construction of the input–output operator, and proof of its injectivity (distinguishability property). Throughout, we rely on key results from the theory of fractional calculus and the spectral theory of Sturm–Liouville operators [23].

3.1 Ordinary Differential Equation of Fractional order

Theorem 3.1.1 *Let $n-1 < \alpha \leq n, n \in \mathbb{N}, \lambda \in \mathbb{R}$ and let $f(x)$ be a given real function defined on $R+ := \{x \in R : x > 0\}$. Then the equation*

$$\partial_t^\alpha y(x) - \lambda y(x) = f(x), x \in \mathbb{R}+ \tag{3.1.1}$$

is solvable, and its general solution is given by

$$y(x) = \int_0^x (x-t)^{\alpha-1} E\alpha, \alpha(\lambda(x-t)^\alpha) f(t)dt + \sum_{j=0}^{l-1} c_j x^j E_{\alpha,j+1}(\lambda t^\alpha) \tag{3.1.2}$$

where $c_j, j = 0, ..., n-1$, *are arbitrary real constants.*

In particular, the general solution to the equation with $0 < \alpha \leq 1$ *and* $1 < \alpha \leq 2$ *have the forms*

$$y(x) = \int_0^x (x-t)^{\alpha-1} E\alpha, \alpha(\lambda(x-t)^\alpha) f(t)dt + c_1 E_\alpha(\lambda t^\alpha) \tag{3.1.3}$$

and

$$y(x) = \int_0^x (x-t)^{\alpha-1} E\alpha, \alpha(\lambda(x-t)^\alpha) f(t)dt + c_1 E_\alpha(\lambda t^\alpha) + c_2 x E_{\alpha,2}(\lambda t^\alpha) \tag{3.1.4}$$

3.2 Analysis of a time-dependent inverse coefficient problem

3.2.1 Direct Problem, Existence and Uniquness

Problem statement

Let $\alpha \in (0,1)$, $T > 0$, and let $\Omega = (0,1)$. This section is related to the modeling of diffusion problems, known as diffusion equation as given in (1):

$$D_t^\alpha u(x,t) = u_{xx}(x,t) - p(t)u(x,t) + F(x,t) \quad 0 < \alpha \leq 1,\ (x,t) \in \Omega_T \tag{3.2.1}$$

$$u(x,0) = g(x) \tag{3.2.2}$$

$$u_x(0,t) = u_x(1,t),\ u(0,t) = \Psi_1(t) \tag{3.2.3}$$

where $\Omega_T = \{(x,t) \in R^2 : 0 < x < 1, 0 < t \leq T\}$ and the fractional derivative $D_t^\alpha u(x,t)$ is defined in Caputo sense $D_t^\alpha u(x,t) = (I^{1-\alpha} u')(t)$, $0 < \alpha \leq 1$, I^α being the Riemann-

Liouville fractional integral,

$$(I^{\alpha}f)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau & 0 < \alpha \leq 1 \\ f(t) & \alpha = 0 \end{cases}$$

Equations (3.2.1)–(3.2.3) indicate an inverse problem with respect to the unknown function $p(t)$. $F(x,t)$ is a source function. This problem is treated in [28]

The left boundary value function $\Psi_1(t)$ belongs to $C[0,T]$. This function $g(x)$ satisfies the following consistency conditions:

$$\begin{array}{ll} \text{(C1)} & g(0) = \Psi_1(0) \\ \text{(C2)} & g'(1) = u_x(0,0) \end{array}$$

Theorem 3.2.1 *Under (C1) and (C2), the initial boundary value problem (3.2.1)–(3.2.3) has the unique solution $u(x,t)$, which is defined in the domain $\Omega_T = \{(x,t) \in R^2 : 0 \leq x \leq 1, 0 \leq t \leq T\}$ and belongs to the space*

$$C(\Omega_T) \cap W_t^1(0,T) \cap C_x^2(0,1)$$

where the solution u is continuous with respect to x and t, u_t is in L^1, u_x and u_{xx} are continuous.

3.3 Transformation to Homogeneous Boundary Data

The inhomogeneous Dirichlet conditions impede direct spectral analysis. We eliminate them by introducing the substitution. Consider the inverse problem with measured output data $h(t)$ at $x = 1$. To formulate the solution for the parabolic problem (3.2.1)–(3.2.3) by using the Fourier method of separation of variables, we first introduce another formulation of the problem

Proposition 3.3.1 *The problem (3.2.1)–(3.2.3) can be written as follow form*

$$D_t^{\alpha} v(x,t) - v_{xx}(x,t) = D_t^{\alpha}\Psi_1(t)(1-x) - p(t)v(x,t) - p(t)\Psi_1(t)(1-x) + F(x,t) \quad (3.3.1)$$

$$v(x,0) = g(x) - \Psi_1(0)(1-x) \quad (3.3.2)$$

$$v_x(0,t) - v_x(1,t) = 0 \tag{3.3.3}$$

$$v(0,t) = 0 \tag{3.3.4}$$

Indeed, by the use of an auxiliary function $v(x,t)$.as follows:

$$v(x,t) = u(x,t) - \Psi_1(t)(1-x), \quad x \in [0,1] \quad (7)$$

by which we transform problem (3.2.1)–(3.2.3) into a problem with homogeneous boundary conditions. Therefore, the initial boundary value problem (3.2.1)–(3.2.3) can be rewritten in terms of $v(x,t)$ in the given

Proposition 3.3.2 *The unique solution to the initial-boundary value problem can be represented in the following form [35]:*

$$\begin{gathered} v(x,t) = \sum_{k=1}^{\infty} < \zeta(\theta), X_k(\theta) > E_{\alpha,1}(\lambda_k t^\alpha) Y_k(x) + \\ \sum_{k=1}^{\infty} \left(\int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k s^\alpha) < \zeta(\theta, t-s; p_j(t)), X_k(\theta) > ds \right) Y_k(x) \end{gathered} \tag{3.3.5}$$

where

$$\begin{aligned} \zeta(x) &= g(x) - \Psi_1(0)(1-x) \\ \zeta(x,t) &= -D_t^\alpha \Psi_1(t)(1-x) - p(t)v(x,t) - p(t)\Psi_1(t)(1-x) + F(x,t) \end{aligned} \tag{3.3.6}$$

Moreover, $\langle \zeta(\theta), \Phi_n(\theta) \rangle = \int_0^1 \Phi_n(\theta)\zeta(\theta)d\theta$*;* $E_{\alpha,\beta}$ *is the generalized Mittag-Leffler function, defined by [13]*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + \alpha)}$$

3.4 Solution of the Direct Problem via Eigenfunction Expansion

Spectral Theory of the Spatial Operator

Proposition 3.4.1 *[28] Assume that $X_k(x)$ is the solution to the following Sturm-Liouville problem.*

$$\begin{cases} -\Phi_{xx}(x) = \lambda\Phi(x) & 0 < x < 1 \\ \Phi'(1) = \Phi'(0), & \Phi(0) = 0 \end{cases} \tag{3.4.1}$$

$$\{X_0(x) = 2,\ X_{2k-1}(x) = 4\cos(2\pi kx),\ X_{2k}(x) = 4(1-x)\sin(2\pi kx),\ k = 1, 2, \ldots\}$$

$$\{Y_0(x) = x,\ Y_{2k-1}(x) = x\cos(2\pi kx),\ Y_{2k}(x) = \sin(2\pi kx),\ k = 1, 2, \ldots\}$$

The system of functions Y_n's are biorthonormal bases, that is,

$$\langle\langle Y_i, X_j\rangle\rangle = 0 \text{ otherwise } \langle\langle Y_i, X_j\rangle\rangle = 1 \text{ if } i = j \tag{3.4.2}$$

These are also Riesz bases in L^2.

The Dirichlet type of the measured output data at the boundary $x = 1$ in terms of $v(x,t)$ can be written in the following form [15, 37]:

$$h(t) = u(1,t) = v(1,t) \tag{3.4.3}$$

To simplify (3.3.5), define the following:

$$z_k(t) = \sum_{k=1}^{\infty} \langle\zeta(\theta), X_k(\theta)\rangle E_{\alpha,1}(\lambda_k t^\alpha) \tag{3.4.4}$$

$$w_k(t) = \sum_{k=1}^{\infty} \left(\int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k s^\alpha) \langle\zeta(\theta, t-s; p_j(t)), X_k(\theta)\rangle ds \right) \tag{3.4.5}$$

By using $z_k(t)$ and $w_k(t)$, we can write the solution as follows:

$$v(x,t) = \sum_{k=1}^{\infty} z_k(t) Y_k(x) + \sum_{k=1}^{\infty} w_k(t) Y_k(x) \tag{3.4.6}$$

The analytical solution to the problem [16, 7] in series form is given in (20). Therefore, by substituting $x = 1$,

$$h(t) = v(1,t) = \sum_{k=1}^{\infty} z_k(t) Y_k(1) + \sum_{k=1}^{\infty} w_k(t) Y_k(1) \tag{3.4.7}$$

is obtained.

3.4.1 Derivation of the Modal Equations

Substituting (3.4.6) into the PDE (3.4.6), using $\mathcal{L}_q Y_k = -\lambda_k Y_k$, and applying the orthonormality (3.4.2) by multiplying by $Y_j(x)$ and integrating over Ω, we isolate a fractional ODE for each modal coefficient:

$$\partial_t^\alpha T_k(t) + \lambda_k\, T_k(t) = F_k(t) := \langle \xi(\cdot,t),\, Y_k \rangle, \quad t \in (0,T], \quad k \geq 1, \tag{3.4.8}$$

with initial condition

$$T_k(0) = \langle \zeta,\, Y_k \rangle =: \zeta_k. \tag{3.4.9}$$

Equation (3.4.8) is a *linear Caputo fractional ODE of order* α with a known forcing $F_k(t)$ (which itself depends on p).

3.4.2 Solving the Fractional ODEs

The following theorem gives the explicit solution of the fractional ODE (3.4.8).

Theorem 3.4.1 (Solution of the Fractional Cauchy Problem [23, Theorem 5.1]) *Let $\alpha \in (0,1)$, $\lambda > 0$, and $F \in C[0,T]$. Then the unique solution $T \in C[0,T]$ of*

$$\partial_t^\alpha T(t) + \lambda T(t) = F(t), \quad t \in (0,T], \quad T(0) = T_0, \tag{3.4.10}$$

is given by

$$T(t) = T_0\, E_{\alpha,1}(-\lambda t^\alpha) + \int_0^t s^{\alpha-1}\, E_{\alpha,\alpha}(-\lambda s^\alpha)\, F(t-s)\, ds, \tag{3.4.11}$$

where the two-parameter Mittag-Leffler function is

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta > 0,\ z \in \mathbb{C}. \tag{3.4.12}$$

Proof. Taking the Laplace transform of the equation and applying Lemma 1.2.2:

$$z^\alpha \hat{T}(z) - z^{\alpha-1} T_0 + \lambda \hat{T}(z) = \hat{F}(z), \tag{3.4.13}$$

which gives

$$\hat{T}(z) = \frac{z^{\alpha-1}}{z^\alpha + \lambda}\, T_0 + \frac{1}{z^\alpha + \lambda}\, \hat{F}(z). \tag{3.4.14}$$

The inverse Laplace transforms (see [23, Section 3.4]) give $\mathcal{L}^{-1}\left\{\frac{z^{\alpha-1}}{z^\alpha+\lambda}\right\} = E_{\alpha,1}(-\lambda t^\alpha)$ and $\mathcal{L}^{-1}\left\{\frac{1}{z^\alpha+\lambda}\right\} = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$. Applying the Laplace convolution theorem to the second term yields (3.4.11).Applying Theorem 3.4.1 to each modal equation (3.4.8)–(3.4.9), we obtain:

$$\boxed{T_k(t) = \langle \zeta,\, Y_k\rangle\, E_{\alpha,1}(-\lambda_k t^\alpha) + \int_0^t s^{\alpha-1}\, E_{\alpha,\alpha}(-\lambda_k s^\alpha)\, \langle \xi(\cdot, t-s),\, Y_k\rangle\, ds.} \tag{3.4.15}$$

■

The following example illustrates the role of the Mittag-Leffler function in the solution.

Example 3.4.1 (Free Decay of a Single Mode) *Consider the homogeneous problem ($F_k = 0$) with initial datum $\zeta_k = 1$ for a fixed mode k. Then $T_k(t) = E_{\alpha,1}(-\lambda_k t^\alpha)$. For $\alpha = 1$ (classical diffusion), this reduces to $T_k(t) = e^{-\lambda_k t}$, i.e., exponential decay. For $\alpha \in (0,1)$, the Mittag-Leffler function decays* algebraically*: by [23, Theorem 3.23],*

$$E_{\alpha,1}(-\lambda_k t^\alpha) \sim \frac{1}{\lambda_k\, \Gamma(1-\alpha)}\, t^{-\alpha} \quad \text{as } t \to +\infty. \tag{3.4.16}$$

This power-law decay (slower than exponential) is the mathematical signature of subdiffusion*: energy in each mode dissipates much more slowly than in classical diffusion, reflecting the memory of the fractional operator.*

Remark 3.4.1 (Caputo Derivative of the Mittag-Leffler Function) *A fundamental algebraic identity, noted in [23, Remark 4.6], is that*

$$\partial_t^\alpha E_{\alpha,1}(-\lambda t^\alpha) = -\lambda\, E_{\alpha,1}(-\lambda t^\alpha), \quad t \geq 0. \tag{3.4.17}$$

That is, $E_{\alpha,1}(-\lambda^{\,\alpha})$ is an eigenfunction *of the Caputo derivative ∂_t^α, just as $e^{-\lambda t}$ is an eigenfunction of $\frac{d}{dt}$ in the classical case. This identity confirms that (3.4.15) indeed solves (3.4.8).*

3.4.3 Series Solution and Convergence

Substituting (3.4.15) into (3.4.8), the complete solution of the transformed problem is:

$$v(x,t) = \underbrace{\sum_{k=1}^{\infty} \zeta_k\, E_{\alpha,1}(-\lambda_k t^\alpha)\, Y_k(x)}_{\text{free evolution term}} + \underbrace{\sum_{k=1}^{\infty} \int_0^t s^{\alpha-1}\, E_{\alpha,\alpha}(-\lambda_k s^\alpha)\, \langle \xi(\cdot, t-s),\, Y_k\rangle\, ds\, Y_k(x)}_{\text{forced response term}}. \tag{3.4.18}$$

The decay estimate $|E_{\alpha,\alpha}(-\lambda t^\alpha)| \leq C(1+\lambda t^\alpha)^{-1}$ (see [23, Section 3.4]) ensures absolute convergence of both series in $C([0,T];L^2(\Omega))$ under the conditions $\zeta \in L^2(\Omega)$ and $\xi \in L^2(\Omega \times (0,T))$.

3.5 The Input–Output Mapping

3.5.1 Formal Derivation of the Observation Functional

Evaluating the series (3.4.18) at the observation point $x = 1$:

$$v(1,t) = \sum_{k=1}^{\infty} [z_k(t) + w_k(t)]\, Y_k(1). \tag{3.5.1}$$

The measured boundary output $h(t) = u(1,t) = v(1,t) + \Psi_1(t)$, so that $v(1,t) = h(t) - \Psi_1(t)$. Absorbing Ψ_1 into the known side, we define the *input–output operator*:

$$h(t) = \Psi[p](t) := \sum_{k=1}^{\infty} z_k(t)\, Y_k(1) + \sum_{k=1}^{\infty} w_k(t)\, Y_k(1). \tag{3.5.2}$$

This separates the observation into a known background contribution $\sum_k z_k(t) Y_k(1)$ and the parameter-dependent signal $\sum_k w_k(t) Y_k(1)$.

3.5.2 Definition and Properties of the Operator

As a result, $h(t)$ is analytically determined as a series representation. The right-hand side of (**??**) defines the input-output mapping $\Psi[p]$:

$$\Psi[p] = \sum_{k=1}^{\infty} z_k(t) Y_k(1) + \sum_{k=1}^{\infty} w_k(t) Y_k(1) \tag{3.5.3}$$

The relationship between the functions $p_1(t), p_2(t) \in K$ at $x = 1$ and the corresponding outputs $h_j(t) = u(1,t;p_j), j = 1,2$ are given in the following lemma.

Lemma 3.5.1 *Let $v_1(x,t) = v(x,t;p_1)$ and $v_2(x,t) = v(x,t;p_2)$ be the solutions to the direct problem (3.3.1)–(3.3.4) corresponding to the admissible coefficients $p_1(t), p_2(t) \in K$. If $h_j(t) = u(1,t;p_j)$, $j = 1,2$ are the corresponding outputs [28], then $h_j(t)$ satisfy the following series identity.*

$$\Delta h(t) = \sum_{k=1}^{\infty} \Delta w_k(t) Y_k(1) \tag{3.5.4}$$

for each $t \in (0,T]$ *where* $\Delta h(t) = h_1(t) - h_2(t), \Delta w_k(t) = w_k^1(t) - w_k^2(t)$.

Proof. By using identity (3.5.2), the measured output data $h_j(t) = v(1,t), j = 1,2$ can be written as:

$$\begin{aligned} h_1(t) &= \sum_{k=1}^{\infty} z_k^1(t) Y_k(1) + \sum_{k=1}^{\infty} w_k^1(t) Y_k(1) \\ h_2(t) &= \sum_{k=1}^{\infty} z_k^2(t) Y_k(1) + \sum_{k=1}^{\infty} w_k^2(t) Y_k(1) \end{aligned} \tag{3.5.5}$$

Respectively, since $z_k^1(t) = z_k^2(t)$ from the definition of $z_k(t)$. The difference of these formulas implies the desired result. ■

The lemma and the definitions enable us to reach the following corrolllary.

Corollary 3.5.1 *Let the conditions of Lemma 1 hold. If, in addition,* $\langle \xi(x,t;p_1(t)) - \xi(x,t;p_2(t)), X_k(x) = 0 \; \forall t \in (0,T]$ *holds, then* $h_1(t) = h_2(t) \; \forall t \in (0,T]$.

In fact, Since $Y_k(x) \; \forall k = 0,1,2,\ldots$ forms a basis for the space and $Y_k(1) \neq 0 \; \forall k = 0,1,2,\ldots$, then $\langle \xi_1(x,t;p_1(t)) - \xi_2(x,t;p_2(t)), X_k(x) \rangle \neq 0$, at least for some $k \in \mathbb{N}$. Hence, through the lemma, we can conclude that $h_1(t) \neq h_2(t)$, which leads us to the following consequence: $\Psi[p_1] \neq \Psi[p_2]$ implies that $p_1(t) \neq p_2(t)$.

Theorem 3.5.1 *Let conditions (C1) and (C2) hold. Assume that* $\Psi[\cdot] : K \longrightarrow C[0,T]$ *is the input-output mapping defined by* (3.5.3) *and corresponding to the measured output* $h(t) = u(1,t)$. *In this case, the mapping* $\Psi[p]$ *has the distinguishability property in the class of admissible parameter* K, *i.e.,* $\Psi[p_1] \neq \Psi[p_2] \; \forall p_1, p_2 \in K$ *implies* $p_1(t) \neq p_2(t)$.

3.5.3 Stability Estimate for the Time-Dependent Component

The following theorem, adapted from [23, Theorem 10.6], gives a bilateral stability estimate connecting the source $p(t)$ to the boundary measurement.

Theorem 3.5.2 (Stability of the Inverse Source Problem [23, Theorem 10.6]) *Let* $r \in H^2(\Omega)$ *with* $r(x_0) \neq 0$ *at the observation point* $x_0 = 1$. *Then for any* $p \in C[0,T]$, *the solution* v *of* (3.3.1)–(3.3.4) *with* $\zeta = 0$ *satisfies*

$$c \, \|\partial_t^{\alpha} v(1,\cdot)\|_{L^\infty(0,T)} \leq \|p\|_{L^\infty(0,T)} \leq C \, \|\partial_t^{\alpha} v(1,\cdot)\|_{L^\infty(0,T)}, \tag{3.5.6}$$

for constants $c, C > 0$ *independent of* p.

Proof. From the modal representation, differentiating (**??**) and evaluating at $x = 1$:

$$\partial_t^\alpha v(1,t) = p(t)\, r(1) - \sum_{k=1}^{\infty} \lambda_k\, w_k(t)\, Y_k(1) = p(t)\, r(1) - \mathcal{R}[p](t), \tag{3.5.7}$$

where $\mathcal{R}[p](t) = \sum_{k=1} \lambda_k w_k(t) Y_k(1)$ is a Volterra operator of the second kind. Since $r(1) \neq 0$ and $\mathcal{R}$ is compact from $L^\infty(0,T)$ to itself, the lower bound follows from a standard Volterra inversion argument. The upper bound is a direct L^∞ estimates ■

3.6 Distinguishability and Uniqueness

For two parameters $p_1, p_2 \in K$, let $v^{(i)}$ denote the corresponding solutions of (3.3.1)–(3.3.4), and let $T_k^{(i)} = z_k + w_k^{(i)}$ be their modal decompositions (z_k is the same for both since it is independent of p). The key question is Does $\Psi[p_1] = \Psi[p_2]$ imply $p_1 = p_2$?

An affirmative answer – injectivity of Ψ – constitutes the *distinguishability property* and is a necessary condition for the uniqueness of the inverse problem. The following lemma, analogous to (Lemma 10.5) in [23], is a cornerstone of the uniqueness analysis.

Lemma 3.6.1 *Let $w^{(1)}, w^{(2)} \in C([0,T]; L^2(\Omega))$ be solutions of (3.3.1) with $\xi = 0$ and with initial data $\zeta^{(1)}, \zeta^{(2)} \in L^2(\Omega)$, respectively. Suppose*

$$w^{(1)}(1,t) = w^{(2)}(1,t) \quad \text{for all } t \in (0,T]. \tag{3.6.1}$$

Then $\zeta^{(1)} = \zeta^{(2)}$ in $L^2(\Omega)$, and hence $w^{(1)} \equiv w^{(2)}$ on $\Omega \times [0,T]$.

Proof. Set $w = w^{(1)} - w^{(2)}$, which solves (3.3.1) with $\xi = 0$ and initial data $\zeta = \zeta^{(1)} - \zeta^{(2)}$, and satisfies $w(1,t) = 0$ for all $t \in (0,T]$. The modal coefficients of w are $\tilde{T}_k(t) = \tilde{\zeta}_k\, E_{\alpha,1}(-\lambda_k t^\alpha)$ where $\tilde{\zeta}_k = \langle \zeta^{(1)} - \zeta^{(2)}, Y_k \rangle$. The boundary condition $w(1,t) = 0$ gives

$$\sum_{k=1}^{\infty} \tilde{\zeta}_k\, E_{\alpha,1}(-\lambda_k t^\alpha)\, Y_k(1) = 0 \quad \forall\, t \in (0,T]. \tag{3.6.2}$$

Taking the Laplace transform, using $\mathcal{L}\{E_{\alpha,1}(-\lambda_k t^\alpha)\}(z) = \dfrac{z^{\alpha-1}}{z^\alpha + \lambda_k}$, we obtain

$$\sum_{k=1}^{\infty} \frac{\tilde{\zeta}_k\, Y_k(1)}{z^\alpha + \lambda_k} = 0 \quad \text{for } \mathrm{Re}(z) > 0. \tag{3.6.3}$$

The functions $\{(z^\alpha + \lambda_k)^{-1}\}_{k=1}^{\infty}$ have distinct poles at $z = \lambda_k^{1/\alpha} e^{i\pi/\alpha}$, $k = 1, 2, \ldots$, so by analyticity and the Riemann–Lebesgue lemma applied to the Laplace transform (cf. [23, Lemma 10.5, proof]), we conclude $\tilde{\zeta}_k\, Y_k(1) = 0$ for all $k \geq 1$. The non-degeneracy condition $Y_k(1) \neq 0$ for all k (which holds for generic potentials) then forces $\tilde{\zeta}_k = 0$ for all k, hence $\zeta^{(1)} = \zeta^{(2)}$. ∎

3.6.1 Main Distinguishability Lemma

Lemma 3.6.2 (Difference of Outputs) *For any $p_1, p_2 \in K$:*

$$\big(\Psi[p_1] - \Psi[p_2]\big)(t) = \sum_{k=1}^{\infty} \big[w_k^{(1)}(t) - w_k^{(2)}(t)\big]\, Y_k(1), \quad t \in [0, T]. \tag{3.6.4}$$

Proof. From (3.5.2),

$$\Psi[p_i](t) = \sum_{k=1}^{\infty} z_k(t)\, Y_k(1) + \sum_{k=1}^{\infty} w_k^{(i)}(t)\, Y_k(1), \quad i = 1, 2.$$

Since z_k is the same for both p_1 and p_2 (it depends only on the known data ζ and g), subtracting gives

$$\big(\Psi[p_1] - \Psi[p_2]\big)(t) = \sum_{k=1}^{\infty} \big[w_k^{(1)}(t) - w_k^{(2)}(t)\big]\, Y_k(1). \tag{3.6.5}$$

∎

Remark 3.6.1 *Lemma 3.6.2 shows that the difference of outputs depends* only *on the difference of the w_k-terms, which are precisely the terms carrying the dependence on the unknown p. Thus, the problem of distinguishing p_1 from p_2 is fully encoded in the behaviour of the forced convolution terms $\{w_k^{(i)}\}$.*

Corollary 3.6.1 *If $\Psi[p_1] = \Psi[p_2]$ in $C[0, T]$, then for each $k \geq 1$*

$$\int_0^t s^{\alpha-1}\, E_{\alpha,\alpha}(-\lambda_k s^\alpha)\, \big[p_1(t-s) - p_2(t-s)\big]\, \langle r(\cdot, t-s),\, Y_k\rangle\, ds = 0, \quad t \in [0, T]. \tag{3.6.6}$$

Proof. By Lemma 3.6.2, $\sum_{k=1}^{\infty} [w_k^{(1)} - w_k^{(2)}] Y_k(1) = 0$. Substituting the definition (3.4.5) of $w_k^{(i)}$ and noting that all other terms cancel, we obtain (3.6.6) for each k. ∎

3.6.2 Uniqueness Theorem for the Inverse Source Problem

The following theorem on uniqueness for the inverse source problem is a direct consequence of in [23, Theorem 10.7].

Theorem 3.6.1 (Uniqueness of Source Recovery [23, Theorem 10.7]) *Let the source shape $r(x,t) = r(x)$ be time-independent, with $r \in L^2(\Omega)$. Suppose that the boundary measurement $h(t) = v(1,t)$, $t \in [0,T]$, uniquely determines the initial data (i.e., Lemma 3.6.1 applies). Then the observation $h(t)$ uniquely determines the source term $p(t)$: if two parameters $p_1, p_2 \in K$ generate the same boundary data $\Psi[p_1] = \Psi[p_2]$ on $[0,T]$, then $p_1 = p_2$ on $[0,T]$.*

Proof. Suppose $\Psi[p_1](t) = \Psi[p_2](t)$ for all $t \in [0,T]$. Set $\delta p = p_1 - p_2$. By Corollary 3.6.1, for each $k \geq 1$:

$$r_k \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k s^\alpha)\, \delta p(t-s)\, ds = 0, \quad t \in [0,T], \tag{3.6.7}$$

where $r_k = \langle r,\, Y_k \rangle$. Taking the Laplace transform of (3.6.7):

$$r_k \cdot \widehat{\delta p}(z) \cdot \frac{1}{z^\alpha + \lambda_k} = 0 \quad \text{for } \mathrm{Re}(z) > 0. \tag{3.6.8}$$

Since $(z^\alpha + \lambda_k)^{-1} \neq 0$ for $\mathrm{Re}(z) > 0$, we have $r_k\, \widehat{\delta p}(z) = 0$. Since $r \not\equiv 0$, there exists k_0 with $r_{k_0} \neq 0$, so $\widehat{\delta p}(z) = 0$ for $\mathrm{Re}(z) > 0$. By the injectivity of the Laplace transform, $\delta p = 0$ on $[0,T]$, i.e., $p_1 = p_2$. ∎

3.6.3 Injectivity of the Input–Output Map

Theorem 3.6.2 (Injectivity of Ψ) *Assume that $r \in L^2(\Omega)$ with $\langle r,\, Y_{k_0} \rangle \neq 0$ for some $k_0 \geq 1$. Then the input–output operator $\Psi[\,\cdot\,] : K \to C[0,T]$ is injective, i.e:*

$$\Psi[p_1] = \Psi[p_2] \quad \Longrightarrow \quad p_1 = p_2. \tag{3.6.9}$$

Equivalently, distinct parameters produce distinguishable observations:

$$p_1 \neq p_2 \Rightarrow \Psi[p_1] \neq \Psi[p_2].$$

Proof. This follows directly from Theorem 3.6.1 applied to mode $k = k_0$. ■

The distinguishability property of the input-output mapping $\Psi[\cdot] : K \longrightarrow C[0,T]$ was investigated using measured output data at $x = 1$. The measured output data $h(t)$ were obtained analytically as a series representation. This also leads to the input-output mapping $\Psi[\cdot]$ in an explicit form

3.6.4 Identification of the Fractional Order

A remarkable consequence of the Mittag-Leffler structure is that not only can the source $p(t)$ be recovered from boundary data, but under appropriate conditions the fractional order α itself can be uniquely identified. The following result, which is (Theorem 10.8) in [23], makes this precise.

Theorem 3.6.3 (Joint Uniqueness of Parameter and Fractional Order [23, Theorem 10.8]) *Let $q_1, q_2 \in C^1(\overline{\Omega})$ with $q_i(x) \geq 0$, and let $\alpha_1, \alpha_2 \in (0,1)$. Let $u(\cdot\,;\, q_i, \alpha_i)$ denote the solution of (3.2.1) with potential q_i and fractional order α_i, driven by boundary input $a(t)$ (with $a \not\equiv 0$ on $(0,T)$). If*

$$u(1,t;\, q_1, \alpha_1) = u(1,t;\, q_2, \alpha_2) \quad \text{for all } t \in (0,T], \tag{3.6.10}$$

then $\alpha_1 = \alpha_2$ and $q_1(x) = q_2(x)$ for all $x \in [0,1]$.

This theorem shows that a single boundary time-trace at $x = 1$ carries sufficient information to determine both the unknown potential and the fractional order of the diffusion model, a striking feature with no classical ($\alpha = 1$) analogue.

Remark 3.6.2 *In the classical case $\alpha = 1$, the analogous injectivity result holds with exponential kernel $e^{-\lambda_k s}$ replacing $s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k s^\alpha)$. However, Theorem 3.6.3 has no counterpart in the classical setting: one cannot recover α from boundary data when $\alpha = 1$ is fixed. The fractional model thus offers strictly more identifiability than the classical one, which is one of its principal advantages from an inverse problems perspective [23].*

Remark 3.6.3 (Well-Posedness of the Inverse Problem) *Theorems 3.5.2, 3.6.1, and 3.6.2 together establish three of the four classical Hadamard conditions for the inverse problem:*

1. ***Existence***: *Guaranteed by the direct problem theory (Chapter 6 of [23]).*

2. ***Uniqueness***: *Established by Theorem 3.6.1.*

3. ***Injectivity***: *Established by Theorem 3.6.2.*

4. ***Stability***: *Established by Theorem 3.5.2.*

The inverse problem is thus well-posed in the sense of Hadamard within the admissible class K, though it remains mildly ill-posed (in the sense of requiring regularisation when data are noisy) due to the Volterra structure of Ψ.Numerical Method

3.7 Numerical Method

3.7.1 Finite Difference Discretization

This section considers the inverse problem given by(3.2.1)–(3.2.3) and(3.4.3). We use the finite difference method to discretize this problem. The domain $[0,1]\times[0,T]$ is divided into an $M\times N$ mesh with the spatial step size $k=1/M$ in x direction and the time step size $\tau=T/N$, respectively.

The grid points x_i, t_n are defined by

$$x_i = ik;\ i=0,1,2,\ldots,M;$$

$$t_j = j\tau;\ j=0,1,2,\ldots,N;$$

in which M and N are integers. The notations $u_i^j, F_i^j, p^j, g_i, \psi_1^j$ and h^j denote finite difference approximations of $u(x_i,t_j), F(x_i,t_j), p(t_j), g(x_i), \psi_1(t_j)$ and $h(t_j)$, respectively.

The finite-difference approximation for discretizing problem (3.2.1)–(3.2.3)and (3.4.3) is:

$$\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{j+1}\frac{\Gamma(j-m-\alpha+1)}{(j-m)!}\frac{u_i^m-u_i^{m-1}}{\tau^\alpha}=\frac{1}{h^2}\left(u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}\right)-p^j u_i^{j+1}+F_i^j \tag{3.7.1}$$

$$u_i^0 = g_i \tag{3.7.2}$$

$$u_0^j = \psi_1^j \tag{3.7.3}$$

$$u_{M+1}^{j} = \left(u_{M}^{j} + u_{1}^{j} - \psi_{1}^{j}\right) \tag{3.7.4}$$

where $1 \leq i \leq M$ and $0 \leq j \leq N$.

Now, let us construct the predicting-correcting mechanism. Firstly, if we use the measured output data $u(1,t) = h(t)$, we obtain

$$p(t) = \frac{D_t^{\alpha} h(t) - u_{xx}(1,t) - F(1,t)}{h(t)} \tag{3.7.5}$$

The finite difference approximation of $p(t)$ is

$$p^{j} = \frac{H^{j} - \frac{1}{k^2}\left(u_{M+1}^{j+1} - 2u_{M}^{j+1} + u_{M-1}^{j+1}\right) - F_{M}^{j}}{h^{j}} \tag{3.7.6}$$

where $H^{j} = D_t^{\alpha} h(t_j)$, $j = 0, 1, \ldots, N$.

In numerical computation, since the time step is very small, we can take $p^{j(0)} = p^{j-1}$, $u_i^{j(0)} = u_i^{j-1}$, $j = 0, 1, 2, \ldots, N$, $i = 1, 2, \ldots, M$. At each s-th iteration step, we first determine $p^{j(s)}$ from the formula

$$p^{j(s)} = \frac{H^{j} - \frac{1}{k^2}\left(u_{M+1}^{j+1(s)} - 2u_{M}^{j+1(s)} + u_{M-1}^{j+1(s)}\right) - F_{M}^{j}}{h^{j}} \tag{3.7.7}$$

Then, from (3.7.1)-(3.7.4), we obtain:

$$\begin{aligned} \frac{1}{\Gamma(1-\alpha)} \sum_{m=1}^{j+1} \frac{\Gamma(j-m-\alpha+1)}{(j-m)!} \frac{u_i^{m(s+1)} - u_i^{m-1(s)}}{\tau^{\alpha}} = \frac{1}{h^2}\left(u_{i+1}^{j+1(s+1)} - 2u_i^{j+1(s+1)} + u_{i-1}^{j+1(s+1)}\right) \\ - p^{j(s)} u_i^{j+1(s+1)} + F_i^{j}. \end{aligned} \tag{3.7.8}$$

$$u_0^{j(s)} = \psi_1^{j} \tag{3.7.9}$$

$$u_{M+1}^{j(s)} = \left(u_M^{j(s)} + u_1^{j(s)} - \psi_1^{j}\right) \tag{3.7.10}$$

The system of Equations (3.7.2) and (3.7.8)-(3.7.10) can be solved by the Gauss elimination method and $u_i^{j+1(s+1)}$ is determined. If the difference in values between the two iterations reaches the prescribed tolerance, the iteration is stopped and we accept the corresponding values $p^{j(s)}$, $u_i^{j+1(s+1)}$ $(i = 1, 2, \ldots, N_x)$ as p^{j}, u_i^{j+1} $(i = 1, 2, \ldots, N_x)$ on the j-th time step, respectively. By virtue of this iteration, we can move from level j to level $j+1$.

3.7.2 Example

Consider the inverse problem of identifying the time-dependent coefficient $p(t)$ in the following time-fractional parabolic equation:

$$\begin{aligned} D_t^\alpha u(x,t) &= u_{xx}(x,t) - p(t)u(x,t) + F(x,t), \quad (x,t) \in (0,1) \times (0,T], \\ u(x,0) &= g(x), \\ u_x(0,t) &= u_x(1,t), \quad u(0,t) = \Psi_1(t), \\ h(t) &= u(1,t) \quad \text{(measured output)}. \end{aligned}$$

The fractional derivative D_t^α is understood in the Caputo sense. In this note we set $\alpha = 1/2$ and construct an exact solution that satisfies all the above conditions.

Exact Solution

We choose the following simple functions

$$\boxed{p(t) = 1 + t}, \qquad \boxed{u(x,t) = t^2 \cos(2\pi x)}.$$

From these we derive the remaining data

$$\begin{aligned} g(x) &= u(x,0) = 0, \\ \Psi_1(t) &= u(0,t) = t^2, \\ h(t) &= u(1,t) = t^2 \cos(2\pi) = t^2. \end{aligned}$$

The source term $F(x,t)$ is obtained by substituting the chosen u and p into the PDE:

$$F(x,t) = D_t^\alpha u - u_{xx} + p(t)u.$$

We now compute each component for $\alpha = 1/2$.

Fractional Derivative $D_t^{1/2}u$

Since $u(x,t) = t^2 \cos(2\pi x)$, we have

$$D_t^{1/2}u = \cos(2\pi x)\, D_t^{1/2}(t^2).$$

Using the formula $D_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}$ for $\beta = 2$, $\alpha = 1/2$:

$$\Gamma(3) = 2, \qquad \Gamma(2.5) = \frac{3}{4}\sqrt{\pi},$$

$$D_t^{1/2} t^2 = \frac{2}{\frac{3}{4}\sqrt{\pi}}\, t^{3/2} = \frac{8}{3\sqrt{\pi}}\, t^{3/2}.$$

Hence

$$D_t^{1/2} u = \frac{8}{3\sqrt{\pi}}\, t^{3/2} \cos(2\pi x).$$

Spatial Second Derivative

$$u_{xx} = t^2 \frac{\partial^2}{\partial x^2} \cos(2\pi x) = t^2 \left(-4\pi^2 \cos(2\pi x)\right) = -4\pi^2 t^2 \cos(2\pi x).$$

Thus

$$-u_{xx} = 4\pi^2 t^2 \cos(2\pi x).$$

Reaction Term

$$p(t)u = (1+t) \cdot t^2 \cos(2\pi x) = (t^2 + t^3) \cos(2\pi x).$$

Assembling $F(x,t)$

$$\begin{aligned} F(x,t) &= D_t^{1/2} u - u_{xx} + pu \\ &= \left[\frac{8}{3\sqrt{\pi}} t^{3/2}\right] \cos(2\pi x) + \left[4\pi^2 t^2\right] \cos(2\pi x) + \left[t^2 + t^3\right] \cos(2\pi x) \\ &= \cos(2\pi x) \left[t^3 + (4\pi^2 + 1) t^2 + \frac{8}{3\sqrt{\pi}} t^{3/2}\right]. \end{aligned}$$

Therefore, the source term is given by

$$\boxed{F(x,t) = \cos(2\pi x) \left[t^3 + (4\pi^2 + 1) t^2 + \frac{8}{3\sqrt{\pi}}\, t^{3/2}\right]}.$$

Verification of Consistency Conditions

The problem requires the following consistency conditions (C1) and (C2):

$$\text{(C1)} \quad g(0) = \Psi_1(0), \qquad \text{(C2)} \quad g'(1) = u_x(0,0).$$

With $g(x) = 0$ and $\Psi_1(t) = t^2$ we have $g(0) = 0$, $\Psi_1(0) = 0$ then (C1) holds. $g'(1) = 0$ and $u_x(x,t) = t^2(-2\pi \sin(2\pi x))$ gives $u_x(0,0) = 0$ hence (C2) holds.

Verification of Boundary Conditions

The nonlocal boundary condition requires $u_x(0,t) = u_x(1,t)$. $u_x(x,t) = -2\pi t^2 \sin(2\pi x)$. At $x = 0$: $u_x(0,t) = 0$. At $x = 1$: $\sin(2\pi) = 0$ so $u_x(1,t) = 0$. Thus the condition is satisfied.

The Dirichlet condition at $x = 0$: $u(0,t) = t^2 = \Psi_1(t)$ holds by construction.

The measured output at $x = 1$: $u(1,t) = t^2 \cos(2\pi) = t^2 \longrightarrow$ indeed $h(t) = t^2$.

Numerical Method (Brief Summary)

The numerical scheme uses a finite difference discretisation on a uniform grid with spatial step $k = 1/M$ and time step $\tau = T/N$. The Caputo derivative is approximated by

$$D_t^\alpha u(x_i, t_{j+1}) \approx \frac{1}{\Gamma(1-\alpha)} \sum_{m=1}^{j+1} \frac{\Gamma(j-m-\alpha+1)}{(j-m)!} \frac{u_i^m - u_i^{m-1}}{\tau^\alpha}.$$

For each time step j, an iterative predictor-corrector procedure is used:

1. Start with initial guesses $p^{j,(0)} = p^{j-1}$ and $u_i^{j+1,(0)} = u_i^j$.

2. Compute $p^{j,(s+1)}$ from the formula derived from the PDE at $x = 1$:

$$p^{j,(s+1)} = \frac{H^j - \frac{1}{k^2}\left(u_{M+1}^{j+1,(s)} - 2u_M^{j+1,(s)} + u_{M-1}^{j+1,(s)}\right) - F_M^j}{h^j},$$

   where H^j approximates $D_t^\alpha h(t_j)$.

3. Solve the tridiagonal system for $u_i^{j+1,(s+1)}$ using the updated $p^{j,(s+1)}$.

4. Stop when the change in u is below a tolerance.

The example above is recommended for validation because it yields a smooth, non-constant $p(t) = 1 + t$ and a simple spatial dependence $\cos(2\pi x)$. The numerical solution should recover $p(t)$ and $u(x,t)$ accurately, even in the presence of small measurement noise.

Conclusion

We have provided an independent test problem for the numerical method of Ozbilge et al. (2022) with $\alpha = 0.5$. All problem data are given explicitly, and the exact solution satisfies the nonlocal boundary conditions as well as the consistency requirements. This example can be used to verify the correctness and stability of the finite difference predictor-corrector algorithm. For practical implementation, the fractional derivative of the measured output $h(t) = t^2$ can be computed analytically: $D_t^{1/2} t^2 = \dfrac{8}{3\sqrt{\pi}} t^{3/2}$, which may be used to supply the exact H^j values in a noise-free test.

Bibliography

[1] Robert A. Adams and John J. F. Fournier, *Sobolev spaces*, 2nd ed., Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003.

[2] Agrawal, O.P. Fractional variational calculus and the transversality conditions. *Journal of Physics A: Mathematical and General*, **39**, pp. 10375-10384 (2006).

[3] Agrawal, O.P. A formulation and a numerical scheme for fractional optimal. In *FDA06 - Proceedings of the Second IFAC Workshop on Fractional Differentiation and Applications* (2006).

[4] Amanov, D, Ashyralyev, A. Initial-boundary value problem for fractional partial differential equations of higher order. *Abstr. Appl. Anal.* **2012**, Article ID 9733102 (2012).

[5] Ashyralyev, A, Dal, F. Finite Difference and Iteration Methods for Fractional Hyperbolic Partial Differential Equations with the Neumann Condition. *Discrete Dyn. Nat. Soc.* **2012**, Article ID 434976 (2012).

[6] Ashyralyev, A, Artykov, M, Cakir, Z. A note on fractional parabolic differential and difference equations. In: *AIP Conference Proceedings*, vol. 1611, pp. 251-254 (2014).

[7] Bulut, A.; Hacioglu, I. The energy of all connected cubic circulant graphs. *Linear Multilinear Algebra* **2020**, *68*, 679-685.

[8] Bulut, A.; Hacioglu, I. Asymptotic energy of connected cubic circulant graphs. *Akte Int. J. Graphs Comb.* **2021**, *18*, 25-28.

[9] J. R. Cannon, Determination of an unknown heat source from overspecified boundary data, *SIAM J. Numer. Anal.* **5** (1968), 275–286, DOI 10.1137/0705024.

[10] Cannon, J.R.; Lin, Y. Determination of source parameter in parabolic equations. *Mechanica* **1992**, *27*, 85-94.

[11] J. R. Cannon and Paul DuChateau, Structural identification of an unknown source term in a heat equation, *Inverse Problems* **14** (1998), no. 3, 535–551, DOI 10.1088/0266-5611/14/3/010.

[12] Caputo, M. Linear model of dissipation whose q is almost frequency independent - ii. *Geophys. J.R. Astr. Soc* **13**, pp. 529-539 (1967).

[13] Caepinteri, A.; Mainardi, F. *Fractals and Fractional Calculus in Continuum Mechanics*; Springer: Berlin/Heidelberg, Germany, 1997.

[14] Chen, W.; Fu, Z.J. Boundary particle method for inverse cauchy problems of inhomogeneous helmholtz equations. *J. Mar. Sci. Technol.* **2009**, *17*, 157-163.

[15] Djrbashian, M.M. Differential operators of fractional order and boundary value problems in the complex domain. In *The Gohberg Anniversary Collection*; Springer: Berlin/Heidelberg, Germany, 1989; pp. 153-172.

[16] Francesco, M.; Luchko, Y.; Pagnini, G. The fundamental solution of the space-time fractional diffusion equation. *arXiv* **2007**, arXiv:cond-mat/0702419.

[17] Grad Shteyn, I.S., and Ryzhik, I.M. *Table of Integrals, Series, and Products*. Elsevier Inc, Academic Press Inc, 2nd edition, 1980.

[18] P. Grisvard, *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics, vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[19] Hacioglu, I.; Bulut, A.; Kaskaloglu, K. The minimal and maximal energies of all cubic circulant graphs. *Akte Int. J. Graphs Comb.* **2021**, *18*, 148-153.

[20] Humbert, P. Quelques resultats relatifs a la fonction de mitag-leffler. *C. R. Hebd. Des Séances Acad. Des Sci.* **1953**, *236*, 1467-1468.

[21] Oleg Yu. Imanuvilov and Masahiro Yamamoto, Lipschitz stability in inverse parabolic problems by the Carleman estimate, *Inverse Problems* **14** (1998), no. 5, 1229–1245, DOI 10.1088/0266-5611/14/5/009.

[22] Bangti Jin and William Rundell, A tutorial on inverse problems for anomalous diffusion processes, *Inverse Problems* **31** (2015), no. 3, 035003, 40, DOI 10.1088/0266-5611/31/3/035003.

[23] B. Kaltenbacher, W. Rundell, *Inverse Problems for Fractional Partial Differential Equations*, Grad. Stud. Math., vol. 230, American Mathematical Society, Providence, R. Island, 2023, ISBN: 9781470472450, 9781470472764 (ebook).

[24] Giovanni Leoni, *A first course in Sobolev spaces*, Graduate Studies in Mathematics, vol. 105, American Mathematical Society, Providence, RI, 2009, DOI 10.1090/gsm/105.

[25] Li, Z.; Yamamoto, M. Uniqueness for inverse problems of determining orders of multi-term time-fractional derivatives of diffusion equation. *Appl. Anal.* **2015**, *94*, 570-579.

[26] Mittag-Leffler, G.M. Sur la nouvelle fonction e_a(x). *CR Acad. Sci. Paris* **1903**, *137*, 554-558.

[27] Mittag-Leffler, G., et al. Sue la représentation analytique d'une brache uniforme d'une fonction monogÃ¨ne: ciqiÃ¨me note. *Acta mathematica*, **29**: 101-181 (1905).

[28] Ozbilge E, Kanca F, Özbilge E. Inverse Problem for a Time Fractional Parabolic Equation with Nonlocal Boundary Conditions. *Mathematics*. 2022;10(9):1479. https://doi.org/10.3390/math10091479

[29] Alan Pierce, Unique identification of eigenvalues and coefficients in a parabolic problem, *SIAM J. Control Optim.* **17** (1979), no. 4, 494–499, DOI 10.1137/0317035.

[30] Podlubny, I. *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[31] William Rundell, Determination of an unknown nonhomogeneous term in a linear partial differential equation from overspecified boundary data, *Applicable Anal.* **10** (1980), no. 3, 231–242, DOI 10.1080/00036818008839304.

[32] Samko, S.G., Kilbas, A.A., and Marichev, O.I. *Fractional Integrals and Derivatives and Some of Their Applications.* Nauka i technika, Minsk, 1987.

[33] Stegum, I.A., Abramowitz M. *Handbook of Mathematical Functions.* 9th edition, 1964.

[34] Wei, H.; Chen, W.; Sun, H.; Li, X. A coupled method for inverse source problem of spatial fractional anomalous diffusion equations. *Inverse Probl. Sci. Eng.* **2010**, *18*, 945-956.

[35] Wei, T.; Zhang, Z.Q. Reconstruction of a time-dependent source term in a time-fractional diffusion equation. *Eng. Anal. Bound. Elem.* **2013**, *37*, 23-31.

[36] Wiman, A. Über den fundamentalsatz in der teorie der funktionen $e_a(x)$. *Acta Mathematica*, **29**(1): 191-201 (1905).

[37] Zhang, Y.; Xiang, X. Inverse source problem for a fractional diffusion equation. *Inverse Probl.* **2011**, *27*, 035010.

Abstract

This dissertation investigates an inverse coefficient problem for a time-fractional parabolic equation subject to nonlocal boundary conditions. We analyse the distinguishability property of the input–output mapping, develop a finite-difference based iterative reconstruction algorithm, and present numerical experiments that confirm the theoretical findings
and illustrate the stability (or instability) of the method under noisy data.

Key words : Fractional Derivative ; Inverse Problem ; Sturm Liouville Problem ; Finite Difference Method

الملخص

تبحث هذه الأطروحة في مسألة المعامل العكسي لمعادلة مكافئة كسرية زمنية خاضعة لشروط حدية غير محلية. نقوم بتحليل خاصية قابلية التمييز لتطبيق المدخلات والمخرجات، ونطور خوارزمية إعادة بناء تكرارية تعتمد على الفروق المنتهية، ونقدم تجارب عددية تؤكد النتائج النظرية وتوضح استقرار (أو عدم استقرار) الطريقة في ظل وجود بيانات مشوشة

الكلمات المفتاحية: المشتقة الكسرية، المشكل المعكوس، نظام ستورم ليوفيل، طريقة الفروق المنتهية

Résumé

Cette thèse étudie un problème de coefficient inverse pour une équation parabolique fractionnaire en temps soumise à des conditions aux limites non locales. Nous analysons la propriété de distinguabilité de l'application entrée-sortie, développons un algorithme de reconstruction itératif basé sur les différences finies, et présentons des expériences numériques qui confirment les résultats .théoriques et illustrent la stabilité (ou l'instabilité) de la méthode en présence de données bruitées

Mots clés : Dérivée Fractionaire ; Le problème Inverse ; Problème de Sturm Liouville ; Méthode des Différence Finis .