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Contents

Notations

$\mathbb{K}$	The either field $\mathbb{R}$ of the real numbers or the field $\mathbb{C}$ of the complex numbers.
$\mathbb{N}$	The set of all natural numbers $\{0, 1, ..., \}$.
p^*	The extended real number satisfying that $1/p + 1/p^* = 1$.
B_X	The closed unit ball of the Banach space X (i.e., $\{x \in X : \|x\|_X \leq 1\}$).
$\mathcal{L}(X, Y)$	The set of all continuous linear mappings between X and Y.
$\mathcal{K}(X, Y)$	The set of all compact linear mappings between X and Y.
$\mathcal{W}(X, Y)$	The set of all wekly compact linear mappings between X and Y.
X^*	The dual space of the Banach space X.
T^*	The adjoint of the operator $T \in \mathcal{L}(X, Y)$.
$\ell_{p,\omega}(X)$	The Banach space of all weakly p-summable sequences in X.
$\ell_p(X)$	The Banach space of all absolutely p-summable sequences in X.
$\ell_p \langle X \rangle$	The Banach space of all stongly p-summable sequences in X.
$\ell_{p,q} \langle X \rangle$	The Banach space of all stongly (p, q)-summable sequences in X.
$\ell_{m(s,p)}(X)$	The Banach space of all mixed (s, p)-summable sequences in X.
$\ell^{p\sigma}(X)$	The vector space of all (p, σ)-weakly summable sequences in X.
$\ell_p^{q\sigma} \langle X \rangle$	The Banach space of all stongly (p, q, σ)-summable sequences in X.
$C(K)$	The space of all scalar-valued (i.e., real or complex-valued), bounded, continuous functions on the topological space K.
$\Pi_{p,q}$	The set of all linear absolutely (p, q)-summing operators.
$\mathcal{D}_{p,q}$	The set of all linear strongly (p, q)-summing operators.
$\mathcal{N}_{p,q}$	The set of all linear Cohen (p, q)-nuclear operators.
$\Pi_{r,p,q}$	The set of all linear absolutely (r, p, q)-summing operators.
$\mathcal{L}_{(p,m(s,q))}$	The set of all linear $(p, m(s, q))$-summing operators.
$\mathcal{L}_{(m(s,q),p)}$	The set of all linear $(m(s, q), p)$-summing operators.
$\mathcal{L}_{(<p,q>,r)}$	The set of all linear $(< p, q >, r)$-summing operators.
$\mathcal{L}_{(r,<p,q>)}$	The set of all linear $(r, < p, q >)$-summing operators.
$\Pi_{p,\sigma}$	The set of all linear (p, σ)-absolutely continuous operators.
$\mathcal{D}_p^\sigma$	The set of all linear strongly (p, σ)-continuous operators.
$\mathcal{N}_{p,\sigma,q,\nu}$	The set of all linear Cohen (p, σ, q, ν)-nuclear operators.
$\mathcal{D}_{p,\sigma,q,\nu}$	The set of all linear (p, σ, q, ν)-dominated operators.

Introduction

The main topic treated in this thesis is the study of some sequence spaces with values in a Banach space. These spaces of sequences are intimately related to the summability of operators between Banach spaces. For example, the absolutely p-summing operators, introduced by Pietsch [26], are the continuous operators which take weakly p-summable sequences into absolutely p-summable sequences (see [16]).

Many authors take this approach to characterize some class of linear operators that are defined by a summability property. In what follow we mention the most important, Cohen in 1973 [11] defined the Banach space $\ell_p \langle X \rangle$ of strongly p-summable sequences and used it to characterize the ideal of strongly p-summing operators $\mathcal{D}_p$. In 1976, Apiola studied the duality relation between the sequences spaces of weakly p-summable, the absolutely p-summable, and strongly p-summable (see [5]), and applied this relation to characterizing the adjoints of absolutely (p,q)-summing and Cohen (p,q)-nuclear operators.

The paper written by Rushdi Khalil [17] in 1982 is another cornerstone in this line of thought. He introduced there the Banach space of strongly (p,q)-summable sequences, extending the space of strongly p-summable sequences in a natural way, and found his dual. In 2002, Arregui and Blasco published the paper [6], describing some properties of this space but under the name of (p,q)-summing sequences. In the famous book [27] we find another interesting sequence space: the space of mixed (s,p)-summable sequences. In [21] Matos studied the Banach space $\ell_{m(s,q)}(X)$ of mixed (s,q)-summable sequences and used it together with the spaces of absolutely p-summable sequences $\ell_p(X)$, and weakly p-summable sequences $\ell_{p,\omega}(X)$ to define and study the classes of $(p, m(s,q))$-summing operators, and $(m(s,q),p)$-summing operators, (are a natural generalization of the class $\Pi_{p,q}$ of absolutely (p,q)-summing operators). Moreover, if $p=q$ then $(m(s,p),p)$-summing operators is exactly the class of (s,p)-mixing operators introduced by

Pietsch in [27].

In the nineties, Molina and Sánchez-Pérez studied the factorization properties and the trace duality for the class of the (p, σ)-absolutely continuous operators in a series of papers, introducing the class of tensor norms that represent these operators (see [18], [19] and [28]). These last authors introduced the normed space of (p, σ)-weakly summable sequences in order to study the (p, σ)-absolutely continuous operators. In 2014, Achour, Dahia, Sánchez-Pérez and Rueda introduced in [3] the notion of strongly (p, σ)-continuous operators to characterize those operators whose adjoints are (p^*, σ)-absolutely continuous operators.

Recently, Botelho and Campos in [8] introduce provide a new unifying approach to study the Banach ideals of linear and multi-linear operators defined, or characterized, by the transformation of vector-valued sequences.

The main goal of this thesis is to introduce some new Banach spaces, of vector-valued sequences and to study and characterize some ideals of linear operators related to these sequence spaces.

The thesis consists of four chapters. In the preliminaries (Chapter 1) we establish the notation of the thesis. We introduce some important results concerning sequences Banach spaces and we recall the main definitions and properties of the theory of operator ideals that we will use later. Also, we recall the most important results for the classes of the absolutely (p, q)-summing operators.

In Chapter 2 of this thesis, we continue the study of the Banach space of strongly (p, q)-summable sequences. We shall begin by showing that this space coincides with the one of (p, q)-summing sequences (presented by Arregui and Blasco). We investigate the duality between the space of strongly (p, q)-summable sequences and the space of mixed (s, p)-summable sequences, obtaining in this way some relevant properties of this space. Also, we give an application to (r, p, q)-summing operators introduced by Pietsch in [27].

In the next chapter (Chapter 3) we use the results obtained in Chapter 2 to characterize the adjoin of the $(p, m(s, q))$-summing linear operators and $(m(s, q), p)$-summing linear operators, by defining and study two new types of linear summing operators. In this chapter, we use a new unifying approach, introduced by Botelho and Campos in [8], to study Banach's representations of linear operators, using the notions of finitely determined and linearly stable sequence classes.

In the last chapter (Chapter 4) we introduce and study the Banach space $\ell_p^{q\sigma}\langle X\rangle$, of vector-valued sequences which are called strongly (p, q, σ)-summable sequences. We present a new class of the (p, σ, q, ν)-nuclear operators that is defined by using a summability property and we characterize this class and the class of strongly (p, σ)-continuous operators by our Banach sequence space $\ell_p^{q\sigma}\langle X\rangle$. We also present some new results concerning this last class of operators.

Chapter 1

Preliminaries

In this chapter, we present the concepts and results used throughout the thesis on some Banach sequence spaces and operator ideals.

We denote by $\mathbb{R}$ the field of the real numbers. The set of all natural numbers $\{0, 1, \dots, \}$ is denoted by $\mathbb{N}$. If $1 \leq p \leq \infty$, we write p^* for the extended real numbers satisfying that $1/p + 1/p^* = 1$. We write X for a Banach space with the norm $\|\cdot\|_X$. The closed unit ball of X is denoted by B_X that is the set $\{x \in X : \|x\|_X \leq 1\}$. If Y is a Banach space, the space $\mathcal{L}(X, Y)$ of all continuous linear mappings is a Banach space with the norm

$$\|T\| = \|T\|_{\mathcal{L}(X,Y)} = \sup_{\|x\| \leq 1} \|T(x)\| ,$$

and if we have $Y = \mathbb{R}$ the Banach space $\mathcal{L}(X, \mathbb{R})$ is denoted by X^* and it called the dual space of X. For $x \in X$, we shall write $\langle x, x^* \rangle$ (or $\langle x^*, x \rangle$) for the action of the ***functional*** x^* on x. The norm of $x^* \in X^*$ is given by

$$\|x^*\|_{X^*} = \sup \{|\langle x, x^* \rangle| : \|x\|_X \leq 1\} .$$

Throughout the thesis, we call ***operator*** to a continuous linear mapping.

A linear operator $T : X \longrightarrow Y$ between two normed spaces X and Y is an ***isomorphism*** if T is a continuous bijection whose inverse T^{-1} is also continuous. In such case the spaces X and Y are said to be ***isomorphic***. Such a mapping T is an ***isometric isomorphism*** when $\|T(x)\|_Y = \|x\|_X$ for all $x \in X$.

A linear operator T is an ***embedding*** of X into Y if T is an isomorphism into its image $T(X)$. In this case we say that X embeds in Y. If $T : X \longrightarrow Y$ is an embedding such that $\|T(x)\|_Y = \|x\|_X$ for all $x \in X$, then T is said to be an ***isometric embedding***.

Given the continuous linear operator $T : X \longrightarrow Y$, the continuous linear operator $T^* : Y^* \longrightarrow X^*$ defined as

$$T^*(y^*)(x) = y^*(T(x)),$$

for every $y^* \in Y^*$ and $x \in X$ is called the ***adjoint*** of T and has the property that

$$\|T^*\|_{\mathcal{L}(Y^*,X^*)} = \|T\|_{\mathcal{L}(X,Y)} .$$

The $C(K)$ space. If K is a topological space, then by $C(K)$ we mean the space of all scalar-valued (i.e., real or complex-valued), bounded, continuous functions on K. This is a Banach space with the norm

$$\|f\|_\infty = \sup_{x \in K} |f(x)| .$$

Clearly, if K is a compact space then $C(K)$ consists of all continuous, scalar-valued functions.

The dual of the space $C(K)$, K compact, equals the space $M(K)$ of all regular Borel measures (scalar-valued, but obviously not necessarily positive) on K. The duality is defined as

$$\langle f, \mu \rangle = \mu(f) = \int_K f d\mu, \quad f \in C(K), \quad \mu \in M(K).$$

1.1 Some Banach sequences spaces

In order to study the behavior of the summability properties of the linear mappings, several spaces of vector-valued sequences are necessary. We introduce them in this section. Let X be a Banach space over $\mathbb{R}$, and $1 \leq p \leq \infty$.

The classical Banach sequence spaces ℓ_p, ℓ_∞ and c_0 are defined by

$$\ell_p = \left\{ (x_i)_{i=1}^\infty \subset \mathbb{R} : \|(x_i)_{i=1}^\infty\|_{\ell_p} = \left(\sum_{i=1}^\infty |x_i|^p \right)^{\frac{1}{p}} < \infty \right\}, \ \ 1 \leq p < \infty,$$
$$\ell_\infty = \left\{ (x_i)_{i=1}^\infty \subset \mathbb{R} : \|(x_i)_{i=1}^\infty\|_{\ell_\infty} = \sup_{i \in \mathbb{N}} |x_i| < \infty \right\}, \ \ p = \infty,$$
$$c_0 = \left\{ (x_i)_{i=1}^\infty \subset \mathbb{R} : \lim_{i \to \infty} |x_i| = 0 \right\}.$$

1.1.1 Absolutely and weakly p-summable sequences

Let $\ell_p(X)$ be the Banach space of all ***absolutely*** p***-summable*** sequences $(x_i)_{i=1}^\infty$ in X with the norm

$$\|(x_i)_{i=1}^\infty\|_{\ell_p(X)} = \left(\sum_{i=1}^\infty \|x_i\|^p \right)^{\frac{1}{p}}. \tag{1.1}$$

We denote by $\ell_{p,\omega}(X)$ the Banach space of all ***weakly*** p***-summable*** sequences $(x_i)_{i=1}^\infty$ in X with the norm

$$\begin{aligned} \|(x_i)_{i=1}^\infty\|_{\ell_{p,\omega}(X)} &= \sup_{\|x^*\|_{X^*} \leq 1} \|(x^*(x_i))_{i=1}^\infty\|_{\ell_p} \\ &= \sup_{\|x^*\|_{X^*} \leq 1} \left(\sum_{i=1}^\infty |x^*(x_i)|^p \right)^{\frac{1}{p}}. \end{aligned} \tag{1.2}$$

Notice that $\ell_p(X)$ is a linear subspace of $\ell_{p,\omega}(X)$ and

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_{p,\omega}(X)} \leq \|(x_i)_{i=1}^{\infty}\|_{\ell_p(X)},$$

for all $(x_i)_{i=1}^{\infty} \in \ell_p(X)$.
If $p = \infty$ we are restricted to the case of bounded sequences and in $\ell_\infty(X)$ we use the sup norm. Then the spaces $\ell_\infty(X)$ and $\ell_{\infty,\omega}(X)$ coincide and

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_\infty(X)} = \|(x_i)_{i=1}^{\infty}\|_{\ell_{\infty,\omega}(X)} \text{ for all } (x_i)_{i=1}^{\infty} \in \ell_\infty(X).$$

The vector subspace of $\ell_\infty(X)$ that is formed by the sequences $(x_i)_{i=1}^{\infty}$, which as usual can be regarded as infinite sequences by completing with zeros, is denoted by $c_{00}(X)$.

If X is finite dimensional with $\dim X = n$, then $\ell_p(X) = \ell_{p,\omega}(X)$ and

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_{p,\omega}(X)} \leq \|(x_i)_{i=1}^{\infty}\|_{\ell_p(X)} \leq n^{\frac{1}{p}} \|(x_i)_{i=1}^{\infty}\|_{\ell_{p,\omega}(X)}. \tag{1.3}$$

for all $(x_i)_{i=1}^{\infty} \in \ell_p(X)$.
If we take $n = 1$ in (1.3), or $X = \mathbb{R}$, then the spaces $\ell_p(\mathbb{R})$ and $\ell_{p,\omega}(\mathbb{R})$ coincide and we denote $\ell_p(\mathbb{R})$ by ℓ_p. In this case we have

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_{p,\omega}} = \|(x_i)_{i=1}^{\infty}\|_{\ell_p} \text{ for all } (x_i)_{i=1}^{\infty} \in \ell_p. \tag{1.4}$$

We know (see [5, Theorem 2.1]) that $\ell_p(X)^* = \ell_{p^*}(X^*)$ isometrically i.e.,

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_p(X)} = \sup\left\{\left|\sum_{i=1}^{\infty} \langle x_i, x_i^*\rangle\right| : \|(x_i^*)_{i=1}^{\infty}\|_{\ell_{p^*}(X^*)} \leq 1\right\}. \tag{1.5}$$

For the particular case $p = 1$ and $X = \mathbb{R}$ we have

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_1} = \sup\left\{\left|\sum_{i=1}^{\infty} \lambda_i x_i\right| : (\lambda_i)_{i=1}^{\infty} \subset \mathbb{R}, \|(\lambda_i)_{i=1}^{\infty}\|_{\ell_\infty} \leq 1\right\}. \tag{1.6}$$

Let $(x_i^*)_{i=1}^{\infty} \subset X^*$. Then it is also known (see [24, Page 1] or [25, Lemma 2.1]) that

$$\begin{aligned}\|(x_i^*)_{i=1}^{\infty}\|_{\ell_{p,\omega}(X^*)} &= \sup_{\|x^{**}\|\leq 1} \left(\sum_{i=1}^{\infty} |\langle x_i^*, x^{**}\rangle|^p\right)^{\frac{1}{p}} \\ &= \sup_{\|x\|\leq 1} \|(\langle x, x_i^*\rangle)_{i=1}^{\infty}\|_{\ell_p}.\end{aligned} \tag{1.7}$$

1.1.2 Cohen strongly p-summable sequences

The space of Cohen strongly p-summable sequences was introduced by Cohen in [11] in order to give a characterization for the class of strongly p-summing linear operators.

A sequence $(x_i)_{i=1}^{\infty}$ in a Banach space X is ***Cohen strongly p-summable*** if the series $\left|\sum_{i=1}^{\infty} \langle x_i, x_i^* \rangle\right|$ converges for all $(x_i^*)_{i=1}^{\infty} \in \ell_{p^*,\omega}(X^*)$. We denote by $\ell_p \langle X \rangle$ the space of Cohen strongly p-summable sequences in X which is a Banach space (see [10, Proposition 2.1.7]) with the norm

$$\begin{aligned} \|(x_i)_{i=1}^{\infty}\|_{\ell_p\langle X\rangle} &:= \sup\left\{\left|\sum_{i=1}^{\infty} \langle x_i, x_i^* \rangle\right|, \|(x_i^*)_{i=1}^{\infty}\|_{\ell_{p^*,\omega}(X^*)} \leq 1\right\} \\ &= \sup\left\{\sum_{i=1}^{\infty} |\langle x_i, x_i^* \rangle|, \|(x_i^*)_{i=1}^{\infty}\|_{\ell_{p^*,\omega}(X^*)} \leq 1\right\}. \end{aligned} \tag{1.8}$$

Notice that

$$\ell_p \langle X \rangle \subset \ell_p(X) \subset \ell_{p,\omega}(X).$$

Moreover, for all $(x_i)_{i=1}^{\infty} \in \ell_p \langle X \rangle$ we have

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_{p,\omega}(X)} \leq \|(x_i)_{i=1}^{\infty}\|_{\ell_p(X)} \leq \|(x_i)_{i=1}^{\infty}\|_{\ell_p\langle X\rangle}. \tag{1.9}$$

If $p = 1$ we get $\ell_1 \langle X \rangle = \ell_1(X)$ with $\|.\|_{\ell_1\langle X\rangle} = \|.\|_{\ell_1(X)}$.

1.2 Normed operator ideals

1.2.1 Definitions and general properties

A linear operator $T \in \mathcal{L}(X, Y)$ is said to be a ***finite rank*** if $T(X)$ is finite-dimensional. The class of all finite rank linear operators between Banach spaces is denoted by $\mathcal{L}_f(X, Y)$. This space is generated by the mappings of the special form

$$x^* \otimes y : x \longmapsto \langle x, x^* \rangle y$$

i.e., if $T \in \mathcal{L}_f(X, Y)$ we have

$$T = \sum_{i=1}^{n} x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^{n} \subset X^*$ and $(y_i)_{i=1}^{n} \subset Y$ (see [27, page 25]).

Definition 1.2.1. An operator ***ideal*** $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces X and Y its components $\mathcal{I}(X,Y) := \mathcal{L}(X,Y) \cap \mathcal{I}$ satisfy

(1) $\mathcal{I}(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$ which contains the finite rank operators.

(2) The ideal property: if $R \in \mathcal{L}(X,Z)$, $S \in \mathcal{I}(Z,K)$ and $T \in \mathcal{L}(K,Y)$, then the composition $T \circ S \circ R$ is in $\mathcal{I}(X,Y)$.

If $\|\cdot\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ satisfies

(i) $(\mathcal{I}(X,Y), \|\cdot\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces X and Y,

(ii) $\|id_{\mathbb{R}} : \mathbb{R} \longrightarrow \mathbb{R}\|_{\mathcal{I}} = 1$,

(iii) If $R \in \mathcal{L}(X,Z)$, $S \in \mathcal{I}(Z,K)$ and $T \in \mathcal{L}(K,Y)$,

$$\|T \circ S \circ R\|_{\mathcal{I}} \leq \|T\| \, \|S\|_{\mathcal{I}} \, \|R\| \, ,$$

then $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is called a normed (Banach) operator ideal.

The operator ideal $\mathcal{I}$ is said to be ***closed*** if each $\mathcal{I}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$ for the sup norm. The ideal $\mathcal{L}_f$ of finite rank linear operators is the smallest operator ideal and $\mathcal{L}$ the largest one [27, Theorem 1.2.2].

Proposition 1.2.2. *[27, Proposition 6.1.4] Let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ be a normed operator ideal. Then*

$$\|T\| \leq \|T\|_{\mathcal{I}} \, ,$$

for all $T \in \mathcal{I}$.

Definition 1.2.3. (injective operator ideal). A normed operator ideal $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is said to be ***injective*** if for every isometric embedding $i : Y \hookrightarrow G$ and every $T \in \mathcal{L}(X,Y)$ it follows from $i \circ T \in \mathcal{I}(X,G)$ that $T \in \mathcal{I}(X,Y)$. Moreover

$$\|i \circ T\|_{\mathcal{I}} = \|T\|_{\mathcal{I}} \, ,$$

i.e., the ideal does not depend on the image space.

Definition 1.2.4. (dual of an operator ideal). Let $\mathcal{I}$ is a normed operator ideal. A linear mapping $T \in \mathcal{L}(X, Y)$ belongs to $\mathcal{I}^{dual}$ if $T^* \in \mathcal{I}(Y^*, X^*)$, where T^* is the adjoint of the operator T. In this case we define

$$\|T\|_{\mathcal{I}^{dual}} = \|T^*\|_{\mathcal{I}}.$$

Proposition 1.2.5. *[15, Page 114] If $\mathcal{I}$ is a normed (Banach) operator ideal, then $\left(\mathcal{I}^{dual}, \|\cdot\|_{\mathcal{I}^{dual}}\right)$ is as well. This normed (Banach) ideal is called the dual of $\mathcal{I}$.*

Some examples

1) Compact linear operators. A linear operator $T \in \mathcal{L}(X, Y)$ is said to be ***compact*** if $T(B)$ is a precompact subset of Y for every bounded subset B of X.

An equivalent formulation is that T is compact if and only if every bounded sequence $(x_i)_{i=1}^{\infty}$ in X has a subsequence $(x_{i_k})_{k=1}^{\infty}$ such that $(T(x_{i_k}))_{k=1}^{\infty}$ converges in Y.

We denote by $\mathcal{K}(X, Y)$ the vector space of all compact linear mappings from X into Y.

2) Weakly compact linear operators. A continuous linear operator $T : X \to Y$ is said to be ***weakly compact***, in symbols $T \in \mathcal{W}(X, Y)$, if T maps B_X into a relatively weakly compact subset of Y. This is equivalent to say that $(T(x_i))_{i=1}^{\infty}$ has a weakly convergent subsequence for every bounded sequence $(x_i)_{i=1}^{\infty}$ in X.

We know that Schauder's theorem asserts that a bounded linear operator between Banach spaces is compact if and only if its adjoint is. Therefore, both $\mathcal{K}$ and $\mathcal{W}$ are fulfilling this theory, i.e., $\mathcal{K} = \mathcal{K}^{dual}$ and $\mathcal{W} = \mathcal{W}^{dual}$ (see [27], [15] or [7]).

1.2.2 The ideal $\Pi_{p,q}$ of absolutely (p, q)-summing operators

By Mitiagin and Pelczyński [23] a linear operator T from a Banach space X into another Banach space Y is said to be ***absolutely*** (p, q)***-summing***, $1 \leq p, q < \infty$ if there is a constant $C \geq 0$ such that for every finite sequence $(x_i)_{i=1}^{n}$ of points in X the inequality

$$\|(T(x_i))_{i=1}^{n}\|_{\ell_p(Y)} \leq C \, \|(x_i)_{i=1}^{n}\|_{\ell_{q,\omega}(X)} \tag{1.10}$$

is satisfied.

The set of all absolutely (p,q)-summing operators is denoted by $\Pi_{p,q}(X,Y)$ and the absolutely (p,q)-summing norm of $T \in \Pi_{p,q}(X,Y)$ is defined as the infimum of the numbers C satisfying the defining inequality (1.10) and it will be denoted by $\pi_{p,q}(T)$.

The theory of absolutely p-summing operators is based on a crucial criterion due to Pietsch [26]. We recall that a linear operator T is said to be ***absolutely*** $\boldsymbol{p}$***-summing*** (or just $\boldsymbol{p}$***-summing***) if it satisfies the inequality (1.10) for $p = q$. In this case, we use the notation $\Pi_p(X,Y) = \Pi_{p,p}(X,Y)$ for all absolutely p-summing linear operators T and $\pi_p(T) = \pi_{p,p}(T)$ for the norm of $T \in \Pi_p(X,Y)$. Under this norm the class Π_p is a injective Banach ideal.

Nowadays classical Pietsch's domination theorem characterizes the p-summability of an operator by means of a norm domination uniform inequality. Concretely, it says that the mapping $T \in \mathcal{L}(X,Y)$ is p-summing if and only if there exists a constant $C \geq 0$ and a regular Borel probability measure μ on B_{X^*} (with the weak star topology) so that

$$\|T(x)\| \leq C \left(\int_{B_{X^*}} |\langle x, x^* \rangle|^p \, d\mu(x^*) \right)^{\frac{1}{p}}, \qquad x \in X. \tag{1.11}$$

In this case, $\pi_p(T)$ is the least of all the constants $C \geq 0$ such that (1.11) holds. This inequality also provides a factorization of T through the natural mapping $C(B_{X^*}) \longrightarrow L^p(\mu)$, which allows proving a lot of important results in the theory of Banach spaces. A detailed factorization schemes for summing linear and nonlinear operators is studied in [4]. The following result –that is called the Dvoretzky-Rogers Theorem– can be proved using this factorization for p-summing operators.

Theorem 1.2.6. *[16, page 50] If $1 \leq p < \infty$, a Banach space X is finite dimensional if and only if the identity mapping $id_X : X \to X$ is p-summing.*

Other easy consequence of the domination of p-summing operators and Hölder's Inequality is the fact that they form a chain, that is, if $1 \leq p \leq q \leq \infty$, then $\Pi_p \subseteq \Pi_q$.

Chapter 2

Strongly (p, q)-summable sequences

In this chapter, we continue the study of the Banach space of strongly (p, q)-summable sequences [17]. Before we show that this space coincides with the one of (p, q)-summing sequences (presented by Arregui and Blasco) [6], we present the definition of the Banach space of mixed (s, p)-summable sequences (see [27] and [21]) with some important characterization. We investigate the duality between the space of strongly (p, q)-summable sequences and the space of mixed (s, p)-summable sequences, obtaining in this way some relevant properties of this space. Also, we give an application to (r, p, q)-summing operators introduced by Pietsch in [27].

2.1 Definitions and properties

2.1.1 Mixed (s,p)-summable sequences

Among the important spaces that we will need later in our research is the space of mixed (s,p)-summable sequences, which we find a definition for it in the famous Pietsch's book (see [27, Page 225]).

Let X be Banach space and $0 < p \leq s \leq +\infty$ and determine r by $\frac{1}{r} + \frac{1}{s} = \frac{1}{p}$. A sequence $x = (x_i)_{i=1}^{\infty} \in X^{\mathbb{N}}$ is said to be ***mixed*** (s,p)***-summable*** if there exists a sequence $\tau = (\tau_i)_{i=1}^{\infty} \in \ell_r$ and a sequence $x^0 = (x_i^0)_{i=1}^{\infty} \in \ell_{s,\omega}(X)$ such that for all $i \in \mathbb{N}$ we have

$$x_i = \tau_i \cdot x_i^0. \tag{2.1}$$

We denote by $\ell_{m(s,p)}(X)$ the Banach space of all mixed (s,p)-summable sequences of elements of X with the norm

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_{m(s,p)}(X)} = \inf \|(\tau_i)_{i=1}^{\infty}\|_{\ell_r} \left\|(x_i^0)_{i=1}^{\infty}\right\|_{\ell_{s,\omega}(X)},$$

where the infimum is taken over all possible representations of x in the form (2.1).
Note that if $1 \leq p, s_1, s_2 \leq \infty$ such that $s_1 \leq s_2$, then

$$\ell_{m(s_1,p)}(X) \subset \ell_{m(s_2,p)}(X), \tag{2.2}$$

with $\|(x_i)_{i=1}^{\infty}\|_{\ell_{m(s_2,p)}(X)} \leq \|(x_i)_{i=1}^{\infty}\|_{\ell_{m(s_1,p)}(X)}$, for all $(x_i)_{i=1}^{\infty} \in \ell_{m(s_1,p)}(X)$.
If $s = p$ we have

$$\ell_{m(p,p)}(X) = \ell_{p,\omega}(X), \tag{2.3}$$

with $\|\cdot\|_{\ell_{m(p,p)}(X)} = \|\cdot\|_{\ell_{p,\omega}(X)}$ and for $s = \infty$ we obtain

$$\ell_{m(\infty,p)}(X) = \ell_p(X), \tag{2.4}$$

with $\|\cdot\|_{\ell_{m(\infty,p)}(X)} = \|\cdot\|_{\ell_p(X)}$.

The relationships between the various sequences spaces mentioned above are given in the following proposition (see [21]).

Proposition 2.1.1. *Let* $0 < p \leq s \leq \infty$ *then*

$$\ell_p(X) \subset \ell_{m(s,p)}(X) \subset \ell_{p,\omega}(X),$$

with

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_{p,\omega}(X)} \leq \|(x_i)_{i=1}^{\infty}\|_{\ell_{m(s,p)}(X)} \leq \|(x_i)_{i=1}^{\infty}\|_{\ell_p(X)},$$

for all $(x_i)_{i=1}^{\infty} \in \ell_p(X)$.

The important characterization for this space is the following theorem that we can see it and its proof in [27, Page 225]. Let K be a compact Hausdorff space. Then the so-called Borel σ-algebra $\mathcal{B}(K)$ is generated by the collection $\mathcal{C}(K)$ of all open subsets. A Borel probability μ of K is a measure defined on $\mathcal{B}(K)$ such that $\mu(K) = 1$. Moreover, μ is said to be regular, if

$$\mu(B) = \inf \{\mu(G) : B \subseteq G, G \text{ open}\}$$

for every $B \in \mathcal{B}(K)$. The set of all regular Borel probabilities is denoted by $W(K)$.

Theorem 2.1.2. *[27, Theorem 16.4.3] Let $0 < p < s < \infty$ such that $\frac{1}{r} + \frac{1}{s} = \frac{1}{p}$. A sequence $(x_i)_{i=1}^{\infty}$ where $x_i \in X$ for all $i \in \mathbb{N}$, is mixed (s, p)-summable if and only if*

$$\left(\left(\int_{B_{X^*}} |x^*(x_i)|^s \, d\mu(x^*)\right)^{\frac{1}{s}}\right)_{i=1}^{\infty} \in \ell_p,$$

for all $\mu \in W(B_{X^*})$. In this case we have

$$\| (x_i)_{i=1}^{\infty} \|_{\ell_{m(s,p)}(X)} = \sup_{\mu \in W(B_{X^*})} \left\| \left(\left(\int_{B_{X^*}} |x^*(x_i)|^s \, d\mu(x^*)\right)^{\frac{1}{s}}\right)_{i=1}^{\infty} \right\|_{\ell_p}. \quad (2.5)$$

Lemma 2.1.3. *If $(x_i)_{i=1}^{\infty} \in \ell_{m(s,p)}(X)$ then $\|x_i\| \leq \| (x_i)_{i=1}^{\infty} \|_{\ell_{m(s,p)}(X)}$ for all $i \in \mathbb{N}$.*

Proof. If $(x_i)_{i=1}^{\infty} \in \ell_{m(s,p)}(X)$ then $x_i = \tau_i x_i^0$ for all $i \in \mathbb{N}$ with $(\tau_i)_{i=1}^{\infty} \in \ell_r$ and $(x_i^0)_{i=1}^{\infty} \in \ell_{s,\omega}(X)$. For all $i \in \mathbb{N}$ we have

$$\begin{aligned}
\|x_i\| = \left\|\tau_i x_i^0\right\| = |\tau_i| \left\|x_i^0\right\| &\leq \left(\sum_{i=1}^{\infty} |\tau_i|^r\right)^{\frac{1}{r}} \left\|x_i^0\right\| \\
&= \|(\tau_i)_{i=1}^{\infty}\|_{\ell_r} \sup_{\|x^*\| \leq 1} \left|x^*\left(x_i^0\right)\right| \\
&\leq \|(\tau_i)_{i=1}^{\infty}\|_{\ell_r} \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^{\infty} \left|x^*\left(x_i^0\right)\right|^s\right)^{\frac{1}{s}} \\
&= \|(\tau_i)_{i=1}^{\infty}\|_{\ell_r} \left\|\left(x_i^0\right)_{i=1}^{\infty}\right\|_{\ell_{s,\omega}(X)}.
\end{aligned}$$

Since the infimum is taken on all possible factorization of $x_i = \tau_i x_i^0$ then

$$\|x_i\| \leq \inf \|(\tau_i)_{i=1}^{\infty}\|_{\ell_r} \left\|\left(x_i^0\right)_{i=1}^{\infty}\right\|_{\ell_{s,\omega}(X)} = \| (x_i)_{i=1}^{\infty} \|_{\ell_{m(s,p)}(X)}.$$

□

Through the next example we show that $\ell_p(X) \neq \ell_{m(s,p)}(X)$.

Example 2.1.4. Let $(e_n)_n$ be the canonical bases of ℓ_2 and $(x_n)_n \in \ell_2$ defined by $x_n = \frac{1}{n}e_n$ for all $n \in \mathbb{N}^*$. Then $(x_n)_n \in \ell_{m(2,1)}(\ell_2)$ but $(x_n)_n \notin \ell_1(\ell_2)$.

Proof. We have

$$\|x_n\|_{\ell_2} = \left\|\left(0, \cdots, \frac{1}{n}, \cdots\right)\right\|_{\ell_2} = \frac{1}{n},$$

and

$$\|(x_n)_n\|_{\ell_1} = \sum_n \frac{1}{n} = +\infty.$$

then $(x_n)_n \notin \ell_1(\ell_2)$. On the other hand we have $\left(\frac{1}{n}\right)_n \in \ell_2$ since

$$\left\|\left(\frac{1}{n}\right)_n\right\|_{\ell_2} = \left(\sum_n \frac{1}{n^2}\right)^{\frac{1}{2}} < +\infty,$$

and $(e_n) \in \ell_{2,\omega}(\ell_2)$ since

$$\begin{aligned}
\|(e_n)_n\|_{\ell_{2,\omega}(\ell_2)} &= \sup_{\|x^*\|\leq 1} \left(\sum_n |x^*(e_n)|^2\right)^{\frac{1}{2}} \\
&= \sup_{\left(\sum_m |x_m|^2\right)^{\frac{1}{2}}\leq 1} \left(\sum_n \left|\sum_m x_m e_m^*(e_n)\right|^2\right)^{\frac{1}{2}} \\
&= \sup_{\left(\sum_m |x_m|^2\right)^{\frac{1}{2}}\leq 1} \left(\sum_m |x_m|^2\right)^{\frac{1}{2}} = 1.
\end{aligned}$$

This is implies that $(x_n)_n \in \ell_{m(2,1)}(\ell_2)$ and

$$\|(x_n)_n\|_{\ell_{m(2,1)}(\ell_2)} \leq \left\|\left(\frac{1}{n}\right)_n\right\|_{\ell_2} \cdot \|(e_n)_n\|_{\ell_{2,\omega}(\ell_2)} < \infty.$$

□

2.1.2 Strongly (p, q)-summable sequences

Roshdi Khalil in [17] introduced the Banach space of strongly (p, q)-summable sequences, $\ell_{p,q}\langle X\rangle$ $(1 \leq p, q \leq \infty)$, naturally extending the space of strongly q-summable sequences which described as follows.

Definition 2.1.5. A sequence $(x_i)_{i=1}^{\infty}$ in X is ***strongly*** (p,q)***-summable*** if $\sum\limits_{i=1}^{\infty} |x_i^*(x_i)|^p < \infty$ for all $(x_i^*)_{i=1}^{\infty} \in \ell_{q^*,\omega}(X^*)$. The norm of $(x_i)_{i=1}^{\infty} \in \ell_{p,q}\langle X \rangle$ is given by

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_{p,q}\langle X\rangle} := \sup_{\left\|(x_i^*)_{i=1}^{\infty}\right\|_{\ell_{q^*,\omega}(X^*)}\leq 1} \left(\sum_{i=1}^{\infty} |x_i^*(x_i)|^p\right)^{\frac{1}{p}}. \tag{2.6}$$

For $p=1$ we have

$$\ell_{1,q}\langle X\rangle = \ell_q\langle X\rangle, \tag{2.7}$$

with $\|\cdot\|_{\ell_{1,q}\langle X\rangle} = \|\cdot\|_{\ell_q\langle X\rangle}$.

Arregui and Blasco in [6] introduced and studied the Banach space, $\ell_{\pi_{p,q}}(X)$, of (p,q)***-summing sequences*** $(1 \leq p,q < \infty)$, to be the space of all sequence in X such that for some constant $C \geq 0$ we have

$$\left(\sum_{i=1}^{n} |x_i^*(x_i)|^p\right)^{\frac{1}{p}} \leq C \sup_{\|x\|\leq 1} \left(\sum_{i=1}^{n} |x_i^*(x)|^q\right)^{\frac{1}{q}}. \tag{2.8}$$

The smallest constant C such that the above inequality holds is the norm of $(x_i)_{i=1}^{\infty} \in \ell_{\pi_{p,q}}(X)$, and is denoted by $\pi_{p,q}\left((x_i)_{i=1}^{\infty}\right)$.

In the following proposition we show that the spaces $\ell_{\pi_{p,q^*}}(X)$ and $\ell_{p,q}\langle X\rangle$ are coincide. The proof is straightforward using the closed graph theorem.

Proposition 2.1.6. *[29, Proposition 3.1] A sequence $(x_i)_{i=1}^{\infty}$ is (p,q^*)-summing if and only if it is strongly (p,q)-summable. Moreover, we have*

$$\|\,(x_i)_{i=1}^{\infty}\,\|_{\ell_{p,q}\langle X\rangle} = \pi_{p,q^*}\left((x_i)_{i=1}^{\infty}\right).$$

Proof. Suppose that $(x_i)_{i=1}^{\infty} \in \ell_{\pi_{p,q^*}}(X)$ then for any $n \in \mathbb{N}^*$ and $(x_i^*)_{i=1}^{\infty} \subset X^*$ we have

$$\begin{aligned}\left(\sum_{i=1}^{n} |x_i^*(x_i)|^p\right)^{\frac{1}{p}} &\leq \pi_{p,q^*}\left((x_i)_{i=1}^{\infty}\right) \sup_{\|x\|\leq 1}\left(\sum_{i=1}^{n} |x_i^*(x)|^{q^*}\right)^{\frac{1}{q^*}} \\ &\leq \pi_{p,q^*}\left((x_i)_{i=1}^{\infty}\right) \sup_{\|x\|\leq 1}\sup_{n}\left(\sum_{i=1}^{n} |x_i^*(x)|^{q^*}\right)^{\frac{1}{q^*}} \\ &= \pi_{p,q^*}\left((x_i)_{i=1}^{\infty}\right) \|(x_i^*)_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(X^*)}.\end{aligned}$$

By taking the supremum over all $n \in \mathbb{N}^*$ we get

$$\|(x_i^*(x_i))_{i=1}^{\infty}\|_{\ell_p} \leq \pi_{p,q^*}\left((x_i)_{i=1}^{\infty}\right) \|(x_i^*)_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(X^*)}.$$

Taking the supremum over all $(x_i^*)_{i=1}^{\infty} \subset X^*$ such that $\| (x_i^*)_{i=1}^{\infty} \|_{\ell_{q^*,\omega}(X^*)} \leq 1$, we get

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_{p,q}\langle X\rangle} \leq \pi_{p,q^*}\left((x_i)_{i=1}^{\infty}\right) < \infty.$$

Conversely, suppose that $(x_i)_{i=1}^{\infty} \in \ell_{p,q}\langle X\rangle$ and define the linear mapping operator T as follows

$$\begin{array}{cccc} T: & \ell_{q^*,\omega}(X^*) & \longrightarrow & \ell_p \\ & (x_i^*)_{i=1}^{\infty} & \longmapsto & (x_i^*(x_i))_{i=1}^{\infty} \end{array}$$

The graph of T is given by

$$G(T) = \left\{\left((x_i^*)_{i=1}^{\infty}, (\alpha_i)_{i=1}^{\infty}\right) \in \ell_{q^*,\omega}(X^*) \times \ell_p : T\left((x_i^*)_{i=1}^{\infty}\right) = (\alpha_i)_{i=1}^{\infty}\right\}.$$

We show that T has a closed graph (i.e: $\overline{G(T)} \subset G(T)$). Indeed, if $(x^*, \alpha) \in \overline{G(T)}$ then there is a sequence $\left(x^{*k}, \alpha^k\right)_k \in G(T)$ such that

$$\begin{cases} \lim\limits_{k\longrightarrow\infty} x^{*k} = (x_i^*)_{i=1}^{\infty} = x^* \\ \lim\limits_{k\longrightarrow\infty} \alpha^k = (\alpha_i)_{i=1}^{\infty} = \alpha \\ T\left(x^{*k}\right) = \alpha^k \end{cases}$$

We show that $T\left((x_i^*)_{i=1}^{\infty}\right) = (\alpha_i)_{i=1}^{\infty}$. Since $\lim\limits_{k\longrightarrow\infty} x^{*k} = (x_i^*)_{i=1}^{\infty}$ in the space $\ell_{q^*,\omega}(X^*)$, we have

$$\lim_{k\longrightarrow\infty} \|x^{*k} - (x_i^*)_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(X^*)} = \lim_{k\longrightarrow\infty}\left\{\sup_{\|x\|\leq 1}\left(\sum_{i=1}^{\infty}\left|\left(x_i^{*k} - x_i^*\right)(x)\right|^{q^*}\right)^{\frac{1}{q^*}}\right\} = 0$$

Hence

$$\lim_{k\longrightarrow\infty}\left|x_i^{*k}(x) - x_i^*(x)\right| = 0, \text{ for all } x \in X \text{ and } i \in \mathbb{N}^*.$$

Since $x \in X$ is arbitrary it follows that

$$\lim_{k\longrightarrow\infty}\left|x_i^{*k}(x_i) - x_i^*(x_i)\right| = 0, \text{ for all } i \in \mathbb{N}^*. \tag{2.9}$$

On the other hand, since

$$(\alpha_i)_{i=1}^{\infty} = \lim_{k\longrightarrow\infty} \alpha^k = \lim_{k\longrightarrow\infty} T\left(x^{*k}\right) = \lim_{k\longrightarrow\infty}\left(x_i^{*k}(x_i)\right)_{i=1}^{\infty},$$

in the space ℓ_p, we have

$$\lim_{k\longrightarrow\infty}\left(\sum_{i=1}^{\infty}\left|x_i^{*k}(x_i) - \alpha_i\right|^p\right)^{\frac{1}{p}} = 0.$$

This implies that

$$\lim_{k\longrightarrow\infty}\left|x_i^{*k}(x_i)-\alpha_i\right|=0, \text{ for all } i\in\mathbb{N}^*. \tag{2.10}$$

Thus according to (2.9) and (2.10) we have

$$x_i^*(x_i)=\alpha_i, \text{ for all } i\in\mathbb{N}^*,$$

which means that $T\left((x_i^*)_{i=1}^\infty\right)=(\alpha_i)_{i=1}^\infty$. It follows that T has a closed graph. Then

$$\|T\|=\sup_{\|(x_i^*)_{i=1}^\infty\|_{\ell_{q^*,\omega}(X^*)}\leq 1}\left(\sum_{i=1}^\infty |x_i^*(x_i)|^p\right)^{\frac{1}{p}}=\|(x_i)_{i=1}^\infty\|_{\ell_{p,q}\langle X\rangle}<\infty.$$

This implies that for all $n\in\mathbb{N}^*$ we have

$$\left(\sum_{i=1}^n |x_i^*(x_i)|^p\right)^{\frac{1}{p}}\leq\|T\|\sup_{\|x\|\leq 1}\left(\sum_{i=1}^n |x_i^*(x)|^{q^*}\right)^{\frac{1}{q^*}},$$

and then

$$\pi_{p,q^*}\left((x_i)_{i=1}^n\right)\leq\|T\|=\|(x_i)_{i=1}^n\|_{\ell_{p,q}\langle X\rangle}.$$

□

Roshdi Khalil in [17, Theorem 1.3] proved that the topological dual of $\ell_{p,q}\langle X\rangle$ is the product space $\ell_{p^*}\cdot\ell_{q^*,\omega}(X^*)$, (i.e., the set of all elements of the form $x.y$ such that $x\in\ell_{p^*}$ and $y\in\ell_{q^*,\omega}(X^*)$). Pietsch in [27, Page 225] mentioned that this set is exactly the Banach space $\ell_{m(q^*,s^*)}(X^*)$ of all mixed (q^*,s^*)-summable sequences.

Theorem 2.1.7. *[17] Let $1\leq p,q,s\leq\infty$ such that $\frac{1}{s^*}=\frac{1}{p^*}+\frac{1}{q^*}$. The space $\ell_{m(q^*,s^*)}(X^*)$ is isometrically isomorphic to $\left(\ell_{p,q}\langle X\rangle\right)^*$ through the mapping ψ given by*

$$\psi\left((x_i^*)_{i=1}^\infty\right)\left((x_i)_{i=1}^\infty\right)=\sum_{i=1}^\infty x_i^*(x_i),$$

for every $(x_i^*)_{i=1}^\infty\in\ell_{m(q^*,s^*)}(X^*)$ and $(x_i)_{i=1}^\infty\in\ell_{p,q}\langle X\rangle$.

Remark 2.1.8. The duality identification $\left(\ell_{p,q}\langle X\rangle\right)^*=\ell_{m(q^*,s^*)}(X^*)$ yields a new formula for the norm $\|\cdot\|_{\ell_{p,q}\langle X\rangle}$,

$$\|(x_i)_{i=1}^\infty\|_{\ell_{p,q}\langle X\rangle}=\sup_{\left\|(x_i^*)_{i=1}^\infty\right\|_{\ell_{m(q^*,s^*)}(X^*)}\leq 1}\left|\sum_{i=1}^\infty x_i^*(x_i)\right|. \tag{2.11}$$

Consequently, we obtain a special case for the strongly (p, q)-summable sequences.

Corollary 2.1.9. *If* $q = 1$ *then* $\ell_{p,1}\langle X\rangle = \ell_p(X)$ *with* $\|\cdot\|_{\ell_{p,1}\langle X\rangle} = \|\cdot\|_{\ell_p(X)}$.

Proof. For all $(x_i)_{i=1}^{\infty} \in \ell_p(X)$ we have

$$\begin{aligned}\|(x_i)_{i=1}^{\infty}\|_{\ell_{p,1}\langle X\rangle} &= \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{\ell_{m(\infty,p^*)}(X^*)}\le 1}\left|\sum_{i=1}^{\infty} x_i^*(x_i)\right| \\ &= \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{\ell_{p^*}(X^*)}\le 1}\left|\sum_{i=1}^{\infty} x_i^*(x_i)\right| \\ &= \|(x_i)_{i=1}^{\infty}\|_{\ell_p(X)} < \infty.\end{aligned}$$

□

We can use (2.2) and (2.11) to establish useful inclusion relations between $\ell_{q,s}\langle X\rangle$.

Proposition 2.1.10. *[29, Proposition 3.5] Let* $1 \le p, q_1, q_2, s_1, s_2 \le \infty$ *such that* $1 + \frac{1}{p} = \frac{1}{q_1} + \frac{1}{s_1} = \frac{1}{q_2} + \frac{1}{s_2}$*, if* $s_1 \le s_2$ *then* $q_2 \le q_1$ *and we have* $\ell_{q_2,s_2}\langle X\rangle \subset \ell_{q_1,s_1}\langle X\rangle$*. In this case we have*

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_{q_1,s_1}\langle X\rangle} \le \|(x_i)_{i=1}^{\infty}\|_{\ell_{q_2,s_2}\langle X\rangle},$$

for all $(x_i)_{i=1}^{\infty} \in \ell_{q_2,s_2}\langle X\rangle$.

Proof. According to (2.2) and Remark 2.1.8 we have $\ell_{m(s_1^*,p^*)}(X^*) \subseteq \ell_{m(s_2^*,p^*)}(X^*)$ for all $1 \le p, q_1, q_2, s_1, s_2 \le \infty$ such that $1 + \frac{1}{p} = \frac{1}{q_1} + \frac{1}{s_1} = \frac{1}{q_2} + \frac{1}{s_2}$ and $s_2^* \le s_1^*$, $q_1^* \le q_2^*$, this implies that $B_{\ell_{m(s_1^*,p^*)}(X^*)} \subseteq B_{\ell_{m(s_2^*,p^*)}(X^*)}$ and for all $(x_i)_{i=1}^{\infty} \in \ell_{q_2,s_2}\langle X\rangle$ we have

$$\begin{aligned}\sup_{\|(x_i^*)_{i=1}^{\infty}\|_{\ell_{m(s_1^*,p^*)}(X^*)}}\left|\sum_{i=1}^{\infty} x_i^*(x_i)\right| &= \|(x_i)_{i=1}^{\infty}\|_{\ell_{q_1,s_1}\langle X\rangle} \\ &\le \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{\ell_{m(s_2^*,p^*)}(X^*)}}\left|\sum_{i=1}^{\infty} x_i^*(x_i)\right| \\ &= \|(x_i)_{i=1}^{\infty}\|_{\ell_{q_2,s_2}\langle X\rangle} < \infty.\end{aligned}$$

□

In the following proposition we prove a relationship between the space of absolutely p-summable sequences, strongly p-summable sequences and strongly (p,q)-summable sequences.

Proposition 2.1.11. *[29, Proposition 3.6] Let* $1 \leq p, q \leq \infty$, *we have the inclusions*

$$\ell_p(X) \subset \ell_{p,q}\langle X\rangle \text{ and } \ell_q\langle X\rangle \subset \ell_{p,q}\langle X\rangle .$$

In addition $\|\cdot\|_{\ell_{p,q}\langle X\rangle} \leq \|\cdot\|_{\ell_p(X)}$ *and* $\|\cdot\|_{\ell_{p,q}\langle X\rangle} \leq \|\cdot\|_{\ell_q\langle X\rangle}$.

Proof. If $(x_i)_{i=1}^{\infty} \in \ell_p(X)$ we have

$$\begin{aligned} \|(x_i)_{i=1}^{\infty}\|_{\ell_{p,q}\langle X\rangle} &\leq \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{\ell_{\infty,\omega}(X^*)}\leq 1} \|(x_i^*(x_i))_{i=1}^{\infty}\|_{\ell_p} \\ &= \|(x_i)_{i=1}^{\infty}\|_{\ell_{p,1}\langle X\rangle} = \|(x_i)_{i=1}^{\infty}\|_{\ell_p(X)} < \infty. \end{aligned}$$

Similarly, if $(x_i)_{i=1}^{\infty} \in \ell_q\langle X\rangle$,

$$\begin{aligned} \|(x_i)_{i=1}^{\infty}\|_{\ell_{p,q}\langle X\rangle} &\leq \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(X^*)}\leq 1} \|(x_i^*(x_i))_{i=1}^{\infty}\|_{\ell_1} \\ &= \|(x_i)_{i=1}^{\infty}\|_{\ell_{1,q}\langle X\rangle} = \|(x_i)_{i=1}^{\infty}\|_{\ell_q\langle X\rangle} < \infty. \end{aligned}$$

□

2.2 The dual space of $\ell_{m(s,p)}(X)$

Now we prove the main theorem which states that the topological dual of $\ell_{m(s,p)}(X)$ is the space $\ell_{q^*,s^*}\langle X^*\rangle$. In order to prove it, we need the following results. The proof of Proposition 2.2.2 can be found in [9, Page 526].

Proposition 2.2.1. *Let* $(x_i)_{i=1}^{\infty} \in \ell_{p,q}\langle X\rangle$. *Then,*

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_{p,q}\langle X\rangle} = \sup_{\|(\alpha_i)_{i=1}^{\infty}\|_{\ell_{p^*}}\leq 1} \|(\alpha_i x_i)_{i=1}^{\infty}\|_{\ell_q\langle X\rangle} . \tag{2.12}$$

Proof. Let $(x_i)_{i=1}^{\infty} \in \ell_{p,q}\langle X\rangle$, by using the duality between the spaces ℓ_p and ℓ_{p^*} also the duality between the spaces $\ell_q\langle X\rangle$ and $\ell_{q^*,\omega}(X^*)$ we obtain

$$
\begin{aligned}
\|(x_i)_{i=1}^{\infty}\|_{\ell_{p,q}\langle X\rangle} &= \sup_{\left\|(x_i^*)_{i=1}^{\infty}\right\|_{\ell_{q^*,\omega}(X^*)}\le 1} \|(x_i^*(x_i))_{i=1}^{\infty}\|_{\ell_p} \\
&= \sup_{\left\|(x_i^*)_{i=1}^{\infty}\right\|_{\ell_{q^*,\omega}(X^*)}\le 1} \sup_{\left\|(\alpha_i)_{i=1}^{\infty}\right\|_{\ell_{p^*}}\le 1} \left|\sum_{i=1}^{\infty} \alpha_i x_i^*(x_i)\right| \\
&= \sup_{\left\|(\alpha_i)_{i=1}^{\infty}\right\|_{\ell_{p^*}}\le 1} \|(\alpha_i x_i)_{i=1}^{\infty}\|_{\ell_q\langle X\rangle} .
\end{aligned}
$$

□

Proposition 2.2.2. *For all* $(x_i^*)_{i=1}^{\infty} \in \ell_p \langle X^* \rangle$ *we have*

$$
\|(x_i^*)_{i=1}^{\infty}\|_{\ell_p\langle X^*\rangle} = \sup_{\left\|(x_i)_{i=1}^{\infty}\right\|_{\ell_{p^*,\omega}(X)}\le 1} \|(x_i^*(x_i))_{i=1}^{\infty}\|_{\ell_1} . \tag{2.13}
$$

Proposition 2.2.3. *For each* $(x_i^*)_{i=1}^{\infty} \in \ell_{p,q} \langle X^* \rangle$ *we have*

$$
\|(x_i^*)_{i=1}^{\infty}\|_{\ell_{p,q}\langle X^*\rangle} = \sup_{\left\|(x_i)_{i=1}^{\infty}\right\|_{\ell_{q^*,\omega}(X)}\le 1} \|(x_i^*(x_i))_{i=1}^{\infty}\|_{\ell_p} . \tag{2.14}
$$

Proof. Let $(x_i^*)_{i=1}^{\infty} \in \ell_{p,q} \langle X^* \rangle$. By (2.12) and (2.13) we get

$$
\begin{aligned}
\|(x_i^*)_{i=1}^{\infty}\|_{\ell_{p,q}\langle X^*\rangle} &= \sup_{\left\|(\alpha_i)_{i=1}^{\infty}\right\|_{\ell_{p^*}}\le 1} \|(\alpha_i x_i^*)_{i=1}^{\infty}\|_{\ell_q\langle X^*\rangle} \\
&= \sup_{\left\|(\alpha_i)_{i=1}^{\infty}\right\|_{\ell_{p^*}}\le 1} \sup_{\left\|(x_i)_{i=1}^{\infty}\right\|_{\ell_{q^*,\omega}(X)}\le 1} \|(\alpha_i x_i^*(x_i))_{i=1}^{\infty}\|_{\ell_1} \\
&= \sup_{\left\|(x_i)_{i=1}^{\infty}\right\|_{\ell_{q^*,\omega}(X)}\le 1} \sup_{\left\|(\alpha_i)_{i=1}^{\infty}\right\|_{\ell_{p^*}}\le 1} \|(\alpha_i x_i^*(x_i))_{i=1}^{\infty}\|_{\ell_1} \\
&= \sup_{\left\|(x_i)_{i=1}^{\infty}\right\|_{\ell_{q^*,\omega}(X)}\le 1} \|(x_i^*(x_i))_{i=1}^{\infty}\|_{\ell_p} .
\end{aligned}
$$

□

Now we are ready to prove the main theorem. This result asserts that the space of mixed (s, p)-summable sequences is a predual of $\ell_{q^*,s^*} \langle X^* \rangle$.

Theorem 2.2.4. *[29, Theorem 3.10] If* $1 \le p, q, s \le \infty$ *such that* $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$ *then we have the isometric isomorphic identification*

$$
\left(\ell_{m(s,p)}(X)\right)^* = \ell_{q^*,s^*} \langle X^* \rangle .
$$

Proof. First note that $\frac{1}{q^*}+\frac{1}{s^*}=1+\frac{1}{p^*}$. Consider the mapping

$$T:\ell_{q^*,s^*}\langle X^*\rangle \longrightarrow \left(\ell_{m(s,p)}(X)\right)^*$$

defined by

$$T\left((x_i^*)_{i=1}^\infty\right)\left((x_i)_{i=1}^\infty\right)=\sum_{i=1}^\infty x_i^*(x_i),$$

for all $(x_i^*)_{i=1}^\infty \in \ell_{q^*,s^*}\langle X^*\rangle$ and $(x_i)_{i=1}^\infty \in \ell_{m(s,p)}(X)$. It is easy to see that the correspondence T is linear. We take $(x_i^*)_{i=1}^\infty \in \ell_{q^*,s^*}\langle X^*\rangle$ and let $(x_i)_{i=1}^\infty=(\tau_i x_i^0)_{i=1}^\infty \in \ell_{m(s,p)}(X)$ where $(\tau_i)_{i=1}^\infty \in \ell_q$ and $(x_i^0)_{i=1}^\infty \in \ell_{s,\omega}(X)$. Hence, by Hölder's inequality it follows that

$$\begin{aligned}
\left|\sum_{i=1}^\infty x_i^*(x_i)\right| &\leq \sum_{i=1}^\infty |\tau_i|\left|x_i^*\left(x_i^0\right)\right| \\
&\leq \|(\tau_i)_{i=1}^\infty\|_{\ell_q}\left\|\left(x_i^*\left(x_i^0\right)\right)_{i=1}^\infty\right\|_{\ell_{q^*}} \\
&\leq \|(\tau_i)_{i=1}^\infty\|_{\ell_q}\left\|\left(x_i^0\right)_{i=1}^\infty\right\|_{\ell_{s,\omega}(X)} \sup_{\|(z_i)_{i=1}^\infty\|_{\ell_{s,\omega}(X)}\leq 1}\|(x_i^*\left(z_i\right))_{i=1}^\infty\|_{\ell_{q^*}} \\
&= \|(\tau_i)_{i=1}^\infty\|_{\ell_q}\left\|\left(x_i^0\right)_{i=1}^\infty\right\|_{\ell_{s,\omega}(X)}\|(x_i^*)_{i=1}^\infty\|_{\ell_{q^*,s^*}\langle X^*\rangle}.
\end{aligned}$$

Since this holds for all possible factorization of the form $x_i=\tau_i x_i^0$, it follows that,

$$\left|T\left((x_i^*)_{i=1}^\infty\right)\left((x_i)_{i=1}^\infty\right)\right| \leq \|(x_i)_{i=1}^\infty\|_{\ell_{m(s,p)}(X)}\|(x_i^*)_{i=1}^\infty\|_{\ell_{q^*,s^*}\langle X^*\rangle}.$$

Since $(x_i)_{i=1}^\infty$ is arbitrary it follows that

$$\left\|T\left((x_i^*)_{i=1}^\infty\right)\right\|_{\left(\ell_{m(s,p)}(X)\right)^*} \leq \|(x_i^*)_{i=1}^\infty\|_{\ell_{q^*,s^*}\langle X^*\rangle}.$$

This is implies that T is well-defined and continuous. Now consider the linear operator $S:\left(\ell_{m(s,p)}(X)\right)^* \longrightarrow \ell_{q^*,s^*}\langle X^*\rangle$ given by $S(g)=(g\circ\varphi_i)_{i=1}^\infty$ where $g\in\left(\ell_{m(s,p)}(X)\right)^*$ and $\varphi_i: X\longrightarrow \ell_{m(s,p)}(X)$ is the linear operator defined by $\varphi_i(x)=(0,\cdots,0,x,0,\cdots)$ with x placed in the i-th position. Using (2.14) and the duality between ℓ_q and ℓ_{q^*} we obtain

$$\begin{aligned}
\|(g\circ\varphi_i)_{i=1}^{\infty}\|_{\ell_{q^*,s^*}\langle X^*\rangle} &= \sup_{\left\|(x_i)_{i=1}^{\infty}\right\|_{\ell_{s,\omega}(X)}\leq 1} \|(g\circ\varphi_i(x_i))_{i=1}^{\infty}\|_{\ell_{q^*}} \\
&= \sup_{\left\|(x_i)_{i=1}^{\infty}\right\|_{\ell_{s,\omega}(X)}\leq 1} \sup_{\left\|(\alpha_i)_{i=1}^{\infty}\right\|_{\ell_q}\leq 1} \left|\sum_{i=1}^{\infty} g\circ\varphi_i\left(\alpha_i x_i\right)\right| \\
&= \sup_{\left\|(x_i)_{i=1}^{\infty}\right\|_{\ell_{s,\omega}(X)}\leq 1} \sup_{\left\|(\alpha_i)_{i=1}^{\infty}\right\|_{\ell_q}\leq 1} \left|g\left((\alpha_i x_i)_{i=1}^{\infty}\right)\right| \\
&\leq \|g\| \sup_{\left\|(x_i)_{i=1}^{\infty}\right\|_{\ell_{s,\omega}(X)}\leq 1} \sup_{\left\|(\alpha_i)_{i=1}^{\infty}\right\|_{\ell_q}\leq 1} \|(\alpha_i x_i)_{i=1}^{\infty}\|_{\ell_{m(s,p)}(X)} \\
&\leq \|g\| \sup_{\left\|(x_i)_{i=1}^{\infty}\right\|_{\ell_{s,\omega}(X)}\leq 1} \sup_{\left\|(\alpha_i)_{i=1}^{\infty}\right\|_{\ell_q}\leq 1} \|(\alpha_i)_{i=1}^{\infty}\|_{\ell_q} \|(x_i)_{i=1}^{\infty}\|_{\ell_{s,\omega}(X)} \\
&\leq \|g\| < \infty.
\end{aligned}$$

This means that $(g\circ\varphi_i)_{i=1}^{\infty} \in \ell_{q^*,s^*}\langle X^*\rangle$ and we can conclude that S is well-defined, continuous and $\|S\| \leq 1$. On the other hand, a straightforward calculation shows that S and T are inverses. Finally, if $(x_i^*)_{i=1}^{\infty} \in \ell_{q^*,s^*}\langle X^*\rangle$ then

$$\begin{aligned}
\|T\left((x_i^*)_{i=1}^{\infty}\right)\|_{\left(\ell_{m(s,p)}(X)\right)^*} &\leq \|(x_i^*)_{i=1}^{\infty}\|_{\ell_{q^*,s^*}\langle X^*\rangle} \\
&= \|S\circ T\left((x_i^*)_{i=1}^{\infty}\right)\|_{\ell_{q^*,s^*}\langle X^*\rangle} \\
&\leq \|T\left((x_i^*)_{i=1}^{\infty}\right)\|_{\left(\ell_{m(s,p)}(X)\right)^*}.
\end{aligned}$$

□

According to the above theorem and Hahn-Banach theorem, we have the following result.

Corollary 2.2.5. *Let $1 \leq p,q,s \leq \infty$ such that $\frac{1}{q}+\frac{1}{s}=\frac{1}{p}$. For every $(x_i)_{i=1}^{\infty} \in \ell_{m(s,p)}(X)$ we have*

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_{m(s,p)}(X)} = \sup_{\left\|(x_i^*)_{i=1}^{\infty}\right\|_{\ell_{q^*,s^*}\langle X^*\rangle}\leq 1} \left|\sum_{i=1}^{\infty} x_i^*(x_i)\right|. \tag{2.15}$$

A direct consequence of Theorems 2.1.7 and 2.2.4 is the following.

Corollary 2.2.6. *We have the two isometric isomorphism identifications* (*i*) $\ell_{p,q}\langle X\rangle^{**} = \ell_{p,q}\langle X^{**}\rangle$.

(*ii*) $\ell_{m(s,p)}(X)^{**} = \ell_{m(s,p)}(X^{**})$.

Remark 2.2.7. If we apply Theorem 2.1.7 and Theorem 2.2.4 for some extreme cases of parameters p, q and s, we obtain the well-known duality identifications for the sequences spaces $\ell_q\langle X\rangle, \ell_p(X)$ and $\ell_{p,\omega}(X)$.

(*i*) In the Theorem 2.1.7 if we take $p = 1$ then by (2.3) and (2.7) we obtain

$$(\ell_q\langle X\rangle)^* = (\ell_{1,q}\langle X\rangle)^* = \ell_{m(q^*,q^*)}(X^*) = \ell_{q^*,\omega}(X^*).$$

(*ii*) In the Theorem 2.1.7 if we take $p = s$ then by (2.4) and Corollary 2.1.9 we obtain

$$(\ell_p(X))^* = (\ell_{p,1}\langle X\rangle)^* = \ell_{m(\infty,p^*)}(X^*) = \ell_{p^*}(X^*).$$

(*iii*) In the Theorem 2.2.4 if take $s = p$ then we obtain

$$(\ell_{p,\omega}(X))^* = \left(\ell_{m(p,p)}(X)\right)^* = \ell_{1,p^*}\langle X^*\rangle = \ell_{p^*}\langle X^*\rangle.$$

Remark 2.2.8. Using the Corollary 2.2.5 we can prove the special cases of the space $\ell_{m(s,p)}(X)$ as shown in the following

i) If $q^* = 1$ then $q = \infty$ and $p = s$ then we have

$$\begin{aligned}\|(x_i)_{i=1}^{\infty}\|_{\ell_{m(p,p)}(X)} &= \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{\ell_{1,p^*}\langle X^*\rangle}\le 1}\left|\sum_{i=1}^{\infty} x_i^*(x_i)\right| \\ &= \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{\ell_{p^*}\langle X^*\rangle}\le 1}\left|\sum_{i=1}^{\infty} x_i^*(x_i)\right| \\ &= \|(x_i)_{i=1}^{\infty}\|_{\ell_{p,\omega}(X)}.\end{aligned}$$

ii) If $s^* = 1$ then $s = \infty$ and $p = q$ then we have

$$\begin{aligned}\|(x_i)_{i=1}^{\infty}\|_{\ell_{m(\infty,p)}(X)} &= \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{\ell_{p^*,1}\langle X^*\rangle}\le 1}\left|\sum_{i=1}^{\infty} x_i^*(x_i)\right| \\ &= \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{\ell_{p^*}(X^*)}\le 1}\left|\sum_{i=1}^{\infty} x_i^*(x_i)\right| \\ &= \|(x_i)_{i=1}^{\infty}\|_{\ell_p(X)}.\end{aligned}$$

Using the principle of local reflexivity and the Corollary 2.2.6 we obtain the following results.

Proposition 2.2.9. *[29, Proposition 3.13] Let* $1 \leq p, q, s \leq \infty$

i) If $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$ *and* $(x_i^*)_{i=1}^{\infty} \in \ell_{m(s,p)}(X^*)$ *then we have*

$$\begin{aligned} \| (x_i^*)_{i=1}^{\infty} \|_{\ell_{m(s,p)}(X^*)} &= \sup_{\|(x_i^{**})_{i=1}^{\infty}\|_{\ell_{q^*,s^*}\langle X^{**}\rangle} \leq 1} \left| \sum_{i=1}^{\infty} x_i^{**}(x_i^*) \right| \\ &= \sup_{\|(x_i)_{i=1}^{\infty}\|_{\ell_{q^*,s^*}\langle X\rangle} \leq 1} \left| \sum_{i=1}^{\infty} x_i^*(x_i) \right|. \end{aligned} \tag{2.16}$$

ii) If $\frac{1}{s^*} = \frac{1}{q^*} + \frac{1}{p^*}$ *and* $(x_i^*)_{i=1}^{\infty} \in \ell_{p,q}\langle X^* \rangle$ *then we have*

$$\begin{aligned} \| (x_i^*)_{i=1}^{\infty} \|_{\ell_{p,q}\langle X^*\rangle} &= \sup_{\|(x_i^{**})_{i=1}^{\infty}\|_{\ell_{m(q^*,s^*)}(X^{**})} \leq 1} \left| \sum_{i=1}^{\infty} x_i^{**}(x_i^*) \right| \\ &= \sup_{\|(x_i)_{i=1}^{\infty}\|_{\ell_{m(q^*,s^*)}(X)} \leq 1} \left| \sum_{i=1}^{\infty} x_i^*(x_i) \right|. \end{aligned} \tag{2.17}$$

Proof.

i) Let $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$ and $(x_i^*)_{i=1}^{\infty} \in \ell_{m(s,p)}(X^*)$. Since

$$\ell_{q^*,s^*}\langle X\rangle \subseteq \ell_{q^*,s^*}\langle X^{**}\rangle = (\ell_{q^*,s^*}\langle X\rangle)^{**}$$

we have

$$\begin{aligned} \| (x_i^*)_{i=1}^{\infty} \|_{\ell_{m(s,p)}(X^*)} &= \sup_{\left\|(x_i^{**})_{i=1}^{\infty}\right\|_{\ell_{q^*,s^*}\langle X^{**}\rangle} \leq 1} \left| \sum_{i=1}^{\infty} x_i^{**}(x_i^*) \right| \\ &= \sup_{\left\|(x_i^{**})_{i=1}^{\infty}\right\|_{\ell_{q^*,s^*}\langle X^{**}\rangle} \leq 1} \sum_{i=1}^{\infty} |x_i^{**}(x_i^*)| \\ &\geq \sup_{\left\|(x_i)_{i=1}^{\infty}\right\|_{\ell_{q^*,s^*}\langle X\rangle} \leq 1} \sum_{i=1}^{\infty} |x_i^*(x_i)| \\ &= \sup_{\|(x_i)_{i=1}^{\infty}\|_{\ell_{q^*,s^*}\langle X\rangle} \leq 1} \left| \sum_{i=1}^{\infty} x_i^*(x_i) \right|. \end{aligned}$$

For the revers inequality we apply the principle of local reflexivity. Let $N \in \mathbb{N}$ and $\varepsilon > 0$ there exist an application $T_N : \text{Span}\{x_1^{**}, \ldots, x_N^{**}\} \longrightarrow$

X such that $\|T_N\| \leq 1$ and $\forall 1 \leq j \leq N : \left|x_j^{**}\left(x_j^*\right) - x_j^*\left(T_N\left(x_j^{**}\right)\right)\right| < \frac{\varepsilon}{N}$. Then we have

$$\begin{aligned}
\sum_{i=1}^{\infty} \left|x_i^{**}\left(x_i^*\right)\right| &\leq \varepsilon + \sum_{i=1}^{\infty} \left|x_i^*\left(T_N\left(x_i^{**}\right)\right)\right| \\
&= \varepsilon + \sum_{i=1}^{\infty} \left|x_i^*\left(T_N\left(x_i^{**}\right)\right)\right| \times \frac{\|(x_i^{**})_{i=1}^{\infty}\|_{\ell_{q^*,s^*}\langle X^{**}\rangle}}{\|(x_i^{**})_{i=1}^{\infty}\|_{\ell_{q^*,s^*}\langle X^{**}\rangle}} \\
&\leq \varepsilon + \|(x_i^{**})_{i=1}^{\infty}\|_{\ell_{q^*,s^*}\langle X^{**}\rangle} \sum_{i=1}^{\infty} \left|x_i^*\left(x_i\right)\right| \ , \qquad x_i = \frac{T_N\left(x_i^{**}\right)}{\|(x_i^{**})_{i=1}^{\infty}\|_{\ell_{q^*,s^*}\langle X^{**}\rangle}}
\end{aligned}$$

It is easy from $T_N \in \mathcal{L}\left(\text{Span}\left\{x_1^{**}, \ldots, x_N^{**}\right\}, X\right)$ to show that $\left(T_N\left(x_i^{**}\right)\right)_{i=1}^{\infty} \in \ell_{q^*,s^*}\langle X\rangle$ and we have

$$\|(T_N\left(x_i^{**}\right))_{i=1}^{\infty}\|_{\ell_{q^*,s^*}\langle X\rangle} = \|T_N\| \, \|(x_i^{**})_{i=1}^{\infty}\|_{\ell_{q^*,s^*}\langle X^{**}\rangle} \leq \|(x_i^{**})_{i=1}^{\infty}\|_{\ell_{q^*,s^*}\langle X^{**}\rangle}$$

i.e., $\left(\frac{T_N\left(x_i^{**}\right)}{\|(x_i^{**})_{i=1}^{\infty}\|_{\ell_{q^*,s^*}\langle X^{**}\rangle}}\right)_{i=1}^{\infty} \in B_{\ell_{q^*,s^*}\langle X\rangle}$, this implies that

$$\begin{aligned}
\|\left(x_i^*\right)_{i=1}^{\infty}\|_{\ell_{m(s,p)}(X^*)} &= \sup_{\|(x_i^{**})_{i=1}^{\infty}\|_{\ell_{q^*,s^*}\langle X^{**}\rangle} \leq 1} \sum_{i=1}^{\infty} \left|x_i^{**}\left(x_i^*\right)\right| \\
&\leq \varepsilon + \sup_{\|(x_i)_{i=1}^{\infty}\|_{\ell_{q^*,s^*}\langle X\rangle} \leq 1} \sum_{i=1}^{\infty} \left|x_i^*\left(x_i\right)\right| .
\end{aligned}$$

Since this holds for every $\varepsilon > 0$, it follows that

$$\|\left(x_i^*\right)_{i=1}^{\infty}\|_{\ell_{m(s,p)}(X^*)} \leq \sup_{\|(x_i)_{i=1}^{\infty}\|_{\ell_{q^*,s^*}\langle X\rangle} \leq 1} \sum_{i=1}^{\infty} \left|x_i^*\left(x_i\right)\right| .$$

The part *ii)* is proved in a similar way. □

In the following proposition we give a relationship between the space of strongly (q, s)-summable sequences and the spaces of absolutely (strongly) p-summable sequences.

Proposition 2.2.10. *[29, Theorem 3.15] Let* $1 \leq p, q, s \leq \infty$ *such that* $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{s}$ *then*

$$\ell_p\langle X\rangle \subset \ell_{q,s}\langle X\rangle \subset \ell_p(X).$$

In this case we have

$$\|(x_i)_{i=1}^\infty\|_{\ell_p(X)} \le \|(x_i)_{i=1}^\infty\|_{\ell_{q,s}\langle X\rangle} \le \|(x_i)_{i=1}^\infty\|_{\ell_p\langle X\rangle},$$

for each $(x_i)_{i=1}^\infty \in \ell_p\langle X\rangle$.

Proof. Since $\frac{1}{p^*} = \frac{1}{q^*} + \frac{1}{s^*}$ we get $\ell_{p^*}(X^*) \subset \ell_{m(s^*,p^*)}(X^*) \subset \ell_{p^*,\omega}(X^*)$. Let $(x_i)_{i=1}^\infty \in \ell_p\langle X\rangle$. From the duality between $\ell_p(X)$ and $\ell_{p^*}(X^*)$ and Equality (2.2.5), we obtain

$$\begin{aligned}
\|(x_i)_{i=1}^\infty\|_{\ell_p(X)} &= \sup_{\|(x_i^*)_{i=1}^\infty\|_{\ell_{p^*}(X^*)}\le 1} \left|\sum_{i=1}^\infty x_i^*(x_i)\right| \\
&\le \sup_{\|(x_i^*)_{i=1}^\infty\|_{\ell_{m(s^*,p^*)}(X^*)}\le 1} \left|\sum_{i=1}^\infty x_i^*(x_i)\right| \\
&= \|(x_i)_{i=1}^\infty\|_{\ell_{q,s}\langle X\rangle} \\
&\le \sup_{\|(x_i^*)_{i=1}^\infty\|_{\ell_{p^*,\omega}(X^*)}\le 1} \left|\sum_{i=1}^\infty x_i^*(x_i)\right| \\
&= \|(x_i)_{i=1}^\infty\|_{\ell_p\langle X\rangle} < \infty.
\end{aligned}$$

□

Regarding the preceding proposition, let us show with an example the difference between $\ell_{q,s}\langle X\rangle$ and $\ell_p(X)$.

Example 2.2.11. Let $(e_i)_{i=1}^\infty$ the unit vector basis of ℓ_2. The sequence $(x_i)_{i=1}^\infty$ defined by $x_i = \frac{1}{\sqrt{i}}e_i$ belongs to $\ell_\infty(\ell_2)$ but it is not in $\ell_{2,2}\langle \ell_2\rangle$. In order to see this, $\|(x_i)_{i=1}^\infty\|_{\ell_\infty(\ell_2)} = \sup_i \dfrac{1}{\sqrt{i}} = 1$. On the other hand, since

$$\|(e_i^*)_{i=1}^\infty\|_{\ell_{2,\omega}(\ell_2)} = \|(e_i)_{i=1}^\infty\|_{\ell_{2,\omega}(\ell_2)} = 1,$$

we have that

$$\|(x_i)_{i=1}^\infty\|_{\ell_{2,2}\langle \ell_2\rangle} \ge \|(e_i^*(x_i))_{i=1}^\infty\|_{\ell_2} = \left(\sum_{i=1}^\infty \frac{1}{i}\right)^{\frac{1}{2}} = \infty.$$

2.3 Applications to (r,p,q)-summing operators

Throughout this section, let $1 \le p,q,r \le \infty$ such that $\frac{1}{r} \le \frac{1}{p} + \frac{1}{q}$. The definition of (r,p,q)-summing operators is due to Pietsch [27, Section 17.1].

Definition 2.3.1. An operator $T \in \mathcal{L}(X,Y)$ is (r,p,q)***-summing***, in symbols $T \in \Pi_{r,p,q}(X,Y)$, if there is $C > 0$ such that

$$\left\|(y_i^*(T(x_i)))_{i=1}^n\right\|_{\ell_r} \leq C \left\|(x_i)_{i=1}^n\right\|_{\ell_{p,\omega}(X)} \left\|(y_i^*)_{i=1}^n\right\|_{\ell_{q,\omega}(Y^*)}, \tag{2.18}$$

for all $n \in \mathbb{N}$, $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$.

Note that $\Pi_{r,p,q}(X,Y)$ is a Banach space equipped with the norm $\pi_{r,p,q}(T)$ which is the smallest constant C satisfying the defining Inequality (2.18).

Let $\mathcal{X} \subset X^{\mathbb{N}}$ and $\mathcal{Y} \subset Y^{\mathbb{N}}$ be spaces of vector-valued sequences. A linear operator $T \in \mathcal{L}(X,Y)$ between Banach spaces, induces a linear operator $\widehat{T}$ mapping $\mathcal{X}$ into $Y^{\mathbb{N}}$ in the following way : $\widehat{T}\left((x_i)_{i=1}^\infty\right) = (T(x_i))_{i=1}^\infty$ for all $(x_i)_{i=1}^\infty \in \mathcal{X}$. In the sequel, if $\widehat{T}(\mathcal{X}) \subset \mathcal{Y}$, we say that T transfers $\mathcal{X}$ into $\mathcal{Y}$.

As in the case of p-summing operators, the natural way of presenting the summability properties of (r,p,q)-summing operators is by defining the corresponding operator $\widehat{T}$ between $\ell_{p,\omega}(X)$ and $\ell_{r,q^*}\langle Y\rangle$.

Proposition 2.3.2. *[29, Proposition 4.2] The operator $T \in \mathcal{L}(X,Y)$ is (r,p,q)-summing if and only if T transfers $\ell_{p,\omega}(X)$ into $\ell_{r,q^*}\langle Y\rangle$.*

Proof. Indeed, starting from (2.18) and pass to the limit for n tending to ∞ we obtain

$$\left\|(T(x_i))_{i=1}^\infty\right\|_{\ell_{r,q^*}\langle Y\rangle} \leq \pi_{r,p,q}(T) \left\|(x_i)_{i=1}^\infty\right\|_{\ell_{p,\omega}(X)}, \tag{2.19}$$

for all $(x_i)_{i=1}^\infty \in \ell_{p,\omega}(X)$. Then it follows that $\widehat{T} : \ell_{p,\omega}(X) \longrightarrow \ell_{r,q^*}\langle Y\rangle$ is well-defined and $\widehat{T}\left(\ell_{p,\omega}(X)\right) \subset \ell_{r,q^*}\langle Y\rangle$. In addition $\widehat{T}$ is continuous with norm $\leq \pi_{r,p,q}(T)$. Suppose conversely that T transfers $\ell_{p,\omega}(X)$ into $\ell_{r,q^*}\langle Y\rangle$, an appeal to the Closed Graph Theorem shows that $\widehat{T}$ is continuous and

$$\left\|(T(x_i))_{i=1}^n\right\|_{\ell_{r,q^*}\langle Y\rangle} \leq \left\|\widehat{T}\right\| \left\|(x_i)_{i=1}^n\right\|_{\ell_{p,\omega}(X)},$$

and so $T \in \Pi_{r,p,q}(X,Y)$ with $\pi_{r,p,q}(T) \leq \left\|\widehat{T}\right\|$. □

Remark 2.3.3. The operator $T \in \mathcal{L}(X,Y)$ is (r,p,q)-summing if and only if T transfers $\ell_{p,\omega}(X)$ into $\ell_{r,q^*}\langle Y\rangle$.

Corollary 2.3.4. *According to the corollary 2.1.9 and (2.7) we have*

$$\Pi_{r,p,\infty} = \Pi_{r,p}$$

the ideal of absolutely (r,p)-summing operators and

$$\Pi_{1,p,q} = \mathcal{N}_{q^*,p}$$

the ideal of Cohen (q^*,p)-nuclear operators (see [5]).

In the next result we give a new characterization for the (r,p,q)-summing operators by using the Banach spaces of strongly q^*-summable and mixed (p,s)-summable sequences obtaining in this way another corresponding operator $\widehat{T}$ of the (r,p,q)-summing operator T.

Theorem 2.3.5. *[29, Theorem 4.3] Let $s \geq 1$ such that $\frac{1}{s} = \frac{1}{r^*} + \frac{1}{p}$. The operator $T \in \mathcal{L}(X,Y)$ is (r,p,q)-summing if and only if there is a constant $C > 0$ such that for any $x_1, ..., x_n \in X$ we have*

$$\left\|(T(x_i))_{i=1}^n\right\|_{\ell_{q^*}\langle Y\rangle} \leq C \left\|(x_i)_{i=1}^n\right\|_{\ell_{m(p,s)}(X)}. \tag{2.20}$$

Proof. Suppose that $T \in \Pi_{r,p,q}(X,Y)$. Let $(y_i^*)_{i=1}^n \subset Y^*$, $(x_i)_{i=1}^n \subset X$ and $\varepsilon > 0$. Choose $(\alpha_i)_{i=1}^n \subset \mathbb{R}$ and $(z_i)_{i=1}^n \subset X$ such that $x_i = \alpha_i z_i$, $i = 1, ..., n$ and

$$\left\| (\alpha_i)_{i=1}^n \right\|_{\ell_{r^*}} \left\|(z_i)_{i=1}^n\right\|_{\ell_{p,\omega}(X)} \leq (1+\varepsilon) \left\|(x_i)_{i=1}^n\right\|_{\ell_{m(p,s)}(X)}.$$

By Hölder's inequality we get

$$\begin{aligned}\left|\sum_{i=1}^n \alpha_i y_i^*(T(z_i))\right| &\leq \left\| (\alpha_i)_{i=1}^n \right\|_{\ell_{r^*}} \left\|(y_i^*(T(z_i)))_{i=1}^n\right\|_{\ell_r} \\ &\leq \pi_{r,p,q}(T) \left\| (\alpha_i)_{i=1}^n \right\|_{\ell_{r^*}} \left\|(z_i)_{i=1}^n\right\|_{\ell_{p,\omega}(X)} \left\|(y_i^*)_{i=1}^n\right\|_{\ell_{q,\omega}(Y^*)}.\end{aligned}$$

By taking the supremum over all $(y_i^*)_{i=1}^n$ such that $\left\|(y_i^*)_{i=1}^n\right\|_{\ell_{q,\omega}(Y^*)} \leq 1$ we obtain

$$\left\|(T(x_i))_{i=1}^n\right\|_{\ell_{q^*}\langle Y\rangle} \leq \pi_{r,p,q}(T)(1+\varepsilon) \left\|(x_i)_{i=1}^n\right\|_{\ell_{m(p,s)}(X)}.$$

Since this holds for every $\varepsilon > 0$, we obtain (2.20).
Suppose conversely that the operator T satisfies the condition (2.20). For all $(y_i^*)_{i=1}^n \subset Y^*$, $(x_i)_{i=1}^n \subset X$ and $(\alpha_i)_{i=1}^n \subset \mathbb{R}$ we have

$$\begin{aligned}\left|\sum_{i=1}^n \alpha_i y_i^*(T(x_i))\right| &= \left|\sum_{i=1}^n y_i^*(T(\alpha_i x_i))\right| \\ &\leq \left\|(y_i^*)_{i=1}^n\right\|_{\ell_{q,\omega}(Y^*)} \left\|(T(\alpha_i x_i))_{i=1}^n\right\|_{\ell_{q^*}\langle Y\rangle} \\ &\leq C \left\|(y_i^*)_{i=1}^n\right\|_{\ell_{q,\omega}(Y^*)} \left\|(\alpha_i x_i)_{i=1}^n\right\|_{\ell_{m(p,s)}(X)} \\ &\leq C \left\|(y_i^*)_{i=1}^n\right\|_{\ell_{q,\omega}(Y^*)} \left\| (\alpha_i)_{i=1}^n \right\|_{\ell_{r^*}} \left\|(x_i)_{i=1}^n\right\|_{\ell_{p,\omega}(X)}.\end{aligned}$$

Taking the supremum over all $(\alpha_i)_{i=1}^n \subset \mathbb{R}$ such that $\|(\alpha_i)_{i=1}^n\|_{\ell_{r^*}} \leq 1$ we get

$$\|(y_i^*(T(x_i)))_{i=1}^n\|_{\ell_r} \leq C \|(x_i)_{i=1}^n\|_{\ell_{p,\omega}(X)} \|(y_i^*)_{i=1}^n\|_{\ell_{q,\omega}(Y^*)} .$$

□

The next corollary and its proof are similar to Proposition 2.3.2 except that (2.20) is used instead of (2.19).

Corollary 2.3.6. *$T \in \Pi_{r,p,q}(X,Y)$ if and only if T transfers $\ell_{m(p,s)}(X)$ into $\ell_{q^*}\langle Y\rangle$. In addition we have*

$$\pi_{r,p,q}(T) = \left\|\widehat{T}\right\| .$$

Corollary 2.3.7. *According to (2.3) and (2.4) we have the cases that Apiola mentioned in [5] given as follows.*

- *If $p = s$ then $r = 1$ and $\Pi_{1,p,q} = \mathcal{N}_{q^*,p}$ the ideal of Cohen (q^*, p)-nuclear operators.*
- *If $p = \infty$ then $s = r^*$ and $\Pi_{r,\infty,q} = \mathcal{D}_{q^*,r^*}$ the ideal of strongly (q^*, r^*)-summing operators.*

Although the following result is essentially already known (it was proved by Pietsch, see [27, Theorem 17.1.5]), we write a new direct proof that highlights the role of the dual space of $\ell_{m(s,p)}(X)$ and $\ell_{p,q}\langle X\rangle$. By using Corollary 2.3.6, Proposition 2.3.2, the identifications

$$\left(\ell_{m(p,s)}(X)\right)^* = \ell_{r,p^*}\langle X^*\rangle \ \text{ and } \ (\ell_{q^*}\langle Y\rangle)^* = \ell_{q,\omega}(Y^*)$$

and taking into account that the adjoint of the operator $\widehat{T} : \ell_{m(p,s)}(X) \longrightarrow \ell_{q^*}\langle Y\rangle$ can be identified with the operator

$$\begin{aligned} \widehat{T^*} : \ \ell_{q,\omega}(Y^*) &\longrightarrow \ell_{r,p^*}\langle X^*\rangle \\ (y_i^*)_{i=1}^\infty &\longmapsto \widehat{T^*}\left((y_i^*)_{i=1}^\infty\right) = \left(T^*\left(y_i^*\right)\right)_{i=1}^\infty \end{aligned}$$

we have the following.

Theorem 2.3.8. *The operator T belongs to $\Pi_{r,p,q}(X,Y)$ if and only if its adjoint T^* belongs to $\Pi_{r,q,p}(Y^*,X^*)$. Furthermore,*

$$\pi_{r,p,q}(T) = \pi_{r,q,p}(T^*).$$

It is easy to prove the following result.

Corollary 2.3.9. *The operator T belongs to $\Pi_{r,p,q}(X,Y)$ if and only if its second adjoint T^{**} belongs to $\Pi_{r,p,q}(X^{**},Y^{**})$. Furthermore,*

$$\pi_{r,p,q}(T) = \pi_{r,p,q}(T^{**}).$$

Chapter 3

Summing operators related to $\ell_{p,q}\langle X\rangle$

In this chapter, we use the results obtained in Chapter 2 to characterize the adjoin of the $(p, m(s, q))$-summing linear operators and $(m(s, p), q)$-summing linear operators, by defining and study two new classes of linear summing operators. We use a new unifying approach, introduced by Botelho and Campos in [8], to study Banach's representations of linear operators, using the notions of finitely determined and linearly stable sequences classes.

The ideal $\mathcal{D}_p$ of strongly p-summing linear operators is introduced by Cohen in [11] to characterize the ideal Π_{p^*} of all absolutely p^*-summing linear operators, also Cohen in [11] define the ideal $\mathcal{N}_p$ of Cohen p-nuclear operators and gives some properties. These ideals were generalized a natural way to $\Pi_{p,q}$ of absolutely (p,q)-summing linear operators by Mitiagin and Pelczyński in [23], and to $\mathcal{D}_{p,q}$ (respectively $\mathcal{N}_{p,q}$) of strongly (p,q)-summing linear operators (respectively Cohen (p,q)-nuclear operators) by Apiola in [5].

It should be noted that the most important result Cohen provided was his use of the duality relationships between the Banach sequences spaces of weakly p-summable, absolutely p-summable and strongly p-summable to determine the dual of Π_p, $\mathcal{D}_p$ and $\mathcal{N}_p$. This finding became a special case only of what Apiola presented in his article [5].

A linear operator $T : X \longrightarrow Y$ is called ***strongly*** (p,q)***-summing***, $1 \leq p,q \leq +\infty$ if there exists a constant $C \geq 0$ such that for all finite sequences $(x_i)_{i=1}^n$ of points in X, the inequality

$$\left\|(T(x_i))_{i=1}^n\right\|_{\ell_p\langle Y\rangle} \leq C\left\|(x_i)_{i=1}^n\right\|_{\ell_q(X)} \tag{3.1}$$

is satisfied.

The set of all strongly (p,q)-summing operators is denoted by $\mathcal{D}_{p,q}(X,Y)$ and the strongly (p,q)-summing norm of $T \in \mathcal{D}_{p,q}(X,Y)$ is defined as the infimum of the numbers C satisfying the defining inequality (3.1) and it will be denoted by $d_{p,q}(T)$.

If instead of the above inequality we have

$$\left\|(T(x_i))_{i=1}^n\right\|_{\ell_p\langle Y\rangle} \leq C\left\|(x_i)_{i=1}^n\right\|_{\ell_{q,\omega}(X)}, \tag{3.2}$$

T is said to be ***Cohen*** (p,q)***-nuclear*** and denoted by $T \in \mathcal{N}_{p,q}(X,Y)$. The infimum of the numbers C satisfying Inequality (3.2) is denoted by $n_{p,q}(T)$.

It is easy to see that each of the classes $\Pi_{p,q}$, $\mathcal{D}_{p,q}$ and $\mathcal{N}_{p,q}$ forms a Banach ideal in the sense of Pietsch [27].

If we equip the spaces $X^{\mathbb{N}}$ and $Y^{\mathbb{N}}$ with the various norms introduced in (1.1), (1.2) and (1.8) we can reformulate the definitions of the operator classes $\Pi_{p,q}$, $\mathcal{D}_{p,q}$ and $\mathcal{N}_{p,q}$ as follows (see [5]).

i) An operator T is belongs to $\Pi_{p,q}(X,Y)$ if and only if T transfer $\ell_{q,\omega}(X)$ into $\ell_p(Y)$. In this case $\pi_{p,q}(T) = \left\|\widehat{T}\right\|$.

ii) An operator T is belongs to $\mathcal{D}_{p,q}(X,Y)$ if and only if T transfer $\ell_q(X)$ into $\ell_p\langle Y\rangle$. In this case $d_{p,q}(T)=\left\|\widehat{T}\right\|$.

iii) An operator T is belongs to $\mathcal{N}_{p,q}(X,Y)$ if and only if T transfer $\ell_{q,\omega}(X)$ into $\ell_p\langle Y\rangle$. In this case $n_{p,q}(T)=\left\|\widehat{T}\right\|$.

Among the most important results that Apiola presented in [5] is the following theorem.

Theorem 3.0.1. *[5] Let $1\leq p,q\leq+\infty$. We have*

i) An operator T is belongs to $\Pi_{p,q}(X,Y)$ if and only if its adjoint T^* is belongs to $\mathcal{D}_{q^*,p^*}(Y^*,X^*)$. Moreover,

$$\pi_{p,q}(T)=d_{q^*,p^*}(T^*).$$

ii) An operator T is belongs to $\mathcal{D}_{p,q}(X,Y)$ if and only if its adjoint T^* is belongs to $\Pi_{q^*,p^*}(Y^*,X^*)$. Moreover,

$$d_{p,q}(T)=\pi_{q^*,p^*}(T^*).$$

iii) An operator T is belongs to $\mathcal{N}_{p,q}(X,Y)$ if and only if its adjoint T^* is belongs to $\mathcal{N}_{q^*,p^*}(Y^*,X^*)$. Moreover,

$$n_{p,q}(T)=n_{q^*,p^*}(T^*).$$

The above theorem gives us straight (see [5, Theorem 3.4]).

Corollary 3.0.2. *Let $1\leq p,q\leq+\infty$. We have*

i) An operator T is belongs to $\Pi_{p,q}(X,Y)$ if and only if its second adjoint T^{**} is belongs to $\Pi_{p,q}(X^{**},Y^{**})$. In this case

$$\pi_{p,q}(T)=\pi_{p,q}(T^{**}).$$

ii) An operator T is belongs to $\mathcal{D}_{p,q}(X,Y)$ if and only if its second adjoint T^{**} is belongs to $\mathcal{D}_{p,q}(X^{**},Y^{**})$. In this case

$$d_{p,q}(T)=d_{p,q}(T^{**}).$$

iii) An operator T is belongs to $\mathcal{N}_{p,q}(X,Y)$ if and only if its second adjoint T^{**} is belongs to $\mathcal{N}_{p,q}(X^{**},Y^{**})$. In this case

$$n_{p,q}(T)=n_{p,q}(T^{**}).$$

Some early Grothendieck's results of the operator ideal theory deal with the coincidence of the class of all linear and continuous operators between some classical Banach spaces and the class of the p-summing operators between these spaces. The relevant Grothendieck's Theorem establishes that the identity from ℓ_1 into ℓ_2 is absolutely 1-summing, and so 2-summing (see for instance [26, page 338]). However, the adjoint operator is not absolutely 2-summing. This well-known fact motivated the analysis of the concept of a strongly p-summing linear operator ($1 < p \leq \infty$). It was introduced by Cohen in [11] as a characterization of linear operators having absolutely p-summing adjoint.

We recall that a linear operator T between two Banach spaces X and Y is ***strongly p-summing*** for $1 < p \leq \infty$ if it satisfies the inequality (3.1) for $p = q$.

The collection of all strongly p-summing linear operators, denoted by $\mathcal{D}_p = \mathcal{D}_{p,p}$ is a Banach ideal with the ideal norm $d_p(T) = d_{p,p}(T)$ for all $T \in \mathcal{D}_p(X, Y)$. For $p = 1$ we have $\mathcal{D}_1(X, Y) = \mathcal{L}(X, Y)$.

The domination theorem for the strongly p-summing linear operators due to Cohen [11, Theorem 2.3.1].

Theorem 3.0.3. *[11] A linear operator $T \in \mathcal{L}(X, Y)$ is strongly p-summing if and only if there is a constant $C > 0$ and a regular Borel probability measure μ on $B_{Y^{**}}$, (with the weak star topology) so that for all $x \in X$ and $y^* \in Y^*$, the inequality*

$$|\langle T(x), y^* \rangle| \leq C \, \|x\| \left(\int_{B_{Y^{**}}} |\langle y^*, y^{**} \rangle|^{p^*} \, d\mu \right)^{\frac{1}{p^*}}, \tag{3.3}$$

holds.

We recall that if the inequality (3.2) is satisfied for $p = q$ then we say about T that it is ***Cohen p-nuclear*** (see [11]). The collection of all Cohen p-nuclear linear operators, denoted by $\mathcal{N}_p = \mathcal{N}_{p,p}$ is a Banach ideal with the ideal norm $n_p(T) = n_{p,p}(T)$ for all $T \in \mathcal{N}_p(X, Y)$. For $p = q$ the results presented in the Theorem 3.0.1 and the Corollary 3.0.2 also remain valid both each $\Pi_p, \mathcal{D}_p$ and $\mathcal{N}_p$.

3.1 The ideals of $(p, m(s,q))$ and $(m(s,q), p)$-summing operators

3.1.1 The ideal of $(p, m(s,q))$-summing operators

The notion of $(p, m(s,q))$-summing operators between the Banach spaces X and Y is introduced by Matos in [20] as a natural generalization of the class of absolutely (p,q)-summing linear operators. Matos does not pretend to give a full exposition of the theory of these operators, but just gave the essentials that motivate the study of the non-linear absolutely summing mappings between Banach spaces.

Definition 3.1.1. [20, Definition 2.2] For $0 < q \leq s \leq \infty$ and $p \geq q$ a linear operator T from X into Y is said to be $(p, m(s,q))$-***summing*** if $(T(x_i))_{i=1}^{\infty} \in \ell_p(Y)$ for each $(x_i)_{i=1}^{\infty} \in \ell_{m(s,q)}(X)$. When $s = q < \infty$ the operator T is absolutely (p,q)-summing.

We denote by $\mathcal{L}_{(p,m(s,q))}$ the class of all $(p, m(s,q))$-summing operators, and by $\|T\|_{(p,m(s,q))}$ the norm of $T \in \mathcal{L}_{(p,m(s,q))}(X,Y)$. If we had $p < q$ in the above definition, the only linear mapping T satisfying the definition would be $T = 0$. The most important characterization of these mappings are mentioned in the following theorem.

Theorem 3.1.2. *[21, Theorem 3.1.2] If T is a linear mapping from X into Y, then the following conditions are equivalent*

(1) T is $(p, m(s,q))$-summing on X,

(2) The operator $\widehat{T}$ is well defined, linear and continuous from $\ell_{m(s,q)}(X)$ into $\ell_p(Y)$,

(3) There is $C \geq 0$ such that

$$\|(T(x_i))_{i=1}^{n}\|_{\ell_p(Y)} \leq C \|(x_i)_{i=1}^{n}\|_{\ell_{m(s,q)}(X)}, \tag{3.4}$$

for every $n \in \mathbb{N}$ and $x_i \in X$,

(4) There is $D \geq 0$ such that

$$\|(T(x_i))_{i=1}^{\infty}\|_{\ell_p(Y)} \leq D \|(x_i)_{i=1}^{\infty}\|_{\ell_{m(s,q)}(X)}, \tag{3.5}$$

for every $(x_i)_{i=1}^{\infty} \in \ell_{m(s,q)}(X)$. In this case

$$\left\|\widehat{T}\right\| = \inf\{C : C \text{ satisfies } (3.4)\} = \inf\{D : D \text{ satisfies } (3.5)\}.$$

Under some requirements, we can show the coincidence between the spaces $\mathcal{L}_{(1,m(q,s))}(X,Y)$ and $\Pi_{r,q}(X,Y)$.

Proposition 3.1.3. *Let* $1 \leq q,r,s \leq \infty$ *such that* $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{s}$ *then the spaces* $\mathcal{L}_{(1,m(q,s))}(X,Y)$ *and* $\Pi_{r,q}(X,Y)$ *are coincide. In this case*

$$\|\cdot\|_{(1,m(q,s))} = \pi_{r,q}(\cdot)$$

Proof. Suppose that $T \in \mathcal{L}_{(1,m(q,s))}(X,Y)$. Let $(x_i)_{i=1}^n \subset X$, we put $x_i = \alpha_i z_i$, $i=1,...,n$ such that $(\alpha_i)_{i=1}^n \in \ell_{r^*}$ and $(z_i)_{i=1}^n \in \ell_{q,\omega}(X)$ then according to Theorem 3.1.2 and the duality relationships between $\ell_1(Y)$ and $\ell_\infty(Y^*)$ we have

$$\begin{aligned}\|(T(x_i))_{i=1}^n\|_{\ell_1(Y)} &= \sup_{\left\|(y_i^*)_{i=1}^n\right\|_{\ell_\infty(Y^*)}\leq 1} \left|\sum_{i=1}^n \alpha_i y_i^*(T(z_i))\right| \\ &\leq \|T\|_{(1,m(q,s))} \|(x_i)_{i=1}^n\|_{\ell_{m(s,q)}(X)} \\ &\leq \|T\|_{(1,m(q,s))} \|(\alpha_i)_{i=1}^n\|_{\ell_{r^*}} \|(z_i)_{i=1}^n\|_{\ell_{q,\omega}(X)}.\end{aligned}$$

By taking the supremum over all $(\alpha_i)_{i=1}^n$ such that $\|(\alpha_i)_{i=1}^n\|_{\ell_{r^*}} \leq 1$ we obtain

$$\begin{aligned}\sup_{\left\|(y_i^*)_{i=1}^n\right\|_{\ell_\infty(Y^*)}\leq 1} \|y_i^*\left(T(z_i)\right)_{i=1}^n\|_{\ell_r} &= \|(T(z_i))_{i=1}^n\|_{\ell_{r,1}\langle Y\rangle} \\ &\leq \|T\|_{(1,m(q,s))} \|(z_i)_{i=1}^n\|_{\ell_{q,\omega}(X)}.\end{aligned}$$

Since $\|(T(z_i))_{i=1}^n\|_{\ell_{r,1}\langle Y\rangle} = \|(T(z_i))_{i=1}^n\|_{\ell_r(Y)}$, this implies that $T \in \Pi_{r,q}(X,Y)$ and $\pi_{r,q}(T) \leq \|T\|_{(1,m(q,s))}$.

Conversely, suppose that $T \in \Pi_{r,q}(X,Y)$. Let $(y_i^*)_{i=1}^n \subset Y^*$, $(x_i)_{i=1}^n \subset X$ and $\varepsilon > 0$. Choose $(\alpha_i)_{i=1}^n \subset \mathbb{R}$ and $(z_i)_{i=1}^n \subset X$ such that $x_i = \alpha_i z_i$, $i=1,...,n$ and

$$\|\left(\alpha_i\right)_{i=1}^n\|_{\ell_{r^*}} \|(z_i)_{i=1}^n\|_{\ell_{q,\omega}(X)} \leq (1+\varepsilon)\,\|(x_i)_{i=1}^n\|_{\ell_{m(q,s)}(X)}.$$

By Hölder's inequality we get

$$\begin{aligned}\left|\sum_{i=1}^n \alpha_i y_i^*(T(z_i))\right| &\leq \|(\alpha_i)_{i=1}^n\|_{\ell_{r^*}} \|(y_i^* T(z_i))_{i=1}^n\|_{\ell_r} \\ &\leq \|(\alpha_i)_{i=1}^n\|_{\ell_{r^*}} \|(y_i^*)_{i=1}^n\|_{\ell_\infty(Y^*)} \|(T(z_i))_{i=1}^n\|_{\ell_r(Y)} \\ &\leq \pi_{r,q}(T) \|(\alpha_i)_{i=1}^n\|_{\ell_{r^*}} \|(y_i^*)_{i=1}^n\|_{\ell_\infty(Y^*)} \|(z_i)_{i=1}^n\|_{\ell_{q,\omega}(X)}.\end{aligned}$$

By taking the supremum over all $(y_i^*)_{i=1}^n$ such that $\|(y_i^*)_{i=1}^n\|_{\ell_\infty(Y)} \leq 1$ we obtain

$$\|(T(x_i))_{i=1}^n\|_{\ell_1(Y)} \leq \pi_{r,q}(T)(1+\varepsilon)\,\|(x_i)_{i=1}^n\|_{\ell_{m(q,s)}(X)}.$$

Since this holds for every $\varepsilon > 0$, we get that $T \in \mathcal{L}_{(1,m(q,s))}(X,Y)$ and

$$\|T\|_{(1,m(q,s))} \leq \pi_{r,q}(T).$$

□

If we set

$$\begin{aligned}\|T\|_{(p,m(s,q))} = \left\|\widehat{T}\right\| &= \inf\{C : C \text{ satisfies } (3.4)\}\\ &= \inf\{D : D \text{ satisfies } (3.5)\},\end{aligned}$$

for every $T \in \mathcal{L}_{(p,m(s,q))}(X,Y)$ then $\left(\mathcal{L}_{(p,m(s,q))}, \|.\|_{(p,m(s,q))}\right)$ is a Banach space for $1 \leq p \leq \infty$ (complete p-normed space, if $0 < p < 1$).

The following result will be referred to as the Dvoretzky-Rogers Theorem for the mixed (s,q)-summable sequences (see [21, Theorem 3.4.8]).

Theorem 3.1.4. *If $0 < p \leq s < \infty$, a Banach space X is finite dimensional if and only if $\ell_{m(s,p)}(X) = \ell_p(X)$.*

3.1.2 The ideal of $(m(s,q),p)$-summing operators

In his famous book [27, Page 279], Pietsch introduced the concept of (s,p)-mixing operators between Banach spaces, in an attempt to give a generalization to the ideal of p-summing linear operators. As such, Matos in [21] introduced the concept of $(m(s,q),p)$-summing linear operators to become a natural generalization –in a manner similar to what was mentioned in the Definition 3.1.1– the class of (q,p)-summing operators.

Definition 3.1.5. [21, Definition 4.1.1] For $0 < q \leq s \leq \infty$ and $p \leq q$ a linear mapping T from X into Y is said to be $(m(s,q),p)$***-summing*** if $(T(x_i))_{i=1}^\infty \in \ell_{m(s,q)}(Y)$ for each $(x_i)_{i=1}^\infty \in \ell_{p,\omega}(X)$. When $s = \infty$ the operator T is absolutely (q,p)-summing. If $q = p$ the operator T is said to be (s,p)***-mixing***.

We denote by $\mathcal{L}_{(m(s,q),p)}$ the class of all $(m(s,q),p)$-summing operators and by $\|T\|_{(m(s,q),p)}$ the norm of $T \in \mathcal{L}_{(m(s,q),p)}(X,Y)$. If we had $p > q$ in the above definition, the only linear mapping T satisfying the definition would be $T = 0$. The following theorem is an important characterization for these mappings.

Theorem 3.1.6. *[21, Theorem 4.1.2] If T is a linear mapping from X into Y, then the following conditions are equivalent*

(1) T is $(m(s,q),p)$-summing on X,

(2) The operator $\widehat{T}$ is well defined and linear from $\ell_{p,\omega}(X)$ into $\ell_{m(s,q)}(Y)$,

(3) The operator $\widehat{T}$ is well defined, linear and continuous from $\ell_{p,\omega}(X)$ into $\ell_{m(s,q)}(Y)$,

(4) There is $C \geq 0$ such that

$$\|(T(x_i))_{i=1}^{n}\|_{\ell_{m(s,q)}(Y)} \leq C \, \|(x_i)_{i=1}^{n}\|_{\ell_{p,\omega}(X)}, \tag{3.6}$$

for every $n \in \mathbb{N}$ and $x_i \in X$,

(5) There is $D \geq 0$ such that

$$\|(T(x_i))_{i=1}^{\infty}\|_{\ell_{m(s,q)}(Y)} \leq D \, \|(x_i)_{i=1}^{\infty}\|_{\ell_{p,\omega}(X)}, \tag{3.7}$$

for every $(x_i)_{i=1}^{\infty} \in \ell_{p,\omega}(X)$. In this case we have

$$\left\|\widehat{T}\right\| = \inf\{C : C \textit{ satisfies } (3.6)\} = \inf\{D : D \textit{ satisfies } (3.7)\}.$$

If we set

$$\|T\|_{(m(s,q),p)} = \left\|\widehat{T}\right\| = \inf\{C : C \text{ satisfies } (3.6)\}$$
$$= \inf\{D : D \text{ satisfies } (3.7)\},$$

for every $T \in \mathcal{L}_{(m(s,q),p)}(X,Y)$ then $\left(\mathcal{L}_{(m(s,q),p)}, \|.\|_{(m(s,q),p)}\right)$ is a Banach space for $1 \leq q \leq \infty$ (complete q-normed space, if $0 < q < 1$).

Pietsch in [27, Theorem 20.1.4] state an important characterization for the (s,p)-mixing operator and Matos proved same characterization for the class of $(m(s,q),p)$-summing operators.

Theorem 3.1.7. *[21, Theorem 4.1.5] An operator T is $(m(s,q),p)$-summing if and only if there exists a constant $C \geq 0$ such that*

$$\left(\sum_{i=1}^{m}\left(\sum_{k=1}^{n}|\langle T(x_i), y_k^*\rangle|^s\right)^{\frac{q}{s}}\right)^{\frac{1}{q}} \leq C \, \|(x_i)_{i=1}^{m}\|_{\ell_{p,\omega}(X)} \, \|(y_k^*)_{k=1}^{n}\|_{\ell_s(Y^*)}, \tag{3.8}$$

for all finite families of elements $x_1, \cdots, x_m \in X$ and functionals $y_1^*, \cdots, y_n^* \in Y^*$. In this case $\|T\|_{(m(s,q),p)} = C$.

3.2 The new unifying approach of Botelho and Campos

In the following, we consider that Banach space X is a vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

3.2.1 Finitely determined sequence class

Definition 3.2.1. [8, Definition 2.1]

A ***class of vector-valued sequences*** $\mathcal{X}$, or simply a ***sequence class*** $\mathcal{X}$, is a rule that assigns to each Banach space X a Banach space $\mathcal{X}(X)$ of X-valued sequences, that is $\mathcal{X}(X)$ is a subspace of $X^{\mathbb{N}}$ with the coordinate-wise operations, such that:

$$c_{00}(X) \subseteq \mathcal{X}(X) \overset{1}{\hookrightarrow} \ell_\infty(X) \text{ and } \|e_i\|_{\mathcal{X}(\mathbb{K})} = 1.$$

For every i, where e_i is the i-th canonical unit scalar-valued sequence.

A sequence class $\mathcal{X}$ is ***finitely determined*** if for every sequence $(x_i)_{i=1}^\infty \in X^{\mathbb{N}}$,

$$(x_i)_{i=1}^\infty \in \mathcal{X}(X) \text{ if and only if } \sup_k \left\|(x_i)_{i=1}^k\right\|_{\mathcal{X}(X)} < \infty,$$

and in this case

$$\|(x_i)_{i=1}^\infty\|_{\mathcal{X}(X)} = \sup_k \left\|(x_i)_{i=1}^k\right\|_{\mathcal{X}(X)}$$

Recall that the symbol $X \overset{1}{\hookrightarrow} Y$ means that X is a linear subspace of Y and $\|x\|_X \leq \|y\|_Y$ for every $x \in X$.

Example 3.2.2. The spaces listed below are a sequences class.

- $\ell_\infty(X)$ = The vector space of all bounded sequences of elements of X with the sup norm.
- $c_0(X)$ = The vector space formed by all sequences $(x_i)_{i=1}^\infty$ in X that converge to 0 with the sup norm.
- $c_{0,\omega}(X)$ = The vector space formed by all sequences $(x_i)_{i=1}^\infty \subset X$ such that the sequence $(x^*(x_i))_{i=1}^\infty$ converge to 0 for all $x^* \in B_{X^*}$ with the sup norm.

- $\ell_p(X) =$ The vector space of all absolutely p-summable sequences $(x_i)_{i=1}^{\infty}$ in X with the norm $\|.\|_{\ell_p(X)}$.
- $\ell_{p,\omega}(X) =$ The vector space of all weakly p-summable sequences $(x_i)_{i=1}^{\infty}$ in X with the norm $\|.\|_{\ell_{p,\omega}(X)}$.
- $\ell_p \langle X \rangle =$ The vector space of all strongly p-summable sequences $(x_i)_{i=1}^{\infty}$ in X with the norm $\|.\|_{\ell_p\langle X\rangle}$.

In addition, the sequences classes $\ell_\infty(X), \ell_p(X), \ell_{p,\omega}(X)$ and $\ell_p \langle X \rangle$ are finitely determined (see [8, Example 2.2]).

The following proposition describes how to works the transformation of vector-valued sequences by linear operators.

Proposition 3.2.3. *[8, Proposition 2.4] Let $\mathcal{X}, \mathcal{Y}$ be sequence classes. The following conditions are equivalent for a given linear operator $T \in \mathcal{L}(X, Y)$:*

(a) $(T(x_i))_{i=1}^{\infty} \in \mathcal{Y}(Y)$ *whenever* $(x_i)_{i=1}^{\infty} \in \mathcal{X}(X)$,

(b) The induced map $\widehat{T} : \mathcal{X}(X) \longrightarrow \mathcal{Y}(Y)$,

$$\widehat{T}\left((x_i)_{i=1}^{\infty}\right) = (T(x_i))_{i=1}^{\infty},$$

is a well-defined continuous linear operator.

The conditions above imply condition (c) below, and they are all equivalent if the sequence classes $\mathcal{X}$ and $\mathcal{Y}$ are finitely determined.

(c) There is a constant $C > 0$ such that

$$\left\|(T(x_i))_{i=1}^{k}\right\|_{\mathcal{Y}(Y)} \leq C \left\|(x_i)_{i=1}^{k}\right\|_{\mathcal{X}(X)}, \tag{3.9}$$

for every $k \in \mathbb{N}$ and all finite sequences $x_i \in X, i = 1, ..., k$. In this case,

$$\left\|\widehat{T}\right\| = \inf\{C : \text{ such that } (3.9) \text{ holds }\} \tag{3.10}$$

Botelho and Campos in [8] give a study of the classes of linear and multi-linear operators satisfying the equivalent conditions of Proposition [8, Proposition 2.4]. we limit ourselves to the linear case.

Definition 3.2.4. [8, Definition 3.1]

Let $\mathcal{X}, \mathcal{Y}$ be sequence classes. A linear operator $T \in \mathcal{L}(X, Y)$ is $(\mathcal{X}, \mathcal{Y})$***-summing*** if the equivalent conditions of the above proposition hold for T, that is $(T(x_i))_{i=1}^{\infty} \in \mathcal{Y}(Y)$ whenever $(x_i)_{i=1}^{\infty} \in \mathcal{X}(X)$. In this case we write $T \in \mathcal{L}_{\mathcal{X},\mathcal{Y}}(X, Y)$ and define

$$\|T\|_{\mathcal{X},\mathcal{Y}} = \|\widehat{T}\|_{\mathcal{L}(\mathcal{X}(X),\mathcal{Y}(Y))}.$$

3.2.2 Linearly stable sequence class

Definition 3.2.5. [8, Definition 3.2]

A sequence class $\mathcal{X}$ is said to be ***linearly stable*** if $\mathcal{L}_{\mathcal{X},\mathcal{X}}(X, Y) = \mathcal{L}(X, Y)$ isometrically isomorphic for every Banach spaces X and Y, that is, for every $T \in \mathcal{L}(X, Y)$, $(T(x_i))_{i=1}^{\infty} \in \mathcal{X}(Y)$ whenever $(x_i)_{i=1}^{\infty} \in \mathcal{X}(X)$ and

$$\left\|\widehat{T} : \mathcal{X}(X) \longrightarrow \mathcal{X}(Y)\right\| = \|T\|.$$

Example 3.2.6. All sequences classes mentioned in Example 3.2.2 are linearly stable (see [8, Example 3.3]).

Theorem 3.2.7. *[8, Theorem 3.6]*

Let $\mathcal{X}, \mathcal{Y}$ be linearly stable sequence classes such that $\mathcal{X}(\mathbb{K}) \overset{1}{\hookrightarrow} \mathcal{Y}(\mathbb{K})$. Then $\left(\mathcal{L}_{\mathcal{X},\mathcal{Y}}, \|\cdot\|_{\mathcal{X},\mathcal{Y}}\right)$ is a Banach ideal of linear operators.

In the following proposition and under certain requirements we show that the space $\ell_{p,q}\langle X\rangle$ fulfills the definitions of Botelho and Campus.

Proposition 3.2.8. *The space $\ell_{p,q}\langle X\rangle$ is a finitely determined and linearly stable sequence class.*

Proof. We apply the Definitions 3.2.1 and 3.2.5. Firstly, we show that $\ell_{p,q}\langle X\rangle$ is a finitely determined sequence class. Since $\ell_p(X)$ is finitely determined and $\ell_p(X) \subseteq \ell_{p,q}\langle X\rangle$ then we have

$$c_{00}(X) \subseteq \ell_{p,q}\langle X\rangle.$$

Let us show that $\ell_{p,q}\langle X\rangle \overset{1}{\hookrightarrow} \ell_{\infty}(X)$. According to [6, Theorem 4] and Proposition 2.1.6 we have $\ell_{p,q}\langle X\rangle \subseteq \ell_{r,s}\langle X\rangle$ for $\frac{1}{s} + \frac{1}{r} \leq \frac{1}{p} + \frac{1}{q}$. Also in [6, Remark 2] by putting $r = \infty$ and $s = q$ we get

$$\ell_{p,q}\langle X\rangle \subseteq \ell_{\infty,q}\langle X\rangle = \ell_{\infty}(X).$$

By using the Proposition 2.2.10 we get

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_\infty(X)} \leq \|(x_i)_{i=1}^{\infty}\|_{\ell_{p,q}\langle X\rangle}.$$

Also, since $\ell_p(X) \subset \ell_{p,q}\langle X\rangle$ and $\|e_i\|_{\ell_p(\mathbb{K})} = \|e_i\|_{\ell_\infty(\mathbb{K})} = 1$ then we have

$$\|e_i\|_{\ell_{p,q}\langle \mathbb{K}\rangle} = 1.$$

So $\ell_{p,q}\langle X\rangle$ is a sequence class.
Let $(x_i)_{i=1}^{\infty} \in \ell_{p,q}\langle X\rangle$. Since ℓ_p is finitely determined we have

$$\begin{aligned}\|(x_i)_{i=1}^{\infty}\|_{\ell_{p,q}\langle X\rangle} &= \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(X^*)}\leq 1} \|(x_i^*(x_i))_{i=1}^{\infty}\|_{\ell_p} \\ &= \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(X^*)}\leq 1} \sup_k \left\|(x_i^*(x_i))_{i=1}^{k}\right\|_{\ell_p} \\ &= \sup_k \left\|(x_i)_{i=1}^{k}\right\|_{\ell_{p,q}\langle X\rangle} < \infty.\end{aligned}$$

Conversely, if the sup on the right side of the above equality is finite then $(x_i)_{i=1}^{\infty} \in \ell_{p,q}\langle X\rangle$. So,

$$(x_i)_{i=1}^{\infty} \in \ell_{p,q}\langle X\rangle \text{ if and only if } \sup_k \left\|(x_i)_{i=1}^{k}\right\|_{\ell_{p,q}\langle X\rangle} < \infty$$

and

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_{p,q}\langle X\rangle} = \sup_k \left\|(x_i)_{i=1}^{k}\right\|_{\ell_{p,q}\langle X\rangle}.$$

Thus, $\ell_{p,q}\langle X\rangle$ is finitely determined.

Secondly, we show that $\ell_{p,q}\langle X\rangle$ is linearly stable sequence class. Let $\widehat{T} \in \mathcal{L}(\ell_{p,q}\langle X\rangle, \ell_{p,q}\langle Y\rangle)$ then T is verified the conditions of Proposition 3.2.3. In the condition (c) in Proposition 3.2.3 if we take $k=1$ we get

$$\|T(x)\|_Y \leq \left\|\widehat{T}\right\| \|x\|_X \text{ for all } x \in X$$

this implies that $T \in \mathcal{L}(X,Y)$ and $\|T\| \leq \left\|\widehat{T}\right\|$. Conversely, suppose that $T \in \mathcal{L}(X,Y)$ and show that $(T(x_i))_{i=1}^{\infty} \in \ell_{p,q}\langle Y\rangle$ for all $(x_i)_{i=1}^{\infty} \in \ell_{p,q}\langle X\rangle$, we have

$$\begin{aligned}\|(T(x_i))_{i=1}^{\infty}\|_{\ell_{p,q}\langle Y\rangle} &= \sup_{\|(y_i^*)_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(Y^*)}\leq 1} \|(y_i^*(T(x_i)))_{i=1}^{\infty}\|_{\ell_p} \\ &= \sup_{\|(y_i^*)_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(Y^*)}\leq 1} \|(T^*y_i^*(x_i))_{i=1}^{\infty}\|_{\ell_p}.\end{aligned}$$

Since $\ell_{p,\omega}(X)$ is linearly stable then $(T^*(y_i^*))_{i=1}^{\infty} \in \ell_{q^*,\omega}(X^*)$ and

$$\|(T^*(y_i^*))_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(X^*)} \leq \|T\| \, \|(y_i^*)_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(Y^*)} \, .$$

Then we obtain

$$\begin{aligned}
\left\|\widehat{T}\left((x_i)_{i=1}^{\infty}\right)\right\|_{\ell_{p,q}\langle Y\rangle} &= \|(T(x_i))_{i=1}^{\infty}\|_{\ell_{p,q}\langle Y\rangle} \\
&= \sup_{\left\|(y_i^*)_{i=1}^{\infty}\right\|_{\ell_{q^*,\omega}(Y^*)}\leq 1} \left\|\left(\left(\frac{T^*y_i^*}{\|(T^*(y_i^*))_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(X^*)}}\right)(x_i)\right)_{i=1}^{\infty}\right\|_{\ell_p} \\
&\times \|(T^*(y_i^*))_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(X^*)} \\
&\leq \|T\| \sup_{\left\|(x_i^*)_{i=1}^{\infty}\right\|_{\ell_{q^*,\omega}(X^*)}\leq 1} \|(x_i^*(x_i))_{i=1}^{\infty}\|_{\ell_p} \\
&= \|T\| \, \|(x_i)_{i=1}^{\infty}\|_{\ell_{p,q}\langle X\rangle} < \infty.
\end{aligned}$$

This implies that $(T(x_i))_{i=1}^{\infty} \in \ell_{p,q}\langle Y\rangle$ and Proposition 3.2.3 assure that $\widehat{T} \in \mathcal{L}\left(\ell_{p,q}\langle X\rangle, \ell_{p,q}\langle Y\rangle\right)$ with $\left\|\widehat{T}\right\| \leq \|T\|$ □

3.3 New ideals of linear summing operators

We are now looking to find the dual of the classes defined by Matos in [21]. For this, we introduce two new classes of linear summing operators, naturally extending the ideal of strongly (q,r)-summing operators. This by using the sequences spaces of strongly (p,q)-summable, absolutely r-summable, and strongly r-summable.

3.3.1 The ideal of $(<p,q>,r)$-summing operators

We present in the next the definition of the first class which generalizes the ideal $\mathcal{D}_{q,r}$ of all strongly (q,r)-summing operator in a natural way.

Definition 3.3.1. Let $1 \leq p,q,r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \frac{1}{r}$. A linear mapping T from X into Y is said to be $(<p,q>,r)$***-summing*** if $(T(x_i))_{i=1}^{\infty} \in \ell_{p,q}\langle Y\rangle$ for each $(x_i)_{i=1}^{\infty} \in \ell_r(X)$. When $p=1$ the mapping T is strongly (q,r)-summing.

The class of all $(<p,q>,r)$-summing operators is denoted by $\mathcal{L}_{(<p,q>,r)}$.

The most important characterizations for these mappings are described in the following theorem.

Theorem 3.3.2. *If T is a linear mapping from X into Y, then the following statements are equivalent*

(a) *T is $(< p, q >, r)$-summing,*

(b) *The induced map $\widehat{T}$ is well defined, linear and continuous from $\ell_r(X)$ into $\ell_{p,q}\langle Y\rangle$,*

(c) *There is $A \geq 0$ such that*

$$\|(T(x_i))_{i=1}^n\|_{\ell_{p,q}\langle Y\rangle} \leq A\,\|(x_i)_{i=1}^n\|_{\ell_r(X)}\,, \tag{3.11}$$

for every $x_i \in X$, $i = 1, ..., n$,

(d) *There is $B \geq 0$ such that*

$$\|(T(x_i))_{i=1}^\infty\|_{\ell_{p,q}\langle Y\rangle} \leq B\,\|(x_i)_{i=1}^\infty\|_{\ell_r(X)}\,, \tag{3.12}$$

for every $(x_i)_{i=1}^\infty \in \ell_r(X)$,

(e) *There is $C \geq 0$ such that*

$$\|(y_i^*(T(x_i)))_{i=1}^\infty\|_{\ell_p} \leq C\,\|(x_i)_{i=1}^\infty\|_{\ell_r(X)}\,\|(y_i^*)_{i=1}^\infty\|_{\ell_{q^*,\omega}(Y^*)}\,, \tag{3.13}$$

for every $(x_i)_{i=1}^\infty \in \ell_r(X)$ and $(y_i^*)_{i=1}^\infty \in \ell_{q^*,\omega}(Y^*)$,

(f) *There is $D \geq 0$ such that*

$$\|(y_i^*(T(x_i)))_{i=1}^\infty\|_{\ell_1} \leq D\,\|(x_i)_{i=1}^\infty\|_{\ell_r(X)}\,\|(y_i^*)_{i=1}^\infty\|_{\ell_{m(q^*,s^*)}(Y^*)}\,, \tag{3.14}$$

for every $(x_i)_{i=1}^\infty \in \ell_r(X)$ and $(y_i^*)_{i=1}^\infty \in \ell_{m(q^*,s^*)}(Y^*)$. In addition

$$\begin{aligned}\left\|\widehat{T}\right\| &= \inf\{A : A \text{ satisfies } (3.11)\}\\ &= \inf\{B : B \text{ satisfies } (3.12)\}\\ &= \inf\{C : C \text{ satisfies } (3.13)\}\\ &= \inf\{D : D \text{ satisfies } (3.14)\}\,.\end{aligned}$$

Proof.

(a) $\Leftrightarrow$ (b) Apply Proposition 3.2.3.

(a) $\Leftrightarrow$ (c) Since $\ell_{p,q}\langle Y\rangle$ and $\ell_r(X)$ are finitely determined then the Proposition 3.2.3 assure the equivalence between (a) and (c) with

$$\left\|\widehat{T}\right\| = \inf\{A : A \text{ satisfies } (3.11)\}\,.$$

(c) $\Rightarrow$ (d) Suppose that (3.11) is satisfied, by passing to limit for n tending to ∞ we get (3.12) with $A = B$.

(d) $\Rightarrow$ (a) Is obvious.

(d) $\Rightarrow$ (b) Is clear.

(d) $\Rightarrow$ (e) Suppose that (3.12) is satisfied, then according to (2.6) we have

$$\begin{aligned}\|(T(x_i))_{i=1}^{\infty}\|_{\ell_{p,q}\langle Y\rangle} &= \sup_{\|(y_i^*)_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(Y^*)\leq 1}} \|(y_i^*(T(x_i)))_{i=1}^{\infty}\|_{\ell_p} \\ &\leq B\,\|(x_i)_{i=1}^{\infty}\|_{\ell_r(X)}\,.\end{aligned}$$

Implies

$$\|(y_i^*(T(x_i)))_{i=1}^{\infty}\|_{\ell_p} \leq B\,\|(x_i)_{i=1}^{\infty}\|_{\ell_r(X)}\,, \tag{3.15}$$

for all $(y_i^*)_{i=1}^{\infty} \in \ell_{q^*,\omega}(Y^*)$ with $\|(y_i^*)_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(Y^*)} \leq 1$.
Let $0 \neq (z_i^*)_{i=1}^{\infty} \in \ell_{q^*,\omega}(Y^*)$, we pose

$$y_i = \frac{z_i}{\|(z_i^*)_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(Y^*)}}$$

then we have $\|(y_i^*)_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(Y^*)} = 1$. By Inequality (3.15) we get

$$\|(z_i^*(T(x_i)))_{i=1}^{\infty}\|_{\ell_p} \leq B\,\|(x_i)_{i=1}^{\infty}\|_{\ell_r(X)}\,\|(z_i^*)_{i=1}^{\infty}\|_{\ell_{q^*,\omega}(Y^*)}\,.$$

Thus, there is $C = B$ satisfying (3.13).

(d) $\Rightarrow$ (f) Suppose that (3.12) is satisfied then according to Remark 2.1.8 we have

$$\begin{aligned}\|(T(x_i))_{i=1}^{\infty}\|_{\ell_{p,q}\langle Y\rangle} &= \sup_{\|(y_i^*)_{i=1}^{\infty}\|_{\ell_{m(q^*,s^*)}(Y^*)\leq 1}} \|(y_i^*(T(x_i)))_{i=1}^{\infty}\|_{\ell_1} \\ &\leq B\,\|(x_i)_{i=1}^{\infty}\|_{\ell_r(X)}\,.\end{aligned}$$

Implies that

$$\|(y_i^*(T(x_i)))_{i=1}^{\infty}\|_{\ell_1} \leq B\,\|(x_i)_{i=1}^{\infty}\|_{\ell_r(X)}\,, \tag{3.16}$$

for all $(y_i^*)_{i=1}^{\infty} \in \ell_{m(q^*,s^*)}(Y^*)$ with $\|(y_i^*)_{i=1}^{\infty}\|_{\ell_{m(q^*,s^*)}(Y^*)} \leq 1$.
Let $0 \neq (z_i^*)_{i=1}^{\infty} \in \ell_{m(q^*,s^*)}(Y^*)$, we pose

$$y_i = \frac{z_i}{\|(z_i^*)_{i=1}^{\infty}\|_{\ell_{m(q^*,s^*)}(Y^*)}}$$

then we have $\|(y_i^*)_{i=1}^\infty\|_{\ell_{m(q^*,s^*)}(Y^*)} = 1$. By Inequality (3.16) we get

$$\|(z_i^*(T(x_i)))_{i=1}^\infty\|_{\ell_1} \leq B\,\|(x_i)_{i=1}^\infty\|_{\ell_r(X)}\,\|(z_i^*)_{i=1}^\infty\|_{\ell_{m(q^*,s^*)}(Y^*)}\,.$$

Thus, there is $D = B$ satisfying (3.14).

(e) $\Rightarrow$ (d) By taking the supremum over all $(y_i^*)_{i=1}^\infty \in \ell_{q^*,\omega}(Y^*)$ with $\|(y_i^*)_{i=1}^\infty\|_{\ell_{q^*,\omega}(Y^*)} \leq 1$ and according to (2.6) we get (3.12) with $C = B$.

(f) $\Rightarrow$ (d) By taking the supremum over all $(y_i^*)_{i=1}^\infty \in \ell_{m(q^*,s^*)}(Y^*)$ such that $\|(y_i^*)_{i=1}^\infty\|_{\ell_{m(q^*,s^*)}(Y^*)} \leq 1$ and according to Remark 2.1.8 we get (3.12) with $D = B$. □

Contrary to usual practice, in the following theorem, we show that the class $\mathcal{L}_{(<p,q>,r)}$ is a Banach operator ideal. We prove this by using the new unifying approach presented by Botelho and Campos in [8].

Theorem 3.3.3. *Let $1 \leq p,q,r \leq \infty$ such that $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$. For all $T \in \mathcal{L}_{(<p,q>,r)}(X,Y)$ if we set*

$$\begin{aligned}\|T\|_{(<p,q>,r)} &= \inf\{A : A \textit{ satisfies } (3.11)\}\\ &= \inf\{B : B \textit{ satisfies } (3.12)\}\\ &= \inf\{C : C \textit{ satisfies } (3.13)\}\\ &= \inf\{D : D \textit{ satisfies } (3.14)\}\\ &= \|\widehat{T}\|,\end{aligned}$$

then the class $\left(\mathcal{L}_{(<p,q>,r)}, \|\cdot\|_{(<p,q>,r)}\right)$ is a Banach ideal of linear operators.

Proof. To prove this theorem we apply Theorem 3.2.7, (i.e., we show that $\ell_r(\cdot), \ell_{p,q}\langle\cdot\rangle$ are linearly stable and $\ell_r(\mathbb{K}) \overset{1}{\hookrightarrow} \ell_{p,q}\langle\mathbb{K}\rangle$ with $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$). We know that $\ell_r(\cdot)$ and $\ell_r\langle\cdot\rangle$ are linearly stable and the Proposition 3.2.8 gives that $\ell_{p,q}\langle\cdot\rangle$ is also linearly stable.
By Definition 3.3.1, Definition 3.3.7 and Definition 3.2.4, a linear operator T is $(<p,q>,r)$-summing if and only if T is $(\ell_r(\cdot);\ell_{p,q}\langle\cdot\rangle)$-summing. It remains to show that $\ell_r(\mathbb{K}) \overset{1}{\hookrightarrow} \ell_{p,q}\langle\mathbb{K}\rangle$ with $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$, for this we use Proposition 2.2.10, we obtain

$$\ell_r\langle\mathbb{K}\rangle \subset \ell_{p,q}\langle\mathbb{K}\rangle \subset \ell_r(\mathbb{K}).$$

With

$$\|(x_i)_{i=1}^\infty\|_{\ell_r(\mathbb{K})} \leq \|(x_i)_{i=1}^\infty\|_{\ell_{p,q}\langle\mathbb{K}\rangle} \leq \|(x_i)_{i=1}^\infty\|_{\ell_r\langle\mathbb{K}\rangle}\,.$$

And since $dim(\mathbb{K}) < \infty$ we have

$$\ell_r \langle \mathbb{K} \rangle = \ell_{p,q} \langle \mathbb{K} \rangle = \ell_r(\mathbb{K}).$$

With

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_r(\mathbb{K})} = \|(x_i)_{i=1}^{\infty}\|_{\ell_{p,q}\langle \mathbb{K} \rangle} = \|(x_i)_{i=1}^{\infty}\|_{\ell_r\langle \mathbb{K} \rangle}.$$

Finally, from Theorem 3.2.7 it follows that $\left(\mathcal{L}_{(<p,q>,r)}, \|\cdot\|_{(<p,q>,r)}\right)$ is a Banach ideal of linear operators. □

Proposition 3.3.4. *If* $1 + \frac{1}{r} < \frac{1}{p} + \frac{1}{q}$ *then* $\mathcal{L}_{(<p,q>,r)}(X,Y) = \{0\}$.

Proof. If $1+\frac{1}{r} < \frac{1}{p}+\frac{1}{q}$ then $\frac{1}{q^*}+\frac{1}{r} = \frac{1}{\frac{rq^*}{r+q^*}} < \frac{1}{p}$ this implies that $\ell_p \subset \ell_{\frac{rq^*}{r+q^*}}$. Let $(\alpha_i)_{i=1}^{\infty} \in \ell_{\frac{rq^*}{r+q^*}} - \ell_p$. We suppose that $T \in \mathcal{L}_{(<p,q>,r)}(X,Y)$ and $T \neq 0$. Then there exist $x \in X$ such that $T(x) \neq 0$. Take $y^* \in Y^*$, then for every $x \in X$ such that $T(x) \neq 0$ we have

$$\begin{aligned}
\left(\sum_{i=1}^{\infty} |y^*, T(\alpha_i x)|^p\right)^{\frac{1}{p}} &= |y^*, T(x)| \, \|(\alpha_i)_{i=1}^{\infty}\|_{\ell_p} \\
&= \left(\sum_{i=1}^{\infty} \left|\alpha_i^{\frac{r}{r+q^*}} y^*, T\left(\alpha_i^{\frac{q^*}{r+q^*}} x\right)\right|^p\right)^{\frac{1}{p}} \\
&\leq \|T\|_{(<p,q>,r)} \left\|\left(\alpha_i^{\frac{q^*}{r+q^*}} x\right)_{i=1}^{\infty}\right\|_{\ell_r(X)} \left\|\left(\alpha_i^{\frac{r}{r+q^*}} y^*\right)_{i=1}^{\infty}\right\|_{\ell_{q^*,\omega}(Y^*)} \\
&= \|T\|_{(<p,q>,r)} \|x\| \, \|y^*\| \left\|\left(\alpha_i^{\frac{q^*}{r+q^*}}\right)_{i=1}^{\infty}\right\|_{\ell_r} \left\|\left(\alpha_i^{\frac{r}{r+q^*}}\right)_{i=1}^{\infty}\right\|_{\ell_{q^*}} \\
&= \|T\|_{(<p,q>,r)} \|x\| \, \|y^*\| \, \|(\alpha_i)_{i=1}^{\infty}\|_{\ell_{\frac{rq^*}{r+q^*}}}.
\end{aligned}$$

By taking the supremum over all $x \in X$ such that $T(x) \neq 0$ and $y^* \in Y^*$ with $\|x\|_X \leq 1$ and $\|y^*\|_{Y^*} \leq 1$ we get

$$\|T\| \, \|(\alpha_i)_{i=1}^{\infty}\|_{\ell_p} \leq \|T\|_{(<p,q>,r)} \|(\alpha_i)_{i=1}^{\infty}\|_{\ell_{\frac{rq^*}{r+q^*}}} < \infty.$$

Therefore, $(\alpha_i)_{i=1}^{\infty} \in \ell_p$ which is a contradiction. □

The coincidence between $\mathcal{L}_{(<p,q>,\infty)}(X,Y)$ and $\mathcal{D}_{q,p^*}(X,Y)$ is shown in the following proposition.

Proposition 3.3.5. *Let* $1 \leq p, q \leq \infty$ *then the spaces* $\mathcal{L}_{(<p,q>,\infty)}(X,Y)$ *and* $\mathcal{D}_{q,p^*}(X,Y)$ *are coincide. In addition we have*

$$\|\cdot\|_{(<p,q>,\infty)} = d_{q,p^*}(\cdot).$$

Proof. If $T \in \mathcal{L}_{(<p,q>,\infty)}(X,Y)$ then for all $(x_i)_{i=1}^{\infty} \in \ell_\infty(X)$ and according to Proposition 2.2.1 we have

$$\begin{aligned}\|(T(x_i))_{i=1}^{\infty}\|_{\ell_{p,q}\langle Y\rangle} &= \sup_{\|(\alpha_i)_{i=1}^{\infty}\|_{\ell_{p^*}}\leq 1} \|(T(\alpha_i x_i))_{i=1}^{\infty}\|_{\ell_q\langle Y\rangle} \\ &\leq \|T\|_{(<p,q>,\infty)} \|(x_i)_{i=1}^{\infty}\|_{\ell_\infty(X)}.\end{aligned}$$

Putting $z_i = \alpha_i x_i$ we obtain

$$\begin{aligned}\|(T(\alpha_i x_i))_{i=1}^{\infty}\|_{\ell_q\langle Y\rangle} &= \|(T(z_i))_{i=1}^{\infty}\|_{\ell_q\langle Y\rangle} \\ &\leq \|T\|_{(<p,q>,\infty)} \|(x_i)_{i=1}^{\infty}\|_{\ell_\infty(X)} \|(\alpha_i)_{i=1}^{\infty}\|_{\ell_{p^*}}.\end{aligned}$$

Taking the infimum over all representation of the form $z_i = \alpha_i x_i$ we get

$$\begin{aligned}\|(T(z_i))_{i=1}^{\infty}\|_{\ell_q\langle Y\rangle} &\leq \|T\|_{(<p,q>,\infty)} \inf \|(x_i)_{i=1}^{\infty}\|_{\ell_\infty(X)} \|(\alpha_i)_{i=1}^{\infty}\|_{\ell_{p^*}} \\ &= \|T\|_{(<p,q>,\infty)} \|(z_i)_{i=1}^{\infty}\|_{\ell_{p^*}(X)}.\end{aligned}$$

Thus $T \in \mathcal{D}_{q,p^*}(X,Y)$ and $d_{q,p^*}(T) \leq \|T\|_{(<p,q>,\infty)}$.
Conversely, suppose that $T \in \mathcal{D}_{q,p^*}(X,Y)$. Let $(x_i)_{i=1}^{\infty} \in \ell_\infty(X)$ and $(\alpha_i)_{i=1}^{\infty} \in \ell_{p^*}$, take $z_i = \alpha_i x_i$ we have

$$\begin{aligned}\|(T(\alpha_i x_i))_{i=1}^{\infty}\|_{\ell_q\langle Y\rangle} &\leq d_{q,p^*}(T) \|(z_i)_{i=1}^{\infty}\|_{\ell_{p^*}(X)} \\ &\leq d_{q,p^*}(T) \|(x_i)_{i=1}^{\infty}\|_{\ell_\infty(X)} \|(\alpha_i)_{i=1}^{\infty}\|_{\ell_{p^*}}.\end{aligned}$$

By taking the supremum over all $(\alpha_i)_{i=1}^{\infty} \in \ell_{p^*}$ with $\|(\alpha_i)_{i=1}^{\infty}\|_{\ell_{p^*}} \leq 1$ and according to Proposition 2.2.1 we get

$$\|(T(z_i))_{i=1}^{\infty}\|_{\ell_{p,q}\langle Y\rangle} \leq d_{q,p^*}(T) \|(z_i)_{i=1}^{\infty}\|_{\ell_\infty(X)}.$$

Thus $T \in \mathcal{L}_{(<p,q>,\infty)}(X,Y)$ and $\|T\|_{(<p,q>,\infty)} \leq d_{q,p^*}(T)$. □

The relationship between our class and the classes presented by Matos in [21] and Apiola in [5] is given in the following proposition.

Proposition 3.3.6. *Let T be a linear operator from a Banach space X into another Banach space Y and $1 \leq p,q,r,s \leq \infty$. We have the following inclusions*

1) $\mathcal{L}_{(p,m(s,r))}(X,Y) \subseteq \mathcal{L}_{(<p,q>,r)}(X,Y)$ *with*

$$\|T\|_{(<p,q>,r)} \leq \|T\|_{(p,m(s,r))},$$

for all $T \in \mathcal{L}_{(p,m(s,r))}(X,Y)$.

2) $\mathcal{D}_{q,r}(X,Y) \subseteq \mathcal{L}_{(<p,q>,r)}(X,Y)$ *with*

$$\|T\|_{(<p,q>,r)} \leq d_{q,r}(T),$$

for all $T \in \mathcal{D}_{q,r}(X,Y)$.

Proof.

1) Let $T \in \mathcal{L}_{(p,m(s,r))}(X,Y)$, according to Theorem 3.1.2 we have

$$\|(T(x_i))_{i=1}^{\infty}\|_{\ell_p(Y)} \leq \|T\|_{(p,m(s,r))} \|(x_i)_{i=1}^{\infty}\|_{\ell_{m(s,r)}(X)},$$

for all $(x_i)_{i=1}^{\infty} \in \ell_{m(s,r)}(X)$. The Proposition 2.2.10 and Proposition 2.1.1 gives

$$\begin{aligned}\|(T(x_i))_{i=1}^{\infty}\|_{\ell_{p,q}\langle Y\rangle} &\leq \|(T(x_i))_{i=1}^{\infty}\|_{\ell_p(Y)} \\ &\leq \|T\|_{(p,m(s,r))} \|(x_i)_{i=1}^{\infty}\|_{\ell_{m(s,r)}(X)} \\ &\leq \|T\|_{(p,m(s,r))} \|(x_i)_{i=1}^{\infty}\|_{\ell_r(X)}.\end{aligned}$$

The Theorem 3.3.2 assure that $T \in \mathcal{L}_{(<p,q>,r)}(X,Y)$ and

$$\|T\|_{(<p,q>,r)} \leq \|T\|_{(p,m(s,r))}.$$

2) Let $T \in \mathcal{D}_{q,r}(X,Y)$ then we have

$$\|(T(x_i))_{i=1}^{\infty}\|_{\ell_q\langle Y\rangle} \leq d_{q,r}(T) \|(x_i)_{i=1}^{\infty}\|_{\ell_r(X)},$$

for all $(x_i)_{i=1}^{\infty} \in \ell_r(X)$. The Proposition 2.1.11 gives

$$\begin{aligned}\|(T(x_i))_{i=1}^{\infty}\|_{\ell_{p,q}\langle Y\rangle} &\leq \|(T(x_i))_{i=1}^{\infty}\|_{\ell_q\langle Y\rangle} \\ &\leq d_{q,r}(T) \|(x_i)_{i=1}^{\infty}\|_{\ell_r(X)}.\end{aligned}$$

By Theorem 3.3.2 we get $T \in \mathcal{L}_{(<p,q>,r)}(X,Y)$ and

$$\|T\|_{(<p,q>,r)} \leq d_{q,r}(T).$$

□

3.3.2 The ideal of $(r, < p, q >)$-summing operators

Now, we present the definition of a second class which also generalizes the ideal $\mathcal{D}_{r,p}$ of all strongly (r,p)-summing operator in a natural way.

Definition 3.3.7. Let $1 \leq p, q, r \leq \infty$ and $p \leq r$. A linear mapping T from X into Y is said to be $(r, < p, q >)$***-summing*** if $(T(x_i))_{i=1}^{\infty} \in \ell_r \langle Y \rangle$ for each $(x_i)_{i=1}^{\infty} \in \ell_{p,q} \langle X \rangle$. When $q = 1$ the mapping T is strongly (r, p)-summing.

The class of all $(r, < p, q >)$-summing operators is denoted by $\mathcal{L}_{(r,<p,q>)}$.
Using a similar argument to the one in Theorem 3.3.2 we present the following result.

Theorem 3.3.8. *If T is a linear mapping from X into Y, then the following statements are equivalent*

(1) T is $(r, < p, q >)$-summing,

(2) The induced map $\widehat{T}$ is well defined, linear and continuous from $\ell_{p,q} \langle X \rangle$ into $\ell_r \langle Y \rangle$,

(3) There is $A \geq 0$ such that

$$\|(T(x_i))_{i=1}^{n}\|_{\ell_r \langle Y \rangle} \leq A \, \|(x_i)_{i=1}^{n}\|_{\ell_{p,q} \langle X \rangle}, \tag{3.17}$$

for every $x_i \in X$, $i = 1, ..., n$,

(4) There is $B \geq 0$ such that

$$\|(T(x_i))_{i=1}^{\infty}\|_{\ell_r \langle Y \rangle} \leq B \, \|(x_i)_{i=1}^{\infty}\|_{\ell_{p,q} \langle X \rangle}, \tag{3.18}$$

for every $(x_i)_{i=1}^{\infty} \in \ell_{p,q} \langle X \rangle$,

(5) There is $C \geq 0$ such that

$$\|(y_i^*(T(x_i)))_{i=1}^{\infty}\|_{\ell_1} \leq C \, \|(x_i)_{i=1}^{\infty}\|_{\ell_{p,q} \langle X \rangle} \, \|(y_i^*)_{i=1}^{\infty}\|_{\ell_{r^*,\omega}(Y^*)}, \tag{3.19}$$

for every $(x_i)_{i=1}^{\infty} \in \ell_{p,q} \langle X \rangle$ and $(y_i^*)_{i=1}^{\infty} \in \ell_{r^*,\omega}(Y^*)$. In addition we have

$$\begin{aligned} \left\|\widehat{T}\right\| &= \inf \{A : A \text{ satisfies } (3.17)\} \\ &= \inf \{B : B \text{ satisfies } (3.18)\} \\ &= \inf \{C : C \text{ satisfies } (3.19)\} \end{aligned}$$

Always using the new unifying approach presented by Botelho and Campos [8] we prove in the following theorem that the class $\mathcal{L}_{(r,<p,q>)}$ is a Banach operator ideal, the proof is a similar way to Theorem 3.3.3.

Theorem 3.3.9. *Let $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. For all $T \in \mathcal{L}_{(r,<p,q>)}(X,Y)$ if we set*

$$\begin{aligned}\|T\|_{(r,<p,q>)} &= \inf\{A : A \text{ satisfies } (3.17)\}\\ &= \inf\{B : B \text{ satisfies } (3.18)\}\\ &= \inf\{C : C \text{ satisfies } (3.19)\}\\ &= \left\|\widehat{T}\right\|,\end{aligned}$$

then the class $\left(\mathcal{L}_{(r,<p,q>)}, \|\cdot\|_{(r,<p,q>)}\right)$ is a Banach ideal of linear operators.

Remark 3.3.10. Since $\ell_{1,r}\langle X\rangle = \ell_r\langle X\rangle$ and $\ell_{p,1}\langle X\rangle = \ell_p(X)$ we have

$$\mathcal{L}_{(<1,r>,p)}(X,Y) = \mathcal{L}_{(r,<p,1>)}(X,Y) = \mathcal{D}_{r,p}(X,Y).$$

Proposition 3.3.11. *If $r < p$ then $\mathcal{L}_{(r,<p,q>)}(X,Y) = \{0\}$.*

Proof. If $r < p$ then $\frac{1}{\frac{pr^*}{p+r^*}} < 1$. Let $(\alpha_i)_{i=1}^{\infty} \in \ell_{\frac{pr^*}{p+r^*}} - \ell_1$. We suppose that $T \in \mathcal{L}_{(r,<p,q>)}(X,Y)$ and $T \neq 0$. Then there exist $x \in X$ such that $T(x) \neq 0$. Take $y^* \in Y^*$, then for every $x \in X$ such that $T(x) \neq 0$ we have

$$\begin{aligned}\sum_{i=1}^{\infty} |\langle y^*, T(\alpha_i x)\rangle| &= |\langle y^*, T(x)\rangle| \,\|(\alpha_i)_{i=1}^{\infty}\|_{\ell_1}\\ &= \sum_{i=1}^{\infty} \left|\alpha_i^{\frac{p}{p+r^*}} y^*, T\left(\alpha_i^{\frac{r^*}{p+r^*}} x\right)\right|\\ &\leq \|T\|_{(r,<p,q>)} \left\|\left(\alpha_i^{\frac{r^*}{p+r^*}} x\right)_{i=1}^{\infty}\right\|_{\ell_{p,q}\langle X\rangle} \left\|\left(\alpha_i^{\frac{p}{p+r^*}} y^*\right)_{i=1}^{\infty}\right\|_{\ell_{r^*,\omega}(Y^*)}\\ &\leq \|T\|_{(r,<p,q>)} \left\|\left(\alpha_i^{\frac{r^*}{p+r^*}} x\right)_{i=1}^{\infty}\right\|_{\ell_p(X)} \left\|\left(\alpha_i^{\frac{p}{p+r^*}} y^*\right)_{i=1}^{\infty}\right\|_{\ell_{r^*,\omega}(Y^*)}\\ &= \|T\|_{(r,<p,q>)} \|x\| \|y^*\| \left\|\left(\alpha_i^{\frac{r^*}{p+r^*}}\right)_{i=1}^{\infty}\right\|_{\ell_p} \left\|\left(\alpha_i^{\frac{p}{p+r^*}}\right)_{i=1}^{\infty}\right\|_{\ell_{r^*}}\\ &= \|T\|_{(r,<p,q>)} \|x\| \|y^*\| \|(\alpha_i)_{i=1}^{\infty}\|_{\ell_{\frac{pr^*}{p+r^*}}}.\end{aligned}$$

By taking the supremum over all $x \in X$ such that $T(x) \neq 0$ and $y^* \in Y^*$ with $\|x\|_X \leq 1$ and $\|y^*\|_{Y^*} \leq 1$ we get

$$\|T\| \|(\alpha_i)_{i=1}^{\infty}\|_{\ell_1} \leq \|T\|_{(r,<p,q>)} \|(\alpha_i)_{i=1}^{\infty}\|_{\ell_{\frac{pr^*}{p+r^*}}} < \infty.$$

Thus $(\alpha_i)_{i=1}^{\infty} \in \ell_1$ which is a contradiction. □

A similar proof as the one in Proposition 3.3.6 allows us to give the following result.

Proposition 3.3.12. *Let T be a linear operator from a Banach space X into another Banach space Y and $1 \leq p, q, r, s \leq \infty$. We have the following inclusion*

$$\mathcal{L}_{(r,<p,q>)}(X, Y) \subseteq \mathcal{D}_{r,p}(X, Y).$$

Moreover,

$$d_{r,p}(T) \leq \|T\|_{(r,<p,q>)},$$

for all $T \in \mathcal{L}_{(r,<p,q>)}(X, Y)$.

3.4 Duality relationships

In this section, we present the duality relationship between the classes of $(p, m(s, q))$-summing linear operators and $(m(s, q), p)$-summing linear operators, and the classes that we have provided above. To prove the following results, we rely on the study carried out in the second chapter, and taking into account that the adjoint of the operator $\widehat{T}$ can be identified with the operator $\widehat{T^*}$.

Theorem 3.4.1. *Let $1 \leq p, q, r, s \leq \infty$ such that $1 + \frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. We have*

i) The operator T is belongs to $\mathcal{L}_{(<p,q>,r)}(X, Y)$ if and only if its adjoint T^* is belongs to $\mathcal{L}_{(r^*,m(q^*,s^*))}(Y^*, X^*)$. Moreover,

$$\|T\|_{(<p,q>,r)} = \|T^*\|_{(r^*,m(q^*,s^*))}.$$

ii) The operator T is belongs to $\mathcal{L}_{(r,<p,q>)}(X, Y)$ if and only if its adjoint T^* is belongs to $\mathcal{L}_{(m(q^*,s^*),r^*)}(Y^*, X^*)$. Moreover,

$$\|T\|_{(r,<p,q>)} = \|T^*\|_{(m(q^*,s^*),r^*)}.$$

Proof.

i) Let $T \in \mathcal{L}_{(<p,q>,r)}(X, Y)$. By Theorem 3.3.2 we have $\widehat{T} \in \mathcal{L}\left(\ell_r(X), \ell_{p,q}\langle Y\rangle\right)$ withe

$$\|T\|_{(<p,q>,r)} = \left\|\widehat{T}\right\|.$$

This is equivalent to say $\widehat{T^*} \in \mathcal{L}\left(\ell_{m(q^*,s^*)}(Y^*), \ell_{r^*}(X^*)\right)$, with

$$\left\|\hat{T}\right\| = \left\|\widehat{T^*}\right\|,$$

and according to Theorem 3.1.2 we get $T^* \in \mathcal{L}_{(r^*,m(q^*,s^*))}(Y^*, X^*)$ with

$$\left\|\widehat{T^*}\right\| = \|T^*\|_{(r^*,m(q^*,s^*))}.$$

ii) Is similar way, applying only Theorem 3.3.8 and Theorem 3.1.6. □

The next theorem and its proof are similar to the previous theorem.

Theorem 3.4.2. *Let* $1 \leq p, q, r, s \leq \infty$ *such that* $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$ *then we have*

i) The operator T *is belongs to* $\mathcal{L}_{(r,m(q,s))}(X, Y)$ *if and only if its adjoint* T^* *is belongs to* $\mathcal{L}_{(<p^*,q^*>,r^*)}(Y^*, X^*)$. *Moreover,*

$$\|T\|_{(r,m(q,s))} = \|T^*\|_{(<p^*,q^*>,r^*)}.$$

ii) The operator T *is belongs to* $\mathcal{L}_{(m(q,s),r)}(X, Y)$ *if and only if its adjoint* T^* *is belongs to* $\mathcal{L}_{(r^*,<p^*,q^*>)}(Y^*, X^*)$. *Moreover,*

$$\|T\|_{(m(q,s),r)} = \|T^*\|_{(r^*,<p^*,q^*>)}.$$

From the Theorems 3.4.1 and 3.4.2 we conclude directly the following corollary.

Corollary 3.4.3. *Let* $1 \leq p, q, r, s \leq \infty$ *such that* $1 + \frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. *We have*

i) The operator T *is belongs to* $\mathcal{L}_{(<p,q>,r)}(X, Y)$ *if and only if its second adjoint* T^{**} *is belongs to* $\mathcal{L}_{(<p,q>,r)}(X^{**}, Y^{**})$.

ii) The operator T *is belongs to* $\mathcal{L}_{(r,<p,q>)}(X, Y)$ *if and only if its second adjoint* T^{**} *is belongs to* $\mathcal{L}_{(r,<p,q>)}(X^{**}, Y^{**})$.

and the norms of T *and* T^{**} *are equal.*

Corollary 3.4.4. *Let* $1 \leq p, q, r, s \leq \infty$ *such that* $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. *We have*

i) The operator T *is belongs to* $\mathcal{L}_{(r,m(q,s))}(X, Y)$ *if and only if its second adjoint* T^{**} *is belongs to* $\mathcal{L}_{(r,m(q,s))}(X^{**}, Y^{**})$.

ii) *The operator T is belongs to $\mathcal{L}_{(m(q,s),r)}(X,Y)$ if and only if its second adjoint T^{**} is belongs to $\mathcal{L}_{(m(q,s),r)}(X^{**},Y^{**})$.*

and the norms of T and T^{**} are equal.

Proof. We prove only point *i)* from Corollary 3.4.3, the proof of the other points is similar. Suppose that T is belongs to $\mathcal{L}_{(<p,q>,r)}(X,Y)$, according to Theorem 3.4.1 T^* is belongs to $\mathcal{L}_{(r^*,m(q^*,s^*))}(Y^*,X^*)$ with

$$\|T\|_{(<p,q>,r)} = \|T^*\|_{(r^*,m(q^*,s^*))},$$

by applying Theorem 3.4.2 we get T^{**} is belongs to $\mathcal{L}_{(<p,q>,r)}(X^{**},Y^{**})$ with

$$\|T^*\|_{(r^*,m(q^*,s^*))} = \|T^{**}\|_{(<p,q>,r)}.$$

□

Remark 3.4.5. For some extreme cases of the parameters p,q,r and s, we give the well-known duality identifications that are showed by Apiola in [5].

(1) If we take $p=1$ in the Theorem 3.4.1 then $p^*=\infty$ and $s^*=q^*$ in this case we obtain

$$T \in \mathcal{L}_{(<1,q>,r)} = \mathcal{D}_{q,r} \text{ if and only if } T^* \in \mathcal{L}_{(r^*,m(q^*,q^*))} = \Pi_{r^*,q^*}$$

(2) If we take $q=1$ in the Theorem 3.4.1 then $q^*=\infty$ and $s^*=p^*$ in this case we obtain

$$T \in \mathcal{L}_{(r,<p,1>)} = \mathcal{D}_{r,p} \text{ if and only if } T^* \in \mathcal{L}_{(m(\infty,p^*),r^*)} = \Pi_{p^*,r^*}$$

(3) If we take $q=\infty$ in the Theorem 3.4.2 then $q^*=1$ and $s=p$ in this case we obtain

$$T \in \mathcal{L}_{(m(\infty,p),r)} = \Pi_{p,r} \text{ if and only if } T^* \in \mathcal{L}_{(r^*,<p^*,1>)} = \mathcal{D}_{r^*,p^*}$$

(4) If we take $p=\infty$ in the Theorem 3.4.2 then $p^*=1$ and $s=q$ in this case we obtain

$$T \in \mathcal{L}_{(r,m(q,q))} = \Pi_{r,q} \text{ if and only if } T^* \in \mathcal{L}_{(<1,q^*>,r^*)} = \mathcal{D}_{q^*,r^*}$$

By using Proposition 2.1.10 we can prove the inclusion theorem for the class $\mathcal{L}_{(<p,q>,r)}$.

Proposition 3.4.6. *If $p_2 \leq p_1$, $q_1 \leq q_2$ and $r_2 \leq r_1$ then*

$$\mathcal{L}_{(<p_1,q_1>,r_1)}(X,Y) \subseteq \mathcal{L}_{(<p_2,q_2>,r_2)}(X,Y).$$

Moreover,

$$\|T\|_{(<p_2,q_2>,r_2)} \leq \|T\|_{(<p_1,q_1>,r_1)},$$

for every $T \in \mathcal{L}_{(<p_1,q_1>,r_1)}(X,Y)$.

Proof. Let $T \in \mathcal{L}_{(<p_1,q_1>,r_1)}(X,Y)$ then for every $(x_i)_{i=1}^{\infty} \in \ell_r(X)$ and according to Proposition 2.1.10 we obtain

$$\begin{aligned}\|(T(x_i))_{i=1}^{\infty}\|_{\ell_{p_2,q_2}\langle Y\rangle} &\leq \|(T(x_i))_{i=1}^{\infty}\|_{\ell_{p_1,q_1}\langle Y\rangle} \\ &\leq \|T\|_{(<p_1,q_1>,r_1)} \|(x_i)_{i=1}^{\infty}\|_{\ell_{r_1}(X)} \\ &\leq \|T\|_{(<p_1,q_1>,r_1)} \|(x_i)_{i=1}^{\infty}\|_{\ell_{r_2}(X)}.\end{aligned}$$

Thus $T \in \mathcal{L}_{(<p_2,q_2>,r_2)}(X,Y)$ and $\|T\|_{(<p_2,q_2>,r_2)} \leq \|T\|_{(<p_1,q_1>,r_1)}$. □

From the above proposition and the Theorem 3.4.2, we immediately derive the following result.

Corollary 3.4.7. *If $p_1 \leq p_2$, $q_2 \leq q_1$, $r_1 \leq r_2$ and $1 \leq s_1, s_2$ such that $\frac{1}{s_1} = \frac{1}{p_1} + \frac{1}{q_1}$ and $\frac{1}{s_2} = \frac{1}{p_2} + \frac{1}{q_2}$ then*

$$\mathcal{L}_{(r_1,m(q_1,s_1))}(X,Y) \subseteq \mathcal{L}_{(r_2,m(q_2,s_2))}(X,Y)$$

Moreover,

$$\|T\|_{(r_2,m(q_2,s_2))} \leq \|T\|_{(r_1,m(q_1,s_1))},$$

for every $T \in \mathcal{L}_{(r_1,m(q_1,s_1))}(X,Y)$.

Proof. If $T \in \mathcal{L}_{(r_1,m(q_1,s_1))}(X,Y)$ then by Theorem 3.4.2 we have

$$T^* \in \mathcal{L}_{(<p_1^*,q_1^*>,r_1^*)}(Y^*,X^*) \text{ with } \|T\|_{(r_1,m(q_1,s_1))} = \|T^*\|_{(<p_1^*,q_1^*>,r_1^*)}$$

and the Proposition 3.4.6 assure that

$$T^* \in \mathcal{L}_{(<p_2^*,q_2^*>,r_2^*)}(Y^*,X^*) \text{ with } \|T^*\|_{(<p_2^*,q_2^*>,r_2^*)} \leq \|T^*\|_{(r_1,m(q_1,s_1))}.$$

Applying Theorem 3.4.2 again we get

$$T \in \mathcal{L}_{(r_2,m(q_2,s_2))}(X,Y) \text{ with } \|T^*\|_{(<p_2^*,q_2^*>,r_2^*)} = \|T\|_{(r_2,m(q_2,s_2))}.$$

□

Also, by using Proposition 2.1.10 we can prove the inclusion theorem for the class $\mathcal{L}_{(r,<p,q>)}$. The proof is similar to the proof of Proposition 3.4.6.

Proposition 3.4.8. *If $p_1 \leq p_2$, $q_2 \leq q_1$ and $r_1 \leq r_2$ then*

$$\mathcal{L}_{(r_1,<p_1,q_1>)}(X,Y) \subseteq \mathcal{L}_{(r_2,<p_2,q_2>)}(X,Y).$$

Moreover,

$$\|T\|_{(r_2,<p_2,q_2>)} \leq \|T\|_{(r_1,<p_1,q_1>)},$$

for every $T \in \mathcal{L}_{(r_1,<p_1,q_1>)}(X,Y)$.

From the above proposition and the Theorem 3.4.2, we immediately obtain the following result.

Corollary 3.4.9. *If $p_2 \leq p_1$, $q_1 \leq q_2$, $r_2 \leq r_1$ and $s_j \geq 1$ such that $\frac{1}{s_j} = \frac{1}{p_j} + \frac{1}{r_j}$, $j = 1,2$ then*

$$\mathcal{L}_{(m(r_1,s_1),q_1)}(X,Y) \subseteq \mathcal{L}_{(m(r_2,s_2),q_2)}(X,Y).$$

Moreover,

$$\|T\|_{(m(r_2,s_2),q_2)} \leq \|T\|_{(m(r_1,s_1),q_1)},$$

for every $T \in \mathcal{L}_{(m(r_1,s_1),q_1)}(X,Y)$.

By using the various relationships between the Banach sequences spaces which were studied in the first and second chapters, we can prove the following composition results.

Proposition 3.4.10. *Let X, Y, Z be a Banach spaces and $1 \leq p, q, r, s \leq \infty$. We have the following*

1) $\mathcal{L}_{(r,<p,q>)} \circ \mathcal{L}_{(p,m(s,r))} \subseteq \mathcal{D}_r$.

2) $\mathcal{L}_{(r,<p,q>)} \circ \mathcal{L}_{(<p,q>,r)} \subseteq \mathcal{D}_r$.

3) $\mathcal{L}_{(r,<p,q>)} \circ \Pi_{p,r} \subseteq \mathcal{N}_r$.

4) $\mathcal{L}_{(r,<p,q>)} \circ \mathcal{D}_{q,r} \subseteq \mathcal{D}_r$.

5) $\mathcal{L}_{(r,<p,q>)} \circ \mathcal{N}_{q,r} \subseteq \mathcal{N}_r$.

Proof.

1) Let $T \in \mathcal{L}_{(r,<p,q>)}(Y,Z)$ and $S \in \mathcal{L}_{(p,m(s,r))}(X,Y)$ then we have

$$\|(T \circ S(x_i))_{i=1}^{\infty}\|_{\ell_r\langle Z\rangle} \leq \|T\|_{(r,<p,q>)} \|(S(x_i))_{i=1}^{\infty}\|_{\ell_{p,q}\langle Y\rangle}.$$

By using the Proposition 2.1.11 we get

$$\begin{aligned}\|(T \circ S(x_i))_{i=1}^{\infty}\|_{\ell_r\langle Z\rangle} &\leq \|T\|_{(r,<p,q>)} \|(S(x_i))_{i=1}^{\infty}\|_{\ell_p(Y)} \\ &\leq \|T\|_{(r,<p,q>)} \|S\|_{(p,m(s,r))} \|(x_i)_{i=1}^{\infty}\|_{\ell_{m(s,r)}(X)}.\end{aligned}$$

According to Proposition 2.1.1 we obtain

$$\|(T \circ S(x_i))_{i=1}^{\infty}\|_{\ell_r\langle Z\rangle} \leq \|T\|_{(r,<p,q>)} \|S\|_{(p,m(s,r))} \|(x_i)_{i=1}^{\infty}\|_{\ell_r(X)}.$$

This implies that $T \circ S \in \mathcal{D}_r(X,Z)$ with

$$d_r(T \circ S) \leq \|T\|_{(r,<p,q>)} \|S\|_{(p,m(s,r))}.$$

2) Let $T \in \mathcal{L}_{(r,<p,q>)}(Y,Z)$ and $S \in \mathcal{L}_{(<p,q>,r)}(X,Y)$ then we have

$$\begin{aligned}\|(T \circ S(x_i))_{i=1}^{\infty}\|_{\ell_r\langle Z\rangle} &\leq \|T\|_{(r,<p,q>)} \|(S(x_i))_{i=1}^{\infty}\|_{\ell_{p,q}\langle Y\rangle} \\ &\leq \|T\|_{(r,<p,q>)} \|S\|_{(<p,q>,r)} \|(x_i)_{i=1}^{\infty}\|_{\ell_r(X)}.\end{aligned}$$

This implies that $T \circ S \in \mathcal{D}_r(X,Z)$ with

$$d_r(T \circ S) \leq \|T\|_{(r,<p,q>)} \|S\|_{(<p,q>,r)}.$$

3) Let $T \in \mathcal{L}_{(r,<p,q>)}(Y,Z)$ and $S \in \Pi_{p,r}(X,Y)$ then we have

$$\|(T \circ S(x_i))_{i=1}^{\infty}\|_{\ell_r\langle Z\rangle} \leq \|T\|_{(r,<p,q>)} \|(S(x_i))_{i=1}^{\infty}\|_{\ell_{p,q}\langle Y\rangle}.$$

Using the Proposition 2.1.11 we get

$$\begin{aligned}\|(T \circ S(x_i))_{i=1}^{\infty}\|_{\ell_r\langle Z\rangle} &\leq \|T\|_{(r,<p,q>)} \|(S(x_i))_{i=1}^{\infty}\|_{\ell_p(Y)} \\ &\leq \|T\|_{(r,<p,q>)} \pi_{p,r}(S) \|(x_i)_{i=1}^{\infty}\|_{\ell_{r,\omega}(X)}.\end{aligned}$$

This implies that $T \circ S \in \mathcal{N}_r(X,Z)$ with

$$n_r(T \circ S) \leq \|T\|_{(r,<p,q>)} \pi_{p,r}(S).$$

4) and 5) are similar to 3), using the Proposition 2.1.11 only. □

We finish this chapter with the following two theorems. We present only the proof of the first theorem, as for the second, its proof is a similar way (using [21, Theorem 4.2.5]).

Theorem 3.4.11. *For $1 \leq q \leq \infty$ and $0 < q, s \leq t \leq r \leq \infty$ such that $\frac{1}{t} + 1 = \frac{1}{q} + \frac{1}{p}$. A mapping $S \in \mathcal{L}(X, Y)$ is in $\mathcal{L}_{(r,<p,q>)}(X, Y)$ if and only if $T \circ S \in \mathcal{D}_{r,s}(X, Z)$ for every $T \in \mathcal{L}_{(<p,q>,s)}(Y, Z)$ and each Banach space Z.*

Proof. First, suppose that $S \in \mathcal{L}_{(r,<p,q>)}(X, Y)$, then for every $T \in \mathcal{L}_{(<p,q>,s)}(Y, Z)$ (Z is a Banach space) and according to Theorem 3.4.1 we get

$$S^* \in \mathcal{L}_{(m(q^*,t^*,),r^*)}(Y^*, X^*) \text{ and } T^* \in \mathcal{L}_{(s^*,m(q^*,t^*,))}(Z^*, Y^*),$$

applying [21, Theorem 4.2.3] we obtain

$$S^* \circ T^* \in \Pi_{s^*,r^*}(Z^*, X^*),$$

by Theorem 3.0.1 this equivalent to $T \circ S \in \mathcal{D}_{r,s}(X, Z)$. Conversely, we suppose that $T \circ S \in \mathcal{D}_{r,s}(X, Z)$ for every $T \in \mathcal{L}_{(<p,q>,s)}(Y, Z)$ and each Banach space Z, then by Theorem 3.0.1 and Theorem 3.4.1 we get

$$S^* \circ T^* \in \Pi_{s^*,r^*}(Z^*, X^*) \text{ and } T^* \in \mathcal{L}_{(s^*,m(q^*,t^*))}(Z^*, Y^*),$$

(Z^* is a Banach space), applying [21, Theorem 4.2.3] we obtain

$$S^* \in \mathcal{L}_{(m(q^*,t^*,),r^*)}(Y^*, X^*),$$

by Theorem 3.4.1 this equivalent to $S \in \mathcal{L}_{(r,<p,q>)}(X, Y)$. □

Theorem 3.4.12. *For $1 \leq q \leq \infty$, $0 < s \leq q, r \leq \infty$ such that $\frac{1}{s} + 1 = \frac{1}{q} + \frac{1}{p}$. A mapping $S \in \mathcal{L}(X, Y)$ is in $\mathcal{L}_{(r,<p,q>)}(X, Y)$ if and only if $T \circ S \in \mathcal{D}_{r,s}(X, Z)$ for every $T \in \mathcal{D}_q(Y, Z)$ and each Banach space Z.*

Chapter 4

Banach space of strongly (p, q, σ)-summable sequences and applications

In this chapter, we introduce and study the Banach space $\ell_p^{q\sigma}\langle X\rangle$, of vector-valued sequences which are called strongly (p, q, σ)-summable sequences [14]. We present a new class of the (p, σ, q, ν)-nuclear operators that is defined by using a summability property and we characterize this class and the class of strongly (p, σ)-continuous operators by our Banach sequence space $\ell_p^{q\sigma}\langle X\rangle$. We also present some new results concerning this last class of operators.

4.1 (p, σ)-weakly summable sequences

The space of (p, σ)-weakly summable sequences was introduced in [18] by Molina and Sánchez-Pérez, we recall some properties of this space. Let $1 \leq p < \infty$ and $0 \leq \sigma < 1$. Let X be a Banach space and $(x_i)_{i=1}^{\infty} \subset X$. Define

$$\delta_{p\sigma}\left((x_i)_{i=1}^{\infty}\right) = \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^{\infty} \left(|\langle x_i, x^* \rangle|^{1-\sigma} \|x_i\|^{\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}},$$

and

$$H_{p,\sigma}(X) = \left\{ (x_i)_{i=1}^{\infty} \subset X : \delta_{p\sigma}\left((x_i)_{i=1}^{\infty}\right) < \infty \right\}.$$

We have that

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_{\frac{p}{1-\sigma},\omega}(X)} \leq \delta_{p\sigma}\left((x_i)_{i=1}^{\infty}\right) \leq \|(x_i)_{i=1}^{\infty}\|_{\ell_{\frac{p}{1-\sigma}}(X)}, \tag{4.1}$$

for all $(x_i)_{i=1}^{\infty} \in \ell_{\frac{p}{1-\sigma}}(X)$.
For the extreme cases $\sigma = 1$ and $p = \infty$, we define also for all $0 \leq \tau \leq 1$ and $1 \leq q \leq \infty$

$$\delta_{q1}\left((x_i)_{i=1}^{n}\right) = \delta_{\infty\tau}\left((x_i)_{i=1}^{n}\right) = \sup_{1 \leq i \leq n} \|x_i\| = \|(x_i)_{i=1}^{n}\|_{\ell_{\infty}(X)}. \tag{4.2}$$

Definition 4.1.1. [18, Definition 1.2] A sequence $(x_i)_{i=1}^{\infty}$ of elements in X is said to be (p, σ)-***weakly summable*** if it belongs to the vector space spanned by $H_{p,\sigma}(X)$.
We denote by $\ell^{p\sigma}(X)$ the vector space of all (p, σ)-weakly summable sequences of X. For $(x_i)_{i=1}^{\infty} \in \ell^{p\sigma}(X)$, we set

$$\|(x_i)_{i=1}^{\infty}\|_{\ell^{p\sigma}(X)} = \inf \sum_{l=1}^{k} \delta_{p\sigma}\left(\left(x_i^l\right)_{i=1}^{\infty}\right), \tag{4.3}$$

where the infimum is taken over all representations of $(x_i)_{i=1}^{\infty}$ of the form

$$(x_i)_{i=1}^{\infty} = \sum_{l=1}^{k} \left(x_i^l\right)_{i=1}^{\infty},$$

with $\left(x_i^l\right)_{i=1}^{\infty} \in H_{p,\sigma}(X), k \in \mathbb{N}$.

We have the following proposition.

Proposition 4.1.2. *[18, Proposition 1.3] On $\ell^{p\sigma}(X)$, the function $\|\cdot\|_{\ell^{p\sigma}(X)}$ defined by (4.3), is a norm. In addition, we have the inclusions*

$$\ell_{\frac{p}{1-\sigma}}(X) \subset \ell^{p\sigma}(X) \subset \ell_{\frac{p}{1-\sigma},\omega}(X), \tag{4.4}$$

with

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_{\frac{p}{1-\sigma},\omega}(X)} \leq \|(x_i)_{i=1}^{\infty}\|_{\ell^{p\sigma}(X)} \leq \|(x_i)_{i=1}^{\infty}\|_{\ell_{\frac{p}{1-\sigma}}(X)},$$

for all $(x_i)_{i=1}^{\infty} \in \ell_{\frac{p}{1-\sigma}}(X)$. Moreover,

$$\|(x_i)_{i=1}^{\infty}\|_{\ell^{p\sigma}(X)} \leq \inf\left\{\sum_{l=1}^{k} \left\|\left(x_i^l\right)_{i=1}^{\infty}\right\|_{\ell_{\frac{p}{1-\sigma}}(X)}^{\sigma} \cdot \left\|\left(x_i^l\right)_{i=1}^{\infty}\right\|_{\ell_{\frac{p}{1-\sigma},\omega}(X)}^{1-\sigma}\right\},$$

where the infimum is taken over all representations of $(x_i)_{i=1}^{\infty} \in \ell_{\frac{p}{1-\sigma}}(X)$ of the form $(x_i)_{i=1}^{\infty} = \sum_{l=1}^{k} \left(x_i^l\right)_{i=1}^{\infty}$ with $\left(x_i^l\right)_{i=1}^{\infty} \in H_{p,\sigma}(X)$.

Now we give a representation for the elements in the completion $\widehat{\ell^{p\sigma}}(X)$, which complements nicely the representation for the elements of $\ell^{p\sigma}(X)$. We will continue denoting by $\|\cdot\|_{\ell^{p\sigma}(X)}$, the norm in $\widehat{\ell^{p\sigma}}(X)$.

Proposition 4.1.3. *[18, Proposition 1.4] Let $1 \leq p < \infty$ and X be Banach space. If $\varphi \in \widehat{\ell^{p\sigma}}(X)$, there exist $x^n = (x_i^n)_{i=1}^{\infty} \in H_{p,\sigma}(X), n \in \mathbb{N}$ such that*

$$\sum_{n=1}^{\infty} \delta_{p\sigma}(x^n) < \infty \text{ and } \varphi = \sum_{n=1}^{\infty} x^n \text{ in } \widehat{\ell^{p\sigma}}(X).$$

Conversely, for $x^n = (x_i^n)_{i=1}^{\infty} \in H_{p,\sigma}(X)$ there exists a unique $\varphi \in \widehat{\ell^{p\sigma}}(X)$ such that $\varphi = \sum_{n=1}^{\infty} x^n$. In both cases

$$\|\varphi\|_{\widehat{\ell^{p\sigma}}(X)} = \inf \sum_{n=1}^{\infty} \delta_{p\sigma}(x^n),$$

where the infimum is taken over all representations of φ of the appropriate form.

4.1.1 (p,σ)-absolutely continuous operators

The class of (p,σ)-absolutely continuous linear operators is due to Matter [22]. In [18] Molina and Sánchez-Pérez presented a characterization for these operators by using the space of (p,σ)-weakly summable sequences. This class of operators is generalized to multi-linear and polynomials setting in [2] and [12] respectively.

Definition 4.1.4. [22] Let $1 \leq p < \infty$ and $0 \leq \sigma < 1$. We say that $T \in \mathcal{L}(X,Y)$ is a (p,σ)-***absolutely continuous operator***, in symbols $T \in \Pi_{p,\sigma}(X,Y)$, if there is a constant $C > 0$ such that for every finite sequence $(x_i)_{i=1}^n \subset X$,

$$\|(T(x_i))_{i=1}^n\|_{\ell_{\frac{p}{1-\sigma}}(Y)} \leq C\delta_{p\sigma}\left((x_i)_{i=1}^n\right). \tag{4.5}$$

The norm of T is defined by

$$\pi_{p,\sigma}(T) = \inf\left\{C : C \text{ verifying the inequality (4.5)}\right\}.$$

By [18, Theorem 1.7] and the paragraph that precedes it, the above definition is equivalent to say that $\widehat{T}\left(\widehat{\ell^{p\sigma}}(X)\right) \subset \ell_{\frac{p}{1-\sigma}}(Y)$ where the operator $\widehat{T}$ is defined by

$$\widehat{T}\left((x_i)_{i=1}^\infty\right) = \left(T(x_i)\right)_{i=1}^\infty,$$

for all $(x_i)_{i=1}^\infty \in \ell^{p\sigma}(X)$ and by

$$\widehat{T}(\varphi) = \left(\lim_{n\to\infty} T(x_i^n)\right)_{i=1}^\infty,$$

for all $\varphi \in \widehat{\ell^{p\sigma}}(X)$ and $\varphi = \lim_{n\to\infty} x_i^n$ with $(x_i^n)_{i=1}^\infty \in \ell^{p\sigma}(X)$.

Theorem 4.1.5. *[22, Theorem 3.2] The class $(\Pi_{p,\sigma}, \pi_{p,\sigma}(.))$ is an injective Banach operator ideal.*

We are going now to compare $\Pi_{p,\sigma}$ with other known ideals. We get the following proposition.

Proposition 4.1.6. *[22, Proposition 4.2] We have the inclusions*

$$\Pi_{\frac{p}{1-\sigma}} \subset \Pi_{p,\sigma} \subset \Pi_{\frac{p}{1-\sigma},p}.$$

Moreover,

$$\pi_{\frac{p}{1-\sigma},p}(T) \leq \pi_{p,\sigma}(T) \leq \pi_{\frac{p}{1-\sigma}}(T),$$

for all $T \in \Pi_{\frac{p}{1-\sigma}}(X,Y)$.

4.2 Strongly (p,q,σ)-summable sequences

Now we introduce the space of strongly (p,q,σ)-summable sequences [14] in order to give a characterization of the classes of (p,σ,q,ν)-nuclear (that we will define later in the last section), and strongly (p,σ)-continuous linear operators.

Definition 4.2.1. [14] Let $1 < p, q < \infty$ and $0 \leq \sigma < 1$ such that $\frac{1}{p} + \frac{1-\sigma}{q} = 1$. We define the space $\ell_p^{q\sigma}\langle X\rangle$ to be the set of all sequence $(x_i)_{i=1}^{\infty}$ in X such that $\left|\sum_{i=1}^{\infty}\langle x_i, x_i^*\rangle\right| < \infty$ for all $(x_i^*)_{i=1}^{\infty} \in \ell^{q\sigma}(X^*)$. In this case we say that $(x_i)_{i=1}^{\infty}$ is ***strongly*** (p, q, σ)***-summable***. For $(x_i)_{i=1}^{\infty} \in \ell_p^{q\sigma}\langle X\rangle$, we put

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_p^{q\sigma}\langle X\rangle} = \sup\left\{\left|\sum_{i=1}^{\infty}\langle x_i, x_i^*\rangle\right|, \|(x_i^*)_{i=1}^{\infty}\|_{\ell^{q\sigma}(X^*)} \leq 1\right\}. \tag{4.6}$$

As it is proved in [10, Lemma 2.1.7], it is possible to interchange the summation and absolute value symbols in (4.6), that is

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_p^{q\sigma}\langle X\rangle} = \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{\ell^{q\sigma}(X^*)}\leq 1} \|(\langle x_i, x_i^*\rangle)_{i=1}^{\infty}\|_{\ell_1}. \tag{4.7}$$

Following the idea of [11, Theorem 1.1.2], we show that the function $\|\cdot\|_{\ell_p^{q\sigma}\langle X\rangle}$ is well-defined.

Proposition 4.2.2. *[29, Proposition 1] If* $(x_i)_{i=1}^{\infty} \in \ell_p^{q\sigma}\langle X\rangle$*, then* $\|(x_i)_{i=1}^{\infty}\|_{\ell_p^{q\sigma}\langle X\rangle} < \infty$.

Proof. Let $(x_i)_{i=1}^{\infty} \in \ell_p^{q\sigma}\langle X\rangle$ and consider the linear form f on $\hat{\ell}^{q\sigma}(X^*)$ given by

$$f(\phi) = \lim_{m\longrightarrow\infty}\sum_{i=1}^{\infty}\langle x_i, \varphi_i^m\rangle,$$

for all $\phi \in \hat{\ell}^{q\sigma}(X^*)$ such that $\phi = \lim_{m\longrightarrow\infty}\varphi_i^m$, with $(\varphi_i^m)_{i=1}^{\infty} \in \ell^{q\sigma}(X^*)$. Now we define a sequence of linear functionals $(f_n)_{n=1}^{\infty}$ on $\hat{\ell}^{q\sigma}(X^*)$ by

$$f_n(\phi) = \lim_{m\longrightarrow\infty}\sum_{i=1}^{n}\langle x_i, \varphi_i^m\rangle.$$

Since the function $\langle x, \cdot\rangle, x \in X$ is continuous we obtain that the functionals f_n are continuous. Also clearly $(f_n)_{n=1}^{\infty}$ is converge to f at each point of $\hat{\ell}^{q\sigma}(X^*)$. An application of Banach-Steinhaus Theorem reveals that f is continuous. Finally,

$$\begin{aligned}\|(x_i)_{i=1}^{\infty}\|_{\ell_p^{q\sigma}\langle X\rangle} &= \sup\left\{\left|\sum_{i=1}^{\infty}\langle x_i, x_i^*\rangle\right| : (x_i^*)_{i=1}^{\infty} \in \ell^{q\sigma}(X^*), \|(x_i^*)_{i=1}^{\infty}\|_{\ell^{q\sigma}(X^*)} \leq 1\right\} \\ &\leq \sup\left\{\left|\lim_{m\longrightarrow\infty}\sum_{i=1}^{\infty}\langle x_i, \varphi_i^m\rangle\right| : \|\phi\|_{\hat{\ell}^{q\sigma}(X^*)} \leq 1, \phi = \lim_{m\longrightarrow\infty}\varphi_i^m\right\} \\ &= \|f\| < \infty.\end{aligned}$$

□

Theorem 4.2.3. *[14, Theorem 1] On $\ell_p^{q\sigma}\langle X\rangle$, the function $\|\cdot\|_{\ell_p^{q\sigma}\langle X\rangle}$, defined by (4.6) or (4.7) is a norm. In addition we have the inclusions*

$$\ell_p\langle X\rangle \subset \ell_p^{q\sigma}\langle X\rangle \subset \ell_p(X). \tag{4.8}$$

Moreover, for all $(x_i)_{i=1}^\infty \in \ell_p\langle X\rangle$ we have

$$\|(x_i)_{i=1}^\infty\|_{\ell_p(X)} \le \|(x_i)_{i=1}^\infty\|_{\ell_p^{q\sigma}\langle X\rangle} \le \|(x_i)_{i=1}^\infty\|_{\ell_p\langle X\rangle}.$$

Proof. It is not difficult to verify the axioms of the norm. Now let $(x_i)_{i=1}^\infty \in \ell_p^{q\sigma}\langle X\rangle$, by using the isometric identification $(\ell_p(X))^* = \ell_{\frac{q}{1-\sigma}}(X^*)$ and the fact that $\|\cdot\|_{\ell_{\frac{q}{1-\sigma}}(X^*)} \ge \|\cdot\|_{\ell^{q\sigma}(X^*)}$ we obtain

$$\begin{aligned}\|(x_i)_{i=1}^\infty\|_{\ell_p(X)} &= \sup_{\|(x_i^*)_{i=1}^\infty\|_{\ell_{\frac{q}{1-\sigma}}(X^*)}\le 1} \left|\sum_{i=1}^\infty \langle x_i, x_i^*\rangle\right| \\ &\le \sup_{\|(x_i^*)_{i=1}^\infty\|_{\ell^{q\sigma}(X^*)}\le 1} \left|\sum_{i=1}^\infty \langle x_i, x_i^*\rangle\right| \\ &= \|(x_i)_{i=1}^\infty\|_{\ell_p^{q\sigma}\langle X\rangle},\end{aligned}$$

and then $(x_i)_{i=1}^\infty \in \ell_p(X)$. The inclusion $\ell_p\langle X\rangle \subset \ell_p^{q\sigma}\langle X\rangle$ follows directly from the inequality $\|\cdot\|_{\ell^{q\sigma}(X^*)} \ge \|\cdot\|_{\ell_{\frac{q}{1-\sigma},\omega}(X^*)}$. Indeed, for all $(x_i)_{i=1}^\infty \in \ell_p\langle X\rangle$ we have

$$\begin{aligned}\|(x_i)_{i=1}^\infty\|_{\ell_p^{q\sigma}\langle X\rangle} &= \sup_{\|(x_i^*)_{i=1}^\infty\|_{\ell^{q\sigma}(X^*)}\le 1} \|(\langle x_i, x_i^*\rangle)_{i=1}^\infty\|_{\ell_1} \\ &\le \sup_{\|(x_i^*)_{i=1}^\infty\|_{\ell_{\frac{q}{1-\sigma},\omega}(X^*)}\le 1} \|(\langle x_i, x_i^*\rangle)_{i=1}^\infty\|_{\ell_1} \\ &= \|(x_i)_{i=1}^\infty\|_{\ell_p\langle X\rangle}.\end{aligned}$$

□

Taking into account the inequality $\|\cdot\|_{\ell_p(X)} \le \|\cdot\|_{\ell_p^{q\sigma}\langle X\rangle}$ and using the fact that $\ell_p(X)$ is a Banach space, we obtain the following theorem.

Theorem 4.2.4. *[14, Theorem 2] For $1 < p, q < \infty$ and $0 \le \sigma < 1$ such that $\frac{1}{p} + \frac{1-\sigma}{q} = 1$, the space $\ell_p^{q\sigma}\langle X\rangle$ with the norm $\|\cdot\|_{\ell_p^{q\sigma}\langle X\rangle}$ is a Banach space.*

Proof. Let $\left\{\xi^k\right\}_{k=1}^\infty$ be a Cauchy sequence in $\ell_p^{q\sigma}\langle X\rangle$, where $\xi^k = \left(x_i^k\right)_{i=1}^\infty$. For an $\varepsilon > 0$, choose a number $k_0 \in \mathbb{N}$ such that

$$\begin{aligned} \left\|\xi^k - \xi^l\right\|_{\ell_p(X)} &\leq \left\|\xi^k - \xi^l\right\|_{\ell_p^{q\sigma}\langle X\rangle} \\ &= \sup_{\left\|(x_i^*)_{i=1}^\infty\right\|_{\ell^{q\sigma}(X^*)}\leq 1} \left\|\left(\left\langle x_i^k - x_i^l, x_i^*\right\rangle\right)_{i=1}^\infty\right\|_{\ell_1} \leq \varepsilon, \end{aligned} \tag{4.9}$$

for every $k, l \geq k_0$. Which implies that $\left\{\xi^k\right\}_{k=1}^\infty$ is a Cauchy sequence in the Banach space $\ell_p(X)$, hence the sequence $\left\{\xi^k\right\}_{k=1}^\infty$ converges to $(x_i)_{i=1}^\infty \in \ell_p(X)$. We will show that $(x_i)_{i=1}^\infty$ is in $\ell_p^{q\sigma}\langle X\rangle$. Since $\left\{\xi^k\right\}_{k=1}^\infty$ is a Cauchy sequence in $\ell_p^{q\sigma}\langle X\rangle$, there exists $C \geq 0$, such that

$$\sup_{\left\|(x_i^*)_{i=1}^\infty\right\|_{\ell^{q\sigma}(X^*)}\leq 1} \sum_{i=1}^n \left|\left\langle x_i^k, x_i^*\right\rangle\right| \leq C, \text{ for all } n, k \in \mathbb{N}.$$

By letting $k \longrightarrow \infty$ we get

$$\sup_{\left\|(x_i^*)_{i=1}^\infty\right\|_{\ell^{q\sigma}(X^*)}\leq 1} \sum_{i=1}^n \left|\left\langle x_i, x_i^*\right\rangle\right| \leq C, \text{ for all } n \in \mathbb{N}.$$

Therefore $\sup_{\left\|(x_i^*)_{i=1}^\infty\right\|_{\ell^{q\sigma}(X^*)}\leq 1} \|(\langle x_i, x_i^*\rangle)_{i=1}^\infty\|_{\ell_1} \leq C$ and $(x_i)_{i=1}^\infty \in \ell_p^{q\sigma}\langle X\rangle$. Now we let $l \to \infty$ in (4.9) to obtain

$$\left\|\xi^k - (x_i)_{i=1}^\infty\right\|_{\ell_p^{q\sigma}\langle X\rangle} \leq \varepsilon, \text{ for every } k \geq k_0.$$

Therefore $\xi^k \longrightarrow (x_i)_{i=1}^\infty$ in $\ell_p^{q\sigma}\langle X\rangle$. □

4.3 Cohen (p, σ, q, ν)-nuclear operators

Before studying the class of $(p, \sigma, q, \nu,)$-nuclear operators we present some new results concerning the class of strongly (p, σ)-continuous linear operators.

4.3.1 Strongly (p, σ)-continuous linear operators

The class of strongly (p, σ)-continuous linear operators is introduced by Achour et al. (see [3]) in order to study the adjoints of the (p, σ)-absolutely continuous linear operators.

Definition 4.3.1. [3] Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$, such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. An operator $T \in \mathcal{L}(X, Y)$ between Banach spaces is ***strongly*** (p, σ)***-continuous*** if there is a constant $C > 0$ such that for every $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$ the following inequality holds

$$\|(\langle T(x_i), y_i^* \rangle)_{i=1}^n\|_{\ell_1} \leq C \, \|(x_i)_{i=1}^n\|_{\ell_r(X)} \, \delta_{p^*\sigma} \left((y_i^*)_{i=1}^n\right). \tag{4.10}$$

The class of all strongly (p, σ)-continuous linear operators from X into Y is denoted by $\mathcal{D}_p^\sigma(X, Y)$ and by $d_p^\sigma(T)$ the strongly (p, σ)-continuous norm which is defined by $d_p^\sigma(T) = \inf C$, where the infimum is taken over all constants C verifying the inequality (4.10).

Actually, if we take $\sigma = 0$ we obtain $(\mathcal{D}_p^0, d_p^\sigma(\cdot)) = (\mathcal{D}_p, d_p(\cdot))$, the class of strongly p-summing operators introduced by Cohen in [11] and generalized to the multilinear setting in [1]. Note that for all $T \in \mathcal{D}_p^\sigma(X, Y)$ we have

$$\|T\| \leq d_p^\sigma(T). \tag{4.11}$$

The following Corollary shows that the strongly (p^*, σ)-continuous linear operators are the adjoints of (p, σ)-absolutely continuous linear operators.

Corollary 4.3.2. *[3, Corollary 3.8] Let $1 < p < \infty$ and $0 \leq \sigma < 1$. Let $T \in \mathcal{L}(X, Y)$ and $T^* \in \mathcal{L}(Y^*, X^*)$ its adjoint. Then T is (p, σ)-absolutely continuous if and only if T^* is strongly (p^*, σ)-continuous.*

The next result describes the relationship between the three classes with different parameters works as one would expect.

Proposition 4.3.3. *[14, Proposition 2] Let $1 \leq p, r < \infty$ and $0 \leq \sigma < 1$ such that $\frac{1}{r} + \frac{1-\sigma}{p} = 1$, then*

$$\mathcal{D}_r(X, Y) \subset \mathcal{D}_{p^*}^\sigma(X, Y) \subset \mathcal{D}_{p^*, r}(X, Y).$$

Moreover,

$$d_{p^*, r}(T) \leq d_{p^*}^\sigma(T) \leq d_r(T),$$

for all $T \in \mathcal{D}_r(X, Y)$.

Proof. Let $T \in \mathcal{D}_r(X, Y)$, by Theorem 3.0.1 and Proposition 4.1.6 we have

$$T^* \in \Pi_{p,\sigma}(Y^*, X^*) \text{ with } \pi_{p,\sigma}(T^*) \leq \pi_{r^*}(T^*).$$

Now using Corollary 4.3.2 we get

$$T \in \mathcal{D}^{\sigma}_{p^*}(X,Y) \text{ and } d^{\sigma}_{p^*}(T) = \pi_{p,\sigma}(T^*) \leq d_r(T).$$

In order to show the second inclusion, if $T \in \mathcal{D}^{\sigma}_{p^*}(X,Y)$ then according to Corollary 4.3.2 and Proposition 4.1.6 we have

$$T^* \in \Pi_{\frac{p}{1-\sigma},p}(Y^*,X^*) \text{ with } \pi_{\frac{p}{1-\sigma},p}(T^*) \leq \pi_{p,\sigma}(T^*).$$

Finally, by Theorem 3.0.1 we obtain

$$T \in \mathcal{D}_{p^*,r}(X,Y) \text{ and } d_{p^*,r}(T) \leq d^{\sigma}_{p^*}(T).$$

□

As in the classical cases, the natural way of presenting the summability properties of the strongly (p,σ)-continuous operators is by defining the corresponding operator between adequate sequence spaces.

Theorem 4.3.4. *[14, Theorem 3] Let* $1 < p, r < \infty$ *and* $0 \leq \sigma < 1$, *such that*

$$\frac{1}{r} + \frac{1-\sigma}{p^*} = 1. \tag{4.12}$$

Then the operator $T \in \mathcal{L}(X,Y)$ *is strongly* (p,σ)*-continuous if and only if* $\widehat{T}\left(\ell_r(X)\right) \subset \ell_r^{p^*\sigma}\langle Y\rangle$.

Proof. For the necessity, let $T \in \mathcal{D}^{\sigma}_{p}(X,Y)$, $(x_i)_{i=1}^{\infty} \in \ell_r(X)$ and $(y_i^*)_{i=1}^{\infty} \in \ell^{p^*\sigma}(Y^*)$. For each $\varepsilon > 0$, there exists $\left(y_{i,j}^*\right)_{i=1}^{\infty} \in H_{p^*,\sigma}(Y^*)$, $j = 1, ..., k$, such that

$$(y_i^*)_{i=1}^{\infty} = \sum_{j=1}^{k} \left(y_{i,j}^*\right)_{i=1}^{\infty} \quad \text{with} \quad \sum_{j=1}^{k} \delta_{p^*\sigma}\left(\left(y_{i,j}^*\right)_{i=1}^{\infty}\right) \leq (1+\varepsilon)\left\|(y_i^*)_{i=1}^{\infty}\right\|_{\ell^{p^*\sigma}(Y^*)}.$$

So we have

$$\begin{aligned}
\left\|(\langle T(x_i), y_i^*\rangle)_{i=1}^{\infty}\right\|_{\ell_1} &= \sup_n \left\|(\langle T(x_i), y_i^*\rangle)_{i=1}^{n}\right\|_{\ell_1} \\
&\leq \sup_n \sum_{j=1}^{k} \left\|\left(\left\langle T(x_i), y_{i,j}^*\right\rangle\right)_{i=1}^{n}\right\|_{\ell_1} \\
&\leq d^{\sigma}_{p}(T) \sup_n \sum_{j=1}^{k} \left\|(x_i)_{i=1}^{n}\right\|_{\ell_r(X)} \delta_{p^*\sigma}\left(\left(y_{i,j}^*\right)_{i=1}^{n}\right) \\
&\leq d^{\sigma}_{p}(T) \left\|(x_i)_{i=1}^{\infty}\right\|_{\ell_r(X)} (1+\varepsilon) \left\|(y_i^*)_{i=1}^{\infty}\right\|_{\ell^{p^*\sigma}(Y^*)}.
\end{aligned}$$

Since this holds for all $\varepsilon > 0$, we obtain

$$\begin{aligned}\|(T(x_i))_{i=1}^{\infty}\|_{\ell_r^{p^*\sigma}\langle Y\rangle} &= \sup_{\left\|(y_i^*)_{i=1}^{\infty}\right\|_{\ell^{p^*\sigma}(Y^*)}\leq 1} \|(\langle T(x_i), y_i^*\rangle)_{i=1}^{\infty}\|_{\ell_1} \\ &\leq d_p^{\sigma}(T)\, \|(x_i)_{i=1}^{\infty}\|_{\ell_r(X)} < \infty,\end{aligned}$$

and then $\widehat{T}\left((x_i)_{i=1}^{\infty}\right) \in \ell_r^{p^*\sigma}\langle Y\rangle$. Furthermore, the last inequality actually implies that the operator $\widehat{T} : \ell_r(X) \longrightarrow \ell_r^{p^*\sigma}\langle Y\rangle$ is continuous with norm $\leq d_p^{\sigma}(T)$. In order to prove sufficiency, suppose $\widehat{T}$ maps $\ell_r(X)$ into $\ell_r^{p^*\sigma}\langle Y\rangle$ and assume that $T \notin \mathcal{D}_p^{\sigma}(X, Y)$. Then for each $n \in \mathbb{N}$, we may choose a finite sequence $(x_{i,j})_{i=1}^{m_j} \subset X$ such that

$$\left\|(x_{i,j})_{i=1}^{m_j}\right\|_{\ell_r(X)} \leq 1 \quad \text{and} \quad \left\|(T(x_{i,j}))_{i=1}^{m_j}\right\|_{\ell_r^{p^*\sigma}\langle Y\rangle} \geq 2^j,$$

which implies

$$\left|\sum_{i=1}^{m_j} y_{i,j}^*(T(x_{i,j}))\right| \geq 2^j, \tag{4.13}$$

for all $\left(y_{i,j}^*\right)_{i=1}^{m_j} \in \ell^{p^*\sigma}(Y^*)$ such that $\left\|\left(y_{i,j}^*\right)_{i=1}^{m_j}\right\|_{\ell^{p^*\sigma}(Y^*)} \leq 1$. Let $(z_j)_{j=1}^{\infty}$ be the sequence

$$\begin{aligned}&\left(\left(\frac{x_{i,1}}{2^{\frac{1}{r}}}\right)_{i=1}^{m_1}, \left(\frac{x_{i,2}}{2^{\frac{2}{r}}}\right)_{i=1}^{m_2}, ..., \left(\frac{x_{i,j}}{2^{\frac{j}{r}}}\right)_{i=1}^{m_j}, ...\right) \\ &= \left(\frac{x_{1,1}}{2^{\frac{1}{r}}}, \frac{x_{2,1}}{2^{\frac{1}{r}}}, ..., \frac{x_{m_1,1}}{2^{\frac{1}{r}}}, \frac{x_{1,2}}{2^{\frac{2}{r}}}, \frac{x_{2,2}}{2^{\frac{2}{r}}}, ..., \frac{x_{m_2,2}}{2^{\frac{2}{r}}}, ..., \frac{x_{1,j}}{2^{\frac{j}{r}}}, \frac{x_{2,j}}{2^{\frac{j}{r}}}, ..., \frac{x_{m_j,j}}{2^{\frac{j}{r}}}, ...\right).\end{aligned}$$

We have

$$\begin{aligned}\left\|(z_j)_j^{\infty}\right\|_{\ell_r(X)} &= \left(\sum_{j=1}^{+\infty}\sum_{i=1}^{m_j}\left\|\frac{x_{i,j}}{2^{\frac{j}{r}}}\right\|^r\right)^{\frac{1}{r}} \\ &= \left(\sum_{j=1}^{+\infty}\frac{1}{2^j}\left\|(x_{i,j})_{i=1}^{m_j}\right\|_{\ell_r(X)}^r\right)^{\frac{1}{r}} \\ &\leq \left(\sum_{j=1}^{+\infty}\frac{1}{2^j}\right)^{\frac{1}{r}} < \infty.\end{aligned}$$

Then, $(z_j)_{j=1}^{\infty} \in \ell_r(X)$. However, $\widehat{T}((z_j)_{j=1}^{\infty}) \notin \ell_r^{p^*\sigma}\langle Y\rangle$. In order to see this, note that given $\varepsilon > 0$ there is a representation $(y_{i,n}^*)_{i=1}^{m_n} = \sum_{l=1}^{k_n}(y_{i,n,l}^*)_{i=1}^{m_n}$ with $(y_{i,n,l}^*)_{i=1}^{m_n} \in H_{p^*,\sigma}(Y^*), 1 \leq l \leq k_n$ such that

$$\sum_{l=1}^{k_n}\delta_{p^*\sigma}((y_{i,n,l}^*)_{i=1}^{m_n}) < \left\|(y_{i,n}^*)_{i=1}^{m_n}\right\|_{\ell^{p^*\sigma}(Y^*)} + \varepsilon \leq 1 + \varepsilon.$$

Consider the sequences

$$\varphi^n := (\varphi_i^n)_{i=1}^\infty := \left(0, 0, ..., 0, \frac{y_{1,n}^*}{2^{\frac{(1-\sigma)n}{p^*}}}, \frac{y_{2,n}^*}{2^{\frac{(1-\sigma)n}{p^*}}}, ..., \frac{y_{m_n,n}^*}{2^{\frac{(1-\sigma)n}{p^*}}}, 0, 0, ...\right), \quad n \in \mathbb{N},$$

and

$$\varphi^{nl} := \left(\varphi_i^{nl}\right)_{i=1}^\infty := \left(0, 0, ..., 0, \frac{y_{1,n,l}^*}{2^{\frac{(1-\sigma)n}{p^*}}}, \frac{y_{2,n,l}^*}{2^{\frac{(1-\sigma)n}{p^*}}}, ..., \frac{y_{m_n,n,l}^*}{2^{\frac{(1-\sigma)n}{p^*}}}, 0, 0, ...\right), \quad 1 \le l \le k_n, \quad n \in \mathbb{N},$$

with 0 in the $\nu_{n-1} := \sum_{j=1}^{n-1} m_j$ first positions. Clearly each $\varphi^{nl} \in H_{p^*,\sigma}(Y^*)$.
Define $\left(\Phi_i^h\right)_{i=1}^\infty := \sum_{n=1}^h (\varphi_i^n)_{i=1}^\infty = \sum_{n=1}^h \sum_{l=1}^{k_n} \varphi^{nl} \in \ell^{p^*\sigma}(Y^*)$. Then

$$\begin{aligned}
\left\|\left(\Phi_i^h\right)_{i=1}^\infty\right\|_{\ell^{p^*\sigma}(Y^*)} &\le \sum_{n=1}^h \sum_{l=1}^{k_n} \delta_{p^*,\sigma}\left((\varphi_i^{nl})_{i=1}^\infty\right) \\
&= \sum_{n=1}^h \sum_{l=1}^{k_n} \sup_{\phi \in B_{Y^{**}}} \left(\sum_{i=1}^{m_n} \left(\left|\frac{\phi(y_{i,n,l}^*)}{2^{\frac{(1-\sigma)n}{p^*}}}\right|^{1-\sigma} \left\|\frac{y_{i,n,l}^*}{2^{\frac{(1-\sigma)n}{p^*}}}\right\|^\sigma\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}} \\
&= \sum_{n=1}^h \frac{1}{2^{\frac{(1-\sigma)n}{p^*}}} \sum_{l=1}^{k_n} \delta_{p^*\sigma}((y_{i,n,l}^*)_{i=1}^{m_n}) \\
&< \sum_{n=1}^h \frac{1}{2^{\frac{(1-\sigma)n}{p^*}}} \left(\left\|(y_{i,n}^*)_{i=1}^{m_n}\right\|_{\ell^{p^*\sigma}(Y^*)} + \varepsilon\right) \\
&\le \sum_{n=1}^\infty \frac{1+\varepsilon}{2^{\frac{(1-\sigma)n}{p^*}}} = \frac{1+\varepsilon}{2^{\frac{(1-\sigma)}{p^*}} - 1}.
\end{aligned}$$

However, since for every $h \in \mathbb{N}$ and $\nu_{j-1} < i \le \nu_j, \ j \le h$ we have

$$\Phi_i^h = \frac{1}{2^{\frac{(1-\sigma)j}{p^*}}} \sum_{l=1}^{k_j} y_{i,j,l}^* = \frac{1}{2^{\frac{(1-\sigma)j}{p^*}}} \ y_{i,j}^*,$$

by (4.12) and (4.13) it turns out that

$$
\begin{aligned}
\left\|\widehat{T}\left((z_j)_{j=1}^{\infty}\right)\right\|_{\ell_r^{p^*\sigma}\langle Y\rangle} &= \sup_{\left\|(w_j)_{j=1}^{\infty}\right\|_{\ell^{p^*\sigma}(Y^*)}\leq 1} \left\|\left(\left\langle T(z_j), w_j\right\rangle\right)_{j=1}^{\infty}\right\|_{\ell_1} \\
&\geq \frac{2^{\frac{(1-\sigma)}{p^*}}-1}{1+\varepsilon}\left\|\left(\left\langle T(z_j), \Phi_j^h\right\rangle\right)_{j=1}^{\nu_h}\right\|_{\ell_1} \\
&\geq \frac{2^{\frac{(1-\sigma)}{p^*}}-1}{1+\varepsilon}\sum_{j=1}^{h}\frac{1}{2^{\frac{j}{r}+\frac{(1-\sigma)j}{p^*}}}\sum_{i=1}^{m_j}\left|\left\langle T(x_{i,j}), y_{i,j}^*\right\rangle\right| \\
&\geq h\frac{2^{\frac{(1-\sigma)}{p^*}}-1}{1+\varepsilon} \longrightarrow \infty \text{ if } h \longrightarrow \infty,
\end{aligned}
$$

which according (4.6) is a contradiction with the fact that $\widehat{T}$ maps $\ell_r(X)$ into $\ell_r^{p^*\sigma}\langle Y\rangle$. □

Corollary 4.3.5. *Consider* $1 < p \leq q < \infty$. *Then,*

$$\mathcal{D}_q^{\sigma}(X,Y) \subset \mathcal{D}_p^{\sigma}(X,Y)$$

4.3.2 Cohen (p,σ,q,ν)-nuclear operators

The class of (p,σ,q,ν)-nuclear operators can be obtained as a particular ideal of (p,σ,q,ν)-dominated operators, by taking $r=\infty$ in the definition given in [18].

Definition 4.3.6. Let $1 < p,q < \infty$ and $0 \leq \sigma,\nu < 1$ such that $\frac{1-\sigma}{p}+\frac{1-\nu}{q}=1$. An operator $T \in \mathcal{L}(X,Y)$ is said to be ***Cohen*** (p,σ,q,ν)***-nuclear*** if there exist a constant $C>0$ such that for every $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$ the following inequality holds

$$\|(\langle T(x_i), y_i^*\rangle)_{i=1}^n\|_{\ell_1} \leq C\delta_{p\sigma}\left((x_i)_{i=1}^n\right)\delta_{q\nu}\left((y_i^*)_{i=1}^n\right). \tag{4.14}$$

In such case, we put

$$N_{p,q}^{\sigma,\nu}(T) = \inf C,$$

where the infimum is taken over all constants C either in (4.14).
We denote by $\left(\mathcal{N}_{p,\sigma,q,\nu}, N_{p,q}^{\sigma,\nu}(\cdot)\right)$ the Banach ideal of (p,σ,q,ν)-nuclear linear operators.

The following theorem gives a characterization for the class of Cohen (p,σ,q,ν)-nuclear linear operators in terms of a summability property and an integral domination. This is a particular case of the general characterization of (p,σ,q,ν)-dominated operators [18, Theorem 2.4], the equivalence with the following *(iv)* is new. The proof of this equivalence is similar to the given one in Theorem 4.3.4, and we will omit it.

Theorem 4.3.7. *Let $T \in \mathcal{L}(X,Y)$. The following are equivalent*

(i) T is Cohen (p,σ,q,ν)-nuclear.

(ii) There exist Banach spaces G, H, linear operators $R \in \Pi_p(X,G)$, $S \in \Pi_q(Y^*,H)$ and a constant $C > 0$ such that

$$|\langle T(x), y^*\rangle| \leq C \|x\|^{\sigma} \|R(x)\|^{1-\sigma} \|y^*\|^{\nu} \|S(y^*)\|^{1-\nu}, \tag{4.15}$$

for all $x \in X$ and $y^* \in Y^*$.

(iii) There exist a constant $C > 0$ and regular Borel probability measures μ and τ on B_{X^*} and $B_{Y^{**}}$, respectively, such that for every $x \in X$ and $y^* \in Y^*$, the following inequality holds

$$\begin{aligned} |\langle T(x), y^*\rangle| \leq \ & C \left(\int_{B_{X^*}} \left(|\langle x, x^*\rangle|^{1-\sigma} \|x\|^{\sigma} \right)^{\frac{p}{1-\sigma}} d\mu \right)^{\frac{1-\sigma}{p}} \\ & \times \left(\int_{B_{Y^{**}}} \left(|\langle y^*, y^{**}\rangle|^{1-\nu} \|y^*\|^{\nu} \right)^{\frac{q}{1-\nu}} d\tau \right)^{\frac{1-\nu}{q}}. \end{aligned} \tag{4.16}$$

(iv) The map $\widehat{T}$ maps $\ell^{p\sigma}(X)$ into $\ell^{q\nu}_{\frac{p}{1-\sigma}}\langle Y\rangle$.

(v) (Factorization theorem). There exist a Banach space G, two operators A, B such that $A \in \Pi_{p,\sigma}(X,G)$, $B^* \in \Pi_{q,\nu}(Y^*,G^*)$ and $T = B \circ A$.

Moreover,

$$N^{\sigma,\nu}_{p,q}(T) = \left\| \widehat{T} \right\| = \inf \pi_{p,\sigma}(A).\pi_{q,\nu}(B^*) = \inf C, \tag{4.17}$$

where the last infimum is taken over all constants C either in (4.16) or in (4.15).

A direct consequence of the previous theorem is the following corollary.

Corollary 4.3.8. *Let X and Y be Banach spaces. Then we have the inclusion*

$$\mathcal{N}_{p,\sigma,q,\nu}(X,Y) \subset \Pi_{p,\sigma}(X,Y),$$

with

$$N^{\sigma,\nu}_{p,q}(\cdot) \geq \pi_{p,\sigma}(\cdot).$$

Proof. Let $T \in \mathcal{N}_{p,\sigma,q,\nu}(X,Y)$. By (4.17), for each $\varepsilon > 0$ there exist a Banach space G and two operators A, B such that $A \in \Pi_{p,\sigma}(X,G)$, $B^* \in \Pi_{q,\nu}(Y^*, G^*)$ and $T = B \circ A$ with

$$\pi_{p,\sigma}(A)\pi_{q,\nu}(B^*) \leq (1+\varepsilon)N_{p,q}^{\sigma,\nu}(T)$$

By the ideal property concerning $\Pi_{p,\sigma}$ we have $\pi_{p,\sigma}(T) \leq \pi_{p,\sigma}(A)\,\|B\|$ and then

$$\begin{aligned}\pi_{p,\sigma}(T) &\leq \pi_{p,\sigma}(A)\,\|B^*\| \\ &\leq \pi_{p,\sigma}(A)\pi_{q,\nu}(B^*) \\ &\leq (1+\varepsilon)N_{p,q}^{\sigma,\nu}(T).\end{aligned}$$

Since this holds for all $\varepsilon > 0$, we obtain $\pi_{p,\sigma}(T) \leq N_{p,q}^{\sigma,\nu}(T)$. $\square$

A natural question is to study the connection between linear operators and their adjoints for the class of Cohen (p,σ,q,ν)-nuclear operators or simply a Schauder type theorem.

Theorem 4.3.9. *[14, Theorem 5] Let $1 < p, q < \infty$ and $0 \leq \sigma, \nu < 1$ such that $\frac{1-\sigma}{p} + \frac{1-\nu}{q} = 1$. Let $T \in \mathcal{L}(X,Y)$ and $T^* \in \mathcal{L}(Y^*, X^*)$ its adjoint. Then T is Cohen (p,σ,q,ν)-nuclear if and only if T^* is Cohen (q,ν,p,σ)-nuclear. Moreover,*

$$N_{q,p}^{\nu,\sigma}(T^*) = N_{p,q}^{\sigma,\nu}(T).$$

Proof. Suppose that $T \in \mathcal{N}_{p,\sigma,q,\nu}(X,Y)$. By *(v)* in the Theorem 4.3.7, select a typical factorization

$$T = B \circ A : X \xrightarrow{A} G \xrightarrow{B} Y,$$

where G is a Banach space, $A \in \Pi_{p,\sigma}(X,G)$ and $B^* \in \Pi_{q,\nu}(Y^*, G^*)$. By [3, Proposition 3.7] we have $A^{**} \in \Pi_{p,\sigma}(X^{**}, G^{**})$ and $\pi_{p,\sigma}(A^{**}) = \pi_{p,\sigma}(A)$. Again by *(v)* in the Theorem 4.3.7 and using the fact that $T^* = A^* \circ B^*$, we obtained $T^* \in \mathcal{N}_{q,\nu,p,\sigma}(Y^*, X^*)$ and

$$N_{q,p}^{\nu,\sigma}(T^*) \leq \pi_{q,\nu}(B^*)\pi_{p,\sigma}(A^{**}) = \pi_{q,\nu}(B^*)\pi_{p,\sigma}(A).$$

Passing to the infimum we arrive at $N_{q,p}^{\nu,\sigma}(T^*) \leq N_{p,q}^{\sigma,\nu}(T)$. The other implication is proved in a similar way [11, Theorem 2.2.4]. $\square$

As a consequence of the above theorem, we obtain the following.

Corollary 4.3.10. *The operator T is belongs to $\mathcal{N}_{p,\sigma,q,\nu}(X,Y)$ if and only if its bi-adjoint T^{**} is belongs to $\mathcal{N}_{p,\sigma,q,\nu}(X,Y)$. In addition*

$$N_{p,q}^{\sigma,\nu}(T) = N_{p,q}^{\sigma,\nu}(T^{**}).$$

Corollary 4.3.11. *Let X and Y be Banach spaces. Then we have the inclusion*

$$\mathcal{N}_{p,\sigma,q,\nu}(X,Y) \subset \mathcal{D}_{q^*}^{\nu}(X,Y),$$

with

$$N_{p,q}^{\sigma,\nu}(\cdot) \geq d_{q^*}^{\nu}(\cdot).$$

Proof. Let $T \in \mathcal{N}_{p,\sigma,q,\nu}(X,Y)$. By the Theorem 4.3.9 and the Corollary 4.3.8 we have $T^* \in \Pi_{q,\nu}(Y^*,X^*)$ with

$$N_{p,q}^{\sigma,\nu}(T) = N_{q,p}^{\nu,\sigma}(T^*) \geq \pi_{q,\nu}(T^*).$$

Now, using the Corollary 4.3.2 we get $T \in \mathcal{D}_{q^*}^{\nu}(X,Y)$ and

$$N_{p,q}^{\sigma,\nu}(T) \geq \pi_{q,\nu}(T^*) = d_{q^*}^{\nu}(T).$$

□

In what follows we prove that under certain conditions we can ensure that the Cohen (p,σ,q,ν)-nuclear operator is compact.

Corollary 4.3.12. *Let X, Y be Banach spaces, X in addition reflexive. If T belongs to $\mathcal{N}_{p,\sigma,q,\nu}(X,Y)$, then T is compact.*

Proof. The operator T is (p,σ)-absolutely continuous (by Corollary 4.3.8) and has a reflexive domain. Then T is compact (see [12, Proposition 5.1] and [13, Corollary 2.1.22]). □

In the following proposition we prove a Dvoretzky-Rogers type theorem for the class of Cohen (p,σ,q,ν)-nuclear operators.

Proposition 4.3.13. *[14, Proposition 3] A Banach space X is finite dimensional if and only if the identity mapping $id_X : X \longrightarrow X$ is Cohen (p,σ,q,ν)-nuclear.*

Proof. If X is finite dimensional it is clear that $id_X \in \mathcal{N}_{p,\sigma,q,\nu}(X,X)$ since it is a finite rank operator [18, Corollary 2.5]. Conversely, assume that id_X is (p,σ,q,ν)-nuclear. Then by (iii) in Theorem 4.3.7 there exist a constant $C > 0$ and regular Borel probability measures $\mu \in C(B_{X^*})^*$ and $\tau \in C(B_{X^{**}})^*$ such that for every $0 \neq x \in X$, we have

$$\begin{aligned}
\|x\| &= \sup_{\|x^*\|\leq 1} |\langle x, x^* \rangle| \\
&\leq C \left(\int_{B_{X^*}} \left(|\langle x, x^* \rangle|^{1-\sigma} \|x\|^{\sigma} \right)^{\frac{p}{1-\sigma}} d\mu \right)^{\frac{1-\sigma}{p}} \\
&\times \sup_{\|x^*\|\leq 1} \left(\int_{B_{X^{**}}} (|\langle x^*, x^{**} \rangle|^{1-\nu} \|x^*\|^{\nu})^{\frac{q}{1-\nu}} d\tau \right)^{\frac{1-\nu}{q}} \\
&= C \|x\|^{\sigma} \left(\int_{B_{X^*}} |\langle x, x^* \rangle|^{p} d\mu \right)^{\frac{1-\sigma}{p}} \sup_{\|x^*\|\leq 1} \|x^*\|^{\nu} \left(\int_{B_{X^{**}}} |\langle x^*, x^{**} \rangle|^{q} d\tau \right)^{\frac{1-\nu}{q}} \\
&\leq C \|x\|^{\sigma} \left(\int_{B_{X^*}} |\langle x, x^* \rangle|^{p} d\mu \right)^{\frac{1-\sigma}{p}} \sup_{\|x^*\|\leq 1} \|x^*\| \left(\int_{B_{X^{**}}} d\tau \right)^{\frac{1-\nu}{q}} \\
&= C \|x\|^{\sigma} \left(\int_{B_{X^*}} |\langle x, x^* \rangle|^{p} d\mu \right)^{\frac{1-\sigma}{p}} .
\end{aligned}$$

This implies that

$$\|x\| \leq C^{\frac{1}{1-\sigma}} \left(\int_{B_{X^*}} |\langle x, x^* \rangle|^{p} d\mu \right)^{\frac{1}{p}} .$$

Hence id_X is p-summing and the result is obtained by the well-known version of the Dvoretzky-Rogers Theorem involving p-summing operators (see [16, Page 50]). □

From the previous proposition, Theorem 4.3.7, the inclusions (4.4) and (4.8), we obtain the following.

Corollary 4.3.14. $\ell^{q\nu}_{\frac{p}{1-\sigma}} \langle X \rangle = \ell^{p\sigma}(X)$ *if and only if the Banach space* X *is finite dimensional.*

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ملخص

في هذه الرسالة قمنا بمواصلة دراسة بعض فضاءات المتتاليات البناخية، حيث أثبتنا وجود علاقة ثنوية بينها، بالإضافة إلى تحديد العلاقة بينها وبين فضاءات متتاليات أخرى مشهورة، كما استعملنا هذه الفضاءات لتعريف ودراسة نوعين جديدين من المثاليات لمؤثرات خطية، واستطعنا إثبات وجود علاقة ثنوية بينها وبين مثاليات معروفة مسبقا. كما قمنا بدراسة بعض الخصائص الجديدة لمثالي آخر معروف وذلك من خلال تعريف ودراسة نوع جديد من فضاءات المتتاليات البناخية.
كلمات مفتاحية: متتاليات (p,q)-جمعية بقوة، متتاليات (s,p)-جمعية مختلطة، متتاليات (p,q,σ)-جمعية بقوة، مؤثرات جمعية، مؤثرات (p,σ,q,ν)-كوهين نووية.

Résumé

Dans cette thèse, nous avons continué à étudier certains espaces de suites de Banach, où nous avons démontré l'existence d'une relation de dualité entre eux, en plus de déterminer la relation entre eux et d'autres espaces de suites de Banach consécutifs bien connus. Nous avons également utilisé ces espaces pour définir et étudier deux idéaux d'opérateurs linéaires sommants, et nous avons pu prouver l'existence d'une relation de dualité entre eux et des idéaux connus auparavant. Nous avons également étudié quelques nouvelles propriétés d'un autre idéal bien connu, en introduisant et en étudiant un nouvel espace des suites de Banach.
Mots clés: Suites fortement (p,q)-sommables, suites mixte (s,p)-sommables, suites fortement (p,q,σ)-sommables, opérateurs sommants, opérateurs Cohen (p,σ,q,ν)-nucléaires.

Abstract

In this thesis, we have continued to study some Banach sequence spaces, where we have demonstrated the existence of a duality relationship between them, in addition to determining the relationship between them and other well-known consecutive Banach sequence spaces. We have also used these spaces to define and study two new types of ideals of linear summing operators, and we were able to prove the existence of a duality relationship between them and previously known ideals. We have also studied some new properties of another well-known ideal, by defining and studying a new type of Banach sequence space.
Keywords: Strongly (p,q)-summable sequences, mixed (s,p)-summable sequences, strongly (p,q,σ)-summable sequences, summing operators, Cohen (p,σ,q,ν)-nuclear operators.