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Thème

La multiplication dans certains espaces fonctionnels de type de Herz

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Introduction

Pointwise multiplication has been a central topic in modern analysis, and are now of increasing applications in many fields of mathematics especially partial differential equations.

The theory of this problem had a remarkable development in recent years. For more details for Besov and Triebel-Lizorkin one can refer to [8] and [18].

In recent years many researchers have modified the classical spaces and have generalized the classical results to these modified ones. For example: Herz-type Besov and Triebel-Lizorkin spaces. They are modeled on Besov spaces and Triebel-Lizorkin spaces, but the underlying norm is of $\dot{K}_q^{\alpha,p}$ type rather than L^p. These function spaces introduced earlier in the papers of J. Xu and D. Yang [23] , [24] and [25].

The purpose of this thesis is then to consider the problem of pointwise multiplication on Herz-type Besov and Triebel-Lizorkin spaces.

Let us present the contents of this thesis. Chapter 1 collects fundamental notation and concepts. We recall some basic facts on Herz spaces and several basic properties of Herz-type Besov and Triebel-Lizorkin spaces. Some necessary tools are given. In Chapter 2 we prove some embeddings of the form

$$\dot{K}B \cdot \dot{K}B \hookrightarrow \dot{K}B$$

where $\dot{K}B$, will be defined the Herz-type Besov spaces.

In Chapter 3 we present some embeddings of the form

$$\dot{K}F \cdot \dot{K}B \hookrightarrow \dot{K}F$$

where $\dot{K}F$ and $\dot{K}B$, will be defined the space of Herz-type Lizorkin-Triebel and the space of Herz-type Besov respectively. We would like to mention that all the results of this thesis are from [6] and [1]

Notation

- $\mathbb{R}^n$ is the n-dimensional real Euclidean space.
- $\mathbb{N}$ is the collection of all natural numbers.
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- $\mathbb{Z}$ is the set of all integer numbers.
- $\mathbb{C}$ is the set of complex numbers.
- $\mathcal{S}(\mathbb{R}^n)$, is the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$.
- $\mathcal{S}'(\mathbb{R}^n)$ is the space of temperate distributions on $\mathbb{R}^n$.
- The Fourier transform is defined by :

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x)dx, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

Its inverse is defined by :

$$\mathcal{F}^{-1}(f)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} f(\xi)d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

- For $0 < p \leq \infty$ and $\Omega \subset \mathbb{R}^n$. The classical Lebesgue space $L^p(\Omega)$ is the class of all measurable functions f on Ω normed by (quassi-normed for $p < 1$):

$$\left\| f \mid L^p(\Omega) \right\| = \begin{cases} \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}} & \text{if} \quad 0 < p < \infty \\ \sup\limits_{x \in \Omega} -ess\, |f(x)| & \text{if} \quad p = \infty \end{cases}.$$

If $\Omega = \mathbb{R}^n$, then we put $L^p(\mathbb{R}^n) = L^p$ and $\left\| f \mid L^p(\mathbb{R}^n) \right\| = \|f\|_p$.

- ℓ^q is the set of sequences $(a_k)_k \subset \mathbb{C}$ such that :

$$\|(a_k)_k\|_{\ell^q} = \left(\sum_{k=0}^{\infty} |a_k|^q \right)^{1/q} < \infty.$$

- Let $0 < p < \infty, 0 < q \leq \infty$, so $\ell^q(L^p)$ is the space of sequence of functions $\{f_k\}_k$ such that:

$$\|\{f_k\}_k\|_{\ell^q(L^p)} = \left(\sum_{k=0}^{\infty} \|f_k\|_p^q \right)^{1/q} < \infty.$$

- p' is the conjugate of p: $\frac{1}{p} + \frac{1}{p'} = 1$.
- Let X and Y be two qussi-normed spaces, we say that $X \hookrightarrow Y$ if there exists $c > 0$ such that:

$$\|f\|_Y \leq c \|f\|_X , \quad \forall f \in X.$$

- Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ supp f is the support of the function f

$$\text{supp } f = \overline{\{x \in \mathbb{R}^n, f(x) \neq 0\}}.$$

- $B(x,r)$ the open ball in $\mathbb{R}^n$ with center x and radius r.

$$B(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\} .$$

- $f \in L^1_{\text{loc}}$: f is locally integrable function.
- $f * g(\cdot) = \int_{\mathbb{R}^n} f(\cdot - y) g(y) dy$ is the convolution of f and g.
- χ_E is the characteristic function on $E \subset \mathbb{R}^n$.

Chapter 1

Herz-type Besov and Triebel-Lizorkin spaces

In this chapter we recall some basic facts on Herz spaces and several basic properties of Herz-type Besov and Triebel-Lizorkin spaces. Also we give some key technical lemmas needed in the proofs of the main statements of this master thesis.

1.1 Herz spaces

For any $u > 0, k \in \mathbb{Z}$ we set $C(u) = \{x \in \mathbb{R}^n : u/2 < |x| \leq u\}$ and $C_k = C(2^k)$. Let χ_k, for $k \in \mathbb{Z}$, denote the characteristic function of the set C_k. In this section we derive several technical lemmas that were used in the thesis. We start by recalling the definition and some of the properties of the homogenous Herz spaces $\dot{K}_q^{\alpha,p}$.

Definition 1.1.1 *Let* $\alpha \in \mathbb{R}, 0 < p, q \leq \infty$. *The homogeneous Herz space* $\dot{K}_q^{\alpha,p}$ *is defined by*

$$\dot{K}_q^{\alpha,p} := \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \; \|f\chi_k\|_q^p \right)^{1/p} < \infty,$$

with the usual modifications made when $p = \infty$ *and/or* $q = \infty$.

Remark 1.1.2 *(i) the spaces* $\dot{K}_q^{\alpha,p}$ *are quasi-Banach spaces and if* $\min(p,q) \geq 1$ *then* $\dot{K}_q^{\alpha,p}$ *are Banach spaces .*
(ii) When $\alpha = 0$ *and* $0 < p = q \leq \infty$ *then* $\dot{K}_q^{\alpha,p}$ *coincides with the Lobesque spaces* $L^p(\mathbb{R}^n)$.
(iii) If $0 < p_1 \leq p_2 \leq \infty$, *we may derive the embedding*

$$\dot{K}_q^{\alpha,p_1} \hookrightarrow \dot{K}_q^{\alpha,p_2}.$$

A detailed discussion of the properties of these spaces my be found in the papers [10], [9], [12], [13], and references therein.

The classical Plancherel-Polya-Nikolskij inequality (cf. [19, 1.3.2/5, Rem. 1.4.1/4]), says that $\|f\|_q$ can be estimated by

$$c\ R^{n(1/p-1/q)}\ \|f\|_p$$

for any $0 < p \leq q \leq \infty$, $R > 0$ and any $f \in L^p(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp} \mathcal{F}f \subset \overline{B}(0,R)$. The constant $c > 0$ is independent of R. This inequality plays an important role in theory of function spaces and PDE's.

The following lemma is the $\dot{K}_q^{\alpha,p}$-version of the Plancherel-Polya-Nikolskij inequality.

Lemma 1.1.3 *Let* $\alpha_1, \alpha_2 \in \mathbb{R}$ *and* $0 < s, p, q, r \leq \infty$. *We suppose that* $\alpha_1 + n/s > 0, 0 < q \leq s \leq \infty$ *and* $\alpha_2 \geq \alpha_1$. *Then there exist a positive constant* $c > 0$ *independent of* R *such that for all* $f \in \dot{K}_q^{\alpha_2,p} \cap \mathcal{S}'(\mathbb{R}^n)$ *with* $\operatorname{supp} \mathcal{F}f \subset \overline{B}(0,R)$, *we have*

$$\|f\|_{\dot{K}_s^{\alpha_1,r}} \leq c\ R^{n/q-n/s+\alpha_2-\alpha_1}\ \|f\|_{\dot{K}_q^{\alpha_2,\theta}},$$

where

$$\theta = \begin{cases} r & \text{if} \quad \alpha_2 = \alpha_1 \\ p & \text{if} \quad \alpha_2 > \alpha_1. \end{cases}$$

Remark 1.1.4 We would like to mention that Lemma 1.1.3 generalizes the classical Plancherel-Polya-Nikolskij inequality by taking $\alpha_1 = \alpha_2 = 0$, $r = s$ and by using the embedding $\ell_q \hookrightarrow \ell_s$.

In the previous lemma we have not treated the case $s \leq q$. The next lemma gives a positive answer.

Lemma 1.1.5 *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $0 < s, p, q, r \leq \infty$. We suppose that $\alpha_1 + n/s > 0, 0 < s \leq q \leq \infty$ and $\alpha_2 > \alpha_1 + n/s - n/q$. Then there exist a positive constant c independent of R such that for all $f \in \dot{K}_q^{\alpha_2,p} \cap \mathcal{S}'(\mathbb{R}^n)$ with* $\operatorname{supp} \mathcal{F}f \subset \overline{B}(0,R)$*, we have*

$$\|f\|_{\dot{K}_s^{\alpha_1,r}} \leq c\ R^{n/q-n/s+\alpha_2-\alpha_1} \|f\|_{\dot{K}_q^{\alpha_2,p}}.$$

For the proof see [3].

1.2 Function spaces

In this section we present the Fourier analytical definition of Herz-type Besov spaces $\dot{K}_q^{\alpha,p}B_\beta^s$ and Herz-type Triebel-Lizorkin spaces $\dot{K}_q^{\alpha,p}F_\beta^s$ and recall their basic properties. We first need the concept of a smooth dyadic resolution of unity.

Definition 1.2.1 *Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\Psi(x) = 1$ for $|x| \leq 1$ and $\Psi(x) = 0$ for $|x| \geq \frac{3}{2}$. We put $\varphi_0(x) = \Psi(x)$, $\varphi_1(x) = \Psi(x/2) - \Psi(x)$ and*

$$\varphi_j(x) = \varphi_1\left(2^{-j+1}x\right) \quad \textit{for} \quad j = 2, 3, ...$$

Then we have $\operatorname{supp} \varphi_j \subset \{x \in \mathbb{R}^n : 2^{j-1} \leq |x| \leq 3 \cdot 2^{j-1}\}$, $\varphi_j(x) = 1$ *for* $3 \cdot 2^{j-2} \leq |x| \leq 2^j$ *and*

$$\Psi(x) + \sum_{j=0}^{\infty} \varphi_j(x) = 1$$

for all $x \in \mathbb{R}^n$. The system of functions $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is called a smooth dyadic resolution of unity. We define the convolution operators Δ_j by the following:

$$\Delta_j f = \mathcal{F}^{-1}\varphi_j * f, \quad j \in \mathbb{N} \quad \textit{and} \quad \Delta_0 f = \mathcal{F}^{-1}\Psi * f, \quad f \in \mathcal{S}'(\mathbb{R}^n).$$

Thus we obtain the Littlewood-Paley decomposition

$$f = \sum_{j=0}^{\infty} \Delta_j f$$

of all $f \in \mathcal{S}'(\mathbb{R}^n)$ *(convergence in* $\mathcal{S}'(\mathbb{R}^n)$*).*

We are now in a position to state the definitions of Herz-type Besov and Triebel-Lizorkin spaces.

Definition 1.2.2 (i) *Let* $\alpha, s \in \mathbb{R}$ *and* $0 < p, q, \beta \leq \infty$*. The Herz-type Besov space* $\dot{K}_q^{\alpha,p} B_\beta^s$ *is the collection of all* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that*

$$\|f\|_{\dot{K}_q^{\alpha,p} B_\beta^s} = \left(\sum_{j=0}^{\infty} 2^{js\beta} \|\Delta_j f\|_{\dot{K}_q^{\alpha,p}}^{\beta} \right)^{1/\beta} < \infty,$$

with the obvious modification if $\beta = \infty$.

(ii) *Let* $\alpha, s \in \mathbb{R}, 0 < p, q < \infty$ *and* $0 < \beta \leq \infty$*. The Herz-type Triebel-Lizorkin space* $\dot{K}_q^{\alpha,p} F_\beta^s$ *is the collection of all* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that*

$$\|f\|_{\dot{K}_q^{\alpha,p} F_\beta^s} = \left\| \left(\sum_{j=0}^{\infty} 2^{js\beta} |\Delta_j f|^{\beta} \right)^{1/\beta} \right\|_{\dot{K}_q^{\alpha,p}} < \infty,$$

with the obvious modification if $\beta = \infty$.

Remark 1.2.3 Let $s \in \mathbb{R}, 0 < p, q, \beta \leq \infty$ (with $0 < p, q < \infty$ for $\dot{K}_q^{\alpha,p} F_\beta^s$ spaces) and $\alpha > -n/q$. The spaces $\dot{K}_q^{\alpha,p} B_\beta^s$ and $\dot{K}_q^{\alpha,p} F_\beta^s$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_{j \in \mathbb{N}_0}$ appearing in theirs definitions (in the sense of equivalent quasi-norms). In particular both $\dot{K}_q^{\alpha,p} B_\beta^s$ and $\dot{K}_q^{\alpha,p} F_\beta^s$ are quasi-Banach spaces and if $p, q, \beta \geq 1$, then both $\dot{K}_q^{\alpha,p} B_\beta^s$ and $\dot{K}_q^{\alpha,p} F_\beta^s$ are Banach spaces. Further results, concerning, for instance, lifting properties, fourier multiplier and local means characterizations can be found in [*3*], [*24*], [*25*] and [*27*].

Now we give the definitions of the spaces $B_{p,\beta}^s$ and $F_{p,\beta}^s$.

Definition 1.2.4 (i) *Let* $s \in \mathbb{R}$ *and* $0 < p, \beta \leq \infty$. *The Besov space* $B^s_{p,\beta}$ *is the collection of all* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that*

$$\|f\|_{B^s_{p,\beta}} = \left(\sum_{j=0}^{\infty} 2^{js\beta} \|\Delta_j f\|_p^{\beta} \right)^{1/\beta} < \infty.$$

(ii) *Let* $s \in \mathbb{R}, 0 < p < \infty$ *and* $0 < \beta \leq \infty$. *The Triebel-Lizorkin space* $F^s_{p,\beta}$ *is the collection of all* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that*

$$\|f\|_{F^s_{p,\beta}} = \left\| \left(\sum_{j=0}^{\infty} 2^{js\beta} |\Delta_j f|^{\beta} \right)^{1/\beta} \right\|_p < \infty.$$

The theory of the spaces $B^s_{p,q}$ and $F^s_{p,\beta}$ has been developed in detail in [19], [20] and [21] but has a longer history already including many contributors; we do not want to discuss this here. In particular, with $p = q = \infty, s > 0$, one recovers Hölder-Zygmund spaces $\mathcal{C}^s = B^s_{\infty,\infty}$, cf. [19, Thm. 2.5.12]. Clearly, for $s \in \mathbb{R}, 0 < p \leq \infty$ ($0 < p < \infty$ for the $\dot{K}^{0,p}_p F^s_\beta$ spaces) and $0 < \beta \leq \infty$,

$$\dot{K}^{0,p}_p B^s_\beta = B^s_{p,\beta} \quad \text{and} \quad \dot{K}^{0,p}_p F^s_\beta = F^s_{p,\beta}.$$

1.3 Embeddings

The following theorem gives basic embeddings of the spaces $\dot{K}^{\alpha,p}_q B^s_\beta$ and $\dot{K}^{\alpha,p}_q F^s_\beta$. For $\dot{K}^{\alpha,p}_q F^s_\beta$ spaces these results are proved in [24]. Their arguments are true for $\dot{K}^{\alpha,p}_q B^s_\beta$ spaces.

Theorem 1.3.1 *Let* $s \in \mathbb{R}, 0 < p, q \leq \infty$ *and* $\alpha > -n/q$.
(i) *If* $0 < \beta_1 \leq \beta_2 \leq \infty$, *then*

$$\dot{K}^{\alpha,p}_q B^s_{\beta_1} \hookrightarrow \dot{K}^{\alpha,p}_q B^s_{\beta_2} \quad \textit{and} \quad \dot{K}^{\alpha,p}_q F^s_{\beta_1} \hookrightarrow \dot{K}^{\alpha,p}_q F^s_{\beta_2}.$$

(ii) *If* $0 < \beta_1, \beta_2 \leq \infty$ *and* $\varepsilon > 0$, *then*

$$\dot{K}^{\alpha,p}_q B^{s+\varepsilon}_{\beta_1} \hookrightarrow \dot{K}^{\alpha,p}_q B^s_{\beta_2} \quad \textit{and} \quad \dot{K}^{\alpha,p}_q F^{s+\varepsilon}_{\beta_1} \hookrightarrow \dot{K}^{\alpha,p}_q F^s_{\beta_2}.$$

(iii) *If* $0 < p_1 \leq p_2 \leq \infty$*, then*

$$\dot{K}_q^{\alpha,p_1} B_\beta^s \hookrightarrow \dot{K}_q^{\alpha,p_2} B_\beta^s \quad \textit{and} \quad \dot{K}_q^{\alpha,p_1} F_\beta^s \hookrightarrow \dot{K}_q^{\alpha,p_2} F_\beta^s.$$

(iv) $0 < q_1 \leq q_2 \leq \infty$*, then*

$$\dot{K}_{q_2}^{\alpha,p} B_\beta^s \hookrightarrow \dot{K}_{q_1}^{r,p} B_\beta^s \quad \textit{and} \quad \dot{K}_{q_2}^{\alpha,p} F_\beta^s \hookrightarrow \dot{K}_{q_1}^{r,p} F_\beta^s,$$

where $r = \alpha - n\left(1/q_1 - 1/q_2\right)$.

The following theorem gives basic embeddings between the spaces $\dot{K}_q^{\alpha,p} B_\beta^s$ and $\dot{K}_q^{\alpha,p} F_\beta^s$.

Theorem 1.3.2 *Let* $s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leq \infty$ *and* $\alpha > -n/q$*. Then*

$$\dot{K}_q^{\alpha,p} B_{\min(\beta,p,q)}^s \hookrightarrow \dot{K}_q^{\alpha,p} F_\beta^s \hookrightarrow \dot{K}_q^{\alpha,p} B_{\max(\beta,p,q)}^s. \tag{1.1}$$

Remark 1.3.3 Theorem 1.3.1 when $\alpha = 0$, $p = q$ generalizes the corresponding results on Besov and Triebel-Lizorkin spaces established in [*19*], Section 2.3.

Theorem 1.3.4 (i) *Let* $s \in \mathbb{R}, 0 < p, q, \beta \leq \infty$ *and* $\alpha > -n/q$*. The Herz-type Besov space* $\dot{K}_q^{\alpha,p} B_\beta^s$ *is a quasi-Banach space. Furthermore,*

$$\mathcal{S}\left(\mathbb{R}^n\right) \hookrightarrow \dot{K}_q^{\alpha,p} B_\beta^s \hookrightarrow \mathcal{S}'\left(\mathbb{R}^n\right). \tag{1.2}$$

If $0 < p, q, \beta < \infty, s \in \mathbb{R}$ *and* $\alpha > -n/q$*, then* $\mathcal{S}\left(\mathbb{R}^n\right)$ *is dense in* $\dot{K}_q^{\alpha,p} B_\beta^s$*.*
(ii) *Let* $s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leq \infty$ *and* $\alpha > -n/q$*. The Herz-type Triebel-Lizorkin space* $\dot{K}_q^{\alpha,p} F_\beta^s$ *is a quasi-Banach space. Furthermore,*

$$\mathcal{S}\left(\mathbb{R}^n\right) \hookrightarrow \dot{K}_q^{\alpha,p} F_\beta^s \hookrightarrow \mathcal{S}'\left(\mathbb{R}^n\right). \tag{1.3}$$

If $s \in \mathbb{R}, 0 < p, q, \beta < \infty$ *and* $\alpha > -n/q$*, then* $\mathcal{S}\left(\mathbb{R}^n\right)$ *is dense in* $\dot{K}_q^{\alpha,p} F_\beta^s$*.*

Chapter 2

Multiplication of type $\dot{K}B \cdot \dot{K}B \hookrightarrow \dot{K}B$

This chapter is concerned with proving some embeddings of the form

$$\dot{K}B \cdot \dot{K}B \hookrightarrow \dot{K}B$$

where $\dot{K}B$, with five indices, will be defined the Herz-type Besov spaces.

2.1 Preparation

Consider the partition of the unity

$$\psi\left(\xi\right) + \sum_{j=1}^{\infty} \phi\left(2^{1-j}\xi\right) = 1 \qquad \left(\xi \in \mathbb{R}^n\right)$$

where $\phi, \varphi \in C^{\infty}$, positive functions, supp $\phi \subset \left\{\xi \in \mathbb{R}^n : 1 \leq |\xi| \leq 3\right\}$ and supp $\psi \subset \left\{\xi \in \mathbb{R}^n : |\xi| \leq \frac{3}{2}\right\}$. We associate the convolution operators Q_j and Δ_k defined as

$$\begin{cases} Q_j f = \mathcal{F}^{-1}\left(\psi(2^{-j}\cdot)\right) * f & \left(j \in \mathbb{N}\right), \\ \Delta_k f = \mathcal{F}^{-1}\left(\phi(2^{-k}\cdot)\right) * f & \left(k \in \mathbb{N} \setminus \{0\}\right). \end{cases}$$

We set $\Delta_0 = Q_0$, thus we obtain the Littlewood-Paley's decomposition $f = \sum_{j=0}^{\infty} \Delta_j f$ (convergence in $\mathcal{S}'\left(\mathbb{R}^n\right)$), with $Q_k f = \sum_{j=0}^{k} \Delta_j f$.

For $g, f \in \mathcal{S}'(\mathbb{R}^n)$ we define the product of these distributions as

$$f \cdot g = \lim_{j \to \infty} Q_j f \cdot Q_j g$$

whenever this limit exists in $\mathcal{S}'$. Related to this definition we introduce the following operators

$$\Pi_1(f,g) = \sum_{j=2}^{\infty} Q_{j-2} g \cdot \Delta_j f,$$

$$\Pi_2(f,g) = \sum_{j=0}^{\infty} \overline{\Delta}_j g \cdot \Delta_j f,$$

and

$$\Pi_3(f,g) = \sum_{j=2}^{\infty} Q_{k-2} f \cdot \Delta_k g,$$

with $\overline{\Delta}_j = \sum_{k=j-1}^{j+1} \Delta_k, j \in \mathbb{N}_0$. The advantage of the above decomposition consists in

$$\mathrm{supp}\mathcal{F}\left(Q_{j-2} f \cdot \Delta_j g\right) \subset \{\xi : 2^{j-3} \le |\xi| \le 2^{j+1}\}, \quad j \ge 2 \tag{2.1}$$

and

$$\mathrm{supp}\mathcal{F}\left(\overline{\Delta}_j g \cdot \Delta_j f\right) \subset \{\xi : |\xi| \le 5 \cdot 2^j\}, \quad j \ge 0. \tag{2.2}$$

Various important results have been proved in the space $\dot{K}_p^{\alpha,q}$ under some assumptions on α, p and q. The conditions $-\frac{n}{p} < \alpha < n(1-\frac{1}{p}), 1 < p < \infty$ and $0 < q \le \infty$ is crucial in the study of the boundedness of classical operators in $\dot{K}_p^{\alpha,q}$ spaces. This fact was first realized by Li and Yang [11] with the proof of the boundedness of the maximal function. In this section we present some results which are useful for us. As usual, we put

$$\mathcal{M}(f)(x) := \sup_Q \frac{1}{|Q|} \int_Q |f(y)|\, dy, \quad f \in L^1_{\mathrm{loc}},$$

where the supremum is taken over all cubes with sides parallel to the axis and $x \in Q$.

Lemma 2.1.1 *Let $1 < p < \infty$ and $0 < q \le \infty$. If f is a locally integrable functions on $\mathbb{R}^n$ and $-\frac{n}{p} < \alpha < n(1-\frac{1}{p})$, then*

$$\left\| \mathcal{M} f \right\|_{\dot{K}_p^{\alpha,q}} \lesssim \left\| f \right\|_{\dot{K}_p^{\alpha,q}}.$$

The next three lemmas are used in the proof of our result, see [16] for the Besov and Triebel-Lizorkin.

Lemma 2.1.2 *Let* $s \in \mathbb{R}, A, B > 0,\ 0 < p, q \leq \infty$ *and* $\alpha > -\frac{n}{p}$. *Let* $\{f_l\}_{l\in\mathbb{N}_0}$ *be a sequence of functions such that*

$$\mathrm{supp}\mathcal{F}f_0 \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq A\}$$

and

$$\mathrm{supp}\mathcal{F}f_l \subseteq \left\{\xi \in \mathbb{R}^n : B2^{l+1} \leq |\xi| \leq A2^{l+1}\right\}.$$

There exists a constant $c > 0$ *such that the following inequalities*

$$\Big\|\sum_{l=0}^{\infty} f_l\Big\|_{\dot{K}_p^{\alpha,q}B_\beta^s} \leq c\Big(\sum_{l=0}^{\infty} 2^{ls\beta}\big\|f_l\big\|_{\dot{K}_p^{\alpha,q}}^{\beta}\Big)^{1/\beta}$$

and

$$\Big\|\sum_{l=0}^{\infty} f_l\Big\|_{\dot{K}_p^{\alpha,q}F_\beta^s} \leq c\Big\|\Big(\sum_{l=0}^{\infty} 2^{ls\beta}\left|f_l\right|^{\beta}\Big)^{1/\beta}\Big\|_{\dot{K}_p^{\alpha,q}}, \quad 0 < p, q < \infty$$

hold.

Lemma 2.1.3 *Let* $A, B > 0,\ 0 < p, q, \beta \leq \infty$ *and* $-\frac{n}{p} < \alpha < n - \frac{n}{p}$. *Let* $\{f_l\}_{l\in\mathbb{N}_0}$ *be a sequence of functions such that*

$$\mathrm{supp}\mathcal{F}f_l \subseteq \left\{\xi \in \mathbb{R}^n : |\xi| \leq A2^{l+1}\right\}.$$

Then it holds that

$$\Big\|\sum_{l=0}^{\infty} f_l\Big\|_{\dot{K}_p^{\alpha,q}B_\beta^s} \lesssim \Big\|\left\{2^{ls}f_l\right\}_l\Big\|_{\ell^\beta(\dot{K}_p^{\alpha,q})}, \quad s > n(\max\{1, \frac{1}{p}\} - 1).$$

and

$$\Big\|\sum_{l=0}^{\infty} f_l\Big\|_{\dot{K}_p^{\alpha,q}F_\beta^s} \leq c\Big\|\Big(\sum_{l=0}^{\infty} 2^{ls\beta}\left|f_l\right|^{\beta}\Big)^{1/\beta}\Big\|_{\dot{K}_p^{\alpha,q}}, \quad s > n(\max\{1, \frac{1}{p}, \frac{1}{q}\} - 1), 0 < p, q < \infty$$

Lemma 2.1.4 *Let* $0 < a < 1$ *and* $0 < q \leq \infty$. *Let* $\{\varepsilon_k\}$ *of positive real numbers, such that* $\|\{\varepsilon_k\}\|_{\ell^q} = A < \infty$. *Then the sequences*

$$\left\{\delta_k : \delta_k = \sum_{j=0}^{k} a^{k-j}\varepsilon_j\right\}$$

and

$$\left\{\eta_k : \eta_k = \sum_{j=k}^{\infty} a^{j-k}\varepsilon_j\right\}$$

are in ℓ^q *with*

$$\|\{\delta_k\}\|_{\ell^q} + \|\{\eta_k\}\|_{\ell^q} \leq cA.$$

c *depends only on* a *and* q.

2.2 Main results

The following results give an extension of the sufficient hypotheses obtained in Besov and Triebel-Lizorkin spaces.

Theorem 2.2.1 *Let* $1 < p, p_1, p_2, q, q_1, q_2 < \infty,\ 1 \leq \beta, \beta_2 \leq \infty,\ -\frac{n}{p_1} < \alpha_1 < n - \frac{n}{p_1}$ *and* $-\frac{n}{p_2} < \alpha_2 < n - \frac{n}{p_2}$. *Assume that*

$$\tfrac{1}{q} = \tfrac{1}{q_1} + \tfrac{1}{q_2}, \quad \alpha = \alpha_1 + \alpha_2, \quad 0 < s < r$$

and

$$\tfrac{1}{p} = \tfrac{1}{p_1} + \tfrac{1}{p_2}.$$

Then the embedding

$$\dot{K}_{p_1}^{\alpha_1, q_1} B_\beta^s \cdot \dot{K}_{p_2}^{\alpha_2, q_2} B_{\beta_2}^r \hookrightarrow \dot{K}_p^{\alpha, q} B_\beta^s$$

holds.

Proof.

- *Estimate of* $\Pi_1(f,g)$. Using Lemma 2.1.2 together with (2.1) we obtain

$$\big\|\Pi_1(f,g)\big\|_{\dot{K}_p^{\alpha,q}B_\beta^s} \lesssim \Big(\sum_{j=2}^{\infty} 2^{js\beta}\Big\|Q_{j-2}g\cdot\Delta_j f\Big\|_{\dot{K}_p^{\alpha,q}}^{\beta}\Big)^{1/\beta}.$$

By Hölder's inequality,

$$2^{k\alpha}\big\|\,(Q_{j-2}g\cdot\Delta_j f)\,\chi_k\big\|_p \le 2^{k\alpha_1}\big\|\,(\Delta_j f)\,\chi_k\big\|_{p_1}2^{k\alpha_2}\big\|\,(Q_{j-2}g)\,\chi_k\big\|_{p_2}$$

for any $k\in\mathbb{Z}$ and $j\ge 2$. Again from Hölder's inequality and the fact that $\alpha=\alpha_1+\alpha_2$ we obtain

$$\Big\|Q_{j-2}g\cdot\Delta_j f\Big\|_{\dot{K}_p^{\alpha,q}} \le \Big\|Q_{j-2}g\Big\|_{\dot{K}_{p_1}^{\alpha_1,q_1}}\Big\|\Delta_j f\Big\|_{\dot{K}_{p_2}^{\alpha_2,q_2}}.$$

We have

$$\begin{aligned}\Big\|Q_{j-2}g\Big\|_{\dot{K}_{p_2}^{\alpha_2,q_2}} &\le \sum_{k=0}^{j-2}\Big\|\Delta_k g\Big\|_{\dot{K}_{p_2}^{\alpha_2,q_2}}\\ &= \sum_{k=0}^{j-2}2^{-rk}2^{rk}\Big\|\Delta_k g\Big\|_{\dot{K}_{p_2}^{\alpha_2,q_2}}\\ &\lesssim \sup_{k\ge 0}\Big(2^{rk}\Big\|\Delta_k g\Big\|_{\dot{K}_{p_2}^{\alpha_2,q_2}}\Big)\sum_{k=0}^{j-2}2^{-rk}\\ &\lesssim \Big\|g\Big\|_{\dot{K}_{p_2}^{\alpha_2,q_2}B_{\beta_2}^r}.\end{aligned}$$

Hence

$$\begin{aligned}\big\|\Pi_1(f,g)\big\|_{\dot{K}_p^{\alpha,q}B_\beta^s} &\lesssim \Big\|g\Big\|_{\dot{K}_{p_2}^{\alpha_2,q_2}B_{\beta_2}^r}\Big(\sum_{j=2}^{\infty}2^{js\beta}\Big\|\Delta_j f\Big\|_{\dot{K}_{p_1}^{\alpha_1,q_1}}^{\beta}\Big)^{1/\beta}\\ &\lesssim \Big\|g\Big\|_{\dot{K}_{p_2}^{\alpha_2,q_2}B_{\beta_2}^r}\Big\|f\Big\|_{\dot{K}_{p_1}^{\alpha_1,q_1}B_{\beta_2}^s}.\end{aligned}$$

- *Estimate of* $\Pi_2(f,g)$. Since $s>0$, by Lemma 2.1.3 together with (2.2) we obtain

$$\big\|\Pi_2(f,g)\big\|_{\dot{K}_p^{\alpha,q}B_\beta^s} \lesssim \Big(\sum_{k=0}^{\infty}2^{ks\beta}\big\|\Delta_k f\cdot\overline{\Delta}_k g\big\|_{\dot{K}_p^{\alpha,q}}^{\beta}\Big)^{1/\beta}.$$

Hölder's inequality yields

$$\begin{aligned}2^{ks}\big\|\Delta_k f\cdot\overline{\Delta}_k g\big\|_{\dot{K}_p^{\alpha,q}} &\lesssim 2^{kr}\,\big\|\overline{\Delta}_k g\big\|_{\dot{K}_{p_1}^{\alpha_1,q_1}}\,2^{k(s-r)}\Big\|\Delta_k f\Big\|_{\dot{K}_{p_1}^{\alpha_1,q_1}}\\ &\lesssim \Big\|g\Big\|_{\dot{K}_{p_2}^{\alpha_2,q_2}B_{\beta_2}^r}2^{k(s-r)}\Big\|\Delta_k f\Big\|_{\dot{K}_{p_1}^{\alpha_1,q_1}}\end{aligned}$$

and hence

$$\big\|\Pi_2(f,g)\big\|_{\dot K_p^{\alpha,q}B_\beta^s} \lesssim \Big\|g\Big\|_{\dot K_{p_2}^{\alpha_2,q_2}B_{\beta_2}^r}\big\|f\big\|_{\dot K_{p_1}^{\alpha_1,q_1}B_\beta^s},$$

because of $s < r$.

- *Estimate of* $\Pi_3(f,g)$. Using Lemma 2.1.3 we obtain

$$\big\|\Pi_3(f,g)\big\|_{\dot K_p^{\alpha,q}B_\beta^s} \lesssim \Big(\sum_{k=2}^{\infty} 2^{ks\beta}\big\|Q_{k-2}f\cdot\Delta_k g\big\|_{\dot K_p^{\alpha,q}}^{\beta}\Big)^{1/\beta}.$$

On the other hand Hölder's inequality yields

$$2^{ks}\big\|Q_{k-2}f\cdot\Delta_k g\big\|_{\dot K_v^{\alpha_2,q}} \le c\,2^{kr}\big\|\Delta_k g\big\|_{\dot K_{p_2}^{\alpha_2,q_2}}2^{k(s-r)}\sum_{j=0}^{k-2}2^{-js}2^{js}\big\|\Delta_j f\big\|_{\dot K_{p_1}^{\alpha_1,q_1}}.$$

By applying inequality of Hölder again in ℓ^β-norm and Lemma 2.1.4, (because $s-r<0$), we obtain the bound

$$c\Big\|g\Big\|_{\dot K_{p_2}^{\alpha_2,q_2}B_{\beta_2}^r}\big\|f\big\|_{\dot K_{p_1}^{\alpha_1,q_1}B_\beta^s}.$$

□

Theorem 2.2.2 *Let* $1<p,p_1,p_2,q,q_1,q_2<\infty,\ 1\le\beta,\beta_2\le\infty,\ -\frac{n}{p_1}<\alpha_1<n-\frac{n}{p_1}$ *and* $-\frac{n}{p_2}<\alpha_2<n-\frac{n}{p_2}$. *Assume that*

$$\tfrac{1}{q}=\tfrac{1}{q_1}+\tfrac{1}{q_2},\quad \alpha=\alpha_1+\alpha_2, 0<s<r$$

and

$$\tfrac{1}{p}=\tfrac{1}{p_1}+\tfrac{1}{p_2}.$$

Then the embedding

$$\dot K_{p_1}^{\alpha_1,q_1}F_\beta^s\cdot\dot K_{p_2}^{\alpha_2,q_2}F_{\beta_2}^r \hookrightarrow \dot K_p^{\alpha,q}F_\beta^s$$

holds.

Proof.

- *Estimate of* $\Pi_1(f,g)$. Using Lemma 2.1.2 combined with (2.1) we obtain

$$\big\|\Pi_1(f,g)\big\|_{\dot K_p^{\alpha,q}F_\beta^s} \lesssim \Big\|\Big(\sum_{j=2}^{\infty}2^{js\beta}\,|Q_{j-2}g\cdot\Delta_j f|^\beta\Big)^{1/\beta}\Big\|_{\dot K_p^{\alpha,q}}$$

Since

$$|Q_{j-2}g\left(x\right)\Delta_{j}f\left(x\right)| \leq \sup_{j\in\mathbb{N}}|Q_{j-2}g\left(x\right)||\Delta_{j}f\left(x\right)| \lesssim \mathcal{M}g\left(x\right)|\Delta_{j}f\left(x\right)|$$

for any $x\in\mathbb{R}^{n}$ and any $j\geq 2$, we obtain

$$\Big\|\left\{2^{js}Q_{j-2}g\cdot\Delta_{j}f\right\}_{j\geq 2}\Big\|_{\ell^{\beta}} \leq c\mathcal{M}g\big\|\left\{2^{js}\Delta_{j}f\right\}_{j\geq 2}\big\|_{\ell^{\beta}}.$$

Therefore,

$$\Big\|\Big(\sum_{j=2}^{\infty}2^{js\beta}\left|Q_{j-2}g\cdot\Delta_{j}f\right|^{\beta}\Big)^{1/\beta}\Big\|_{\dot{K}_{p}^{\alpha,q}}$$

is bounded by

$$\left\|\mathcal{M}g\right\|_{\dot{K}_{p_{2}}^{\alpha_{2},q_{2}}}\left\|\left\{\Delta_{j}f\right\}_{j\geq 2}\right\|_{\dot{K}_{p_{1}}^{\alpha_{1},q_{1}}(\ell^{\beta})} \leq c\left\|g\right\|_{\dot{K}_{p_{2}}^{\alpha_{2},q_{2}}}\left\|f\right\|_{\dot{K}_{p_{1}}^{\alpha_{1},q_{1}}F_{\beta}^{s}}$$

where we have used Lemma 2.1.1. By the embedding

$$\dot{K}_{p_{2}}^{\alpha_{2},q_{2}}B_{\infty}^{r}\hookrightarrow\dot{K}_{p_{2}}^{\alpha_{2},q_{2}},$$

we obtain the the desired term.

- *Estimate of* $\Pi_{2}(f,g)$. Since $s>0$, by Lemma 2.1.3 together with (2.2) we obtain

$$\begin{aligned}\left\|\Pi_{2}(f,g)\right\|_{\dot{K}_{p}^{\alpha,q}F_{\beta}^{s}} &\lesssim \Big\|\Big(\sum_{k=0}^{\infty}2^{ks\beta}|\Delta_{k}f\cdot\overline{\Delta}_{k}g|^{\beta}\Big)^{1/\beta}\Big\|_{\dot{K}_{p}^{\alpha,q}}\\ &\lesssim \Big\|\sup_{k\geq 0}|\overline{\Delta}_{k}g|\Big(\sum_{k=0}^{\infty}2^{ks\beta}|\Delta_{k}f|^{\beta}\Big)^{1/\beta}\Big\|_{\dot{K}_{p}^{\alpha,q}}.\end{aligned}$$

Hölder's inequality yields the last term is bounded by

$$\begin{aligned}&\Big\|\sup_{k\geq 0}|\overline{\Delta}_{k}g|\Big\|_{\dot{K}_{p_{2}}^{\alpha_{2},q_{2}}}\Big\|\Big(\sum_{k=0}^{\infty}2^{ks\beta}|\Delta_{k}f|^{\beta}\Big)^{1/\beta}\Big\|_{\dot{K}_{p_{1}}^{\alpha_{1},q_{1}}}\\ &c\Big\|g\Big\|_{\dot{K}_{p_{2}}^{\alpha_{2},q_{2}}F_{\beta_{2}}^{r}}\left\|f\right\|_{\dot{K}_{p_{1}}^{\alpha_{1},q_{1}}F_{\beta}^{s}}.\end{aligned}$$

- *Estimate of* $\Pi_{3}(f,g)$. Using Lemma 2.1.3 we obtain

$$\left\|\Pi_{3}(f,g)\right\|_{\dot{K}_{p}^{\alpha,q}F_{\beta}^{s}} \lesssim \Big(\Big\|\sum_{k=2}^{\infty}2^{ks\beta}|Q_{k-2}f\cdot\Delta_{k}g|^{\beta}\Big)^{1/\beta}\Big\|_{\dot{K}_{p}^{\alpha,q}}.$$

which is bounded by

$$c\Big\| \sup_{k\geq 0} |\Delta_k g| \Big(\sum_{k=2}^{\infty} 2^{ks\beta} |Q_{k-2} f|^{\beta} \Big)^{1/\beta} \Big\|_{\dot{K}_p^{\alpha,q}}.$$

By Hölder's inequality we estimate the last term

$$\Big\| \sup_{k\geq 0} 2^{kr} |\overline{\Delta}_k g| \Big\|_{\dot{K}_{p_2}^{\alpha_2,q_2}} \Big\| \Big(\sum_{k=0}^{\infty} 2^{k(s-r)\beta} |Q_{k-2} f|^{\beta} \Big)^{1/\beta} \Big\|_{\dot{K}_{p_1}^{\alpha_1,q_1}}. \tag{2.3}$$

We have

$$2^{k(s-r)} |Q_{k-2} f| \leq c\, 2^{k(s-r)} \sum_{j=0}^{k-2} 2^{-js} 2^{js} |\Delta_j f|.$$

By applying Lemma 2.1.4, (because $s - r < 0$), we obtain (2.3) is bounded by

$$c\Big\| g \Big\|_{\dot{K}_{p_2}^{\alpha_2,q_2} F_{\beta_2}^{r}} \big\| f \big\|_{\dot{K}_{p_1}^{\alpha_1,q_1} F_{\beta}^{s}}.$$

□

Chapter 3

Mixed multiplication

This chapter is concerned with proving some embeddings of the form

$$\dot{K}F \cdot \dot{K}B \hookrightarrow \dot{K}F$$

where $\dot{K}F$ and $\dot{K}B$, with five indices, will be defined the space of Herz-type Lizorkin-Triebel and the space of Herz-type Besov respectively. The different embeddings obtained here are under some conditions used by Franke [7] and Johnsen [8].

3.1 Preparation

Theorem 3.1.1 *Let* $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}, 0 < s, p, q, r, \beta \leq \infty, \alpha_1 > -n/s$ *and* $\alpha_2 > -n/q$. *We suppose that*

$$s_1 - n/s - \alpha_1 \leq s_2 - n/q - \alpha_2. \tag{3.1}$$

(i) *Let* $0 < q \leq s \leq \infty$ *and* $\alpha_2 \geq \alpha_1$ *or* $0 < s \leq q \leq \infty$ *and*

$$\alpha_2 + n/q \geq \alpha_1 + n/s. \tag{3.2}$$

Then

$$\dot{K}_q^{\alpha_2,\theta} B_\beta^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,r} B_\beta^{s_1}, \tag{3.3}$$

where

$$\theta = \begin{cases} r & \textit{if} \quad \alpha_2 + n/q = \alpha_1 + n/s, s \leq q \ \textit{or} \ \alpha_2 = \alpha_1, q \leq s \\ p & \textit{if} \quad \alpha_2 + n/q > \alpha_1 + n/s, s \leq q \ \textit{or} \ \alpha_2 > \alpha_1, q \leq s. \end{cases}$$

(ii) *Let* $0 < q \leq s < \infty$ *and* $\alpha_2 \geq \alpha_1$ *or* $0 < s \leq q < \infty$ *and* (3.2). *Then*

$$\dot{K}_q^{\alpha_2,r} F_\theta^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,p} F_\beta^{s_1},$$

where

$$\theta = \begin{cases} \beta & \textit{if} \quad 0 < s \leq q < \infty \ \textit{and} \ \alpha_2 + n/q = \alpha_1 + n/s \\ \infty & \qquad\qquad \textit{otherwise.} \end{cases}$$

The proof is given in [3] and [4].

Theorem 3.1.2 *Let* $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}$, $0 < s, p \leq \infty, 0 < q, r < \infty, \alpha_1 > -\frac{n}{s}$ *and* $\alpha_2 > -\frac{n}{q}$. *We suppose that*

$$s_1 - \frac{n}{s} - \alpha_1 = s_2 - \frac{n}{q} - \alpha_2. \tag{3.4}$$

Under the following assumptions

$$0 < q < s \leq \infty, \quad q \leq r \quad \textit{and} \quad \alpha_2 > \alpha_1 \tag{3.5}$$

or

$$0 < q < \min(s,p), \quad q \leq r \leq \min(s,p) \quad \textit{and} \quad \alpha_2 = \alpha_1 \tag{3.6}$$

or

$$0 < s \leq q < \infty, \quad \alpha_2 + \frac{n}{q} > \alpha_1 + \frac{n}{s} \tag{3.7}$$

or

$$0 < s \leq q < \infty, \quad q \leq r \leq p \leq \infty \quad \textit{and} \quad \alpha_2 + \frac{n}{q} = \alpha_1 + \frac{n}{s},$$

we have

$$\dot{K}_q^{\alpha_2,r} F_\theta^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,p} B_r^{s_1}, \tag{3.8}$$

where

$$\theta = \begin{cases} r & \textit{if} \quad 0 < s \leq q < \infty, \ q \leq r \leq p \leq \infty \ \textit{and} \ \alpha_2 + \frac{n}{q} = \alpha_1 + \frac{n}{s} \\ \infty & \qquad\qquad \textit{otherwise.} \end{cases}$$

Theorem 3.1.3 *Let* $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}$, $0 < s, p, q < \infty, 0 < \theta \leq \infty, \alpha_1 > -\frac{n}{s}$ *and* $\alpha_2 > -\frac{n}{q}$. *We suppose that*

$$s_1 - \tfrac{n}{s} - \alpha_1 = s_2 - \tfrac{n}{q} - \alpha_2.$$

Let

$$0 < q < s < \infty, \alpha_2 \geq \alpha_1, \tag{3.9}$$

or

$$0 < s \leq q < \infty \text{ and } \alpha_2 + \tfrac{n}{q} > \alpha_1 + \tfrac{n}{s}.$$

Then

$$\dot{K}_q^{\alpha_2,p} B_p^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,p} F_\theta^{s_1}.$$

For the proof see [5].

3.2 Main results

The following results give an extension of the sufficient hypotheses obtained in Besov and Triebel-Lizorkin spaces.

Theorem 3.2.1 *Let* $1 < p, p_1, p_2, q, q_1 < \infty$, $1 < q_2 < \infty, 0 < \beta, \beta_2 \leq \infty$ *and* $r > 0$. *Assume that*

$$\tfrac{1}{q} \leq \tfrac{1}{q_1} + \tfrac{1}{q_2}, \quad p_1 \leq q_1, \quad \frac{1}{\frac{1}{p_2} - \frac{r}{n}} > 1, \tfrac{n}{p_1} - r - \tfrac{n}{q_1} < 0.$$

and

$$-r + \max\left(0, \tfrac{n}{p_1} + \tfrac{n}{p_2} - n\right) < s < \min\left(\tfrac{n}{p_1}, r\right).$$

Then the embedding

$$\dot{K}_{p_1}^{0,q} F_\beta^s \cdot \dot{\mathsf{K}}\mathsf{B}_r \hookrightarrow \dot{K}_p^{0,q} F_\beta^s$$

holds, where

$$\dot{\mathsf{K}}\mathsf{B}_r = \begin{cases} \dot{K}_{p_2}^{0,q_2} B_{\beta_2}^r, & \text{if } \ r < \frac{n}{p_2},\ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{r}{n} \text{ and } \frac{1}{\beta_2} \geq \frac{1}{p_2} - \frac{r}{n} \\ \dot{K}_{p_2}^{0,q_2} B_\infty^{\frac{n}{p_2}} \cap L^\infty, & \text{if } \qquad\qquad r = \frac{n}{p_2} \text{ and } p = p_1. \end{cases}$$

Proof. *Step 1. The first case for* $\dot{\mathsf{K}}\mathsf{B}_r$:

- *Estimate of* $\Pi_1(f,g)$. Using Lemma 2.1.2 together with (2.1) we obtain

$$\big\|\Pi_1(f,g)\big\|_{\dot{K}_p^{\alpha,q}F_\beta^s} \lesssim \Big\|\Big(\sum_{j=2}^{\infty} 2^{js\beta}\,|Q_{j-2}g\cdot\Delta_j f|^\beta\Big)^{1/\beta}\Big\|_{\dot{K}_p^{\alpha,q}}.$$

Since

$$|Q_{j-2}g\,(x)\,\Delta_j f\,(x)\,| \le \sup_{j\in\mathbb{N}}|Q_{j-2}g\,(x)\,||\Delta_j f\,(x)\,| \lesssim \mathcal{M}g\,(x)\,|\Delta_j f\,(x)\,|$$

for any $x\in\mathbb{R}^n$ and any $j\ge 2$, we obtain

$$\Big\|\,\big\{2^{js}Q_{j-2}g\cdot\Delta_j f\big\}_{j\ge 2}\,\Big\|_{\ell^\beta} \le c\mathcal{M}g\big\|\,\big\{2^{js}\Delta_j f\big\}_{j\ge 2}\,\big\|_{\ell^\beta}. \tag{3.10}$$

We put $b=\big(\frac{1}{p_2}-\frac{r}{n}\big)^{-1}$, which implies that the left hand-side of (3.10), in $\dot{K}_p^{\alpha,q}$-norm, is bounded by

$$c\big\|\mathcal{M}g\big\|_{\dot{K}_b^{0,q_2}}\big\|\,\{\Delta_j f\}_{j\ge 2}\,\big\|_{\dot{K}_{p_1}^{0,q_1}\left(\ell^{s,\beta}\right)} \lesssim \big\|g\big\|_{\dot{K}_b^{0,q_2}}\big\|\,\{\Delta_j f\}_{j\ge 2}\,\big\|_{\dot{K}_{p_1}^{0,q_1}(\ell^{s,\beta})}, \tag{3.11}$$

where we have used Hölder's inequality and Lemma 2.1.1. The embedding

$$\dot{K}_{p_2}^{0,q_2}B_{q_2}^r \hookrightarrow \dot{K}_b^{0,q_2},$$

see Theorem 3.1.3, yields the desired estimate.

- *Estimate of* $\Pi_2(f,g)$. We study two cases.

Case 1. $\frac{1}{p_1}+\frac{1}{p_2}\le 1$. Let $t\in\mathbb{R}$ be such that

$$\max\big(0,\tfrac{1}{p_1}-\tfrac{r}{n},\tfrac{1}{p_1}-\tfrac{r+s}{n}\big)<\tfrac{1}{t}<\tfrac{1}{p_1}. \tag{3.12}$$

We set

$$d=\big(\tfrac{1}{p_2}+\tfrac{1}{t}\big)^{-1},\quad \eta=s-\tfrac{n}{p}+\tfrac{n}{d}\quad\text{and}\quad \delta=s-\tfrac{n}{p_1}+\tfrac{n}{t}.$$

Then

$$\dot{K}_d^{0,q}B_q^\eta \hookrightarrow \dot{K}_p^{0,q}F_\beta^s,$$

see Theorem 3.1.3. Since $\delta+r>0$, by Lemma 2.1.3 together with (2.2) we obtain

$$\big\|\Pi_2(f,g)\big\|_{\dot{K}_p^{0,q}F_\beta^s} \lesssim \Big(\sum_{k=0}^{\infty} 2^{k\eta q}\big\|\Delta_k f\cdot\overline{\Delta}_k g\big\|_{\dot{K}_d^{0,q}}^q\Big)^{1/q}.$$

Hölder's inequality yields

$$2^{k\eta}\big\|\Delta_k f\cdot\overline{\Delta}_k g\big\|_{\dot K^{0,q}_d}\lesssim 2^{kr}\,\big\|\overline{\Delta}_j g\big\|_{\dot K^{0,q_2}_{p_2}}\,2^{k\delta}\,\|\Delta_j f\|_{\dot K^{0,q_1}_t}$$

and hence

$$\big\|\Pi_2(f,g)\big\|_{\dot K^{0,q}_p F^s_\beta}\lesssim\big\|g\big\|_{\dot K^{0,q_2}_{p_2}B^r_{q_2}}\big\|f\big\|_{\dot K^{0,q_1}_t B^\delta_{q_1}}.$$

We conclude our disired estimate by the following embedding

$$\dot K^{0,q}_{p_1}F^s_\beta\hookrightarrow\dot K^{0,q_1}_t B^\delta_{q_1},$$

see Theorem 3.1.2.

Case 2. $\frac{1}{p_1}+\frac{1}{p_2}>1$. Let $v\in\mathbb{R}$ be such that

$$\max\big(0,1-\tfrac{1}{p_2},\tfrac{1}{p_1}-\tfrac{r}{n}\big)<\tfrac{1}{v}<\min(\tfrac{1}{p_1},\tfrac{1}{q_1}).\tag{3.13}$$

We set

$$w=\big(\tfrac{1}{p_2}+\tfrac{1}{v}\big)^{-1}\quad\text{and}\quad\gamma=s-\tfrac{n}{p}+\tfrac{n}{w}.$$

This guarantees

$$\dot K^{0,q}_w B^\gamma_q\hookrightarrow\dot K^{0,q}_p F^s_\beta.$$

see Theorem 3.1.3. Since $\gamma>\max(0,\frac{n}{w}-n)$, by Lemma 2.1.3 combined with (2.2) we obtain

$$\big\|\Pi_2(f,g)\big\|_{\dot K^{0,q}_p F^s_\beta}\lesssim\Big(\sum_{k=0}^{\infty}2^{k\gamma q}\big\|\Delta_k f\cdot\overline{\Delta}_k g\big\|^q_{\dot K^{0,q}_w}\Big)^{1/q}.$$

Hölder's inequality yields

$$\begin{aligned}2^{k\gamma}\big\|\Delta_k f\cdot\overline{\Delta}_k g\big\|_{\dot K^{0,q}_w}&\lesssim 2^{k\gamma}\big\|\overline{\Delta}_k g\big\|_{\dot K^{0,q_2}_{p_2}}\big\|\Delta_k f\big\|_{\dot K^{0,q_1}_v}\\&\lesssim 2^{kr}\big\|\overline{\Delta}_k g\big\|_{\dot K^{0,q_2}_{p_2}}2^{k\varrho}\big\|\Delta_k f\big\|_{\dot K^{0,q_1}_v},\end{aligned}$$

where $\varrho=s-\frac{n}{p_1}+\frac{n}{v}$. Therefore

$$\big\|\Pi_2(f,g)\big\|_{\dot K^{0,q}_p F^s_\beta}\lesssim\big\|g\big\|_{\dot K^{0,q_2}_{p_2}B^r_{q_2}}\big\|f\big\|_{\dot K^{0,q_1}_v B^\varrho_{q_1}}.$$

We conclude our disired estimate by the following embedding

$$\dot{K}_{p_1}^{0,q_1}F_{\beta}^{s} \hookrightarrow \dot{K}_{v}^{0,q_1}B_{q_1}^{\varrho},$$

see again Theorem 3.1.2 since $v \geq q_1$.

- *Estimate of* $\Pi_3(f,g)$. Let $u \in \mathbb{R}$ be such that

$$\max\left(0, \tfrac{1}{p_1} - \tfrac{r}{n}\right) < \tfrac{1}{u} < \min\left(\tfrac{1}{p_1}, \tfrac{1}{q_1}, \tfrac{1}{p_1} - \tfrac{s}{n}\right).$$

We set

$$v = \left(\tfrac{1}{p_2} + \tfrac{1}{u}\right)^{-1}, \quad \sigma = s - \tfrac{n}{p} + \tfrac{n}{v} \quad \text{and} \quad \varrho = s - \tfrac{n}{p_1} + \tfrac{n}{u}. \tag{3.14}$$

Then we have the embeddings

$$\dot{K}_{v}^{0,q}B_{q}^{\sigma} \hookrightarrow \dot{K}_{p}^{0,q}F_{\beta}^{s} \quad \text{and} \quad \dot{K}_{p_1}^{0,q_1}F_{\infty}^{s} \hookrightarrow \dot{K}_{u}^{0,q_1}B_{q_1}^{\varrho}. \tag{3.15}$$

since $u \geq q_1$, see Theorems 3.1.2 and 3.1.3. Using Lemma 2.1.3 we obtain

$$\big\|\Pi_3(f,g)\big\|_{\dot{K}_{p}^{0,q}F_{\beta}^{s}} \lesssim \Big(\sum_{k=2}^{\infty} 2^{ksq}\big\|Q_{k-2}f \cdot \Delta_k g\big\|_{\dot{K}_{p}^{0,q}}^{q}\Big)^{1/q}.$$

On the other hand Hölder's inequality yields

$$2^{k\sigma}\big\|Q_{k-2}f \cdot \Delta_k g\big\|_{\dot{K}_{v}^{0,q}} \leq c\, 2^{kr}\big\|\Delta_k g\big\|_{\dot{K}_{p_2}^{0,q_2}} 2^{k\varrho}\sum_{j=0}^{k-2} 2^{-j\varrho}2^{j\varrho}\big\|\Delta_j f\big\|_{\dot{K}_{u}^{0,q_1}}.$$

By applying successively inequality of Hölder again in ℓ^p-norm and Lemma 2.1.4, (because $\beta < 0$), we obtain the bound

$$c\big\|g\big\|_{\dot{K}_{p_2}^{0,q_2}B_{q_2}^{r}}\big\|f\big\|_{\dot{K}_{u}^{0,q_1}B_{q_1}^{\varrho}}.$$

So (3.15) and (3.15) give

$$\big\|\Pi_3(f,g)\big\|_{\dot{K}_{p}^{0,q}F_{\beta}^{s}} \lesssim \big\|g\big\|_{\dot{K}_{p_2}^{0,q_2}B_{q_2}^{r}}\big\|f\big\|_{\dot{K}_{p_1}^{0,q_1}F_{\infty}^{s}}.$$

Step 2. The second case for B_r: We only estimate $\{\Pi_1(f,g)\}$; it is sufficient to see that

$$|Q_{j-2}g\,(x)\,\Delta_j f\,(x)\,| \leq \sup_{j\in\mathbb{N}}|Q_{j-2}g\,(x)\,||\Delta_j f\,(x)\,| \lesssim \|g\|_{\infty}\,|\Delta_j f\,(x)\,|.$$

□

Theorem 3.2.2 *Let* $1 < p, p_1, p_2, q, q_1 < \infty,\ 1 < q_2 < \infty, 0 < \beta, \beta_2 \leq \infty,$ *and* $r > 0.$ *Under some suitable assumptions on* $p, p_1, p_2, q, q_1, \alpha, \alpha_2, \alpha_1$ *we have*

$$\dot{K}_{p_1}^{\alpha_1,q}F_\beta^s \cdot \dot{\mathsf{KB}}_r^{\alpha_2} \hookrightarrow \dot{K}_p^{\alpha,q}F_\beta^s,$$

where

$$\dot{\mathsf{KB}}_r = \dot{K}_{p_2}^{\alpha_2,q_2}B_{\beta_2}^r, or\ \dot{\mathsf{KB}}_r = \dot{K}_{p_2}^{\alpha_2,q_2}B_{\beta_2}^r \cap L^\infty.$$

Dimonstration. For the proof see [6]. ■

Remark 3.2.3 *More multiplocations are given in [6] and [1].*

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ملخص المذكرة:

في هذه المذكرة قمنا بدراسة فضاءات هارز، فضاءات هارز-بيزوف و فضاءات هارز-فضاءات ليزوركين -تريبل حيث أعطينا بعض خصائص الاحتواءات لهذه الفضاءات. ثم قمنا بدراسة الضرب و الضرب المختلط لهذه الفضاءات.

الكلمات المفتاحية : فضاءات هارز، فضاءات هارز –بيزوف، فضاءات هارز ليزوركين – تريبل ، الاحتواءات ، الضرب المختلط .

Résumé de thèse:

Dans cette thèse, nous avons étudié les espaces Herz, les espaces de Herz-Besov et les espaces de Herz-Lizorkin-Triebel ou nous avons donné certaines propriétés des inclusions de base de ces espaces. Puis nous avons étudié la multiplication et la multiplication mixte de ces espaces.

Mots clés : Espaces Herz, Espaces de Herz-Besov, Espaces de Herz-Lizorkin-Triebel, Les inclusions, La multiplication mixte.

Abstract of thesis:

In this thesis, we studied the Herz spaces and the Herz-type-Besov and Triebel-Lizorkin spaces where we gave some basic embeddings properties of these spaces. After that we studied the multiplication and mixed multiplication of these spaces.

Key words : Herz spaces, Herz-type Besov spaces, Herz-type Triebel-Lizorkin spaces, Embeddings, The mixed multiplication.