



جامعة محمد بوضياف - المسيلة
Université Mohamed Boudiaf - M'sila

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ON THE WEYL AND BROWDER THEOREMS OF BOUNDED LINEAR OPERATOR

Persented by: **Bouguerra Malika**

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Abdelhamid TALLAB	M.C.A.	University of M'sila	Chairperson
Toufik HERAIZ	M.C.B.	University of M'sila	Supervisor
Khaled HAMIDI	M.C.B .	University of M'sila	Examiner

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Notations

$B(X, Y)$	the algebra of all bounded linear operators from X into Y
$\ker(T)$	The null space of T
$R(T)$	The range space of T
$\mathcal{B}(X)$	The set of all the bounded linear operators from X into itself
$\mathcal{L}(X)$	The set of all continuous linear operators from X into itself
$\sigma(T)$	The spectrum of T
$\rho(T)$	The resolvent set of T
$\alpha(T)$	The nullity of T is defined as the dimension of $N(T)$
$\beta(T)$	The deficiency of T is defined as the codimension of $R(T)$
I	Operator of identity
$\oplus(X, Y)$	The set of Fredholm operators
$\Phi_+(X, Y)$	The set of upper semi-Fredholm operators
$\Phi_-(X, Y)$	The set of lower semi-Fredholm operators
$\mathcal{R}(X)$	The set of Riesz operators
$K(T)$	The set of all compact operators on X
$L(X, Y)$	The algebra of all bounded linear operators from X into Y
$\sigma_w(T)$	Weyl spectrum of T
$\sigma(T)$	The spectrum of T
σ_a	approximate point spectrum
$\Phi_{\pm}(X, Y)$	The set of semi-Fredholm operators
p	The ascent of T
q	The descent of T
σ_{LD}	Drazin invertible
$\mathcal{F}(X, Y)$	The set of Fredholm perturbations
$F(X)$	The set of Fredholm operators

$\sigma(T)$	Spectrum of T
σ_{BW}	B -Weyl Spectrum
$E(T)$	Isolated point
$E^a(T)$	Isolated point in the approximate point spectrum $\sigma(T)$
$E_0(T)$	The set of all eigenvalues λ of finite multiplicity isolated in $\sigma(T)$
$E_0^a(T)$	The set of all eigenvalues λ of finite multiplicity isolated approximate point spectrum
$\Pi(T)$	The set of the poles of T
$\Pi_0(T)$	The set of all poles of T finite rank
$\Pi^a(T)$	The set of left poles of T
$\Pi_0^a(T)$	The set of all left poles of finite rank

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Introduction

- Local Spectrum Theory is a mathematical framework that focuses on understanding the behavior of operators on Hilbert spaces, particularly within a localized region. It deals with the spectral properties of operators, such as eigenvalues and eigenvectors, in a specific neighborhood or domain. This theory is essential in various areas of mathematics and physics, including quantum mechanics, partial differential equations, and functional analysis. By examining the spectral characteristics of operators in local regions, researchers can gain insights into the behavior of complex systems and phenomena. In 1909, H. Weyl [44] examined the spectra of all compact perturbations $T + K$ of a Hermitian operator T operating on a Hilbert space. He demonstrated that a complex number λ belongs to $\sigma_w(T)$ if and only if it is not an isolated point of finite multiplicity in $\sigma(T)$. This fundamental finding, originally by Weyl, is now referred to as Weyl's theorem. and it has been extended to several classes of operators acting in Banach spaces."This work is devoted to study the Weyl and Browder theorems of bounded linear operators. by using Weyl's and Browder's theorem which is stand as pivotal landmarks in the realm of functional analysis, offering profound insights into the spectral behavior structural properties of bounded linear operator. This work is composed of three chapters.

The first chapter: we recall the basic properties of spectral theory of bounded linear operators in a vector space, continuity and bounded operators, then we will also discuss about compact operator, fredholm operators, and the spectrum.

The second chapter: we study the Weyl's theorem : statement and proof, then we discuss about related results and application. We will also study the Browder's theorem statement and proof , then we will shows connection to other theorems in operator theory .

The third chapter: we will analysing similarities and defferences between the Weyl and Browder theorems, and we will also discuss about investigating extentions of the theorems to broader operator classes.

Chapitre 1

Generalities about linear operators

Definition 1.0.1 Let X and Y be two vectors space over a field \mathbb{k} . we say that The operator A is a linear operator from X to Y if

$$A(x + y) = A(x) + A(y), A(\lambda x) = \lambda A(x), \text{ for all } x, y \in X \text{ and } \lambda \in \mathbb{k}$$

- Let $L(X, Y)$ the set of all linear operators from X on Y , and $L(X, X) = L(X)$
 - The **null spaces** of a linear operator A is $N(A) = \{x \in X : Ax = 0\}$, it is also called the kernel of A and denoted by $\ker(A)$.
 - The **image** of a linear operator A is $R(A) = \{y \in Y : y = A(x)\}$, it is also called the range of A and denoted by $\text{Ran}(A)$

Definition 1.0.2 Let $A \in L(X, Y)$ and $x_0 \in X$, the operator A is said continuous at x_0 if and only if

$$\lim_{x \rightarrow x_0} A(x) = A(x_0)$$

That's mean, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|x - x_0\|_X < \delta \Rightarrow \|Ax - Ax_0\|_Y < \varepsilon$$

We say the operator A is continuous if it's continuous at any $x \in X$.

- Let $\mathcal{L}(X, Y)$ the set of all continuous linear operators from X on Y , and $\mathcal{L}(X, X) = \mathcal{L}(X)$.

Definition 1.0.3 *Let X and Y be normed vector spaces, the linear operator $A : X \rightarrow Y$ is called bounded if there is a constant c such that*

$$\|Ax\| \leq c \|x\|, \text{ for every } x \in X.$$

- The norm of such an operator is defined by

$$\|A\| = \sup_{x \in X} \frac{\|Ax\|}{\|x\|}$$

- We note by $B(X, Y)$ the set of all the bounded linear operators from X on Y , and by $B(X)$ if $X = Y$

Proposition 1.0.1 *Let X and Y be normed linear spaces and let A be a linear operator with domain X and range in Y . The following statements are equivalent:*

- A is continuous at a point $x_0 \in X$
- A is bounded. i.e, there exists a constant C such that for all $x \in X$

$$\|Ax\| \leq C \|x\|$$

1.1 Properties of the space of bounded linear operators

Proposition 1.1.1 $\mathcal{L}(X, Y)$, the space of all bounded linear operators from X to Y , is a normed vector space.

If Y is Banach, then so is $\mathcal{L}(X, Y)$, from which it follows that dual spaces are Banach.

Theorem 1.1.1 *For any A in $\mathcal{L}(X, Y)$, $\ker(A)$ is a closed linear subspace of X .*

Example 1.1.1 Many integral transforms are bounded linear operators, for instance if $K : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is a continuous function

then the operator L defined on the space of continuous functions on $[a, b]$ endowed with the uniform norm and with values in the space with given by the formula

$$(LF) = \int_a^b k(x, y) f(x) dx$$

is bounded.

1.2 Compact operator

Definition 1.2.1 Let X and Y be Banach spaces then $A : X \rightarrow Y$ is called a compact linear operator or completely continuous linear operator if for every bounded subset G of X ; the image $A(G)$ is relatively compact in Y . In other words, the closure $\overline{A(G)}$ is compact.

In other word, A is compact if and only if for every bounded sequence $(x_n)_n$ in X , the sequence $(Ax_n)_n$ has a subsequence converges in Y .

Theorem 1.2.1 Let $A, A_n \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, Z)$ with X, Y and Z are Banach spaces

(a) The sequence A_n of compact operators defined from a normed space X into a Banach space Y converges uniformly to an operator A , say

$$\lim \|A_n - A\| = 0$$

Then the limit operator A is compact..

(b) The product AB is compact if the operator A or B is compact.

Proof. (a) Let $(x_m)_m$ in $\mathcal{B}(0, 1)$. for all n there existe a subsequence $(x_{\varphi_n(m)})_m$ such that $(Ax_{\varphi_n(m)})_m$ is convergente because A_n is compact ■

$$\|Ax_{\varphi n(n)} - Ax_{\varphi m(m)}\| \leq \|A - A_n\| + \|A_m - A\| + \|A_n x_{\varphi n(n)} - A_m x_{\varphi m(m)}\|_{n,m \rightarrow \infty} \rightarrow 0$$

(b) Let x_n be a bounded sequence in X ; then if we consider B as a bounded operator the sequence $Bx_n(x)$ is bounded, and from the compactness of the operator A gives a convergent subsequence $A(Bx_{n_k}(x))$ of $A(Bx_n(x))$. Hence the operator AB is compact.

On the other hand, if we consider B as a compact, one can extract from $Bx_n(x)$ a convergent subsequence $Bx_{n_k}(x)$, and from the boundedness of the operator A gives the convergence of the sequence $A(Bx_{n_k}(x))$. Hence the operator AB is compact.

Theorem 1.2.2 *The linear combination $A = \alpha A_1 + \beta A_2$ of compact operators A_1 and A_2 is a compact operator, for every scalars α and β .*

1.3 Adjoint operators

Recall that when X a Banach space, the dual space $X^* := \mathcal{L}(X; \mathbb{C})$, consists of the bounded linear functionals x^* on X is a Banach space with the norm

$$\|x^*\|_{X^*} = \inf \{|x^*(x)| : x \in X, \|x\| = 1\}$$

When $A : X \rightarrow Y$ is densely defined, we can define the adjoint operator $A^* : Y^* \rightarrow X^*$ as follows:

the domain $D(A^*)$ consists of the $y^* \in Y^*$, for which the linear functional $x \rightarrow y^*(Ax)$, $x \in D(A)$

is continuous, this means that there is a constant C such that

$$|y^*(Ax)| \leq C \|x\|_X, \quad \text{for all } x \in D(A)$$

since $D(A)$ is dense in X , the mapping extends by continuity to X , so there is a uniquely determined $x^* \in X^*$ so that

$$y^*(Ax) = x^*(x), \quad \text{for all } x \in D(A)$$

when x^* is determined from y^* , we can define the operator A^* from Y^* to X^* , by:

$$A^*y^* = x^*, \quad \text{for all } y^* \in D(A^*)$$

Theorem 1.3.1 *Let A be a bounded operator defined from a Hilbert space H_1 into a Hilbert space H_2 . Then there exists an adjoint operator of A denoted by A^* defined from*

H_1 into H_2 such that

$$\langle Ax, y \rangle_{H_2} = \langle x, A^*y \rangle_{H_1}$$

For all $x \in H_1$ and $y \in H_2$.

Besides, we have

$$\|A\| = \|A^*\|$$

Theorem 1.3.2 *Let A be a compact operator defined from a Hilbert space H_1 into a Hilbert space H_2 then, the adjoint operator A^* defined from H_2 into H_1 is also a compact operator.*

- For nonempty sets $M \subseteq X$ and $N \subseteq X^*$ we define the annihilators

$$M^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in M\}$$

$$N^\perp = \{x \in X : f(x) = 0 \text{ for all } x \in N\}$$

Even if M and N are not subspaces and M^\perp and N^\perp are closed subspaces of X^* and X respectively, We have $M^\perp = X^*$ (resp. $N^\perp = X$) if only if $M = \{0\}$ (resp. $N = \{0\}$)

1.3.1 Normal operator and Self-adjoint operator

Definition 1.3.1 *Let A be a linear operator from a Hilbert space H to itself*

- if $AA^* = A^*A$ then the operator A is said normal operator
- If $A = A^*$ then the operator A is said self-adjoint operator

1.4 Fredholm operator

Definition 1.4.1 Let X and Y be Banach spaces and let $A : X \rightarrow Y$ be a bounded linear operator, $A \in \mathcal{L}(X, Y)$, we say that

1. A is a **Fredholm operator** if the following three conditions satisfied

i) $\alpha(A) = \dim(N(A)) < \infty$.

ii) $R(A)$ is closed.

iii) $\beta(A) = \text{co dim}(R(A)) < \infty$

The set of Fredholm operator from X on Y denoted by $\Phi(X, Y)$ and $\Phi(X)$ when $X = Y$

2. Let $\Phi_+(X)$ denoted the set of **upper semi-Fredholm** operators defined by

$$\Phi_+(A) = \{A \in B(X) \text{ such that } \alpha(A) < \infty \text{ and } R(A) \text{ is closed in } X\}$$

3. Let $\Phi_-(X)$ denoted the set of **lower semi-Fredholm** operators defined by

$$\Phi_-(X) = \{A \in B(X) \text{ such that } \beta(A) < \infty \text{ and } R(A) \text{ is closed in } X\}$$

4. Let $\Phi_{\pm}(X)$ denoted the set of **semi-Fredholm operator** such that

$$\Phi_{\pm}(X) = \Phi_+(X) \cup \Phi_-(X)$$

(i) $\Phi(X, Y) = \Phi_+(X, Y) \cap \Phi_-(X, Y)$

(ii) $\Phi(X)$ is a non-empty set as it contains the identity. On the other hand, $\Phi(X, Y)$ may be empty when $X \neq Y$

Definition 1.4.2 For all operator $A \in \mathcal{L}(X, Y)$, the number $i(A) = \alpha(A) - \beta(A)$ is called the index of A

$$i : \mathcal{L}(X, Y) \rightarrow \mathbb{Z}$$

$$A \rightarrow i(A) = \alpha(A) - \beta(A)$$

And in this case, we have $i(A) \in \mathbb{Z} \cup \{\pm\infty\}$, such that

- If $A \in \Phi(X)$ so, $i(A) < \infty$.
- If $A \in \Phi_+(X) \setminus \Phi(X)$ so, $i(A) = -\infty$.
- If $A \in \Phi_-(X) \setminus \Phi(X)$ so, $i(A) = +\infty$.

Theorem 1.4.1 *Let $A \in \mathcal{L}(X, Y)$ a closed image operator*

- a) A is **upper semi-Fredholm** if and only if A^* is **lower semi-Fredholm**.
- b) A is **lower semi-Fredholm** if and only if A^* is **upper semi-Fredholm**.
- c) A est **semi-Fredholm** if and only if A^* is **semi-Fredholm**.
- d) A is **Fredholm** if and only if A^* is **Fredholm**.

And we have with all cases $i(A^*) = -i(A)$

Proof. For $R(A)$ so $R(A^*)$ is closed, such that ■

$$R(A) = N(A^*)^\perp \text{ and } R(A^*) = N(A)^\perp$$

So, $\alpha(A^*) = \dim N(A^*) = \text{co dim } R(A) = \beta(A)$ and $\beta(A^*) = \text{co dim } R(A^*) = \dim N(A) = \alpha(A)$.

For $\dim N(A^*)$ and $\text{co dim } R(A^*) < \infty$, alors A^* is Fredholm.

$$i(A^*) = \alpha(A^*) - \beta(A^*) = \beta(A) - \alpha(A) = -i(A)$$

Theorem 1.4.2 *Let $A \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(X, Y)$ such that*

1. If $A \in \Phi_+(X, Y)$ and $S \in \Phi_+(Y, Z)$, so $SA \in \Phi_+(X, Z)$.
2. If $A \in \Phi_-(X, Y)$ and $S \in \Phi_-(Y, Z)$, so $SA \in \Phi_-(X, Z)$.
3. If $A \in \Phi(X, Y)$ and $S \in \Phi(Y, Z)$, so $SA \in \Phi(X, Z)$.

Lemma 1.4.1 *If $A, B \in \Phi(X)$ then we have $BA \in \Phi(X)$. Suth that,*

$$i(BA) = i(A) + i(B)$$

Theorem 1.4.3 (Fredholm Alternative) *Let $A \in \mathcal{L}(X)$ a compact operator, for all $\lambda \in \mathbb{C}^*$*

1. $N(\lambda I - A) < \infty$
2. $R(\lambda I - A)$ is closed
3. $N(\lambda I - A) = \{0\} \Leftrightarrow R(\lambda I - A) = X$.

1.4.1 Fredholm perturbations

Definition 1.4.3 *Let X and Y be two Banach spaces and let $A \in \mathcal{L}(X)$.*

- A is called a **Fredholm perturbation** if $A + U \in \Phi$, whenever $U \in \Phi(X, Y)$
- A is called a **upper semi-Fredholm perturbation** (resp. a lower semi-Fredholm perturbation) if $A + U \in \Phi_+(X, Y)$ (resp. Φ_-) for all $U \in \Phi_+(X, Y)$ (resp. $\Phi_-(X)$).

The set of Fredholm perturbations, upper semi-Fredholm perturbation (resp. a lower semi-Fredholm perturbation), denoted by $\mathcal{F}(X, Y)$, $\mathcal{F}_+(X, Y)$, $\mathcal{F}_-(X, Y)$

Proposition 1.4.1 *Let X, Y and Z be three Banach spaces. If at least of the sets $\Phi(X, Y)$ and $\Phi(Y, Z)$ is not empty, then*

- (i) $F \in \mathcal{F}(X, Y)$, $A \in \mathcal{L}(Y, Z)$ imply $AF \in \mathcal{F}(X, Z)$.
- (ii) $F \in \mathcal{F}(Y, Z)$, $A \in \mathcal{L}(X)$ imply $FA \in \mathcal{F}(X, Z)$.

Proposition 1.4.2 *Let X be a complex Banach space of infinite dimension and let $F \in \mathcal{F}(X)$, such that*

$$i(A + F) = i(A) \text{ for all } A \in \mathcal{L}(X).$$

Lemma 1.4.2 *Let $A \in \mathcal{L}(X)$ and $U \in \mathcal{L}(X)$. then*

- i) if $A \in \Phi(X)$ and $U \in \mathcal{F}(X)$, so $A + U \in \Phi(X)$ and $i(A + U) = i(A)$

ii) if $A \in \oplus_+(X)$ and $U \in \mathcal{F}^+(X)$, so $A + U \in \Phi_+^b(X)$ and $i(A + U) = i(A)$

iii) if $A \in \oplus_-(X)$ and $U \in \mathcal{F}^-(X)$, so $A + U \in \Phi_-^b(X)$ and $i(A + U) = i(A)$

Definition 1.4.4 (*Riesz operator*) Let X be a Banach space and $A \in \mathcal{L}(X)$, A is said to be a Riesz operator (and we note $A \in \mathcal{R}(X)$) if $\lambda \in \mathbb{C} \setminus \{0\}$ is fredholm.

- The family of Riesz operator is not an ideal of $\mathcal{L}(X)$.
- It is proved that $\mathcal{F}(X)$ is the largest ideal of $\mathcal{L}(X)$ contained in the family of Riesz operator.

1.5 Spectrum

Definition 1.5.1 Let A be a closable linear operator in a Banach space X . The resolvent set and the spectrum of A are, respectively, defined as

$$\rho(A) = \{\lambda \in \mathbb{C} \text{ such that } A - \lambda \text{ is injective and } (\lambda - A)^{-1} \in \mathcal{L}(X)\}.$$

We call $\sigma(A) = \mathbb{C} \setminus \rho(A)$ the spectrum of A .

and the point spectrum, continuous, and the residual spectrum are defined as

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \text{ such that } A \text{ is not injective}\}$$

$$\sigma_c(A) = \left\{ \lambda \in \mathbb{C} \text{ such that } A \text{ is injective } \overline{R(\lambda - A)} = X, R(\lambda - A) \neq X \right\}$$

$$\sigma_r(A) = \left\{ \lambda \in \mathbb{C} \text{ such that } A \text{ is injective, } \overline{R(\lambda - A)} \neq X \right\}$$

Note that, if $\rho(A) \neq \emptyset$ then A is closed. In fact, if $\lambda \in \rho(A)$ then $(\lambda - A)^{-1}$ is closed, which is also valid for $\lambda - A$. Then, according to the closed graph, we deduce that

$$\rho(A) = \{\lambda \in \mathbb{C} \text{ such that } A \text{ is bijective}\}$$

And hence

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$$

Theorem 1.5.1 *Let A be a bounded linear operator with a nonempty resolvent set $\rho(A)$. For each $\lambda_0 \in \rho(A)$ we have*

$$\sigma((\lambda_0 - A)^{-1}) = (\lambda_0 - \sigma(A))^{-1}$$

1.6 Essential spectrum of wolf and weyl

In Banach space X , the Wolf essential spectrum of the operator $A \in \mathcal{C}(X)$ is defined by

$$\sigma_f(A) := \mathbb{C} \setminus \{\lambda \in \mathbb{C} : \lambda - A \in \Phi(X)\}$$

The Weyl essential spectrum of the operator $A \in \mathcal{C}(X)$ is defined by

$$\sigma_w(A) := \bigcap_{k \in \mathcal{K}(X)} \sigma(A + k)$$

where $\mathcal{K}(X)$ stands for the ideal of all compact operator on X .

Proposition 1.6.1 *Let $A \in \mathcal{L}(X)$. Then $\lambda \notin \sigma_w(A)$ if and only if $\lambda - A \in \Phi(X)$ and $i(\lambda - A) = 0$.*

Remark 1.6.1 1. *The relation ship between the Weyl essential spectrum and the Wolf essential spectrum of A defined by:*

$$\sigma_w(A) = \sigma_f(A) \cup \{\lambda \in \mathbb{C} : i(\lambda - A) \neq 0\}$$

2. *We recall that for $\Theta \in \{\sigma_w(A), \sigma_f(A)\}$, $A \in \mathcal{R}(X)$ if and only if $\Theta = \{0\}$.*

1.7 Local spectrum and SVEP

1.7.1 Local spectrum

Definition 1.7.1 *Given an arbitrary operator $T \in L(X)$, X a Banach space, let $\rho_T(x)$ denoted the set of all $\lambda \in \mathbb{C}$ for wick there exists an open neighborhood \mathcal{U}_λ of λ in \mathbb{C} and an analytic function $f : \mathcal{U}_\lambda \rightarrow X$ such that the equation*

$$(\mu I - T) f(\mu) = x \quad \text{holds for all } \mu \in \mathcal{U}_\lambda.$$

Remark 1.7.1 *If the function f is defined on the set $\rho_T(x)$ then it is called a local resolvent function of T at x . The set $\rho_T(x)$ is called the local resolvent of T at x . The local spectrum $\sigma_T(x)$ of T at the point $x \in X$ is defined to be the set*

$$\sigma_T(x) := \mathbb{C} \setminus \rho_T(x).$$

1.7.2 SVEP Property

Definition 1.7.2 *Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have the single-valued extension property at $\lambda_0 \in \mathbb{C}$, abbreviated T has the SVEP at λ_0 , if for every neighbourhood \mathcal{U} of λ_0 the only analytic function $f : \mathcal{U} \rightarrow X$ which satisfies the equation*

$$(\lambda I - T) f(\lambda) = 0$$

is the constant function $f \equiv 0$.

The operator T is said to have the SVEP if T has the SVEP at every $\lambda \in \mathbb{C}$.

The single valued extension property was introduced in the early years of local spectral theory by Dunford [20], [21] and plays an important role in spectral theory [35], [1, Theorem 3.65].

Remark 1.7.2 (a) *The SVEP ensures the consistency of the local solutions of equation (32), in the sense that if $x \in X$ and T has the SVEP at $\lambda_0 \in \rho_T(x)$ then there exists a neighborhood \mathcal{U} of λ_0 and an unique analytic function $f : \mathcal{U} \rightarrow X$ satisfying the equation $(\lambda I - T) f(\lambda) = x$ for all $\lambda \in \mathcal{U}$.*

Another important consequence of the SVEP is the existence of a maximal analytic extension \tilde{f} of $R(\lambda, T)x := (\lambda I - T)^{-1}x$ to the set $\rho_T(x)$ for every $x \in X$. This function identically verifies the equation

$$(\mu I - T) \tilde{f}(\mu) = x \quad \text{for every } \mu \in \rho_T(x)$$

and obviously

$$\tilde{f}(\mu) = (\mu I - T)^{-1}x \quad \text{for every } \mu \in \rho(T).$$

(b) It is immediate to verify that the SVEP is inherited by the restrictions on invariant subspaces, i.e. if T/M has the SVEP at λ_0 . Moreover,

$$\sigma_T(x) \subseteq \sigma_{T/M}(x) \quad \text{for every } x \in M.$$

(c) Let $\sigma_p(T)$ denote the point spectrum of $T \in L(X)$, i.e.

$$\sigma_p(T) := \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T\}.$$

It is easy to see the implication :

$$\sigma_p(T) \text{ does not cluster at } \lambda_0 \Rightarrow T \text{ has the SVEP } \lambda_0.$$

1.7.3 Ascent, Descent

For any $T \in \mathcal{L}(X)$ and any $n \in \mathbb{N} \cup \{0\}$, we have the ascending sequence

$$\ker(T^n) \subseteq \ker(T^{n+1}) \subseteq \ker(T^{n+2}) \cdots \subseteq \bigcup_n \ker(T^n)$$

and the descending sequence

$$\bigcap_n R(T^n) \subseteq \cdots \subseteq R(T^{n+1}) \subseteq R(T^n)$$

Definition 1.7.3 *Given a linear operator T on a vector space X , T is said to have finite ascent if $\mathcal{N}^\infty(T) = \ker T^k$ for some positive integer k . Clearly, in such a case there is a smallest positive integer $p = p(T)$ such that $\ker T^p = \ker T^{p+1}$. The positive integer p is called the ascent of T . If there is no such integer we set $p(T) = \infty$. Analogously, T is said to have finite descent if $T^\infty(X) = T^k(X)$ for some k . The smallest integer $q = q(T)$ such that $T^{q+1}(X) = T^q(X)$ is called the descent of T . If there is no such integer we set $q(T) := \infty$.*

So we have

$$\begin{aligned} p(T) &= \inf \{n; \ker(T^n) = \ker(T^{n+1})\} \\ q(T) &= \inf \{n; R(T^n) = R(T^{n+1})\} \end{aligned}$$

Clearly $p(T) = 0$ if and only if T is injective and $q(T) = 0$ if and only if T is surjective. The classical Riesz-Schauder theory asserts that $p(\lambda I - T) = q(\lambda I - T) < \infty$ for every compact operator T on a Banach space X , see Heuser [159, Chapter VI].

1.7.4 Quasi-nilpotent part of an operator

Definition 1.7.4 Let $T \in L(X)$ X a Banach space. The quasi-nilpotent part of T is defined to be the set

$$H_0(T) := \left\{ x \in X : \lim_{n \rightarrow \infty} \|T^n x\|^{\frac{1}{n}} = 0 \right\}.$$

As usual, $T \in L(X)$ is said to be quasi-nilpotent if its spectral radius

$$r(T) := \inf_{n \in \mathbb{N}} \|T^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}}$$

is zero.

Clearly $H_0(T)$ is a linear subspace of X , generally not closed. In the following theorem we collect some elementary properties of $H_0(T)$.

Chapitre 2

Weyl's Theorem and Browder's Theorem

2.1 Classical weyl's theorem

Theorem 2.1.1 [44] *If T is Hermitian operator acting on a Hilbert space H , then the Weyl spectrum is the set of all points in $\sigma(T)$ except for the set $E_0(T)$ of all eigenvalues λ of finite multiplicity ($0 < \alpha(T - \lambda I) < \infty$) isolated in $\sigma(T)$ that is*

$$\sigma_W(T) = \sigma(T) \setminus E_0(T)$$

$E_0(T)$ is the set of all eigenvalues of T which are an isolated point of finite multiplicity in $\sigma(T)$

2.2 Generalized Weyl's Theorem

2.2.1 B-Fredholm and Semi-B-Fredholm operator

Definition 2.2.1 [46] *For a bounded linear operator T and a nonnegative integer n define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[n]} = T$). If for some integer n the range space $R(T^n)$ is closed and $T_{[n]}$ is*

an upper (resp. a lower) semi-Fredholm operator, then T is called an upper (resp. lower) semi-B-Fredholm operator.

- Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a B-Fredholm operator. A semi-B-Fredholm is an upper or a lower semi-B-Fredholm operator. In [14, Theorem 2.7] it is shown that if H is a Hilbert space, then an operator $T \in L(H)$ is a semi-B-Fredholm operator if and only if $T = T_0 \oplus T_1$, where T_0 is a semi-Fredholm operator and T_1 is nilpotent.

Let T be a semi-B-Fredholm operator and let d be the degree of the stable iteration of T . Definition ?? It follows from [14, Proposition 2.1] that $T_{[n]}$ is a semi-Fredholm operator, and $ind(T_{[m]}) = ind(T_d)$ for each $m \geq d$. This enables us to define the index of a semi-B-Fredholm operator T as the index of the semi-Fredholm operator $T_{[d]}$. \square

2.2.2 B-Weyl operator and B-Weyl spectrum

Definition 2.2.2 An operator $T \in L(X)$ is called a B-Weyl operator if it is a B-Fredholm operator of index 0. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\} \quad (1.1)$$

Theorem 2.2.1 [46] [12, Theorem 4.5] if T is a normal operator acting on a Hilbert space H then we have

$$\sigma_{BW}(T) = \sigma(T) \setminus E(T),$$

where $\sigma_{BW}(T)$ is the B-Weyl spectrum of T and $E(T)$ is the set of all eigenvalues of T which are isolated in the spectrum of T .

- This result gives a generalization of the classical Weyl's theorem.

Let $SF_+(X)$ be the class of all upper semi-Fredholm operators, $SF_-(X)$ the class of all $T \in L(X)$ with $ind(T) \leq 0$, and for any $T \in L(X)$ let

$$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} / T - \lambda I \notin SF_+^-(X)\}.$$

Let $E_0^a(T)$ be the set of all eigenvalues of T of finite multiplicity which are isolated in the approximate point spectrum

$$\sigma_a(X) = \left\{ \lambda \in \mathbb{C} : \inf_{\|x\|=1} \|(T - \lambda I)(x)\| = 0 \right\}$$

. Following [40], we say that T satisfies a-Weyl's theorem if $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T)$. It follows from [40, Corollary 2.5] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem.

Similarly, let $SBF_+(X)$ be the class of all upper semi-B-Fredholm operators, and $SBF_+^-(X)$ the class of all $T \in SBF_+^-(X)$ such that $ind(T) \leq 0$, and

$$\sigma_{SBF_+^-}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin SBF_+^-(X) \}$$

2.2.3 Drazin invertible

Definition 2.2.3 [46] [30, Definition 4.1] we say that $T \in L(X)$ is Drazin invertible (with a finite index) if there exist $B, U \in L(X)$ such that U is nilpotent and

$$TB = BT, \quad BTB = B, \quad TBT = T + U.$$

- Recall that the concept of Drazin invertibility was originally introduced by Drazin in [19] where elements of an associative semigroup satisfying an equivalent relation were called pseudo-invertible. It is well known that T is Drazin invertible if and only if it has a finite ascent and descent,
- It is shown in [15] that Drazin invertibility is a good tool for the investigating of the class of B -Fredholm operators and of the induced B -Weyl spect.

Definition 2.2.4 [46] If X a Banach space and $T \in \mathcal{L}(X)$, $\lambda \in \mathbb{C}$ is said a pole of T if $\lambda I - T$ is Drazin invertible.

- $\Pi(T)$ denoted the set of all the poles of T and $\Pi_0(T)$ the set of all poles of finite rank.

2.2.4 Left Drazin invertible, left Drazin spectrum and left pole

Definition 2.2.5 A non-empty subset $\mathfrak{R} \subset L(X)$ is called a regularity if it satisfies the following conditions:

1. If $A \in L(X)$ and $n \geq 1$, then $A \in \mathfrak{R}$ if and only if $A^n \in \mathfrak{R}$.
2. If $A, B, C, D \in L(X)$ are mutually commuting operators satisfying $AC + BD = I$, then $AB \in \mathfrak{R}$ if and only if $A, B \in \mathfrak{R}$.

A regularity \mathfrak{R} induces the spectrum

$$\sigma_{\mathfrak{R}}(T) = \{\lambda \in \mathbb{C}, T - \lambda I \notin \mathfrak{R}\}, \text{ for any } T \in L(X)$$

- Define a set $LD(X)$ by

$$LD(X) = \{T \in L(X) : a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed}\}.$$

Definition 2.2.6 [46] Let X be a Banach space. Then $T \in L(X)$ is called left Darzin invertible if $T \in LD(X)$; the left Darzin spectrum $\sigma_{LD}(T)$ of T is defined as the spectrum induced by the regularity $\mathfrak{R}_8 = LD(X)$.

Definition 2.2.7 [46, Theorem 2.7] We will say that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I \in LD(X)$, and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda I) < \infty$.

We will denote by $\Pi^a(T)$ the set of all left poles of T , and by $\Pi_0^a(T)$ the set of all left poles of T of finite rank.

Remark 2.2.1 If $\lambda \in \Pi^a(T)$, then it's easily seen that $T - \lambda I$ is an operator of topological uniform descent. Therefore from [23, Theorem 4.7], it follows that λ is isolated in $\sigma_a(T)$.

Theorem 2.2.2 [9] Let $T \in L(X)$ and $\lambda \in \mathbb{C}$ be isolated in $\sigma_a(T)$. Then $\lambda \in \Pi^a(T)$ if and only if $\lambda \notin \sigma_{SBF_+^-}(T)$, and $\lambda \in \Pi_0^a(T)$ if and only if $\lambda \notin \sigma_{SF_+^-}(T)$.

Proof. [46] Without loss of generality we can assume $\lambda = 0$. If $0 \in \Pi^a(T)$, then $d = a(T) < \infty$ and $R(T^{d+1})$ is closed. From [36, Lemma 12] it follows that $R(T^d)$ is closed and $T_{[d]}$ is an upper semi-Fredholm operator such that $\text{ind}(T_{[d]}) \leq 0$. Therefore $0 \notin \sigma_{SBF_+^-}(T)$.

Conversely suppose that $0 \notin \sigma_{SBF_+^-}(T)$. Then T is upper semi-B-Fredholm, and, in particular, an operator of topological uniform descent. Since 0 is isolated in $\sigma_a(T)$, from [23, Theorem 4.7] it follows that for n large enough we have $c_n(T) = 0$. So $a(T) < \infty$. Since T is semi-B-Fredholm, it is also an operator of topological uniform descent, and by [?, Lemma 12], $R(T^{a(T)+1})$ is closed. Now it is easily seen that an upper semi-B-Fredholm operator is upper semi-Fredholm if and only if $\alpha(T) < \infty$. Using this fact we can show analogously that $0 \in \Pi_0^a(T)$ if and only if $0 \notin \sigma_{SBF_+^-}(T)$. ■

Proposition 2.2.1 *Let H be a Hilbert space. Then $T \in L(H)$ is an upper semi-B-Fredholm operator if and only if $T = T_1 \oplus T_2$, where T_1 is upper semi-Fredholm and T_2 nilpotent. Moreover, in this case we have $\text{ind}(T) = \text{ind}(T_1)$.*

Proof. We know from [14, Theorem 2.6] that T is an upper semi-B-Fredholm operator if and only if $T = T_1 \oplus T_2$, where T_1 is upper semi-Fredholm and T_2 nilpotent. We only have to show that if such a decomposition exist for T , then $\text{ind}(T) = \text{ind}(T_1)$. If T_1 is a B-Fredholm operator then T is also B-Fredholm, and from [11, Proposition 2.1] it follows that $\text{ind}(T) = \text{ind}(T_1)$. If T_1 is not B-Fredholm, let m be such that $T_2^m = 0$, and let $d \geq \max(\text{dis}(T), m)$. Then $R(T^d) = R(T^{(d+1)})$. Therefore $\dim(R(T^d)/R(T^{(d+1)})) = \dim(R(T_1^d)/R(T^{(d+1)}))$. Since T_1 is upper semi-Fredholm operator which is not a B-Fredholm operator, then $\dim(R(T_1^d)/R(T_1^{(d+1)})) = \infty = \dim(R(T^d)/R(T^{(d+1)}))$. Consequently $\text{ind}(T_1) = -\infty = \text{ind}(T)$. We have immediately the following corollary: ■

Corollary 2.2.1 [46] *Let H be a Hilbert space and $T \in L(H)$. then $T \in SBF_+^-(H)$ if and only if $T = T_1 \oplus T_2$, where T_1 is an upper semi-Fredholm operator with $\text{ind}(T_1) \leq 0$, and T_2 is nilpotent.*

In this case of a Hilbert space H we can give yet characterization of isolated left poles of an operator.

Theorem 2.2.3 [46] *Let H be a Hilbert space, $T \in L(H)$, and let λ be an isolated point in $\sigma_a(T)$. Then the following properties are equivalent:*

- a) λ is a left pole of T ,
- b) There exist T -invariant subspaces M and N of H such that $H = M \oplus N$, $(T - \lambda I)|_M$ is bounded below and $(T - \lambda I)|_N$ is nilpotent.

Proof. Let λ be a left pole of T and let $S = T - \lambda I$. Then $d = a(S) < \infty$ and both $R(S^{d+1})$ and $R(S^d)$ are closed, which implies that $N(S_{[d]}) = \{0\}$ and $R(S_{[d]}) = R(S^{d+1})$ is closed in $R(S^d)$; hence $S_{[d]}$ is bounded below. From [16, Corollary 3.13] it follows that $S = S_1 \oplus S_2$, where S_1 is bounded below and S_2 nilpotent. Conversely, suppose there exist T -invariant subspace M and N of H such that $H = M \oplus N$, $(T - \lambda I)|_M$ is bounded below and $(T - \lambda I)|_N$ is nilpotent. Then from the previous corollary it follows that $T - \lambda I \in SBF_+^-(H)$. By Theorem 2.2.2, λ is a left pole of T . ■

Lemma 2.2.1 *Let X be a Banach space and let $T \in L(X)$. Then*

$$\sigma_{SBF_+^-}(T) \subset \sigma_{LD}(T) \subset \sigma_a(T)$$

Proof. If $\lambda \notin \sigma_a(T)$, then $\inf_{\|x\|=1} \|(T - \lambda I)x\| > 0$. So there exist $\varepsilon > 0$ such that $\|(T - \lambda I)x\| \geq \varepsilon \|x\|$ for all $x \in X$. Hence $N(T - \lambda I) = \{0\}$, and $a(T - \lambda I) = 0$.

Moreover since $\|(T - \lambda I)x\| \geq \varepsilon \|x\|$, $R(T - \lambda I)$ is closed. Consequently $\lambda \notin \sigma_{LD}(T)$ and $\sigma_{LD}(T) \subset \sigma_a(T)$. If $\lambda \notin \sigma_{LD(X)}(T)$, then $d = a(T - \lambda I) < \infty$ and $R(T^{d+1})$ is closed. From [?, Lemma 12] we deduced that $R(T^d)$ is closed and $T_{[d]}$ is an upper semi-Fredholm operator of negative index. So $\lambda \notin \sigma_{SBF_+^-}(T)$ and $\sigma_{SBF_+^-}(T) \subset \sigma_{LD}(T) \subset \sigma_a(T)$. ■

2.3 Browder operator and Browder Spectrum

We begin by some definitions motivated by [16] and [23].

Definition 2.3.1 For a Banach space X , the class of all Browder operators is defined

$$\mathcal{B}(X) := \{T \in \Phi(X) : p(T), q(T) < \infty\}.$$

Obviously

$$\mathcal{B}(X) \subseteq W(X)$$

- The browder spectrum of $T \in L(X)$, noted by $\sigma_b(T)$ is defined by

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}(X)\},$$

We have

$$\sigma_b(T) = \sigma_W(T) \cup \text{acc}\sigma(T)$$

where $\text{acc}K$ denote the set of accumulation points of the set $K \subseteq \mathbb{C}$, see [1, Theorem 3.65].

2.4 a -Weyl's Theorem

Definition 2.4.1 For any $T \in L(X)$ we define the sequences $(c_n(T))$, $(c'_n(T))$ and $(k_n(T))$ as follows:

1. $c_n(T) = \dim(R(T^n)/R(T^{n+1}))$.
2. $c'_n(T) = \dim(N(T^{n+1})/N(T^n))$.
3. $k_n(T) = \dim[(R(T^n) \cap N(T))/R(T^{n+1}) \cap N(T)]$.

- Clearly,

$$p(T) = \inf \{n : c'_n(T) = 0\}$$

$$q(T) = \inf \{n : c_n(T) = 0\}$$

Definition 2.4.2 (See [?].) Let $T \in L(X)$ and let

$$\Delta(T) = \{n \in \mathbb{N} : \forall m \in \mathbb{N} \ m > n \implies (R(T^n) \cap N(T)) \subset (R(T^m) \cap N(T))\}.$$

Then the *degree of stable iteration* $\text{dis}(T)$ of T is defined as $\text{dis}(T) = \inf \Delta(T)$.

Definition 2.4.3 (See [23].) Let $T \in L(X)$ and let $d \in \mathbb{N}$. Then T has a uniform descent for $n \geq d$ if $R(T) + N(T^n) = R(T) + N(T^d)$ for all $n \geq d$ (equivalently, $k_n(T) = 0$ for all $n \geq d$). If, in addition, $R(T) + N(T^d)$ is closed, then T is said to have a topological uniform descent for $n \geq d$.

In [32] Kordula and Muller defined the concept of regularity as follows

Definition 2.4.4 Let $T \in L(X)$. We will say that:

1. T satisfies Weyl's theorem if $\sigma_W(T) = \sigma(T) \setminus E_0(T)$.
2. T satisfies generalized Weyl's theorem if $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$.
3. T satisfies Browder's theorem if $\sigma_W(T) = \sigma(T) \setminus \Pi_0(T)$.
4. T satisfies generalized Browder's theorem if $\sigma_{WB}(T) = \sigma(T) \setminus \Pi(T)$.
5. T satisfies a-Weyl's theorem if $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T)$.
6. T satisfies generalized a-Weyl's theorem if $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T)$.
7. T satisfies a-Browder's theorem if $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \Pi_0^a(T)$.
8. T satisfies generalized a-Browder's theorem if $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \Pi^a(T)$.

2.5 Weyl type theorems for Banach space operators

In the following section a characterization of operators obeying generalized a-Weyl's theorem or a-Weyl's theorem. This characterization extends results of Barnes

[10, Theorem 4], proved in the case of Weyl's theorem, and Barkani [13, Theorem 2.5], proved in the case of generalized Weyl's theorem.

Theorem 2.5.1 [46] Let X be a Banach space and let $T \in L(X)$. Then the following are true:

1. $\sigma_{SBF_+^-}(T) \subseteq \sigma_a(T) \setminus E^a(T)$ if and only if $E^a(T) = \Pi^a(T)$.

2. $\sigma_{SBF_+^-}(T) \supseteq \sigma_a(T) \setminus E^a(T)$ if and only if $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$.

Proof. 1) Suppose that $\sigma_{SBF_+^-}(T) \subseteq \sigma_a(T) \setminus E^a(T)$ and let $\lambda \in E^a(T)$. Then λ is isolated in $\sigma_a(T)$, and $\lambda \notin \sigma_{SBF_+^-}(T)$. So $T - \lambda I$ is in $SBF_+^-(X)$. From Theorem 2.2.2 it follows that λ is a left pole of T , and so $\lambda \in \Pi^a(T)$. As we have always $\Pi^a(T) \subset E^a(T)$, then $E^a(T) = \Pi^a(T)$.

Conversely if $E^a(T) = \Pi^a(T)$ and $\lambda \in E^a(T)$, then $(T - \lambda I) \in SBF_+^-(X)$. Therefore $\lambda \notin \sigma_{SBF_+^-}(T)$ and so $\sigma_{SBF_+^-}(T) \subseteq \sigma_a(T) \setminus E^a(T)$.

2. Suppose that $\sigma_{SBF_+^-}(T) \supseteq \sigma_a(T) \setminus E^a(T)$, and let $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Then $\lambda \in E^a(T)$, in particular λ is isolated in $\sigma_a(T)$. Moreover $T - \lambda I$ is upper semi-B-Fredholm operator and $ind(T - \lambda I) \leq 0$. From Theorem 2.2.2 it follows that $T - \lambda I$ is left Drazin invertible and $\sigma_{LD}(T) \subset \sigma_{SBF_+^-}(T)$. As it is always true that $\sigma_{SBF_+^-}(T) \subset \sigma_{LD}(T)$, then $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$. Conversely suppose that $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$. Let $\lambda \notin \sigma_{SBF_+^-}(T)$. Let $\lambda \notin \sigma_{SBF_+^-}(T)$, then $\lambda \notin \sigma_{LD}(T)$. So $T - \lambda I$ is left Drazin invertible and $\lambda \in E^a(T)$. Hence $\sigma_{SBF_+^-}(T) \supseteq \sigma_a(T) \setminus E^a(T)$. From this theorem we obtain immediately the following corollaries. ■

Corollary 2.5.1 *Let X be a Banach space and let $T \in L(X)$. Then T satisfies generalized a -Weyl's theorem if and only if $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \Pi^a(T)$ and $E^a(T) = \Pi^a(T)$.*

Corollary 2.5.2 *Let X be a Banach space and let $T \in L(X)$. If T satisfies generalized a -Weyl's theorem, then it satisfies a -Browder's theorem.*

Chapitre 3

Characterization of operators satisfying Weyl's and Browder's Theorems

3.1 Characterization of operators satisfying a-Weyl's theorem

Theorem 3.1.1 . *Let X be a Banach space and let $T \in L(X)$. Then T satisfies a-Weyl's theorem if and only if $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \Pi^a(T)$ and $E_0^a(T) = \Pi_0^a(T)$.*

As a consequence of this theorem we have the following result

Corollary 3.1.1 *If $T \in L(X)$ satisfies a-Weyl's theorem, then it satisfies a-Browder's theorem.*

Remark 3.1.1 *In [40, Theorem 1.1] $T \in L(X)$ satisfies a-Weyl's theorem if and only if the following conditions are satisfied*

1. If $\lambda \in \Pi_0^a(T)$, then $R(T - \lambda I)$ is closed.
2. If $T - \lambda I \in SF_+^-(X)$, then λ is not an interior point of $\sigma_a(T)$.

M. Berkani [11] introduced a characterization of operators obeying a-Weyl's theorem appears to have a simpler formulation than Rakočević's Theorem.

Theorem 3.1.2 [11, Theorem 3.7]. *If $T \in L(X)$ satisfies generalized a-Weyl's theorem, then it satisfies generalized Weyl's theorem.*

Proof. Suppose that T satisfies generalized a-Weyl's theorem and $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T - \lambda I$ is a B-Fredholm operator of index 0 and hence also upper semi-B-Fredholm with index 0. Therefore $\lambda \notin \sigma_{SBF_+^-}(T)$. Let $\lambda \notin \sigma_a(T)$. Since $T - \lambda I$ is an operator of topological uniform descent, then there exist $\varepsilon > 0$ such that if $0 < |\lambda - \mu| < \varepsilon$, then for n large enough we have $c_n(T - \lambda I) = c_0(T - \mu I)$ and $c'_n(T - \lambda I) = c'_0(T - \mu I)$. Since $T - \lambda I$ is a B-Weyl operator, then for n large enough we have $c_n(T - \lambda I) = c'_n(T - \lambda I)$. Since $\lambda \notin \sigma_a(T)$ we have $c'_0(T - \lambda I) = 0$. It follows then that $c_0(T - \lambda I) = c'_0(T - \lambda I) = 0$. Consequently $\lambda \notin \sigma(T)$, which is contradiction. Hence $\lambda \in \sigma_a(T)$. satisfies generalized a-Weyl's theorem, then $\lambda \in E^a(T)$. If $\mu \neq \lambda$ and $|\lambda - \mu|$ is small enough, then $\mu \notin \sigma_a(T)$. So $c'_0(T - \mu I) = 0$. Hence $c_n(T - \lambda I) = 0$ for n large enough. Since $ind(T - \lambda I) = 0$, then $T - \mu I$ is invertible. Consequently $\lambda \in E(T)$. Conversely, if $\lambda \in E(T) \subset E^a(T)$, then $\lambda \notin \sigma_{SBF_+^-}(T)$. Hence $T - \lambda I$ is an operator of topological uniform descent. So there is $\varepsilon > 0$, such that if $0 < |\lambda - \mu| < \varepsilon$, then $c_n(T - \lambda I) = c_n(T - \mu I)$ and $c'_n(T - \lambda I) = c'_n(T - \mu I)$ for n large enough. Since λ is isolated in $\sigma(T)$, if ε is chosen small enough, then $c_n(T - \mu I) = c'_n(T - \mu I) = 0$. So $T - \lambda I$ is draizing invertible. Therefore $\lambda \notin \sigma_{BW}(T)$ and $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$. ■

Theorem 3.1.3 [46] *If $T \in L(X)$ satisfies generalized a-Browder's theorem, then it also satisfies the generalised Browder's theorem.*

Further we have

Theorem 3.1.4 [46] *If $T \in L(X)$ satisfies generalized Weyl's theorem, then it also satisfies Weyl's theorem.*

Proof. Suppose that T satisfies generalized Weyl's theorem, and let $\lambda \in \sigma(T)$ and $\lambda \notin \sigma(T) \setminus E_0(T)$. Then $\lambda \in E_0(T) \subset E(T)$. Since T satisfies generalized Weyl's Theorem,

$T - \lambda I$ is a-B-Fredholm operator of index 0. As is isolated from [16, Theorem 4.5] that λ is a pole of T . Hence there is an integer n such that $X = N((T - \lambda I)^n) \oplus R((T - \lambda I)^n)$. Since $\alpha(T - \lambda I) < \infty$, then from [43, Lemma 3.3(a)] we have $\alpha((T - \lambda I)^n) < \infty$. Therefore $(T - \lambda I)^n$ is a Fredholm operator of index 0, and so is $T - \lambda I$. Hence $\lambda \notin \sigma_W(T)$. Conversely, if $\lambda \in \sigma(T)$ and $\lambda \notin \sigma_W(T)$, then $T - \lambda I$ is Fredholm operator of index 0. Therefore $\lambda \notin \sigma_{BW}(T)$. Since T satisfies the generalized Weyl theorem, we have $\lambda \in E(T)$, while $\alpha(T - \lambda I) < \infty$. So $\lambda \in E_0(T)$ and $\sigma_W(T) = \sigma(T) \setminus E_0(T)$. ■

Corollary 3.1.2 *If $T \in L(X)$ satisfies generalized a-Weyl's theorem, then it satisfies Weyl's theorem.*

Theorem 3.1.5 *If $T \in L(X)$ satisfies generalized a-Weyl's theorem, then it satisfies a-Weyl's theorem.*

Proof. Let $T \in L(X)$. Suppose that $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E^a(T)$ and let us show that $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T)$. If $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$, then $T - \lambda I$ is an upper semi-Fredholm operator and $ind(T - \lambda I) \leq 0$. Therefore $\lambda \notin \sigma_{SBF_+^-}(T)$. Since T satisfies the generalized a-Weyl theorem, then $\lambda \in E_0^a(T)$. As $\alpha(T - \lambda I) < \infty$, then $\lambda \in E_0^a(T)$. Conversely suppose that $\lambda \in E_0^a(T)$. Then $\lambda \notin \sigma_{SBF_+^-}(T)$, and $T - \lambda I$ is an upper semi-B-Fredholm operator with $ind(T - \lambda I) \leq 0$. Since $\alpha(T - \lambda I) < \infty$, then $T - \lambda I$ is an upper semi-Fredholm operator with $ind(T - \lambda I) \leq 0$. So $\lambda \notin \sigma_{SF_+^-}(T)$. The converse of the preceding theorem is not true as shown by the following example There is an operator $T \in L(X)$ which obeys a-Weyl's theorem and Weyl's theorem, but does not obey generalized Weyl's theorem or a-Weyl's theorem.

Let \mathcal{Q} be defined for each $x = (\zeta_i) \in \ell^1$ by

$$\mathcal{Q}(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_k, \dots) = (0, \alpha_1 \zeta_1, \alpha_2 \zeta_2, \dots, \alpha_{k-1} \zeta_{k-1}, \dots),$$

where (α_i) is a sequence of complex numbers such that $0 < |\alpha_i| \leq 1$ and $\sum_{i=0}^{\infty} |\alpha_i| < \infty$.

We observe that

$$\overline{R(\mathcal{Q}^n)} \neq R(\mathcal{Q}^n), \quad n = 1, 2, \dots$$

Indeed, for a given $n \in \mathbb{N}$ let $x_k^{(n)} = (1, \dots, 1, 0, 0, \dots)$ (with $n+k$ zeros). Then the limit $y^{(n)} = \lim_{k \rightarrow \infty} x_k^{(n)}$ exists and lies in $\overline{R(\mathcal{Q}^n)}$. However, there is no element

$x^{(n)} \in \ell^1$ satisfying the equation $\mathcal{Q}^n x^{(n)} = y^{(n)}$ as the algebraic solution to this equation is $(1, 1, 1, \dots) \notin \ell^1$. Define T on $X = \ell^1 \oplus \ell^1$ by $T = \mathcal{Q} \oplus 0$. Then $N(T) = \{0\} \oplus \ell^1$, $\sigma(T) = \sigma_a(T) = \{0\}$, $E(T) = \{0\}$, $E_0^a(T) = \emptyset$. Since $R(T^n) = R(\mathcal{Q}^n) \oplus \{0\}$, $R(T^n)$ is not closed for any $n \in \mathbb{N}$; so T is not a B-Weyl operator, and $\sigma_{BW}(T) = \{0\}$. Further, $T \notin SF_+^-(X)$, and $\sigma_{SF_+^-}(T) = \{0\}$. We then have

$$\sigma_{BW}(T) \neq \sigma(T) \setminus E(T), \quad \sigma_{SF_+^-}(T) \setminus E_0^a(T).$$

Hence T satisfies a-Weyl's theorem but not generalized Weyl's theorem. Then T does not obey generalized a-Weyl's theorem Theorem 3.1.2, but obeys Weyl's theorem [40, Corollary 2.5] ■

Theorem 3.1.6 *If $T \in L(X)$ satisfies generalized a-Browder's theorem, then it satisfies a-Browder's theorem.*

Proof. Suppose that $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \Pi^a(T)$, and let us show that $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \Pi_0^a(T)$. If $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$, then $T - \lambda I$ is an upper semi-Fredholm operator and $\text{ind}(T - \lambda I) \leq 0$. Then $\lambda \notin \sigma_{SBF_+^-}(T)$. Since T satisfies generalized a-Browder's theorem, then $\lambda \in \Pi^a(T)$. As $\alpha(T - \lambda I) < \infty$, we have $\lambda \in \Pi_0^a(T)$. Conversely suppose that $\lambda \in \Pi_0^a(T)$. Then $\lambda \notin \sigma_{SBF_+^-}(T)$. Therefore $T - \lambda I$ is an upper semi-B-Fredholm operator with $\text{ind}(T - \lambda I) \leq 0$. Since $\alpha(T - \lambda I) < \infty$, $T - \lambda I$ is an upper semi-Fredholm operator with $\text{ind}(T - \lambda I) \leq 0$. So $\lambda \notin \sigma_{SF_+^-}(T)$. Combining Corollary 2.5.2 with the preceding theorem we obtain the following result (which also follows Theorem 3.1.6 and Corollary 3.1.1). ■

Corollary 3.1.3 *If $T \in L(X)$ satisfies generalized a-Weyl's theorem, then it satisfies a-Browder's theorem.*

Theorem 3.1.7 *If $T \in L(X)$ satisfies generalized Browder's, then it also satisfies Browder's theorem.*

Proof. If $\lambda \in \sigma(T) \setminus \sigma_W(T)$, then $T - \lambda I$ is a Fredholm operator of index 0 and also a-B-Fredholm operator of index 0. So λ is a pole of T , and by [34, Theorem 2.1] there is an integer n such that $X = N((T - \lambda I)^n) \oplus R((T - \lambda I)^n)$. Since $\alpha(T - \lambda I) < \infty$, then from

[43, Lemma 3.3(a)] we have $\alpha((T - \lambda I)^n) < \infty$. From [10, Proposition 2] it follows that λ is a pole of finite rank of T . Conversely, if λ is a pole of finite rank of T , then $T - \lambda I$ is a B-Fredholm operator of index 0 since T satisfies generalized Browder's theorem. So there is an integer n such that $X = N((T - \lambda I)^n) \oplus R((T - \lambda I)^n) < \infty$. Therefore $T - \lambda I$ is also a Fredholm operator of index 0. Hence we have $\sigma_W(T) = \sigma(T) \setminus \Pi_0(T)$. ■

3.2 Finite rank perturbations for Hilbert space operators

In this section H denotes a Hilbert space and $F_0(H)$ the set of finite rank operators in $L(H)$. Classical characterizations of Weyl's and Browder's spectrum state that, for every $T \in L(H)$,

$$\sigma_W(T) = \bigcap_{F \in F_0(H)} \sigma(T + F), \quad \sigma_B(T) = \bigcap_{F \in F_0(H), TF=FT} \sigma(T + F).$$

Many extensions of these results have been obtained (for instance, [10], [31], [?], [?]). In the following we present some results analogous to the above characterizations of Weyl's and Browder's spectra, and also show that the validity of generalized a-Browder's theorem and of a-Browder's theorem is preserved under commuting finite rank perturbations.

Theorem 3.2.1 *Let $T \in L(H)$. Then*

$$\sigma_{SBF_+^-}(T) = \bigcap_{F \in F_0(H)} \sigma_{LD}(T - F).$$

Proof. If $\lambda \notin \sigma_{SBF_+^-}(T)$, then $T - \lambda I$ is a semi-B-Fredholm operator and $\text{ind}(T - \lambda I) \leq 0$. Therefore $H = H_1 \oplus H_2$, and $T - \lambda I = T_1 \oplus T_2$ relatively to this decomposition, with T_1 in $SF_+^-(H)$ and T_2 a nilpotent operator. By [?, Corollary 2.7] there exists a finite rank operator F_0 such that $T_1 \oplus F_0$ is bounded below. Let $F = F_0 \oplus 0$. Then F is finite rank operator, $T - \lambda I + F = T_1 + F \oplus T_2$ is in $LD(X)$ and $\lambda \notin \bigcap \{\sigma_{LD}(T + F) : F \in F_0(X)\}$. Conversely suppose that $\lambda \notin \bigcap \{\sigma_{LD}(T + F) : F \in F_0(X)\}$. Then there exists a finite rank

operator F such that $T - \lambda I + F$ is in $LD(X)$. In particular $T - \lambda I + F$ is semi-B-Fredholm with $ind(T - \lambda I + F) \leq 0$. As F is of finite rank, then by [14, Proposition 2.7], $T - \lambda I$ is also an upper semi-B-Fredholm operator. Moreover there exists an integer n_0 such that for any $n \geq n_0$, $T - \lambda I - \frac{1}{n}I$ and $T - \lambda I + F - \frac{1}{n}I$ are semi-Fredholm operators, $ind(T - \lambda I - \frac{1}{n}I) = ind(T - \lambda I)$ and $ind(T - \lambda I + F - \frac{1}{n}I) = ind(T - \lambda I + F)$. Since F is a finite rank operator and $T - \lambda I - \frac{1}{n}I$ is semi-Fredholm, by the known properties of the index we have $ind(T - \lambda I + F - \frac{1}{n}I) = ind(T - \lambda I - \frac{1}{n}I)$. So $ind(T - \lambda I) = ind(T - \lambda I + F) \leq 0$. ■

Theorem 3.2.2 *Let $T \in L(H)$. then*

$$\sigma_{LD}(T) = \bigcap_{F \in F_0(X), FT=TF} \sigma_{LD}(T + F).$$

Proof. Let $\lambda \notin \sigma_{LD}(T)$. Then $T + 0 - \lambda I$ is left Drazin invertible. Since 0 is a finite rank operator, it follows that $\lambda \notin \bigcap \{\sigma_{LD}(T + F) : F \in F_0(X), FT = TF\}$. Conversely suppose that $\lambda \notin \bigcap \{\sigma_{LD}(T + F) : F \in F_0(X), FT = TF\}$. Then there exists a finite rank operator F commuting with T such that $T + F - \lambda I$ is left Drazin invertible. Hence from [14, Proposition 2.7], $T - \lambda I = (T - \lambda I + F) - F$ is a semi-B-Fredholm operator. In particular the two operators $T - \lambda I$ and $T - \lambda I + F$ are of topological uniform descent. By [23, Theorem 5.8], for n large enough we have $c_n(T - \lambda I) = c_n(T - \lambda I + F)$ and $c'_n(T - \lambda I) = c'_n(T - \lambda I + F)$. So $T - \lambda I + F$ is left Drazin invertible, then for n large enough we have $c'_n(T - \lambda I + F) = 0$. So for n large enough we have $c'_n(T - \lambda I) = 0$. Hence $T - \lambda I$ is Drazin invertible. ■

Theorem 3.2.3 [46] *If $T \in L(H)$ satisfies generalized a-Browder's theorem and if F is a finite rank operator such that $TF = FT$, then $T + F$ satisfies generalized a-Browder's theorem.*

Proof. From the characterization of $\sigma_{SBF_+^-}(T)$ it follows that if F is a finite rank operator, then $\sigma_{SBF_+^-}(T + F) = \sigma_{SBF_+^-}(T)$. Moreover, if F commutes with T , it follows from Theorem 3.2.2 that $\sigma_{LD}(T + F) = \sigma_{LD}(T)$. If T satisfies generalized a-Browder's theorem, then $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$. Hence $\sigma_{SBF_+^-}(T + F) = \sigma_{LD}(T + F)$, and so $T + F$ satisfies generalized a-Browder's theorem. ■

Similarly, using properties of $\sigma_{SBF_+^-}(T)$ for an operator T acting on a Banach space X , we have the following:

Theorem 3.2.4 *Let X be a Banach space. If $T \in L(X)$ satisfies a-Browder's theorem and F is a finite rank operator such that $TF = FT$, then $T + F$ satisfies a-Browder's theorem.*

We give a generalization of a result involving poles of finite rank [10, Fact 10], and arbitrary poles [13, Theorem 2.9], which in general holds for Banach space operators.

Theorem 3.2.5 *Let Γ be a nonempty connected subset of \mathbb{C} such that $T - \alpha I$ is an upper semi-B-Fredholm operator for all $\alpha \in \Gamma$. If there is $\alpha \in \Gamma$ such that $T - \alpha I$ is left Drazin invertible, then every point of $\sigma(T) \cap \Gamma$ is a left pole of T and $\sigma_a(T) \cap \Gamma$ is a countable discrete set.*

Proof. Since $T - \alpha I$ is left Drazin invertible, for n large enough we have $c'_n(T - \alpha I) = 0$. Let $\Omega = \{\mu \in \Gamma : T - \mu I \text{ is left Drazin invertible}\}$. Then $\alpha \in \Omega$ and $\Omega \neq \emptyset$. If $\lambda \in \Omega$, Since $T - \lambda I$ is left Drazin invertible, then there is an open neighborhood $B(\lambda, \varepsilon)$ such that $B(\lambda, \varepsilon) \setminus \{\lambda\} \subset \rho_a(T)$, is the complement set of $\sigma_a(T)$. Therefore $B(\lambda, \varepsilon) \cap \Gamma \subset \Omega$, and Ω is open in Γ . Now let $\lambda \in \overline{\Omega} \cap \Gamma$, where $\overline{\Omega}$ is the closure of Ω . In particular, $T - \lambda I$ is an upper semi-B-Fredholm operator. Hence there is $\varepsilon > 0$ such that if $|\lambda - \mu| < \varepsilon$ then for n large enough we have $c_n(T - \lambda I) = c_n(T - \mu I)$,

$c'_n(T - \lambda I) = c'_n(T - \mu I)$. Since $\lambda \in \overline{\Omega}$, then $B(\lambda, \varepsilon) \cap \Gamma \neq \emptyset$. So there is $\mu \in B(\lambda, \varepsilon) \cap \Omega$. Hence $c'_n(T - \lambda I) = 0$, and so $\lambda \in \Omega$. Therefore Ω is closed in Γ . Since Γ is connected, then $\Omega = \Gamma$. Moreover, if $\lambda \in \sigma_a(T) \cap \Gamma$, then λ is a pole of T , and an isolated point of $\sigma_a(T)$. Since $\sigma_a(T)$ is a compact set, $\sigma_a(T) \cap \Gamma$ is a discrete set. ■

Theorem 3.2.6 *Let $T \in L(H)$. Suppose that $\sigma_{SBF_+^-}(T)$ is simply connected. Then $T + F$ satisfies generalized a-Browder's theorem for every $F \in F_0(X)$.*

Proof. Suppose that $\lambda \in \sigma_a(T)$ and $(T - \lambda I) \in SBF_+^-(X)$. Let $\Gamma = \{\alpha \in \mathbb{C} : (T - \alpha I) \in SBF_+^-(X)\}$. Then Γ is connected. Since $\Gamma \cap \sigma_a(T)$ consists entirely of left poles of T . So $\lambda \in \Pi^a(T)$ and T satisfies generalized a-Browder's theorem. If $F \in F_0(X)$, is a finite rank operator, then $\sigma_{SBF_+^-}(T + F) = \sigma_{SBF_+^-}(T)$. Thus $\sigma_{SBF_+^-}^-(T + F)$ is simply connected, and so $T + F$ satisfies generalized Browder's theorem. ■

3.3 Characterisation of operators satisfies Browder's Theorem.

Evidently, an operator $T \in L(X)$ has *SVEP* at every point of the resolvent $p(T) := C \setminus \sigma(T)$. Moreover, the identity theorem for analytic function implies that $T \in L(X)$ has *SVEP* at every point of the boundary $\partial\sigma(T)$ of the spectrum .

In particular, both T and dual T^* have *SVEP* at every isolated point of $\sigma(T)$.

Note that

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has } SVEP \text{ at } \lambda,$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has } SVEP \text{ at } \lambda,$$

see [1], Theorem 3.8]

Let us consider the quasi-nilpotent part of T , i.e. is the set

$$H_0(T) := \left\{ x \in X : \lim_{n \rightarrow \infty} \|T^n x\|^{\frac{1}{n}} = 0 \right\}.$$

It is easily seen that $\ker(T^m) \subseteq H_0(T)$ for every $m \in \mathbb{N}$. Moreover, T is quasi-nilpotent if and only if $H_0(T) = X$, see [1, Theorem 1.68].

Analytic core

Definition 3.3.1 *If $T \in L(X)$, the analytic core $K(T)$ is the set of all $x \in X$ such that there exist a constant $c > 0$ and a sequence of elements $x_n \in X$ such that $x_0 = x$, $Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$,*

- See [1] for informations on the subspace $H_0(T)$, $K(T)$. The subspace $H_0(T)$ and $K(T)$ may be not closed. We have

$$H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has } SVEP \text{ at } \lambda,$$

see [5]

3.3.1 SVEP property for semi-Fredholm operators

In the following theorem some characterizations of SVEP property for semi-Fredholm operators. These characterizations have been proved in [5], [1, Chapter 3]

Theorem 3.3.1 *Suppose that $\lambda_0 I - T \in \Phi_{\pm}(X)$. Then the following statements are equivalent:*

1. T has SVEP at λ_0 ;
2. $p(\lambda_0 I - T) < \infty$;
3. $H_0(\lambda_0 I - T)$ is closed.
4. $H_0(\lambda_0 I - T)$ is finite-dimensional.

Dually, if $\lambda_0 I - T \in \Phi_{\pm}(X)$ then the following statement are equivalent:

5. T^* has SVEP at λ_0 ;
6. $q(\lambda_0 I - T) < \infty$.
7. $K(\lambda I - T)$ is finite-codimension

Following Harte and W.Y.Lee [24] we shall say that T satisfies Browder's theorem if

$$\sigma_W(T) = \sigma_b(T),$$

or equivalently, by (1), if

$$\text{acc}\sigma(T) \subseteq \sigma_W(T).$$

Browder's Theorem, an approximate point spectrum variant of it, a-Browder's theorem, has been studied by several authors ([18], [25], [45], [2], [4]). In the following section a characterization of operators satisfying Weyl's theorem. The methods here employed are essentially those from local spectral theory. In particular we shall see that Browder's theorem for T (or, it is the same, for T^*) is equivalent to localized SVEP at the point of some $\mathbb{C} \setminus \sigma_W(T)$. This characterization allows us to give a precise description of some parts of the spectrum. Browder's theorem and Weyl's theorem are also described by means of the discontinuity of some mappings at the points of certain sets $\Delta(T)$ and $\Delta_{00}(T)$, respectively, as

well as by means of properties of the quasi-nilpotent part $H_0(\lambda I - T)$ as λ ranges in these sets.

3.3.2 Characterization of Browder's Theorem via SVEP

Let write $iso K$ for the set of all isolated points of $K \subseteq \mathbb{C}$. For a bounded operator $T \in L(X)$ define

$$\pi_{00}(T) := \{\lambda \in iso \sigma(T) : \lambda I - T < \infty\},$$

and

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T) = \{\lambda \in \sigma(T) : \lambda I - T \in B(T)\},$$

the set of all Riesz points in $\sigma(T)$. Finally, let us consider the following set

$$\Delta(T) := \sigma(T) \setminus \sigma_W(T).$$

If $\lambda \in \Delta(T)$ then $\lambda I - T \in W(X)$ and since $\lambda \in \sigma(T)$ it follows that $\alpha(\lambda I - T) = \beta(\lambda I - T) > 0$, so we can write

$$\Delta(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in W(X), 0 < \alpha(\lambda I - T)\}.$$

The set $\Delta(T)$ has been recently studies in [25], where the points of $\Delta(T)$ are called generalized Riesz points

Lemma 3.3.1 *For every $T \in L(X)$ we have $p_{00}(T) \subseteq \pi_{00}(T) \cap \Delta(T)$.*

Proof.

1. If $\lambda \in p_{00}(T)$ then $\lambda I - T \in B(X)$ and $p(\lambda I - T) = q(\lambda I - T) < \infty$, so λ is isolated in $\sigma(T)$. Furthermore, since $\lambda I - T \in W(X)$ we have $0 < \alpha(\lambda I - T) < \infty$, otherwise by [1, Theorem 3.4] we would have $\alpha(\lambda I - T) = \beta(\lambda I - T) = 0$, hence $\lambda \notin \sigma(T)$, a contradiction.

■

It is known that the SVEP for T or T^* entails Browder's theorem

[1, Corollary 3.53]. But the followig example shows that SVEP for T or T^* is not necessary for Browder's theorem.

Example 3.3.1 Let $T := L \oplus L^* \oplus Q$, where L is the unilateral left shift on $\ell^2(\mathbb{N})$, defined by

$$L(x_1, x_2, \dots) := (x_2, x_3, \dots), \quad (x_n) \in \ell^2(\mathbb{N})$$

and Q is any quasi-nilpotent operator. L does not have SVEP, see [1, proposition 71], so also T and T^* do not have SVREP, see [1, Theorem 2.9]. On the other hand, we have $\sigma_b(T) = \sigma_W(T) = D$, where D is the closed unit disc in \mathbb{C} , so Browder's theorem holds for T .

The following result shows that Browder's theorem is equivalent to the localized SVEP at the points of the complement in \mathbb{C} of $\sigma_W(T)$.

Theorem 3.3.2 [9] For an operator $T \in L(X)$ the following statement hold:

1. $p_{00}(T) = \Delta(T)$;
2. T satisfies Browder's theorem;
3. T^* satisfies Browder's theorem;
4. T has SVEP at every $\lambda \notin \sigma(T)$;
5. T^* has SVEP at every $\lambda \notin \sigma_W(T)$.

Proof. **1** \Rightarrow **2** Suppose that $p_{00}(T) = \Delta(T)$. Let $\lambda \notin \sigma_W(T)$. We show that $\lambda \notin \sigma_b(T)$. If $\lambda \notin \text{acc}\sigma(T)$ then by the equality (1) we have $\lambda \notin \sigma_b(T)$. Consider the other case $\lambda \in \text{acc}\sigma(T)$. Since $\sigma(T)$ is closed then $\lambda \in \sigma(T)$ and since $\lambda I - T \in W(X)$ it must be $0 < \alpha(\lambda I - T) = \beta(\lambda I - T)$. Therefore $\lambda \in \Delta(T) = p_{00}(T) = \sigma(T) \setminus \sigma_b(T)$, thus $\lambda \notin \sigma_b(T)$. Hence $\sigma_b(T) \subseteq \sigma_W(T)$ and since the reverse inclusion is satisfied by every operator we then conclude that $\sigma_b(T) = \sigma_W(T)$, i.e. T satisfies Browder's theorem. **2** \Leftrightarrow **3** Obvious, since $\sigma_b(T) = \sigma_b(T^*)$ and $\sigma_W(T) = \sigma_W(T^*)$.

2 \Rightarrow **4** Suppose that $\sigma_b(T) = \sigma_W(T)$. If $\lambda \notin \sigma_W(T)$ then $\lambda I - T \in B(X)$ so $p(\lambda I - T) < \infty$ and hence T has SVEP at λ . **4** \Rightarrow **5** Suppose that T has SVEP at every point

$\lambda \in \mathbb{C} \setminus \sigma_W(T)$. For every $\lambda \notin \sigma_W(T)$ then $\lambda I - T \in W(X)$, and the SVEP at λ implies by Theorem 1.2 that $p(\lambda I - T) < \infty$. Since $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$ it then

follows by [1, Theorem 3.4] that $q(\lambda I - T) < \infty$, and consequently, by Theorem 3.3.1, T^* has SVEP at λ . Suppose that $\lambda \in \Delta(T)$. We have $\lambda I - T \in W(X)$ and hence $ind(\lambda I - T) = 0$. By Theorem 3.3.1 the SVEP of T^* at λ implies that $q(\lambda I - T) < \infty$ and hence, again by [1, Theorem 3.4], also $p(\lambda I - T)$ is finite. Therefore $\lambda \in \sigma(T) \setminus \sigma_b(T) = p_{00}(T)$. This shows that $\Delta(T) \subseteq p_{00}(T)$ and by Lemma 3.3.1 we then conclude that equality $p_{00}(T) = \Delta(T)$ holds. ■

Remark 3.3.1 Theorem 3.3.2 provides a somewhat useful tool in order to prove that Browder's theorem holds, or not holds, for a bounded operator. For instance, let $T := R \oplus L$, where R and L are the unilateral right shift and the unilateral left shift, respectively, on $\ell^p(\mathbb{N})$. By [1, Theorem 2.9] then T does not have SVEP at 0, since L fails SVEP at 0, and $0 \notin \sigma_W(T) = D$, D the unit disc in \mathbb{C} . Hence Browder's theorem fails for T .

Let $K(X)$ denote the set of all compact operators on X . Denote by $\sigma_a(T)$ the classical approximate point spectrum of T , and by $\sigma_s(T)$ surjectivity spectrum of T . The approximate point Weyl spectrum is defined by

$$\sigma_{wa}(T) := \bigcup_{K \in K(X)} \sigma_a(T + K),$$

whilst the surjectivity Weyl spectrum of T is defined by

$$\sigma_{ws}(T) := \bigcup_{K \in K(X)} \sigma_s(T + K),$$

The spectrum $\sigma_{wa}(T)$ coincides with the complement of the set all $\lambda \in \mathbb{C}$ such that $\lambda I - T \in \Phi_+(T)$ with $ind(\lambda I - T) \leq 0$, whilst $\sigma_{ws}(T)$ coincides with the complement of the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T \in \Phi_-(X)$ with $ind(\lambda I - T) \geq 0$, see [1, Theorem 3.65]. The upper semi-Browder spectrum $\sigma_{ub}(T)$ is defined as the complement of set all $\lambda \in \mathbb{C}$ such that $\lambda I - T \in \Phi_+(X)$ with $p(\lambda I - T) < \infty$. Clearly, $\sigma_{wa}(T) \subseteq \sigma_{ub}(T)$ for every $T \in L(X)$.

A bounded operator $T \in L(X)$ is said to satisfying a-Browder's theorem have been investigated in several articles, for instance [18] and [4]. In particular, the operators satisfying a-Browder's theorem admits a similar characterization to that established in Theorem 3.3.2 for operators satisfying Browder's theorem, see [4]:

(3) a-Browder's theorem holds for $T \iff T$ has SVEP at all $\lambda \notin \sigma_{wa}(T)$.

Since $\sigma_{wa}(T) \subseteq \sigma_w(T)$, from the result of Theorem 3.3.2 it is obvious that a-Browder's theorem for T implies that Browder's theorem holds for T . In fact, these two theorem correspond to the SVEP at two subsets, where one is in general larger than the other.

Set $\Lambda(T) := \sigma_w(T) \setminus \sigma_{wa}(T)$. It is easily seen that

$$\Lambda(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_+(X), \text{ind}(\lambda I - T) < 0\}.$$

Indeed, if $\lambda \in \sigma_w(T) \setminus \sigma_{wa}(T)$ then $\lambda I - T \in \Phi_+(X)$ and $\text{ind}(\lambda I - T) \leq 0$. Since $\lambda I - T$ is not Weyl there are two possibilities: $\lambda I - T \notin \Phi(X)$ or $\lambda I - T \in \Phi(X)$ with $\text{ind}(\lambda I - T) \neq 0$. In the first case $\lambda I - T \notin \Phi_-(X)$, so $\text{ind}(\lambda I - T) = -\infty$, whilst in the second case we obviously have $\text{ind}(\lambda I - T) < 0$. Therefore the inclusion $\Lambda(T) \subseteq \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_+(X), \text{ind}(\lambda I - T) < 0\}$ holds. The reverse inclusion is obvious.

Corollary 3.3.1 *Suppose that T satisfies Browder's theorem. Then the following statement are equivalent:*

1. a-Browder's theorem holds for T ;
2. T has SVEP at every $\lambda \in \Lambda(T)$;
3. $\sigma_a(T) \cap \Lambda(T) \subseteq \text{iso } \sigma_a(T)$.

Proof. We have $\sigma_w(T) = \sigma_{wa}(T) \cup \Lambda(T)$, so the equivalence **1** \Leftrightarrow **2** follows from Theorem 3.3.2 and equivalence (3). To show the equivalence **2** \Leftrightarrow **3** it suffices to observe that if $\lambda \in \Lambda(T)$ then $\lambda I - T \in \Phi_+(X)$, and T has the SVEP at λ if and only if $\sigma_a(T)$ does not cluster at λ , see [7, Theorem 2.2]. ■

The set $\Lambda(T)$ may be empty. For instance

$$T^* \text{ has SVEP} \Rightarrow \Lambda(T) = \emptyset,$$

and

$$T \text{ has SVEP} \Rightarrow \Lambda(T^*) = \emptyset.$$

To show the implication (4) observe that if $\Lambda(T) \neq \emptyset$ and T^* has SVEP at $\lambda \in \Lambda(T)$ then, by Theorem 3.3.1, $q(\lambda I - T) < \infty$ and hence, see [11, Theorem 3.4], $\text{ind}(\lambda I - T) \geq 0$, a contradiction. Analogously, if $\Lambda(T^*) \neq \emptyset$ and T has SVEP at $\lambda \in \Lambda(T^*)$ then, by duality, $\lambda I -$

$T \in \Phi_-(X)$ and hence by Theorem 1.2 $p(\lambda I - T) < \infty$. This implies that $\text{ind}(\lambda I - T) \leq 0$, again by [1, Theorem 3.4], and consequently $\text{ind}(\lambda I - T^*) = -\text{ind}(\lambda I - T) \geq 0$, which is impossible.

Corollary 3.3.2 *If either T or T^* has SVEP then a-Browder's theorem holds for both T and T^* .*

Proof. If T has SVEP (respectively, T^* has SVEP) the equivalence (3) entails that a-Browder's theorem holds for T (respectively, T^*). If T has SVEP then T^* satisfies Browder's theorem and this, together $\Lambda(T^*) = \emptyset$, implies that a-Browder's theorem for T^* . Analogously, if T^* has SVEP then Browder's theorem holds for T and this, together $\Lambda(T) = \emptyset$, implies that T satisfies a-Browder's theorem ■

3.4 Perturbation results of operator satisfies Browder's Theorem

Proposition 3.4.1 *From Fredholm theory we know that if $K \in K(X)$ then $\sigma_{wa}(T + K) = \sigma_{wa}(T)$. If $TK = KT$ we also have $\sigma_{ub}(T + K) = \sigma_{ub}(T)$, see [1, Theorem 3.43]. Hence T satisfies a-Browder's theorem if and only if $T + K$ satisfies a-Browder's theorem.*

- In some special cases the commutativity is not required Let Ω be a nonempty compact subset of \mathbb{C} . A hole is any bounded component of $\mathbb{C} \setminus \Omega$.

Theorem 3.4.1 *Suppose that $\sigma_{wa}(T)$ is simply connected (hence has not holes). Then $T + K$ satisfies a-Browder's theorem for all $K \in K(X)$.*

Proof. Let $\lambda \notin \sigma_{wa}(T)$ be arbitrary given. We show first that T has SVEP at λ , i.e. T satisfies a-Browder's theorem. Let

$$\Phi_{\pm}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_{\pm}(X)\}$$

denote the semi-Fredholm region of T . It is well known that the set $\Phi_{\pm}(T)$ is an open set of \mathbb{C} , so it can be decomposed in maximal connected components. Since $\lambda I - T \in W(X)$ there

is component Ω of $\Phi_{\pm}(T)$ containing the point λ . Since by assumption $\sigma_{wa}(T)$ is simple connected then Ω intersects the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. By [1, Theorem 3.63] then T has SVEP at every point of Ω , and in particular at λ . The equivalence (3) then implies that a-Browder's theorem holds for T . ■

Now suppose that $k \in K(X)$. Then $\sigma_{wa}(T) = \sigma_{wa}(T + k)$, which has no hole. By the result proved in the first part of the proof then $T + k$ satisfies a-Browder's theorem.

Example 3.4.1 *The previous result applies to some shift operators. Let $1 \leq p \leq \infty$ be given, and let $w := (w_n)$ be a bounded sequence of strictly positive real numbers. The corresponding unilateral weighted right shift on $\ell^p(\mathbb{N})$ is defined by*

$$Tx := \sum_{n=1}^{\infty} w_n x_n e_{n+1} \text{ for all } x = (x_n) \in \ell^p(\mathbb{N}),$$

where (e_n) is the standard basis of $\ell^p(\mathbb{N})$. In this case the spectral radius of T is given by

$$r(T) = \limsup_{n \rightarrow \infty} \sup_{k \in \mathbb{N}} (w_k \dots w_{k+n-1})^{\frac{1}{n}}.$$

Define

$$i(T) := \liminf_{n \rightarrow \infty} \inf_{k \in \mathbb{N}} (w_k \dots w_{k+n-1})^{\frac{1}{n}}$$

It is known that $\sigma_a(T) = \{\lambda \in \mathbb{C} : i(T) \leq |\lambda| \leq r(T)\}$, [35], so $iso \sigma_a(T) = \emptyset$. Moreover, T has SVEP [1, Theorem 2.82] and by [4, Corollary 2.10] we have $\sigma_{wa}(T) = \sigma_a(T)$. If $i(T) = 0$ then $\sigma_{wa}(T)$ has no holes, so a-Browder's theorem holds for every compact perturbation $T + K$ of T .

3.5 Characterizations of operators satisfying Browder's theorem via $\Delta(T)$

Theorem 3.5.1 [9] *For an operator $T \in L(X)$ the following statements are equivalent.*

1. T satisfies Browder's theorem;
2. Every $\lambda \in \Delta(T)$ is an isolated point of $\sigma(T)$;

3. $\Delta(T) \subseteq \partial\sigma(T), \partial\sigma(T)$ the topological boundary of $\sigma(T)$;

4. $\text{int}\Delta(T) = \emptyset$;

5. $\sigma(T) = \sigma_w(T) \cup \text{iso } \sigma(T)$.

Proof. **1** \Rightarrow **2** If T satisfies Browder's theorem then $\Delta(T) = p_{00}(T)$, and in particular every $\lambda \in \Delta(T)$ is an isolated point of $\sigma(T)$. ■

2 \Rightarrow **3** Obvious.

3 \Rightarrow **4** Clear, since $\text{int } \partial\sigma(T) = \emptyset$

4 \Rightarrow **5** Suppose that $\text{int}\Delta(T) = \emptyset$. Let $\lambda_0 \in \Delta(T) = \sigma(T) \setminus \sigma_w(T)$. We show first that $\lambda_0 \in \partial\sigma(T)$. Suppose that $\lambda_0 \notin \partial\sigma(T)$. Then there exists an open disc centred at λ_0 contained in the spectrum. Since $\lambda_0 I - T \in W(X)$ by the classical punctured neighborhood theorem there exists another open disc \mathbf{D} centered at λ_0 such that $\lambda I - T \in W(X)$ for all $\lambda \in \mathbf{D}$. Therefore $\lambda_0 \in \text{int}\Delta(T)$, which is impossible. This argument shows that $\sigma(T) = \sigma_w(T) \cup \partial\sigma(T)$.

Now, if $\lambda \in \partial\sigma(T)$ and $\lambda \notin \sigma_w(T)$ then $\lambda I - T \in W(X)$ and, since both T and T^* have SVEP at every point of $\partial\sigma(T) = \partial\sigma(T^*)$, by Theorem [3.3.1](#) we have $p(\lambda I - T) = q(\lambda I - T) < \infty$. Therefore λ is an isolated point of $\sigma(T)$ and this entails that $\sigma(T) = \sigma_w(T) \cup \text{iso } \sigma(T)$.

4 \Rightarrow **1** Suppose that $\sigma(T) = \sigma_w(T) \cup \text{iso } \sigma(T)$. Let $\lambda \notin \sigma(T) \setminus \sigma_w(T)$. Then $\lambda \in \text{iso } \sigma(T)$ (otherwise, $\lambda \notin \sigma_w(T) \cup \text{iso } \sigma(T) = \sigma(T)$, a contradiction). Since T and T^* have SVEP at every isolated point of $\sigma(T)$ and $\lambda I - T \in W(X)$ it then follows that $p(\lambda I - T) = q(\lambda I - T) < \infty$, so $\lambda \notin \sigma_b(T)$. Therefore $\sigma_b(T) = \sigma_w(T)$.

It is easily seen from definition that $\sigma_{wa}(T) \subseteq \sigma_a(T)$ for every $T \in L(X)$. From definition of SVEP we also have

(6)

$$\lambda \notin \text{acc } \sigma_a(T) \Rightarrow T \text{ has SVEP at } \lambda.$$

The following useful result shows that a-Browder's theorem holds for T precisely when $\sigma_a(T)$ is obtained by adding to $\sigma_{wa}(T)$ the isolated points of $\sigma_a(T)$.

Theorem 3.5.2 $T \in L(X)$ satisfies a-Browder's theorem if and only if

$$\sigma_a(T) = \sigma_{wa}(T) \cup iso \sigma_a(T). \quad (7)$$

Analogously, a-Browder's theorem holds for T^* if and only if

$$\sigma_s(T) = \sigma_{ws}(T) \cup iso \sigma_a(T). \quad (8)$$

Proof. The inclusion $\sigma_{wa}(T) \cup iso \sigma_a(T) \subseteq \sigma_a(T)$ holds for every $T \in L(X)$, so we need only to prove the reverse inclusion. Suppose that a-Browder's theorem holds. If $\lambda \in \sigma_a(T) \setminus \sigma_{wa}(T)$ then the equivalence (3) entails that T has SVEP at λ , and hence by [7, Theorem 2.2] $\lambda \in iso \sigma_a(T)$. Therefore $\sigma_a(T) \subseteq \sigma_{wa}(T) \cup iso \sigma_a(T)$, so the equality (7) is proved. Conversely, suppose that $\sigma_a(T) = \sigma_{wa}(T) \cup iso \sigma_a(T)$. Let $\lambda \notin \sigma_{wa}(T)$. There are two possibilities: $\lambda \in iso \sigma_a(T)$ or $\lambda \notin iso \sigma_a(T)$. By (6) if $\lambda \in iso \sigma_a(T)$ then T has SVEP at λ . In the other case $\lambda \notin \sigma_{wa}(T) \cup iso \sigma_a(T)$, and again by (6) we know that T has SVEP at λ . From the equivalence (4) we then conclude that a-Browder's theorem holds for T . The second assertion follows by duality, since $\sigma_s(T) = \sigma_a(T^*)$ and $\sigma_{ws}(T) = \sigma_{wa}(T^*)$ for every $T \in L(X)$. ■

Example 3.5.1 The operator $T := L \oplus L^* \oplus Q$ considered before Theorem 3.3.2 does not satisfy a-Browder's theorem since $\sigma_{wa}(T) = \partial \mathbf{D} \cup \{0\}$, $\sigma_a(T) = \mathbf{D}$ and hence $\sigma_a(T) \neq \sigma_{wa}(T) \cup iso \sigma_a(T)$.

Definition 3.5.1 We recall that reduced minimum modulus of a non-zero operator T is defined by

$$\gamma(T) := \inf_{x \notin \ker T} \frac{\|Tx\|}{\|x\|}.$$

Proposition 3.5.1 It is well-known that $T(X)$ is closed if and only if $\gamma(T) > 0$, see for instance [1, Theorem 1.13]

Definition 3.5.2 Let M, N denoted two closed linear subspaces of a Banach space X and define

$$\delta(M, N) := \sup \{ \text{dist}(u, N) : u \in M, \|u\| = 1 \}$$

in the case $M \neq \{0\}$, otherwise we put $\delta(\{0\}, N) = 0$ for any subspace N

- The gap between M and N defined by

$$\widehat{\delta}(M, N) := \max \{ \delta(M, N), \delta(N, M) \}.$$

- Note that the function $\widehat{\delta}$ is a metric on the set of all linear closed subspace of X , see [?, Chapter 2] and the convergence $M_n \rightarrow M$ is obviously defined by $\widehat{\delta}(M_n, M) \rightarrow 0$ as $n \rightarrow \infty$.

We recall some results from Fredholm theory. Suppose that $\lambda_0 \in \Delta(T)$. Then $\lambda_0 I - T \in \Phi(X)$ with $\text{ind}(\lambda_0 I - T) = 0$ and $0 < \alpha(\lambda_0 I - T)$. By the punctured neighborhood theorem [28, Theorem 5.31] there exists $\varepsilon > 0$ such that $\lambda I - T \in \Phi(X)$, $\alpha(\lambda I - T)$ is constant for all $0 < |\lambda - \lambda_0| < \varepsilon$,

$$\alpha(\lambda I - T) \leq \alpha(\lambda_0 I - T) = 0 \text{ for all } |\lambda - \lambda_0| < \varepsilon,$$

and

$$\text{ind}(\lambda I - T) = \text{ind}(\lambda_0 I - T) = 0 \text{ for all } |\lambda - \lambda_0| < \varepsilon.$$

Theorem 3.5.3 [9] *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

1. T satisfies Browder's theorem;
2. the mapping $\lambda \rightarrow \ker(\lambda I - T)$ is not continuous at every $\lambda \in \Delta(T)$ in the gap metric;
3. the mapping $\lambda \rightarrow \gamma(\lambda I - T)$ is not continuous at every $\lambda \in \Delta(T)$;
4. the mapping $\lambda \rightarrow (\lambda I - T)(X)$ is not continuous at every $\lambda \in \Delta(T)$ in the gap metric.

Proof. $1 \Rightarrow 2$ By Theorem 3.5.1 if T satisfies Browder's theorem then $\Delta(T) \subseteq \text{iso } \sigma(T)$. For every $\lambda_0 \in \Delta(T)$ we have $\alpha(\lambda_0 I - T) > 0$ and since λ_0 is an isolated point of $\sigma(T)$ there exists an open $\text{disc } \mathbf{D}(\lambda_0, \varepsilon)$ such that $\alpha(\lambda I - T) = 0$ for all $\lambda \in \mathbf{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. Therefore the mapping $\lambda \rightarrow \ker(\lambda I - T)$ is not continuous at λ_0 in the gap metric. $2 \Rightarrow 1$ Let $\lambda_0 \in \Delta(T)$ be arbitrary. By the punctured neighborhood theorem there exists an open

disc $\mathbf{D}(\lambda_0, \varepsilon)$ such that, $\lambda I - T \in \Phi(X)$ for all $\lambda \in \mathbf{D}(\lambda_0, \varepsilon)$, $\alpha(\lambda I - T)$ is constant as λ ranges on $\mathbf{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, $ind(\lambda I - T) = ind(\lambda_0 I - T)$ for all $\lambda \in \mathbf{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$,

$$ind(\lambda I - T) = ind(\lambda_0 I - T) \quad \text{for all } \lambda \in \mathbf{D}(\lambda_0, \varepsilon).$$

and

$$0 < \alpha(\lambda I - T) \leq \alpha(\lambda_0 I - T) \quad \text{for all } \lambda \in \mathbf{D}(\lambda_0, \varepsilon).$$

The discontinuity of the mapping $\lambda \rightarrow \ker(\lambda I - T)$ at every $\lambda \in \Delta(T)$ implies that

$$0 \leq \alpha(\lambda I - T) < \alpha(\lambda_1 I - T) \quad \text{for all } \lambda \in \mathbf{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}.$$

We claim that $\alpha(\lambda I - T) = 0$ for all $\lambda \in \mathbf{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. To see this, suppose that there is $\lambda_1 \in \mathbf{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$ such that $\alpha(\lambda_1 I - T) > 0$. Clearly, $\lambda_1 \in \Delta(T)$, so arguing as for λ_0 we obtain a $\lambda_2 \in \mathbf{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0, \lambda_1\}$ such that

$$0 < \alpha(\lambda_2 I - T) < \alpha(\lambda_1 I - T),$$

and this is impossible since $\alpha(\lambda I - T)$ is constant for all $\lambda \in \mathbf{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. Therefore $0 = \alpha(\lambda I - T)$ for $\lambda \in \mathbf{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, and since $\lambda I - T \in W(X)$ for all $\lambda \in \mathbf{D}(\lambda_0, \varepsilon)$ we conclude that $\alpha(\lambda I - T) = \beta(\lambda I - T) = 0$ for all $\lambda \in \mathbf{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. Hence $\lambda_0 \in iso \sigma(T)$, thus T satisfies Browder's theorem by Theorem [3.5.1](#). To show the equivalences of assertions 1, 3 and 4 observe first that since for every $\lambda_0 \in \Delta(T)$ we have $\lambda_0 I - T \in \Phi(X)$ and hence the range $(\lambda I - T)(X)$ is closed for all λ near to λ_0 . The equivalences $2 \iff 3 \iff 4$ then follow from [\[1, Theorem 1.38\]](#) ■

Definition 3.5.3 A bounded operator $T \in L(X)$ is said to be semi-regular (or also a Kato operator) if $\ker T^n \subseteq T(X)$ for all $n \in \mathbb{N}$ and $T(X)$ is closed. The Kato spectrum is defined by

$$\sigma_{se}(T) := \{\lambda I - T \text{ is not semi-regular}\}.$$

Note that $\sigma_{se}(T)$ is non-empty compact subset of \mathbb{C} , since it contains the boundary of the spectrum, see [\[1, Theorem 1.38\]](#)

A bounded operator $T \in L(X)$ is said to admit a generalized inverse $S \in L(X)$ if $TST = T$. It is well known that T admits a generalized inverse if and only if both the

subspaces $\ker T$ and $T(X)$ are complemented in X . It is well-known that every Fredholm operator admits a generalized inverse, see [1, Theorem 7.3]. A "complemented" version of Kato operators is given by the Saphar operators: $T \in L(X)$ is said to be Saphar if T is semi-regular and admits a generalized inverse. The Saphar spectrum is defined by

$$\sigma_{sa}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Saphar}\}.$$

Clearly, $\sigma_{se}(T) \subseteq \sigma_{sa}(T)$, so $\sigma_{sa}(T)$ is nonempty, for other properties on Saphar operators see [38, Chapter 2]

Theorem 3.5.4 [9] *For a bounded operator T each of the following statements is equivalent to Browder's theorem:*

1. $\Delta(T) \subseteq \sigma_{se}(T)$;
2. $\Delta(T) \subseteq iso \sigma_{se}(T)$;
3. $\Delta(T) \subseteq \sigma_{sa}(T)$;
4. $\Delta(T) \subseteq iso \sigma_{sa}(T)$.

Proof. By [1, Theorem 7.3] the equivalent conditions of Theorem 3.5.3 are equivalent to saying that $\lambda I - T$ is not semi-regular for all $\lambda \in \Delta(T)$. $1 \Leftrightarrow 2$ The implication $2 \Rightarrow 1$ is obvious. To show that $1 \Rightarrow 2$ suppose that $\Delta_a(T) \subseteq \sigma_{se}(T)$. If $\lambda_0 \in \Delta_a(T)$ then $\lambda_0 I - T \in \Phi_+(X)$ so $\lambda_0 I - T$ is essentially semi-regular, in particular of Kato type. By [1, Theorem 1.44] there exists an open *dis* $\mathbf{D}(\lambda_0, \varepsilon)$ such that $\lambda I - T$ is semi-regular for all $\lambda \in \mathbf{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. But $\lambda_0 \in \sigma_{se}(T)$, so $\lambda_0 \in iso \sigma_{se}(T)$.

$1 \Leftrightarrow 3$ The implication $1 \Rightarrow 3$ is immediate, since $\sigma_{se}(T) \subseteq \sigma_{sa}(T)$. To show the implication $3 \Rightarrow 1$ suppose that $\Delta(T) \subseteq \sigma_{sa}(T)$. Let $\lambda \in \Delta(T)$. Then $\alpha(\lambda I - T) < \infty$ and since $\lambda I - T \in W(X)$ it follows that $\beta(\lambda I - T) < \infty$. Clearly, $\ker(\lambda I - T)$ is complemented, since it is a finite-dimensional, and $(\lambda I - T)(X)$ is complemented, since it is closed and finite-codimensional. Therefore T admits a generalized inverse, and from $\lambda \in \sigma_{sa}(T)$ it then follows that $\lambda I - T$ is not semi-regular. Thus $\Delta(T) \subseteq \sigma_{se}(T)$.

$4 \Rightarrow 3$ Obvious

$2 \Rightarrow 4$ Let $\lambda_0 \in \Delta(T)$. Since $\lambda_0 I - T \in W(X)$, it there exists an open *disc* \mathbf{D} centered at λ_0 such that $\lambda I - T \in W(X)$ for all $\lambda \in \mathbf{D}$, so $\lambda I - T$ is Fredholm and hence admits a generalized inverse for all $\lambda \in \mathbf{D}(\lambda_0, \varepsilon)$. On the other hand, λ_0 is isolated in $\sigma_{se}(T)$, so $\lambda_0 \in iso \sigma_{sa}(T)$. ■

We now establish some characterization of operators satisfying Browder's theorem in terms of the quasi-nilpotent parts $H_0(\lambda I - T)$.

Theorem 3.5.5 *For a bounded opertor $T \in L(X)$ Browder's theorem holds precisely when one of the following statements holds*

1. $H_0(\lambda I - t)$ is finite-dimentional for every $\lambda \in \Delta(T)$;
2. $H_0(\lambda I - T)$ is closed for all $\lambda \in \Delta(T)$;
3. $K(\lambda I - T)$ is finite-codimentional for all $\lambda \in \Delta(t)$.

Proof. Suppose that T satisfies Browder's theorem. By Theorem 3.3.2 then

$$\Delta(T) = p_{00}(T) = \sigma(T) \setminus \sigma_b(T).$$

If $\lambda \in \Delta(T)$ then $\lambda I - T \in B(X)$, so λ is isolated in $\sigma(T)$ and hence T has SVEP at λ . We also have that $\lambda I - T \in \Phi(X)$, so, from [5, Theorem 2.6] and [5, Theorem 2.8] we conclude that $H_0(\lambda I - T)$ is finite-dimentional. Conversely, suppose that $H_0(\lambda I - T)$ is finite-dimentional for every $\lambda \in \Delta(T)$. By Theorem 3.3.1 then T has SVEP at every $\lambda \in \Delta(T)$ and since $\lambda I - T \in W(X)$ we also have, again by Theorem 3.3.1 that $p(\lambda I - T) < \infty$. Since $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$ it then follows that $q(\lambda I - T) < \infty$, hence $\lambda I - T \in B(X)$ for all $\lambda \in \Delta(T)$. Hence $\lambda \notin \sigma_b(T)$. On the other hand, $\Delta(T) \subseteq \sigma(T)$, so $\Delta(T) \subseteq p_{00}(T)$. By Lemma 3.3.1, it follows that $\Delta(T) = p_{00}(T)$, and by Theorem 3.3.2 we then conclude that T satisfies Browder's Theorm. The equivalence $a \Leftrightarrow b$ is clear by Theorem 3.5.3. To show the equivalence $a \Leftrightarrow c$ observe that, by Theorem 3.5.1, Browder's theorem is equivalent to saying that every $\lambda \in \Delta(T)$ is an isolated point of $\sigma(T)$. Let P denote the spectral projection associated with λ . Then $P(X) = H_0(\lambda I - T)$ and $\ker P = k(\lambda I - T)$, see [1, Theorem 3.74]. Hence $H_0(\lambda I - T)$ is finite-dimentional if and only if $K(\lambda I - T)$ is

finite-codimensional. According to Coburn [?, Theorem 3.74] a bounded operator $T \in L(X)$ satisfies Weyl's theorem if

$$\Delta(T) = \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T).$$

■

- A large number of the commonly considered classes of operators satisfying Weyl's theorem, amongst them hyponormal operators, paranormal operators, multipliers of commutative semi-simple Banach algebras, see for instance [8],[45]. The following result is known, we shall give a proof for sake of completeness.

Theorem 3.5.6 [9] *If a bounded operator $T \in L(X)$ satisfies Weyl's theorem then*

$$p_{00}(T) = \pi_{00}(T) = \Delta(T).$$

Proof. Suppose that T satisfies Weyl's theorem. By definition then $\Delta(T) = \pi_{00}(T)$. We show now the equality $p_{00}(T) = \pi_{00}(T)$. It suffices to prove the inclusion $\pi_{00}(T) \subseteq p_{00}(T)$. Let λ be an arbitrary point of $\pi_{00}(T)$. Since λ is isolated in $\sigma(T)$ then T has SVEP at λ and from the equality $\pi_{00}(T) = \sigma(T) \setminus \sigma_w(T)$ we know that $\lambda I - T \in W(X)$. Hence $\lambda I - T \in \Phi(X)$ and the SVEP at λ by [5, Corollary 2.7] implies that $p(\lambda I - T) < \infty$, so $\lambda \in p_{00}(T)$. ■

Example 3.5.2 *From Theorem 3.5.6 it easily follows that for an operator Weyl's theorem implies Browder's theorem. It is not difficult to find an example of operator satisfying Browder's theorem but not Weyl's theorem. For instance, if $T \in L(\ell^2(\mathbb{N}))$ is defined by*

$$T(x_0, x_1, \dots) = \left(\frac{1}{2}x_1, \frac{1}{3}x_2, \dots \right) \text{ for all } (x_n) \in \ell^2(\mathbb{N}),$$

then T is quasi-nilpotent, so T has SVEP and consequently satisfies Browder's theorem. On the other hand T does not satisfy Weyl's theorem, since $\sigma(T) = \sigma_w(T) = \{0\}$ and $\pi_{00}(T) = \{0\}$.

Let us define

$$\Delta_{00}(T) := \Delta(T) \cup \pi_{00}(T).$$

Theorem 3.5.7 [9] Let $T \in L(X)$. Then the following statements are equivalent:

1. T satisfies Weyl's theorem;
2. T satisfies Browder's theorem and $p_{00}(T) = \pi_{00}(T)$;
3. the map $\lambda \rightarrow \gamma(\lambda I - T)$ is not continuous at every $\lambda \in \Delta_{00}(T)$;

Proof. The implication $1 \Rightarrow 2$ is clear, from Theorem 3.5.6, whilst the implication $2 \Rightarrow 1$ follows immediately from Theorem 3.3.2. Therefore 1 and 2 are equivalent.

$1 \Rightarrow 3$ By Theorem 3.5.6 we have that $\Delta_{00}(T) = \Delta(T)$ and T satisfies Browder's theorem. Therefore, by Theorem 3.5.3, the mapping $\lambda \rightarrow \gamma(\lambda I - T)$ is not continuous at every point $\lambda \in \Delta_{00}(T)$.

$3 \Rightarrow 2$ Suppose that $\lambda \rightarrow \gamma(\lambda I - T)$ is not continuous at every $\lambda \in \Delta_{00}(T) = \Delta(T) \cup \pi_{00}(T)$. The discontinuity at the points of $\Delta(T)$ entails, by Theorem 3.5.3, that T satisfies Browder's theorem, whilst the discontinuity at the points of $\pi_{00}(T)$ is equivalent to saying $\pi_{00}(T) \cap \sigma_b(T) = \emptyset$, see [3]. The last equality obviously implies that $\pi_{00}(T) \subseteq p_{00}(T)$, hence $\pi_{00}(T) = p_{00}(T)$.

The equivalent $1 \Leftrightarrow 2$ of Theorem 3.5.7 is owed to [10]. From Theorem 3.5.3 and Theorem 3.5.7 we see that Browder's theorem and a-Weyl's theorem equivalent to the discontinuity of the mapping $\lambda \rightarrow \gamma(\lambda I - T)$ at the points of two sets $\Delta(T)$ and $\Delta_{00}(T)$, respectively, with $\Delta_{00}(T)$ large than $\Delta(T)$. Note that the discontinuity of the mapping $\lambda \rightarrow \ker(\lambda I - T)$ at every $\lambda \in \Delta_{00}(T) = \Delta(T) \cup \pi_{00}(T)$ does not imply Weyl's theorem. In fact, since every point of $\pi_{00}(T)$ is an isolated points of $\sigma(T)$, it is evident that that the map $\lambda \rightarrow \ker(\lambda I - T)$ is not continuous at every $\lambda \in \pi_{00}(T)$ for all operators $T \in L(X)$. therefore, the discontinuity of the mapping $\lambda \rightarrow \ker(\lambda I - T)$ at every $\lambda \in \Delta_{00}(T)$ is equivalent to the discontinuity of the same mapping at the points of $\Delta(T)$, i.e. to Browder's theorem for T . ■

Theorem 3.5.8 [9] For $T \in L(X)$ the following statements are equivalent:

1. T satisfies Weyl's theorem;
2. $H_0(\lambda I - T)$ is finite-dimensional for all $\lambda \in \Delta_{00}(T)$;

3. $K(\lambda I - T)$ is finite-dimensional for all $\lambda \in \Delta_{00}(T)$.

Proof. $1 \Rightarrow 2$ If T satisfies Weyl's theorem then T satisfies Browder's theorem and $\pi_{00}(T) = p_{00}(T) = \Delta(T)$, by Theorem 3.5.6. By Theorem 3.5.5 then $H_0(\lambda I - T)$ is finite-dimensional for all $\lambda \in \Delta_{00}(T) = \Delta(T)$.

$2 \Rightarrow 1$ Since $\Delta(T) \subseteq \Delta_{00}(T)$ then Browder's theorem holds for T by Theorem 3.5.5. From $\pi_{00}(T) \subseteq \Delta_{00}(T)$ we know that $H_0(\lambda I - T)$ is finite-dimensional for every $\lambda \in \pi_{00}(T)$. Since every $\lambda \in \pi_{00}(T)$ is an isolated point $\sigma(T)$ by [1, Theorem 3.77] it then follows that $\lambda I - T$ is Browder's. Therefore $\pi_{00}(T) = p_{00}(T)$, so by Theorem 3.5.7 satisfies Weyl's theorem.

$3 \Leftrightarrow 2$ The equivalence follows from Theorem 3.5.5 and from the fact that for any point of $\lambda \in \pi_{00}(T)$ we have $X = H_0(\lambda I - T) \oplus K(\lambda I - T)$. In contrast with Theorem 3.5.5 the condition 2 of Theorem 3.5.8 cannot be replaced by the formally weaker condition that $H_0(\lambda I - T)$ is closed for all $\lambda \in \Delta_{00}(T)$. Indeed, if T is defined as Example 3.5.2 then $0 \in \pi_{00}(T)$, T does not satisfy Weyl's theorem, whilst $H_0(\lambda I - T) = \ell^2(\mathbb{N})$ is closed, since

T is quasi-nilpotent ■

Bibliographie

- [1] P. Aiena Fredholm and local spectral theory, with application to multipliers. Kluwer Acad. Publishers (2004).
- [2] P. Aiena, M. T. Biondi Some spectral mapping theorem for semi-Browder spectra through local spectral theory. *Rend. Circ. Mat. Palermo* **53**, (2004), 165-184.
- [3] P. Aiena, C. Carpintero Weyl's theorem, a-Weyl's theorem and single-valued extension property, (2003) To appear on *Extracta Math.*
- [4] P. Aiena, C. Caprinto, E. Rosas Some characterizations of operators satisfying a-Browder theorem. (2004), submitted.
- [5] P. Aiena, M. L. Colasante, M. Gonzalez Operators which have a closed quasi-nilpotent part, *Proc. Amer. Math. Soc.* **130**, (9) (2002), 2701-2710.
- [6] P. Aiena, T. L. Miller, M. M. Newmann On a localized single-valued extension property, *Math. Proc. Royal Ir. Acad.* **104A** 1, (2004), p. 17-34.
- [7] P. Aiena, E. Rosas The single-valued extension property at the points
- [8] P. Aiena, F. Villafañe Weyl's theorem for some classes of operators, *Int. Eq. and Oper. Theory*, (2003)
- [9] P. Aiena, B., Maria, Browder's Theorems through Localized SVEP. *Mediterranean Journal of Mathematics.* 2. 137-151. (2005)
- [10] B.A Barnes, Riesz points and Weyl's theorem, *Integral Equation Operators Theory* 34(1999), 187-196.

-
- [11] M. Barkani, On a class of quasi-Fredholm operators, *Integral Equations Operator Theory* **34**(1999), 244-249.
- [12] M. Berkani, Index of B-Fredholm operators and generalization of a Weyl's theorem, *Proc.Amer. Math. Soc.* to appear.
- [13] M. Berkani, B-Weyl spectrum and poles of the resolvent, submitted.
- [14] M.Berkani and M. Sarih, On semi-B-Fredholm operators, *Glasgow Math. J.*, to appear.
- [15] Berkani and M.Sarih, An Atkinson-type theorem for B-Fredholm operators, *Studia Math.*, to appear.
- [16] M. Berkani, Restriction of operators to the range of its powers, *Studia Math.***140** (2000), 163-175.
- [17] L. A. Coburn Weyl's theorem for nonnormal operators *Michigan Math. J.* **20**
- [18] S. V. Djordjevic and Y. Han, Browders theorem and spectral continuity, *Glasgow Math. J.* **42**(2000), 479-486.
- [19] M. P. Drazin, Pseudoinverse in associative rings and semigroups, *Amer. Math. Monthly* **65** (1958), 506-514.
- [20] N. Dunford Spectral theory I. Resolution of identity *Pacific J. Math.* **2** (1952), 559-614.
- [21] N. Dunford spectral operators *Pacific J. Math.* **4** (1954), 321-354.
- [22] J. K. Finch The single valued extension property on a Banach space *Pacific J. Math.* **58** (1975), 61-69.
- [23] S. Grabiner, Uniform ascent and descent of bounded operators, *J. Math. Soc. Japan* **34** (1982), 317-337.
- [24] R.Harte, Woo Yong Lee Another note on Weyl's theorem *Trans.Amer. Math. Soc.* **349** (1997), 2115-2124.

-
- [25] R. Harte, Woo Young Lee, L. L. Littlejohn On generalized Riesz points *J Operator Theory* **47** (2002), 187-196.
- [26] H. Heuser *Functional Analysis* (1982), Marcel Dekker, New York.
- [27] A. Jeribi, *Spectral theory and application of linear operators and blocks operator Matrices*, University of Sfax, Sfax, Tunisia, ISBN 978-3-319-17566-9(ebook)(2016).
- [28] T. Kato Perturbation theory for nullity, deficiency and other quantities of linear operators. *J. anal. Math.* **6** (1958), 261-322.
- [29] J. J. Koliha, Isolated spectral points, *Proc. Amer. Math. Soc.* **124** (1996), 3417-3424.
- [30] J. J. Koliha, A generalized Drazin inverse, *Glasgow Math.* **38** (1996), **367-381**.
- [31] J. J. Koliha and P. W. Poon, On West and Stampfli decomposition, *Acta Sci. M(Szeged)* **63** (1997), 181-194.
- [32] V. Kolrdula and V. Müller, On the axiomatic theory of the spectrum, *Studia Math.* **11** (1996), 109-128.
- [33] J. Ph. Labrousse, Les opérateurs quasi-Fredholm: une généralisation des opérateurs semi-Fredholm, *Rend. Circ. Math. Palermo (2)* **29** (1980), **161-258**.
- [34] D. C. Lay, Spectral analysis using ascent, descent, nullity and defect, *Math. Ann.* **184** (1970), 197-214.
- [35] K. B. Laursen, M. M. Neumann introduction to local spectral theory. Clarendon Press, Oxford 2000.
- [36] M. Mbekhta and V. Müller, On the axiomatic theory of the spectrum, **II**, *Studia Math.* **119** (1999), 129-147.
- [37] M. Mbekhta Sur la théorie spectrale local et limite des nilpotent. *Proc. Amer. Math. Soc.* **110** (1990), 621-631.
- [38] M. Muller Spectral theory of linear operators. *Operators Theory, Advances and Applications* **139**, Birkhäuser 2003.

- [39] V. Rakočević, Approximate point spectrum and commuting compact perturbations, Glasgow Math. J. **28** (1986), 193-198.
- [40] V. Rakočević, Operators obeying a-Weyl's theorem, Rev. Roumaine Math. Pures Appl. **10** (1989), 915-919.
- [41] M. Shechter, Principles of functional analysis, Academic Press, New York,
- [42] J. G. Stampfli, Compact perturbations, normal eigenvalues, and a problem of Salinas, J. London Math. Soc. (2) **9** (1974), 165-175.
- [43] A. E. Taylor, Theorems on ascent, descent, nullity and defect of linear operators, Math. Ann. **163** (1966), 18-49.
- [44] H. Weyl, Über beschränkte quadratische Formen, deren Differenz vollstetig ist, Rend. Circ. Mat. Palermo **27** (1909), 373-392.
- [45] M. Oudghiri, Weyl's and Browder's theorem for operators satisfying the SVEP. Studia Math. **163**, 1, (2004), 85-101.
- [46] M. Berkani, J. Koliha, Weyl type theorems for bounded linear operators. Acta Sci. Math. (Szeged). **69**, 359-376, (2003).

Abstract

The Weyl and Browder theorems stand as pivotal landmarks in the realm of functional analysis, offering profound insights into the spectral behavior and structural properties of bounded linear operators. Pietro Aiena's seminal contributions have significantly enriched our understanding of these theorems, unraveling their intricate connections and extending their applicability to various classes of operators. In this research endeavor, we embark on a comprehensive exploration of Aiena's work, delving deep into the nuanced implications and far-reaching consequences of the Weyl and Browder theorems. Through a meticulous examination of Aiena's results, we elucidate the underlying principles governing the spectral decomposition, essential spectra, and related spectral properties of bounded linear operators. Moreover, we investigate the interplay between these theorems and other fundamental concepts in functional analysis, such as compact operators, spectral mapping theorems, and perturbation theory. By synthesizing Aiena's insights with contemporary developments in the field, we aim to provide a unified framework for understanding the spectral theory of bounded linear operators, with implications for diverse areas including operator theory, mathematical physics, and dynamical systems. Our research not only consolidates and extends Aiena's seminal contributions but also sheds light on new avenues for theoretical exploration and practical applications in functional analysis and its myriad interdisciplinary ramifications.

المخلص

تعتبر نظريات ويل و براودر بمثابة معالم محورية في مجال التحليل الوظيفي , حيث تقدم رؤى عميقة حول السلوك الطيفي والخصائص الهيكلية للعوامل الخطية المحددة. لقد اثرت مساهمات بيترو ايينا الأساسية في فهمنا لهذه النظريات بشكل كبير, وكشفت اتصالاتها المعقدة ووسعت نطاق تطبيقها على فئات مختلفة من المشغلين في هذا المسعى البحثي نبدا في استكشاف شامل لعمل ايينا ونتعمق في الاثار الدقيقة والعواقب بعيدة المدى لنظريات ويل وبراودر. من خلال الفحص الدقيق لنتائج ايينا نوضح المبادئ الأساسية التي تحكم التحلل الطيفي والاطياف الأساسية والخصائص الطيفية ذات الصلة للمشغلين الخطيين المحددين. علاوة على ذلك فاننا ندرس التفاعل بين هذه النظريات والمفاهيم الأساسية الأخرى في التحليل الوظيفي مثل العوامل المدمجة ونظريات رسم الخرائط الطيفية ونظرية الاضطراب. من خلال تجميع رؤى ايينا مع التطورات المعاصرة في هذا المجال, فاننا نهدف الى توفير اطار موحد لفهم النظرية الطيفية للمشغلين الخطيين المحددين, مع اثار على مجالات متنوعة بما في ذلك نظرية المشغل, الفيزياء الرياضية والأنظمة الديناميكية. ان بحثنا لا يعزز ويوسع مساهمات ايينا الأساسية فحسب بل يسלט الضوء على طرق جديدة للاستكشاف النظري والتطبيقات العملية في التحليل الوظيفي وتشبعاته المتعددة التخصصات التي لا تعد ولا تحصى.

Résumé

Les théorèmes de Weyl et Browder constituent des jalons essentiels dans le domaine de l'analyse fonctionnelle, offrant des informations approfondies sur le comportement spectral et les propriétés structurelles des opérateurs linéaires bornés. Les contributions fondamentales de Pietro Aiena ont considérablement enrichi notre compréhension de ces théorèmes, démêlant leurs liens complexes et étendant leur applicabilité à diverses classes d'opérateurs. Dans cet effort de recherche, nous nous

lançons dans une exploration globale des travaux d'Aiena, en approfondissant les implications nuancées et les conséquences profondes des théorèmes de Weyl et de Browder. Grâce à un examen minutieux des résultats d'Aiena, nous élucidons les principes sous-jacents régissant la décomposition spectrale, les spectres essentiels et les propriétés spectrales associées des opérateurs linéaires bornés. De plus, nous étudions l'interaction entre ces théorèmes et d'autres concepts fondamentaux de l'analyse fonctionnelle, tels que les opérateurs compacts, les théorèmes de cartographie spectrale et la théorie des perturbations. En synthétisant les connaissances d'Aiena avec les développements contemporains dans le domaine, nous visons à fournir un cadre unifié pour comprendre la théorie spectrale des opérateurs linéaires bornés, avec des implications pour divers domaines, notamment la théorie des opérateurs, la physique mathématique et les systèmes dynamiques. Notre recherche non seulement consolide et étend les contributions fondamentales d'Aiena, mais met également en lumière de nouvelles voies d'exploration théorique et d'applications pratiques en analyse fonctionnelle et ses innombrables ramifications interdisciplinaires.