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Title

PRINCIPAL FUZZY IDEALS AND FILTERS ON A LATTICE

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Contents

Introduction

In 1965, Zadeh introduced the concept of "Fuzzy Set" as an extension of the classical concept of set. In the classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition, an element either belongs or does not belong to the set. While fuzzy sets theory permits the gradual assessment of the membership of elements in a set; this is described with the help of a membership function in the real unit interval [0 1] or on a bounded poset.

The notions of ideals and filters is one of the most important concepts in the lattices theory. These notions are mainly used to translate connections between properties on algebraic structures and to define congruence relations and quotient algebras. They are played a central role in Stone representation theorem for Boolean lattice and in the representation of a distributive lattice. Also, in topology like completeness and compactness in metric spaces. In fuzzy setting, for the same purposes, several authors introduced and investigated the concepts of ideals and filters on lattice, on BL-algebras, on ordered ternary semigroups and on fuzzy structures.

In this work, we generalize the notion of principal ideal (resp. filter) on a lattice to the setting of fuzzy sets and investigate their various characterizations and properties. More specifically, we show that any principal fuzzy ideal (resp. filter) coincides with a fuzzy down-set (resp. up-set) generated by a fuzzy singleton. Afterwards, for a given fuzzy set, we introduce two fuzzy sets: its fuzzy down-set and up-set, and we investigate their interesting properties.

The memoir is devaided in to 3 chapters

• **In chapter 1, we recall generalities of fuzzy sets, operations of fuzzy sets, characteristics of fuzzy sets, cartesian product, T-norms and T-conorms. Next, we recall a basic notion of fuzzy relations, their properties and operations.**

• **In chapter 2, we study the concept of fuzzy ideals and filters on a lattice, where the first part contains general concepts about ideals and filters on a lattice, while**

the secend part contains fuzzy lattices, where we recall the notion of fuzzy ideals and filters on a lattice, characterizations of fuzzy ideals and filters on a lattice.

• In chapter 3, we generalize the notion of principal ideal (resp. filter) on a lattice to the setting of fuzzy sets and investigate their various characterizations and properties. More specifically, we show that any principal fuzzy ideal (resp. filter) coincides with a fuzzy down-set (resp. up-set) generated by a fuzzy singleton. Afterwards, for a given fuzzy set, we introduce two fuzzy sets: its fuzzy down-set and up-set, and we investigate their interesting properties.

Chapter 1

Generalities on fuzzy sets and fuzzy relations

The purpose of this chapter is to provide to the generalities of fuzzy sets, operations of fuzzy sets, characteristics of fuzzy sets, cartesian product, T-norms and T-conorms. Next, we recall a basic notion of fuzzy relations, there properties and operations. Many of the properties of these concepts will be used in the next chapters.

1.1 Generalities on fuzzy sets

In this section, we introduce basic notion of fuzzy set, operation of fuzzy sets, characteristics of fuzzy sets, cartesian product, T-norms and T-conorms.

1.1.1 Crisp sets

The concept of a set is fundamental in mathematics and intuitively can be described as a collection of objects possibly linked through some properties. The crisp set has clear boundaries, i.e., $x \in A$ or $x \notin A$ exclude any other possibility.

Definition 1.1. *The set can be defined by:*

(i) Writing of all its elements, whose elements are $\alpha_1, \alpha_2, ..., \alpha_n$, and we write, $A = \{\alpha_1, \alpha_2, ..., \alpha_n\}$.

(ii) A property or properties are satisfied by its elements, and we write, $A = \{x \mid P(x)\}$. Where the symbol "$|$" means the sentence "such that" and $P(x)$ a proposal of the form "x a P property".

(iii) *Let* X *be a set and* A *be a subset on* X $(A \subseteq X)$. *Then the function*

$$\begin{aligned} \chi_A : X &\longrightarrow \{0,1\} \\ x &\longmapsto \begin{cases} 0 & if \quad x \notin A; \\ 1 & if \quad x \in A, \end{cases} \end{aligned}$$

$$\begin{aligned} \chi_A : X &\longrightarrow \{0,1\} \\ x &\longmapsto \begin{cases} 0 & if \quad x \notin A; \\ 1 & if \quad x \in A, \end{cases} \end{aligned}$$

is called the characteristic function of the set $A \in X$.

Classical sets and their operations can be represented by their characteristic function.

Definition 1.2 (Operations on crisp sets)**.** *Let* X *be a universal set, let* A *and* B *are a subsets on* X.

(i) *Inclusion:* $A \subset B$ *if and only if any* $x \in X$, $(x \in A) \Rightarrow (x \in B)$,*i,e.,*$\chi_A(x) \leq \chi_B(x)$.

(ii) *Equality:* $A = B$ *if and only if any* $A \subseteq B$ *and* $B \subseteq A$ *i.e.,* $(\chi_A(x) = \chi_B(x))$.

(iii) *Complement:* $A^c = \{x \in X \mid x \notin X\}$ *i.e.,* $(\chi_{A^c}(x) = 1 - \chi_A(x))$.

(iv) *Intersection:* $A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}$*i.e.,* $\chi_{A\cap B}(x) = min(\chi_A(x), \chi_B(x))$.

(v) *Union :* $A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}$*i.e.,* $\chi_{A\cup B}(x) = max(\chi_A(x), \chi_B(x))$.

(vi) *Relative complement:* $A \setminus B = A - B = A \cap B^c = \{x \in X \mid x \in A \text{ and } x \notin B\}$*i.e.,*

$$\chi_{A-B}(x) = \chi_{A\cap B^c}(x) = min(\chi_A(x), \chi_{B^c}(x)).$$

Example 1.1. *Let* $X = \{2, 3, 5, 7, 11, 13, 17, 19\}$ *be a set, let* A *and* B *be two subsets on* X *such that* $A = \{2, 11, 17\}$ *and* $B = \{3, 5, 7, 13, 17\}$.*Then*

$$A^c = \{3, 5, 7, 13, 19\};$$

$$B^c = \{2, 11, 19\};$$

$$A \cap B = \{17\};$$

$$A \cup B = \{2, 3, 5, 7, 11, 13, 17\};$$

$$B \setminus A = \{3, 5, 7, 13\}.$$

1.1.2 Fuzzy sets

This subsection contains the basic definitions and properties of fuzzy sets, and several operations on fuzzy sets. The notion of fuzzy set was introduced in 1965 by L.A. Zadeh [28].

Definition 1.3. [28] *Let* X *be a non empty set. A fuzzy set* $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ *is characterized by a membership function* $\mu_A : X \to [0,1]$*, where* $\mu_A(x)$ *is interpreted as the degree of membership of the element* x *in the fuzzy subset* A *for each* $x \in X$.

Example 1.2. *Let* $A = X = \{a, b, c\}$ *. We define the fuzzy subset* A *on* X *by:*

(1) $\{\langle a, 0.4\rangle, \langle b, 0.1\rangle, \langle c, 0.7\rangle$.

(2) In this example, we consider the expression "young"in the context "a young person" in order to exemplify how linguistic expression can be modeled using fuzzy sets. The fuzzy set $A : [40, 100] \to [0,1]$

$$A(x) = \begin{cases} 1 \text{ , } if \ 40 \leqslant x < 50, \\ \dfrac{60-x}{10} \text{ , } if \ \ 50 \leqslant x < 60, \\ 0, \ if \ 60 \le x \le 100, \end{cases}$$

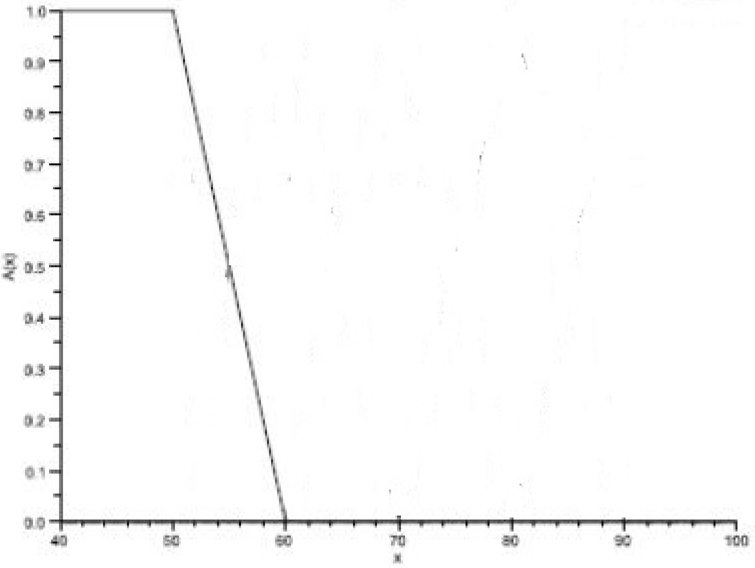

Figure 1.1: Graph of fuzzy set for modeling the expresion young person

1.1.3 Operations of fuzzy sets

Definition 1.4 (**Inclusion**). [28] *Let X be a non empty set and let A and B be two fuzzy subsets, we say that $A \subseteq B$, if and only if $\mu_A(x) \leq \mu_B(x)$, for any $x \in X$.*

Definition 1.5 (**Equality**). [28] *Let X be a non empty set and let A and B be two fuzzy subsets, we say that $A = B$, if and only if $\mu_A(x) = \mu_B(x)$, for any $x \in X$.*

Definition 1.6 (**Intersection**). [28] *Let X be a non empty set and let A and B be two fuzzy subsets, the intersection is defined for any $x \in X$by*

$$\mu_{A\cap B}(x) = min\,\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \wedge \mu_B(x).$$

Definition 1.7 (**Union**). [28] *Let X be non empty set and let A and B be two fuzzy subsets, the union is defined for any $x \in X$ by*

$$\mu_{A\cup B}(x) = max\,\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \vee \mu_B(x).$$

Definition 1.8 (**Complement**). [28] *The complement of a fuzzy set A denoted by $C(A)$ and is defined by for any $x \in X$*

$$\mu_{C(A)}(x) = 1 - \mu_A(x).$$

Definition 1.9 (**Sum**). [28] *Let X be non empty set and let A and B be two fuzzy subsets, the sum defined by for any $x \in X$*

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x).$$

Definition 1.10 (**Product**). [28] *Let X be non empty set and let A and B be two fuzzy subsets, the product defined by for any $x \in X$*

$$\mu_{A\times B}(x) = \mu_A(x)\mu_B(x).$$

Example 1.3. *Let $X = \{a, b, c\}$, let $A = \{\langle a, 0.2\rangle; \langle b, 0.9\rangle; \langle c, 0.5\rangle\}$, and $B = \{\langle a, 0.7\rangle; \langle b, 0.1\rangle; \langle c, 1\rangle\}$. Then*

1. $A \cap B = \{\langle a, 0.2\rangle; \langle b, 0.1\rangle; \langle c, 0.5\rangle\}$;
2. $A \cup B = \{\langle a, 0.7\rangle; \langle b, 0.9\rangle; \langle c, 1\rangle\}$;
3. $A \times B = \{\langle a, 0.14\rangle; \langle b, 0.09\rangle; \langle c, 0.5\rangle\}$;
4. $A + B = \{\langle a, 0.76\rangle; \langle b, 0.94\rangle; \langle c, 1\rangle\}$;
5. $C(A) = \{\langle a, 0.8\rangle; \langle b, 0.1\rangle; \langle c, 0.5\rangle\}$.

1.1.4 Characteristics of fuzzy sets

The characteristics of a fuzzy set A of X most useful in describing it are that which shows how it differs from a classical set of X.

Definition 1.11 (**Support**)**.** [28, 29] *Let A be a fuzzy set on a set X. The support of A is the crisp subset on X given by*

$$Supp(A) = \left\{ x \in X \mid \mu_A(x) > 0 \right\}.$$

Definition 1.12 (**Kernel**)**.** [28, 29] *Let A be a fuzzy set on a set X. The kernel of A is the crisp subset on X given by*

$$Ker(A) = \left\{ x \in X \mid \mu_A(x) = 1 \right\}.$$

Definition 1.13 (**Height**)**.** [28, 29] *Let A be a fuzzy set on a set X. The height of A is the highest value taken by its membership function given by*

$$H(A) = sup \left\{ \mu_A \mid x \in X \right\}.$$

Definition 1.14 (**Cardinality**)**.** [28, 29] *The cardinality of a finite fuzzy set A, denoted $\mid A \mid$ is defined as*

$$\mid A \mid = \sum_{x \in X} \mu_A(x).$$

Definition 1.15 (**α-cuts**)**.** [28, 29] *Let A be a fuzzy set on a set X. The α-cut of A is a crisp subset, denoted A_α.*

$$A_\alpha = \left\{ x \in X \mid \mu_A(x) \geq \alpha \right\}.$$

Where $\alpha \in [0, 1]$, $A_0 = X$ and $A_1 = Ker(A)$.

Example 1.4. *(1) Let $X = \{a, b, c, d, e, f\}$,*

and $A = \{\langle a, 0.0\rangle, \langle b, 0.2\rangle, \langle c, 1\rangle, \langle d, 0.4\rangle, \langle e, 1\rangle, , \langle f, 0.9\rangle\}$ $Supp(A) = \{b, c, d, e, f\}$;

$Ker(A) = \{c, e\}$; $H(A) = 1$; $|A| = 3.5$.

(2) Let $X = [0, 35]$ with $\alpha \in [0, 1]$. We defined the fuzzy set A on X by:

$$\mu_A(x) = \begin{cases} 1 \ , \ if \ x \in [20, 30] \\ 0 \ , \ if \ \ x \geq 35 and x \leq 15, \\ \alpha \ , \ if \ \ x \in]15, 20[and x \in]30, 35[, \end{cases}$$

Then, $KerA = [20, 30]$, $SuppA = [15, 35]$ and $H(A) = 1$.

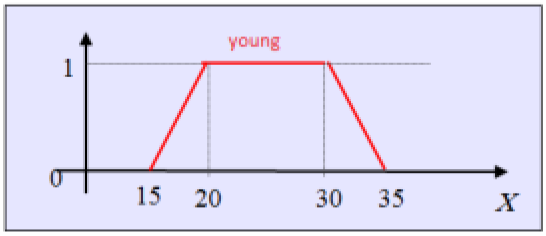

Figure 1.2: young fuzzy subset .

Proposition 1.1. *Let X be a nonempty set and let A be a fuzzy subset on X. Then*

(i) $Supp(A^c) = (Ker(A))^c = X - Ker(A)$;

(ii) $Ker(A^c) = (Supp(A))^c = X - Supp(A)$.

Proof. (i)

$$\begin{aligned} Supp(A^c) &= \{x \in X | A^c(x) \neq 0\}; \\ &= \{x \in X | 1 - A(x) \neq 0\}; \\ &= \{x \in X | x \notin ker(A)\}; \\ &= X - Ker(A). \end{aligned}$$

(ii)

$$\begin{aligned} Ker(A^c) &= \{x \in X | A^c(x) = 1\}; \\ &= \{x \in X | 1 - A(x) = 1\}; \\ &= \{x \in X | A(x) = 0\}; \\ &= \{x \in X | x \notin Supp(A)\}; \\ &= X - Supp(A). \end{aligned}$$

□

Theorem 1.1. [15] *Any fuzzy subset A of the reference set X is defined from its α-cute for any x on X.*

$$\mu_A(x) = \sup_{x \in]0,1]} (\alpha . \chi_{A_\alpha}(x)).$$

where χ_{A_α} is the characteristic function of A_α.

Proof. Let the characteristic function:

$$\chi_{A_\alpha} = \begin{cases} 1 \;\; if \; \mu_A(x) \geq \alpha \; ; \\ \\ 0 \;\; if \;\; otherwise. \end{cases}$$

By multipling each member by a real number α, we obtain

$$\alpha\chi_{A_\alpha} = \begin{cases} \alpha \;\; if \; \mu_A(x) \geq \alpha \; ; \\ \\ 0 \;\; if \;\; otherwise. \end{cases}$$

By introducing the operator "sup" in each member, we get $\underset{x\in]0,1]}{sup}\,(\alpha\chi_{A_\alpha}(x)) = \underset{x\in]0,1]}{sup}\,\{\mu_A(x) \geq \alpha\}$, this implies that $\underset{x\in]0,1]}{sup}\,(\alpha\chi_{A_\alpha}(x)) = \underset{x\in]0,1]}{sup}\,\{\alpha \leq \mu_A(x) \geq \alpha\}$ We conclude that:

$$\mu_A(x) = \underset{x\in]0,1]}{sup}\,(\alpha.\chi_{A_\alpha}(x)).$$

□

Example 1.5. *Let* $X = \{1, 2, 3, ..., 10\}$ *and* $A = \{\langle 1, 0.2\rangle; \langle 2, 0.5\rangle; \langle 3, 0.8\rangle; \langle 4, 1\rangle; \langle 5, 0.7\rangle; \langle 6, 0.3\rangle; \langle 7, 0\rangle; \langle 8, 0$
We have for any level α *in* $[0, 1]$.

$$A_0 = X$$

$$A_1 = \{x \in X \mid \mu_A(x) \geq 1\} = \{4\};$$

$$A_{0.2} = \{x \in X \mid \mu_A(x) \geq 0.2\} = \{1, 2, 3, 4, 5, 6\};$$

$$A_{0.3} = \{x \in X \mid \mu_A(x) \geq 0.3\} = \{2, 3, 4, 5, 6\}.;$$

$$A_{0.5} = \{x \in X \mid \mu_A(x) \geq 0.5\} = \{2, 3, 4, 5\};$$

$$A_{0.7} = \{x \in X \mid \mu_A(x) \geq 0.2\} = \{3, 4, 5\}.$$

$$A_{0.8} = \{x \in X \mid \mu_A(x) \geq 0.2\} = \{3, 4\}.$$

Also, we get

$$\mu_A(1) = max(1 \times 0, ..., 0.2 \times 1, ..., 0 \times 1) = 0.2;$$

$$\mu_A(2) = max(1 \times 0, ..., 0.5 \times 1, ..., 0 \times 1) = 0.5;$$

$$\mu_A(3) = max(1 \times 0, ..., 0.8 \times 1, ..., 0 \times 1) = 0.8;$$

$$\mu_A(4) = max(1 \times 0, ..., 0 \times 1) = 1;$$

$$\mu_A(5) = max(1 \times 0, ..., 0.7 \times 1, ..., 0 \times 1) = 0.7.$$

$$\mu_A(6) = max(1 \times 0, ..., 0.3 \times 1, ..., 0 \times 1) = 0.3.$$

$$\mu_A(7) = max(1 \times 0, ..., 0 \times 1, ..., 0 \times 1) = 0.0.$$

$$\mu_A(8) = max(1 \times 0, ..., 0 \times 1, ..., 0 \times 1) = 0.$$

$$\mu_A(9) = max(1 \times 0, ..., 0 \times 1, ..., 0 \times 1) = 0.$$

$$\mu_A(10) = max(1 \times 0, ..., 0 \times 1, ..., 0 \times 1) = 0.$$

1.1.5 Cartesian product and projection on fuzzy sets

The cartesian product of the fuzzy subsets is the minimum of these degrees of belonging and these projection is the maximun of these cartesian product.

Definition 1.16 (Cartesian product). [15] *The cartesian product applied to n fuzzy sets can be defined as follows. Let $\mu_{A_1}, \mu_{A_2}, ...\mu_{A_n}$, be membership functions of $A = A_1, A_2, ..., A_n$. Then, the membership degree of $(x_1, x_2, ...x_n) \in X_1 \times X_2 \times ... \times X_n$ on the fuzzy set $A = A_1 \times A_2 \times ... \times A_n$ is given by*

$$\mu_A(x_1, x_2, ..., x_n) = min \left\{ \mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n) \right\}.$$

Which provides the set A.

Example 1.6. *Let $X_1 = \{a, b, c, d\}$, $X_2 = \{\alpha, \beta\}$, let A_1, A_2 be two fuzzy subsets respectively defined on X_1 and X_2 by:*
$A_1 = \{\langle a, 0.1\rangle; \langle b, 0.4\rangle; \langle c, 0.8\rangle; \langle d, 0.5\rangle\}$;
$A_2 = \{\langle \alpha, 0.2\rangle; \langle \beta, 0.6\rangle\}$.
So, we get:

$$A_1 \times A_2 = \{\langle (a, \alpha), 0.1\rangle; \langle (a, \beta), 0.1\rangle; \langle (b, \alpha), 0.2\rangle; \langle (b, \beta), 0.4\rangle; \langle (c, \alpha), 0.2\rangle; \langle (c, \beta), 0.6\rangle; \langle (d, \beta), 0.5\rangle\}.$$

Definition 1.17 (Projection). [15] *The projection on X_1 of the fuzzy set A of $X_1 \times X_2 \times ... \times X_n$ is the fuzzy set $Proj_{X_1}(A)$ of X_1, whose membership function is defined by: for any $x_1 \in X_1$,*

$$\mu_{ProjX_{1(A)}}(x_1) = sup_{x_2 \in X_2, x_3 \in X_3, ..., x_n \in X_n}(\mu_A(x_1, x_2, ..., x_n)).$$

Example 1.7. *Let $X = X_1 \times X_2$ the set of reference such that X_1 and X_2 are two subsets of X, we consider $A_1 \times A_2 = A$ defined by:*

$$A = \{\langle(a,\alpha),0.1\rangle;\langle(a,\beta),0.1\rangle;\langle(b,\alpha),0.2\rangle;\langle(b,\beta),0.4\rangle;\langle(c,\alpha),0.2\rangle;\langle(c,\beta),0.6\rangle\};$$

Therefore,

$$\begin{aligned} Proj_{X_1}(A) &= \{\langle a, max(0.1,0.1)\rangle;\langle b, max(0.2,0.4)\rangle;\langle c, max(0.2,0.6)\rangle;\} \\ &= \{\langle a,0.1\rangle;\langle b,0.4\rangle;\langle c,0.6\rangle\}; \end{aligned}$$

1.1.6 T-norms and T-conorms

The history of triangular norms(t-norm) started with the paper [14]. The main idea was to construct metric spaces where probability distributions rather than numbers are used in order to describe the distance between two elements. Schweizer and Abe Sklar in [24] provided the axioms of t-norm, as they are used today.

Definition 1.18. [14] *A t-norms on* $[0,1]$ *is function* $T : [0,1]^2 \longrightarrow [0,1]$ *satisfies the following four axioms:*

(T1) Commutativity: $(\forall x, y \in [0,1])(T(x,y) = T(y,x))$.

(T2) Associativity: $(\forall x, y, z \in [0,1])(T(x,T(y,z)) = T(T(x,y),z))$.

(T3) Monotonicity: $(\forall x, y, z \in [0,1])(x \leq y \Rightarrow T(x,y) \leq T(y,z))$.

(T4) Boundary condition: $(\forall x \in [0,1])(T(x,1) = x)$.

Condition $(T3)$ *and* $(T4)$ *imply that for any t-norms* T *it hold that* $T(x,y) \leq x$, $T(x,y) \leq y$, $T(x,y) \leq Min(x,y)$ *and* $T(x,0) = x$.

Example 1.8.

(T5) Minimum: $T(x,y) = min\{x,y\}$.

(T6) Lukasiewicz: $T(x,y) = max\{x+y-1,0\}x.y$.

(T7) Einstein: $T(x,y) = xy/(2-x-y+xy)$.

(T8) Algebraic or probaliste: $T(x,y) = xy$.

(T9) Hamacher: $T(x,y) = xy/(x+y-xy)$.

(T10) Drastic product:

$$T_A(x,y) = \begin{cases} 0 & if \quad \{x,y\} \in [0,1]^2 \\ min\{x,y\} & otherwise \end{cases}$$

Definition 1.19. [14] *A triangular conorm is a binary operation S on the unit interval $[0,1]$, i.e., it is a function $S : [0,1]^2 \longrightarrow [0,1]$: the following four axioms are satisfied*

(S1) Commutativity : $S(x,y) = S(y,x)$.

(S2) Associativity : $S(x, S(y,z)) = S(S(x,y), z)$.

(S3) Monotonicity : $S(x,y) \leq S(x,z)$ whenever $y \leq z$.

(S4) Boundary condition : $S(x,0) = x$.

Example 1.9.

(1) Einstein: $S(x,y) = x + y/(1 + xy)$.

(2) Hamacher: $S(x,y) = x + y - 2xy/(1 - xy)$.

(3) Dubois and parade $\alpha \in [0,1] : S(x,y) = x + y + xy - min(x,y,1-\alpha)/(max(1-\alpha;1-y,\alpha))$.

Example 1.10. *Let $X = \{a,b,c\}$,let A and B be two fuzzy subsets on X as $A = \{\langle a,0.4\rangle;\langle b,0.8\rangle;\langle c,0.3\rangle\}, B = \{\langle a,0.5\rangle;\langle b,0.2\rangle;\langle c,0.7\rangle\}$. Lukasiewicz operators can be used to define intersection and union by:*

(i) $\mu_{A\cap_T B}(x) = max(\mu_A(x) + \mu_B(x) - 1, 0), \forall x \in X$;

(ii) $\mu_{A\cup_S B}(x) = min(\mu_A(x) + \mu_B(x), 1), \forall x \in X$.
Then, we get

(i) $A \cap_T B = \{\langle a,0\rangle;\langle b,0\rangle;\langle c,0\rangle\}$;

(ii) $A \cup_S B = \{\langle a,0.9\rangle;\langle b,1.0\rangle;\langle c,1\rangle\}$.

1.2 Fuzzy relations

This section contains the basic definitions and proprieties of fuzzy relation, fuzzy relations, fuzzy order relations and several operations on fuzzy relations. The notion of fuzzy relations was first introduced by Zadeh [29] as a natural extension of fuzzy set and plays an important role in the theory of such sets and their applications.

1.2.1 Basic definition of fuzzy relation

Definition 1.20. [29] *Let X and Y be two nonempty sets. A binary fuzzy relation from X to Y, is a fuzzy subset of $X \times Y$ characterized by a membership function μ_R which associates with each pair (x, y) its grade of membership $R(x, y)$ in the interval* $[0, 1]$.

Example 1.11. *Let R fuzzy relation on $x = \{1, 2, 3\}$,*

R	*1*	*2*	*3*
1	*0.7*	*0.9*	*0.4*
2	*0.1*	*0.3*	*0.5*
3	*0.2*	*0.1*	*0*

Example 1.12. *The fuzzy relation R "x approximately equal to 2"can be defined on $R \times R$ by the membership function:*

$$R(x, 2) = \frac{1}{1 + (x - 2)^2}.$$

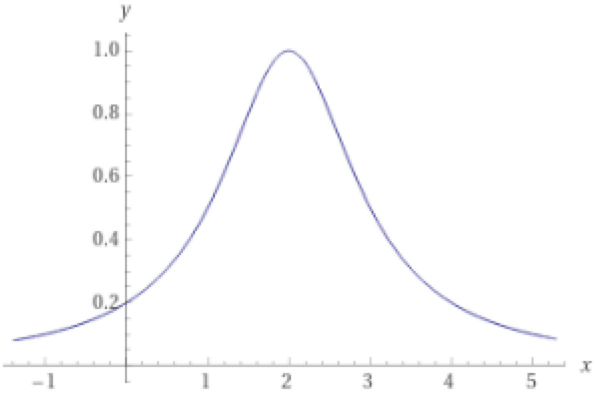

Figure 1.3: The membership function .

1.2.2 Operations on fuzzy relations

Let R, P be two fuzzy relations. R is said to be contained in P(or we say that P contains R), denoted by $R \subseteq P$, if for any $(x, y) \in X \times Y$ it hold that $R(x, y) \leq P(x, y)$.

The transpose (or the inverse) R^t of R is the fuzzy relation from Y to X defined by:

$$R^t = \{\langle (x, y), R^t(x, y) \rangle \mid (x, y) \in X \times Y\}.$$

The intersection of two fuzzy relations R and P is defined as

$$R \cap P = \{\langle (x,y), min(R(x,y), P(x,y) \rangle \mid (x,y) \in X \times Y\}.$$

The union of two fuzzy relations R and P is defined as

$$R \cup P = \{\langle (x,y), max(R(x,y), P(x,y) \rangle \mid (x,y) \in X \times Y\}.$$

Example 1.13. *Let R and P be two fuzzy relations on $X \times Y$, $X = \{x, y, z\}$,*

R	x	y	z
x	1	0.8	0.7
y	0.8	1	0.7
z	0.7	0.7	0.6

P	x	y	z
x	0.6	0.23	0.7
y	0.9	0.0	1.0
z	0.1	0.7	0.6

$R \cup P$	x	y	z
x	1	0.8	0.7
y	0.9	1	1
z	0.7	0.7	0.6

$R \cap P$	x	y	z
x	0.6	0.2	0.7
y	0.8	0.0	0.7
z	0.1	0.7	0.6

Proposition 1.2. [28, 29] *let R, P and Q be three fuzzy relations from a universe X to a universe Y*

(1) if $R \subset P$, then $R^t \subset P^t$;

(2) $(R \cup P)^t = R^t \cup P^t$;

(3) $(R \cap P)^t = R^t \cap P^t$;

(4) $(R^t)^t = R$;

(5) $R \cap (P \cup Q) = (R \cap P) \cup (R \cap Q)$ and $R \cup (P \cap Q) = (R \cup P) \cap (R \cup Q)$;

(6) $R \subseteq R \cup P, P \subseteq R \cup P, R \cup P \subseteq R, R \cup P \subseteq P$;

(7) if $P \subseteq R$ and $Q \subseteq R$, then $P \cup Q \subseteq R$;

(8) if $R \subseteq P$ and $R \subseteq Q$, then$R \subseteq P \cap Q$.

Definition 1.21. [28, 29] *Let R be a fuzzy relation (fuzzy relation on X, for short). The following properties are :*

(i) Reflexivity: If $R(x, x) = 1$*, for any* $x \in X$*.*

(ii) Symmetry: If $R(x, y) = R(y, x)$*, for all* $x, y \in X$*.*

(iii) Antisymmetry: If $x \neq y \quad R(x, y) = 0 \vee R(y, x) = 0$*, for all* $x, y \in X$*.*

(iv) Transitivity: if $R(x, z) \geq max\{min\{R(x, y), R(y, z)\}\}$*, for all* $x, y, z \in X$*.*

Definition 1.22. *Let* X *be a nonempty crisp set and* R *be a fuzzy relation on* X*.* R *is called fuzzy order or partial fuzzy order if it is reflexive, transitive and antisymmetric.*
A nonempty set X *with a fuzzy order* R *defined on it is called a fuzzy ordered set and is denoted by* (X, R)*. It easily follows that each partially ordered set* $(X, \leq)$ *and each fuzzy ordered set* (X, R) *can be viewed as fuzzy ordered sets.*

Example 1.14. *Let* $X = \{a, b, c, \}$*, then the fuzzy relation* R *defined on* X *by*

$$R = \{\langle (x, y), R(x, y) \rangle \mid (x, y) \in X\}$$

R(.,.)	*a*	*b*	*c*
a	*1*	*0.0*	*0.0*
b	*0.3*	*1*	*0*
c	*0.2*	*0.6*	*1.0*

Then, R *is a fuzzy order relation.*

Example 1.15. *Let* $x, y \in \mathbb{N}$*. Then the following fuzzy relation* R *on* $\mathbb{N}$ *is a fuzzy order relation, where*

$$R(x, y) = \begin{cases} 1, \ if \ x = y; \\ 1 - \frac{x}{y} \ , \quad if \ x < y; \\ 0 \ , \quad if \quad x > y. \end{cases}$$

On the basis of the above definition of antisymmetry we define a complete (or total) fuzzy order as follows.

Definition 1.23. *A fuzzy order* R *on a universe* X *is called complete (or total) if for all* $x, y \in X$ *it holds that*

$$[R(x, y) > 0 \quad or \quad (R(y, x) > 0)].$$

Example 1.16. *Let* R*a fuzzy relation on* $X = \{x, y, z\}$ *given by:*

R	x	y	z
x	*1*	*0.4*	*0.5*
y	*0*	*1*	*0.8*
z	*0*	*0*	*1*

R is a fuzzy order total.

Definition 1.24. *A fuzzy ordered set (X, R) in which R is linear is called a linearly fuzzy ordered set or a fuzzy chain.*

Chapter 2

Fuzzy ideals and filters on a lattice

In this chapter, we study the concept of fuzzy ideals and filters on a lattice, where the first part contains general concepts about ideals and filters on a lattice, while the secend part contains fuzzy lattices, fuzzy lattices, fuzzy filters on a lattice and characterization of fuzzy ideals and filters on a lattice .

2.1 Ideals and filters on a lattice

In this section, we recall some definitions and notions of ideals and filters and on a lattice.

Definition 2.1. [7] *Lattice*

Let $(P, \leqslant)$ is a nonempty poset. Then

(i) $(P, \leqslant)$ *is called a lattice if* $\sup\{x, y\}$ *and* $inf\{x, y\}$ *exist for all* $x, y \in P$.

(ii) $(P, \leq)$ *is called a complete lattice if* $\sup S$ *and* $inf S$ *exist for all* $S \subseteq P$.

Given a lattice L, we may define binary operation join and meet on the nonempty set L by for all $x, y \in L$ $x \wedge y = \inf\{x, y\}$ *and* $x \vee y = \sup\{x, y\}$.

Example 2.1.

Figure 2.1: graph of the diagrams are lattices

Definition 2.2. *Maximal element and Minimal element:*

- (•) *An element k of E is said to be maximal if for all $x \in E : k \nless x$ (which means: either $x \leq k$, or x and k are incomparable).*
- (•) *An element k of E is said to be minimal if for all $x \in E : k \ngtr x$ (which means: either $k \leq x$, or x and k are incomparable).*

Definition 2.3. *sup,inf Let $(E, \leq)$ be an ordered set and A a nonempty part of E.*

- (•) *We say that $s \in E$ is an upper bound of A in E if s is the smallest majorant of A. Then simply: $s = min(Major(A)) = min z \in E : a \leq z, a \in A$. Note that such an element, if it exists, is unique. If a part A of E admits an upper bound s, this will be denoted: $s = sup_E A$.*
- (•) *We say that $i \in E$ is a lower bound of A in E if i is the largest minorant of A. Then simply: $i = max(Minore(A)) = max z \in E : z \leq a, a \in A$. Note that such an element, if it exists, is unique. If a part A of E admits a lower bound i, this will be denoted: $i = inf_E A$.*

Definition 2.4. [11] *Ideal Let $L = (L, \bigwedge, \bigvee)$ a lattice, a nonempty subset I is called an Ideal of L if for all $x, y \in L$*

(i) if $y \in I$ and $y \geqslant x$,then $x \in I$.

(ii) if $x, y \in I$,implies $x \bigvee y \in I$.

Example 2.2. *in the N5 lattices, the parts in bold are ideals*

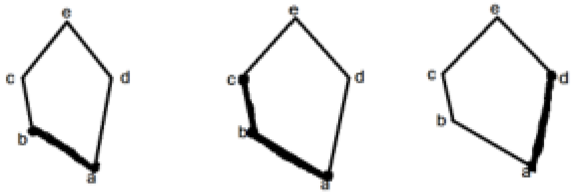

Figure 2.2: the N5 lattices

Proposition 2.1. *Let L be a lattice.*

(i) *An ideal I of L is called proper if it does not coincide withe L.*

(ii) *An ideal I of a lattice with 1 is propre if and only if* $1 \notin I$

(iii) *An ideal I of L is prime if and only if* $x, y \in L$ *and* $x \bigwedge y \in L$ *imply that* $x \in I$ *or* $y \in I$.

(iv) *For each* $x \in L$ *the set* $x = \{y \in L, x \geqslant y\}$ *is an ideal.*

Definition 2.5. [17] *Filter} Let* $L = (L, \bigwedge, \bigvee)$ *a lattice, a nonempty subset F is called a filter on L if for all* $x, y \in L$.

(i) *if* $y \in F$ *and* $y \leq x$, *then* $x \in F$;

(ii) *if* $x, y \in F$, *implies* $x \bigwedge y \in F$.

Example 2.3. *In N5, the parts in bold are filters*

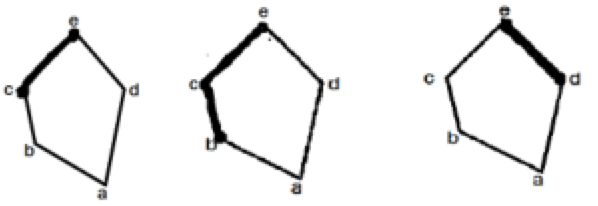

Figure 2.3: filters on N5

Proposition 2.2. *Let L be a lattice. Then*

(i) *A filter F of L is called proper if it does not coincide with L.*

(ii) *A filter F of a lattice with 0 is proper if and only if* $0 \notin F$.

(iii) *A filter F of L is prime if and only if* $x, y \in L$ *and* $x \bigvee y \in F$ *imply that* $x \in F$ *or* $y \in F$.

(iv) *For each* $x \in L$ *the set* $\uparrow x = \{y \in L, y \geqslant x\}$ *is a filter.*

2.2 Fuzzy lattices

The concept of a fuzzy lattice on a lattice was introduced by Ajmal and Thomas [1] as a fuzzy set on a crisp lattice stable by the supremum and the infimum of the binary operations meet a join.

To avoid any confusion or misunderstanding in some formulas, we use the notation $(\leq, \sqcap, \sqcup)$ to refer the (order, min, max) on the lattice L and $(\leq, \wedge, \vee)$ to refer the (usual order, min, max) on the real interval $[0, 1]$.

Definition 2.6. [1] *Let L be a lattice and $A = \{\langle x, \mu_A(x)\rangle | x \in L\}$ be a fuzzy subset on L. Then A is called a fuzzy sub-lattice (fuzzy lattice, for short) if for any $x, y \in L$, the following conditions are satisfied:*

(i) $\mu_A(x \sqcup y) \geq \mu_A(x) \wedge \mu_A(y)$;

(ii) $\mu_A(x \sqcap y) \geq \mu_A(x) \wedge \mu_A(y)$.

Example 2.4. *Consider the lattice L of divisor of 15, that is $L = \{1, 3, 5, 15\}$. given by $A = \{< 1, 0.5 >, < 3, 0.4 >, < 5, 0.4 >, < 15, 0.7 >\}$ then A is a fuzzy lattice.*

The notion of fuzzy ideal (resp. filter) on a lattice was first introduced by Ajmal and Thomas [1].

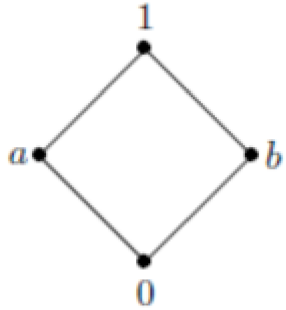

Figure 2.4: Hasse diagram of a lattice (L,$\leq, \sqcap, \sqcup$) with L=0,a,b,1

.

Definition 2.7. [1] *Let L be a lattice and $I = \{\langle x, \mu_I(x)\rangle | x \in L\}$ be a fuzzy subset on L. Then I is called a fuzzy ideal on L (F-ideal, for short) if for all $x, y \in L$, the following conditions are satisfied:*

(i) $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$;

(ii) $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$.

Example 2.5. *Consider the lattice* $L = \{1, 2, 3, 4, 6, 12\}$ *of divisors of 12 we define by :*$A = \{< 1, 0.7 >, < 2, 0.5 >, < 3, 0.6 >, < 4, 0.4 >, < 6, 0.5 >, < 12, 0.7 >\}$ *is a fuzzy ideal.*

Definition 2.8. [1] *Let* L *be a lattice and* $F = \{\langle x, \mu_F(x)\rangle | x \in L\}$ *be a fuzzy subset on* L*. Then* F *is called a fuzzy filter on L (F-filter, for short) if for any* $x, y \in L$*, the following conditions are satisfied:*

(i) $\mu_F(x \sqcup y) \geq \mu_F(x) \vee \mu_F(y)$;

(ii) $\mu_F(x \sqcap y) \geq \mu_F(x) \wedge \mu_F(y)$.

Example 2.6. *Consider the lattice* $L = \{1, 2, 3, 4\}$*, with* $\leq$*is the useal order in N, we define by :* $F = \{< 1, 0.1 >, < 2, 0.3 >, < 3, 0.4 >, < 4, 0.7 >\}$ *is a fuzzy filter.*

Remark 2.1. *Notice that every fuzzy ideal on L is a fuzzy lattice, but the converse is not true in general. Indeed, let L be the lattice given by the Hasse diagram in Figure 2.4 and* $A \in FS(L)$ *defined by* $A = \{< 0, 0.3 >, < a, 0.4 >, < b, 0.4 >, < 1, 0.7 >\}$*. Then A is a fuzzy lattice, but since* $\mu_A(a) = \mu_A(a \sqcap 1) = 0,4 \ngeq max\{0.4; 0.7\}$*, then it hold that A is not a fuzzy ideal on L. As well since* $\mu_A(0) = \mu_A(a \sqcap b) = 0.3 \ngeq min\{0.4; 0.4\}$*, then it hold that A is not a fuzzy filter on L.*

The following results will be needed throughout this chapter.

Proposition 2.1. *Let L be a lattice,* L^d *be its order-dual lattice and* $A \in FS(L)$*. Then it holds that A is a fuzzy ideal on L if and only if A is a fuzzy filter on* L^d *and conversely.*

Proposition 2.2. [25] *Let L be a lattice, A and B are two fuzzy sets on L. Then it holds that*

(i) if A and B are two fuzzy ideals on L, then $A \cap B$ *is a fuzzy ideal on L;*

(ii) if A and B are two fuzzy filters on L, then $A \cap B$ *is a fuzyy filter on L.*

Proof. (i) Let $A = \{< x, \mu_A(x) > | x \in L\}$ and $B = \{< x, \mu_B(x) > | x \in L\}$ be two fuzzy sets. Then $A \cap B = \{< x, \mu_{A\cap B}(x) > | x \in L\}$, where $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$.

$$\begin{aligned}
\mu_{A\cap B}(x \sqcup y) &= \{\mu_A(x \sqcup y) \wedge \mu_B(x \sqcup y)\} \\
&\geq \{\{\mu_A(x) \wedge \mu_A(y)\} \wedge \{\mu_B(x) \wedge \mu_B(y)\}\} \\
&\geq \{\{\mu_A(x) \wedge \mu_B(x)\} \wedge \{\mu_A(y) \wedge \mu_B(y)\}\} \\
&\geq \mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y).
\end{aligned}$$

By that same method, we prove that $\mu_{A\cap B}(x \sqcap y) \geq \mu_{A\cap B}(x) \vee \mu_{A\cap B}(y)$. Hence, $A \cap B$ is a fuzzy ideal on L.

(ii) Follows from Proposition 2.2 and (i).

□

Remark 2.2. *The Union of two fuzzy ideals (resp. filters) does not necessarily be a fuzzy ideals (resp. filters).*

2.3 Characterization of fuzzy ideals and filters on a lattice

In this section, we provide interesting charactirizations of fuzzy ideals and filters on a lattice in terms of the lattice operations, and in terms of their α-level sets.

2.3.1 Characterization of fuzzy ideals and filters in terms of lattice operations

Milles et al [21] have characterize the notion of intuitionistic fuzzy ideals and intuitionistic fuzzy filters on a lattice in terms of the lattice-operations. Here, we use this characterization in fuzzy setting.

Theorem 2.1. [21] *Let L be a lattice and $A \in$ fuzzy subset (L). Then for any $x, y \in L$ the following two statements hold*

(i) $(\mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y))$ *if and only if* $(x \leq y \Rightarrow \mu_A(x) \geqslant \mu_A(y))$;

(ii) $(\mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y))$ *if and only if* $(x \leq y \Rightarrow \mu_A(x) \leq \mu_A(y))$.

Proof. Let $x, y \in L$.

(i) Suppose that $\mu_A(x \sqcap y) \geqslant \mu_A(x) \bigvee \mu_A(y)$. If $x \leq y$ then $x \sqcap y = x$. Since $\mu_A(x \sqcap y) \geqslant \mu_A(x) \bigvee_A(y)$, it follows that $\mu_A(x) = \mu_A(x \sqcap y) \geqslant \mu_A(x) \bigvee \mu_A(y)$. Hence, $\mu_A(x) \geqslant \mu_A(y)$. Conversely, suppose that $(x \leq y \Longrightarrow \mu_A(x) \geqslant \mu_A(y))$. Then it follows that $\mu_A(x \sqcap y) \geqslant \mu_A(x)$ and $\mu(x \sqcap y) \geqslant \mu_A(y)$. Hence, $\mu_A(x \sqcap y) \geqslant \mu_A(x) \bigvee \mu_A(y)$.

(ii) Suppose that $\mu_A(x \sqcup y) \geqslant \mu_A(x) \bigvee \mu_A(y)$. If $x \leq y$ then $x \sqcup y = y$. Since $\mu_A(x) \geqslant \mu_A(x) \bigvee \mu_A(y)$, it follows that $\mu_A(y) = \mu_A(x) \geqslant \mu_A(x) \bigvee A(y)$. Hence, $\mu_A(x) \leq \mu_A(y)$. Conversely, suppose that $(x \leq y \implies \mu_A(x) \leq \mu_A(y))$. Then it follows that $\mu_A(x) \leq \mu(x \sqcup y)$ and $\mu_A(y) \leq \mu_A(x \sqcup y)$. Hence, $\mu_A(x \sqcup y) \geqslant \mu_A(x) \bigvee \mu_A(y)$.

(iii) The proof is similar to (i).

(iv) The proof is similar to (ii).

As corollaries, we obtain the following interesting properties of fuzzy ideals and fuzzy filters. □

Corollary 2.1. *Let L be a lattice and I be a fuzzy ideal on L. Then for any $x, y \in L$ it holds that*

If $x \leq y$, then $\mu_I(x) \geq \mu_I(y)$, (i.e., the map $\mu_I : L \to [0,1]$ is antitone).

Remark 2.3. *The converse of the above implications are not necessarily hold. Indeed, let us consider the lattice L given by the Hasse diagram in Figure 2.4 and I the fuzzy ideal on L given by $I = \{< 0, 0.5 >, < a, 0.4 >, < b, 0.1 >, < 1, 0.1 >\}$. It is easy to verify that $\mu_I(a) = 0.4 \geq \mu_I(b) = 0.1$, but a, b are incomparable elements.*

Corollary 2.2. *Let L be a lattice and F be a fuzzy filter on L. Then for any $x, y \in L$ it holds that*

If $x \leq y$, then $\mu_F(x) \leq \mu_F(y)$, (i.e., the map $\mu_F : L \to [0,1]$ is monotone).

Remark 2.4. *The converse of the above implications are not necessarily hold. Indeed, let us consider the lattice L given by the Hasse diagram in Figure 2.4 and F the fuzzy filter on L given by $F = \{< 0, 0.1 >, < a, 0.2 >, < b, 0.1 >, < 1, 0.4 >\}$. It is easy to verify that $\mu_F(b) = 0.1 \leq \mu_F(a) = 0.2$, but a, b are incomparable elements.*

In the following theorem, we apply the characterization theorem given by Milles et al. [21] to the fuzzy setting.

Theorem 2.2. *Let L be a lattice and I is a fuzzy subset on L. Then it holds that I is a fuzzy ideal on L if and only if the following condition is satisfied:*

$$\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$$

for $x, y \in L$.

Proof. Suppose that I is a fuzzy ideal on L. Then for any $x, y \in L$ it hold that $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$. Since $x \leq x \sqcup y$ and $y \leq x \sqcup y$, from Corollary 2.1 it follows that

$$\mu_I(x) \geq \mu_I(x \sqcup y)$$

and

$$\mu_I(y) \geq \mu_I(x \sqcup y).$$

Hence, $\mu_I(x) \wedge \mu_I(y) \geq \mu_I(x \sqcup y)$. Thus, $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$.
Conversely, suppose that $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$, for any $x, y \in L$. Then it is easy to see that

$$\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$$

Next, we show that $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$ for $x, y \in L$. Let $x, y \in L$. Since $x \sqcup (x \sqcap y) = x$ and $y \sqcup (x \sqcap y) = y$ then it holds that $\mu_I(x \sqcup (x \sqcap y)) = \mu_I(x)$ and $\mu_I(y \sqcup (x \sqcap y)) = \mu_I(y)$. From Definition 2.2 (hypothesis (i) and (ii)) it follows that

$$\mu_I(x) \wedge \mu_I(x \sqcap y) = \mu_I(x)$$

and

$$\mu_I(y) \wedge \mu_I(x \sqcap y) = \mu_I(y).$$

Hence, $\mu_I(x \sqcap y) \geq \mu_I(x)$ and $\mu_I(x \sqcap y) \geq \mu_I(y)$. Thus, $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$, for any $x, y \in L$. Therefore, I is a fuzzy ideal on L.

□

In the same manner, the following theorem provides a basic characterization of fuzzy filter on a lattice.

Theorem 2.3. *Let L be a lattice and F is a fuzzy subset on L. Then it holds that F is a fuzzy filter on L if and only if the following condition is satisfied:*

$$\mu_F(x \sqcap y) = \mu_F(x) \wedge \mu_F(y)$$

for $x, y \in L$.

Proof. The proof is a direct application of Proposition 2.1 and Theorem 2.2 □

In the following theorem, we will show that the image of an fuzzy ideal (resp. fuzzy filter) is a fuzzy ideal (resp. fuzzy filter).

Theorem 2.4. *Let L, L' be two lattices and $f : L \to L'$ be a lattices-homomorphism. Then it holds that*

(i) If A is an fuzzy ideal on L, then $f(A)$ is a fuzzy ideal on L',

(ii) If A is an fuzzy filter on L, then $f(A)$ is a fuzzy filter on L'.

Proof. (i) Let A be a fuzzy ideal on L. For any $y, z \in L'$, it holds that

$$\begin{array}{rcl} f(\mu_A)(y \sqcup z) & = & \sup\{\mu_A(x) \mid x \in f^{-1}(y \sqcup z)\} \\ & = & \sup\{\mu_A(u \vee v) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ & = & \sup\{(\mu_A(u) \wedge \mu_A(v)) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ & = & \sup\{\mu_A(u) \mid u \in f^{-1}(y)\} \wedge \sup\{\mu_A(v) \mid v \in f^{-1}(z)\} \\ & = & f(\mu_A)(y) \wedge f(\mu_A)(z). \end{array}$$

Thus, we can conclude that $f(A)$ is a fuzzy ideal on L'.

(ii) Follows from Proposition 2.1 and (i).

□

In the following theorem, we will show that the inverse image of a fuzzy ideal (resp. fuzzy filter) is a fuzzy ideal (resp. fuzzy filter).

Theorem 2.5. *Let L, L' be two lattices and $f : L \to L'$ be a lattices-homomorphism. Then it holds that*

(i) If A' is a fuzzy ideal on L', then $f^{-1}(A')$ is a fuzzy ideal on L,

(ii) If A' is a fuzzy filter on L', then $f^{-1}(A')$ is a fuzzy filter on L.

Proof. (i) Let A' be a fuzzy ideal on L'. For any $x, y \in L$ it holds that

$$\begin{array}{rcl} f^{-1}(\mu_{A'})(x \sqcup y) & = & \mu_{A'}(f(x \sqcup y)) \\ & = & \mu_{A'}(f(x)) \wedge \mu_{A'}(f(y)) \\ & = & f^{-1}(\mu_{A'})(x) \wedge f^{-1}(\mu_{A'})(y) \end{array}$$

Therefore, $f^{-1}(A')$ is a fuzzy ideal on L.

(ii) Follows from Proposition 2.1 and (i).

□

2.3.2 Characterization of fuzzy ideals and filters in terms of their level sets

In this subsection, we provide some interesting characterizations and properties of fuzzy ideals and fuzzy filters in terms of their level sets. These characterizations are a direct results from the paper [21].

Proposition 2.3. *Let L be a lattice and A is a fuzzy subset on L. The following statements hold*

(i) if A is a fuzzy ideal, then its support $Supp(A)$ is an ideal on L;

(ii) if A is a fuzzy filter, then its support $Supp(A)$ is a filter on L.

Proof. Let $A \in$ fuzzy subset (L).

(i) Suppose that $A \in$ fuzzy subset (L) is a fuzzy ideal. We show that $Supp(A)$ is an ideal on L.

  (a) Let $x \in Supp(A)$ and $y \leq x$, then it hold that $\mu_A(x) > 0$.
  To consider that ($y \leq x$ and $\mu_A(x) > 0$). We suppose that $y \leq x$ and $\mu_A(x) > 0$. Since $y \leq x$, then it holds that $x \sqcup y = x$. This implies that $\mu_A(x) = \mu_A(x \sqcup y) > 0$. From Theorem 2.2 (i), it follows that $\mu_A(x \sqcup y) = \mu_I(x) \wedge \mu_I(y) > 0$. Hence, $\mu_A(y) > 0$. Thus, $y \in Supp(A)$.

  (b) Let $x, y \in Supp(A)$. We show now that $x \sqcup y \in Supp(A)$. We have that $\mu_A(x) > 0$ and $\mu_A(y) > 0$. Since A is a fuzzy ideal, then from Theorem 2.2 (i) it follows that $\mu_A(x \sqcup y) = \mu_I(x) \wedge \mu_I(y) > 0$. Hence, $x \sqcup y \in Supp(A)$. Thus, $Supp(A)$ is an ideal on L.

(ii) Follows from Proposition 2.1 and (i).

□

Remark 2.5. *The converse of the above implications are not necessarily hold. Indeed, let us consider the lattice L given by the Hasse diagram in Figure 2.1 and A is a fuzzy subset on L given by $A = \{< 0, 0.5 >, < a, 0.4 >, < b, 0.4 >, < 1, 0.7 >\}$. It is easy to verify that Supp$(A) = L$ is an ideal and a filter on L, but A is neither a fuzzy ideal nor a fuzzy filter on L.*

The following theorem provides a characterization of fuzzy ideal (resp. filter) in terms of their level sets.

Theorem 2.6. *Let L be a lattice and A is a fuzzy subset on L. The following statements hold*

(i) A is a fuzzy ideal if and only if their level set is an ideal on L;

(ii) A is a fuzzy filter if and only if their level set is a filters on L.

Proof. Let $A \in$ fuzzy subset on L and A_α their level set, where $\alpha \in [0, 1]$.

(i) suppose that A is a fuzzy ideal on L. We show that A_α is an ideal on L for $\alpha \in [0, 1]$.

  (a) Let $\alpha \in [0, 1]$, $x \in A_\alpha$ and $y \in L$ such that $y \leq x$. Since $x \in A_\alpha$, then it hold that $\mu_A(x) \geq \alpha$. Since $y \leq x$, from Corollory 2.1 it follows that $\mu_A(y) \geq \mu_A(x)$. This implies that $\mu_A(y) \geq \alpha$. Hence, $y \in A_\alpha$, for any $\alpha \in [0, 1]$.

  (b) Let $\alpha \in [0, 1]$ and $x, y \in A_\alpha$. Then it holds that $\mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \alpha$. From theorem 2.2 it follows that $\mu_A(x \sqcup y) = \mu_A(x) \wedge \mu_A(y) \geq \alpha$. Hence, $x \sqcup y \in A_\alpha$ for $\alpha \in [0, 1]$.
  Thus, A_α is an ideal on L for $\alpha \in [0, 1]$.

Conversely, suppose that all level sets of A are ideals on L. We show that A is a fuzzy ideal on L. Let $x, y \in L$, $\alpha = \mu_A(x) \wedge \mu_A(y)$. Then it follows that $\mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \alpha$. The case $\alpha = 0$ is obvious. Let $\alpha \in [0, 1]$ and $x, y \in A_\alpha$. Since A_α is an ideal on L, then it holds that $x \sqcup y \in A_\alpha,\ \alpha \in [0, 1]$. This implies that $\mu_A(x \sqcup y) \geq \alpha$. Hence, $\mu_A(x \sqcup y) \geq \mu_A(x) \wedge \mu_A(y)$.
On other hand, let $\alpha = \mu_A(x \sqcup y)$. The case $\alpha = 0$ is also obvious. Otherwise $\alpha \in [0, 1]$, $x \sqcup y \in A_\alpha$. Since A_α is an ideal on L, $x \leq x \sqcup y$ and $y \leq x \sqcup y$, it follows that $\mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \alpha$. Hence, $\mu_A(x) \wedge \mu_A(y) \geq \mu_A(x \sqcup y)$. Therefore, Theorem 2.2 guarantees that A is a fuzzy ideal on L.

(ii) Follows from Proposition 2.1 and (i).

□

Chapter 3

Principal fuzzy ideals and filters on a lattice

In this chapter, we generalize the notion of principal ideal (resp. filter) on a lattice to the setting of fuzzy sets and investigate their various characterizations and properties. More specifically, we show that any principal fuzzy ideal (resp. filter) coincides with a fuzzy down-set (resp. up-set) generated by a fuzzy singleton. Afterwards, for a given fuzzy set, we introduce two fuzzy sets: its fuzzy down-set and up-set, and we investigate their interesting properties.

3.1 Fuzzy down-sets and up-sets

In this section, we introduce the notion of fuzzy down-set (resp. fuzzy up-set) on a lattice analogously to the crisp down-set (resp. up-set), and then we show their interesting properties.

3.1.1 Definitions

Let L be a lattice and S be a subset on L. S is called a down-set (alternative terms include lower-set) if $y \in S$ implies $x \in S$ for all $x \leqslant y$. Dually, S is called an up-set (alternative terms include upper-set) if $y \in S$ implies $x \in S$ for all $y \leqslant x$. For a given subset S on L, we denote by $\downarrow S$ the set of all elements smaller than or equal to some element of S, i.e.,

$$\downarrow S = \{x \in L \mid x \leqslant y, \text{ for some } y \in S\},$$

and $\uparrow S$ the set of all elements bigger than or equal to some element of S, i.e.,

$$\uparrow S = \{x \in L \mid y \leqslant x, \text{ for some } y \in S\}.$$

It is easily to check that $\downarrow S$ (resp. $\uparrow S$) is the smallest down-set (resp. the smallest up-set) containing S. $\downarrow S$ (resp. $\uparrow S$) is called the down-set (resp. the up-set) of S. Similarly, for a given element x on a lattice L, the down-set $\downarrow \{x\}$ ($\downarrow x$, for short) and the up-set $\uparrow \{x\}$ ($\uparrow x$, for short) are defined as

$$\downarrow x = \{y \in L \mid y \leqslant x\} \text{ (resp. } \uparrow x = \{y \in L \mid x \leqslant y\}).$$

Note that if S is a down-set (resp. an up-set), then $\downarrow S$ (resp. $\uparrow S$) coincides with S.

Analogously to a crisp down-set and up-set on a lattice L, we introduce the notions of a fuzzy down-set and a fuzzy up-set. Also, the $\Downarrow S$ and $\Uparrow S$, for any fuzzy set S on L.

Definition 3.1. *Let L be a lattice and $S \in FS(L)$.*

(i) S is called a fuzzy down-set (fuzzy-down-set, for short) if $\mu_S(x) \geq \mu_S(y)$ and for all $x \leqslant y$.

(ii) Dually, S is called a fuzzy up-set (fuzzy-up-set, for short) if $\mu_S(x) \leq \mu_S(y)$ for all $x \leqslant y$.

Definition 3.2. *For a given fuzzy set S on a lattice L we denote by:*

(i) $\Downarrow S$ the fuzzy set associated to S defined as

$$\mu_{\Downarrow S}(x) = \sup_{y \in \uparrow x} \mu_S(y),$$

(ii) $\Uparrow S$ the fuzzy set associated to S defined as

$$\mu_{\Uparrow S}(x) = \sup_{y \in \downarrow x} \mu_S(y),$$

Remark 3.1. *For any crisp set S on a given lattice L, it holds that*

(i) $\Downarrow S = \downarrow S$;

(ii) $\Uparrow S = \uparrow S$.

For a given lattice L and $S \in FS(L)$, it is clear that

(i) $\mu_{\Downarrow S}$ is an antitone mapping and ;

(ii) $\mu_{\Uparrow S}$ is a monotone mapping .

3.1.2 Properties of Fuzzy-down-sets and Fuzzy-up-sets

In this subsection, we show some interesting properties of fuzzy down-sets and up-sets on a lattice. We start with the easier one.

Proposition 3.1. *Let L be a lattice, L^d be its order-dual lattice and $S \in FS(L)$. The following statements hold:*

(i) S is a fuzzy-down-set on L if and only if S is an fuzzy-up-set on L^d;

(ii) S is a fuzzy-up-set on L if and only if S is a fuzzy-down-set on L^d;

(iii) $\Downarrow S$ on L coincides with $\Uparrow S$ on L^d ;

(iv) $\Uparrow S$ on L coincides with $\Downarrow S$ on L^d.

The following proposition shows that Fuzzy-down-sets (resp. Fuzzy-up-sets) on a lattice are closed under the union and intersection of fuzzy sets.

Proposition 3.2. *Let L be a lattice and $R, S \in FS(L)$. It holds that*

(i) If R and S are fuzzy-down-sets, then $R \cup S$ and $R \cap S$ are Fuzzy-down-sets;

(ii) If R and S are fuzzy-up-sets, then $R \cup S$ and $R \cap S$ are Fuzzy-up-sets.

Proof. (i) We only show that $R \cup S$ is a fuzzy-down-set, as $R \cap S$ can be proved analogously. Let $x, y \in L$ such that $x \leqslant y$. Since R and S are fuzzy-down-sets, and $\mu_{R\cup S}(x) = \mu_R(x) \vee \mu_S(x)$, it follows that $\mu_{R\cup S}(x) \geq \mu_R(y) \vee \mu_S(y) = \mu_{R\cup S}(y)$. Thus, $R \cup S$ is a fuzzy-down-set.

(ii) Follows from Proposition 3.1 and (i).

□

Proposition 3.3. *Let L be a lattice and $S \in FS(L)$. It holds that*

(i) $\Downarrow S$ is the smallest fuzzy-down-set containing S;

(ii) $\Uparrow S$ is the smallest fuzzy-up-set containing S.

Proof. (i) At first we show $S \subseteq \Downarrow S$. Let $a \in L$ and $b \in \uparrow a$, then it holds that $\mu_S(a) \leq \sup\limits_{b \in \uparrow a} \mu_S(b) = \mu_{\Downarrow S}(a)$. Hence, $S \subseteq \Downarrow S$. Now, we show that $\Downarrow S$ is a fuzzy-down-set. Let

$x, y \in L$ such that $x \leqslant y$. We have $\mu_{\Downarrow S}(x) = \sup\limits_{t\in\uparrow x}\mu_S(t) \geq \sup\limits_{t\in\uparrow y}\mu_S(t)$. Hence, $\mu_{\Downarrow S}(x) \geq \mu_{\Downarrow S}(y)$. Thus, $\Downarrow S$ is a fuzzy-down-set. Furthermore, let R be a fuzzy-down-set containing S. This implies that $\mu_S(x) \leq \mu_R(x)$ and $\nu_S(x) \geq \nu_R(x)$, for any $x \in L$. Hence, $\sup\limits_{t\in\uparrow x}\mu_S(t) \leq \sup\limits_{t\in\uparrow x}\mu_R(t)$. Thus, $\Downarrow S \subseteq \Downarrow R$. Finally, we conclude that $\Downarrow S$ is the smallest fuzzy-down-set containing S.

(ii) Follows from Proposition 3.1 and (i).

□

From Proposition 3.3, we obtain the following corollary. It shows a characterization of fuzzy-down-sets and fuzzy-up-sets.

Corollary 3.1. *Let L be a lattice and $S \in FS(L)$. The following equivalences hold:*

(i) S is a fuzzy-down-set if and only if $S = \Downarrow S$;

(ii) S is a fuzzy-up-set if and only if $S = \Uparrow S$.

The following propositions list some properties of fuzzy-down and fuzzy-up sets.

Proposition 3.4. *Let L be a lattice and $R, S \in FS(L)$. The following statements hold:*

(i) If $S \subseteq R$, then $\Downarrow S \subseteq \Downarrow R$;

(ii) $\Downarrow(\Downarrow S) = \Downarrow S$;

(iii) $\Downarrow(S \cup R) = \Downarrow S \cup \Downarrow R$;

(iv) $\Downarrow(S \cap R) \subseteq \Downarrow S \cap \Downarrow R$.

Proof. (i) Since $R \subseteq \Downarrow R$, it holds that $S \subseteq \Downarrow R$. From Proposition 3.3, it trivially holds that $\Downarrow S \subseteq \Downarrow R$.

(ii) Follows from Proposition 3.3 and Corollary 3.1.

(iii) On the one hand, we easily verify from (i) that $\Downarrow S \cup \Downarrow R \subseteq \Downarrow (S \cup R)$. On the other hand, since $\Downarrow S \cup \Downarrow R$ is an IF-down-set and $S \cup R \subseteq \Downarrow S \cup \Downarrow R$, it follows from Proposition 3.3 that $\Downarrow(S \cup R) \subseteq \Downarrow S \cup \Downarrow R$. Thus, $\Downarrow(S \cup R) = \Downarrow S \cup \Downarrow R$.

(iv) Follows from Proposition 3.3 and (i).

□

In the same direction, a dual version of Proposition 3.4 can also obtained for fuzzy up-sets. Its proof follows from Propositions 3.1 and 3.4.

Proposition 3.5. *Let L be a lattice and $R, S \in FS(L)$. The following statements hold:*

(i) *If* $S \subseteq R$, *then* $\Uparrow S \subseteq \Uparrow R$;

(ii) $\Uparrow (\Uparrow S) = \Uparrow S$;

(iii) $\Uparrow (S \cup R) = \Uparrow S \cup \Uparrow R$;

(iv) $\Uparrow (S \cap R) \subseteq \Uparrow S \cap \Uparrow R$.

The following result follows immediately from Propositions 3.3, 3.4 and 3.5.

Proposition 3.6. *Let L be a lattice. Then the mappings $\Downarrow$ and $\Uparrow$ define topological closures on the set $FS(L)$ of fuzzy sets on L.*

The following proposition shows the interaction of the *Support* and the *Kernel* with the notions of fuzzy-down-set and fuzzy-up-set.

Proposition 3.7. *Let L be a lattice and $S \in FS(L)$. It holds that*

(i) $Supp(\Downarrow S) = \downarrow Supp(S)$ *and* $Ker(\Downarrow S) = \downarrow Ker(S)$;

(ii) $Supp(\Uparrow S) = \uparrow Supp(S)$ *and* $Ker(\Uparrow S) = \uparrow Ker(S)$.

□

Proof. (i) First, we prove that $Supp(\Downarrow S) = \downarrow Supp(S)$. On the one hand, let $x \in Supp(\Downarrow S)$. Then it holds that

$$\mu_{\Downarrow S}(x) > 0 \ .$$

Since $\mu_{\Downarrow S}(x) > 0$, then $\sup\limits_{y \in \uparrow x} \mu_S(y) > 0$. This implies that there exists $t \in \uparrow x$ such that $\mu_S(t) > 0$. Hence, $t \in Supp(S)$. Since $t \in \uparrow x$, it follows that $x \in \downarrow Supp(S)$.

Thus, $Supp(\Downarrow S) \subseteq \downarrow Supp(S)$. On the other hand, let $x \in \downarrow Supp(S)$. Then it holds that

$$\mu_S(x) > 0 \ .$$

Since $\mu_S(x) > 0$, then $\sup\limits_{y\in\uparrow x}\mu_S(y) = \mu_{\Downarrow S}(x) > 0$. Hence, $x \in Supp(\Downarrow S)$. Thus, $\downarrow Supp(S) \subseteq Supp(\Downarrow S)$. Therefore, $Supp(\Downarrow S) = \downarrow Supp(S)$.

The proof of $Ker(\Downarrow S) = \downarrow Ker(S)$ is analogous.

(ii) Follows from Proposition 3.1 and (i).

□

In the following result, we show that any fuzzy-ideal (resp. fuzzy-filter) on a lattice L is a fuzzy-down-set (resp. fuzzy-up-set) on L.

Theorem 3.1. *Let L be a lattice and $S \in FS(L)$. The following implications hold:*

(i) If S is a fuzzy-ideal, then S is a fuzzy-down-set.

(ii) If S is a fuzzy-filter, then S is a fuzzy-up-set.

Proof. (i) Let $x, y \in L$ such that $x \leqslant y$. Since S is a fuzzy-ideal, it follows that $\mu_S(x) = \mu_S(x \sqcap y) \geq \mu_S(x) \vee \mu_S(y)$. Hence, $\mu_S(x) \geq \mu_S(y)$. Thus, S is a fuzzy-down-set.

(ii) Follows (i) and Proposition 3.1.

□

Combining Theorem 3.1 and Corollary 3.1 leads to the following corollary.

Corollary 3.2. *Let L be a lattice and $S \in FS(L)$. The following implications hold:*

(i) If S is a fuzzy-ideal, then $\Downarrow S = S$.

(ii) If S is a fuzzy-filter, then $\Uparrow S = S$.

Remark 3.2. *The converse of the implications in the above Theorem 3.1 and Corollary 3.2 does not necessarily hold. Indeed, consider L the lattice given by the Hasse diagram in Figure 1 and $S \in FS(L)$ given by $S = \{< 0, 0.7 >, < a, 0.4 >, < b, 0.3 >, < c, 0.2 >, < 1, 0.1 >\}$. We easily verify that*

x	0	a	b	c	1
$\mu_{\Downarrow S}(x)$	0.7	0.4	0.3	0.2	0.1

Then $\Downarrow S = \{< 0, 0.7 >, < a, 0.4 >, < b, 0.3 >, < c, 0.2 >, < 1, 0.1 >\}$. Hence, $\Downarrow S = S$, i.e., S is an IF-down-set. But, $\mu_S(1) = \mu_S(a \sqcup b) \ngeq \min\{0.4, 0.3\}$, which implies that S is not a fuzzy ideal on L .

3.2 Principal fuzzy-ideals and fuzzy-filters on a lattice

In this section, we introduce the notion of principal fuzzy-ideal (resp. fuzzy-filter) on a lattice. Similarly to the crisp case, we characterize these notions in terms of a down set and an up set generated by fuzzy singletons. First, we need to recall the following definition of crisp principal ideal (resp. filter), and the definition of fuzzy singleton.

Definition 3.3. [7] *Let L be a lattice. Then*

(i) the principal ideal generated by an element $x \in L$ is the smallest ideal contains x, and is given by

$$\downarrow x = \{y \in L \mid y \leqslant x\};$$

(ii) the principal filter generated by an element $x \in L$ is the smallest filter contains x, and is given by

$$\uparrow x = \{y \in L \mid x \leqslant y\}.$$

Definition 3.4. *Let L be a lattice. For any $x \in L$, a fuzzy singleton (fuzzy- singleton, for short) $\tilde{x}$ is a fuzzy set on L given by $\tilde{x} = \{\langle t, \mu_{\tilde{x}}(t)\rangle \mid t \in L\}$, where*

$$\mu_{\tilde{x}}(t) = \begin{cases} 1 \ , \ if \ x = t \\ f(t) \ , \ \ otherwise, \end{cases}$$

and

$$\nu_{\tilde{x}}(t) = \begin{cases} 0 \ , \ if \ x = t \\ g(t) \ , \ \ otherwise, \end{cases}$$

such that f is a monotone mapping on $[0, 1[$, for any $t \in L$.

Definition 3.5. *Let L be a lattice, Then*

(i) the principal fuzzy-ideal generated by a fuzzy-singleton $\tilde{x}$ is the smallest fuzzy-ideal contains $\tilde{x}$;

(ii) the principal fuzzy-filter generated by a fuzzy-singleton $\tilde{x}$ is the smallest fuzzy-filter contains $\tilde{x}$.

Next, we characterize the principal fuzzy-ideals (resp. fuzzy-filters) on a lattice in terms of the down-sets (resp. up-sets) generated by fuzzy-singletons on that lattice. The following theorem shows that the fuzzy-down-set (resp. the fuzzy-up-set) generated by a fuzzy singleton on a lattice L is a fuzzy-ideal (resp. is a fuzzy-filter) on L.

Theorem 3.2. *Let L be a lattice and x be an element on L. Then it holds that*

(i) $\Downarrow \tilde{x}$ is a fuzzy-ideal on L;

(ii) $\Uparrow \tilde{x}$ is a fuzzy-filter on L.

Proof. (i) Show for any $x, y \in L$ that

$$\mu_{\Downarrow\tilde{x}}(x \sqcup y) = \mu_{\Downarrow\tilde{x}}(x) \wedge \mu_{\Downarrow\tilde{x}}(y) \text{ and } \nu_{\Downarrow\tilde{x}}(x \sqcup y) = \nu_{\Downarrow\tilde{x}}(x) \vee \nu_{\Downarrow\tilde{x}}(y).$$

Let $a, b \in L$. On the one hand, by Proposition 3.3, $\Downarrow \tilde{x}$ is a fuzzy-down-set, which implies that $\mu_{\Downarrow\tilde{x}}(a) \geq \mu_{\Downarrow\tilde{x}}(a \sqcup b)$ and $\mu_{\Downarrow\tilde{x}}(b) \geq \mu_{\Downarrow\tilde{x}}(a \sqcup b)$. Hence, $\mu_{\Downarrow\tilde{x}}(a) \wedge \mu_{\Downarrow\tilde{x}}(a) \geq \mu_{\Downarrow\tilde{x}}(a \sqcup b)$. On the other hand, since $\mu_{\tilde{x}}$ is a monotone mapping, it holds that $\mu_{\tilde{x}}(a) \leq \mu_{\tilde{x}}(a \sqcup b)$ and $\mu_{\tilde{x}}(b) \leq \mu_{\tilde{x}}(a \sqcup b)$. This implies that $\sup\limits_{a \leqslant t} \mu_{\tilde{x}}(t) \leq \sup\limits_{a \sqcup b \leqslant t} \mu_{\tilde{x}}(t)$ and $\sup\limits_{b \leqslant t} \mu_{\tilde{x}}(t) \leq \sup\limits_{a \sqcup b \leqslant t} \mu_{\tilde{x}}(t)$. Hence, $\sup\limits_{a \leqslant t} \mu_{\tilde{x}}(t) \wedge \sup\limits_{b \leqslant t} \mu_{\tilde{x}}(t) \leq \sup\limits_{a \sqcup b \leqslant t} \mu_{\tilde{x}}(t)$. Thus, $\mu_{\Downarrow\tilde{x}}(a) \wedge \mu_{\Downarrow\tilde{x}}(b) \leq \mu_{\Downarrow\tilde{x}}(a \sqcup b)$. Therefore, $\mu_{\Downarrow\tilde{x}}(a \sqcup b) = \mu_{\Downarrow\tilde{x}}(a) \wedge \mu_{\Downarrow\tilde{x}}(b)$, for all $a, b \in L$. Finally, we conclude that $\Downarrow \tilde{x}$ is a fuzzy-ideal on L.

(ii) Follows dually by using Proposition 3.1 .

□

In the following result, we show a characterization of a principal fuzzy-ideal (resp. fuzzy-filter) in terms of a down-set (resp. up-set) generated by a fuzzy-singleton.

Theorem 3.3. *Let L be a lattice and I (resp. F) be a fuzzy-ideal (resp. fuzzy-filter) on L. Then it holds that*

(i) I is a principal fuzzy-ideal on L if and only if there exists $x \in L$ such that $I = \Downarrow \tilde{x}$;

(ii) F is a principal fuzzy-filter on L if and only if there exists $x \in L$ such that $F = \Uparrow \tilde{x}$.

Proof. We only prove (i), as (ii) can be proved analogously by using Proposition 3.1. Suppose that I is a principal fuzzy-ideal on L. Then there exists a fuzzy-singleton $\tilde{x}$ such that I is the smallest fuzzy-ideal contains $\tilde{x}$. Since $\tilde{x} \subseteq I$, it follows from Proposition 3.4 that $\Downarrow \tilde{x} \subseteq \Downarrow I = I$.

On the other hand, Theorem 3.2 guarantees that $\Downarrow \tilde{x}$ is an ideal. Then the fact that I is the smallest ideal contains $\tilde{x}$ implies that $I \subseteq \Downarrow \tilde{x}$. Thus, $I = \Downarrow \tilde{x}$.

Conversely, $I = \Downarrow \tilde{x}$ is a fuzzy-ideal contains $\tilde{x}$. Now, suppose that J is an other fuzzy-ideal contains $\tilde{x}$. From Proposition 3.4, it holds that $\Downarrow \tilde{x} \subseteq \Downarrow J = J$. Hence, $I = \Downarrow \tilde{x}$ is the smallest IF-ideal contains $\tilde{x}$. Thus, I is a principal fuzzy-ideal. □

In the following proposition, we show that the kernel of a principal fuzzy-ideal (resp. fuzzy-filter) is a crisp principal ideal (resp. filter).

Proposition 3.8. *Let L be a lattice and x be an element on L. Then it holds that*

(i) $Ker(\Downarrow \tilde{x}) = \downarrow x$;

(ii) $Ker(\Uparrow \tilde{x}) = \uparrow x$.

Proof. (i) From Proposition 3.7, it holds that $Ker(\Downarrow \tilde{x}) = \downarrow Ker(\tilde{x})$. This means that

$$Ker(\Downarrow \tilde{x}) = \downarrow \{t \in L \mid \mu_{\tilde{x}}(t) = 1\}.$$

By the definition of fuzzy-singleton, we know that $\mu_{\tilde{x}}(t) = 1$ if and only if $t = x$. Hence,

$$Ker(\Downarrow \tilde{x}) = \downarrow \{x\} = \downarrow x.$$

(ii) Follows from Proposition 3.1 and (i).

Conclusion

In this memoir, we have studied special type of fuzzy ideal and fuzzy filter witch is the principal fuzzy ideal and the principal fuzzy filter with their fundamental properties. More specifically, we have shown that any principal fuzzy ideal (resp. filter) coincides with a fuzzy down-set (resp. up-set) generated by a fuzzy singleton. Afterwards, for a given fuzzy set, we have introduced two fuzzy sets: its fuzzy down-set and up-set, and we have investigated their interesting properties.

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ملخص

في هذاالعمل ، نقوم بتعميم مفهوم المثالي الرئيسي (المرشح الرئيسي على التوالي) على شبكة في المجموعات الضبابية وندرس مختلف خصائصها ومميزاتها. أكثر دقة نقوم بإثبات أن أي مثالي ضبابي رئيسي (مرشح على التوالي) متطابق مع المجموعة الضبابية السفلية(مجموعة الضبابية العلوية) المولدة بالمجموعة الأحادية الضبابية. ثم من أجل مجموعة ضبابية معطاة نقوم بتعريف مجموعتين ضبابيتين : المجموعة الضبابية السفلية والمجموعة الضبابية العلوية ونقوم بدراسة خصائصها المهمة.

كلمات مفتاحية

شبكة، مجموعة ضبابية، المثالية الضبابية الرئيسية، مرشح ضبابي رئيسي .

Abstract

In this work, we generalize the notion of principal ideal (resp. filter) on a lattice to the setting of fuzzy sets and investigate their various characterizations and properties. More specifically, we show that any principal fuzzy ideal (resp. filter) coincides with a fuzzy down-set (resp. up-set) generated by a fuzzy singleton. Afterwards, for a given fuzzy set, we introduce two fuzzy sets : its fuzzy down-set and up-set, and we investigate their interesting properties.

Key words :

Lattice, fuzzy set, Principal fuzzy ideal, Principal fuzzy filter.

Résumé

Dans ce travail, nous généralisons la notion d'idéal principal (resp.filtre) sur un treillis au cadre des ensembles flous et nous étudions leurs diverses caractérisations et propriétés. Plus précisément, nous montrons que tout idéal (resp. filtre) principal flou coïncide avec un down-set (resp. up-set) flou généré par un singleton flou. Ensuite, pour un ensemble flou donné, nous introduisons deux ensembles flous : son down-set et son up-set flous, et nous étudions leurs propriétés intéressantes.

Mot-clés :

Treillis, ensemble flou, idéal flou principal, filter flou principal.