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Dedication

I dedicate this modest work :

-To my parents. -To my brothers and sisters.

-To my aunts.

-To all my family.

-To all friends and all my department family.

-To all my adorable ones that i have known during all my life . . .

Nouibat wassila

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Notation

We introduce the necessary notations and definition which are used in the sequel.

$\mathcal{H}$	Hilbert space.
$\Omega \subset \mathbb{R}^N$	Open set in $\mathbb{R}^N$
X'	Topological dual of X.
A^c	Complement of the set A.
$(.,.)$	Scalar product.
$\mathbb{R}^N$	Euclidean space of dimension N, where N is a nonzero natural number.
x	Vector in $\mathbb{R}^N$, $x = (x_1, x_2, ..., x_N)$, $x_i \in \mathbb{R}$, $1 \leq i \leq N$
dx	Lebesgue measure in N-dimensional space.
$\partial\Omega = \Gamma$	Boundary of Ω.
$\|E\|$	Or mes (E) measure of the set E.
$B(x_0, r)$	Open ball of radius r centered at x_0.
B_E	$= \Big\{x \in E; \|x\| \leq 1\}$.
$\|.\|$	Hilbert norm.
χ_E	Characteristic function of set E.
$\langle .,. \rangle$	Duality bracket between X and its dual space.
$\int_\Omega f(x)\, dx$	Integral of f in Ω with respect to the Lebesgue measure.
supp u	Support of the function u.
$\frac{\partial u}{\partial n}$	Outward normal derivative.
$\nabla u = Du$	$= (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N})$ Gradient of the function u.
div u	Divergence of the vector u, $divu = \frac{\partial u}{dx_1} + \frac{\partial u}{dx_2} + ... + \frac{\partial u}{dx_N}$
Δu	$= \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$ Laplacian of u.
$f_n \to f$	Denotes that the sequence $\{f_n\}$ converge to f.
$f_n \rightharpoonup f$	Denotes that the sequence $\{f_n\}$ converge weakly to f.
$L^p(\Omega)$	$= \{u : \Omega \to \mathbb{R}$ measurable and $\Big(\int_\Omega \|u(x)\|^p\Big)^{1/p} dx < +\infty$ such that $1 \leq p < \infty\}$.
$\|u\|_p$	$= \Big[\int_\Omega \|u(x)\|^p dx\Big]^{1/P} = \|u\|_{L^p}$.

$L^\infty(\Omega)$ $= \{u : \Omega \to \mathbb{R} \text{ measurable}, \exists M > 0 \mid |u(x)| \leq M \text{a.e. } on\ \Omega.$

$\|u\|_{L^\infty}$ $= \inf\{C : |u(x)| \leq C \text{ a.e on } \Omega\}.$

q Hölder conjugate of p : $q = \frac{p}{p-1}$ if $p > 1$ and $q = \infty$ if $p = 1$

$L^q(\Omega) \subset L^p(\Omega), \quad \forall\, 1 \leq p \leq q < \infty$

$\|u\|_{C_c(\Omega)}$ $= \sup_{x\in\Omega} |u(x)| = \max_{x\in\Omega} |u(x)|$

$C_c(\Omega)$ Space of continuous functions with compact.

$M(\Omega)$ The topological dual of the continuous functions space with compact support in Ω.

$\mathcal{D}(\Omega)$ Space of infinitely differentiable functions on Ω with compact support in Ω.

$W^{1,p}(\Omega)$ $= \left\{u \in L^p(\Omega) \mid \nabla u \in (L^p(\Omega))^N\right\}.$

$W_0^{1,p}(\Omega)$ $= \left\{u \in W^{1,p}(\Omega), \quad \text{with} \quad u = 0 \quad \text{on} \quad \partial\Omega\right\}.$

$(a+b)^p \leq 2^{p-1}(a^p + b^p) \quad \text{for} \quad 1 \leq p < \infty \quad \text{and} \quad a, b \geq 0.$

a.e. Almost everywhere.

General Introduction

Partial differential equations, which is abbreviated as "PDE" in the following, constitute an important branch of applied mathematics and this field is becoming increasingly important in modern times. EDPs have great utility in mod- delusion of many phenomena of different natures such as physics, chemistry, biology and other sciences. In other words, EDPs intervene in many other fields: in chemistry to model reactions, in economics to study market behavior, in finance to study derivative products and in image processing to restore damage. These problems boil down to mathematical models usually written in the form

$$L(u) = f, \qquad (1)$$

where L is a definite operator from a function space E to a function space F, u is the unknown function and f a given function. PDEs probably first appeared during the 17th century. In-the field of PDEs has been enriched as the sciences have developed and especially physics. However, the systematic study of PDEs is much more recent and it is only during of the 20th century that mathematicians began to develop and indeed this theory has recently experienced a great theoretical and practical advancement. The mathematical analysis of these partial differential equations requires an appropriate choice of functional spaces and a clear definition of the notion of solution (the existence and sometimes uniqueness). One of the things to keep in mind about PDEs is to ask the question: can we get solutions explicitly?. So what mathematics can do on the other hand, is to say if one or more solutions exist, and to describe sometimes very precisely met some properties of these solutions. So usually an exact solution of problem (1) is not easy to find and sometimes we cannot even find the explicit solution. This leads to the introduction of the notion of the weak solution

which appeared in 1934 in the work of Jean Leray. Therefore, in most cases it is very difficult, if not impossible, to exhibit the solutions of a PDE. In some cases we manage to try to show that the problem admits a unique solution (it is said to be well-posed). But sometimes we can calculate numerical approximations of the solutions.

Moreover, the work presented in this dissertation concerns a particular case of the equations to partial derivatives of the elliptic type involving the divergent operator

$$Au = -\mathrm{div}(a(x, \nabla u))$$

where $a = |\nabla u|^{p-2}\nabla u$ a Caratheodory function, verifies certain hypotheses.

We are interested in this memoir to study the following problem:

$$(P) \qquad \begin{cases} Au &= f \text{ on } \Omega; \\ u &= 0 \text{ on } \partial\Omega, \end{cases}$$

Ω is a bounded open set of $\mathbb{R}^N$ with $N > 2$, we assume that $f \in L^1(\Omega)$ or $f = \mu \in M(\Omega)$.

We first illustrate the main difficulties that may arise when studying of the problem (P)

1) The non-linearity of the differential operator $A(u) = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$

2) The irregularity in the case where $f \in L^1(\Omega)$ or $f = \mu \in M(\Omega)$.; i.e. the second member of (P) does not belong to space $W^{-1,p'}(\Omega)$.

Technique

We are going to study the existence and regularity of the weak and entropy solutions of the problem (P). To do this, we will approximate the problem (P) by a sequence of approximate problems (P_n) given in L^∞ whose existence of the solution approximate is guaranteed by Theorem 1.10. Then we will study some estimates uniform on the sequence of solutions of these problems (P_n) and their partial derivatives. Once this is done, the linearity of the operator with respect to the gradient as well as the boundedness and the continuity of a will make it possible to pass to the limit.; Thus finding the solution.

Memoir plan

This memoir is divided into three chapters as follows: The first chapter is dedicated to giving some basic definitions and results with functional analysis tools essential to the achievement of the objectives for the study of the (P) problem. For example we recall functional spaces (Lebesgue, Sobolev) and we give brief reminders of weak and weak star convergence and we recall some definitions on the operators (Bounded, himecontinuous, Monotone and coercive). We also present the method of monotonic (pseudo-monotonic) operators in the general framework to prove the existence of a solution for the equation

$$Au = f.$$

According to hypotheses on the functions $a = |\nabla u|^{p-2}\nabla u$ and f $(f \in W^{-1,p'}(\Omega))$ and for to prove the existence and regularity of the solution we use the Theorem 1.9.

In the second chapter, we study the weak solution for a class of nonlinear elliptic problem with measure data $Au = \mu$ and $u = 0$ with μ Radon bounded measures. We limit ourselves to the case where $1 < p < N$ because in the case $p > N$, we have the inclusion $W_0^{1,p}(\Omega) \subset C_c(\Omega)$ which gives $(C_c(\Omega))' \subset (W_0^{1,p}(\Omega))'$. As $(C_c(\Omega))' = M(\Omega)$, the measures are then elements of $W^{-1,p'}(\Omega)$ and we are then in the usual variational framework.

In the case $p > 2 - \frac{1}{N}$ with μ a measure of Radon. The weak solution u of this problem be in the space $W_0^{1,q}(\Omega)$, for all $q \in [1, \frac{N}{N-1}(p-1)[$. As a prototype example, we consider the model problem $Au = \delta$ where δ is the Dirac measure.

Finaly, we define the notion of entropy solutions of problem (P) where $f \in L^1(\Omega)$. Next, we show that these solutions are in the space $u \in W_0^{1,q}(\Omega)$, for all $q \in [1, \frac{N}{N-1}(p-1)[$ if $2 - \frac{1}{N} < p$. The idea is to consider the truncation $T_k(u)$ of the solution u and to work with the derivative $\nabla T_k(u)$ instead of ∇u.

Chapter 1

Preliminaries and basic tools

The purpose of this chapter is to recall a number of analytical tools from the theory of Sobolev spaces that will be used throughout this memoir . We will also take the opportunity to introduce the main notations.

1.1 Reminders and some definitions

Throughout this chapter Ω will denote a bounded domain in $\mathbb{R}^N$,$N \geq 1$, i.e.a connected and bounded open subset of $\mathbb{R}^N$. Its boundary will be denoted by Γ or $\partial\Omega$ and its closure by $\overline{\Omega}$.

Let $u = u(x_1, ..., x_N)$ be a function defined in $\Omega \subset \mathbb{R}^N$. Assuming that exists, the gradient of u at point x is defined as the vector:

$$\nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), ..., \frac{\partial u}{\partial x_N}(x)\right)$$

The Euclidean norm of ∇u is denoted by $|\nabla u|$:

$$|\nabla u| = \left[\left|\frac{\partial u}{\partial x_1}\right|^2 + ... + \left|\frac{\partial u}{\partial x_N}\right|^2\right]^{1/2}$$

Definition 1.1. ***(Space of test functions)****: The space $\mathcal{D}(\Omega)$ or $C_0^\infty(\Omega)$ is the set of infinitely differentiable functions $\varphi : \Omega \longrightarrow \mathbb{R}$ (of class $C^\infty(\Omega)$) with compact support in Ω. $\mathcal{D}(\Omega)$ is a vector space, and each element of this space is called a test function.*

- *supp* $\varphi = \overline{\{x \in \Omega,\ \varphi(x) \neq 0\}}$,
- $\mathcal{D}(\Omega) = C^\infty(\Omega) \cap C_c(\Omega)$.

Definition 1.2. ***Separable Spaces*** *A Banach space E is said to be separable if there exists a countable set that is dense in E.*

Definition 1.3. ***Dual Space*** *The topological dual of a vector space E over the field $\mathbb{K}, (where \mathbb{K} = \mathbb{R}$ or $\mathbb{C})$ is defined as the set of continuous linear forms on E,i.e. continuous linear maps form E to $\mathbb{K}$. We denote by $E' = \mathbb{L}(E, \mathbb{K})$.*

E' equipped with the norm $\|.\|$ defined by:

$$\|u\|_{E'} = \sup_{x \in E, \|x\|=1} |u(x)| = \sup_{x \in E, \|x\| \leq 1} |u(x)| = \sup_{x \in E, \|x\| \neq 0} \frac{|u(x)|}{\|x\|}$$

Proposition 1.1. *1. $E_1 \subset E_2$ so $E_2' \subset E_1'$,*

2. $\langle u, x \rangle_{E'E} \leq \|u\|_{E'} \|x\|_E$.

1.2 Functional spaces

1.2.1 L^p Spaces

Let Ω be an open subset of $\mathbb{R}^N$ and let p be a real number greater than or equal to 1. The Lebesgue Space $L^p(\Omega)$ is defined as:

$$L^p(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \quad \text{is measurable and} \quad \int_\Omega \mid u(x) \mid^p < \infty \Big\}.$$

We equip it with the following norm

$$\|u\|_{L_p(\Omega)} = \|u\|_p = \Big(\int_\Omega |u(x)|^p \Big)^{\frac{1}{p}}, \quad p \geq 1.$$

By $L^\infty(\Omega)$ we denote the set

$$L^\infty(\Omega) = \{ u : \Omega \to \mathbb{R} \text{ measurable}, \exists M > 0 \mid |u(x)| \leq M \quad \text{a.e.} \}$$

It is equipped with the following norm:

$$\|u\|_{\infty,\Omega} = \inf \quad \{ M > 0 :\mid u \mid \leq M \quad \text{a.e.in} \quad \Omega \}$$

The space $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{2,\Omega} = \int_\Omega uv dx.$$

Remark 1.1. *Let $1 \leq p \leq \infty$, be denoted by p' the conjugate exponent of p*

$$i.e. \quad \frac{1}{p} + \frac{1}{p'} = 1$$

Let $1 \leq p < \infty$. then the dual space of $L^p(\Omega)$ is $L^{p'}(\Omega)$.

1.2.2 Main inequalities

Let $1 \leq p \leq q \leq \infty$ be two real numbers; and let $p^{'}$ be the conjugate exponent of p.

Lemma 1.1 (Hölder's inequality). *If $f \in L^p(\Omega)$, and $g \in L^{p^{'}}(\Omega)$, then $f \cdot g \in L^1(\Omega)$ and*

$$\| f \cdot g \|_{L^1(\Omega)} \leq \| f \|_{L^p(\Omega)} \| g \|_{L^{p'}(\Omega)}$$

If more $| \Omega | < \infty$ and $f \in L^q(\Omega)$, so $f \in L^p(\Omega)$ and

$$\|f\|_{L^p(\Omega)} \leq | \Omega |^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(\Omega)}$$

In particular,

$$L^q(\Omega) \subset L^p(\Omega), \quad \forall\, 1 \leq p \leq q < \infty$$

Lemma 1.2. *Let $p_i \in [1, +\infty]$ be exponents with $1 \leq i \leq k$ such that:*
$1/p = 1/p_1 + ... + 1/p_k \leq 1$. Then, for all functions $f_i \in L^{p_i}(\Omega)$, We have $f = f_1 ... f_k \in L^p(\Omega)$ and generalized Hölder inequality

$$\|f\|_p \leq \| f \|_{p_1} \cdots \| f \|_{p_k} .$$

Lemma 1.3 (Young's inequality). *For all non-negative real numbers a, b and every $1 < p < \infty$, then we have*

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad p' = \frac{p}{p-1}. \tag{1.1}$$

Corollary 1.1. *Generalized Young's inequality: for $1 < p < \infty$, $p' = \frac{p}{p-1}$ and any positive ϵ we have:*

$$ab \leq \epsilon^p \frac{a^p}{p} + \frac{1}{\epsilon^{p'}} \frac{b^{p'}}{p'}.$$

Theorem 1.1 (Fisher-Riesz)**.** *L^p is a Banach space for all $1 \leq p \leq \infty$.*

Lemma 1.4 (**Divergence and Green's formulas**)**.** *Let Ω be a domain in $\mathbb{R}^N$, and $n(x)$ its exterior normal. Let u and v be two regular functions, and w be a vector field defined on Ω. Then,*

$$\int_\Omega \mathrm{div} w \quad dx = \int_{\partial\Omega} w \cdot n d\sigma \quad \textit{(divergence formula)}$$

$$\int_\Omega (\Delta u) v \quad dx = -\int_\Omega \nabla u \cdot \nabla v \quad dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \quad d\sigma \quad \textit{(green formula)}$$

Theorem 1.2 (Rellich-Kondrachov[3])**.** *Let Ω be a bounded open set of $\mathbb{R}^N$ and $p \geq 1$. If $p < N$ then for any q such that $1 \leq q \leq p^*$ (with $1/p^* = 1/p - 1/N$), the injection of $W_0^{1,p}(\Omega)$ in $L^q(\Omega)$ is continuous and for all q such that $1 \leq q < p^*$ the injection is compact, that is, the bounded of $W_0^{1,p}(\Omega)$ are relatively compact in $L^q(\Omega)$.*

1.2.3 Convergence theorems

In this section, we recall some definitions and results on the convergence sequences of measurable functions

Definition 1.4. *Let (u_n) be a sequence of measurable functions on Ω and u a measurable function on Ω.*

- *The sequence (u_n) converges almost everywhere on Ω to u if and only if*

$$meas\{x \in \Omega : u_n(x) \text{ does not converge to } \quad u(x)\} = 0,$$

- *The sequence (u_n) is said to be Cauchy in measure if for every $\varepsilon > 0$ and every $\eta > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$, so*

$$meas\{x \in \Omega : |u_n(x) - u(x)| > \eta\} \leq \varepsilon$$

Lemma 1.5. *[7] Let (u_n) be a sequence of measurable functions of in Ω and $\mathbb{R}^N$. If (u_n) of Cauchy in measure then there exists a sub-sequence of (u_n) converge almost everywhere*

Lemma 1.6. *[8] Let f be a strictly positive measurable function. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every measurable $A \subset \Omega$, we have*

$$\int_A f dx \leq \delta \quad \Rightarrow \quad meas(A) \leq \varepsilon. \tag{1.2}$$

Lemma 1.7 ([2])**.** *Let (u_n) the sequence in $L^p(\Omega)$ with $1 < p < \infty$.*

Suppose that

- *(u_n) is bounded in $L^p(\Omega)$;*
- *$u_n \to u$ a.e. on Ω.*

Then, $u_n \to u$ in $L^q(\Omega)$, for all $1 \leq q < p$ and weakly in $L^p(\Omega)$, i.e.,

$$\lim_{n\to+} \int_\Omega u_n v dx = \int_\Omega uv dx, \text{ for all } \quad v \in L^{p'}(\Omega)$$

Theorem 1.3 (Fatou's Lemma [3])**.** *Let (u_n) be a sequence of non-negative measurable function on Ω. Then*

$$\int_\Omega \liminf_{n\to\infty} u_n(x) dx \leq \liminf_{n\to\infty} \int_\Omega u_n(x) dx.$$

Theorem 1.4 (Giuseppe Vitali convergence theorem [4])**.** *Let $p \in [1, \infty[$ and (f_n) be a sequence of functions in $L^p(\Omega)$ such that*

- *$f_n \to f \quad$ a.e. in $\quad \Omega$*
- *(f_n) equal-integration in Ω, i.e. $\forall \varepsilon > 0, \exists \delta > 0, \forall E \subset \Omega$ with $|E| \leq \delta$, such that*

$$\int_E |f_n(x)|^p dx \leq \varepsilon, \quad \forall n \in \mathbb{N}$$

Or

$$\limsup_{|E|\to 0} \int_E |f_n(x)|^p dx = 0, \quad \forall n \in \mathbb{N}$$

Then

$$f \in L^p(\Omega) \quad \text{and} \quad f_n \to f \quad \text{in} \quad L^p(\Omega) \quad \text{strongly.}$$

Theorem 1.5 (Lebesgue's Dominated Convergence Theorem[3]). *Let $p \in [1, \infty)$, and (u_n) be a sequence of functions in $L^p(\Omega)$ such that*

- *$u_n \to u$ a.e in Ω*
- *there is a function $v \in L^p(\Omega)$ satisfying*

$$|u_n(x)| \leq v(x) \quad a.e \text{ in } \quad \Omega \quad \forall n \in \mathbb{N}$$

Then $u_n \to u$ in $L^p(\Omega)$, that is

$$u \in L^p(\Omega), \quad \text{and} \quad \|u_n - u\|_{L^p(\Omega)} \to 0 \quad \text{as} \quad n \to \infty.$$

1.3 Sobolev Spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 1.5. *The Sobolev space $W^{1,p}(\Omega)$, is defined by*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega), \mid \nabla u \in (L^p(\Omega))^N \right\}$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\| u \|_{W^{1,p}(\Omega)} = \| u \|_p + \| \nabla u \|_p \tag{1.3}$$

and for $p = \infty$

$$\|u\|_{W^{1,\infty}} = \max(\|u\|_{L^\infty}, \|\nabla u\|_{L^\infty}).$$

Definition 1.6. *the space $D(\Omega)$ denotes the set of functions in $C^\infty(\Omega)$ with compact support in Ω, we define $1 \leq p < \infty$*

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega), \quad \text{with} \quad u = 0 \quad \text{on} \quad \partial\Omega \right\}.$$

Remark 1.2. *Since, for $1 \leq p < \infty$, the space $D(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ by definition, we can identify the dual of $W_0^{1,p}(\Omega)$ with a subspace of the space of distributions $D'(\Omega)$. we denote this as: $W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))'$*

In the particular case where $p = 2$, $W^{1,2}(\Omega)$, $W_0^{1,2}(\Omega)$, $W^{-1,2}(\Omega)$, *are respectively denoted* $H^1(\Omega)$, $H_0^1(\Omega)$, $H^{-1}(\Omega)$

The space H^1 *is a Hilbert* [1] *space equipped with the scalar product.*

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + (u', v')_{L^2(\Omega)}$$

The associated norm

$$\|u\|_{H^1(\Omega)} = (\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)})^{1/2}$$

is a norm equivalent to that of $W^{1,2}(\Omega)$

The following proposition represents the fundamental properties of the spaces $W^{1,p}, W_0^{1,p}$. For the proof we refer the reader to reference [3].

Proposition 1.2.

- *The space* $W^{1,p}$ *is a Banach space for* $1 \leq p \leq \infty$.
- *The space* $W^{1,p}$ *is a separable space for* $1 \leq p < \infty$.
- *The space* $W^{1,p}$ *is a reflexive space for* $1 < p < \infty$.
- *The space* $W_0^{1,p}$ *is a separable space . Also reflexive for* $1 < p < \infty$.

1.4 Sobolev injections

Sobolev injections are widely used. they provide inequalities between the norms of sobolev spaces and L^p norms. This compactness result is a powerful tool in the study of PDEs, allowing us to transition from a sobolev space to a space of Lebesgue.

Theorem 1.6 (**compactness theorem**). *Let* Ω *a bounded open subset of* $\mathbb{R}^N$ *with* $N \geq 1$ *and* $1 \leq p < \infty$, *any bounded subset of* $W_0^{1,p}(\Omega)$ *is relatively compact in* $L^p(\Omega)$, *this implies that for any bounded sequence in* $W_0^{1,p}(\Omega)$, *we can extract a sub-sequence that converges in* $L^p(\Omega)$.

[1] David Hilbert (1862-1943) is a German mathematician. He is often considered one of the greatest mathematicians of the 20th century.

Theorem 1.7 (**continuous Injection**). *Let Ω be a bounded open subset of $\mathbb{R}^N$ with $N \geq 1$ and $1 \leq p < \infty$.*

- *If $1 \leq p < N$ then $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ with $p^* = \frac{Np}{N-p} > p$ and the injection is continues, i.e $\exists C \geq 0$ (depending only on p, N) such that*

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W_0^{1,p}(\Omega)}$$

- *If $p > N$ then $W_0^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{N}{p}}$. where for $\alpha > 0$, $C^{0,\alpha}$ is the set of Holder continuous functions with exponent α, defined as*

$$C^{0,\alpha} = \{u \in C(\Omega, \mathbb{R})/\exists K \geq 0, \quad |u(x) - u(y)| \leq K\|x - y\|^{\alpha}, \forall (x, y) \in \Omega^2\}$$

- *If $p = N$ then $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q \geq 1$*

Lemma 1.8 (**Poincare's inequality**). *Suppose that $1 \leq p < \infty$, and Ω is a bounded open set. Then there exists a constant C (depending on Ω and p) such that*

$$\| u \|_{L^p(\Omega)} \leq C \| \nabla u \|_{L^p(\Omega)} . \tag{1.4}$$

Moreover , by the density of $\mathcal{D}(\Omega)$ in $W_0^{1,p}(\Omega)$, the inequality (1.4) stay true for any function $u \in W_0^{1,p}(\Omega)$.

1.5 Weak convergence and (weak*) convergence

1.5.1 Weak convergence

Definition 1.7. *Let E be a Banach space, E' its dual and $\langle ., . \rangle$ the duality bracket EE'.*

We say that the sequence (x_n) in E weakly converges to $x \in E$ if and only if:

$$\langle f, x_n \rangle \longrightarrow^{n \longrightarrow +\infty} \langle f, x \rangle, \quad \forall f \in E'.$$

we denote this as

$$x_n \rightharpoonup x \quad in \quad E.$$

Proposition 1.3.

- *if* $x_n \longrightarrow x$ *strongly then* $x_n \rightharpoonup x$ *in* E.
- *if* $x_n \rightharpoonup x$ *in* E *then* $\|x_n\|_E$ *is bounded.*
- $$\begin{cases} If \quad x_n \rightharpoonup x, \ weakly\ in\ E \\ And \quad f_n \longrightarrow f\ strongly\ in\ E' \end{cases}$$

then $\langle f_n, x_n \rangle \longrightarrow \langle f, x \rangle$ *as* $n \longrightarrow +\infty$.

Theorem 1.8. *Let* E *be a reflexive Banach space. Then, for any bounded sequence* $(x_n)_n \subset E$*, there exists a sub-sequence on* $(x_{nk})_{nk}$ *that converges weakly in* E.

1.5.2 (weak*) convergence

Definition 1.8. *We say that the sequence* (f_n) *in* E' *weakly* converges to* $f \in E'$ *if and only if:*

$$\langle f_n, g \rangle \longrightarrow^{n \longrightarrow +\infty} \langle f, g \rangle, \quad \forall g \in E$$

we denote this as

$$f_n \rightharpoonup *f \quad in \quad E.$$

Proposition 1.4.

$$f_n \longrightarrow f \quad strongly \quad \Longrightarrow f_n \rightharpoonup f \quad inE'$$

For more details, look at the book[3].

1.6 Some notions and results on operators

1.7 Monotone Operators

V denotes a real Banach space, V' denotes its topological dual of V.

Definition 1.9. *An operator* $A : V \to V'$ *is said to be*

- *Monotone if* $\langle Au - Av, u - v\rangle \geq 0, \forall u, v \in V$
- *strictly monotone if* $\langle Au - Av, u - v\rangle > 0, \forall u, v \in V, u \neq v$

Example 1.1. *The operator* $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ *defined by* $Au = -\Delta u$ *is monotone, where* $H_0^1(\Omega)$ *is equipped with the norm of the gradient. Indeed, for* $u, v \in H_0^1(\Omega)$, *we have*

$$\begin{aligned} \langle Au - Av, u - v\rangle &= \int_\Omega \nabla(u - v).\nabla(u - v) \\ &= \|u - v\|^2_{H_0^1(\Omega)} \\ &\geq 0. \end{aligned}$$

1.8 Bounded operator

Definition 1.10. *Let* V *and* W *be two Banach spaces and let* $A : V \to W$ *be an operator. We say that* A *is bounded if it maps every bounded set in* V *to a bounded set in* W*; i.e*

$$\forall \rho > 0, \quad \exists C_\rho > 0 : \quad A(B_V(0, \rho)) \subset B_W(0, C_\rho)$$

1.9 Hemicontnuous Operators

Definition 1.11. *An operator* $A : V \to W$ *is said to be hemicontinuous at the point* u_∞ *in* V *if, for any sequence* $\{u_n\}$ *converging to* u_∞*, the sequence* $\{Au_n\}$ *converges weakly to* Au_∞ *in* W*, in other words*

$$\forall v \in V, \quad \forall \{\lambda_n\} \subset \mathbb{R}, \quad \lambda_n \to 0, \quad A(u_\infty + \lambda_n v) \rightharpoonup Au_\infty.$$

If A *is hemicontnuous at every point in* V*, we say that it is hemicontinuous on* V*. In reflexive spaces and when* $W = V'$*, and passing from sequential to continuous, we can define hemicontinuitly on* V *by requiring that*

$$\forall u, v, w \in V \quad \textit{the map} \quad \mathbb{R} \ni \lambda \mapsto \langle A(u + \lambda v), w\rangle \in \mathbb{R}$$

is continuous.

Example 1.2. *The operator* $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ *defined by* $Au = -\Delta u = -\mathrm{div}(\nabla u)$ *is hemicontinuous. Indeed, let* $u, v, w \in H_0^1(\Omega)$ *and* $\lambda \in \mathbb{R}$*, we have*

$$\begin{aligned} \langle A(u+\lambda v), w\rangle &= \int_\Omega \nabla(u+\lambda v).\nabla w \\ &= \int_\Omega \nabla u.\nabla w + \lambda \int_\Omega \nabla v \nabla w \\ &= a + b\lambda. \end{aligned}$$

This shows that $\lambda \longrightarrow \langle A(u+\lambda v), w\rangle$ *is continuous.*

1.10 The p-Laplacian operator

The p-Laplacian operator is a quasi-linear second-order elliptic partial differential operator defined by

$$\Delta_p(u) = \mathrm{div}(\mid \nabla u \mid^{p-2} \nabla u), \quad 1 < p < \infty$$

this operator takes the form of a divergence when $p \neq 2$ and for $p = 2$, the p-Laplacian coincides with the usual laplacian Δ

1.11 Properties of the p-Laplacian operator

Let Ω be a bounded open subset of $\mathbb{R}^N$, and let the operator

$$A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$$

defined by

$$Au = -\mathrm{div}(\mid \nabla u \mid^{p-2} \nabla u), \quad 1 < p < \infty$$

According to the green's formula, we have

$$\forall \varphi \in W_0^{1,p}(\Omega) : \langle Au, \varphi\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u.\nabla \varphi$$

Proposition 1.5. *The operator $A = -\triangle_p$ is bounded from $W_0^{1,p}(\Omega)$ in $W^{-1,p'}(\Omega)$*

Proof. From the expression of the norm in a dual space, let $\rho > 0$, for $u \in B_V(0,\rho)$, we can write

$$\| Au \|_{V'} = \sup_{\substack{\varphi \in V \\ \|\varphi\| \leq 1}} |\langle Au, \varphi \rangle| = \sup_{\substack{\varphi \in V \\ \|\varphi\| \leq 1}} \left| \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi \right|.$$

Using Hölder's inequality we obtain

$$\begin{aligned} \left| \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi \right| &\leq \int_\Omega |\nabla u|^{p-1} . |\nabla \varphi| \\ &\leq \left(\int_\Omega |\nabla u||^p \right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla \varphi|^p \right)^{\frac{1}{p}} \\ &= \| u \|_V^{p-1} \| \varphi \|_V \\ &\leq \rho^{p-1} \end{aligned}$$

Hence $\| Au \|_{V'} \leq \rho^{p-1}$. This shows that $A(B_V(0,\rho)) \subset B_{V'}(0, \rho^{p-1})$

□

Proposition 1.6. *The operator A is hemicontinuous from V to V'.*

Proof. Let $v \in V$, and $\{\lambda_n\}$ be a real sequence converging to 0.

$$\forall g \in V, \quad \langle A(u_\infty + \lambda_n v), g \rangle \to \langle A u_\infty, g \rangle$$

we have

$$\langle A u_\infty, g \rangle = \int_\Omega |\nabla u_\infty|^{p-2} \nabla u_\infty . \nabla g$$

then

$$\begin{aligned} \langle A(u_\infty + \lambda_n v), g \rangle &= \int_\Omega |\nabla (u_\infty + \lambda_n v)|^{p-2} \nabla (u_\infty + \lambda_n v) \nabla g \\ &= \int_\Omega |(\nabla u_\infty + \lambda_n \nabla v)|^{p-2} (\nabla u_\infty + \lambda_n \nabla v) \nabla g \end{aligned}$$

1. $|\nabla (u_\infty + \lambda_n \nabla v)|^{p-2} (\nabla u_\infty + \lambda_n \nabla v) \nabla g \to^{p.p} |\nabla u_\infty|^{p-2} (\nabla u_\infty) \nabla g$

we can assume $|\lambda_n| \leq 1$, then

$$\begin{aligned}
\Big||\nabla(u_\infty + \lambda_n \nabla v)|^{p-2}(\nabla u_\infty + \lambda_n \nabla v)\nabla g\Big| &= |\nabla(u_\infty + \lambda_n \nabla v|^{p-1}|\nabla g| \\
&\leq \Big[|\nabla u_\infty| + |\lambda_n||\nabla v|\Big]^{p-1}|\nabla g| \\
&\leq \Big[|\nabla u_\infty| + |\nabla v|\Big]^{p-1}|\nabla g|
\end{aligned}$$

we recall that

$$(\sum_{i=1}^{N} a_i)^\alpha \leq \max\{1, N^{\alpha-1}\}\sum_{i=1}^{N} a_i^\alpha, \quad \forall a_i \geq 0,\ \alpha > 0 \tag{1.5}$$

Using Young's inequality (1.3) and (1.5), we can write

$$\begin{aligned}
\Big||\nabla(u_\infty + \lambda_n \nabla v)|^{p-2}(\nabla u_\infty + \lambda_n \nabla v)\nabla g\Big| &\leq \frac{|\nabla g|^p}{p} + \frac{\Big[|\nabla u_\infty + |\nabla v|\Big]^p}{p'} \\
&\leq \frac{1}{p}|\nabla g|^p + \frac{max(1, 2^{p-1})}{p'}\Big(|\nabla u_\infty|^p + |\nabla v|^p\Big).
\end{aligned}$$

the fact that $g, v, u_\infty \in W_0^{1,p}(\Omega)$, implies that

$$\frac{1}{p}|\nabla g|^p + \frac{2^{p-1}}{p'}\Big(|\nabla u_\infty|^p + |\nabla v|^p\Big) \in L^1(\Omega)$$

According to Lebesgue's dominated convergence theorem

$$\lim_{n\to+\infty}\langle A(u_\infty + \lambda_n v), g\rangle = \langle A(u_\infty), g\rangle$$

Which shows the hemicontinuity of A. □

Proposition 1.7. *The operator A is coercive from V to V'.*

Proof.

$$\begin{aligned}
\lim_{|v|_V\to+\infty}\frac{\langle Av, v\rangle}{\|v\|_V} &= \lim_{|v|_V\to+\infty}\frac{\int_\Omega |\nabla v|^p}{\| v \|_V} \\
&= \lim_{|v|_V\to+\infty}\frac{\| v \|_V^p}{\| v \|_V} \\
&= \lim_{|v|_V\to+\infty}\| v \|^{p-1} = +\infty \quad car \quad 1 < p < +\infty.
\end{aligned}$$

□

Proposition 1.8. *The operator A is monotone from V to V'.*

Proof. we recall that

$$\forall x, y \in \mathbb{R}^{\mathbb{N}} : (|x|^{p-2}x - |y|^{p-2}y)(x-y) \geq \alpha|x-y|^p, \alpha > 0.$$

So, for all $\forall u, v \in V$

$$\begin{aligned} \langle Au - Av, u - v\rangle &= \int_\Omega \Big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\Big)\nabla(u-v) \\ &\geq \int_\Omega \alpha|\nabla u - \nabla v|^p \\ &\geq 0 \end{aligned}$$

□

1.12 Monotonic operator theorems

1.12.1 General result

Let V be a reflexive and separable Banach space, and $A : V \to V'$ be an operator.

Theorem 1.9 (**Existence theorem**). *Suppose that $A : V \longrightarrow V'$ is an operator*

- *bounded*
- *hemicontinuous*
- *coercive, meaning that*

$$\lim_{|v|_V \to +\infty} \frac{\langle Av, v\rangle}{\|v\|_V} = +\infty$$

- *monotone*

Let $f \in V'$. Then, there exists a function $u \in V$ and only one such that

$$Au = f$$

For the proof this theorem, look at to the book [5]

Theorem 1.10. *Let $f \in L^{\infty}(\Omega)$. Then the problem $(*)$ defined by*

$$(*) \qquad \begin{cases} -\Delta_p u = -\mathrm{div}(\mid \nabla u \mid^{p-2} \nabla u) & = & f & on & \Omega; \\ u & = & 0 & on & \partial\Omega; \end{cases}$$

has a unique solution $u \in W_0^{1,p}(\Omega)$, for all $1 < p \leq +\infty$.

Proof. According to the previous propositions, we have the principal operator

$$Au = -\mathrm{div}(\mid \nabla u \mid^{p-2} \nabla u)$$

which is coercive and bounded, hemicountinous , monotone, therefore, according to theorem 1.9. The problem $(*)$ has a unique solution $u \in W_0^{1,p}(\Omega)$, for all $1 < p \leq +\infty$. □

Chapter 2

Elliptic Problem with measures data weak solution

In this chapter, we study existence and regularity of weak solutions for a class of non-linear boundary value problems of elliptic type associated with p-Laplacian operator but with a non-regular second member. i.e, not belonging to a dual space. we consider second members in the space of bounded radon measures.

2.1 The $M(\Omega)$ space

We denote by $M(\Omega)$ the space of bounded radon measures on Ω. By definition, $M(\Omega) = [C_c(\Omega)]'$ = which is the topological dual of the space of continuous functions with compact support in Ω equipped with the norm of uniform convergence

$$\|u\|_{C_c(\Omega)} = \sup_{x\in\Omega} |u(x)| = \max_{x\in\Omega} |u(x)|$$

Remark 2.1. *We can verify that the map T from $L^1(\Omega)$ to $M(\Omega)$, which associates the summable function f with the measure Tf defined by*

$$\langle If, \varphi\rangle = \int_\Omega f\varphi, \quad \forall\varphi \in C_c(\Omega)$$

2.2 The problem (P)

We are interested in problems of the type

$$(P) \qquad \begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) &= \mu \text{ on } D'(\Omega); \\ u &= 0 \text{ on } \partial\Omega; \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N > 2)$ with boundary denoted by $\partial\Omega$ and p is a real number such that $p \geq 1$, μ is Radon's bounded measure on Ω.

2.2.1 Weak solutions

Definition 2.1. *u is a weak solution to problem (P) if: $u \in W_0^{1,1}(\Omega)$ and*

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int \mu \varphi d\mu, \ \forall \varphi \in \mathcal{D}(\Omega). \tag{2.1}$$

The result proved in this chapter is the following theorem

Theorem 2.1. *Let $2 - \frac{1}{N} < p \leq N$ and μ a bounded Radon measure ($\mu \in M(\Omega)$). Then The problem (P) has at least one weak solution u which is in $W_0^{1,q}(\Omega)$, for all*

$$q \in \left[1, q_c = \frac{N}{N-1}(p-1)\right[.$$

2.2.2 Model Example

Let $\Omega = B(0,1)$, the open Euclidean unit ball centered at zero with radius 1. find $u : \overline{\Omega} \to \mathbb{R}$ as a solution of

$$(P_0) \qquad \begin{cases} -\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2} \nabla u) = \delta \text{ in } D'(\Omega); \\ u = 0 \text{ on } \partial\Omega; \end{cases}$$

where $p > 1$ is a real number and δ is the Dirac distribution at the origin.

the problem (P_0) has a solution (model) u_{mod} is given by (see [1])

$$u_{mod} = \begin{cases} C(|x|^{\frac{p-N}{p-1}} - 1) & \text{if } p \neq N; \\ C \log |x| & \text{if } p = N; \end{cases}$$

2.2.3 Why the condition $2 - \frac{1}{N} < p \leq N$

In which space -or set- should we look for a solution of (P) ?

- if $p > N$ and Ω is a bounded domain, then we have

$$W_0^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$$

so that

$$M(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$$

Therefor, if $p > N$, we have $M(\Omega) \subset W^{-1,p'}(\Omega)$, which means that this case falls within the classical framework treated in the chapter 1.

The space where we look for solutions in this case is the space $W_0^{1,p}(\Omega)$.

- if $1 < p \leq N$

To make sense of $u = 0$ on $\Gamma = \partial\Omega$ and the gradient ∇u almost everywhere in Ω, it is natural to search for u in Sobolev space. For $u \in W_0^{1,p}(\Omega)$ we have $Au \in W^{-1,p'}(\Omega)$, so, since in the case $1 < p \leq N$, there exist measures $\mu \in M(\Omega)$ such that $\mu \notin W^{-1,p'}(\Omega)$, we have $Au \neq \mu, \quad \forall u \in W_0^{1,p}(\Omega)$. Therefore, The possible solutions of (P) must be sought in a space or set larger than $W_0^{1,p}(\Omega)$. If we stay within the framework of Sobolev spaces, the largest of these spaces (for bounded Ω) is $W_0^{1,1}(\Omega)$. To find out if this space is rich enough to contain the possible solutions of the problem (P), we take up the model equation (P_0) which admits the solution $u_{mod} =$ so, in the case $p \neq N$, we have

$$|\nabla u_{mod}| = C_1|x|^{\frac{p-N}{p-1}-1} = C_1|x|^{\frac{1-N}{p-1}}$$

then

$$\begin{aligned}
\int_\Omega |\nabla u_{mod}|dx &= C_1 \int_{B(0,1)} |x|^{\frac{1-N}{p-1}} dx \\
&= C_2 \int_0^1 r^{\frac{1-N}{p-1}} r^{N-1} dr \\
&= C_2 \int_0^1 r^{\frac{1-N}{p-1}+N-1} dr
\end{aligned}$$

This integral is finite if $\frac{1-N}{p-1} + N - 1 > -1$. Thus, if $p > 2 - \frac{1}{N}$, this calculation shows that:

$$u_{mod} \quad \text{is in} \quad W_0^{1,1}(B(0,1)) \quad \text{only if} \quad p > 2 - \frac{1}{N}.$$

Suppose that $p > 2 - \frac{1}{N}$ and we are looking for critical value of the exponent $q \geq 1$ so that $u_{mod} \in W_0^{1,q}(B(0,1))$ we have :

$$\int_\Omega |\nabla u_{mod}|^q dx = C_1 \int_0^1 r^{\frac{q(1-N)}{p-1}+N-1} dr dx.$$

Our integral will be finite if $\frac{q(1-N)}{p-1} + N - 1 > -1$, which gives

$$u_{mod} \in W_0^{1,q}(B(0,1)) \quad \text{for all } q \text{ such that} \quad 1 \leq q < \frac{N}{N-1}(p-1).$$

Now we give the following general property of measures, which will allow us to approach the problem with measure data by problems with regular data.

Theorem 2.2 (Density theorem[1]). *Let $\mu \in M(\Omega)$, Then there exists a sequence $\mu_n \subset \mathcal{D}(\Omega)$ such that*

$\mu_n \to \mu$ in the vague sense of measures,i.e,

$$\forall \varphi \in C_c(\Omega), \quad \langle \mu_n, \varphi \rangle \to \langle \mu, \varphi \rangle \quad (in \quad \mathbb{R})$$

and

$$\|\mu_n\|_{L^1(\Omega)} \leq \|\mu\|_{M(\Omega)}, \quad \forall n \geq 1$$

Now, the proof of the Theorem 2.1 needs several steps: First, we approximate the problem (P) with sequence of problems (P_n) having smooth solutions (u_n). Then, after deriving uniform estimates onu_n, finally we pass to the limit using a compactness results as in [9].

2.3 Approximation of (P)

By Theorem 2.2, let μ_n be a sequence of $D(\Omega)$ which converges to μ in $D'(\Omega)$ and which verifies the inequalities

$$\|\mu_n\|_{L^1(\Omega)} \leq \|\mu\|_{M(\Omega)} < +\infty, \quad \forall n \geq 1.$$

We define the problems (P_n) by

$$(P_n) \qquad \begin{cases} -\text{div}(\mid \nabla u_n \mid^{p-2} \nabla u_n) & = & \mu_n \quad \text{in} \quad \Omega \\ u_n & \in & W_0^{1,p}(\Omega) \quad (\quad \text{so} \quad u_n = 0 \quad \text{on} \quad \Gamma = \partial\Omega); \end{cases}$$

The existence of a solution u_n in $W_0^{1,p}(\Omega)$ of the problem (P_n) is classical, since $\mu_n \in W^{-1,p'}(\Omega)$ and the fact that the operator $Au_n = -\text{div}(\mid \nabla u_n \mid^{p-2} \nabla u_n)$ is coercive and is a monotone (see Theorem 1.9), i.e, there exists a function u_n solution of problem

which is its equivalent weak form

$$\begin{cases} u_n \in W_0^{1,p}(\Omega) \\ \int_\Omega \mid \nabla u_n \mid^{p-2} \nabla u_n \nabla \varphi \ = \int_\Omega \mu_n \varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega). \end{cases}$$

2.4 Uniform Estimates of Approximate solutions

Since we only have $\|\mu_n\|_{L^1(\Omega)} \leq C, \quad \forall n \geq 1$ in order to to bound the term $\int_\Omega \mu_n \varphi$ uniformly with respect to n, we need to choose a test function

$$\varphi_n \quad \text{a function of } u_n \text{ such that} \quad \|\varphi_n\|_{L^\infty(\Omega)} \leq C, \forall n \geq 1 \quad \text{with} \quad \varphi_n \in W_0^{1,p}(\Omega)$$

To choose the test functions, we use the following result due to Stampacchia

Lemma 2.1 (Stampacchia)**.** *Let $T : \mathbb{R} \to \mathbb{R}$ be a globally lipschitz function,i.e.*

$$\exists C > 0 \quad \textit{such that} \quad |T(s) - T(t)| \leq C|s - t|, \quad \forall s, t \in \mathbb{R},$$

such that $T(0) = 0$. then, $\forall v \in W_0^{1,p}(\Omega)$ with $1 \leq p \leq \infty$ we have

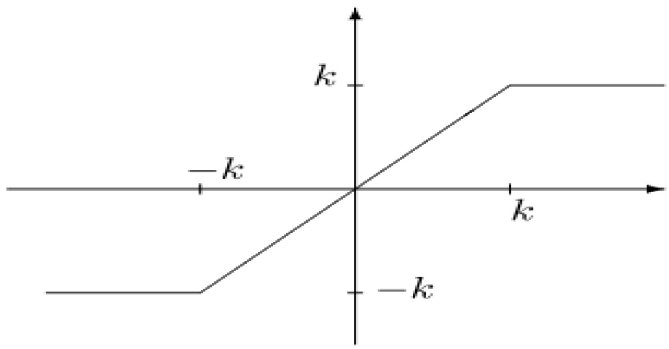
$$T(v) \in W_0^{1,p}(\Omega) \quad \textit{and} \quad \nabla T(v) = T'(v)\nabla v \quad \textit{in} \quad \mathcal{D}'(\Omega) \quad \textit{and} \quad \textit{a.e.} \quad \textit{in} \quad \Omega$$

Example 2.1. *Let $k > 0$. The truncation at levels $-k$ and k is defined by the function T_k of $\mathbb{R}$ from $\mathbb{R}$ given by*

$$T_k(r) = \begin{cases} k, & \textit{if } r \geq k, \\ r, & \textit{if } |r| < k, \\ -k, & \textit{if } r \leq k. \end{cases}$$

It can be verified that the function T_k is a globally Lipschitz function.

The graph of T_k is



We can verify that the function T_k is a globally Lipschitz function satisfying $|T_k(r)| \leq k$ and $|T_k(r)| \leq |r|$ and its primitive $\Theta_k : \mathbb{R} \to \mathbb{R}^+$ defined by

$$\Theta_k(r) = \int_0^r T_k(t)dt = \begin{cases} \frac{r^2}{2}, & \text{if } |r| \leq k \\ k|r| - \frac{k^2}{2}, & \text{if } |r| > k \end{cases}$$

Lemma 2.2. *Let (u_n) be a sequence of solutions of (P_n), the sequence $\{T_k(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$, i.e.*

$$\exists C > 0, \quad \|T_k(u_n)\|_{W_0^{1,p}(\Omega)} \leq C, \quad \forall n \geq 1.$$

Proof. Taking

$$\phi(t) = \int_0^t |T_k'(\sigma)|^p d\sigma$$

as test function in (P_n), since

$$\forall n \geq 1, \quad |\psi(t)| \leq \int_{-\infty}^{+\infty} |T_k'(\sigma)|^p d\sigma = M$$

and $\phi(0) = 0$, we can say that the function ϕ is globally Lipschitz. Then, we obtain

$$\int_\Omega | \nabla u_n |^{p-2} \nabla u_n \nabla \phi(u_n) dx = \int_\Omega \mu_n \phi(u_n) dx.$$

Using that $\nabla \phi(u_n) = |T_k'(u_n)|^p \nabla(u_n)$, we find

$$\begin{aligned} \int_\Omega | \nabla u_n |^{p-2} \nabla u_n |T_k'(u_n)|^p \nabla(u_n) dx &\leq \int_\Omega |\mu_n||\phi(u_n)| dx \\ &\leq M \int_\Omega |\mu_n| dx \end{aligned}$$

so,

$$\int_\Omega |T_k'(u_n)|^p |\nabla(u_n)|^p dx \leq M \int_\Omega |\mu_n| dx, \quad \forall n \geq 1$$

Thus,

$$\int_\Omega |\nabla T_k(u_n)|^p dx \leq M\|\mu_n\|_{M(\Omega)}$$

Then, we have

$$\|T_k(u_n)\|_{W_0^{1,p}(\Omega)} \leq C.$$

□

Lemma 2.3. *Let $\delta > 0$, there exists a constant C_δ independent of n such that*

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{1+\delta}} dx \leq C_\delta, \quad \forall n \geq 1.$$

Proof. Let $\delta > 0$. the function ψ_δ given by

$$\psi_\delta(t) = \int_0^t \frac{d\sigma}{(1+|\sigma|)^{1+\delta}} = \frac{1}{\delta}\left(1 - \frac{1}{(1+|t|)^\delta}\right)\text{sign}(t), \quad t \in \mathbb{R}$$

is globally Lipschitz $\psi_\delta(0) = 0$ and $|\psi_\delta(t)| \leq \frac{1}{\delta}, \quad \forall t \in \mathbb{R}$. Taking $\psi_\delta(u_n)$ as a test function in (P_n), we obtain

$$\langle Au_n, \psi_\delta(u_n)\rangle = \langle \mu_n, \psi_\delta(u_n)\rangle, \quad Au_n = -\text{div}(\mid \nabla u_n \mid^{p-2} \nabla u_n)$$

so that

$$\int_\Omega \mid \nabla u_n \mid^{p-2} \nabla u_n \nabla \psi_\delta(u_n) dx = \int_\Omega \mu_n \psi_\delta(u_n) dx$$

Using the fact that $\nabla\psi_\delta(u_n) = \nabla u_n \psi'_\delta(u_n)$ with $\psi'_\delta(u_n) = \frac{1}{(1+|u_n|)^{1+\delta}}$, we have

$$\begin{aligned}\int_\Omega \mid \nabla u_n \mid^{p-2} \nabla u_n \nabla \psi_\delta(u_n) dx &= \int_\Omega \mid \nabla u_n \mid^{p-2} \nabla u_n \nabla u_n \psi'_\delta(u_n) dx \\ &= \int_\Omega \mid \nabla u_n \mid^{p-2} \nabla u_n \frac{\nabla u_n}{(1+|u_n|)^{1+\delta}} dx,\end{aligned}$$

Therefore,

$$\int_\Omega \mid \nabla u_n \mid^{p-2} \nabla u_n \frac{\nabla u_n}{(1+|u_n|)^{1+\delta}} dx = \int_\Omega \mu_n \psi_\delta(u_n) dx \leq \frac{1}{\delta}\|\mu_n\|_{M(\Omega)},$$

then, we obtain

$$\begin{aligned}\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{1+\delta}} dx &\leq \frac{1}{\delta}\|\mu_n\|_{M(\Omega)}, \quad \forall n \geq 1 \\ &\leq C\end{aligned}$$

□

Lemma 2.4. *Let p and q be such that*

$$2-\frac{1}{N}<p\leq N \quad \textit{and} \quad q\in[1,\frac{N}{N-1}(p-1)[.$$

Then,

- *The sequence (∇u_n) remains bounded in $L^q(\Omega)$.i,e*

there exists a constant $C=C(q)>0$ such that

$$\int_\Omega |\nabla u_n|^q dx \leq C$$

Proof. Note that

$$\frac{N}{N-1}(p-1)-p=\frac{p-N}{N-1}\leq 0 \Longrightarrow 1\leq q<\frac{N}{N-1}(p-1)\leq p\leq N.$$

Let q^* be the sobolev exponent associated with q, i.e.,$\frac{1}{q^*}=\frac{1}{q}-\frac{1}{N}\Leftrightarrow q^*=\frac{Nq}{N-q}$. For

$$\delta=q^*\frac{p-q}{q}-1\Leftrightarrow q^*=(1+\delta)\frac{q}{p-q}$$

Lemma 2.3 implies that there exists a constant $C=C(\delta)>0$ such that

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{1+\delta}}dx\leq C, \quad \forall n\geq 1$$

we can write

$$y_n=\int_\Omega |\nabla u_n|^q dx \quad = \quad \int_\Omega \frac{|\nabla u_n|^q}{(1+|u_n|)^{(1+\delta)q/p}}(1+|u_n|)^{(1+\delta)q/p}dx.$$

Using Hölder's inequality, we obtain

$$\begin{aligned} y_n=\int_\Omega |\nabla u_n|^q dx \quad &\leq \quad \left(\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{(1+\delta)}}\right)^{q/p}\cdot\left(\int_\Omega (1+|u_n|)^{(1+\delta)\frac{q}{p-q}}\right)^{1-q/p} \\ &\leq \quad C^{q/p}\left(\int_\Omega (1+|u_n|)^{q^*}\right)^{1-q/p}. \end{aligned} \tag{2.2}$$

Using the inequality

$$(a+b)^p\leq 2^{p-1}(a^p+b^p) \quad \text{for} \quad 1\leq p<\infty \quad \text{and} \quad a,b\geq 0, \tag{2.3}$$

we can write

$$\begin{aligned} y_n = \int_\Omega |\nabla u_n|^q dx &\leq C^{q/p}(2^{q^*-1})^{1-q/p}\left(\int_\Omega 1dx + \int_\Omega |u_n|^{q^*}\right)^{1-q/p} \\ &\leq C^{q/p}(2^{q^*-1})^{1-q/p}(2^{-q/p})\left\{|\Omega|^{1-q/p} + \left(\int_\Omega |u_n|^{q^*}\right)^{1-q/p}\right\}. \end{aligned}$$

But, by the sobolev Poincare inequality, we obtain

$$\left(\int_\Omega |u_n|^{q^*}\right)^{1/q^*} \leq C_{q,N}\|\nabla u_n\|_{L^q(\Omega)} = C_{q,N} y_n^{1/q}.$$

Therefore,

$$y_n \leq a + b y_n^{\frac{q^*}{q}(1-\frac{q}{p})} \tag{2.4}$$

where a and b are constants depending on q, N and $|\Omega|$. Since, in the case $p < N$, we have

$$\begin{aligned} \frac{q^*}{q}(1-\frac{q}{p}) - 1 &= \frac{Nq}{N-q}\frac{p-q}{qp} - 1 \\ &= \frac{N(p-q)}{(N-q)p} - 1 \\ &= \frac{q}{p}\frac{p-N}{N-q} < 0. \end{aligned}$$

that is to say $\frac{q^*}{q}(1-\frac{q}{p}) < 1$, then the previous inequality (2.4), satisfied by y_n, shows that this sequence is bounded.

In the case $p = N$, we repeat the calculation, but choosing $\delta = \alpha\frac{N-q}{q} - 1$ with $\max\{1, \frac{q}{N-q}\} < \alpha < q^*$. We use the continuity of the injection from $W_0^{1,q}(\Omega)$ in $L^\alpha(\Omega)$ and we obtain

$$y_n \leq a + b y_n^{\frac{\alpha}{q}(1-\frac{q}{N})}, \quad \forall n \geq 1 \quad \text{with} \quad \frac{\alpha}{q}(1-\frac{q}{N}) < \frac{q^*}{q}(1-\frac{q}{p}) = 1$$

This implies that y_n is also bounded in $\mathbb{R}$ in this case. □

Proposition 2.1. *For all $q \in [1, q_c = \frac{N}{N-1}(p-1)[$ and since $(u_n)_n$ is bounded in the space $W_0^{1,q}(\Omega)$, according to Rellich-Kondrachov theorem, we can extract from the sequence (u_n) a sub-sequence, also denote by (u_n) such that*

$$u_n \rightharpoonup u \quad \textit{weakly in} \quad W_0^{1,q}(\Omega), \tag{2.5}$$

$$u_n \longrightarrow u \quad \textit{strongly in} \quad L^q(\Omega). \tag{2.6}$$

Therefore

$$u_n \longrightarrow u \quad a.e \textit{ in} \quad \Omega, \tag{2.7}$$

Proof. For the proof of this proposition, look at the book [3] □

Now, we have the operator $-\Delta_p$ depends on the gradient ∇u, so we need more convergence on the gradients ∇u_n. Let's set $u_n^k = T_k(u_n)$ and $u^k = T_k(u)$

Proposition 2.2. *[1] Let u be the function constructed in proposition 2.1. Then, we have*

$$u^k = T_k(u) \in W_0^{1,p}(\Omega).$$

Lemma 2.5. *For all $k > 0$, we have*

$$\int_{\{u\in\Omega, |u_n - u^k| \leq \varepsilon\}} \mid \nabla u_n \mid^{p-2} \nabla u_n . \nabla(u_n - u^k) \leq \varepsilon \|\mu\|_{M(\Omega)}, \quad \forall \varepsilon > 0$$

Proof. Let $k > 0$ be fixed. For $\varepsilon > 0$, Let T_ϵ be the truncation at levels $-\varepsilon$ and ε. According to proposition 2.2, $u^k = T_k(u) \in W_0^{1,p}(\Omega)$, the function $v_n = T_\varepsilon(u_n - u^k)$ is in $W_0^{1,p}(\Omega)$. Taking v_n as a test function in (P_n), and since $|v_n| = |T_\varepsilon(u_n - u^k)| \leq \varepsilon$, we obtain

$$\begin{aligned} \int_{\{x\in\Omega, |u_n - u^k| \leq \varepsilon\}} \mid \nabla u_n \mid^{p-2} \nabla u_n . \nabla(u_n - u^k) &= \int_\Omega \mu_n T_\varepsilon(u_n - u^k) \\ &\leq \varepsilon \int_\Omega |\mu_n| \leq \varepsilon \|\mu\|_{M(\Omega)} \end{aligned}$$

□

2.4.1 A fundamental compactness Lemma

Lemma 2.6 ([9]). *Let $(u_n)_n$ be a sequence in $W_0^{1,p}(\Omega)$ satisfying the following properties.*

- *(i) There exists $q \in]1, p]$, such that $(u_n)_n$ remains in a bounded set of $W_0^{1,q}(\Omega)$ and (u_n) weakly converges in $W_0^{1,q}(\Omega)$ to a function u, and almost everywhere to a function u in Ω.*

- *(ii) For all $k > 0$, $(u_n^k)_n = (T_k(u_n))_n$ remains in a bounded set of $W_0^{1,p}(\Omega)$ when $n \to +\infty$.*

- *(iii) For all $k > 0$, there exist a function θ_k such that for all $\varepsilon > 0$, we have*

$$\limsup_n \int_{\{u\in\Omega, |u_n - u^k| \le \varepsilon\}} (\mid \nabla u_n \mid^{p-2} \nabla u_n).\nabla(u_n - u^k)\, dx \le \theta_k(\varepsilon),$$

with $\lim_{\varepsilon\to 0} \theta_k(\varepsilon) = 0$.

Then, there exists a sub-sequence (u_n)(denoted the same) such that

$$\nabla u_n \to \nabla u \quad \textit{almost everywhere} \quad \textit{in} \quad \Omega$$

2.5 Passage to the limit for the approximate problems

Let $q \in]1, q_c = \frac{N}{N-1}(p-1)[$. On one hand, according to lemma 2.4, the sequence of approximate solutions (u_n) is bounded in $W_0^{1,q}(\Omega)$, and we have $u_n \to u$ almost everywhere in Ω and according to proposition 2.1, $u_n \rightharpoonup u$ weakly in $W_0^{1,q}(\Omega)$. On the other hand, according to the Lemma 2.2, (u_n^k) remains bounded in $W_0^{1,p}(\Omega)$ and according to Lemma 2.5, we have, for all $k > 0$:

$$\int_{\{u\in\Omega, |u_n - u^k| \le \varepsilon\}} (\mid \nabla u_n \mid^{p-2} \nabla u_n).\nabla(u_n - u^k) \le \varepsilon \|\mu\|_{M(\Omega)}, \quad \forall \varepsilon > 0$$

This shows that all the hypotheses of lemma 2.6 are satisfied by (u_n) and u. Therefore, we can find a sub-sequence (u_n) such that

$$u_n \to u \quad \text{and} \quad \nabla u_n \to \nabla u \quad \text{almost everywhere in } \Omega$$

Now let us fix $\varphi \in \mathcal{D}(\Omega)$. We have

$$\int_\Omega \mid \nabla u_n \mid^{p-2} \nabla u_n \nabla \varphi \quad = \quad \int_\Omega \mu_n \varphi.$$

1) Limit Passage in $\int_\Omega \mu_n \varphi$. We have

$$\lim_{n\to+\infty} \int_\Omega \mu_n \varphi = \langle \mu, \varphi \rangle$$

2) Limit Passage in $\int_\Omega \mid \nabla u_n \mid^{p-2} \nabla u_n \nabla \varphi \, dx$.

As $\nabla u_n \to \nabla u$ almost everywhere in Ω, we have

$$\mid \nabla u_n \mid^{p-2} \nabla u_n \to \mid \nabla u \mid^{p-2} \nabla u \quad \text{almost everywhere in} \quad \Omega. \tag{2.8}$$

Since

$$\frac{N}{N-1}(p-1) - (p-1) = (p-1)\left(\frac{N}{N-1} - 1\right) = \frac{p-1}{N-1} > 0$$

So that

$$\frac{N}{N-1}(p-1) > p-1$$

Now, for any measurable subset for $E \subset \Omega$ we have

$$\begin{aligned}
\int_E \mid \mid \nabla u \mid^{p-2} \nabla u \nabla \varphi \mid dx &\leq \|\nabla \varphi\|_{L^\infty(E)} \int_E |\nabla u_n|^{p-1} dx \\
&\leq \|\nabla \varphi\|_{L^\infty(\Omega)} \int_E |\nabla u_n|^{p-1} dx \\
&\leq C \int_E |\nabla u_n|^{p-1} dx
\end{aligned}$$

We can choose $q > p-1$ and according to Hölder inequality, we have

$$\begin{aligned}
\int_E \mid \mid \nabla u_n \mid^{p-2} \nabla u_n \nabla \varphi \mid dx &\leq C \left(\int_E |\nabla u_n|^q dx\right)^{(p-1)/q} . \left(\int_E 1 dx\right)^{1-(p-1)/q} \\
&\leq C \left(\int_E |\nabla u_n|^q dx\right)^{(p-1)/q} . |E|^{1-(p-1)/q}
\end{aligned}$$

Using lemma 2.4 we have (∇u_n) is bounded in $L^q(\Omega)$, we have

$$\int_E \mid \mid \nabla u_n \mid^{p-2} \nabla u_n \nabla \varphi \mid dx \leq C|E|^{1-(p-1)/q} = C'$$

This proves the equi-integrability of the sequence $(\mid \nabla u_n \mid^{p-2} \nabla u_n \nabla \varphi)_n$. It follows from Vitali's convergence theorem 1.4 that

$$\lim_{n \to +\infty} \int_\Omega \mid \nabla u_n \mid^{p-2} \nabla u_n \nabla \varphi \, dx = \int_\Omega \mid \nabla u \mid^{p-2} \nabla u \nabla \varphi \, dx$$

It has therefore been proven that

$$\int_\Omega \mid \nabla u \mid^{p-2} \nabla u \nabla \varphi = \langle \mu, \varphi \rangle, \quad \forall \varphi \in D(\Omega)$$

Then u is a solution of the problem

$$\begin{cases} u & \in & W_0^{1,q}(\Omega), \quad q \in [1, \frac{N}{N-1}(p-1)[\\ Au & = & \mu \quad \text{on} \quad D'(\Omega); \end{cases}$$

We therefore have to prove that u is a solution to problem (P). This finishes the proof of Theorem 2.1.

Chapter 3

Elliptic Problem with L^1 data Entropy solution

In this chapter, we study the entropy solution for a class of nonlinear elliptic equations in Ω with $p \geq 1$ and L^1 data.

3.1 Positions of the problem

Let Ω a bounded open set of $\mathbb{R}^N$ and (P) is in Ω of the following type

$$(P) \qquad \begin{cases} -\Delta_p u = -\mathrm{div}(\mid \nabla u \mid^{p-2} \nabla u) & = & f \text{ on } \Omega; \\ u & = & 0 \text{ on } \partial\Omega. \end{cases}$$

where $f \in L^1(\Omega)$, Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with boundary denoted by $\partial\Omega$

Definition 3.1. *u is a weak solution to problem (P) if: $u \in W_0^{1,1}(\Omega)$ and*

$$\int_\Omega \mid \nabla u \mid^{p-2} \nabla u \nabla \varphi \, dx \quad = \quad \int_\Omega f \varphi dx, \ \forall \varphi \in \mathcal{D}(\Omega). \tag{3.1}$$

Theorem 3.1. *Let $2 - \frac{1}{N} < p$. Then the problem (P) has at least one weak solution $u \in W_0^{1,q}(\Omega)$, for all $q \in [1, \frac{N}{N-1}(p-1)[$.*

Proof. The proof of the Theorem 3.1 is similar to that of Theorem 2.1 in chapter 2 □

3.2 Entropy solutions

We denote

$$\tau_0^{1,p}(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \quad \text{measurable}, \quad T_k(u) \in W_0^{1,p}(\Omega) \quad \text{for every} \quad k > 0 \Big\}.$$

it is clear that for every $p \in [1, \infty)$ we have the inclusions $W_0^{1,p}(\Omega) \subset \tau_0^{1,p}(\Omega)$ and in this case we have

$$\nabla T_k(u) = 1_{\{|u|<k\}} \nabla u.$$

where 1_A denotes the characteristic function of a measurable set $A \subset \mathbb{R}^N$.

Proposition 3.1. *[6] For every measurable function $u \in \tau_0^{1,1}(\Omega)$, there exists a unique measurable function $v : \Omega \longrightarrow \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|\leq k\}}, \text{ a.e. in } \Omega \text{ and for every } k > 0.$$

Furthermore, $u \in W_0^{1,1}(\Omega)$ and only if $v \in L^1(\Omega)$ and then $v \equiv \nabla u$ in the usual weak sense (in the sense of distribution).

Lemma 3.1. *If $u \in \tau_0^{1,p}(\Omega)$ is an entropy solution of (P) then for every $k > 0$, there exists a positive constant C such that*

$$\|\nabla T_k(u)\|_{L^p(\Omega)} \leq C.$$

Proof. Taking $T_k(u)$ as test function in (P), we obtain

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla T_k(u) dx = \int_\Omega f T_k(u) dx.$$

The fact that $f \in L^1(\Omega)$, we have

$$\begin{aligned} \int_{\{|u|<k\}} |\nabla T_k(u)|^p \, dx &\leq k \int_\Omega |f| dx \\ &\leq k\|f\|_{L^1(\Omega)} \leq C. \end{aligned}$$

Hence, we obtain the following bound in $L^p(\Omega)$

$$\|\nabla T_k(u)\|_{L^p(\Omega)} \leq C.$$

□

Definition 3.2. *Let $f \in L^1(\Omega)$. We will call an entropy solution of (P) a function $u \in \tau_0^{1,p}(\Omega)$ such that for all $k > 0$, we have*

$$\int_\Omega |\nabla u|^{p-2}\nabla u\, \nabla T_k(u-\varphi)dx \leq \int_\Omega f T_k(u-\varphi)dx, \tag{3.2}$$

where $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$

Remark 3.1. *2- We observe that each term in (3.2) is well defined, it is clear for the right side (because $f \in L^1(\Omega)$ and T_k is bounded). While for the left side, we have*

$$\int_\Omega |\nabla u|^{p-2}\nabla u\, \nabla T_k(u-\varphi)dx = \int_\Omega |\nabla T_M(u)|^{p-2}\nabla T_M(u)\, \nabla T_k(u-\varphi)dx$$

where $M = k + \|\varphi\|_{L^\infty}$. Indeed as $\varphi \in L^\infty(\Omega)$

$$\begin{aligned} |u-\varphi| < k &\implies |u| - |\varphi| < |u-\varphi| < k \\ &\implies |u| < |\varphi| + k \\ &\implies |u| < \|\varphi\|_{L^\infty} + k \\ &\implies |u| < M \end{aligned}$$

What is implied $\{|u-\varphi| < k\} \subset \{|u| < M\}$. Actually, for all $u \in \tau_0^{1,p}(\Omega)$, the first integral in (3.2) is zero on the set where $|u| > M$, so we can write

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2}\nabla u\, \nabla T_k(u-\varphi)dx &= \int_{\{|u-\varphi|<k\}} |\nabla u|^{p-2}\nabla u\, \nabla(u-\varphi)dx \\ &= \int_{\{|u-\varphi|<k\}} |\nabla T_M(u)|^{p-2}\nabla T_M(u), \nabla(u-\varphi)dx \\ &= \int_\Omega |\nabla T_M(u)|^{p-2}\nabla T_M(u)\, \nabla T_k(u-\varphi)dx. \end{aligned}$$

to have, that

$$||\nabla T_M(u)|^{p-2}\nabla T_M(u)| = |\nabla T_M(u)|^{p-1} \in L^{p'}(\Omega), \quad p' = \frac{p}{p-1}. \tag{3.3}$$

The result proved here is the following theorem

Theorem 3.2. *Let $f \in L^1(\Omega)$. Then, the problem (P) has at least one entropy solution u. Moreover, if $p > 1 - \frac{2}{N}$ then $u \in W_0^{1,q}(\Omega)$ for all $1 \leq q < \frac{N}{N-1}(p-1)$.*

3.2.1 Proof of Theorems 3.2

We proceed by approximation. We consider the approximate problem

$$(P_n) \qquad \begin{cases} -\mathrm{div}(\mid \nabla u_n \mid^{p-2} \nabla u_n) &= f_n \text{ on } \Omega; \\ u_n &= 0 \text{ on } \partial\Omega; \end{cases}$$

with (f_n), $f_n = T_n(f)$ be a sequence of bounded functions defined in Ω, such that

$$\begin{cases} \|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}, \\ f_n \to f \text{ strongly in } L^1(\Omega) \quad . \end{cases} \tag{3.4}$$

The existence of the approximate solution u_n in $W_0^{1,p}(\Omega)$ of (P_n) is proved as Theorem 1.9 and u_n verifies

$$\int_\Omega \mid \nabla u_n \mid^{p-2} \nabla u_n \nabla \varphi = \int_\Omega f_n \varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{3.5}$$

Lemma 3.2. *The sequence (u_n) remains in a bounded of $W_0^{1,q}(\Omega)$ for all*

$$q \in [1, \frac{N}{N-1}(p-1)[.$$

and that $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ for all $k > 0$

Proof. The proof of the Lemma 3.2 is similar to that of Lemma 2.4 and the Lemma 2.2 □

Remark 3.2. *thanks to the lemma 3.2, there exists u such that*

$$u_n \rightharpoonup u \quad \textit{weakly in} \quad W_0^{1,q}(\Omega), \tag{3.6}$$

$$u_n \longrightarrow u \quad \textit{strongly in} \quad L^q(\Omega). \tag{3.7}$$

and

$$T_k(u_n) \rightharpoonup T_k(u) \quad \textit{weakly in} \quad W_0^{1,p}(\Omega). \tag{3.8}$$

3.3 passage to the limit

Lemma 3.3. *The sequence (∇u_n) converges almost everywhere to ∇u(possibly up to a sub-sequence).*

Proof. We show that (∇u_n) is Cauchy in measure, which implies that $\nabla u_n \longrightarrow \nabla u$ almost everywhere, for a sub-sequence. This consists of proving that

$$\forall \delta > 0, \forall \varepsilon > 0, \exists n_0 \text{ such that } \quad \forall p, q \geq n_0 \quad \text{meas}\{x \in \Omega |\, |(\nabla u_p - \nabla u_q)(x)| \geq \delta\} \leq \varepsilon$$

To do this, let us fix $\delta > 0$ and $\varepsilon > 0$, and note that for $\lambda > 0$ and $\eta > 0$, we have

$$\{x \in \Omega |\, |(\nabla u_p - \nabla u_q)(x)| \geq \delta\} \subset E_1 \cup E_2 \cup E_3 \cup E_4$$

where

$$E_1 = \{x \in \Omega |\, |\nabla u_p| \geq \lambda\}, \quad E_2 = \{x \in \Omega |\, |\nabla u_q| \geq \lambda\}$$

$$E_3 = \{x \in \Omega |\, |u_p - u_q| \geq \eta\}$$

and

$$E_4 = \{|\nabla u_p - \nabla u_q| \geq \delta,\ |\nabla u_p| \leq \lambda,\ |\nabla u_q| \leq \lambda,\ |u_p - u_q| \leq \eta\}.$$

And since the sequence (∇u_n) remains in a bounded of $L^q(\Omega)$ for all $q \in \left[1, p - \frac{N}{N+1}\right[$ [See lemma 3.2], by choosing λ large, we can make meas(E_1) and meas(E_2) arbitrarily small. For example

$$\text{meas}(E_1) = \int_{E_1} 1 dx = \frac{1}{\lambda} \int_{E_1} \lambda dx \leq \frac{1}{\lambda} \int_{E_1} |\nabla u_p| dx \leq \frac{1}{\lambda} \int_{\Omega} |\nabla u_p| dx \leq \frac{C}{\lambda} \leq \varepsilon.$$

Then,

$$\text{meas}(E_1) \to 0 \quad \text{if} \quad \lambda \to +\infty,$$

hence

$$\text{meas}(E_1) \leq \varepsilon.$$

for meas(E_3), we have

$$\text{meas}(E_3) \leq \frac{1}{\eta} \int_{E_3} |u_p - u_q| dx \leq \frac{1}{\eta} \int_{\Omega} |u_p - u_q| dx$$

Since (u_n) is a Cauchy sequence in $L^1(\Omega)$ [see (3.7)], then for fixed $\eta > 0$, we see that

$$\text{meas}(E_3) \to 0 \quad \text{if} \quad p, q \to +\infty$$

hence

$$\text{meas}(E_3) \leq \varepsilon.$$

It remains to control meas(E_4). According to the monotonicity of the function

$$a : (x, \nabla u) \longmapsto a(x, \nabla u) = \mid \nabla u \mid^{p-2} \nabla u \quad \text{(see chapter 1)},$$

we have $\Big(a(x,\xi_1) - a(x,\xi_2)\Big)(\xi_1 - \xi_2) > 0$ and the set

$$\{(\xi_1, \xi_2)|,\ |\xi_1| \leq \lambda,\ |\xi_2| \leq \lambda,\ |\xi_1 - \xi_2| \leq \delta\}$$

is a compact set and a is continuous in ξ, then $\Big(a(x,\xi_1) - a(x,\xi_1)\Big)(\xi_1 - \xi_2)$ attains its minimum on this compact set, which we will denote as $\mu(x)$ such that

$$(a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2) \geq \mu(x) > 0.$$

Therefore, by (1.2) for any $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that

$$\int_{E_4} \mu(x)dx < \varepsilon' \Rightarrow meas(E_4) < \varepsilon. \tag{3.9}$$

To obtain $meas(E_4) < \varepsilon$, it suffices to show that $\int_{E_4} \mu(x)dx < \varepsilon'$. Using the definition of $\mu(x)$ and E_4, we can write

$$\int_{E_4} \mu(x)dx \ \leq \ \int_{E_4} \Big(|\nabla u_p|^{p-2}\nabla u_p - |\nabla u_q|^{p-2}\nabla u_q\Big)\nabla(u_p - u_q)\mathbf{1}_{\{|u_p - u_q| \leq \eta\}}dx.$$

Moreover, the integral term is positive and $\nabla T_\eta(u_p - u_q) = \nabla(u_p - u_q)\mathbf{1}_{\{|u_p - u_q| \leq \eta\}}$, so we have

$$\int_{E_4} \mu(x)dx \ \leq \ \int_{E_4} \Big(|\nabla u_p|^{p-2}\nabla u_p - |\nabla u_q|^{p-2}\nabla u_q\Big)\nabla T_\eta(u_p - u_q)dx. \tag{3.10}$$

Where T_η is the truncation function at levels η and T'_η is defined by

$$T'_\eta(\sigma) = \begin{cases} 1, & |\sigma| \leq \eta; \\ 0, & |\sigma| > \eta. \end{cases}$$

By taking $T_\eta(u_p - u_q)$ as a test function in (P_n) for u_p and u_q, we have

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \nabla T_\eta(u_p - u_q)dx = \int_\Omega f_n T_\eta(u_p - u_q)dx \tag{3.11}$$

and

$$\int_\Omega |\nabla u_q|^{p-2}\nabla u_q \nabla T_\eta(u_p - u_q)dx = \int_\Omega f_n T_\eta(u_p - u_q)dx. \tag{3.12}$$

Then, subtracting the inequality resulting from (3.11) and (3.12), we obtain

$$\int_\Omega (|\nabla u_p|^{p-2}\nabla u_p - |\nabla u_q|^{p-2}\nabla u_q)\nabla T_\eta(u_p - u_q)dx = 0.$$

So, we can write that

$$\begin{aligned}\int_\Omega (|\nabla u_p|^{p-2}\nabla u_p - |\nabla u_q|^{p-2}\nabla u_q)\nabla T_\eta(u_p - u_q)dx \\ \leq \eta \to^{\eta\to 0} 0(\text{uniformly in } p \text{ and } q).\end{aligned} \tag{3.13}$$

For η sufficiently small, (3.10) and (3.13) imply

$$\int_{E_4} \mu(x)dx < \varepsilon',$$

and also by (3.9) we have $meas(E_4) \leq \varepsilon$. Therefore for all $p, q \geq n_0$ we have

$$meas\{x \in \Omega |\, |(\nabla u_p - \nabla u_q)(x)| \geq \delta\} \leq 4\varepsilon.$$

Thus, we have the convergence of ∇u_n to ∇u in measure, as well as the lemma 1.5 (after extraction of a sequence)

$$\nabla u_n \to \nabla u \quad \text{almost everywhere in} \quad \Omega.$$

□

3.4 Existence of entropy solution

We can now show that the limit u of the sequence (u_n), defined previously, is an entropy solution of (P), that is u verifies (3.2).

The solution u obtained in the previous paragraph is the limit of (u_n), which verifies (3.5), that is,

$$\int_\Omega |\, \nabla u_n \,|^{p-2}\, \nabla u_n \nabla \psi \, dx \;\; = \;\; \int_\Omega f_n \psi dx. \tag{3.14}$$

for all $\psi \in W_0^{1,p}(\Omega)$.

Let us choose $\psi = T_k(u_n - \varphi)$ for $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. This choice of ψ in (3.14) is possible because $\psi \in L^p(\Omega)$, $\nabla\psi \in L^p(\Omega)$ since

$$\nabla\psi = \nabla T_k(u_n - \varphi) = \nabla(u_n - \varphi)1_{\{|u_n-\varphi|\leq k\}}.$$

Then, we obtain

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla T_k(u_n - \varphi)dx \quad = \quad \int_\Omega f_n T_k(u_n - \varphi)dx. \tag{3.15}$$

(this is the entropy formulation for u_n with an equality).

let's study the limit passage as $n \longrightarrow \infty$ for each term. Using remark 3.1, We have seen that

$$T_k(u_n - \varphi) = T_k(T_M(u_n) - \varphi) \quad \text{with} \quad M = k + \|\varphi\|_{L^\infty(\Omega)}.$$

So it suffices to show that

$$T_k(T_M(u_n) - \varphi) \rightharpoonup T_k(T_M(u) - \varphi) \quad \text{in} \quad W_0^{1,p}(\Omega).$$

The weak convergence in $W_0^{1,p}(\Omega)$ implies that

$$\nabla T_k(T_M(u_n) - \varphi) \rightharpoonup \nabla T_k(T_M(u) - \varphi) \quad \text{in} \quad L^p(\Omega).$$

This is indeed the case since

$$\nabla T_k(T_M(u_n) - \varphi) = \nabla(T_M(u_n) - \varphi)1_{\{|T_M(u_n)-\varphi|\leq k\}}$$

and since (3.8), we have that $\nabla T_M(u_n)$ converges weakly in $L^p(\Omega)$, so, up to a sub-sequence $T_M(u_n)$ converges almost everywhere to $T_M(u)$ in Ω. therefore,

$$\begin{aligned}\int_\Omega |\nabla T_M(u_n)|^{p-2}\nabla T_M(u_n)\nabla T_k(T_M(u_n) - \varphi) &= \int_\Omega |\nabla T_M(u_n)|^{p-2}\nabla T_M(u_n)\nabla T_M(u_n)1_{\{|T_M(u_n)-\varphi|\leq k\}} \\ &- \int_\Omega |\nabla T_M(u_n)|^{p-2}\nabla T_M(u_n)\nabla\varphi 1_{\{|T_M(u_n)-\varphi|\leq k\}}.\end{aligned}$$

Using (3.8) and dominated convergence , we have

$$\nabla\varphi 1_{\{|T_M(u_n)-\varphi|\leq k\}} \longrightarrow \nabla\varphi 1_{\{|T_M(u)-\varphi|\leq k\}} \quad \text{in} \quad L^p(\Omega).$$

Since $T_M(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ and by (3.3), we have

$$|\nabla T_M(u_n)|^{p-2}\nabla T_M(u_n) \quad \text{is bounded in} \quad L^{p'}(\Omega), \quad p' = \frac{p}{p-1}.$$

Thus, according to lemma 1.7, we deduce

$$|\nabla T_M(u_n)|^{p-2}\nabla T_M(u_n) \rightharpoonup |\nabla T_M(u)|^{p-2}\nabla T_M(u) \quad \text{in} \quad L^{p'}(\Omega)$$

Therefore,

$$\lim_{n\longrightarrow+\infty}\int_\Omega |\nabla T_M(u_n)|^{p-2}\nabla T_M(u_n)\nabla\varphi 1_{\{|T_M(u_n)-\varphi|\leq k\}} = \int_\Omega |\nabla T_M(u)|^{p-2}\nabla T_M(u)\nabla\varphi 1_{\{|T_M(u)-\varphi|\leq k\}}$$

The other term being non negative,, Fatou's lemma gives us (since we have shown the almost everywhere convergence of (∇u_n) which implies the convergence of $\nabla T_M(\nabla u_n)$)

$$\begin{aligned}\int_\Omega |\nabla T_M(u)|^{p-2}\nabla T_M(u)\nabla T_M(u) 1_{\{|T_M(u)-\varphi|\leq k\}} \\ \leq \liminf_{n\to\infty}\int_\Omega |\nabla T_M(u_n)|^{p-2}\nabla T_M(u_n)\nabla T_M(u_n) 1_{\{|T_M(u_n)-\varphi|\leq k\}}.\end{aligned}$$

Finally, it remains to pass to the limit in $\int_\Omega f_n T_k(u_n-\varphi)dx$. It is clear that

$$f_n \longrightarrow f \text{ in} \quad L^1(\Omega),$$

and as $|T_k(u_n-\varphi)| \leq k$, we have

$$T_k(u_n-\varphi) \text{ converges to} \quad T_k(u-\varphi) \quad \text{in} \quad L^\infty(\Omega) \text{ weak*}.$$

By dominated convergence, we find

$$\lim_{n\longrightarrow+\infty}\int_\Omega f_n T_k(u_n-\varphi)dx = \int_\Omega f T_k(u-\varphi)dx$$

Finally, we pass to the limit in the formulation (3.15), we obtain

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla T_k(u-\varphi) \leq \int_\Omega f T_k(u-\varphi).$$

Remark 3.3. *One can, in fact, prove the strong convergence of $\nabla T_h(u_n)$ toward $\nabla T_h(u)$ in $L^p(\Omega)$, which leads to an equality in the entropy formulation.*

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Abstract

In this work, we study the existence and regularity of a weak and entropy solutions of elliptic problem (P) defined by

$$(P) \qquad \begin{cases} Au &= f \text{ on } \Omega; \\ u &= 0 \text{ on } \partial\Omega, \end{cases}$$

the operator $Au = -\text{div}(\mid \nabla u \mid^{p-2} \nabla u), \quad 1 < p < \infty$ is a pseudo-monotone operator, $f \in M(\Omega)$ or $f \in L^1(\Omega)$. The method of solving our problem consist of obtaining local estimates for suitable approximate problems and then passing to the limit.

keywords: Sobolev spaces, pseudo-monotone, operator nonlinear, elliptic equation, weak solution, entropy solution.

Résumé

Dans ce travail, nous étudions l'existence et la régularité d'une solution faible et entropique d'un problème elliptique non régulière définie par

$$(P) \qquad \begin{cases} Au &= f \text{ sur } \Omega; \\ u &= 0 \text{ sur } \partial\Omega, \end{cases}$$

L'opérateur $Au = -\text{div}(\mid \nabla u \mid^{p-2} \nabla u), \quad 1 < p < \infty$ est un opérateur pseudo-monotone, $f \in M(\Omega)$ ou $f \in L^1(\Omega)$. Les étapes principales de la preuve consister à approcher par une suite de problèmes à donnée dans L^∞, ensuite obtenir des estimations uniformes et locales pour la suite des solutions approchées u_n et ∇u_n, puis le passage à la limite.

mots-clés: Espace de Sobolev, pseudo-monotone, opérateur non linéaire, équation elliptique, solution faible, solution entropique .

ملخص

في هذه المذكرة نسعى الى دراسة وجود وصقالة الحلول الضعيفة و الانتروبية للمعادلة التناقصية وغير الخطية المعرفة كما يلي: $Au = f$ تحت الشرط التالي $u = 0$ على الحافة حيث ان الطرف الايمن f ينتمي الى الفضاء $L^1(\Omega)$ أو الى $M(\Omega)$ المؤثر $Au = -\mathrm{div}(\mid \nabla u \mid^{p-2} \nabla u), \quad 1 < p < \infty$ معرف بشكل جيد بين الفضاء $W_0^{1,p}$وفضاءه الثنوي، الخطوات الرئيسية للاثبات تتمحور فيمايلي:

نقرب الجملة (P) بمتتالية جمل(P_n) ثم نبرهن ان الجملة المقربة تتمتع بحل ضعيف محدود u_n ثم الحصول على التقديرات لمتتالية الحلول التقريبية u_n و ∇u_n. وفي الاخير نمر بالنهاية في المسالة التقريبية للحصول على حل للجملة (P)

كلمات مفتاحية: معادلات تناقصية غير خطية ، مؤثر غير خطي، فضاء سوبولاف، حل ضعيف، حل انتروبي