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Theme

Types of fuzzy ideals and filters on a lattice

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خلاصة

في هذه المذكرة، قمنا بدراسة مفهوم المثاليات والمرشحات الضبابية داخل شبكة كتعميم لمفهوم المثاليات والمرشحات الكلاسيكية.

كما قمنا بدراسة أنواع هذه المجموعات الخاصة مثل المثاليات والمرشحات الأولية الضبابية، والمثاليات والمرشحات الأعظمية الضبابية وأيضا المثاليات والمرشحات الأساسية الضبابية .

كلمات مفتاحية :مجموعة ضبابية، شبكة، مثالي، مرشح، مثالي(مرشح) اولي، مثالي (مرشح) اعظمي، مثالي (مرشح) اساسي.

Abstract

In this memory, we have studied the concept of fuzzy ideals and filters on lattice as a generalization of the concept of crisp ideals and filters. Moreover, we studied some types of these ideals and filters such as prime fuzzy ideals (filters), maximal fuzzy ideals (filters), and principal fuzzy ideals (filters).

Keywords: fuzzy sets, lattice, ideal, filter, prime ideal (filter), maximal ideal (filter), principal ideal (filter).

Résumé

Dans cette mémoire, nous avons étudié les concepts d'idéaux et des filtres flous dans un treillis comme une généralisation des concepts d'idéaux et des filtres classiques. Nous avons également étudié quelques types de ces d'idéaux et des filtres tels que les idéaux (filtres) flous premiers, les idéaux (filtres) flous maximaux , et les idéaux (filtres) flous principaux.

Mots-clés: ensembles flous, treillis, idéal, filtre, idéal premier (filtre), idéal maximal (filtre), idéal principal (filtre).

Types of fuzzy ideals and filetrs on a lattice

INES *HAMDI-PACHA*

September 25, 2020

بسم الله الرحمن الرحيم

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Contents

Introduction

The idea of a class of sets with a continuum the grade of membership, ranging between zero and one, was first introduced by Zadeh [29] in 1965. A larger degree of membership of an object reflects a stronger sense of belonging to a set. If A is a set in the ordinary sense of the term, then its membership takes only two values, 0 and 1. The notions of inclusion, union, intersection, complement, relation and convexity can be extended to such sets. Fuzzy logic has been used in numerous applications such as facial pattern recognition, air conditioners, knowledge-based systems for multi objective optimization of power systems, weather forecasting systems, medical diagnosis and treatment plans, and stock trading. Fuzzy logic has been successfully used in numerous fields such as control systems engineering, image processing, power engineering, industrial automation, robotics, consumer electronics, optimization and so on.

The notions of ideals and filters are one of the most important concepts in the lattices theory. These notions are mainly used to translate connections between properties on algebraic structures and to define congruence relations and quotient algebras [28, 23]. They are played a central role in the Stone representation theorem for Boolean lattice [25] and in the representation of a distributive lattice [10, 13, 24]. Also, In topology like completeness and compactness in metric spaces [8]. In fuzzy setting, for the same purposes, several authors introduced and investigated the concepts of ideals and filters on the lattice [1, 7, 11, 15, 27], on BL-algebras [16], on ordered ternary semigroups [2, 5, 9] and on fuzzy structures [17].

Due to the usefulness of these concepts, the first aim of this memory is to investigate fuzzy ideals and fuzzy filters on a crisp lattice. The second aim is to study some types of fuzzy ideals and filters on a crisp lattice. Moreover, we show the relationship between them.
This memory is structured as follows.

- **In chapter 1, we recall generalities on fuzzy sets, t-norms and t-conorms, and the Cartesian product and projection of fuzzy subsets.**

• In chapter 2, we investigate the notion of fuzzy ideal and fuzzy filter on a crisp lattice and some of their fundamental properties. We present interesting characterizations of these notions in terms of lattice operations and in terms of their α-level sets.

• In chapter 3, we treat the notion of prime fuzzy ideal (resp. filter) on a lattice, the notion of maximal fuzzy ideal (resp. filter), and the notion of principal fuzzy ideal (resp. filter) on a crisp lattice with their characterizations and properties.

Chapter 1

Generalities on fuzzy sets

The purpose of this first chapter is to provide a basic introduction to the fuzzy sets, operations on fuzzy sets, characteristics of fuzzy sets, triangular norms and conorms, cartesian product and projection of fuzzy sets. Many of the properties of these concepts will be used in the next chapters.

1.1 Definitions

By a crisp set, or a classical set, or simply a set we mean a collection of distinct well-defined objects. These objects are said to be elements or members of the set. We usually denote the sets by capital letters A, B, C, etc., and the members by a, b, c, etc. To denote a is an element of A we write $a \in A$. The negation of $a \in A$ is written $a \notin A$ and means that a does not belong to A. A set with no elements is called an empty set and will be denoted by ϕ.

A set A with element $a_1, a_2, ..., a_n$ is denoted by $A = \{a_1, a_2, ..., a_n\}$ and in this case, we say that A is a finite. There are several ways to denote sets describing their elements. For instance, the set of even natural numbers is denoted by $E = \{2, 4, 6, ...\}$ or equivantely $E = \{2k | k \in \mathbb{N}\}$ where $\mathbb{N}$ is the set of natural numbers. A set A is infinite or has an infinitely many elements, if it is not finite. A set is denumerable if it is in a one-to-one correspondence with the set of natural numbers. A set is countable if it is finite or denumerable.

When we talking about the sets, it is assumed that all sets are subsets of a given set called universal set, usually denoted by X. Then a universal set is a set which contains all the possible elements we need for a particular discussion or application.

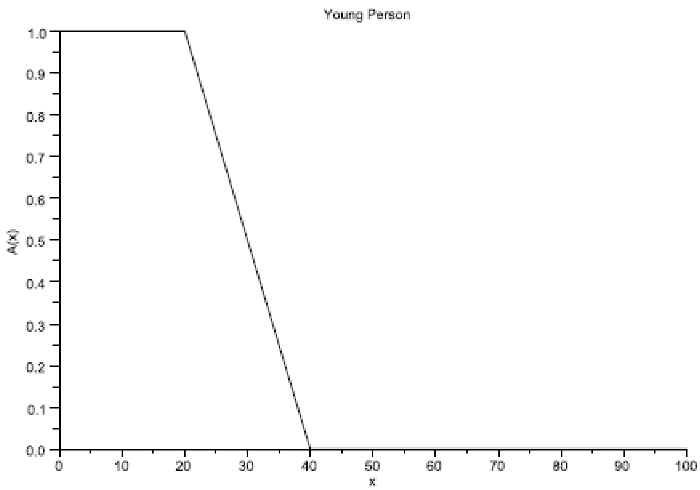
1.1.1 Fuzzy sets

The notion of fuzzy sets was first introduced by Zadeh [29].

Definition 1.1. [29] *Let X be a nonempty set. A fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ is characterized by a membership function $\mu_A : X \to [0,1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for each $x \in X$.*

Example 1.1. *In this example, we consider the expression "young" in the context "a young person" in order to exemplify how linguistic expression can be modeled using fuzzy sets. The fuzzy set $A : [0,100] \to [0,1]$,*

$$A(x) = \begin{cases} 1 & if \quad 0 \leq x \leq 20 \\ \frac{40-x}{20} & if \quad 20 \leq x \leq 40; \\ 0 & if \quad otherwise \end{cases}$$



Example of a fuzzy set for modeling the expression young person.

Example 1.2. *Let us consider the fuzzy set $A : \mathbb{R} \to [0,1], A(x) = \frac{1}{1+x^2}$. This fuzzy set can model the linguistic "real numbers near 0".*

Example 1.3. *Fuzzy sets can be used to express subjective perceptions in a mathematical form. Let $X = [40, 100]$ be the interval of temperatures for a room. Fuzzy sets $A_1, A_2, ..., A_5$ can be*

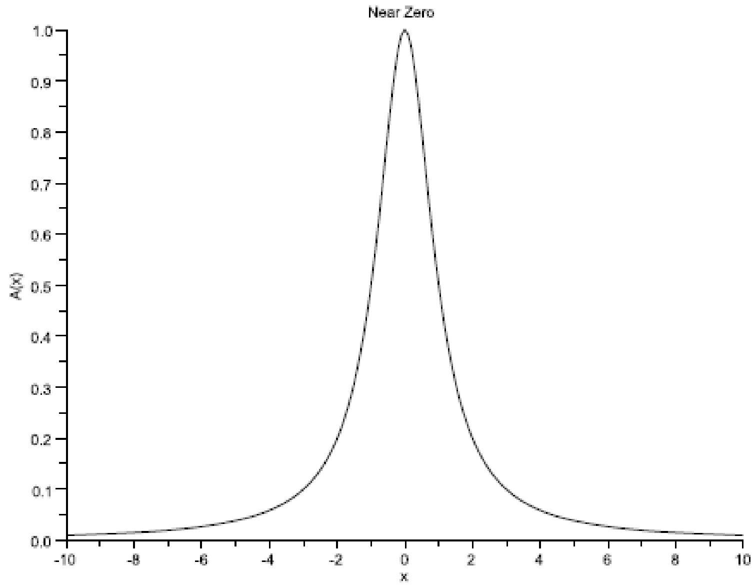


Figure 1.1: Fuzzy set that models a real number near 0.

used to model the perception: cold, cool, just, right, warm, and hot.

cold:

$$A_1(x) = \begin{cases} 1 & if \quad 40 \leq x < 50 \\ \frac{60-x}{10} & if \quad 50 \leq x < 60 \\ 0 & if \quad 60 \leq x \leq 100 \end{cases}$$

cool:

$$A_2(x) = \begin{cases} 0 & if \quad 40 \leq x < 50 \\ \frac{x-50}{10} & if \quad 50 \leq x < 60 \\ \frac{70-x}{10} & if \quad 60 \leq x < 70 \\ 0 & if \quad 70 \leq x \leq 100 \end{cases}$$

hot:

$$A_{(5)}(x) = \begin{cases} 0 & if \quad 40 \leq x < 80 \\ \frac{x-80}{10} & if \quad 80 \leq x < 90 \\ 1 & if \quad 90 \leq x \leq 100 \end{cases}$$

1.1.2 Operations of fuzzy sets

Several operations on fuzzy sets are defined (see, e.g., [1, 6, 29]). Here we will present only those which are related to the present work.

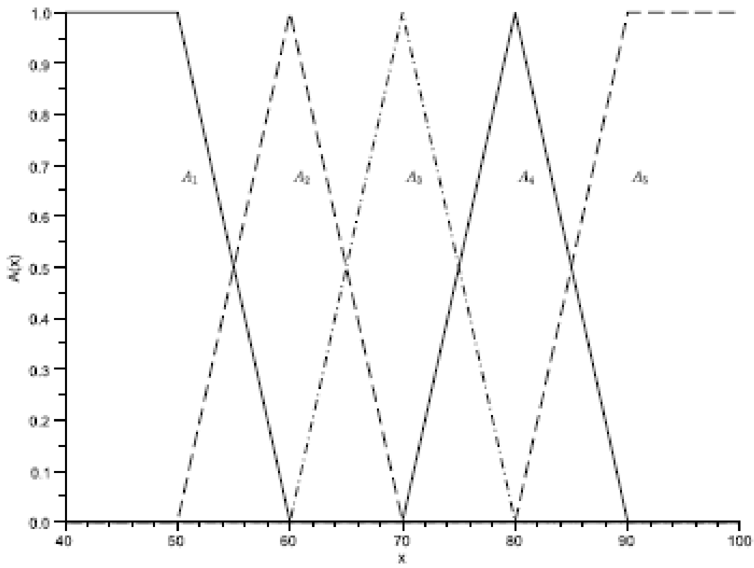


Figure 1.2: Temperature example.

Definition 1.2. ***(Inclusion)*** *Let $A, B \in F(X)$. We say that the fuzzy set A is included in B if*

$$A(x) \leq B(x), \forall x \in X.$$

We denote $A \leq B$. The empty fuzzy set $\emptyset$ is defined as $\emptyset(x) = 0$, $\forall x \in X$, and the total set X is $X(x) = 1$, $\forall x \in X$.

Definition 1.3. ***(Intersection)*** *Let $A, B \in F(X)$. The intersection of A and B is the fuzzy set C with*

$$C(x) = min\{A(x), B(x)\} = A(x) \wedge B(x), \forall x \in X.$$

We denote $C = A \wedge B$.

Definition 1.4. ***(Union)*** *Let $A, B \in F(X)$. The union of A and B is the fuzzy set C, where*

$$C(x) = max\{A(x), B(x)\} = A(x) \vee B(x), \forall x \in X.$$

We denote $C = A \vee B$.

Definition 1.5. ***(Complementation)*** *Let $A, B \in F(X)$ be a fuzzy set. The complement of A is the fuzzy set B where*

$$B(x) = 1 - A(x), \forall x \in X.$$

We denote $B = \bar{A}$.

Example 1.4. *(Finite case) Let $X = \{1, 2, 3, 4\}$, and let A, B two fuzzy subsets in X given by:*

$$\begin{aligned} A &= \{(1,0.2);(2,0.4);(3,0.1);(4,0.8)\}; \\ B &= \{(1,0.3);(2,0.7);(3,0.1);(4,0.5)\}. \end{aligned}$$

So, we get:

$$\begin{aligned} A \cap B &= \{(1,0.2);(2,0.4);(3,0.1);(4,0.5)\}; \\ A \cup B &= \{(1,0.3);(2,0.7);(3,0.1);(4,0.8)\}; \\ \overline{A} &= \{(1,0.8);(2,0.6);(3,0.9);(4,0.2)\}; \\ \overline{B} &= \{(1,0.7);(2,0.3);(3,0.9);(4,0.5)\}. \end{aligned}$$

Example 1.5. *(Infinite case) [6] If we consider the fuzzy sets*

$$A_1(x) = \begin{cases} 1 & if \quad 40 \le x < 50 \\ 1 - \frac{x-50}{10} & if \quad 50 \le x < 60 \\ 0 & if \quad 60 \le x \le 100 \end{cases},$$

$$A_2(x) = \begin{cases} 0 & if \quad 40 \le x < 50 \\ \frac{x-50}{10} & if \quad 50 \le x < 60 \\ 1 - \frac{x-60}{10} & if \quad 60 \le x < 70 \\ 0 & if \quad 70 \le x \le 100 \end{cases}$$

then their union is

$$A_1 \vee A_2(x) = \begin{cases} 1 & if \quad 40 \le x < 50 \\ 1 - \frac{x-50}{10} & if \quad 50 \le x < 55 \\ \frac{x-50}{10} & if \quad 55 \le x \le 60 \\ 1 - \frac{x-60}{10} & if \quad 60 \le x \le 70 \\ 0 & if \quad 70 < x \le 100 \end{cases}$$

The intersection can be expressed as

$$A_1(x) \wedge A_2(x) = \begin{cases} 0 & if \quad 40 \le x < 50 \\ \frac{x-50}{10} & if \quad 50 \le x < 55 \\ 1 - \frac{x-50}{10} & if \quad 55 \le x \le 60 \\ 0 & if \quad 60 < x \le 100 \end{cases}$$

The complement of A_1 can be written

$$\bar{A}_1(x) = \begin{cases} 0 & si \quad 40 \leq x < 50 \\ \frac{x-50}{10} & si \quad 50 \leq x < 60 \\ 1 & si \quad 60 \leq x \leq 100 \end{cases}$$

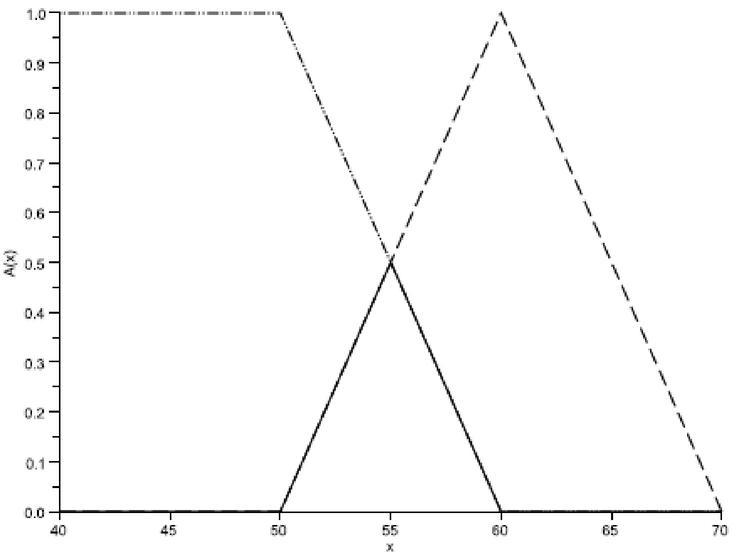


Figure 1.3: Fuzzy Intersection

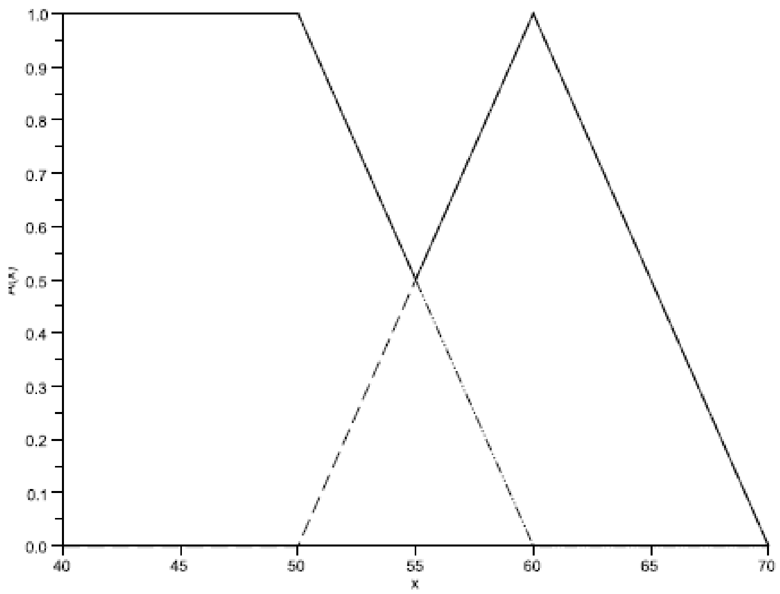


Figure 1.4: Union of two fuzzy sets

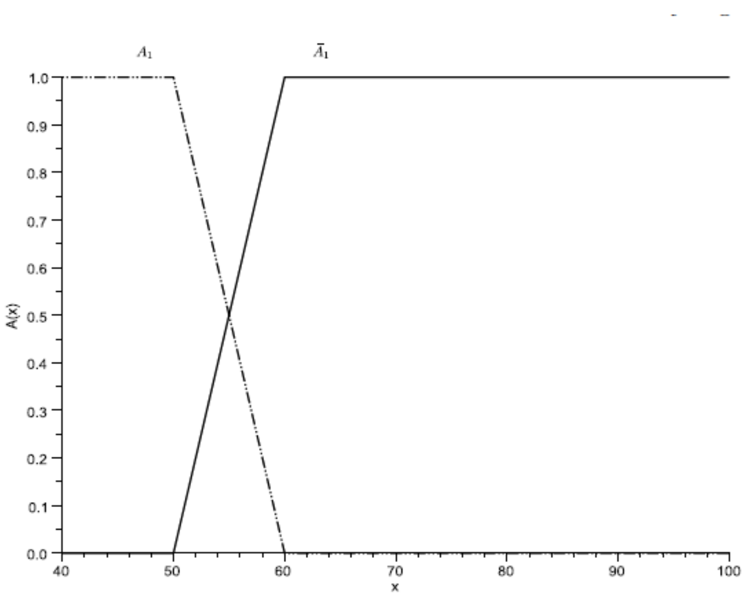


Figure 1.5: The complement of a fuzzy set

Proposition 1.1. *[6] Considering the basic connectives in fuzzy set theory, the following properties hold*

(i) Associativity

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C$$

$$A \vee (B \vee C) = (A \vee B) \vee C$$

(ii) Commutativity

$$A \vee B = B \vee A$$

$$A \wedge B = B \wedge A$$

(iii) Identity;

$$A \wedge X = A$$

$$A \vee \emptyset = A$$

(iv) Absorption by $\emptyset$ and X

$$A \wedge \emptyset = \emptyset$$

$$A \vee X = X$$

(v) Idempotence

$$A \wedge A = A$$

$$A \vee A = A$$

(vi) De Morgan Laws

$$\overline{A \wedge B} = \bar{A} \vee \bar{B}$$

$$\overline{A \vee B} = \bar{A} \wedge \bar{B}$$

(vii) Distributivity

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

(viii) Involution

$$\bar{\bar{A}} = A$$

(ix) Absorption

$$A \wedge (A \vee B) = A$$

$$A \vee (A \wedge B) = A$$

Remark 1.1. *If A is a fuzzy set $A : X \longrightarrow [0, 1]$. Then*

$$A \wedge \bar{A} \neq \emptyset$$

$$A \vee \bar{A} \neq X.$$

1.1.3 Characteristics of fuzzy sets

In this section, we recall the definitions for some characteristics of fuzzy sets: level sets of a fuzzy set, support of a fuzzy set, kernel of a fuzzy set.

Definition 1.6. [29] *Let $A : X \longrightarrow [0, 1]$ be a fuzzy set. The **level sets** of A are defined as the classical sets*

$$A_\alpha = \{x \in X | A(x) \geq \alpha\},$$

$0 < \alpha \leq 1$.

$$A_1 = \{x \in X | A(x) = 1\}$$

*is called the **kernel** of the fuzzy set A, while*

$$SuppA = \{x \in X | A(x) > 0\}$$

*is called the **support** of the fuzzy set A.*

Example 1.6. *Let us consider the cool fuzzy set as in the previous example.*

$$A_2(x) = \begin{cases} 0 & if \quad 40 \leq x < 50 \\ \frac{x-50}{10} & if \quad 50 \leq x < 60 \\ 1 - \frac{x-60}{10} & if \quad 60 \leq x < 70 \\ 0 & if \quad 70 \leq x \leq 100 \end{cases}$$

Its kernel is $(A_2)_1 = \{60\}$*, the* $\frac{1}{2}$*-level set is* $(A_2)_{\frac{1}{2}} = [55, 65]$*, the* α*-level set is* $(A_2)_\alpha = [50 + 10\alpha, 70 - 10\alpha]$*,* $0 < \alpha \leq 1$ *and the support is* $SuppA_2 = (50, 70)$*.*

Remark 1.2. *If the universe of discourse is a finite set* $X = \{x_1, x_2, ..., x_n\}$ *then a fuzzy set* $A : X \to [0, 1]$ *can be represented formally as*

$$A = \frac{A(x_1)}{x_1} + \frac{A(x_2)}{x_2} + ... + \frac{A(x_n)}{x_n}.$$

Example 1.7. *[6] Let us consider the expression "good level in mathematics". This expression can be represented as a fuzzy set* $G : \{A, B, C, D, F\} \longrightarrow [0, 1]$*,* $G = \frac{1}{A} + \frac{0.7}{B} + \frac{0.3}{C} + \frac{0}{D} + \frac{0}{F}$*. The kernel of* G *is* $G_1 = \{A\}$*, the support is* $SuppG = \{A, B, C\}$ *and the* $\frac{1}{2}$*-level set is* $G_{\frac{1}{2}} = \{A, B\}$*.*

Proposition 1.2. *The kernel and the support of a fuzzy subset verify the following properties:*

(i) $Supp(A^c) = X - Ker(A)$;

(ii) $Ker(A^c) = X - Supp(A)$.

Proof. (i)

$$\begin{aligned} Supp(A^c) &= \{x \in X | A^c(x) \neq 0\}; \\ &= \{x \in X | 1 - A(x) \neq 0\}; \\ &= \{x \in X | A(x) \neq 1\}; \\ &= \{x \in X | x \notin Ker(A)\}; \\ &= X - Ker(A). \end{aligned}$$

(ii)

$$\begin{aligned} Ker(A^c) &= \{x \in X | A^c(x) = 1\}; \\ &= \{x \in X | 1 - A(x) = 1\}; \\ &= \{x \in X | A(x) = 0\}; \\ &= \{x \in X | x \notin Supp(A)\}; \\ &= X - Supp(A). \end{aligned}$$

□

1.2 T-normes and T-conorms

Triangular norms and conorms generalize the basic connectives between fuzzy sets. They were first introduced in the theory of probabilistic metric spaces. Later these were found to be very suitable to be used with fuzzy sets.

Definition 1.7. [22] ***(T-norm)*** *A t-norm T on $[0,1]$ is a function $T : [0,1]^2 \longrightarrow [0,1]$ satisfies the following four axioms:*

(T1) Commutativity: $(\forall x, y \in [0,1])(T(x,y) = T(y,x))$;

(T2) Associativity: $(\forall x, y, z \in [0,1])(T(x, T(y,z) = T(T(x,y),z)))$;

(T3) Monotonicity: $(\forall x, y, z \in [0,1])(x \leq y \Rightarrow T(x,z) \leq T(y,z)$;

(T4) Boundary condition: $(\forall x \in [0,1])(T(x,1) = x)$.

Condition $(T4)$ and $(T3)$ imply that for any t-norm T it holds that $T(x,y) \leq x, T(x,y) \leq y, T(x,y) \leq Min(x,y)$ and $T(x,0) = 0$.

Example 1.8. *The following four operations are the most common t-norms:*

(T5) Minimum: $T_M(x,y) = min\{x,y\}$

(T6) Product: $T_P(x,y) = x.y$

(T7) Lukasiewicz: $T_L(x,y) = max\{x+y-1, 0\}$

(T8) Drastic product:

$$T_D(x,y) = \begin{cases} x & if \quad y=1 \\ y & if \quad x=1 \\ 0 & if \quad x,y<1. \end{cases}$$

Triangular conorms (t-conorms) are dual operations of t-norms, we recall the following definition of conorms.

Definition 1.8. *[22]* ***(T-conorm)*** *A t-conorm S is a function* $S:[0,1]^2 \longrightarrow [0,1]$ *that for any* $x,y,z \in [0,1]$ *satisfies* $(T1)-(T3)$ *and the following boundary condition*

$$S(x,0) = S(0,x) = x, S(x,1) = S(1,x) = 0.$$

Remark 1.3. *[19] Given a t-norm T, we find the associated dual t-conorm S by* $S(x,y) = 1 - T(1-x, 1-y)$.

The dual t-conorms w.r.t. T_M, T_P, T_L *and* T_D *are given by:*

(S1) Maximum: $S_M(x,y) = max\{x,y\}$

(S2) Probabilistic sum: $S_P(x,y) = x+y-xy$

(S3) Lukasiewicz: $S_L(x,y) = min\{x+y,1\}$

(S4) Drastic sum:

$$S_D(x,y) = \begin{cases} 1 & if \quad (x,y) \in [0,1]^2 \\ max\{x,y\}, & otherwise \end{cases}$$

1.3 Cartisian product and Projection of fuzzy subsets

The Cartesian product of the fuzzy subsets is the minimum of these degrees of belonging and these projections is the maximum of these Cartesian products.

Definition 1.9. ***(Cartesian product of fuzzy subsets)*** *Let the fuzzy subsets $A_1, A_2, ..., A_n$ respectively defined on $X_1, X_2, ..., X_n$, we define their Cartesian product $A = A_1 \times A_2 \times ... \times A_n$, as a fuzzy subset of X of membership function defined for all $x = (x_1, x_2, ..., x_n) \in X$ by:*

$$\mu_A(x) = min(\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)).$$

Example 1.9. *Lets $X_1 = \{a, b, c, d\}, X_2 = \{\alpha, \beta\}$ and lets A_1, A_2 two fuzzy subset respectively defined on X_1 and X_2 given by:*

$$\begin{aligned} A_1 &= \{< a, 0.1 >; < b, 0.4 >; < c, 0.8 >; < d, 0.5 >\}; \\ A_2 &= \{< \alpha, 0.2 >; < \beta, 0.6 >\}. \end{aligned}$$

So, we get:

$A_1 \times A_2 = \{< (a, \alpha), 0.1 >; < (a, \beta), 0.1 >; < (b, \alpha), 0.2 >; < (b, \beta), 0.4 >; < (c, \alpha), 0.2 >; < (c, \beta), 0.6 >\}; < (d, \alpha), 0.2 >\}; < (d, \beta), 0.5 >\}$

Definition 1.10. ***(Projection of fuzzy subsets)*** *The projection on X_1 of the fuzzy set A of $X_1 \times X_2$ is the fuzzy set $Proj_{X_1}(A)$ of X_1, whose membership function is defined by: $\forall x_1 \in X_1$,*

$$\mu_{ProjX_{1(A)}}(x_1) = sup_{x_2 \in X_2}(\mu_A(x_1, x_2)).$$

We define analogously the projection of A on X_2

Example 1.10. *Let $X = X_1 \times X_2$ the set of reference such that X_1 and X_2 two sets of previous example, we consider $A_1 \times A_2 = A$ given by:*

$A = \{< (a, \alpha), 0.1 >; < (a, \beta), 0.1 >; < (b, \alpha), 0.2 >; < (b, \beta), 0.4 >; < (c, \alpha), 0.2 >; < (c, \beta), 0.6 >\}; < (d, \alpha), 0.2 >\}; < (d, \beta), 0.5 >\}.$

So, we get:

$$\begin{aligned} Proj_{X_1}(A) &= \{< a, max(0.1, 0.1) >; < b, max(0.2, 0.4) >; < c, max(0.2, 0.6) >; < d, max(0.2, 0.5) >\}; \\ &= \{< a, 0.1 >; < b, 0.4 >; < c, 0.6 >; < d, 0.5 >\}. \\ Proj_{X_2}(A) &= \{< \alpha, max(0.1, 0.2, 0.2, 0.6) >; < \beta, max(0.1, 0.4, 0.6, 0.5) >\}; \\ &= \{< \alpha, 0.6 >; < \beta, 0.6 >\}. \end{aligned}$$

Chapter 2

Fuzzy lattices

In this chapter, we recall the basic definitions and properties of fuzzy lattices and fuzzy ideals (resp. filters) and some related notions that will be needed throughout the next chapter.

2.1 Definitions

The concept of a fuzzy lattice on a lattice was introduced by Ajmal and Thomas [1] as a fuzzy set on a crisp lattice stable lattice by the supremum and the infimum of the binary operation meet a join.
To avoid any confusion or misunderstanding in some formulas, we use the notation $(\leq, \sqcap, \sqcup)$ to refer the (order, min, max) on the lattice L and $(\leq, \wedge, \vee)$ to refer the (usual order, min, max) on the real interval $[0, 1]$.

Definition 2.1. [1] *Let L be a lattice and $A = \{\langle x, \mu_A(x)\rangle | x \in L\}$ be a fuzzy subset on L. Then A is called a fuzzy sub-lattice (fuzzy lattice, for short) if for any $x, y \in L$, the following conditions are satisfied:*

(i) $\mu_A(x \sqcup y) \geq \mu_A(x) \wedge \mu_A(y)$;

(ii) $\mu_A(x \sqcap y) \geq \mu_A(x) \wedge \mu_A(y)$.

Example 2.1. *Figure 2.1 shows the Hasse diagram of a lattice $L = \{0, a, b, 1\}$. The fuzzy set A on L given by $A = \{< 0, 0.5 >, < a, 0.4 >, < b, 0.4 >, < 1, 0.7 >\}$ is a fuzzy lattice.*

2.2 Fuzzy ideals and filters on a lattice

The notion of fuzzy ideal (resp. filter) on a lattice was first introduced by Ajmal and Thomas [1].

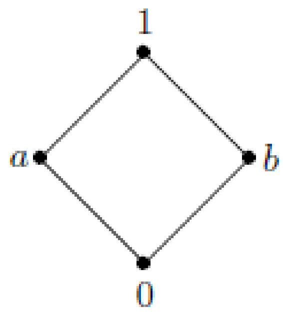


Figure 2.1: Hasse diagram of a lattice $(L, \leq, \sqcap, \sqcup)$ with $L = \{0, a, b, 1\}$.

Definition 2.2. [1] *Let L be a lattice and $I = \{\langle x, \mu_I(x)\rangle | x \in L\}$ be a fuzzy subset on L. Then I is called a fuzzy ideal on L (F-ideal, for short) if for all $x, y \in L$, the following conditions are satisfied:*

(i) $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$;

(ii) $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$.

Example 2.2. *Let L be the lattice given by the Hasse diagram in Figure 2.1.The fuzzy set I on L defined by $I = \{< 0, 0.5 >, < a, 0.4 >, < b, 0.1 >, < 1, 0.1 >\}$ is a fuzzy ideal.*

Definition 2.3. [1] *Let L be a lattice and $F = \{\langle x, \mu_F(x)\rangle | x \in L\}$ be a fuzzy subset on L. Then F is called a fuzzy filter on L (F-filter, for short) if for any $x, y \in L$, the following conditions are satisfied:*

(i) $\mu_F(x \sqcup y) \geq \mu_F(x) \vee \mu_F(y)$;

(ii) $\mu_F(x \sqcap y) \geq \mu_F(x) \wedge \mu_F(y)$.

Example 2.3. *Let L be the lattice given by the Hasse diagram in Figure 2.1.The fuzzy set F on L defined by $F = \{< 0, 0.1 >, < a, 0.2 >, < b, 0.1 >, < 1, 0.4 >\}$ is a fuzzy filter.*

Remark 2.1. *Notice that every fuzzy ideal on L is a fuzzy lattice, but the converse is not true in general. Indeed, let L be the lattice given by the Hasse diagram in Figure 2.1 and $A \in FS(L)$ defined by $A = \{< 0, 0.3 >, < a, 0.4 >, < b, 0.4 >, < 1, 0.7 >\}$. Then A is a fuzzy lattice, but since $\mu_A(a) = \mu_A(a \sqcap 1) = 0,4 \not\geq max\{0.4; 0.7\}$, then it holds that A is not a fuzzy ideal on L. As well since $\mu_A(0) = \mu_A(a \sqcap b) = 0.3 \not\geq min\{0.4; 0.4\}$, then it holds that A is not a fuzzy filter on L.*

The following results will be needed throughout this chapter.

Proposition 2.1. *Let L be a lattice, L^d be its order-dual lattice and $A \in FS(L)$. Then it holds that A is a fuzzy ideal on L if and only if A is a fuzzy filter on L^d and conversely.*

Proposition 2.2. [26] *Let L be a lattice, A and B are two fuzzy sets on L. Then it holds that*

(i) if A and B are two fuzzy ideals on L, then $A \cap B$ is a fuzzy ideal on L;

(ii) if A and B are two fuzzy filters on L, then $A \cap B$ is a fuzyy filter on L.

Proof. (i) Let $A = \{< x, \mu_A(x) > |x \in L\}$ and $B = \{< x, \mu_B(x) > |x \in L\}$ be two fuzzy sets. Then $A \cap B = \{< x, \mu_{A\cap B}(x) > |x \in L\}$, where $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$.

$$\begin{aligned}\mu_{A\cap B}(x \sqcup y) &= \{\mu_A(x \sqcup y) \wedge \mu_B(x \sqcup y)\}\\ &\geq \{\{\mu_A(x) \wedge \mu_A(y)\} \wedge \{\mu_B(x) \wedge \mu_B(y)\}\}\\ &\geq \{\{\mu_A(x) \wedge \mu_B(x)\} \wedge \{\mu_A(y) \wedge \mu_B(y)\}\}\\ &\geq \mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y).\end{aligned}$$

By that same method, we prove that $\mu_{A\cap B}(x \sqcap y) \geq \mu_{A\cap B}(x) \vee \mu_{A\cap B}(y)$. Hence, $A \cap B$ is a fuzzy ideal on L.

(ii) Follows from Proposition 2.2 and (i).

□

Remark 2.2. *The Union of two fuzzy ideals (resp. filters) does not necessarily be a fuzzy ideal (resp. filter).*

2.3 Characterisation of fuzzy ideals and filters on a lattice

In this section, we provide interesting characterizations of fuzzy ideals and filters on a lattice in terms of the lattice operations, and in terms of their α-level sets.

2.3.1 Characterisation of fuzzy ideals and filters in terms of lattice operations

Milles et al. [20] have characterized the notion of fuzzy ideals and fuzzy filters on a lattice in terms of the lattice-operations. Here, we use this characterization in a fuzzy setting.

Theorem 2.1. [20] *Let L be a lattice and $A \in$ fuzzy subset (L). Then for any $x, y \in L$ the following two statements hold*

(i) $(\mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y))$ *if and only if* $(x \leq y \Rightarrow \mu_A(x) \geqslant \mu_A(y))$;

(ii) $(\mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y))$ *if and only if* $(x \leq y \Rightarrow \mu_A(x) \leq \mu_A(y))$.

As corollaries, we obtain the following interesting properties of fuzzy ideals and fuzzy filters.

Corollary 2.1. *Let L be a lattice and I be a fuzzy ideal on L. Then for any $x, y \in L$ it holds that*

If $x \leq y$, then $\mu_I(x) \geq \mu_I(y)$, (i.e., the map $\mu_I : L \to [0, 1]$ is antitone).

Remark 2.3. *The converse of the above implications are not necessarily hold. Indeed, let us consider the lattice L given by the Hasse diagram in Figure 2.1 and I the fuzzy ideal on L given by $I = \{< 0, 0.5 >, < a, 0.4 >, < b, 0.1 >, < 1, 0.1 >\}$. It is easy to verify that $\mu_I(a) = 0.4 \geq \mu_I(b) = 0.1$, but a, b are incomparable elements.*

Corollary 2.2. *Let L be a lattice and F be a fuzzy filter on L. Then for any $x, y \in L$ it holds that*

If $x \leq y$, then $\mu_F(x) \leq \mu_F(y)$, (i.e., the map $\mu_F : L \to [0, 1]$ is monotone).

Remark 2.4. *The converse of the above implications are not necessarily hold. Indeed, let us consider the lattice L given by the Hasse diagram in Figure 2.1 and F the fuzzy filter on L given by $F = \{< 0, 0.1 >, < a, 0.2 >, < b, 0.1 >, < 1, 0.4 >\}$. It is easy to verify that $\mu_F(b) = 0.1 \leq \mu_F(a) = 0.2$, but a, b are incomparable elements.*

In the following theorem, we apply the characterization theorem given by Milles et al. [20] to the fuzzy setting.

Theorem 2.2. *Let L be a lattice and I is a fuzzy subset on L. Then it holds that I is a fuzzy ideal on L if and only if the following condition is satisfied:*

$$\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$$

for $x, y \in L$.

Proof. Suppose that I is a fuzzy ideal on L. Then for any $x, y \in L$ it holds that $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$. Since $x \leq x \sqcup y$ and $y \leq x \sqcup y$, from corollary 2.1 it follows that

$$\mu_I(x) \geq \mu_I(x \sqcup y)$$

and

$$\mu_I(y) \geq \mu_I(x \sqcup y).$$

Hence, $\mu_I(x) \wedge \mu_I(y) \geq \mu_I(x \sqcup y)$. Thus, $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$.
Conversely, suppose that $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$, for any $x, y \in L$. Then it is easy to see that

$$\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$$

Next, we show that $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$ for $x, y \in L$. Let $x, y \in L$. Since $x \sqcup (x \sqcap y) = x$ and $y \sqcup (x \sqcap y) = y$ then it holds that $\mu_I(x \sqcup (x \sqcap y)) = \mu_I(x)$ and $\mu_I(y \sqcup (x \sqcap y)) = \mu_I(y)$. From Definition 2.2 (hypothesis (i) and (ii)) it follows that

$$\mu_I(x) \wedge \mu_I(x \sqcap y) = \mu_I(x)$$

and

$$\mu_I(y) \wedge \mu_I(x \sqcap y) = \mu_I(y).$$

Hence, $\mu_I(x \sqcap y) \geq \mu_I(x)$ and $\mu_I(x \sqcap y) \geq \mu_I(y)$. Thus, $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$, for any $x, y \in L$. Therefore, I is a fuzzy ideal on L.

□

In the same manner, the following theorem provides a basic characterization of fuzzy filter on a lattice.

Theorem 2.3. *Let L be a lattice and F is a fuzzy subset on L. Then it holds that F is a fuzzy filter on L if and only if the following condition is satisfied:*

$$\mu_F(x \sqcap y) = \mu_F(x) \wedge \mu_F(y)$$

for $x, y \in L$.

Proof. The proof is a direct application of Proposition 2.1 and Theorem 2.3.1 □

In the following theorem, we will show that the image of a fuzzy ideal (resp. fuzzy filter) is a fuzzy ideal (resp. fuzzy filter).

Theorem 2.4. *Let L, L' be two lattices and $f : L \to L'$ be a lattices-homomorphism. Then it holds that*

(i) If A is a fuzzy ideal on L, then $f(A)$ is a fuzzy ideal on L',

(ii) If A is a fuzzy filter on L, then $f(A)$ is a fuzzy filter on L'.

Proof. (i) Let A be a fuzzy ideal on L. For any $y, z \in L'$, it holds that

$$\begin{aligned} f(\mu_A)(y \sqcup z) &= \sup\{\mu_A(x) \mid x \in f^{-1}(y \sqcup z)\} \\ &= \sup\{\mu_A(u \vee v) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{(\mu_A(u) \wedge \mu_A(v)) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{\mu_A(u) \mid u \in f^{-1}(y)\} \wedge \sup\{\mu_A(v) \mid v \in f^{-1}(z)\} \\ &= f(\mu_A)(y) \wedge f(\mu_A)(z). \end{aligned}$$

Thus, we can conclude that $f(A)$ is a fuzzy ideal on L'.

(ii) Follows from Proposition 2.1 and (i).

□

In the following theorem, we will show that the inverse image of a fuzzy ideal (resp. fuzzy filter) is a fuzzy ideal (resp. fuzzy filter).

Theorem 2.5. *Let L, L' be two lattices and $f : L \to L'$ be a lattices-homomorphism. Then it holds that*

(i) If A' is a fuzzy ideal on L', then $f^{-1}(A')$ is a fuzzy ideal on L,

(ii) If A' is a fuzzy filter on L', then $f^{-1}(A')$ is a fuzzy filter on L.

Proof. (i) Let A' be a fuzzy ideal on L'. For any $x, y \in L$ it holds that

$$\begin{aligned} f^{-1}(\mu_{A'})(x \sqcup y) &= \mu_{A'}(f(x \sqcup y)) \\ &= \mu_{A'}(f(x)) \wedge \mu_{A'}(f(y)) \\ &= f^{-1}(\mu_{A'})(x) \wedge f^{-1}(\mu_{A'})(y) \end{aligned}$$

Therefore, $f^{-1}(A')$ is a fuzzy ideal on L.

(ii) Follows from Proposition 2.1 and (i).

□

2.3.2 Characterisation of fuzzy ideals and filters in terms of their level sets

In this subsection, we provide some interesting characterizations and properties of fuzzy ideals and fuzzy filters in terms of their level sets. These characterizations are direct results from the paper [20].

Proposition 2.3. *Let L be a lattice and A is a fuzzy subset on L. The following statements hold*

(i) if A is a fuzzy ideal, then its support $Supp(A)$ is an ideal on L;

(ii) if A is a fuzzy filter, then its support $Supp(A)$ is a filter on L.

Proof. Let $A \in$ fuzzy subset (L).

(i) Suppose that $A \in$ fuzzy subset (L) is a fuzzy ideal. We show that $Supp(A)$ is an ideal on L.

(a) Let $x \in Supp(A)$ and $y \leq x$, then it hold that $\mu_A(x) > 0$.
To consider that ($y \leq x$ and $\mu_A(x) > 0$). We suppose that $y \leq x$ and $\mu_A(x) > 0$. Since $y \leq x$, then it holds that $x \sqcup y = x$. This implies that $\mu_A(x) = \mu_A(x \sqcup y) > 0$. From Theorem 2.3.1 (i), it follows that $\mu_A(x \sqcup y) = \mu_I(x) \wedge \mu_I(y) > 0$. Hence, $\mu_A(y) > 0$. Thus, $y \in Supp(A)$.

(b) Let $x, y \in Supp(A)$. We show now that $x \sqcup y \in Supp(A)$. We have that $\mu_A(x) > 0$ and $\mu_A(y) > 0$. Since A is a fuzzy ideal, then from Theorem 2.3.1 (i) it follows that $\mu_A(x \sqcup y) = \mu_I(x) \wedge \mu_I(y) > 0$. Hence, $x \sqcup y \in Supp(A)$. Thus, $Supp(A)$ is an ideal on L.

(ii) Follows from Proposition 2.1 and (i).

□

Remark 2.5. *The converse of the above implications is not necessarily held. Indeed, let us consider the lattice L given by the Hasse diagram in Figure 2.1 and A is a fuzzy subset on L given by $A = \{< 0, 0.5 >, < a, 0.4 >, < b, 0.4 >, < 1, 0.7 >\}$. It is easy to verify that $Supp(A) = L$ is an ideal and a filter on L, but A is neither a fuzzy ideal nor a fuzzy filter on L.*

The following theorem provides a characterization of fuzzy ideal (resp. filter) in terms of their level sets.

Theorem 2.6. *Let L be a lattice and A is a fuzzy subset on L. The following statements hold*

(i) A is a fuzzy ideal if and only if their level set is an ideal on L;

(ii) A is a fuzzy filter if and only if their level set is a filters on L.

Proof. Let $A \in$ fuzzy subset on L and A_α their level set, where $\alpha \in [0, 1]$.

(i) suppose that A is a fuzzy ideal on L. We show that A_α is an ideal on L for $\alpha \in [0, 1]$.

(a) Let $\alpha \in [0, 1]$, $x \in A_\alpha$ and $y \in L$ such that $y \leq x$. Since $x \in A_\alpha$, then it holds that $\mu_A(x) \geq \alpha$. Since $y \leq x$, from Corollary 2.1 it follows that $\mu_A(y) \geq \mu_A(x)$. This implies that $\mu_A(y) \geq \alpha$. Hence, $y \in A_\alpha$, for any $\alpha \in [0, 1]$.

(b) Let $\alpha \in [0, 1]$ and $x, y \in A_\alpha$. Then it holds that $\mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \alpha$. From theorem 2.3.1 it follows that $\mu_A(x \sqcup y) = \mu_A(x) \wedge \mu_A(y) \geq \alpha$. Hence, $x \sqcup y \in A_\alpha$ for $\alpha \in [0, 1]$.

Thus, A_α is an ideal on L for $\alpha \in [0, 1]$.

Conversely, suppose that all level sets of A are ideals on L. We show that A is a fuzzy ideal on L. Let $x, y \in L$, $\alpha = \mu_A(x) \wedge \mu_A(y)$. Then it follows that $\mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \alpha$. The case $\alpha = 0$ is obvious. Let $\alpha \in [0, 1]$ and $x, y \in A_\alpha$. Since A_α is an ideal on L, then it holds that $x \sqcup y \in A_\alpha$, $\alpha \in [0, 1]$. This implies that $\mu_A(x \sqcup y) \geq \alpha$. Hence, $\mu_A(x \sqcup y) \geq \mu_A(x) \wedge \mu_A(y)$.

On other hand, let $\alpha = \mu_A(x \sqcup y)$. The case $\alpha = 0$ is also obvious. Otherwise $\alpha \in [0, 1]$, $x \sqcup y \in A_\alpha$. Since A_α is an ideal on L, $x \leq x \sqcup y$ and $y \leq x \sqcup y$, it follows that $\mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \alpha$. Hence, $\mu_A(x) \wedge \mu_A(y) \geq \mu_A(x \sqcup y)$. Therefore, Theorem 2.3.1 guarantees that A is a fuzzy ideal on L.

(ii) Follows from Proposition 2.1 and (i).

□

Chapter 3

Types of fuzzy ideals and filters on a lattice

In this chapter, we treat some types of fuzzy ideals (resp. filters) on a lattice such as prime fuzzy ideal (resp. filter), maximal fuzzy ideal (resp. filter) and principal fuzzy ideal (resp. filter). Moreover, we provide interesting characterizations of these types in terms of their α-level sets and support.

3.1 Prime fuzzy ideals (resp. filters) on a lattice

In this section, we introduce and characterize the notion of prime fuzzy ideal (resp. filter) on a lattice.

Definition 3.1. [11] *A fuzzy ideal I on a lattice L is called a prime fuzzy ideal if, for any $x, y \in L$,*

$$\mu_I(x \sqcap y) \leq \mu_I(x) \vee \mu_I(y).$$

Example 3.1. *Let L be a lattice given by the Hasse diagram in Figure 2.1. The fuzzy set I on L defined by $I = \{(0, 0.2), (a, 0.1), (b, 0.2), (1, 0.1)\}$ is a fuzzy prime ideal.*

Definition 3.2. [11] *A fuzzy filter F on a lattice L is called a prime fuzzy filter if for $x, y \in L$,*

$$\mu_F(x \sqcup y) \leq \mu_F(x) \vee \mu_F(y).$$

Example 3.2. *Let L be a lattice given by the Hasse diagram in Figure 2.1. The fuzzy set F on L defined by $F = \{(0, 0), (a, 1), (b, 0), (1, 1)\}$ is a fuzzy prime filter.*

Proposition 3.1. *Let $(A_i)_{i \in I}$ be a family of fuzzy sets on a lattice L. Then it holds that* *(i) If A_i is a prime fuzzy ideal on L, for any $i \in I$, then $\bigcap_{i\in I} A_i$ is a prime fuzzy ideal on L.*

(ii) If A_i is a prime fuzzy filter on L, for any $i \in I$, then $\bigcap_{i\in I} A_i$ is a prime fuzzy filter on L.

Proof. (i) Suppose that for any $i \in I$, A_i is a prime fuzzy ideal on L. From Proposition 2.2, it follows that $\bigcap_{i\in I} A_i$ is a fuzzy ideal on L. It remains to show that $\bigcap_{i\in I} A_i$ is prime. Let $x, y \in L$ such that $x \sqcap y \in \bigcap_{i\in I} A_i$. Then it follows that $x \sqcap y \in A_i$, for any $i \in I$. Since for any $i \in I$, A_i is a prime fuzzy ideal, it follows that

$$\mu_{A_i}(x \sqcap y) \leq \mu_{A_i}(x) \vee \mu_{A_i}(y),$$

for any $i \in I$. This implies that

$$\mu_{\bigcap_{i\in I} A_i}(x \sqcap y) \leq \mu_{A_i}(x \sqcap y) \leq \mu_{A_i}(x) \vee \mu_{A_i}(y),$$

and for any $i \in I$. Hence,

$$\mu_{\bigcap_{i\in I} A_i}(x \sqcap y) \leq \bigwedge_{i\in I} (\mu_{A_i}(x) \vee \mu_{A_i}(y)),$$

Thus,

$$\mu_{\bigcap_{i\in I} A_i}(x \sqcap y) \leq \mu_{\bigcap_{i\in I} A_i}(x) \vee \mu_{\bigcap_{i\in I} A_i}(y),$$

Therefore, $\bigcap_{i\in I} A_i$ is a prime fuzzy ideal on L.

(ii) Follows from Proposition 2.1 and (i).

□

A combination of Theorem 2.3.1 and Definition 2.2 leads to the following characterization of prime fuzzy ideals.

Proposition 3.2. *Let L be a lattice and $I \in$ fuzzy subset (L). Then it holds that I is a prime fuzzy ideal on L if and only if the following conditions hold:*

(i) $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$, for any $x, y \in L$;

(ii) $\mu_I(x \sqcap y) = \mu_I(x) \vee \mu_I(y)$, for any $x, y \in L$.

Similarly, Theorem 2.3 and Definition 2.2 leads to the following characterization of prime fuzzy filters.

Proposition 3.3. *Let L be a lattice and F is a fuzzy subset on L. Then it holds that F is a prime fuzzy filter on L if and only if the following conditions hold:*

(i) $\mu_F(x \sqcup y) = \mu_F(x) \vee \mu_F(y)$*, for any* $x, y \in L$*;*

(ii) $\mu_F(x \sqcap y) = \mu_F(x) \wedge \mu_F(y)$*, for any* $x, y \in L$*.*

The following propositions discuss the relationship between fuzzy ideal (resp. fuzzy filter) and its complement.

Proposition 3.4. *Let* L *be a lattice and* $A \in$ *fuzzy subset* (L)*. Then it holds that*

(i) A *is a prime fuzzy ideal if and only if* $\overline{A}$ *is a prime fuzzy filter on* L*.*

(ii) A *is a prime fuzzy filter if and only if* $\overline{A}$ *is a prime fuzzy ideal on* L*.*

Proof. (i) Suppose that A is a prime fuzzy ideal, then for any $x, y \in L$ it follows

$$\begin{aligned} \mu_{\overline{A}}(x \sqcup y) &= 1 - \mu_A(x \sqcup y) \\ &= 1 - (\mu_A(x) \wedge \mu_A(y)) \\ &= (1 - \mu_A(x)) \wedge (1 - \mu_A(y)) \\ &= \mu_{\overline{A}}(x) \wedge \mu_{\overline{A}}(y) \end{aligned}$$

By the same method, we get that $\mu_{\overline{A}}(x \sqcap y) = \mu_{\overline{A}}(x) \vee \mu_{\overline{A}}(y)$. Therefore, $\overline{A}$ is a prime fuzzy filter on L.

(ii) Follows from the fact that $A = \overline{\overline{A}}$ and (i).

□

In the following theorem, we will show that the image of a prime fuzzy ideal (resp. prime fuzzy filter) is a prime fuzzy ideal (resp. prime fuzzy filter).

Theorem 3.1. *Let* L*,* L' *be two lattices and* $f\colon L \to L'$ *be a lattices-homomorphism. Then it holds that*

(i) If A *is a prime fuzzy ideal on* L*, then* $f(A)$ *is a prime fuzzy ideal on* L'*,*

(ii) If A *is a prime fuzzy filter on* L*, then* $f(A)$ *is a prime fuzzy filter on* L'*.*

Proof. (i) Let A be a prime fuzzy ideal on L. Theorem 2.4 guarantees that $f(A)$ is a fuzzy ideal on L'. Next, we show that $f(A)$ is prime. The fact that A is a prime fuzzy ideal implies that for any $y, z \in L$,

$$\begin{aligned} f(\mu_A)(y \sqcap z) &= \sup\{\mu_A(x) \mid x \in f^{-1}(y \sqcap z)\} \\ &= \sup\{\mu_A(u \wedge v) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{(\mu_A(u) \vee \mu_A(v)) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{\mu_A(u) \mid u \in f^{-1}(y)\} \vee \sup\{\mu_A(v) \mid v \in f^{-1}(z)\} \\ &= f(\mu_A)(y) \vee f(\mu_A)(z). \end{aligned}$$

We conclude that $f(A)$ is a prime fuzzy ideal on L'.

(ii) Follows from Proposition 2.1 and (i).

□

In the following theorem, we will show that the inverse image of a prime fuzzy ideal (resp. prime fuzzy filter) is a prime fuzzy ideal (resp. prime fuzzy filter).

Theorem 3.2. *Let L, L' be two lattices and $f\colon L \to L'$ be a lattices-homomorphism. Then it holds that*

(i) If A' is a prime fuzzy ideal on L', then $f^{-1}(A')$ is a prime fuzzy ideal on L,

(ii) If A' is a prime fuzzy filter on L', then $f^{-1}(A')$ is a prime fuzzy filter on L.

Proof. (i) Let A' be a prime fuzzy ideal on L'. Theorem 2.5 guarantees that $f^{-1}(A')$ is a fuzzy ideal on L. Next, we show that $f^{-1}(A')$ is prime. Since A' is a prime fuzzy ideal, it follows that for any $x, y \in L$,

$$\begin{aligned} f^{-1}(\mu_{A'})(x \sqcap y) &= \mu_{A'}(f(x \sqcap y)) \\ &= \mu_{A'}(f(x)) \vee \mu_{A'}(f(y)) \\ &= f^{-1}(\mu_{A'})(x) \vee f^{-1}(\mu_{A'})(y) \end{aligned}$$

Therefore, we conclude that $f^{-1}(A')$ is a prime fuzzy ideal on L.

(ii) Follows from Proposition 2.1 and (i).

□

The following proposition shows that the support of a prime fuzzy ideal (resp. filter) on a lattice is a prime ideal (resp. filter) on that lattice.

Proposition 3.5. *Let L be a lattice and A is a fuzzy subset on L. Then it holds that*

(i) if A is a prime fuzzy ideal, then its support $Supp(A)$ is a prime ideal on L;

(ii) if A is a prime fuzzy filter, then its support $Supp(A)$ is a prime filter on L.

Proof. (i) Suppose that A is a prime fuzzy ideal on a lattice L. From Proposition 2.3.2, it holds that $Supp(A)$ is an ideal on L. Next, we prove that $Supp(A)$ is prime.
Let $x, y \in L$ such that $x \sqcap y \in Supp(A)$. Then $\mu_A(x \sqcap y) > 0$.

If $\mu_A(x \sqcap y) > 0$, then the fact that A is prime fuzzy ideal on L implies that

$$\mu_A(x) \vee \mu_A(y) = \mu_A(x \sqcap y) > 0.$$

This implies that either $\mu_A(x) > 0$ or $\mu_A(y) > 0$. Hence, either $x \in Supp(A)$ or $y \in Supp(A)$. Therefore, Supp(A) is a prime ideal on L.

(ii) Follows by using Proposition 2.1 and (i).

□

In the same manner, we get the following theorem which provides a characterization of prime fuzzy ideals (resp. filters) in terms of their level sets.

Theorem 3.3. *Let L be a lattice and A is a fuzzy subset on L. Then it holds that*

(i) A is a prime fuzzy ideal if and only if their level set is prime ideal;

(ii) A is a prime fuzzy filter if and only if their level set is prime filter.

Proof. (i) From Theorem 3.3, A is a fuzzy ideal on L if and only if A_α is an ideal on L, for $\alpha \in [0, 1]$. Suppose that A is a prime fuzzy ideal on L. Let $x, y \in L$ such that $x \sqcap y \in A_\alpha$. Then from Proposition 3.1, it follows that

$$\mu_A(x \sqcap y) = \mu_A(x) \vee \mu_A(y) \geq \alpha.$$

This implies that either $\mu_A(x) \geq \alpha$ or $\mu_A(y) \geq \alpha$. Hence, either $x \in A_\alpha$ or $y \in A_\alpha$. Thus, A_α is a prime ideal for any $\alpha \in [0, 1]$.
Conversely, suppose that A_α is a prime ideal for any $\alpha \in [0, 1]$ and A is not a prime fuzzy ideal on L. Then it hold that there exist $x, y \in L$ such that $\mu_A(x \sqcap y) > \mu_A(x) \vee \mu_A(y)$. This imply that $\mu_A(x \sqcap y) > \mu_A(x)$ and $\mu_A(x \sqcap y) > \mu_A(y)$. If we put $\mu_A(x \sqcap y) = \alpha$, then it follows that $\mu_A(x) < \alpha$ and $\mu_A(y) < \alpha$. Hence, $x \sqcap y \in A_\alpha$ and $x, y \notin A_\alpha$. That is a contradiction with the fact that A_α is a prime ideal on L, for any $\alpha \in [0, 1]$. Hence, A is a prime fuzzy ideal on L.

(ii) Follows from Proposition 2.1 and (i).

□

3.2 Maximal fuzzy ideals (resp. filters) on a lattice

In this section, we introduce and characterize the maximal fuzzy ideals (resp. filters) on a lattice.

Definition 3.3. *Let L be a lattice and let I be a fuzzy subset on L. We say that I is a maximal fuzzy ideal if there is no fuzzy ideal on L containing I.*

The following proposition shows that the support of a maximal fuzzy ideal (resp. filter) on a lattice is a maximal ideal (resp. filter) on that lattice.

Proposition 3.6. *Let L be a lattice and $I \in$ fuzzy subset (L). Then it holds that*

(i) if I is a maximal fuzzy ideal, then its support $Supp(I)$ is a maximal ideal on L;

(ii) if I is a maximal fuzzy filter, then its support $Supp(I)$ is a maximal filter on L.

Proof. (i) Suppose that I is a maximal fuzzy ideal on a lattice L and we will show that $Supp(I)$ is a maximal ideal on L. We suppose that $Supp(I)$ is not a maximal i.e., there exists a subset K such that $Supp(I) \subset K$, this means that there exists a fuzzy ideal J on R such that $Supp(J) = K$ satisfies $\mu_J(x) = \mu_I(0)$, then it holds that $I \subset J$. That is a contradiction with the fact that I is a fuzzy maximal ideal. Hence, $Supp(I)$ is a maximal ideal on L.

(ii) Follows from Proposition 2.1 and (i).

□

In the same manner, we get the following theorem which provides a characterization of maximal fuzzy ideal (resp. filter) in terms of their level sets.

Theorem 3.4. *Let L be a lattice and I is a fuzzy subset on L. Then it holds that*

(i) I is a maximal fuzzy ideal if and only if their level sets are maximal ideals;

(ii) I is a maximal fuzzy filter if and only if their level sets are maximal filters.

Proof. (⇒)

Suppose that I is a maximal fuzzy ideal on a lattice L and we will show that I_α is a maximal ideal on L, for $\alpha \in [0,1]$. We Suppose that I_α is not a maximal ideal i.e., there exists an ideal J on R such that $I_\alpha \subset J$. We can consider the set J as one of the level sets of another fuzzy set J', for example, J'_{β_0} such that $\beta_0 \geq \alpha$ and $\beta_0 \in [0,1]$. From the definition of level sets, it follows that $\mu_I(x) \leq \mu_{J'}(x)$, for any $x \in R$. That is a contradiction with the fact that I is a fuzzy maximal ideal on L. Hence, I_α are a maximal ideals on L, for all $\alpha \in [0,1]$.

(⇐)

Suppose that I_α are maximal ideals on L for all $\alpha \in [0,1]$ and we will show that I is a fuzzy maximal ideal on L. We Suppose that I is not maximal fuzzy ideal i.e., there exists a fuzzy ideal J on L such that $I \subset J$. From the properties of $\alpha-$cuts, we get that $I_\alpha \subset J_\alpha$. That is a contradiction with the fact that I_α are a maximal ideals on L. Hence, I is a fuzzy maximal ideal. □

In the following theorem, we show the relationship between maximal fuzzy ideal and prime fuzzy ideal on a lattice.

Theorem 3.5. *Let L be a lattice and I is a fuzzy subset on L. Then it holds that*

(i) If I is a maximal fuzzy ideal, then I is a prime fuzzy ideal;

(ii) If I is a maximal fuzzy filter, then I is a prime fuzzy filter;

Proof. (i) Suppose that I is a maximal fuzzy ideal on a lattice L. We show that I is prime fuzzy ideal on L. Since I is a maximal fuzzy ideal on a lattice L, then Theorem 3.4 guarantees that I_α are a maximal ideals on L, for all $\alpha \in [0,1]$. From the crisp case, it holds that I_α are a prime ideals on R, for all $\alpha \in [0,1]$. Therefore, Theorem 3.4 guarantees that I is a prime fuzzy ideal on a lattice L.

(ii) Follows from Proposition 2.1 and (i).

□

Remark 3.1. *The converse of the above implications is not necessarily held. Indeed, let us consider the lattice $L = \{0, a, b, c, 1\}$ represented by the following Hasse diagram*

Let $I = \{< 0, 1 >, < a, 1 >, < b, 0.5 >, < c, 0.5 >, < 1, 0.5 >\}$ be a fuzzy subset on L. Therefore, A is a prime fuzzy ideal on L, but not maximal

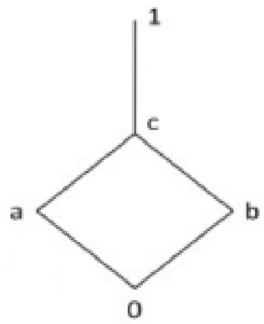


Figure 3.1

3.3 Principal fuzzy ideals (resp. filters) on a lattice

In this section, we introduce the notion of principal fuzzy ideal (resp. fuzzy filter) on a lattice. Similarly to the crisp case, we characterize these notions in terms of a down set and an up set generated by fuzzy singletons. First, we need to recall the following definition of crisp principal ideal (resp. filter), and the definition of a fuzzy singleton.

Definitions

Let L be a lattice and S be a subset on L. S is called a down-set (alternative terms include lower-set) if $y \in S$ implies $x \in S$ for all $x \leqslant y$. Dually, S is called an up-set (alternative terms include upper-set) if $y \in S$ implies $x \in S$ for all $y \leqslant x$. For a given subset S on L, we denote by $\downarrow S$ the set of all elements smaller than or equal to some element of S, i.e.,

$$\downarrow S = \{x \in L \mid x \leqslant y, \text{ for some } y \in S\},$$

and $\uparrow S$ the set of all elements bigger than or equal to some element of S, i.e.,

$$\uparrow S = \{x \in L \mid y \leqslant x, \text{ for some } y \in S\}.$$

It is easy to check that $\downarrow S$ (resp. $\uparrow S$) is the smallest down-set (resp. the smallest up-set) containing S. $\downarrow S$ (resp. $\uparrow S$) is called the down-set (resp. the up-set) of S. Similarly, for a given element x on a lattice L, the down-set $\downarrow \{x\}$ ($\downarrow x$, for short) and the up-set $\uparrow \{x\}$ ($\uparrow x$, for short) are defined as

$$\downarrow x = \{y \in L \mid y \leqslant x\} \text{ (resp. } \uparrow x = \{y \in L \mid x \leqslant y\}).$$

Note that if S is a down-set (resp. an up-set), then $\downarrow S$ (resp. $\uparrow S$) coincides with S.

Analogously to the notion of crisp down-set (resp. up-set) on a lattice L, we introduce the notion of a fuzzy down-set (resp. a fuzzy up-set).

Definition 3.4. *Let L be a lattice and $S \in$ fuzzy subset of L.*

(i) S is called a fuzzy down-set if $\mu_S(x) \geq \mu_S(y)$ for all $x \leqslant y$.

(ii) Dually, S is called a fuzzy up-set if $\mu_S(x) \leq \mu_S(y)$ for all $x \leqslant y$.

Definition 3.5. *For a given fuzzy set S on a lattice L we denote by:*

(i) $\Downarrow S$ the fuzzy set associated to S defined as

$$\mu_{\Downarrow S}(x) = \sup_{y \in \uparrow x} \mu_S(y),$$

(ii) $\Uparrow S$ the fuzzy set associated to S defined as

$$\mu_{\Uparrow S}(x) = \sup_{y \in \downarrow x} \mu_S(y),$$

Proposition 3.7. *Let L be a lattice, L^d be its order-dual lattice and $S \in$ fuzzy subset of L. The following statements hold:*

(i) S is a fuzzy down-set on L if and only if S is a fuzzy up-set on L^d;

(ii) S is a fuzzy up-set on L if and only if S is a fuzzy down-set on L^d;

(iii) $\Downarrow S$ on L coincides with $\Uparrow S$ on L^d ;

(iv) $\Uparrow S$ on L coincides with $\Downarrow S$ on L^d.

The following propositions list some properties of fuzzy down and fuzzy up sets.

Proposition 3.8. [3] [4] *Let L be a lattice and $R, S \in$ fuzzy subset of L. The following statements hold:*

(i) If $S \subseteq R$, then $\Downarrow S \subseteq \Downarrow R$;

(ii) $\Downarrow (\Downarrow S) = \Downarrow S$;

(iii) $\Downarrow (S \cup R) = \Downarrow S \cup \Downarrow R$;

(iv) $\Downarrow (S \cap R) \subseteq \Downarrow S \cap \Downarrow R$.

In the same direction, a dual version of Proposition 3.8 can also obtained for fuzzy up-sets. Its proof follows from Propositions 3.7 and 3.8.

Proposition 3.9. [21] *Let L be a lattice and $R, S \in$ fuzzy subset of L. The following statements hold:*

(i) If $S \subseteq R$, then $\Uparrow S \subseteq \Uparrow R$;

(ii) $\Uparrow (\Uparrow S) = \Uparrow S$;

(iii) $\Uparrow (S \cup R) = \Uparrow S \cup \Uparrow R$;

(iv) $\Uparrow (S \cap R) \subseteq \Uparrow S \cap \Uparrow R$.

Definition 3.6. [21] *Let L be a lattice. For any $x \in L$, a fuzzy singleton (F- singleton, for short) $\tilde{x}$ is a fuzzy set on L given by $\tilde{x} = \{\langle t, \mu_{\tilde{x}}(t)\rangle \mid t \in L\}$, where*

$$\mu_{\tilde{x}}(t) = \begin{cases} 1 \ , \ if \ x = t \\ f(t) \ , \quad otherwise, \end{cases}$$

such that f (resp. g) is a monotone (resp. antitone) mapping on $[0, 1[$ and $f(t) + g(t) < 1$, for any $t \in L$.

Definition 3.7. *Let L be a lattice, Then*

(i) the principal fuzzy ideal generated by a fuzzy singleton $\tilde{x}$ is the smallest fuzzy ideal contains $\tilde{x}$;

(ii) the principal fuzzy filter generated by a fuzzy singleton $\tilde{x}$ is the smallest fuzzy filter contains $\tilde{x}$.

The following theorem shows that the fuzzy down-set (resp. the fuzzy up-set) generated by a fuzzy singleton on a lattice L is a fuzzy ideal (resp. is a fuzzy filter) on L.

Theorem 3.6. *Let L be a lattice and x be an element on L. Then it holds that*

(i) $\Downarrow \tilde{x}$ is an fuzzy ideal on L;

(ii) $\Uparrow \tilde{x}$ is an fuzzy filter on L.

Proof. (i) From Theorem 3.1, it suffices to show for any $x, y \in L$ that

$$\mu_{\Downarrow \tilde{x}}(x \sqcup y) = \mu_{\Downarrow \tilde{x}}(x) \wedge \mu_{\Downarrow \tilde{x}}(y) \ .$$

Let $a, b \in L$. On the one hand, by Proposition 3.7, $\Downarrow \tilde{x}$ is a fuzzy down-set, which implies that $\mu_{\Downarrow\tilde{x}}(a) \geq \mu_{\Downarrow\tilde{x}}(a \sqcup b)$ and $\mu_{\Downarrow\tilde{x}}(b) \geq \mu_{\Downarrow\tilde{x}}(a \sqcup b)$. Hence, $\mu_{\Downarrow\tilde{x}}(a) \wedge \mu_{\Downarrow\tilde{x}}(a) \geq \mu_{\Downarrow\tilde{x}}(a \sqcup b)$. On the other hand, since $\mu_{\tilde{x}}$ is a monotone mapping, it holds that $\mu_{\tilde{x}}(a) \leq \mu_{\tilde{x}}(a \sqcup b)$ and $\mu_{\tilde{x}}(b) \leq \mu_{\tilde{x}}(a \sqcup b)$. This implies that $\sup\limits_{a \leqslant t} \mu_{\tilde{x}}(t) \leq \sup\limits_{a \sqcup b \leqslant t} \mu_{\tilde{x}}(t)$ and $\sup\limits_{b \leqslant t} \mu_{\tilde{x}}(t) \leq \sup\limits_{a \sqcup b \leqslant t} \mu_{\tilde{x}}(t)$. Hence, $\sup\limits_{a \leqslant t} \mu_{\tilde{x}}(t) \wedge \sup\limits_{b \leqslant t} \mu_{\tilde{x}}(t) \leq \sup\limits_{a \sqcup b \leqslant t} \mu_{\tilde{x}}(t)$. Thus, $\mu_{\Downarrow\tilde{x}}(a) \wedge \mu_{\Downarrow\tilde{x}}(b) \leq \mu_{\Downarrow\tilde{x}}(a \sqcup b)$. Therefore, $\mu_{\Downarrow\tilde{x}}(a \sqcup b) = \mu_{\Downarrow\tilde{x}}(a) \wedge \mu_{\Downarrow\tilde{x}}(b)$, for all $a, b \in L$. Finally, we conclude that $\Downarrow \tilde{x}$ is a fuzzy ideal on L.

(ii) Follows dually by using Proposition 3.7, (i) and Proposition 2.1.

□

In the following result, we show a characterization of a principal fuzzy ideal (resp. fuzzy filter) in terms of a down-set (resp. up-set) generated by a fuzzy singleton.

Theorem 3.7. *Let L be a lattice and I (resp. F) be a fuzzy ideal (resp. fuzzy filter) on L. Then it holds that*

(i) I is a principal fuzzy ideal on L if and only if there exists $x \in L$ such that $I = \Downarrow \tilde{x}$;

(ii) F is a principal fuzzy filter on L if and only if there exists $x \in L$ such that $F = \Uparrow \tilde{x}$.

Proof. We only prove (i), as (ii) can be proved analogously by using Proposition 3.7 and Proposition 2.2. Suppose that I is a principal fuzzy ideal on L. Then there exists a fuzzy singleton $\tilde{x}$ such that I is the smallest fuzzy ideal contains $\tilde{x}$. Since $\tilde{x} \subseteq I$, it follows from Proposition 3.8 that $\Downarrow \tilde{x} \subseteq \Downarrow I = I$. On the other hand, Theorem 3.6 guarantees that $\Downarrow \tilde{x}$ is an ideal. Then the fact that I is the smallest ideal contains $\tilde{x}$ implies that $I \subseteq \Downarrow \tilde{x}$. Thus, $I = \Downarrow \tilde{x}$.

Conversely, $I = \Downarrow \tilde{x}$ is a fuzzy ideal contains $\tilde{x}$. Now, suppose that J is an other fuzzy ideal contains $\tilde{x}$. From Proposition 3.8, it holds that $\Downarrow \tilde{x} \subseteq \Downarrow J = J$. Hence, $I = \Downarrow \tilde{x}$ is the smallest fuzzy ideal contains $\tilde{x}$. Thus, I is a principal fuzzy ideal. □

Conclusion

In this memory, we characterized the notions of a fuzzy ideal and fuzzy fiters on a lattice in terms of level sets and support and in terms of lattice operations. Also, we treat some types of fuzzy ideals and fuzzy filter on a lattice such as the prime fuzzy ideal (resp. filter), the maximal fuzzy ideal (resp. filter) and the principal fuzzy ideal (resp. filter). Moreover, we studied the relationship between some of them.

Future work is anticipated in multiple directions. We think it makes sense to study the notions of fuzzy ideal and fuzzy filter for other structure such as residuated lattices. Moreover, we intend to extend this work to other types of fuzzy ideals and fuzzy filters.

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