PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTERY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

MOHAMED BOUDIAF UNIVERSITY -M'SILA
Faculty of Mathematics and Computer Science
Department of Mathematics

Master's degree in Mathematics

Domain : Mathematics and Informatics
Speciality : Mathematics
Option : Functional analysis

Titled

On n-Bounded and n-Continuous Operators in n-Normed space

Presented by :
KECHIDA Marwa

Defended on: 11/06/2023.

In front of the jury composed of:

Chairperson:	*HAMIDI* Khaled	M.C.B, Med Boudiaf University-M'Sila
Supervisor:	*TALLAB* Abdelhamid	M.C.A, Med Boudiaf University-M'Sila
Examiner:	*MAZOUZ* Ahmed	M.A.A, Med Boudiaf University-M'Sila
Examiner:	*HERAIZ* Toufik	M.C.B, Med Boudiaf University-M'Sila

University year: 2023 /2024

Acknowledgments

Above all, the greatest thanks go to ALLAH who alone guides our steps in the right diction during our life.

I would especially like to express my sincere thanks to my supervisor, Dr. TALLAB Abdelhamid, for all his advice and constant support during the preparation of this work from its beginning, and for the trust he placed in me, respecting me and my work, and giving me the energy and inspiration to overcome all difficulties.

My sincere gratitude to all the members of the jury for the honor they do me in agreeing to chair and examine this work.

I also have to thank my parents, all my friends and my colleagues.

Finally, I thank everyone I know and all the administrative staff of the mathematics department.

Dedication

The engine of my research has faced many obstacles and difficulties. Nevertheless, I have tried to overcome all these challenges with great perseverance, thanks to God Almighty and thanks to my dear parents.

I dedicate this humble work to my dear parents, may ALLAH prolong their lives, and to my sister who has never hesitated to support me.

List of Symbols

Symbol	Notation
$\mathbb{K}$	The field of reel $\mathbb{R}$ or complex numbers $\mathbb{C}$.
$\|\cdot,...,\cdot\|$	n-norm.
$(X,\|\cdot,...,\cdot\|)$	n-normed space.
$\mathcal{L}(X,Y)$	The vector space of linear operators of X in Y.
$\mathcal{L}(X)$	The collection of all linear operators from X into X.
$B(X,Y)$	The space of all bounded linear operators from X to Y.
$D(T)$	The domain of $T($ T is linear operator).
$B(X^n,Y)$	The space of all bounded n-linear operators from X^n into Y.
$B_n(X,X)$	The space of all n-bounded linear operators from X into X^n.
$B_n^n(X,X)$	The space of all n-bounded of type-I operators from X^n into X^n.
$C^1\left[0,1\right]$	The space of continuous functions on $[0,1]$.
F_T	The set of all fixed points of T.
$Lip_n(T)$	The lipschitzian constant.
$Lip_n(X,Y)$	The space of all n-lipschitz operators to X into Y.
$Lip_{n.0}(X,Y)$	The space of all n-lipschitz that vanishing at 0.

Contents

Introduction

In functional analysis, an n-normed space is a generalization of a normed space that uses a function called an n-norm instead of a norm. Specifically, an n-normed space is a pair $(X, \|.\|_n)$, where X is a vector space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\|\cdot\|_n$ is a function from X to the non negative real numbers that satisfies certain properties. The concept of an n-normed space was introduced by the mathematician L. Soenjaya in 1963 [7]. As a way to study spaces of functions that are not necessarily continuous or differentiable. In particular, n-normed spaces have been used in the study of various types of nonlinear partial differential equations and in the theory of approximation of functions. One of the key differences between an n-norm and a norm is that an n-norm need not be homogeneous, meaning that $\|ax\|_n$ may not be equal to $|a|\,\|x\|_n$ for all scalar values a and all vectors x in X. This property allows n-normed spaces to capture more general properties of functions than traditional normed spaces. Overall, the theory of n-normed spaces has provided mathematicians with a powerful tool for studying a wide range of mathematical problems in analysis and functional analysis. After many authors began to popularize some concepts such as normed space on an n-normed spaces, bounded on an n-bounded and continuous on an n-continuous.

Several author start to work in thesis was. for example, in 2001 H. Gunawan and Mashadi ([4], [5]) detailed the n-normed spaces and On finite-dimensional 2-normed spaces, then in 2014 M. Kir and H. Kiziltunc [6] follows fixed point theorems for contraction mappings in n-normed spaces

In this memory, we will detail the paper [7] which deals with On n-bounded and n-continuous operator in n-normed space and a fixed point on this space introduced.

This memory is divided into four chapters as follows.

For the first chapter, we will present reminders, definitions and properties on normed and banach spaces, linear and n-linear operator and continuity and boundedness.

The second chapter is devoted to the continuous, bounded n-linear operators between n-normed spaces and normed space. Also, we presented the definition of n-normed and n-Banach spaces, continuous bounded n-linear operator and the space of all bounded n-linear operators from X^n into Y.

The third chapter is devoted to the n- continuous, n-bounded linear operators between normed space and n-normed spaces, we define the n- continuous n-bounded linear operator and the space of all n-bounded linear operators from X into X.

The last chapter, we devote this chapter to other notions of n-continuous, n-bounded Operator and contraction operators and their fixed point theorems in n-normed space.

Chapter 1

Preliminaries

In this chapter, we mention some definitions that we will rely on in subsequent chapters.

1.1 Normed and Banach spaces

Definition 1.1.1 *(Norm)*

Let X be a vector space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A norm on vector space X is a function

$$\|.\| : X \to \mathbb{R}_+$$

such that:

(i)$\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x =$ 0. *(Positive definiteness)*

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for all $\alpha \in \mathbb{K}$ and $x \in X$. *(scalar multiplication)*

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$. *(Triangle inequality)*

In this case, the pair $(X, \|.\|)$ is called a normed space. Also, the distance between the elements x and y in X is defined by $\|x - y\|$.

Example 1.1.1 *1) Consider $X = \mathbb{K}^n$. Put*

$$\|x\|_p := (\sum_1^n |x_i|^p)^{1/p} \quad and \quad \|x\|_\infty := \max_{i=1,\ldots,N} |x_i|$$

for $1 \leq p < \infty$ and $x = (x_1, ..., x_n) \in \mathbb{K}^n$. Then $\|.\|_p$ called the usual norm as $p = 2$ and $\|.\|_\infty$ called the sup-norm all are norms on $\mathbb{K}^n$.

2) Put

$$c_0 := \{(x(i)) : x(i) \in \mathbb{K}, \quad \lim |x(i)| = 0\} \quad (called\ the\ null\ sequence\ space).$$

and

$$\ell^\infty := \left\{(x(i)) : x(i) \in \mathbb{K}, \quad \sup_i |x(i)| < \infty\right\}.$$

Then c_0 *is a subspace of* ℓ^∞. *The sup-norm* $|.\|_\infty$ *on* ℓ^∞ *is defined by*

$$\|x\|_\infty := \sup_i |x(i)|$$

For $x \in \ell^\infty$.

Let

$$c_{00} := \{(x(i)) : there\ are\ only\ finitly\ many\ x(i)'s\ are\ non-zero\}.$$

Also, c_{00} *is endowed with the* $\sup-$*norm defined above and is called the finite sequence space.*

Definition 1.1.2 *(Sequences) Let* $(\xi_n)_n$ *be a sequence of real numbers and* $k \in \mathbb{R}$. *We say that the sequence* $(\xi_n)_n$ *converges to* k *and we denote* $\lim\limits_{n\longrightarrow\infty} \xi_n = k$ *or* $\xi_n \longrightarrow k$ *whenever* $n \longrightarrow \infty$ *if and only if*

$$\forall\varepsilon > 0, \exists M \in \mathbb{N} : \forall n \geq M, \quad |\xi_n - k| \leq \varepsilon.$$

Proposition 1.1.1 *The sequence* $(\xi_n)_n$ *of real numbers converges to* $k \in \mathbb{R}$ *if and only if the sequence* $(|\xi_n - k|)_n$ *converges to* 0.

Definition 1.1.3 *(Cauchy Sequences) Any sequence* $(\xi_n)_{n\in\mathbb{N}}$ *will be called Cauchy if*

$$\forall\varepsilon > 0, \exists M \in \mathbb{N} : \forall n, m \geq M, \quad |\xi_n - \xi_m| < \varepsilon.$$

Proposition 1.1.2 *Any convergent sequence is a Cauchy sequence.*

Definition 1.1.4 *(Banach space) We say that vector space* X *is complete for the norm* $\|\cdot\|_X$ *if all a Cauchy sequence is convergent. Also called Banach space.*

Remark 1.1.1 *Any finite-dimensional norm vector space is a Banach space.*

Example 1.1.1 *Let* $\alpha = (\alpha_n)_{n\in\mathbb{N}^*}$ *be a positive sequence and let* $1 \leq p < \infty$, *we note*

$$\ell^p_\alpha := \{x = (x_n)_{n\in\mathbb{N}^*}, x_n \in \mathbb{C} : (\sum_{i=1}^{\infty} \alpha_n |x_n|^p)^{\frac{1}{p}} < \infty,$$

The space ℓ^p_α *provided with the norm* $\|x\|_{p,\alpha} = (\sum_{i=1}^{\infty} \alpha_n |x_n|^p)^{\frac{1}{p}}$.

(i) For $x = (x_n)_{n\in\mathbb{N}^*}, y = (y_n)_{n\in\mathbb{N}^*} \in \ell^p_\alpha$, *we note* $X = (X_n)_{n\in\mathbb{N}^*}, X_n = \alpha_n^{\frac{1}{p}} x_n$ *and* $Y = (Y_n)_{n\in\mathbb{N}^*}, Yn = \alpha_n^{\frac{1}{p}} y_n$. *By using Minkowski inequality, we define*

$$\|x + y\|_{p,\alpha} \leq \|x\|_{p,\alpha} + \|y\|_{p,\alpha}$$

Then ℓ^p_α *is a norm space.*

(ii) Let $1 \leq p < \infty$ *and* $(\xi_m)_{m\in\mathbb{N}^*}$ *be a Cauchy sequence of* ℓ^p_α*, we note* $\xi_m(n)$ *is the term of rang* n *of* $\xi_m \in \ell^p_\alpha$.

Let $\varepsilon > 0$, $\exists M(\varepsilon) \in \mathbb{N}^*$ *such that*

$$\ell, k \geq M(\varepsilon) \Longrightarrow \|\xi_\ell - \xi_k\|^p_{p,\alpha} = \sum_{n=1}^{\infty} \alpha_n \left|\xi_\ell(n) - \xi_k(n)\right|^p \leq \varepsilon^p \tag{1.1}$$

This implies $|\xi_\ell(n) - \xi_k(n)| \leq \alpha_n^{-\frac{1}{p}}\varepsilon$, $\forall n \in \mathbb{N}^*$ *and* $\ell, k \geq M(\varepsilon)$

For all $n \in \mathbb{N}^*$*, the sequence* $(\xi_m(n))_{m\in\mathbb{N}^*}$ *is a Cauchy in* $\mathbb{C}$ *, then it's converge to* $\xi(n)$ *when* $m \longrightarrow \infty$.

For $\ell, k \geq M(\varepsilon)$*, and* $n \in \mathbb{N}^*$*, we have according to (1.1):*

$$\sum_{n=1}^{\infty} \alpha_n \left|\xi_\ell(n) - \xi_k(n)\right|^p \leq \|\xi_\ell - \xi_k\|^p_{p,\alpha} \leq \varepsilon^p, \tag{1.2}$$

Passing to the limit in (1.2), when $\ell \longrightarrow +\infty$ *and* $k \geq M(\varepsilon)$*, gives*

$$\sum_{n=1}^{\infty} \alpha_n \left|\xi(n) - \xi_k(n)\right|^p \leq \varepsilon^p, \tag{1.3}$$

By making N *tend towards infinity in (1.3), we obtain*

$$k \geq M(\varepsilon) \Longrightarrow \sum_{n=1}^{\infty} \alpha_n \left|\xi(n) - \xi_k(n)\right|^p \leq \varepsilon^p \tag{1.4}$$

This shows that $\xi - \xi_k \in \ell^p_\alpha$*, which is a vector space, and therefore* $\xi \in \ell^p_\alpha$*. (1.4) is interpreted then like this:*

$$k \geq M(\varepsilon) \Longrightarrow \|\xi - \xi_k\|^p_{p,\alpha} \leq \varepsilon^p.$$

Then ℓ^p_α *is complete, i.e., is a Banach space*

Example 1.1.2 *Let* $p = \infty$ *and* $(\xi_m)_{m\in\mathbb{N}^*}$ *be a Cauchy sequence of* ℓ^∞_α*. Let* $\varepsilon > 0$ *given,* $\exists M(\varepsilon) \in \mathbb{N}^*$ *such that*

$$\ell, k \geq M(\varepsilon) \Longrightarrow \sup_{n\geq 1}(\alpha_n \left|\xi_\ell(n) - \xi_k(n)\right|) = \|\xi_\ell - \xi_k\|_{\infty,\alpha} \leq \varepsilon.$$

This implies that

$$\forall n \in \mathbb{N}^*, \quad \left|\xi_\ell(n) - \xi_k(n)\right| \leq \alpha_n^{-1}\varepsilon. \tag{1.5}$$

So the sequence $(\xi_m(n))_{m\in\mathbb{N}^*}$ *is of Cauchy in* $\mathbb{C}$*, then it's convergent to* $\xi(n)$ *whenever* $m \longrightarrow \infty$*.*
Passing to the limit in (1.5), when $\ell \longrightarrow +\infty$ *and* $k \geq M(\varepsilon)$*, gives successively*

$$\alpha_n \left|\xi(n) - \xi_k(n)\right| \leq \varepsilon, \xi - \xi_k \in \ell^\infty_\alpha, \xi \in \ell^\infty_\alpha, \|\xi - \xi_k\|_{\ell^\infty_\alpha} \leq \varepsilon.$$

Then ℓ^∞_α *is a Banach space.*

Definition 1.1.5 *(Inner product) Let X be a vector space. An inner product on X is a rule that assigns to each pair $x, y \in X$ a real number $\langle x, y\rangle$ such that,for all $x, y, z \in X$ and $\alpha \in \mathbb{R}$,*

1. $\langle x, x\rangle \geqslant 0$, *for all x in X .*
2. $\langle x, x\rangle = 0 \Leftrightarrow x = 0$, *with equality if and only if $x = 0$.*
3. $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$.
4. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.
5. $\overline{\langle x, y\rangle} = \langle y, x\rangle$, *for all $x, y \in X$.*

1.2 Linear and n-linear operator

Definition 1.2.1 *(Operator) Let X and Y be two normed vector spaces. We say that an application T defined on $X_0 \subset X$ into Y is an operator if X_0 is linear subspace of X.*

Definition 1.2.2 *(Linear operator) Let X and Y be two vector spaces on $\mathbb{K}$. Let T be an operator from X into Y. We say that T is linear if, for all x and y in X and for all $\lambda \in \mathbb{K}$, we have*

$$\begin{aligned} T(x+y) &= T(x) + T(y) \\ T(\lambda x) &= \lambda T(x) \end{aligned}$$

We denote $\mathcal{L}(X, Y)$ for the vector space of linear operators of X in Y.

Special cases:

1. *The linear mapping T from X to X is called endomorphism of X. The vector space of endomorphisms of x is denoted $\mathcal{L}(X)$.*
2. *The linear mapping T from X to $\mathbb{K}$ is called linear form on X. The vector space of linear forms on X is denoted $X' = \mathcal{L}(X, Y)$.*

Definition 1.2.3 *(n-Linear operator) Let n is a natural number and $X_1, ..., X_n, Y$ be a Banach spaces. An application $T : X_1, ..., X_n \longrightarrow Y$ is n-linear operator if*

$$T(x_1, ..., \alpha x_j + \beta y_j, ..., x_n) = \alpha T(x_1, ..., x_j, ..., x_n) + \beta T(x_1, ..., y_j, ..., x_n),$$

for all $j = \overline{1.n}$ and $x_j, y_j \in X_j$, $\alpha, \beta \in \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). If $Y = \mathbb{K}$, then T is an n-linear form.

It's easy see that,

Example 1.2.1 *The mapping*

$$\begin{aligned} T : \mathbb{K}^n &\longrightarrow \mathbb{K} \\ (x_1, x_2, ..., x_n) &\longmapsto x_1 x_2 \cdots x_n \end{aligned}$$

is a n-linear on $\mathbb{K}$.

1.3 Continuity and boundedness

Definition 1.3.1 *(Bounded Linear Operator) Let X and Y be normed spaces and $T : X \longrightarrow Y$ be a linear operator. The operator T is said to be bounded if there is a real number $c > 0$ such that*

$$\|T(x)\| \leq c \|x\|, \quad \forall x \in X.$$

The space of all bounded linear operators from X to Y denoted $B(X, Y)$.

Theorem 1.3.1 *[7] Let X and Y be normed spaces. Then $(B(X,Y), \|.\|)$ is a normed space, where*

$$\|T\| := \sup_{x \in X, \|x\|=1} \|T(x)\|$$

if Y is a Banach space, then $B(X, Y)$ is a Banach space.

Example 1.3.1 *The identity operator is bounded. Indeed, let I the identity operator such that*

$$\begin{aligned} I : X &\longrightarrow X \\ x &\longmapsto I(x) = x \end{aligned}$$

We have

$$\|I\| = \sup_{x \in D(I) - \{0\}} \frac{\|I(x)\|}{\|x\|} = \sup_{x \in D(I) - \{0\}} \frac{\|x\|}{\|x\|} = 1$$

Definition 1.3.2 *(Continuous linear operator) Let X and Y be normed spaces over filed $\mathbb{K}$ and $T : X \longrightarrow Y$ be a linear operator, we say that T is continuous at $x_0 \in X$ if*

$$\forall \epsilon > 0, \exists \delta > 0 : \|T(x) - T(x_0)\| < \epsilon \quad \textit{whenever} \quad \|x - x_0\| < \delta.$$

Remark 1.3.1 *T is continuous if T is continuous at every $x \in D(T)$.*

Theorem 1.3.1 *(Continuity and Boundedness) Let X, Y are normed spaces and $T : X \longrightarrow Y$ is a linear operator. Then,*

(i) T is continuous if, and only if T is bounded,

(ii) If T is continuous at a single point, then it's continuous.

Proof (i) Let T is continuous, then we have to prove that is bounded by contradiction.

Let if possible T is not bounded, then $\|T(x_n)\| > m\|x_n\|, \quad \forall x_n \in X$. Then

$$\frac{\|T(x_n)\|}{m\|x_n\|} > 1.$$

Hence

$$\left\|T(\frac{x_n}{m\|x_n\|})\right\| > 1.$$

This implies

$$\|T(y_n)\| > 1, \ \ where \ \ y_n = \frac{x_n}{m\|x_n\|}. \tag{1.6}$$

As $n \longrightarrow \infty$, we have

$$\begin{aligned} y_n = \frac{x_n}{m\|x_n\|} \Longrightarrow \|y_n\| = \frac{\|x_n\|}{m\|x_n\|} = \frac{1}{n} \quad &\Longrightarrow \quad \|y_n\| = \frac{1}{n} \longrightarrow 0, \\ &\Longrightarrow \quad \|y_n\| \longrightarrow 0, \\ &\Longrightarrow \quad y_n \longrightarrow 0, \\ &\Longrightarrow \quad T(y_n) \longrightarrow T(0), \\ &\Longrightarrow \quad T(y_n) \longrightarrow 0, \\ &\Longrightarrow \quad \|T(y_n)\| \longrightarrow 0, \\ &\Longrightarrow \quad \|T(y_n)\| \longrightarrow 0. \end{aligned} \tag{1.7}$$

So relations (1.6) and (1.7) contradiction each other. Hence T is bounded.

Conversely, let

$$T \ \ is \ \ bounded \Longrightarrow \exists k > 0 : \|T(x)\| \leq k\|x\|, \quad \forall x \in X. \tag{1.8}$$

We have to prove that T is continuous. Let $x_n \longrightarrow x, in\, X \Longrightarrow x_n - x \longrightarrow 0$

this implies $\|x_n - x\| \longrightarrow 0$, $x_n, x \in X \Longrightarrow (x_n - x) \in X$, from (1.8), we find

$$\|T(x_n - x)\| \leq k\,\|x_n - x\| \longrightarrow 0 \Longrightarrow \|T(x_n) - T(x)\| \longrightarrow 0$$

$$\Longrightarrow$$

$$T(x_n) - T(x) \longrightarrow 0 \Longrightarrow T(x_n) \longrightarrow T(x) \;\; in \;\; Y, \quad as \;\; n \longrightarrow \infty.$$

Then T is continuous. So T is continuous $\Longrightarrow T$ is bounded.

(ii) Let T is continuous at $x_0 \in X$. Then $\|x - x_0\| < \delta \Longrightarrow \|T(x) - T(x_0)\| < \varepsilon, \quad \forall x \in X$

Let $x_k \in X$ is an arbitrary point. Then we will prove that T is continuous at x_k.

For this, $\|x - x_k\| < \delta \Longrightarrow \|x - x_k + x_0 - x_0\| < \delta, \quad \forall x \in X$. Then

$$\|(x + x_0 - x_k) - x_0\| < \delta, \quad \forall x \in X \Longrightarrow \|T(x + x_0 - x_k) - T(x_0)\| < \varepsilon,$$

this implies that

$$\|T(x) - T(x_k)\| < \varepsilon, \quad \forall x \in X.$$

So

$$\|x - x_k\| < \delta \Longrightarrow \|T(x) - T(x_k)\| < \varepsilon \;\; \forall x \in X.$$

This results that, T is continuous at $x_k \in X$. Hence T is continuous at all points of X. □

Chapter 2

Continuous, bounded n-linear operators between n-normed spaces and normed spaces

2.1 n-Normed and n-Banach spaces

Definition 2.1.1 *(n-Norm) Let X be a real vector space with $\dim(X) \geq n$ where n is a positive integer. We allow $\dim(X)$ to be infinite, a real-valued function*

$$\|\cdot, ..., \cdot\| : X^n \to \mathbb{R}$$

is called an n-norm on X^n if the following conditions hold

1. $\|x_1, ..., x_n\| = 0$ *if and only if* $x_1, ..., x_n$ *are linearly dependent,*
2. $\|x_1, ..., x_n\| = \|x_{i_1}, ..., x_{i_n}\|$ *for every permutation* $(i_1, ..., i_n)$ *of* $(1, ..., n)$*,*
3. $\|\lambda x_1, x_2, ..., x_n\| = |\lambda| \, \|x_1, x_2, ..., x_n\|$ *for every real number* λ *and* $x_1, ..., x_n \in X$*,*
4. $\|x + y, x_2, ..., x_n\| \leq \|x, x_2, ..., x_n\| + \|y, x_2, ..., x_n\|$ *for all* $x, y, x_2, ..., x_n \in X$*.*

In this case, the pair $(X, \|\cdot, ..., \cdot\|)$ is called a n-normed space. The n-norm is always non-negative.

Example 2.1.1 *1) Let $X = \mathbb{R}^3$ with vector addition and scalar multiplication defined component wise and with a n-norm defined as follows.*
For all $x_1 = (a_1, b_1, c_1), x_2 = (a_2, b_2, c_2), \ldots, x_n = (a_n, b_n, c_n)$,

$$\|x_1, ..., x_n\| = \max \left\{ |a_1 b_1 - a_2 b_2 - \cdots - a_n b_n| \, , \ldots \right\},$$

Therefore $\|x_1, ..., x_n\|$ *is clearly* n*-norm.*

2) Let $X = \mathbb{R}^n$ *is equipped with the following Euclidean* n*-norm:*

$$\|x_1, ..., x_n\|_E := abs\left(\begin{vmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{vmatrix}\right),$$

where $x_i = (x_{i_1}, ..., x_{i_n}) \in \mathbb{R}^n$ *for each* $i = 1, ..., n$.

Definition 2.1.2 *[4]. A sequence* $\{\xi_n\}$ *in an* n*-normed space* $(Y, \|\cdot, ..., \cdot\|)$ *is said to converge to an* $x \in X$ *if* $\forall x_1, ..., x_{n-1} \in X$ *, we have*

$$\lim_{k \longrightarrow \infty} \|x_1, ..., x_{n-1}, \xi_k - \xi\| = 0.$$

Definition 2.1.3 *[4]. A sequence* $\{\xi_n\}$ *in an* n*-normed space* $(Y, \|\cdot, ..., \cdot\|)$ *is a Cauchy sequence if* $\forall x_1, ..., x_{n-1} \in X$*, we have*

$$\lim_{k,\ell \longrightarrow \infty} \|x_1, ..., x_{n-1}, \xi_k - \xi_\ell\| = 0.$$

Definition 2.1.4 *(n-Banach space) [4]. If every Cauchy sequence in an* n*-normed space* $(Y, \|\cdot, ..., \cdot\|)$ *converges to* $x \in X$*, then* X *is said to be complete. A complete* n*-normed space is called an* n*-Banach space.*

Lemma 2.1.1 *[3] A sequence in* ℓ^p *is convergent in* n*-norm* $\|\cdot, ..., \cdot\|_p$ *if and only if it is convergent in the usual norm* $\|\cdot\|_p$*. Similarly, a sequence in* ℓ^p *is Cauchy with respect to* $\|\cdot, ..., \cdot\|_p$ *if and only if it is Cauchy with respect to* $\|\cdot\|_p$*.*

Example Let $(x_n)_n$ be Cauchy in ℓ^p with respect to $\|\cdot, \cdot\|_p$. Then by Lemma 2.1.1 , $(x_n)_n$ is cauchy in the usual norm $\|\cdot\|_p$. But we know that ℓ^p is banach space with respect to $\|\cdot\|_p$, and so $(x_n)_n$ must converge to some $x \in X$ in $\|\cdot\|_p$. By another application of Lemma 2.1.1 , $(x_n)_n$ also converges to x in $\|\cdot, \cdot\|_p$. This shows that ℓ^p is complete with respect to the 2-norm $\|\cdot, \cdot\|_p$.

Then the space ℓ^p is 2-banach space equipped with the 2-norm $\|\cdot, \cdot\|_p$.

2.2 The infinity n-norm

We will utilize our findings to investigate convergence and completeness in n-normed spaces, to be defined subsequently. This will allow us to establish a fixed-point theorem for select n-normed spaces.

The case $n = 2$ was previously studied in [5].

Definition 2.2.1 *(Preliminary results) Suppose here after that* $n \geq 2$ *and* $(X, \|\cdot, ..., \cdot\|)$ *is an* n*- normed space of dimension* $d \geq n$*. Take a linearly independent set* $a_1, ..., a_n$ *in* X*. With respect to* $a_1, ..., a_n$*, define the following function* $\|\cdot, ..., \cdot\|_\infty$ *on* X^{n-1} *by*

$$\|x_1, ..., x_{n-1}\|_\infty := \max\left\{\|x_1, ..., x_{n-1}, a_i\| : i = \overline{1.n}\right\}.$$

Then we have the following result.

Theorem 2.2.1 *The function* $\|\cdot, ..., \cdot\|_\infty$ *defines an* $(n-1)$*-norm on* X*.*

Proof We will verify that $\|\cdot, ..., \cdot\|_\infty$ satisfies the four properties of an $(n-1)$-norm

(1) If $x_1, ..., x_{n-1}$ are linearly dependent, then $\|x_1, ..., x_{n-1}\| = 0$ for each $i = \overline{1.n}$ and hence $\|x_1, ..., x_{n-1}\|_\infty = 0$. Conversely, if $\|x_1, ..., x_{n-1}\|_\infty = 0$, then $\|x_1, ..., x_{n-1}, a_i\| = 0$ and accordingly $x_1, ..., x_{n-1}, a_i$ are linearly dependent for each $i = \overline{1.n}$. But this can only happen when $x_1, ..., x_{n-1}$ are linearly dependent.

(2) Since $\|x_1, ..., x_{n-1}, a_i\|$ is invariant under any permutation of $\{x_1, ..., x_{n-1}\}$, we find that $\|x_1, ..., x_{n-1}\|_\infty$ is also invariant under any permutation.

(3) Observe that for $i = \overline{1.n}$

$$\begin{aligned}\|x_1, ..., x_{n-2}, \lambda x_{n-1}\|_\infty &= \max\{\|x_1, ..., x_{n-2}, \lambda x_{n-1}, a_i\|\}\\ &= |\lambda| \max\{\|x_1, ..., x_{n-2}, x_{n-1}, a_i\|\}\\ &= |\lambda| \|x_1, ..., x_{n-2}, x_{n-1}\|_\infty.\end{aligned}$$

(4) Observe that

$$\begin{aligned}\|x_1, ..., x_{n-2}, Y + Z\|_\infty &= \max\left\{\|x_1, ..., x_{n-2}, y + z, a_i\| : i = \overline{1.n}\right\}\\ &\leq \max\left\{\|x_1, ..., x_{n-2}, y, a_i\| : i = \overline{1.n}\right\} + \max\left\{\|x_1, ..., x_{n-2}, z, a_i\| : i = \overline{1.n}\right\}\\ &= \|x_1, ..., x_{n-2}, y, a_i\|_\infty + \|x_1, ..., x_{n-2}, z, a_i\|_\infty.\end{aligned}$$

Therefore $\|\cdot,...,\cdot\|_\infty$ defines an $(n-1)$-norm on X. $\square$

Corollary 2.2.1 *Every n-normed space is an $(n-r)$-normed space for all $r = 1,...,n-1$. In particular, every n-normed space is a normed space.*

Definition 2.2.2 *(The finite-dimensional case) For finite-dimensional n-normed space $(X, \|\cdot,...,\cdot\|)$, we can in general derive an $(n-1)$-norm from the n-norm in the following way. Take a linearly independent set $\{a_1,...,a_m\}$ in X, with $n \leq m \leq d$. With respect to $\{a_1,...,a_m\}$, define the following function $\|\cdot,...,\cdot\|_\infty$ on X^{n-1} by*

$$\|x_1,...,x_{n-1}\|_\infty := \max\{\|x_1,...,x_{n-1},a_i\|\}$$

Then, as in Theorem 2.2.1, the function $(X, \|\cdot,...,\cdot\|)_\infty$ defines an $(n-1)$-norm on X. As we will see later, we can obtain a better $(n-1)$-norm by using a set of d, rather than just n, linearly independent vectors in X (that is, by using a basis for X).

2.3 Continuous, bounded n-linear operator

Definition 2.3.1 *[7](Bounded n-linear operator) Let $(X, \|\cdot,...,\cdot\|)$ be an n-normed space and $(Y, \|.\|)$ be a normed space and let T be an n-linear operator on X such that, $T : X^n \longrightarrow Y$. We say that T is bounded if there is a non-negative constant c such that $\forall(x_1,...,x_n) \in X^n$, we have*

$$\|T(x_1,...,x_n)\| \leq c\,\|x_1,...,x_n\|\,.$$

Corollary 2.3.1 *[7] If T is bounded, then $\|T\|$ can be evaluated by*

$$\|T\| := \sup_{\|x_1,...,x_n\|\neq 0} \frac{\|T(x_1,...,x_n)\|}{\|x_1,...,x_n\|}.$$

Or

$$\|T\| = \sup_{\|x_1,...,x_n\|=1} \|T(x_1,...,x_n\|).$$

Remark 2.3.1 *[7] When taking $n = 1$, the concept of Boundedness in n-normed spaces is reduced to the usual concept in boundedness in normed space.*

Example 2.3.1 *[7] Let $\mathbb{R}^n$ equipped with the Euclidean n-norm. Let T be an operator defined as*

$$\begin{aligned} T: \quad \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (x_1, ..., x_n) &\longmapsto T(x_1, ..., x_n) = \det[\alpha_{ij},] \end{aligned}$$

where $x_i = \sum_{j=1}^{n} \alpha_{ij} e_j$, for $i = \overline{1.n}$, with $\{e_1, ..., e_n\}$ the canonical basis. Then T is bounded with $\|T\| = 1$.

Proposition 2.3.1 *[7] Let $T : X^n \longrightarrow Y$ be an n-linear operator T is bounded if and only if for all $(x_1, ..., x_n)$, $(y_1, ..., y_n) \in X^n$,*

$$\begin{aligned} &\|T(x_1, ..., x_n) - T(y_1, ..., y_n)\| \\ &\le c(\|x_1 - y_1, x_2, ..., x_n\| + \|y_1, x_2 - y_2, ..., x_n\| + ... + \|y_1, y_2, ..., x_{n-1} - y_{n-1}, x_n\| \\ &+ \|y_1, y_2, ..., y_{n-1}, x_n - y_n\|). \end{aligned} \tag{2.1}$$

Proof Suppose that (2.1) holds. Take $(y_1, ..., y_n) = (0, ..., 0)$.

Conversely, if T is bounded, then using n-linearity and triangle inequality,

$$\begin{aligned} &\|T(x_1, ..., x_n) - T(y_1, ..., y_n)\| \\ &= \|T(x_1 - y_1, x_2, ..., x_n) + T(y_1, x_2 - y_2, ..., x_n) + ... + T(y_1, y_2, ..., x_n - y_n)\| \\ &\le \|T(x_1 - y_1, x_2, ..., x_n)\| + \|T(y_1, x_2 - y_2, ..., x_n)\| + ... + \|T(y_1, y_2, ..., x_n - y_n)\| \\ &\le c(\|x_1 - y_1, x_2, ..., x_n\| + \|y_1, x_2 - y_2, ..., x_n\| + ... + \|y_1, y_2, ..., x_n - y_n\| .) \end{aligned}$$

Observantly, if T is a bounded n-linear operator, and $x_1, ..., x_n$ are linearly dependent, then $T(x_1, ..., x_n) = 0$. □

The following proposition gives equivalent formulae for $\|T\|$.

Proposition 2.3.2 *[7] Let T be a bounded n-linear operator. Then*

$$\begin{aligned} \|T\| &= \inf\{c : \|T(x_1, ..., x_n)\| \le c\,\|x_1, ..., x_n\|, \quad (x_1, ..., x_n) \in X^n\} \\ &= \sup_{\|x_1, ..., x_n\| \le 1} \|T(x_1, ..., x_n)\| \\ &= \inf\{c : (2.1) \text{ holds}\} \end{aligned}$$

The following definition introduced by L. Soenjaya in [7] that expands on the one provided by White in reference [8] for a 2-normed space. It is important to observe that when n equals 1, the definition mentioned above simplifies to the standard concept of continuity in the normed space.

Definition 2.3.2 *(Continuous n-Linear Operator) Let $(X, \|\cdot, ..., \cdot\|)$ be an n-normed space and $(Y, \|.\|)$ be a normed space, and let T be an n-linear operator on X such that, $T : X^n \longrightarrow Y$. We say that T is continuous at $(x_1, ..., x_n) \in X^n$ if*

$$\forall \varepsilon > 0, \exists \delta > 0 : \|T(x_1, ..., x_n) - T(y_1, ..., y_n)\| < \varepsilon.$$

Whenever

$$\|x_1 - y_1, x_2, ..., x_{n-1}, x_n\| < \delta \quad \textit{and} \quad \|y_1, x_2 - y_2, ..., x_{n-1}, x_n\| < \delta \ \textit{and...}$$

$$\textit{...and} \ \ \|y_1, y_2, ..., x_{n-1} - y_{n-1}, x_n\| < \delta \quad \textit{and} \quad \|y_1, y_2, ..., y_{n-1}, x_n - y_n\| < \delta$$

Or

$$\|x_1 - y_1, y_2, ..., y_{n-1}, y_n\| < \delta \quad \textit{and} \quad \|x_1, x_2 - y_2, ..., y_{n-1}, y_n\| < \delta \ \textit{and...}$$

$$\textit{...and} \ \ \|x_1, x_2, ..., x_{n-1} - y_{n-1}, y_n\| < \delta \quad \textit{and} \quad \|x_1, x_2, ..., x_{n-1}, x_n - y_n\| < \delta$$

where $(y_1, ..., y_n) \in X^n$.

Remark 2.3.2 *(1) T is continuous (on X^n) if it is continuous at every $(x_1, ..., x_n) \in X^n$.*

(2) When taking $n = 1$, the concept of continuity in n-normed spaces is reduced to the usual concept in continuity in normed space.

Theorem 2.3.1 *(Continuity and Boundedness) Let X^n be an n-normed spaces and Y be a normed spaces, and let an n-linear operator $T : X^n \longrightarrow Y$, the following statement are equivalents*

(1) T is continuous;

(2) T is continuous at $(0, ..., 0) \in X^n$;

(3) T is bounded.

Proof (1)$\Longrightarrow$(2) Suppose that T is continuous on X_n, then it's continuous at$(0,...,0) \in X^n$.

(2) $\Longrightarrow$(3) According to [7], we have by supposing T is continuous at $(0,...,0) \in X^n$. Then by definition,

$$\exists \delta > 0 : \|T(u_1,...,u_n)\| < 1 \ \ whenever \ \ \|u_1,...,u_n\| < \delta.$$

Let $(x_1,...,x_n) \in X^n$. Consider the case when $\|x_1,...,x_n\| = 0$. By the continuity at $(0,...,0) \in X^n$ note that there is $\delta_k > 0$ such that $\|T(x_1,...,x_n)\| < \dfrac{1}{k}$ whenever $\|x_1,...,x_n\| < \delta_k$. Then since $\|x_1,...,x_n\| = 0 < \delta_k$, we have $\|T(x_1,...,x_n)\| = 0$, i.e., $T \equiv 0$.
If $\|x_1,...,x_n\| \neq 0$, then let $u_i = \left(\dfrac{\delta}{4\|x_1,...,x_n\|}\right)^{\frac{1}{n}} x_i$ for $i = \overline{1.n}$. Note that $\|u_1,...,u_n\| < \dfrac{\delta}{4} < \delta$. Then

$$\|T(u_1,...,u_n)\| = \frac{\delta}{4\|x_1,...,x_n\|}\|T(x_1,...,x_n)\|.$$

Therefore

$$\|T(x_1,...,x_n)\| = \frac{4}{\delta}\|x_1,...,x_n\|\|T(u_1,...,u_n)\| \leq \frac{4}{\delta}\|x_1,...,x_n\|.$$

Hence T is bounded.

(3) $\Longrightarrow$ (1) : Since T is bounded and by [7, Proposition2.3.1], we have

$$\begin{aligned} &\|T(x_1,...,x_n) - T(y_1,...,y_n)\| \\ &\leq \|T\|(\|x_1 - y_1, x_2,...,x_n\| + \|y_1, x_2 - y_2,...,x_n\| + ... + \|y_1, y_2,...,x_n - y_n\|) \end{aligned} \tag{2.2}$$

Let $\varepsilon > 0$ be given. Take $\delta = \dfrac{\varepsilon}{1 + n\|T\|}$.

If each of the term in the brackets on the right-hand side of (2.2) is less than δ, then

$$\|T(x_1,...,x_n) - T(y_1,...,y_n)\| \leq \varepsilon.$$

Then T is continuous. □

Definition 2.3.3 *(Banach steinhaus theorem) Let* $(X\|\cdot,...,\cdot\|)$ *is n-banach space,* $(Y\|\cdot\|)$ *is a* $\mathbb{K}$*-vector normed space and* $(F_i)_{i\in I}$ *is a family of* $\mathcal{L}(X^n, Y)$ *if*

$$\sup_{i\in I}\|F_i(x_1,...,x_n)\| < \infty \quad for \quad all \quad (x_1,...,x_n) \in X$$

Then

$$\sup_{i\in I}\|F_i\| < \infty.$$

2.4 The space $B(X^n, Y)$

Afterward, we will explore the related set of operators. Let $B(X^n, Y)$ denotes the space of all bounded n-linear operators from X^n into Y.

Theorem 2.4.1 *$(B(X^n, Y), \|\cdot\|)$ is a normed space with norm given by*

$$\|T\| := \sup_{\|x_1,...,x_n\| \neq 0} \frac{\|T(x_1, ..., x_n)\|}{\|x_1, ..., x_n\|}.$$

Proof By the definition of $\|\cdot\|$ we have $\|\lambda T\| = |\lambda| \, \|T\|$. For the triangular inequality for a norm, we have

$$\begin{aligned} \|T_1 + T_2\| &\leq \sup_{\|x_1,...,x_n\| \neq 0} \frac{\|T_1(x_1, ..., x_n)\| + \|T_2(x_1, ..., x_n)\|}{\|x_1, ..., x_n\|} \\ \\ &\leq \sup_{\|x_1,...,x_n\| \neq 0} \frac{\|T_1(x_1, ..., x_n)\| \, |}{\|x_1, ..., x_n\|} + \sup_{\|x_1,...,x_n\| \neq 0} \frac{\|T_2(x_1, ..., x_n)\|}{\|x_1, ..., x_n\|} \\ \\ &= \|T_1\| + \|T_2\| \, . \end{aligned}$$

Finally, for the first condition of the norm, we have

$$\|T\| = 0 \Longrightarrow T(x_1, ..., x_n) = 0,$$

if $\|x_1, ..., x_n\| \neq 0$.

If $\|x_1, ..., x_n\| = 0$, then $x_1, ..., x_n$ are linearly dependent, hence $T(x_1, ..., x_n) = 0$ by the **Proposition 2.1**. Hence $T \equiv 0$. Therefore, $\|\cdot\|$ is a norm. $\square$

Theorem 2.4.2 *[7] . If $(Y, \|\cdot\|)$ is a Banach space, then $(B(X^n, Y), \|\cdot\|)$ is a Banach space.*

Proof

Let $\{\xi_n\}$ be a Cauchy sequence in $B(X^n, Y)$. Then,

$$\forall \varepsilon > 0, \exists N > 0 : \|\xi_n - \xi_m\| \leq \frac{\varepsilon}{2}, \quad \forall n, m > N.$$

We have by definition,

$$\|\xi_n(x_1, ..., x_n) - \xi_m(x_1, ..., x_n)\| \leq \|\xi_n - \xi_m\| \, \|x_1, ..., x_n\| \, . \tag{2.3}$$

$\forall\ n, m > N$, we have

$$\|\xi_n(x_1, ..., x_n) - \xi_m(x_1, ..., x_n)\| \leq \frac{\varepsilon}{2} \|x_1, ..., x_n\|. \tag{2.4}$$

Since $\{\xi_n\}$ is Cauchy sequence and Y is a Banach space and by (2.3), we may define

$\xi(x) = \lim_{n \longrightarrow \infty} \xi_n(x)$. Then

$$\|\xi_M(x_1, ..., x_n) - \xi(x_1, ..., x_n)\| \leq \frac{\varepsilon}{2} \|x_1, ..., x_n\|. \tag{2.5}$$

By (2.4)and (2.5), $\forall n > M$, we have

$$\begin{aligned} \|\xi_n(x_1, ..., x_n) - \xi(x_1, ..., x_n)\| &\leq \|\xi_n(x_1, ..., x_n) - \xi_M(x_1, ..., x_n)\| + \|\xi_M(x_1, ..., x_n) - \xi(x_1, ..., x_n)\| \\ &\leq \varepsilon \|x_1, ..., x_n\|. \end{aligned}$$

This implies $\|\xi_n - \xi\| < \varepsilon$, i.e., $\xi_n \to \xi$. □

Chapter 3

n-Continuous, n-bounded linear operator

3.1 n-Bounded, n-continuous linear operator

The following definition introduced by [7, Definition 3.1.1] which generalizes the concept of bounded operator by introducing the notion of n-bounded operator.

Definition 3.1.1 *(n-Bounded Linear Operator) Let* $(X, \|\cdot\|)$ *be a normed space and* $(X, \|\cdot, ..., \cdot\|)$ *be an n-normed space and let an operator* $T : (X, \|\cdot\|) \longrightarrow (X, \|\cdot, ..., \cdot\|)$ *. We say* T *is n-bounded if there is a constant* c *such that for all* $x_1, ..., x_n \in X$*,*

$$\|T(x_1), x_2, ..., x_n\| + \|x_1, T(x_2), ..., x_n\| + ... + \|x_1, x_2, ..., T(x_n)\| \leq c \|x_1\| \|x_2\| ... \|x_n\| . \quad (3.1)$$

Corollary 3.1.1 *If* T *is n- bounded, then* $\|T\|_n$ *define by*

$$\|T\|_n := \inf \{c : (3.1) \textit{ holds}, \quad x_1, x_2, ..., x_n \in X\} \quad (3.2)$$

Remark 3.1.1 *When taking* $n = 1$*, the definition of n-bounded operator reduces to the usual concept of bounded operator.*

Example 3.1.1 *[7] (1) Let* $X = \mathbb{R}^2$ *be equipped with the* ℓ_1*-norm and the Euclidean 2 -norm. Define operators* T *and* T' *by*

$$T((x_1, x_2)) = (x_1, x_2)$$

and

$$T'((x_1, x_2)) = (0, x_2),$$

where $(x_1, x_2) \in \mathbb{R}^2$*. Then* $\|T\|_2 = 2$ *and* $\|T'\|_2 = 1$*.*

(2) Let $(X, \|\cdot\|)$ *be a real inner product space and define*

$$\|x, y\| = (\|x\|^2 \|y\|^2 - \langle x, y\rangle^2)^{\frac{1}{2}}.$$

If $T : X \longrightarrow X$ *is a bounded linear operator, then*

$$\|T(x), y\| + \|x + T(y)\| \leq 2\,\|T\|\,\|x\|\,\|y\|\,.$$

Hence T *is a 2-bounded linear operator.*

Proposition 3.1.1 *Let* $T : X \longrightarrow X$ *be an n-bounded linear operator. Then for* $x_1, x_2, ..., x_n \in X$*, we have*

$$\begin{aligned}\|T\|_n &= \sup_{\|x_1\|...\|x_n\|=1} \left(\|T(x_1), ..., x_n\| + ... + \|x_1, ..., T(x_n)\|\right)\\ &= \sup_{\|x_1\|...\|x_n\|\neq 0} \left(\frac{\|T(x_1), ..., x_n\| + ... + \|x_1, ..., T(x_n)\|}{\|x_1\|...\|x_n\|}\right).\end{aligned}$$

Proof (i) Let $M = \|T(x_1), ..., x_n\| + ... + \|x_1, ..., T(x_n)\|$. Then by **Definition 3.1.1**, we have $\|T\|_n \geq M$.

Let $y_j = \dfrac{x_j}{\|x_j\|}, \forall j = \overline{1.n}$. Then $M \geq \|T(y_1), ..., y_n\| + ... + \|y_1, ..., T(y_n)\|$. Then $\|T\|_n \leq M$, hence $\|T\|_n = M$.

(ii) Let $N = \dfrac{\|T(x_1), ..., x_n\| + ... + \|x_1, ..., T(x_n)\|}{\|x_1\|...\|x_n\|}$, then we have

$$\frac{\|T(x_1), ..., x_n\|}{\|x_1\|...\|x_n\|} + ... + \frac{\|x_1, ..., T(x_n)\|}{\|x_1\|...\|x_n\|}$$
$$= \left\|T(\frac{x_1}{\|x_1\|}), ..., \frac{x_n}{\|x_n\|}\right\| + ... + \left\|\frac{x_1}{\|x_1\|}, ..., T(\frac{x_n}{\|x_n\|})\right\|.$$

The resulting $M \geq N$. Moreover, M is written as follows

$$M = \sup_{\|x_1\|...\|x_n\|=1} \left(\|T(x_1), ..., x_n\| + ... + \|x_1, ..., T(x_n)\|\right).$$

We see that $M \leq N$ as the set over which the supremum is taken is bigger for N. □

We will now give the develop properties of n-bounded operator that introduced by [7]. Such that, he introduced the concept of n-continuity as follows.

Definition 3.1.2 *(n-Continuous Linear Operator) Let* $T : (X, \|\cdot\|) \longrightarrow (X, \|\cdot, ..., \cdot\|)$ *be an operator. We say that* T *is n-continuous at* $x \in X$ *if*

$$\begin{aligned}\forall \varepsilon > 0, \exists \delta > 0 :\ &\|T(x_1) - T(x), x_2 - x, ..., x_n - x\| + \|x_1 - x, T(x_2) - T(x), ..., x_n - x\|\\ &+ ... + \|x_1 - x, x_2 - x, ..., T(x_n) - T(x)\| < \varepsilon\end{aligned} \tag{3.3}$$

whenever $\|x_1 - x\|\,\|x_2 - x\| ... \|x_n - x\| < \delta$ *, where* $x_1, ..., x_n \in X$*.*

Corollary 3.1.2 *If T is n-continuous at every $x \in X$, then it is n-continuous on X.*

Remark 3.1.2 *When taking $n = 1$, the definition of n-continuous operator reduces to the usual concept of continuous operator.*

Theorem 3.1.1 *(n-Continuity and n-Boundedness). Let $T : (X, \|\cdot\|) \longrightarrow (X, \|\cdot, ..., \cdot\|)$ be an operator. The following statements are equivalent:*

(1) T is n- continuous;

(2) T is n-continuous at $0 \in X$;

(3) T is n-bounded.

Proof (1)$\Longrightarrow$(2) is trivial.

(2) $\Longrightarrow$(3) Suppose T is n- continuous at $0 \in X$, then by definition,

$$\exists \delta > 0 : \|T(x_1), ..., x_n\| + ... + \|x_1, ..., T(x_n)\| < 1 \;\; \textit{whenever} \;\; \|x_1\| ... \|x_n\| < \delta.$$

Let $(v_1, ..., v_n) \in X$. If $\|v_1\| ... \|v_n\| = 0$, then

$$\|T(v_1), ..., v_n\| + ... + \|v_1, ..., T(v_n)\| = 0.$$

If $\|v_1\| ... \|v_n\| \neq 0$, then let $x_i = \left(\frac{\delta}{4}\right)^{\frac{1}{n}} \frac{v_i}{\|v_i\|}$, for $i = \overline{1, n}$.

Note that $\|x_1\| ... \|x_n\| < \delta$. Then we have

$$\|T(x_1), ..., x_n\| + ... + \|x_1, ..., T(x_n)\| = \frac{\delta}{4 \|v_1\| ... \|v_n\|} \left(\|T(v_1), ..., v_n\| + ... + \|v_1, ..., T(v_n)\|\right).$$

Therefore

$$\begin{aligned} &\|T(v_1), ..., v_n\| + ... + \|v_1, ..., T(v_n)\| \\ &= \frac{4 \|v_1\| ... \|v_n\|}{\delta} \left(\|T(x_1), ..., x_n\| + ... + \|x_1, ..., T(x_n)\|\right) < \frac{4}{\delta} \|v_1\| ... \|v_n\| . \end{aligned}$$

Hence T is n-bounded.

(3)$\Longrightarrow$ (1) Since T is n-bounded, then

$$\|T(v_1 - x), ..., v_n - x\| + ... + \|v_1 - x, ..., T(v_n - x)\| \leq \|T\|_n \|v_1 - x\| ... \|v_n - x\| .$$

Let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{1 + \|T\|_n}$, hence

$$\|T(v_1) - T(x), ..., v_n - x\| + ... + \|v_1 - x, v_2 - x, ..., T(v_n) - T(x)\| < \varepsilon \;\; \textit{whenever} \; \|v_1 - x\| ... \|v_n - x\| .$$

So T is n-continuous. $\square$

3.2 The space $B_n(X, X)$

Let $B_n(X, X)$ denotes the space of all n-bounded linear operators from $(X, \|\cdot\|)$ into $(X, \|\cdot, ..., \cdot\|)$.

Theorem 3.2.1 *$(B_n(X, X), \|\cdot\|_n)$ is a normed space with norm given by*

$$\|T\|_n := \inf \{c : (3.1) \text{ holds}, \quad x_1, x_2, ..., x_n \in X\}.$$

Proof We have $\|\cdot\|_n$ satisfies

1. $\|\lambda T\|_n = |\lambda| \|T\|_n$ and $\|T_1 + T_2\|_n \leq \|T_1\|_n + \|T_2\|_n$.
2. $\|T\|_n = 0 \Longrightarrow \|T(x_1), ..., x_n\| = 0, \qquad \forall x_1, ..., x_n \in X \Longrightarrow T \equiv 0$

Therefore, $(B_n(X, X), \|\cdot\|_n)$ is a normed space. □

Theorem 3.2.2 *[4] $(B_n(X, X), \|\cdot\|_n)$ is a Banach space if $(X, \|\cdot, ..., \cdot\|)$ is an n- Banach space.*

Proof Let $\{\xi_k\}$ be a Cauchy sequence in $B_n(X, X)$. Then

$$\forall \varepsilon > 0, \exists N > 0 : \|\xi_k - \xi_l\|_n \leq \frac{\varepsilon}{2}, \quad \forall k, l > N.$$

We have by definition,

$$\|(\xi_k - \xi_l)(x_1), ..., x_n\| + ... + \|x_1, ..., (\xi_k - \xi_l)(x_n)\| \leq \|\xi_k - \xi_l\|_n \|x_1\| ... \|x_n\| \tag{3.4}$$

Therefore, for $\forall k, l > N$, we have

$$\|\xi_k(x_1) - \xi_l(x_1), ..., x_n\| + ... + \|x_1, ..., \xi_k(x_n) - \xi_l(x_n)\| \leq \frac{\varepsilon}{2} \|x_1\| ... \|x_n\| \tag{3.5}$$

Using (3.4) and since $\{\xi_k\}$ is Cauchy and by definition of $\|\cdot\|_n$, we have $\forall x_1, ..., x_n \in X$,

$$\lim_{k,l \longrightarrow \infty} \|\xi_k(x_1) - \xi_l(x_1), ..., x_n\| = 0, ..., \lim_{k,l \longrightarrow \infty} \|x_1, ..., \xi_k(x_n) - \xi_l(x_n)\| = 0$$

$\{\xi_k\}$ is Cauchy in $(X, \|\cdot, ..., \cdot\|)$ $\forall x \in X$.

$(X, \|\cdot, ..., \cdot\|)$ is an n-Banach space, we may define $\xi(x) = \lim_{k \longrightarrow \infty} \xi_k(x)$ in the sense of n-norm. By definition of convergence, $\exists M = (x_1, ..., x_n) > N$ such that $\forall x_1, ..., x_n \in X$, we have

$$\|\xi_M(x_1) - \xi(x_1), ..., x_n\| + ... + \|x_1, ..., \xi_M(x_n) - (x_n)\| \leq \frac{\varepsilon}{2} \|x_1\| ... \|x_n\|. \tag{3.6}$$

By (3.5), (3.6) and triangle inequality for n-norm, $\forall k > M$ and $x_1, ..., x_n \in X$,

$$\|\xi_k(x_1) - \xi(x_1), ..., x_n\| + ... + \|x_1, ..., \xi_k(x_n) - (x_n)\| \leq \varepsilon \|x_1\| ... \|x_n\|, \quad \forall x_1, ..., x_n \in X.$$

Then $\|\xi_k - \xi\| < \varepsilon$, i.e., $\xi_k \longrightarrow \xi$.

Chapter 4

Other notions of n-continuous, n-bounded operator

4.1 n-bounded , n-nontinuous of type-I operator

There exists a several concepts of continuity in n-normed spaces that differs from those discussed earlier. In their work [2], Chu et al. introduced the idea of a 2-continuous mapping to investigate the Aleksandrov problem within 2-normed spaces. Inspired by this research, we aim to introduce the extension of this notion beyond the traditional understanding of continuity in normed spaces [7][Definition4.1.1].

Definition 4.1.1 *(n-Bounded of type-I operator) Let* $(X, \|\cdot, ..., \cdot\|)$ *and* $(Y, \|\cdot, ..., \cdot\|)$ *be two n-normed spaces, and let* T *be an operator such that,* $T : X^n \longrightarrow Y^n$. *We say that* T *is n-bounded of type-I if there is a constant* c *such that* $\forall (x_1, ..., x_n) \in X^n$, *we have*

$$\|T(x_1), ..., T(x_n)\| \leq c \|x_1, ..., x_n\| .$$

Corollary 4.1.1 *If* T *is n-Bounded of type-I operator,we define* $[T]_n$ *by*

$$[T]_n := \sup_{\|x_1,...,x_n\| \neq 0} \frac{\|T(x_1), ..., T(x_n)\|}{\|x_1, ..., x_n\|}.$$

Example 4.1.1 *[7] (1) Let* $T : (X, \|\cdot, ..., \cdot\|) \longrightarrow (Y, \|\cdot, ..., \cdot\|)$ *be a dilation, i.e.,* $T(x) = cx$ *for all* $x \in X$, *where* $c \in \mathbb{R}$. *Then* T *is n-bounded of type-I.*

(2) Let $X = C^1[0, 1]$, *equipped with 2-norm defined by for* $f, g \in X$

$$\|f, g\| = \sup_{t \in [0,1]} W(f, g)(t)$$

, where $W(f, g)$ *is the Wronskian of* f *and* g. *Let* $T : X \longrightarrow X$ *be defined by* $T(x) = y$, *where* $y(t) = tx(t)$. *Then for all* $f, g \in X$

$$\|(f), T(g)\| \leq \|f, g\| .$$

So T is 2-bounded.

We will now give the develop properties of n-bounded of type-I operator that introduced by [7]. Such that, he introduced the concept of n-continuity as follows.

Definition 4.1.2 *(n-continuous of type-I Operator) Let $(X, \|\cdot, ..., \cdot\|)$ and $(Y, \|\cdot, ..., \cdot\|)$ be two n-normed spaces, and let T be an operator. We say that T is n-continuous of type-I at $(x_1, x_2, ..., x_n) \in X$, if*

$$\forall \varepsilon > 0, \exists \delta > 0 : \|T(x_1) - T(x), T(x_2) - T(x), ..., T(x_n) - T(x)\| < \varepsilon. \tag{4.1}$$

whenever $\|x_1 - x, x_2 - x, ..., x_n - x\| < \delta$, where $(x_1, x_2, ..., x_n) \in X$.

Remark 4.1.1 *T is n-continuous of type-I (on X) if it's n-continuous of type-I at every $(x_1, x_2, ..., x_n) \in X$.*

Remark 4.1.2 *(1) When taking $n = 1$, the concept of n-continuity in n-normed spaces is reduced to the usual concept in continuity in normed space.*

(2) When taking $n = 2$, the concept of n- continuity in n-normed spaces is reduced to the usual concept in 2-continuity in normed space.

Theorem 4.1.1 *(n-Continuity and n-Boundedness of type-I) Let $T : (X, \|\cdot, ..., \cdot\|) \longrightarrow (Y, \|\cdot, ..., \cdot\|)$ be a linear operator, the following statement are equivalents*

(1) T is n- continuous of type-I;

(2) T is n-continuous of type-I at $0 \in X$;

(3) T is n-bounded of type-I.

Proof (1)$\Longrightarrow$(2) is trivial.

(2) $\Longrightarrow$(3) Suppose T is n- continuous of type-I at $0 \in X$. Then by definition,

$$\exists \delta > 0 : \|T(x_1), ..., T(x_n)\| < 1 \text{ whenever } \|x_1\| ... \|x_n\| < \delta.$$

Let $(v_1, ..., v_n) \in Y$,

- If $\|v_1, ..., v_n\| = 0$, then $v_1, ..., v_n$ are linearly dependent. By linearity of $T, T(v_1), ..., T(v_n)$ are linearly dependent, hence $\|T(v_1), ..., T(v_n)\| = 0$.

- If $\|v_1, ..., v_n\| \neq 0$, then let $x_i = \left(\dfrac{\delta}{4\|v_i\|}\right)^{\frac{1}{n}} v_i$, for $i = \overline{1.n}$.

Note that $\|x_1, ..., x_n\| < \delta$. Then we have,

$$\|T(x_1), ..., T(x_n)\| = \frac{\delta}{4\|v_1, ..., v_n\|}\|T(v_1), ..., T(v_n)\|.$$

Therefore

$$\|T(v_1), ..., T(v_n)\| = \frac{4\,\|v_1, ..., v_n\|}{\delta}\left(\|T(x_1), ..., T(x_n)\|\right) < \frac{4}{\delta}\,\|v_1, ..., v_n\|\,.$$

Hence T is n-bounded of type-I.

(3) $\Longrightarrow$ (1) Since T is n-bounded of type-I, then

$$\|T(v_1 - x), ..., v_n - x\| + ... + \|v_1 - x, ..., T(v_n - x)\| \leq [T]_n\,\|v_1 - x\| ... \|v_n - x\|\,.$$

Let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{1 + [T]_n}$, then

$$\|T(v_1) - T(x), ..., v_n - x\| + ... + \|v_1 - x, v_2 - x, ..., T(v_n) - T(x)\| < \varepsilon \;\; \textit{whenever}\; \|v_1 - x\| ... \|v_n - x\|\,.$$

So T is n-continuous of type-I. □

4.2 The space $B_n^n(X, X)$ of type-I

Let $B_n^n(X, X)$ denotes the space of all n-bounded of type-I operators from $(X, \|\cdot, ..., \cdot\|)$ into $(X, \|\cdot, ..., \cdot\|)$.

Theorem 4.2.1 $(B_n^n(X, X), |||T|||_n)$ *is an n-normed space with norm given by*

$$|||T|||_n := \sup_{\|x_1, ..., x_n\| \neq 0} \frac{\|T(x_1), ..., T(x_n)\|}{\|x_1, ..., x_n\|}.$$

Proof We have $[T]_n$ satisfies

- $[\lambda T]_n = |\lambda|\,[T]_n$ and $[T_1 + T_2]_n \leq [T_1]_n + [T_2]_n$.
- $[T]_n = 0 \Longrightarrow [T(x_1), ..., x_n]_n = 0, \quad \forall x_1, ..., x_n \in X \Longrightarrow T \equiv 0.$

Therefore $(B_n(X, X), [T]_n)$ is an n-normed space. □

4.3 n-Bounded, n-continuous of type-II operator

Definition 4.3.1 *[7](n-Bounded of type-II operator) Let* $T : X^n \longrightarrow Y^n$ *be an operator. We say that T is n-bounded of type-II if there is a constant c such that* $\forall (x_1, ..., x_n) \in X^n$, *we have*

$$\|T(x_1), x_2, ..., x_n\| + \|x_1, T(x_2), ..., x_n\| + ... + \|x_1, x_2, ..., T(x_n)\| \leq c\,\|x_1, x_2, ..., x_n\| \tag{4.2}$$

Corollary 4.3.1 *[7] If T is n-Bounded of type-II operator, then* $|||T|||_n$ *be*

$$|||T|||_n := \sup_{\|x_1, x_2, ..., x_n\| \neq 0} \frac{\|T(x_1), x_2, ..., x_n\| + ... + \|x_1, x_2, ..., T(x_n)\|}{\|x_1, x_2, ..., x_n\|}. \tag{4.3}$$

Example 4.3.1 *[7] Let $T : X^n \longrightarrow Y^n$ be a dilation, i.e. $T(x_1, x_2, ..., x_n) = k(x_1, x_2, ..., x_n)$, $(x_1, x_2, ..., x_n) \in X^n$, where k is a real number. Then T is n-Bounded of type-II.*

We will now give the develop properties of n-bounded of type-II operator that introduced by [7]. Such that, he introduced the concept of n-continuity as follows.

Definition 4.3.2 *(n-continuous of type-II Operator) Let $(X, \|\cdot, ..., \cdot\|)$ and $(Y, \|\cdot, ..., \cdot\|)$ be two n-normed spaces, and let T be an operator. We say that T is n-continuous of type-II at $(x_1, x_2, ..., x_n) \in X$, if $\forall \varepsilon > 0, \exists \delta > 0$:*

$$\|T(x_1) - T(x), ..., x_n - x\| + \|x_1 - x, T(x_2) - T(x), ..., x_n - x\| + ... + \|x_1 - x, ..., T(x_n) - T(x)\| < \varepsilon$$

whenever $\|x_1 - x, x_2 - x, ..., x_n - x\| < \delta$.

Remark 4.3.1 *T is n-continuous of type-II (on X) if it's n-continuous of type-II at every $(x_1, x_2, ..., x_n) \in X$.*

Theorem 4.3.1 *(n-Continuity and n-Boundedness of type-II) Let X, Y are n-normed spaces and let T be a linear operator to X into Y, the following statement are equivalents*

(1) T is n- continuous of type-II;

(2) T is n-continuous of type-II at $0 \in X$;

(3) T is n-bounded of type-II.

Proof (1)$\Longrightarrow$(2) is trivial.

(2) $\Longrightarrow$(3) Suppose T is n- continuous of type-II at $0 \in X$. Then by definition,

$$\exists \delta > 0 : \|T(x_1), ..., x_n\| + ... + \|x_1, ..., T(x_n)\| < 1 \text{ whenever } \|x_1, ..., x_n\| < \delta.$$

Let $(v_1, ..., v_n) \in Y$, and let $x_i = \left(\frac{\delta}{4\|v_i\|} \right)^{\frac{1}{n}} v_i$, for $i = \overline{1.n}$.
Note that $\|x_1, ..., x_n\| < \delta$. Then we have,

$$\|T(x_1), ..., x_n\| + ... + \|x_1, ..., T(x_n)\| = \frac{\delta}{4\|v_1, ..., v_n\|} \|T(v_1), ..., v_n\| + ... + \|v_1, ..., T(v_n)\|$$

Therefore

$$\begin{aligned} \|T(v_1), ..., v_n\| + ... + \|v_1, ..., T(v_n)\| &= \frac{4\|v_1, ..., v_n\|}{\delta} (\|T(x_1), ..., x_n\| + ... + \|x_1, ..., T(x_n)\|) \\ &< \frac{4}{\delta} \|v_1, ..., v_n\| \end{aligned}$$

Hence T is n-bounded of type-II.

(3) $\Longrightarrow$ (1) Since T is n-bounded of type-II, then

$$\|T(v_1-x),...,v_n-x\|+...+\|v_1-x,...,T(v_n-x)\|\leq \|\|T\|\|_n\,\|v_1-x\|...\|v_n-x\|.$$

Let $\varepsilon>0$ and $\delta=\dfrac{\varepsilon}{1+\|\|T\|\|_n}$, then *whenever* $\|v_1-x\|...\|v_n-x\|$

$$\|T(v_1)-T(x),...,v_n-x\|+...+\|v_1-x,v_2-x,...,T(v_n)-T(x)\|<\varepsilon.$$

So T is n-continuous of type-II.

4.4 The space $B_n^n(X,X)$ of type-II

Let $B_n^n(X,X)$ denotes the space of all n-bounded of type-II operators from $(X,\|\cdot,...,\cdot\|)$ into $(X,\|\cdot,...,\cdot\|)$.

Theorem 4.4.1 *$(B_n^n(X,X),\|\|T\|\|_n)$ is an n-normed space with norm given by*

$$\|\|T\|\|_n:=\sup_{\|x_1,x_2,...,x_n\|\neq 0}\frac{\|T(x_1),x_2,...,x_n\|+...+\|x_1,x_2,...,T(x_n)\|}{\|x_1,x_2,...,x_n\|}.$$

Proof We have $\|\|T\|\|_n$ satisfies

- $\|\|\lambda T\|\|_n=|\lambda|\,\|\|T\|\|_n$ and $\|\|T_1+T_2\|\|_n\leq\|\|T_1\|\|_n+\|\|T_2\|\|_n$.
- $\|\|T\|\|_n=0\Longrightarrow\|\|T(x_1),...,x_n\|\|_n=0\ ,\qquad\forall x_1,...,x_n\in X\Longrightarrow T\equiv 0.$

Therefore, $(B_n(X,Y),\|\cdot\|_n)$ is an n-normed space. ☐

Theorem 4.4.2 *[4] $(B_n^n(X,Y),\|\|T\|\|_n)$ is a Banach space when $(Y,\|\cdot,...,\cdot\|)$ is an n-Banach spaces.*

Proof Let $\{\xi_k\}$ be a Cauchy sequence in $B_n(X,Y)$. Then

$$\forall\varepsilon>0,\exists N>0:\|\xi_k-\xi_l\|_n\leq\frac{\varepsilon}{2},\quad\forall k,l>N.$$

We have by definition,

$$\|(\xi_k-\xi_l)(x_1),...,x_n\|+...+\|x_1,...,(\xi_k-\xi_l)(x_n)\|\leq\|\xi_k-\xi_l\|_n\,\|x_1\|...\|x_n\|.$$

Therefore, $\forall k,l>N$, we have

$$\|\xi_k(x_1)-\xi_l(x_1),...,x_n\|+...+\|x_1,...,\xi_k(x_n)-\xi_l(x_n)\|\leq\frac{\varepsilon}{2}\|x_1\|...\|x_n\|,\tag{4.4}$$

The sequence $\{\xi_k\}$ is Cauchy and by definition of $\|\cdot\|_n$, we have $\forall x_1, ..., x_n \in X$,

$$\lim_{k,l \longrightarrow \infty} \|\xi_k(x_1) - \xi_l(x_1), ..., x_n\| = 0, ..., \lim_{k,l \longrightarrow \infty} \|x_1, ..., \xi_k(x_n) - \xi_l(x_n)\| = 0,$$

Hence $\{\xi_k\}$ is Cauchy in $(Y, \|\cdot, ..., \cdot\|)$ $\forall x \in X$. This implies that $(Y, \|\cdot, ..., \cdot\|)$ is an n-Banach space, we may define $\xi(x) = \lim_{k \longrightarrow \infty} \xi_k(x)$ in the sense of n-norm. By definition of convergence, $\exists M = (x_1, ..., x_n) > N$:

$$\|\xi_M(x_1) - \xi(x_1), ..., x_n\| + ... + \|x_1, ..., \xi_M(x_n) - (x_n)\| \leq \frac{\varepsilon}{2} \|x_1\| ... \|x_n\|, \quad \forall x_1, ..., x_n \in X. \tag{4.5}$$

By (4.4), (4.5) and triangle inequality for n-norm, $\forall k > M$ and $x_1, ..., x_n \in X$,

$$\|\xi_k(x_1) - \xi(x_1), ..., x_n\| + ... + \|x_1, ..., \xi_k(x_n) - (x_n)\| \leq \varepsilon \|x_1\| ... \|x_n\|, \quad \forall x_1, ..., x_n \in X,$$

implies that $\|\xi_k - \xi\| < \varepsilon$, i.e., $\xi_k \longrightarrow \xi$.

4.5 Contraction operators and their fixed point theorems in n-normed space

Definition 4.5.1 *[1] Let X be a nonempty set and $T : X \longrightarrow X$ a selfmap. We say that $x \in X$ is a fixed point of T if $T(x) = x$ and denote by F_T or $Fix(T)$ the set of all fixed points of T.*

Definition 4.5.2 *(Closed subset) Let $X_0 \subseteq X$ be nonempty set. Then we say that X_0 is closed if for every sequence $\{x_m\}_{m=0}^{\infty}$ in X_0 which converges in X , its limit is in X_0.*

Definition 4.5.3 *(c-Lipschitz operator in n-normed space) Let X, Y be a metric spaces with respect to the n-norm $\|\cdot, ..., \cdot\|$. We say that the operator $T : X \longrightarrow Y$ is c-Lipschitz if there is a constant $c > 0$ such that $\forall x_1, ..., x_n \in X$, we have*

$$\|T(x) - T(y), x_2, ..., x_n\| \leq c \|x - y, x_2, ..., x_n\| . \tag{4.6}$$

If $c \in [0, 1)$, we call that T a contraction and if $c = 1$ the operator T is called non-expansive.

Remark 4.5.1 *For a c-lipschitz operator T we define the Lipschitz constant by*

$$\|T\|_{Lip} = Lip(T) := \sup_{\|x-y,x_2,...,x_n\| \neq 0} \frac{\|T(x) - T(y), x_2, ..., x_n\|}{\|x - y, x_2, ..., x_n\|}.$$

Definition 4.5.4 [6]*(n-Lipschitz operators in n-normed space) Let X be a linear n-normed space. We call T is n-Lipschitz operator if there is a $c > 0$ such that $\forall x_0, x_1, ..., x_n \in X$, we have*

$$\|T(x_1) - T(x_0), T(x_2) - T(x_0), ..., T(x_n) - T(x_0)\| \leq c \|x_1 - x_0, x_2 - x_0, ..., x_n - x_0\| .$$

If $c \in [0, 1)$, the operator T is called n-contraction and if $c = 1$ the operator T is called n-non-expansive.

The space of all n-Lipschitz operators to X into Y denoted by $Lip_n(X, Y)$ and the space of n-Lipschitz that vanishing at 0 denoted by $Lip_{n.0}(X, Y)$

Remark 4.5.2 *For a n-lipschitz operator T we define the n-Lipschitz constant by*

$$\|T\|_{Lip_n} = Lip_n(T) := \sup_{\|x_1-x_0,x_2-x_0,...,x_n-x_0\|\neq 0} \frac{\|T(x_1) - T(x_0), T(x_2) - T(x_0), ..., T(x_n) - T(x_0)\|}{\|x_1 - x_0, x_2 - x_0, ..., x_n - x_0\|}.$$

Proposition 4.5.1 *(i) Let X, Y are n-normed spaces and let T_1, T_2 are n-Lipschitz operators from X into Y. Then*

$$Lip_n(T_1 + T_2) \leq Lip_n(T_1) + Lip_n(T_2)$$

(ii) Let X be an n-normed spaces and let T a n-Lipschitz operator from X into Y. Then $\forall \lambda \in \mathbb{R}$, we have

$$Lip_n(\lambda T) = |\lambda| \, Lip_n(T)$$

(iii) Let X, Y, Z are n-normed spaces and let $T_1 : X \longrightarrow Y$ and $T_2 : Y \longrightarrow Z$ are n-Lipschitz operators. Then

$$Lip_n(T_1 \circ T_2) = Lip_n(T_1) Lip_n(T_2)$$

Proof

(i) $\|(T_1 + T_2)(x_1) - (T_1 + T_2)(x_0), ..., (T_1 + T_2)(x_n) - (T_1 + T_2)(x_0)\|$

$= \|T_1(x_1) + T_2(x_1) - T_1(x_0) - T_2(x_0), ..., T_1(x_n) + T_2(x_n) - T_1(x_0) - T_2(x_0)\|$

$= \|T_1(x_1) - T_1(x_0) + T_2(x_1) - T_2(x_0), ..., , ..., T_1(x_n) - T_1(x_0) + T_2(x_n) - T_2(x_0)\|$

$\leq \|T_1(x_1) - T_1(x_0), ..., T_1(x_n) - T_1(x_0)\| + \|T_2(x_1) - T_2(x_0), ..., T_2(x_n) - T_2(x_0)\|$

$\leq Lip_n(T_1) \|x_1 - x_0, ..., x_n - x_0\| + Lip_n(T_2) \|x_1 - x_0, ..., x_n - x_0\|$

$\leq (Lip_n(T_1) + Lip_n(T_2)) \|x_1 - x_0, ..., x_n - x_0\|$.

Then $Lip_n(T_1 + T_2) \leq Lip_n(T_1) + Lip_n(T_2)$.

(ii)

$$\begin{aligned}
Lip_n(\lambda T) &= \sup_{\|x_1-x_0,\dots,x_n-x_0\|\neq 0} \frac{\|\lambda T(x_1)-\lambda T(x_0),\dots,\lambda T(x_n)-\lambda T(x_0)\|}{\|x_1-x_0,\dots,x_n-x_0\|} \\
&= \sup_{\|x_1-x_0,\dots,x_n-x_0\|\neq 0} \frac{\|\lambda(T(x_1)-T(x_0),\dots,T(x_n)-T(x_0))\|}{\|x_1-x_0,\dots,x_n-x_0\|} \\
&= |\lambda| \sup_{\|x_1-x_0,\dots,x_n-x_0\|\neq 0} \frac{\|(T(x_1)-T(x_0),\dots,T(x_n)-T(x_0))\|}{\|x_1-x_0,\dots,x_n-x_0\|} \\
&= |\lambda|\, Lip_n(T).
\end{aligned}$$

(iii)

$\|(T_1\circ T_2)(x_1)-(T_1\circ T_2)(x_0),\dots,(T_1\circ T_2)(x_n)-(T_1\circ T_2)(x_0)\|$

$= \|T_1(T_2(x_1))-T_1(T_2(x_0)),\dots,T_1(T_2(x_n))-T_1(T_2(x_0))\|$

$\leq Lip_n(T_1)\,\|T_2(x_1)-T_2(x_0),\dots,T_2(x_n)-T_2(x_0)\|$

$\leq Lip_n(T_1)Lip_n(T_2)\,\|x_1-x_0,\dots,x_n-x_0\|$.

Then $Lip_n(T_1\circ T_2) = Lip_n(T_1)Lip_n(T_2)$.

Theorem 4.5.1 *$(Lip_{n.0}(X,Y), Lip_{n.0}(\cdot))$ is an n-normed space.*

Kir and Kiziltunc's definition of bounded sets

Definition 4.5.5 *[6](Bounded sets in an n-normed space) Let X_0 be a nonempty subset of X. Then X_0 is called bounded with respect to $\mathcal{S}=\{s_1,\dots,s_n\}$ if there is $M>0$ such that for every $x_0\in X_0$ and $\{i_2,\dots,i_n\}\subset\{1,\dots,n\}$, we have*

$$\|x_0, s_{i_2},\dots,s_{i_n}\| \leq M.$$

Remark 4.5.3 *A set may be bounded according to Kir and Kiziltunc's definition and is unbounded according to ours.*

Example 4.5.1 *[6]*

Let $\varepsilon=\{e_1,\dots,e_n\}$ be the set of the first n basis vectors in $X=\mathbb{R}^d$. Then one may observe that for every $z\in X$ and $\{i_2,\dots,i_n\}\subset\{1,\dots,n\}$, we have

$$\|z, e_{i_2},\dots,e_{i_n}\|_S = |z_{i_1}|\,.$$

Since $z=t_z x$ for some $t_z\in\mathbb{R}$, we have $|z_{i_1}|\longrightarrow\infty$ as $t_z\longrightarrow\infty$. Consequently, L_x is not a bounded set.

The lemma below is employed to demonstrate when a vector equals zero, a crucial step in verifying our future theorems.

Lemma 4.5.1 *[6] Let $x_0 \in X$. If $\|x_0, s_{i_2}, ..., s_{i_n}\| = 0$ for every $\{i_2, ..., i_n\} \subset \{1, ..., n\}$, then $x_0 = 0$*

Proof [6] If $\|x_0, s_{i_2}, ..., s_{i_n}\| = 0$ for every $\{i_2, ..., i_n\} \subset \{1, ..., n\}$, then x_0 is in the span of $\{s_{i_2}, ..., s_{i_n}\}$ for every $\{i_2, ..., i_n\} \subset \{1, ..., n\}$. This can only happen if $x_0 = 0$. $\square$

The fixed point theorem applies to mappings that exhibit contraction properties on a closed and bounded subset in relation to $\mathcal{S}$.

Theorem 4.5.1 *[6] Let $(X, \|\cdot, ..., \cdot\|)$ be a complete n-normed space and $X_0 \subset X$ be a nonempty, closed, and bounded with respect to $\mathcal{S}$. If $T : X_0 \longrightarrow X_0$ is a contraction operator with respect to $\mathcal{S}$, then T has a unique fixed point in $\mathcal{S}$.*

Proof [6] Let $x_0 \in X_0$. We first construct an iterative sequence $\{x_m\}_{m=0}^{\infty}$ where $x_m = T^m x_0$ for $m = 0, 1, 2, ...$

Second, we show that $\{x_m\}_{m=0}^{\infty}$ is a Cauchy sequence with respect to $\mathcal{S}$. Since T is contraction, there is $c \in (0, 1)$ such that for any two consecutive terms in $\{x_m\}_{m=0}^{\infty}$, we have

$$\begin{aligned}
\|x_m - x_{m+1}, s_{i_2}, ..., s_{i_n}\| &= \|T(x_{m-1}) - T(x_m), s_{i_2}, ..., s_{i_n}\| \\
&\leq c\,\|x_{m-1} - x_m, s_{i_2}, ..., s_{i_n}\| \\
&= c\,\|T(x_{m--2}) - T(x_{m-1}), s_{i_2}, ..., s_{i_n}\| \\
&\leq c^2\,\|x_{m-2} - x_{m-1}, s_{i_2}, ..., s_{i_n}\| \\
&\vdots \\
&\leq c^m\,\|x_0 - x_1, s_{i_2}, ..., s_{i_n}\|,
\end{aligned}$$

for every $\{i_2, ..., i_n\} \subset \{1, ..., n\}$. Hence by using the triangle inequality and the formula for the sum of a geometric progression, we obtain for $m > l$,

$$\begin{aligned}
&\|x_m - x_1, s_{i_2}, ..., s_{i_n}\| \\
&= \|x_m - x_{m-1} + x_{m-1} - ... + x_{l+1} - x_l, s_{i_2}, ..., s_{i_n}\| \\
&\leq \|x_m - x_{m-1}, s_{i_2}, ..., s_{i_n}\| + \|x_{m-1} - x_{m-2}, s_{i_2}, ..., s_{i_n}\| + ... + \|x_{l+1} - x_l, s_{i_2}, ..., s_{i_n}\| \\
&\leq c^{m-1}\,\|x_0 - x_1, s_{i_2}, ..., s_{i_n}\| + c^{m-2}\,\|x_0 - x_1, s_{i_2}, ..., s_{i_n}\| + ... + c^l\,\|x_0 - x_1, s_{i_2}, ..., s_{i_n}\| \\
&= \left(c^{m-1} + c^{m-2} + ... + c^l\right)\|x_0 - x_1, s_{i_2}, ..., s_{i_n}\| \\
&< \frac{c^l}{1-c}\,\|x_0 - x_1, s_{i_2}, ..., s_{i_n}\|
\end{aligned}$$

for every $\{i_2, ..., i_n\} \subset \{1, ..., n\}$. Because $c \in (0, 1)$ and $\|x_0 - x_1, s_{i_2}, ..., s_{i_2}\|$ is bounded, we can make the right hand side of above inequality as small as we like, by taking l sufficiently large. Since this holds for every $\{i_2, ..., i_n\} \subset \{1, ..., n\}$, the sequence $\{x_m\}_{m=0}^{\infty}$ is Cauchy with respect to $\mathcal{S}$. As X is a complete n-normed space and X_0 is closed, there exists $x_0 \in X_0$ such that $x_m \overset{\mathcal{S}}{\to} x$.

Third, we prove that x is a fixed point of T, that is $T(x) = x$. By using the triangle inequality and contraction operator (4.6), we have

$$\begin{aligned} \|T(x) - x, s_{i_2}, ..., s_{i_n}\| &\leq \|T(x) - x_m, s_{i_2}, ..., s_{i_n}\| + \|x_m - x, s_{i_2}, ..., s_{i_n}\| \\ &\leq c\,\|x - x_{m-1}, s_{i_2}, ..., s_{i_n}\| + \|x_m - x, s_{i_2}, ..., s_{i_n}\| \end{aligned}$$

for every $\{i_2, ..., i_n\} \subset \{1, ..., n\}$. By taking a sufficiently large m, the sum in the second line can be made smaller than any preassigned $\varepsilon > 0$, because $x_m \overset{\mathcal{S}}{\to} x$. We conclude that $\|T(x) - x, s_{i_2}, ..., s_{i_n}\| = 0$ for every $\{i_2, ..., i_n\} \subset \{1, ..., n\}$. By Lemma 4.5.1, we have $T(x) = x$.

Fourth, we prove that T has no other fixed points. Let $y \in X$ be another fixed point of T, so that $T(y) = y$. We obtain

$$\begin{aligned} \|x - y, s_{i_2}, ..., s_{i_n}\| &= \|T(x) - T(y), s_{i_2}, ..., s_{i_n}\| \\ &\leq c\,\|x - y, s_{i_2}, ..., s_{i_n}\| \end{aligned}$$

for every $\{i_2, ..., i_n\} \subset \{1, ..., n\}$. Since $c \in (0, 1)$, we have $\|x - y, s_{i_2}, ..., s_{i_n}\| = 0$ for every $\{i_2, ..., i_n\} \subset \{1, ..., n\}$. By Lemma 4.5.1, we conclude that $x = y$. □

Conclusion

In this memory, we detailed the paper [7] by A. L. Soenjaya going through the following steps

The first chapter we presented reminders, definitions and properties on normed and banach spaces, linear and n-linear operator and continuity and boundedness. **The second chapter** is devoted to the Continuous, bounded n-linear operators between n-normed spaces and normed space, we presented the definition of n-normed and n-banach spaces, continuous bounded n-linear operator and the space of all bounded n-linear operators from X^n into Y. **The last chapter** we devoted this chapter to Other Notions of n-Continuous, n-Bounded Operator and Contraction operators and their fixed point theorems in n-normed space was devoted to the n- Continuous, n-bounded linear operators between normed space and n-normed spaces, we defined the n- continuous n-bounded linear operator and the space of all n-bounded linear operators from X into X. **The last chapter** we devoted this chapter to Other Notions of n-Continuous, n-Bounded Operator and Contraction operators and their fixed point theorems in n-normed space.

Bibliography

[1] V. Berinde, Iterative Approximation of Fixed Points, Springer, (2006).

[2] H. Y. Chu, S. H. Ku and D. S. Kang, Characterizations on 2-isometries, J. Math. Anal. Appl. 340 (2008), 621–628.

[3] H.Gunawan, The space of p-summable sequences, Bull. Austral. Math. Soc. 64(2001), 137-147.

[4] H. Gunawan and Mashadi, On n-normed spaces, Int. J. Math. Math. Sci. 27 (2001), 631–639. Soc. 18(2012), 45-56.

[5] H. Gunawan and Mashadi, On finite-dimensional 2-normed spaces, to appear in Soochow J. Math.

[6] M. Kir and H. Kiziltunc, On fixed point theorems for contraction mappings in n-normed spaces, Applied Math. Information Sci. Letters 2 (2014), 59–64.

[7] A. L. Soenjaya, On n-bounded and n-continuous operator in n-normed space. J. Indones. Math. 1963

[8] A. White, 2-Banach spaces, Math. Nachr. 42 (1969), 43–60.

الملخص:

تتناول هذه المذكرة مفاهيم التطبيقات متعددة الخطية المحدودة (n-محدودة) والمستمرة (n-مستمرة) في الفضاءات n-ناظمية كما قمنا بدراسة خصائص الفضاءات المقابلة للحصول على نتائج مشابهة لحالة الفضاء النظيمي كما قمنا كذلك بدراسة بعض المفاهيم الاخرى للاستمرارية والمحدودية.

الكلمات المفتاحية:

تطبيقات ليبشيتزية، تطبيقات متعددة الخطية، التطبيقات متعددة الخطية، n-محدودة، الفضاءات n-ناظمية، n-مستمرة.

Résumé

Ce mémoire traite des concepts d'opérateurs n-linéaires bornés (n-bornés) et continus (n- continus) dans les espaces n-normés introduit par De plus, nous étudions les propriétés des espaces correspondants d'opérateurs pour obtenir des résultats analogues au cas des espaces normés, aussi on a étudie autre notions de la continuité de ces opérateurs.

Mots clés: Opérateurs Lipschitziens, opérateurs multi-linéaires, n- bornés, espaces n-normés, n- continus.

Abstract

This memory deals with the concepts of bounded(n-bounded) and continuous(n-continuous) n-linear operators in n-normed spaces introduced by. Also, we study the properties of the corresponding spaces of operators to obtain results analogous to the case of normed space, also we have studied other notions of continuity of these operators.

Keywords: Lipschitz operators, multi-linear operator, n-bounded, n-normed spaces, n-continuous.