PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESERACH

Mohamed Boudiaf University of M'sila
Faculty of Mathematics and Informatics
Departement of Mathematics

Master of Mathematics

Mathematics and Informatics
Specialty: Mathematics
Option : Mathematical and Numerical Analysis

Theme

Modified Homotopy Perturbation Method For Solving Fractional Differential Equations

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Publicly presented on : 20/06/2022.

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University years: 2021/2022.

Acknowledgement

I can't begin and finish my work without thanking my praise and gratitude the greatest Allah the almighty who granted me strength and patience to complete this work

First and foremost,I am indebted to my supervisor **DR.KHIRANI Amina** for here patience,valuable advice and guidance and consistent encouragement that she provided throughout this research,and for the excellent example she has provided as a successful female mathematician and Doctor,I am honored to have here throughout my thesis studies.

I am also deeply grateful to **Dr.ABDELKEBIR saad** for his help during the realization of this work and his remarks and suggestions which allowed me to finalize this thesis.Furthermore, my sincere thanks also go to the Jury members who will be kind enough to read and judge this modest work.

Finally,I would like to extend my thanks and appreciation To **KOUIDER khaoula** and to everyone who extended a helping hand to stir up our path, to everyone who taught us useful knowledge, beginning with my teachers Primary school up to university professors, spacial thanks and gratitude to my family for their help,support and patience,to all my loved ones my friends specially **BOUDEROUAZE khadidja** and to everyone who contributed to the completion of this work from near or from far,who did not skimp on us with his information and capabilities to prepare this thesis

Dedicating

To the one who woven my happiness with threads woven from her heart

" My Mother"

To the one who keeps the thorns away from my path to facilitate the path of knowledge for me

"My Father"

To those who shared my childhood with them and loved me with sincerity and my support in life,

"My Brothers and Sisters"

To those who walked with me together as we paved the road to success together, here we are picking the flower of our effort to

"My Friends and Colleagues"

To those who from their knowledge made me a path and from their thoughts a beacon illuminating the path of knowledge and success for me

" My Honorable Professors"

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Introduction

Fractional Differential Equations FDEs has recently evolved as an interesting and popular field of research, In fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes, FDEs play important roll in different area , for example the nonlinear oscillation of earthquake can be modeled with fractional derivatives and the fluid dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow.

The importance of obtaining the exact and approximate solutions of FDEs in mathematics is still a significant problem that needs new methods to discover exact and approximate solutions. In recent years, many research workers have paid attention to study the solutions of FDEs by using various methods like Laplace transformation LT and Variational Iteration method VIM and so many others, Among these methods the Homotopy Perturbation Method HPM

The Homotopy Perturbation Method proposed first by He in 1997 and systematical description in 2000 which is in fact a coupling of the traditional perturbation method and homotopy in topology,The purpose of this method is to obtain an approximate solution for FDEs, Odibat and Momani (2007)applied modifications of He's HPM for solving nonlinear FDEs,these modifications reduces the nonlinear FDEs to a set of linear ordinary FDEs,the importance of these modifications is create a more efficient solution approach which move within rapid iteration process in order to get faster solution function.

The work in this thesis is divided into three chapter, the first chapter we define some primary definitions and basic concepts related to fractional calculus and their proprieties in addition to that we introducing some fractional differential equations that we will use it in the other chapters , we also studied the existence and uniqueness of the solution after that we show two kind of resolution methods, analytic methods (Laplace transformation Method LTM and Variational Iteration method VIM) and numerical methods(Rectangle Method and Trapezoidal Method),in the second chapter we present the Homotopy Perturbation Method and how it's work ,next we applied two modifications of Odibat and Momani on this method. Finaly,in the third chapter we applied these modifications on linear(nonlinear)ordinary (partial) differential equation with fractional order

Chapter 1

Fractional Differential Equations

In this chapter we give some primary definitionswith their proprieties and basic concepts related to fractional calculus ,also we introduce some fractional differential Equations that we need it in this work,after that we prove the existence and uniqueness of the solution of the FDEs, finally we give two kind of resolution methods(Analytical,Numerical).

1.1 Fractional Calculus

We give some basic definitions and properties of fractional calculus theory which will be used in this work

1.1.1 Special Functions of Fractional Calculus :

1. ***Euler Gamma Function :***

Definition 1.1 *The Euler gamma function is defined by the so-called Euler integral of the second kind and is given by:*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt = 0$$

where $t^{z-1} = e^{(z-1)\ln t}$*. this integral is converge for all complex* z*,*$(\Re(z) > 0)$*.*

Proprieties 1.2 *We give some important properties of the Euler gamma function:*
1. $\Gamma(1) = 1, \Gamma(0^+) = +\infty, \Gamma(\frac{1}{2}) = \sqrt{\pi}$
2. $\Gamma(\alpha)$*is a monotonous and strictly decreasing for* $0 < \alpha \leqslant 1$

3. $\Gamma(\alpha)$*is monotonous and strictly increasing function for* $\alpha \geqslant 2$.

4. By integrating by part, we obtain

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \quad \Re(\alpha) > 0$$

5. this is one of the basic properties of the Euler gamma function :

$$\Gamma(n+1) = n\Gamma(n) = n! \quad \forall n \in \mathbb{N}$$

2. ***Mittag-Leffler Function :***

Definition 1.3 *The standard definition of the Mittag-Leffler function is given by:*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad \alpha > 0, \beta \in \mathbb{R}$$

Spacial Values:

1. $E_{0,1}(z) = \dfrac{1}{1-z}$
2. $E_{1,1}(z) = \sum_{k=0}^{\infty} \dfrac{z^k}{\Gamma(k+1)} = e^z$
3. $E_{1,m}(z) = \dfrac{1}{z^{m-1}} \left[e^z - \sum_{k=0}^{m-2} \dfrac{z^k}{k!} \right]$
4. $E_{2,1}(z) = \cosh\sqrt{z}$

1.1.2 Basic Fractional Integrals and Derivatives :

1. ***Fractional Integral :***

Generally speaking, the fractional integral mainly means (fractional) Riemann-Liouville integral.

Riemann-Liouvill fractional integral RLFI :

Definition 1.4

$$_aI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0 \quad , x > a$$

Proprieties 1.5 *we have:*

1. $_aI_x^{\alpha} cf(x) = c\, _aI_x^{\alpha} f(x)$

2. $_aI_x^{\alpha}(_aI_x^{\beta} f(x)) = {_aI_x^{\alpha+\beta}} f(x) = {_aI_x^{\beta}}(_aI_x^{\alpha} f(x))$

3. $_aI_x^{\alpha}\{f(x) + g(x)\} = {_aI_x^{\alpha}} f(x) + {_aI_x^{\alpha}} g(x)$

RLFI of special functions:

1. ${}_aI_x^\alpha(C) = \dfrac{C(x-a)^\alpha}{\alpha\Gamma(\alpha)}$
2. ${}_aI_x^\alpha(x^n) = \dfrac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}x^{n+\alpha}$
3. ${}_{-\infty}I_x^\alpha(e^{kx}) = \dfrac{e^{kx}}{k^\alpha}$
4. ${}_0I_x^\alpha(\cos(x)) = \sum_{k=0}^{\infty}\dfrac{(-1)^k x^{2k+1}}{\Gamma(2k+\alpha+1)}, \quad \alpha = 1: \quad {}_0I_x^1(\cos(x)) = \sin(x)$
5. ${}_0I_x^\alpha(\sin(x)) = \sum_{k=0}^{\infty}\dfrac{(-1)^k x^{2k+1+\alpha}}{\Gamma(2k+\alpha+2)}, \quad \alpha = 1: \quad {}_0I_x^1(\sin(x)) = -\cos(x)+1$

2. ***Fractional Derivatives:***

2.1. Riemann-Liouvill Fractional Derivative RLFD:

Definition 1.6 *the Riemann-Liouville fractional derivative or Riemann-Liouville fractional differential operator of order α is The popular definition of fractional derivative defined as:*

$$ {}_a^{RL}D_x^\alpha f(x) = \frac{d^n}{dx^n}\left[{}_aI_x^{n-\alpha}f(x)\right] = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\left(\int_a^x (x-t)^{n-\alpha-1}f(t)dt\right) \quad \alpha \in \mathbb{R}^+, $$
$$ n-1 \leqslant \alpha < n, \quad x > a. $$

Proprieties 1.7 *we have:*

1. ${}_a^{RL}D_x^\alpha(C) = \dfrac{C}{\Gamma(1-\alpha)(x-a)^\alpha} \neq 0 \quad a > 0$

2. ${}_a^{RL}D_x^\alpha(x^n) = \dfrac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha} \quad a > 0, \quad n > -1$

3. ${}_{-\infty}^{RL}D_x^\alpha(e^{kx}) = k^\alpha e^{kx} \quad \alpha > 0, \quad k > 0.$

4. ${}_{-\infty}^{RL}D_x^\alpha(\sin(kx)) = k^\alpha \sin(kx + \alpha\dfrac{\pi}{2}) \quad \alpha > 0$

5. ${}_{-\infty}^{RL}D_x^\alpha(\cos(kx)) = k^\alpha \cos(kx + \alpha\dfrac{\pi}{2}) \quad \alpha > 0$

2.2. Caputo Fractional Derivative CFD :

Definition 1.8 *caputo fractional derivative or caputo fractional differential operator of order α is The second popular definition of fractional derivative defined as:*

$$ {}_a^cD_x^\alpha f(x) = {}_aI_x^{n-\alpha}\left[\frac{d^n}{dx^n}f(x)\right] \quad \alpha \in \mathbb{R}^+, \quad n-1 \leqslant \alpha < n, \quad x > a. $$

Proprieties 1.9 *we have:*

1. ${}_0^cD_x^\alpha(C) = 0 \quad \alpha > 0$

2. ${}_0^cD_x^\alpha(x^n) = \dfrac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha} \quad a > 0, \quad n > -1$

3. ${}_0^cD_x^\alpha(e^{kx}) = k^n x^{n-\alpha} E_{1,n-\alpha}(kx) \quad \alpha > 0, \quad k > 0.$
4. ${}_0^cD_x^\alpha(\sin(kx)) = -\frac{i}{2}(ik)^n x^{n-\alpha}[E_{1,n-\alpha+1}(ikx) - (-1)^n E_{1,n-\alpha+1}(-ikx)]$
5. ${}_0^cD_x^\alpha(\cos(kx)) = \frac{i}{2}(ik)^n x^{n-\alpha}[E_{1,n-\alpha+1}(ikx) + (-1)^n E_{1,n-\alpha+1}(-ikx)]$

1.1.3 The Laplace Transform:

Definition 1.10 *Consider a function $f(t)$,the Laplace transform of $f(t)$ is defined as :*

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s) \quad s > 0.$$

where s is the transform variable and $F(s)$ is called Laplace image of the function $f(t)$.the sufficient conditions for the existence of $L\{f(t)\}$ are that :

- *the defining integral converges as $t \to \infty$*
- *the function $f(t)$ is piecewise continuous on the interval $0 \leqslant t < \infty$*

Proprieties 1.11 *we have some (LT)proprieties:*

1. $L\{f(t) \star g(t)\} = L\{f(t)\}L\{g(t)\} = F(s)G(s)$ *with* $(f \star g)(t) = \int_0^\infty f(t-\mu)g(\mu)d\mu$
2. $L\{\sum_{k=0}^\infty a_k f_k(t)\} = \sum_{k=0}^\infty a_k F_k(s)$ *if* $(\sum_{k=0}^\infty a_k f_k(t))$ *is uniformly convergent*
3. $L\{\frac{d^n f(t)}{dt^n}\} = s^n F(s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(0)$
4. $L\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s}$

The Laplace Transform of Some Functions:

1. $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} \quad s > 0.$
2. $L\{x^{\beta-1}E_{\alpha,\beta}(\lambda x^\alpha)\}$

The Laplace Transform of fractional integral and derivatives:

1. Laplace transform of RLFI :

$$L\{{}_0I_x^\alpha c f(x)\} = s^{-\alpha}F(s)$$

2. Laplace transform of RLFD :

$$L\{{}_0^{RL}D_x^\alpha f(x)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k {}_0D_x^{\alpha-k-1} f(0)$$

3. Laplace transform of CFD :

$$L\{{}_0^cD_x^\alpha f(x)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k {}_0D_x^{\alpha-k-1} f^{(k)}(0) \tag{1.1}$$

Inverse Laplace Transform Formula :

$$L^{-1}\left\{\frac{1}{(s^\alpha + \alpha s^\beta)^{n+1}}\right\} = t^{\alpha(n+1)-1} \sum_{k=0}^{\infty} \frac{(-\alpha)^k \binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha)} t^{k(\alpha-\beta)},$$
$$\alpha \in \mathbb{R}, \quad \alpha \geqslant \beta > 0, \quad s^{\alpha-\beta} > |\alpha| \tag{1.2}$$

1.2 Fractional Differential Equations FDEs

FDEs involve fractional derivatives of the form d^α/dt^α , which are defined for $\alpha > 0$, where α is not necessarily an integer. They are generalizations of the ordinary differential equations to a random (noninteger) order.
in this section we define some FDEs that we use in the 3^{rd} Chapter

1.2.1 Fractional Initial-Value Problems FIVPs :

FIVPs is very important class of FDEs, written in the form :

$$D^\alpha u(t) = f(t, u(t)), \quad u^{(k)}(0) = u_0^{(k)}, \quad k = 0, 1, \ldots, n-1.$$

where f is an arbitrary function, D^α denotes the fractional differential derivative in the sense of Caputo, $u^{(k)}(t)$ is the k^{th} derivative of u and $u_0^{(k)}$ are the specified initial conditions.

Remark 1.12 *we note some definitions as a Remark :*

1. *A FDE is called linear if it is linear in the unknown and its derivatives*
2. *A FDE without any linearity properties is called fully nonlinear, and possesses non-linearities on one or more of the highest-order derivatives*
3. *we called FDE is homogeneous if $f(t) = 0$*
4. *we called FDE is non-homogeneous if $f(t) \neq 0$*

1.2.2 Fractional Boundary Value Problems FBVPs

the boundary value problem for a fractional differential equation is at the form:

$$D_t^\alpha u(t) = f(t, u(t)), \quad t \in [a, b]$$

with boundary condition:

$$u^{(k)}(a) = u_k, \quad k = 0, 1, \ldots, n-2, \quad u^{(n-1)}(b) = u_b$$

where D_t^α is the Caputo fractional derivative of order$n - 1 < \alpha \leqslant n$, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is the given continuous function and $u_0, u_1, \ldots, u_{n-2}, u_b$ are the real constants.

1.2.3 Time-Fractional Differential Equations :

1. ***Time-Fractional Wave Equations:*** The traditional wave equation

$$\frac{\partial^2}{\partial t^2} p(x, t) = \Delta_x p(x, t) \tag{1.3}$$

where $p(x, t)$ is the plane wave with replaces the second time derivative in (1.3) by a fractional derivative,we obtain a simple time-fractional wave equation :

$$\frac{\partial^\alpha}{\partial t^\alpha} p(x, t) = \Delta_x p(x, t), \quad 1 < \alpha < 2.$$

2. ***Time-Fractional Heat Equations:*** let the heat conduction equation:

$$\frac{\partial T}{\partial t} = a \int_0^t K(t - \tau) \Delta T(\tau) d\tau$$

where $\tau = (t - u)$ and $K(t)$is the kernel function.
the time-fractional heat equation is on the form

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \Delta T, \quad 0 < \alpha \leqslant 2$$

3. ***Time-Fractional Advection Equation:*** The advection equation is expressed mathematically by:

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

where $u = u(x, t)$,and $x \in \mathbb{R}$,and c a nonzero constant velocity the time-fractional advection equation is at the form:

$$D_t^{\alpha} u(x,t) + u(x,t)u_x(x,t) = f(x,t) \quad 0 < \alpha \leqslant 1$$

D_t^{α} denotes the fractional differential derivative in the sense of Caputo

1.2.4 Fractional Klein-Gordon Equation:

The standard Klein-Gordon equation KGE is written as

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = f(x,t), \quad x > 0, \quad t > 0.$$

where u indicates an unknown function in variables x and t, and $f(x,t)$ stands for the source term.
by replacing integer order time derivative by fractional order Caputo derivative we obtain time fractional Klein-Gordon Equation :

$$D_t^{\alpha} u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) = f(x,t), \quad 1 < \alpha \leqslant 2.$$

having the initial conditions:

$$u(x,O) = f_1(x), \quad \frac{\partial u(x,0)}{\partial t} = f_2(x), \quad x \geqslant 0, \quad t \leqslant 1$$

1.3 The Existence and Uniqueness of the solution

We see in this section the existence and uniqueness of solution of FDE of a General Form.
let us consider the intial -value problem

$$ {}_0D_t^{\sigma_n}u(t) = f(t,u) \tag{1.4}$$

with the initial condition:

$$ {}_0D_t^{\sigma_k-1}u(0) = b_k \quad k=1,...,n. \tag{1.5}$$

where,

$$\begin{aligned}
{}_aD_t^{\sigma_k} &= {}_aD_t^{\alpha_k}\,{}_aD_t^{\alpha_{k-1}}\dots{}_aD_t^{\alpha_1}\\
{}_aD_t^{\sigma_k-1} &= {}_aD_t^{\alpha_k-1}\,{}_aD_t^{\alpha_{k-1}}\dots{}_aD_t^{\alpha_1}\\
\sigma_k &= \sum_{j=1}^{k}\alpha_j, \quad (k=1,2,\dots,n)\\
0 <&\alpha_j \leqslant 1, \quad (j=1,2,\dots,n)
\end{aligned}$$

let us suppose that $f(t,u)$ is defined in domain G of a plane (t,u) and define a regio, $R(h,K)\subset G$ as a set of points $(t,u)\in G$,which satisfy the following inequalities

$$0<t<h, \quad \left|t^{1-\sigma_1}u(t)-\sum_{i=1}^{n}b_i\frac{t^{\sigma_i-\sigma_1}}{\Gamma(\sigma_i)}\right| \leqslant K,$$

where h and K are constant.

Theorem 1.13 *let $f(t,u)$be a real-valued continuous function defined in the domain , satisfying in G the Lipschitz condition with respect to u i,e:*

$$|f(t,u_1)-f(t,u_2)| \leqslant A|u_1-u_2|$$

such that

$$|f(t,u)| \leqslant M \quad forall \quad (t,u)\in G$$

Let also

$$K \geqslant \frac{Mh^{\sigma_n-\sigma_1+1}}{\Gamma(1+\sigma_n)}$$

Then there exists in the region $R(h, K)$ a unique and continuous solution $u(t)$ of the problem (1.4) (1.5)

Proof. First, let us reduce the problem (1.4)-(1.5) to an equivalent fractional iritegral equation. Using this formula

$$u(t) = \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} f(\tau) d\tau + \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i - 1}$$

or performing subsequently the fractional integration of order $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$ with the help of the composition rule which is :

$${}_aD_t^{-p}({}_aD_t^p f(t)) = f(t) - \sum_{j=1}^{k} [{}_aD_t^{p-j} f(t)]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(p-j+1)}$$

we obtain

$$u(t) = \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i - 1} + \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} f(\tau, u(\tau)) d\tau \tag{1.6}$$

we see that if $u(\tau)$ satisfies (1.4)(1.5),then it also satisfies the equation(1.6).
now let us define the sequence of functions $u_0(t), u_1(t), u_2(t), \ldots$ by he folowing relationships

$$u_0(t) = \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i - 1} \tag{1.7}$$

$$u_m(t) = \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i - 1} + \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} f(\tau, u_{m-1}(\tau)) d\tau, \quad m = 1, 2, 3, \ldots \tag{1.8}$$

we will show that $\lim_{m\to\infty} u_m(t)$ excite and gives the required solution $u(t)$ of the equation (1.6) First, it can be shown by induction that for $0 < t \leqslant h$ we have $u_m(t) \in R(h, K)$ for all m. Indeed,

$$\begin{aligned} \left| t^{1-\sigma_1} u_m(t) - \sum_{i=1}^{n} b_i \frac{t^{\sigma_i - \sigma_1}}{\Gamma(\sigma_i)} \right| &= \left| \frac{t^{1-\sigma_1}}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} f(\tau, u_{m-1}(\tau)) d\tau \right| \\ &\leqslant \frac{M t^{\sigma_n - \sigma_1 + 1}}{\Gamma(1+\sigma_n)} \leqslant \frac{M h^{\sigma_n - \sigma_1 + 1}}{\Gamma(1+\sigma_n)} \leqslant K \end{aligned} \tag{1.9}$$

and for the same reasons we have the same inequality for $u_1(t)$

$$\left| t^{1-\sigma_1} u_1(t) - \sum_{i=1}^{n} b_i \frac{t^{\sigma_i - \sigma_1}}{\Gamma(\sigma_i)} \right| \leqslant \frac{M h^{\sigma_n - \sigma_1 + 1}}{\Gamma(1+\sigma_n)} \leqslant K$$

Further it can be shown by induction that for all m

$$|u_m(t) - u_{m-1}(t)| \leqslant \frac{MA^{m-1}t^{m\sigma_n}}{\Gamma(1+m\sigma_n)} \tag{1.10}$$

Indeed, using (1.9): we have for $m = 1$:

$$|u_1(t) - u_0(t)| \leqslant \frac{Mt^{\sigma_n}}{\Gamma(1+\sigma_n)} \quad (0 < t \leqslant h)$$

Let us suppose that

$$|u_{m-1}(t) - u_{m-2}(t)| \leqslant \frac{MA^{m-2}t^{(m-1)\sigma_n}}{\Gamma(1+(m-1)\sigma_n)} \quad (0 < t \leqslant h) \tag{1.11}$$

Then. using (1.8) and (1.11). and recalling the Riemann-Liouville fractional derivative of the power function , we have

$$\begin{aligned}
|u_m(t) - u_{m-1}(t)| &\leqslant \frac{A}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n} |u_{m-1}(\tau) - u_{m-2}(\tau)| d\tau \\
&\leqslant \frac{MA^{m-1}}{\Gamma(1+(m-1)\sigma_n)} \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} \tau^{(m-1)\sigma_n} d\tau \\
&= \frac{MA^{m-1}}{\Gamma(1+(m-1)\sigma_n)} {}_0D_t^{-\sigma_n} t^{(m-1)\sigma_n} \\
&= \frac{MA^{m-1}}{\Gamma(1+(m-1)\sigma_n)} \frac{\Gamma(1+(m-1)\sigma_n)t^{(m-1)\sigma_n} + \sigma_n}{\Gamma(1+(m-1)\sigma_n + \sigma_n)} \\
&= \frac{MA^{m-1}t^{m\sigma_n}}{\Gamma(1+m\sigma_n)}
\end{aligned}$$

This means that (1.10) holds for all m. Now let us consider the series

$$u^*(t) = \lim_{m\to\infty} \Big(u_m(t) - u_0(t) \Big) = \sum_{j=1}^{\infty} \Big(u_j(t) - u_{j-1}(t) \Big) \tag{1.12}$$

According to the estimate (1.10), for $0 < t \leqslant h$ the absolute value of its terms is less than the corresponding terms of the convergent numeric series

$$M \sum_{j=1}^{\infty} \frac{MA^{j-1}h^{j\sigma_n}}{\Gamma(1+j\sigma_n)} = \frac{M}{A} \Big(E_{\sigma_n,1}(Ah_n^{\sigma}) - 1 \Big)$$

where $E_{\lambda,\mu}(z)$ is the Mittag-Leffler function . This means that the series (1.12) converges uniformly. Obviously, each term $\Big(u_j(t) - u_{j-1}(t) \Big)$ of the series (1.12) is a continuous function of t for $0 \leqslant t \leqslant h$ Therefore, the sum of the series (1.12), $u^*(t)$, is a continuous function for $0 \leqslant t \leqslant h$ and

$$u(t) = \lim_{m \to \infty} u_m(t) = u_0(t) + u^*(t)$$

is a continuous function for $0 < t \leqslant h$.

The uniform convergence of the sequence of $u_m(t)$ allows us to take $m \to \infty$ in the relationship (1.8). This gives the equation (1.6). showing that $u(t)$, the limit function of the process defined by (1.7) and (1.8), is the solution of (1.6).

Finally, let us prove the uniqueness of the solution. Let us suppose that $\overline{u}(t)$ is another solution of the equation (1.6). which is continuous in the interval $0 < t \leqslant h$. Then it follows from (1.6) that the function $z(t) = u(t) - \overline{u}(t)$ satisfies the equation

$$z(t) = \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} f(\tau, z(\tau)) d\tau \tag{1.13}$$

from which it follows that $z(0) = 0$. Therefore. $z(t)$ is continuous for $0 \leqslant t \leqslant h$. Then $|z(t)| < B$ for $0 \leqslant t \leqslant h$, where B is constant, and we obtain from the equation (1.13) that

$$|z(t)| \leqslant \frac{ABt^{\sigma_n}}{\Gamma(1+\sigma_n)}, \quad (0 \leqslant t \leqslant h)$$

Repeating this estimates j times, we obtain

$$|z(t)| \leqslant \frac{A^j B t^{j\sigma_n}}{\Gamma(\sigma_n)}, \quad j = 1, 2, \ldots \tag{1.14}$$

In the right-hand side we recognize up to the constant multiplier B the general term of the series for the Mittag-Leffler function $E_{\sigma_n,1}(At_n^{\sigma})$, and therefore for all t

$$\lim_{j \to \infty} \frac{A^j t^{j\sigma_n}}{\Gamma(\sigma_n)} = 0$$

Taking the limit of (1.14) as $j \to \infty$, we conclude that $z(t) = 0$, and $\overline{u}(t) = u(t)$ for $0 < t \leqslant h$, $\square$

1.4 Methods For Solving FDEs

In this section we divide some methods of resolution in two types Analytical Methods(Laplace Transform Method,Variational Iteration Method) and Numerical Methods (Rectangle Method,Trapezoid Method)

1.4.1 Analytical Methods for Solving FDEs:

1. ***The Laplace Transformer Method LTM :***

   LTM is one of the most powerful methods of solving Linear FDEs switch constant coefficients. On the other hand it is useless for Linear FDEs with general variable coefficients or for nonlinear FDEs .

   Consider the fractional differential equation of the form

$$D_t^{\alpha} u(t) = f(t)$$

   with initial conditions

$$u^{(k)}(0) = b_k, \quad k = 0, 1, 2, \ldots, n-1$$

   Where D_t^{α} notes Caputo derivative and n is the smallest integer greater than α Suppose that $f(t)$ a sufficiently good function i.e. Laplace transform of $f(t)$ exist Applying the Laplace transform on both the sides of equation we have,

$$L\{D_t^{\alpha} u(t)\} = L\{f(t)\}$$

   Applying The Laplace transform of Caputo Fractional differential operator of order α (1.1) we get

$$s^{\alpha} U(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} u^{(k)}(0) = F(s)$$

   Since Laplace transform is linear equation can be solved with respect to $U(s)$ as follows

$$U(s) = \frac{F(s) + \sum_{k=0}^{n-1} s^{\alpha-k-1} u^{(k)}(0)}{s^{\alpha}}$$

   by using initial conditions

$$U(s) = \frac{F(s) + \sum_{k=0}^{n-1} s^{\alpha-k-1} b_k}{s^{\alpha}}$$

follows by using the Laplace transform of the two – parameter function of Mittag-Leffler type:

$$L\{x^{\alpha m+\beta-1}E^{m}_{\alpha,\beta}(\lambda x^{\alpha})\}\} = \frac{m! s^{\alpha-\beta}{}^{m=1}}{(s^{\alpha})\lambda}$$

with the linearity property we obtain:

$$U(s) = \frac{F(s)}{s^{\alpha}} + \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^{\alpha}} b_k$$

$$U(s) = \frac{F(s)}{s^{\alpha}} + \sum_{k=0}^{n-1} L\{t^k E_{\alpha,k+1}(t^{\alpha})\} b_k = \frac{F(s)}{s^{\alpha}} + L\left\{\sum_{k=0}^{n-1} b_k t^k E_{\alpha,k+1}(t^{\alpha})\right\}$$

Then using the inverse Laplace transform $u(t)$ (1.2) can be found as

$$L^{-1}\{U(s)\} = L^{-1}\left\{\frac{F(s)}{s^{\alpha}} + L\left\{\sum_{k=0}^{n-1} b_k t^k E_{\alpha,k+1}(t^{\alpha})\right\}\right\}$$

$$u(t) = L^{-1}\left\{\frac{F(s)}{s^{\alpha}}\right\} + \sum_{k=0}^{n-1} b_k t^k E_{\alpha,k+1}(t^{\alpha}) = t^{\alpha-1}E_{\alpha,\alpha}(t^{\alpha}) \times f(t) + \sum_{k=0}^{n-1} b_k t^k E_{\alpha,k+1}(t^{\alpha})$$

which is a required solution

2. ***Variational Iteration Method VIM :***

The principles of the VIM and its applicability for various kinds of differential, Ji-Huan He showed that the variational iteration method is also valid for fractional differential equations. He applied the method to obtain analytical solution for the fractional differential equation

$$D^{\alpha}_{t} u = f(x,t), \quad u(a) = b, \quad 1 < \alpha < 2$$

we extend the application of the variational iteration method to solve the time fractional differential equation:

$$D^{\alpha}_{t} u(x,t) = R(x)u(x,t) + q(x,t), \quad t > 0, x \in \mathbb{R} \tag{1.15}$$

where $R(x)$ is a differential operator in x, subject to the initial and boundary conditions

$$u(x,0) = f(x), \quad 0 < \alpha \leqslant 1$$
$$u(x,t) \to 0 \quad as \quad |x| \to \infty, t > 0,$$

and

$$u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = f(x), \quad 0 < \alpha \leqslant 2$$
$$u(x,t) \to 0, \quad as \quad |x| \to \infty, t > 0,$$

where $f(x), g(x)$ and $q(x,t)$ all are continuous functions and, $m-1 < \alpha \leqslant m$ is a parameter describing the order of the time-fractional derivative in the Caputo sense. According to the variational iteration method, we can construct the correction functional for Eq. (1.15) as:

$$\begin{aligned} u_{k+1}(x,t) &= u_k(x,t) + I_t^{\beta}\left[\lambda\Big(D_t^{\alpha}u_k(x,t) - R(x)\overline{u}_k(x,t) - q(x,t)\Big)\right]. \\ &= u_k(x,t) + \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\lambda(\tau)\Big(D_t^{\alpha}u_k(x,\tau) - R(x)\overline{u}_k(x,\tau) - q(x,\tau)\Big)\Bigg] d\tau \end{aligned} \tag{1.16}$$

where I_t^{β} is the Riemann-Liouville fractional integral operator of order $\beta = \alpha - floor(\alpha)$,that is $\beta = \alpha + 1 - m$ with respect to the variable t and λ is a general Lagrange multiplier, which can be identified optimally via variational theory. To identify approximately Lagrange multiplier, some approximation must be made. The correction functional (1.16) can be approximately expressed as follows

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \left[\lambda(\tau)\Big(D_t^{m}u_k(x,\tau) - R(x)\overline{u}_k(x,\tau) - q(x,\tau)\Big)\right] d\tau$$

Here we apply restricted variations to the nonlinear term $R(x)$, in this case we can easily determine the multiplier. Making the above functional stationary, noticing that $\sigma\overline{u}_k = 0$,

$$\sigma u_{k+1}(x,t) = \sigma u_k(x,t) + \sigma\int_0^t \lambda(\tau)\Big(D_t^{m}u_k(x,\tau) - R(x)u_k(x,\tau) - q(x,\tau)\Big) d\tau$$

yields the following Lagrange multipliers

$$\lambda = -1 \quad for \quad m = 1$$
$$\lambda = \tau - t \quad for \quad m = 2$$

Therefore, for $m = 1 \;\; (0 < \alpha \leqslant 1)$, we substitute $\lambda = -1$ to the functional (1.16) to obtain the following iteration formula:

$$u_{k+1}(x,t) = u_k(x,t) - I_t^\alpha \left[D_t^\alpha u_k(x,t) - R(x)u_k(x,t) - q(x,t) \right]$$

for $m = 2 \;\; (1 < \alpha \leqslant 2)$, we substitute $\lambda = \tau - t$ to the functional (1.16) to get

$$\begin{aligned} u_{k+1}(x,t) &= u_k(x,t) + I_t^\alpha \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^{\alpha-2}(\tau - t)\left(D_t^\alpha u_k(x,\tau) - R(x)u_k(x,\tau) - q(x,\tau) \right) d\tau. \\ &= u_k(x,t) - \frac{\alpha - 1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\alpha-1}\left(D_t^\alpha u_k(x,\tau) - R(x)u_k(x,\tau) - q(x,\tau) \right) d\tau \end{aligned}$$

So, we obtain the following iteration formula

$$u_{k+1}(x,t) = u_k(x,t) - (\alpha - 1) I_t^\alpha \left[D_t^\alpha u_k(x,t) - R(x)u_k(x,t) - q(x,t) \right]$$

The initial approximation u_0 can be freely chosen if it satisfies the initial and boundary conditions of the problem. However the success of the method depends on the proper selection of the initial approximationu_0. Finally, we approximate the solution $u(x,t) = \lim\limits_{k \to \infty} u_N(x,t)$ by the N^{th} term $u_N(x,t)$.

1.4.2 Numerical Methods For Solving FDES :

we considers two methods to solve the equation:

$$\begin{cases} {}_0^cD_t^\alpha u(t) = f(u,t) \quad T > t > 0. \\ u^{(k)}(0) = u_0^{(k)} \end{cases} \tag{1.17}$$

this equation is equivalent to:

$$u(t) - u(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds \tag{1.18}$$

Now we look into more detail on how to solve equation (1.18) with different schemes for $f(s, u(s))$ We will be looking at solving this equation over an interval of $[0, T]$ where $0 = t_0 < t_1 < t_2 < ... < t_n = T$

1. ***Rectangle Method :***

   Firstly we work with equation

   $$u(t) - u(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds$$

   We want to find a value of u_n where $t = t_n$ so the equation becomes:

   $$u_n(t) - u_0 = \frac{1}{\Gamma(\alpha)} \int_0^t (t_n - s)^{\alpha-1} f(s, u(s)) ds \tag{1.19}$$

   Here the integral can be split into:

   $$\int_0^{t_1} (t_n-s)^{\alpha-1} f(s, u(s)) ds + \int_{t_1}^{t_2} (t_n-s)^{\alpha-1} f(s, u(s)) ds + ... + \int_{t_{n-2}}^{t_{n-1}} (t_n-s)^{\alpha-1} f(s, u(s)) ds$$

   which equates to:

   $$\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} f(s, u(s)) ds$$

   this gives the result for y_n

   $$u_n(t) - u_0 = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} f(s, u(s)) ds \tag{1.20}$$

   In order to find the rectangle scheme we use the rectange rule to approximate $f(s, u(s))$ on the interval $[t_n, t_n + 1]$ by $P_0(s)$ where:

$P_0(s) = f(t_k, u(t_k)) \in [t_k, t_{k+1}]$

Now if we refer back to equation(1.20),substituting in the rectangle approximation for $f(s, u(s))$ the equation becomes

$$u_n(t) - u_0 \approx \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} f(t_k, u(t_k)) ds \tag{1.21}$$

Now we wish to solve equation1.21

$$\begin{aligned}
u_n(t) - u_0 &\approx \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} f(t_k, u(t_k)) ds \\
&= \sum_{k=0}^{n-1} \left[\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} ds \right] f(t_k, u(t_k)) \\
&= \sum_{k=0}^{n-1} \left[\frac{1}{\Gamma(\alpha)} \times \frac{1}{\alpha} \left[(t_n - s)^{\alpha} \right]_{t_k}^{t_{k+1}} \right] f(t_k, u(t_k)) \\
&= \sum_{k=0}^{n} \left[\frac{1}{\Gamma(\alpha+1)} \left[(t_{n+1} - t_{k+1})^{\alpha} - (t_{n+1} - t_k)^{\alpha} \right] \right] f(t_k, u(t_k)) \\
&= \sum_{k=0}^{n} \omega_{k,n} f(t_k, u(t_k))
\end{aligned}$$

Finally we have the Rectangle scheme to be:

$$u_n = u_0 + \sum_{k=0}^{n} \omega_{k,n} f(t_k, u(t_k)) \tag{1.22}$$

where

$$\omega_{k,n} = \frac{1}{\Gamma(\alpha+1)} \left[(t_{n+1} - t_{k+1})^{\alpha} - (t_{n+1} - t_k)^{\alpha} \right] \quad k = 0, 1, ..., n$$

As we are only looking at values for α to be $0 < \alpha < 1$, we only need to be given the initial value u_0 so solving equation (1.22)we have:

$$\begin{aligned}
u_0 &\to given \\
u_1 &= u_0 + \omega_{0,1} f(t_0, u(t_0)) \\
u_2 &= u_0 + \omega_{0,2} f(t_0, u(t_0)) + \omega_{1,2} f(t_1, u(t_1)) \\
&\vdots \\
u_n &= u_0 + \omega_{0,n} f(t_0, u(t_0)) + \omega_{1,n} f(t_1, u(t_1)) + \ldots + \omega_{n-1,n} f(t_{n-1}, u(t_{n-1}))
\end{aligned}$$

This is an explicit method. Which is fairly easy to solve, now we want to find the error of this method. This is found by taking the modulus of the exact value minus the approximate value for each n giving:

$$|u_n - u(t_n)| = \epsilon$$

2. ***Trapezoidal Method :***

   we look at how to find the trapezoidal method to solve equation 1.18 Again we begin by working with:

$$u(t) - u(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds$$

We want to find a value of u_{n+1} where $t = t_{n+1}$ so the equation becomes:

$$u_{n+1} - u_0 = \frac{1}{\Gamma(\alpha)} \int_0^t (t_{n+1}-s)^{\alpha-1} f(s, u(s)) ds$$

The integral is again split into:

$$\int_0^{t_1} (t_{n+1}-s)^{\alpha-1} f(s,u(s))ds + \int_{t_1}^{t_2} (t_{n+1}-s)^{\alpha-1} f(s,u(s))ds + \ldots + \int_{t_n}^{t_{n+1}} (t_{n+1}-s)^{\alpha-1} f(s,u(s))ds$$

this equates to:

$$\sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (t_{n+1}-s)^{\alpha-1} f(s, u(s)) ds$$

which gives us the result for u_{n+1} to be:

$$u_{n+1} - u_0 = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (t_{n+1}-s)^{\alpha-1} f(s, u(s)) ds$$

In order to find the rectangle scheme we need to use the rectangle rule to approximate $f(s, u(s))$ on the interval $[t_k, t_{k+1}]$ by $P_1(s)$ where

$$P_1(s) = \frac{s - t_{k+1}}{t_k - t_{k+1}} f(t_k, u(t_k)) + \frac{s - t_k}{t_{k+1} - t_k} f(t_{k+1}, u(t_{k+1})) \quad s \in [t_k, t_{k+1}]$$

thus we get:

$$u_{n+1} - u_0 \approx \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (t_{n+1} - s)^{\alpha-1} [P_1(s)]$$
$$= \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (t_{n+1} - s)^{\alpha-1} \left[\frac{s - t_{k+1}}{t_k - t_{k+1}} f(t_k, u(t_k)) + \frac{s - t_k}{t_{k+1} - t_k} f(t_{k+1}, u(t_{k+1})) \right]$$

we solve the integral

$$\frac{1}{\Gamma(\alpha)} \int_{x_k}^{x_{k+1}} (x_{n+1} - s)^{\alpha-1} [\frac{s - x_{k+1}}{x_k - x_{k+1}} + \frac{s - x_k}{x_{k+1} - x_k}] ds = \frac{1}{\Gamma(\alpha)} \int_{x_k}^{x_{k+1}} (x_{n+1} - s)^{\alpha-1} \frac{s - x_{k+1}}{x_k - x_{k+1}} ds + \frac{1}{\Gamma(\alpha)} \int_{x_k}^{x_{k+1}} (x_{n+1} - s)^{\alpha-1} \frac{s - x_k}{x_{k+1} - x_k} ds \tag{1.23}$$

In order to solve these we need to use the following identity from

$$J^j_{n,k} = \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_n} (t_{n+1} - s)^{\alpha-1} (s - t_k)^j ds = \frac{(t_{n+1} - t_k)^{\alpha+j}}{\Gamma(\alpha + 1 + j)}$$

Firstly we begin by solving the first part of the equation1.23 we need to also use

$$\frac{1}{t_{t+1} - t_k} \left[J^{(1)}_{n,k} + (t_{k+1} - t_k) J^0_{n,k} - J^1_{n,k+1} \right]$$

$$\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_{k+1}} (t_{n+1} - s)^{\alpha-1} \frac{s - t_{k+1}}{t_k - t_{k+1}} = \frac{1}{t_{k+1} - t_k} \left[J^{(1)}_{n,k} + (t_{k+1} - t_k)_{n,k} J^0 - J^{(1)}_{n,k+1} \right]$$
$$J^{(1)}_{n,k} = \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_n} (t_{n+1} - s)^{\alpha-1} (s - t_k) ds$$
$$= \frac{(t_{n+1} - t_k)^{\alpha+1}}{\Gamma(\alpha + 2)}$$
$$J^{(0)}_{n,k} = \frac{(t_{n+1} - t_k)^{\alpha}}{\Gamma(\alpha + 1)}$$
$$J^{(1)}_{n,k+1} = \frac{(t_{n+1} - t_{k+1})^{\alpha+1}}{\Gamma(\alpha + 2)}$$
$$\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_{k+1}} (t_{n+1} - s)^{\alpha-1} \frac{s - t_{k+1}}{t_k - t_{k+1}} = \frac{1}{t_{k+1} - t_k} \left[\frac{(t_{n+1} - t_k)^{\alpha+1}}{\Gamma(\alpha + 2)} + (t_{k+1} - t_k) \frac{(t_{n+1} - t_k)^{\alpha}}{\Gamma(\alpha + 1)} + \frac{(t_{n+1} - t_{k+1})^{\alpha+1}}{\Gamma(\alpha + 2)} \right]$$
$$= \frac{1}{\Gamma(\alpha + 2)(t_{k+1} - t_k)}$$

Chapter 2

Application of Standard and Modified Homotopy Perturbation Method on FDEs

In this chapter we had a look on perturbation theory and how the Homotopy perturbation technique works after tha we applied two modifications on (HPM)

2.1 Perturbation Theory

Perturbation theory comprises mathematical methods which are used to find the approximate solution to a problem which cannot be a solved accurately, by starting from the accurate solution of a related problem. Perturbation theory leads to an expression for the desired solution in terms of a formal power series in small parameter (ϵ) known as perturbation series that quantifies the deviation from the exactly solvable problem and further terms describe the deviation in the solution.

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$$

Here x_0 be the known solution to the exactly solvable initial problem and $x_1, x_2, \dots$ are the higher order terms. For small ϵ these higher order terms are successively smaller.

2.2 Homotopy Perturbation Technique

we consider the following non-linear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{2.1}$$

With boundary conditions:

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma \tag{2.2}$$

Where A is general differential operator,B is boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω .
The operator A can be divided into two parts L and N, Where L is linear and N is nonlinear operator. Then equation (2.1) can be written as follows:

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega \tag{2.3}$$

the He's homotopy perturbation technique defines the homotopy
$v(r, p) : \Omega \times [0, 1] \to \mathbb{R}$ which satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0 \tag{2.4}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{2.5}$$

where $r \in \Omega, p \in [0, 1]$ is an impeding parameter and u_0 is an initial approximation which satisfies the boundary conditions. Obviously, from (2.4)and (2.5),we have:

$$H(v, 0) = L(v) - L(u_0) = 0 \tag{2.6}$$

$$H(v, 1) = L(v) + N(v) - f(r) = 0 \tag{2.7}$$

The changing process of p from zero to unity is just of $v(r, p)$ from $u_0(r)$ to $u(r)$ .In topology, this is called deformation and $L(v) - L(u_0)$ and $L(v) + N(v) - f(r)$ are homo topic.
He assumes that the solutions of (2.5) and(2.4) can be expressed as the power series of p:

$$v = v_0 + pv_1 + p^2 v_2 + ...$$

the approximate solution of (2.2) and (2.3),therefore can be readily obtained:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ... \tag{2.8}$$

The series (2.8) is convergent for most cases, however, the convergent rate depends upon the nonlinear operator $A(v)$.

2.3 Standard Homotopy Perturbation Method For FDEs

we consider the following nonlinear differential equation of fractional order:

$$D^{\alpha} u(t) + L(u(t)) + N(u(t)) = f(t), \quad t > 0, \quad x \in \mathbb{R}, m-1 < \alpha \leqslant m, \tag{2.9}$$

subject to initial condition:

$$u^{(k)}(0) = b_k, \quad k = 0, 1, \ldots, m-1, \tag{2.10}$$

where $D^{\alpha} u(t)$ is the Caputo fractional derivative of order α, L is a linear operator which might include other fractional derivatives of order less than α, N is a nonlinear operator which also might include other fractional derivatives of order less than α, , $f(t)$ is a known analytic function
In view of the homotopy technique, we can construct the following homotopy:

$$(1-p) D^{\alpha} u(t) + p[D^{\alpha} u(t) + L(u(t)) + N(u(t)) - f(t)] = 0 \tag{2.11}$$

or

$$D^{\alpha} u(t) + p[L(u(t)) + N(u(t))] - f(t)] = 0, \tag{2.12}$$

where $p \in [0, 1]$.The homotopy parameter p always changes from zero to unity. In case $p = 0$, the equations (2.11)and(2.12) becomes

$$D^{\alpha} u = 0$$

and when $p = 1$ both Equations (2.11)and (2.12) turns out to be the original fractional differential equation (2.9). The homotopy perturbation method admits a solution in the form:

$$u(t) = p^0 u_0(t) + p^1 u_1(t) + p^2 u_2(t) + \ldots \tag{2.13}$$

Setting $p = 1$ results in the solution of Eq. (2.9)

$$u(t) = u_0(t) + u_1(t) + u_2(t) + \ldots \tag{2.14}$$

Substituting (2.13) in (2.12) and equating the terms with having identical power of p, we obtain a series of equations of the following form

$$\begin{aligned} p^0 &: D^\alpha u_0 = 0 \\ p^1 &: D^\alpha u_1 = -L(u_0) - N(u_0) + f(t) \\ p^2 &: D^\alpha u_2 = -L(u_1) - N(u_0, u_1) \\ p^3 &: D^\alpha u_3 = -L(u_2) - N(u_0, u_1, u_2) \\ &\vdots \end{aligned} \tag{2.15}$$

Applying the operator I^α,the inverse operator of D^α, on both sides of the above linear equations, with considering the initial conditions, the first few terms of the HPM solution can be given by

$$u_0 = \sum_{k=0}^{n-1} u^{(k)}(0) \frac{t^k}{k!} = \sum_{k=0}^{n-1} b_k \frac{t^k}{k!} \tag{2.16}$$

$$\begin{aligned} u_1 &= -I^\alpha[L(u_0)] - I^\alpha[N(u_0)] + I^\alpha(f(t)) \\ u_2 &= -I^\alpha[L(u_1)] - I^\alpha[N(u_0, u_1)] \\ u_3 &= -I^\alpha[L(u_2)] - I^\alpha[N(u_0, u_1, u_2)] \\ &\vdots \end{aligned} \tag{2.17}$$

2.4 Modified Homotopy Perturbation Method For FDEs

The homotopy perturbation method under development. recently Many modified various appeared ,in this section we will mention two modifications:

2.4.1 The First Modification :

HPM method is modified by Odibat and Momani into Modified Homotopy Perturbation Method (MHPM) by including u^m into both sides of the Homotopy equation (2.12). Let the Fractional Differential Equation Model is given below:

$$D^{\alpha}u(t) + L(u(t)) + N(u(t)) = f(t), \quad t > 0, \quad m-1 < \alpha \leqslant m, \tag{2.18}$$

where D^{α} is the Caputo fractional derivative of order α,L is a linear operator, N is a nonlinear operatorand $f(t)$ is function of t. the initial conditions:

$$u^k(0) = b_k, \quad k = 0, 1, 2, \ldots, m-1 \tag{2.19}$$

From (2.18), we obtained the Homotopy equation:

$$u^m + L(u) - f(t) = p[u^m - N(u) - D^{\alpha}u], \quad p \in [0,1] \tag{2.20}$$

or

$$u^m - f(t) = p[u^m - L(u) - N(u) - D^{\alpha}u], \quad p \in [0,1] \tag{2.21}$$

Hence the solution of (2.20) and (2.21) in form of p-power series is:

$$u = p^0u_0 + p^1u_1 + p^2u_2 + \ldots \tag{2.22}$$

By substituting (2.22) to (2.21) and take $p = 1$, then solution function will be:

$$u = \sum_{n=0}^{\infty} u_n \tag{2.23}$$

where u_n obtained from integral result of derivative function as follows:

$$\begin{aligned}
p^0 &: \frac{\partial^m u_0}{\partial t^m} = f(t), \quad u^k(0) = b_k \\
p^1 &: \frac{\partial^m u_1}{\partial t^m} = \frac{\partial^m u_0}{\partial t^m} - L_0(u_0) - N_0(u_0) - D^\alpha u_0, \quad u^k(0) = b_k \\
p^2 &: \frac{\partial^m u_2}{\partial t^m} = \frac{\partial^m u_1}{\partial t^m} - L_1(u_0, u_1) - N_1(u_0, u_1) - D^\alpha u_1, \quad u^k(0) = b_k \\
&\vdots
\end{aligned} \tag{2.24}$$

respectively, where the terms $L_0, L_1, L_2, \ldots$ and $N_0, N_1, N_2, \ldots$ satisfy the following equations

$$L(u_0 + pu_1 + p^2u_2 + \ldots) = L_0(u_0) + pL_1(u_0, u_1) + p^2L_2(u_0, u_1, u_2) + \ldots$$
$$N(u_0 + pu_1 + p^2u_2 + \ldots) = N_0(u_0) + pN_1(u_0, u_1) + p^2N_2(u_0, u_1, u_2) + \ldots$$

Setting $p = 1$ in Eq. (2.22) yields the solution of Eq. (2.18). It obvious that the linear equations in (2.24) are easy to solve, and the components u_n, $n \geqslant 0$ of the homotopy perturbation method can be completely determined, and the series solutions are thus entirely determined. Finally, we approximate the solution $u = \sum_{n=0}^{\infty} u_n$ by the truncated series:

$$u_N = \sum_{n=0}^{N-1} u_n$$

Reliable Algorithm :

We introduce a reliable algorithm to handle in a realistic and efficient way the linear (nonlinear) partial differential equations of fractional order.First, rewrite Eq. (2.18) in the form

$$D_t^\alpha u(x,t) = L(u, u_x, u_{xx}) + N(u, u_x, u_{xx}) + f(x,t) \quad t > 0, \tag{2.25}$$

where L is a linear operator, N is a nonlinear operator, f is a known analytic function, and D^α is the Caputo fractional derivative of order α with $m - 1 < \alpha < m$,, subject to the initial conditions:

$$u^{(k)}(x, 0) = g_k(x), \quad k = 0, 1, ..., m - 1. \tag{2.26}$$

In view of the modified homotopy technique, we can construct the following homotopy:

$$\frac{\partial u^m}{\partial t^m} - L(u, u_x, u_{xx}) - f(x,t) = p\left[\frac{\partial u^m}{\partial t^m} + N(u, u_x, u_{xx}) - D_t^\alpha u\right] \tag{2.27}$$

or

$$\frac{\partial u^m}{\partial t^m} - f(x,t) = p\left[\frac{\partial u^m}{\partial t^m} + L(u, u_x, u_{xx}) + N(u, u_x, u_{xx}) - D_t^\alpha u\right] \tag{2.28}$$

where $p \in [0, 1]$. The homotopy parameter p always changes from zero to unity. In case $p = 0$, Eq. (2.27) becomes the linearized equation:

$$\frac{\partial u^m}{\partial t^m} = L(u, u_x, u_{xx}) + f(x,t) \tag{2.29}$$

and Eq. (2.28) becomes the linearized equation

$$\frac{\partial u^m}{\partial t^m} = f(x,t) \tag{2.30}$$

and when it is one, Eq. (2.27) or Eq. (2.28) turns out to be the original fractional differential equation (2.25). The basic assumption is that the solution of Eq. (2.27) or Eq. (2.28) can be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + ...$$

Finally, we approximate the solution $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ by the n-term truncated series:

$$u_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t)$$

2.4.2 The Second Modification:

The modified form of the HPM proposed by Odibat can be established based on the assumption that the function $f(t)$ in (2.3) can be divided into two parts, namely, f_0 and, f_1 as

$$f(t) = f_0(t) + f_1(t)$$

or on the assumption that the function $f(t)$ can be replaced by a series of infinite components. Under this assumption we suggest that $f(t)$ be expressed in Taylor series:

$$f(t) = \sum_{n=0}^{\infty} f_n(t) \tag{2.31}$$

According to this assumption, $f(t) = f_0(t) + f_1(t)$ we can construct the the following homotopy :

$$(1-p)D^{\alpha}u(t) + p[D^{\alpha}u(t) + L(u) + N(u) - f_1(t)] = f_0(t) \tag{2.32}$$

or

$$D^{\alpha}u(t) + p[L(u) + N(u) - f_1(t)] = f_0(t) \tag{2.33}$$

Here, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part f_0 be assigned to zeroth component u_0 , whereas the remaining part f_1 be combined with the component u_1. If we set $f_1(t) = f(t)$ and $f_0(t) = 0$, then the homotopy (2.32) or (2.33) reduces to the homotopy (2.11) or (2.12), respectively. However the success of the method depends on the proper selection of the functions f_0 and f_1 .
According to the second assumption $f(t) = \sum_{n=0}^{\infty} f_n(t)$ we can construct the folowing homotopy:

$$(1-p)D^{\alpha}u(t) + p[D^{\alpha}u(t) + L(u) + N(u)] = \sum_{n=0}^{\infty} p^n f_n(t) \tag{2.34}$$

or

$$D^{\alpha}u(t) + p[L(u) + N(u)] = \sum_{n=0}^{\infty} p^n f_n(t) \tag{2.35}$$

If we set $f_1(t) = f(t)$, $f_n(t) = 0$ for $n = 0$ or $n \geqslant 2$, then the homotopy (2.34) or (2.35) reduces to the homotopy (2.11) or (2.12), respectively. The form of homotopy (2.35) allows us to obtain the individual terms $u_0, u_1, \ldots$ in (2.13). Substituting (2.13) in (2.35) and collecting the terms with the same powers of p, we obtain

$$\begin{aligned}
&p^0 : D^\alpha u_0 = f_0(t)\\
&p^1 : D^\alpha u_1 = f_1(t) - L(u_0) - N(u_0)\\
&p^2 : D^\alpha u_2 = f_2(t) - L(u_1) - N(u_0, u_1)\\
&p^3 : D^\alpha u_3 = f_3(t) - L(u_2) - N(u_0, u_1, u_2)\\
&\quad\vdots
\end{aligned} \tag{2.36}$$

by applying the operator I^α on both sides of the above linear equations, the first few terms of the HPM solution can be given by

$$u_0 = \sum_{k=0}^{n-1} u^{(k)}(0)\frac{t^k}{k!} + I^\alpha[f_0(t)] = \sum_{k=0}^{n-1} b_k \frac{t^k}{k!} + I^\alpha[f_0(t)] \tag{2.37}$$

$$\begin{aligned}
u_1 &= I^\alpha[f_1(t)] - I^\alpha[L(u_0)] - I^\alpha[N(u_0)]\\
u_2 &= I^\alpha[f_2(t)] - I^\alpha[L(u_1)] - I^\alpha[N(u_0, u_1)]\\
u_3 &= I^\alpha[f_3(t)] - I^\alpha[L(u_2)] - I^\alpha[N(u_0, u_1, u_2)]\\
&\vdots
\end{aligned} \tag{2.38}$$

If $f(t)$ consists of two terms only then the homotopy (2.34) or (2.35) reduces to the homotopy (2.32) or (2.33), respectively. In this case the term f_0 is combined with the component u_0; f_1 is combined with the component u_1, f_2 is combined with the component u_2 and so on. This suggestion will facilitate the calculations of the terms $u_0, u_1, u_2, \ldots$ and hence accelerate the rapid convergence of the series solution. It is easily to observe that the algorithm of the new modification of the HPM, based on the homotopy given in the Eqs. (2.32)–(2.35), reduces the number of terms involved in each component and hence the size of calculations is minimized compared to the standard HPM. Moreover this reduction of terms in each component facilitates the construction of the homotopy perturbation solution.

Reliable Algorithm:

We introduce a reliable algorithm to handle in a realistic and efficient way the linear (non-linear) partial differential equations of fractional order. Consider the general fractional differential equation:

$$D_t^{\alpha} = L(u, u_x, u_{xx}) + N(u, u_x, u_{xx}) + f(x,t) \quad t > 0,$$

where L is a linear operator, N is a nonlinear operator, f is a known analytic function, and D^{α} is the Caputo fractional derivative of order α,with $m-1 < \alpha < m$, subject to the initial conditions:

$$u^{(k)}(x,0) = h_k(x), \quad k = 0, 1, ..., m-1. \tag{2.39}$$

In view of the homotopy technique, we can construct the following homotopy:

$$D_t^{\alpha} u - L(u, u_x, u_{xx}) - f(x,t) = p[N(u, u_x, u_{xx})] \tag{2.40}$$

or

$$D_t^{\alpha} u - f(x,t) = p[L(u, u_x, u_{xx}) + N(u, u_x, u_{xx})] \tag{2.41}$$

In view of the modified homotopy technique, if we write $f(x,t) = f_0(x,t) + f_1(x,t)$, we can construct the following homotopy:

$$D_t^{\alpha} u - L(u, u_x, u_{xx}) - f_0(x,t) = p[N(u, u_x, u_{xx}) + f_1(x,t)] \tag{2.42}$$

or

$$D_t^{\alpha} u - f_0(x,t) = p[L(u, u_x, u_{xx}) + N(u, u_x, u_{xx}) + f_1(x,t)] \tag{2.43}$$

where $p \in [0,1]$. The homotopy parameter p always changes from zero to unity. The basic assumption is that the solution of (2.40) or (2.41) and (2.42) or (2.43) can be written as a power series in p :

$$u = u_0 + pu_1 + p^2 u_2 + ... \tag{2.44}$$

Finally, we approximate the solution $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ by the n-term truncated series:

$$u_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t)$$

Chapter 3

Arithmetic Application of Modified Homotopy Perturbation Method on FDEs

In this chapter we applied MHPM on linear and non linear FDEs

3.1 Linear Problems

To incorporate the 2^{nd} chapter discussion, four linear examples will be studied. The MHPM is used to obtain the approximate and exact solutions of the problems.

Example 3.1 *We apply in this Example the* 1^{st}*Modification.*
We consider time-fractional derivative heat equation in one-dimension defined as:

$$D_t^{\alpha} u = \alpha \frac{\partial^2 u}{\partial x^2}, \quad 0 < \alpha \leqslant 1, \quad t > 0, \quad x \in \mathbb{R} \tag{3.1}$$

with the initial condition

$$u(x,0) = c \sin\left(\frac{bx}{l}\right), \quad c, b, l \in \mathbb{R} \tag{3.2}$$

where $u(x,t)$ *is the heat conduction in one-dimensional isotropic medium and* α *is a thermal diffusion.*

Based on Modified Homotopy Perturbation Method, homotopy for equation (3.1) became:

$$\frac{\partial u}{\partial t} = p\left[\frac{\partial u}{\partial t} + \alpha\frac{\partial^2 u}{\partial x^2} - D_t^{\alpha} u\right] \tag{3.3}$$

Substituting (2.22) and the initial condition (3.2) into the homotopy (3.3) and equating the terms with identical powers of p, we obtain the following set of linear partial differential equations:

$$\begin{aligned}
\frac{\partial u_0}{\partial t} &= 0, \quad u_0(x,0) = c\sin\left(\frac{bx}{l}\right) \\
\frac{\partial u_1}{\partial t} &= \frac{\partial u_0}{\partial t} + \alpha\frac{\partial^2 u_0}{\partial x^2} - D_t^{\alpha} u_0, \quad u_1(x,0) = 0 \\
\frac{\partial u_2}{\partial t} &= \frac{\partial u_1}{\partial t} + \alpha\frac{\partial^2 u_1}{\partial x^2} - D_t^{\alpha} u_1, \quad u_2(x,0) = 0 \\
\frac{\partial u_3}{\partial t} &= \frac{\partial u_2}{\partial t} + \alpha\frac{\partial^2 u_2}{\partial x^2} - D_t^{\alpha} u_2, \quad u_3(x,0) = 0 \\
&\vdots
\end{aligned}$$

Consequently, the first few components of the Modified homotopy perturbation solution for Eq (3.1) are derived in the form:

$$\begin{aligned}
u_0(x,t) &= c\sin\left(\frac{bx}{l}\right) \\
u_1(x,t) &= -\frac{c\alpha b^2}{l^2}\sin\left(\frac{bx}{l}\right)t \\
u_2(x,t) &= \frac{c\alpha b^2}{l^2}\sin\left(\frac{bx}{l}\right)\left[\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} - t\right] + \frac{c\alpha^2 b^4}{l^4}\sin\left(\frac{bx}{l}\right)\frac{t^2}{2} \\
u_3(x,t) &= \frac{c\alpha b^2}{l^2}\sin\left(\frac{bx}{l}\right)\left[\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - t - \frac{4^{\alpha-1}\sqrt{\pi}t^{3-2\alpha}}{(3-2\alpha)\Gamma(1.5-\alpha)\Gamma(2-\alpha)}\right] \\
&\quad + \frac{c\alpha^2 b^4}{l^4}\sin\left(\frac{bx}{l}\right)\left[t^2 - \frac{2t^{3-\alpha}}{\Gamma(4-\alpha)}\right] - \frac{c\alpha^3 b^6}{l^6}\sin\left(\frac{bx}{l}\right)\frac{t^3}{6} \\
&\vdots
\end{aligned}$$

and so on,The four-term approximate solution for Eq (3.1) is given by:

$$\begin{aligned} u(x,t) =& c\sin\left(\frac{bx}{l}\right)+\frac{c\alpha b^2}{l^2}\sin\left(\frac{bx}{l}\right)\left[\frac{3t^{2-\alpha}}{\Gamma(3-\alpha)}-3t-\frac{4^{\alpha-1}\sqrt{\pi}t^{3-2\alpha}}{(3-2\alpha)\Gamma(1.5-\alpha)\Gamma(2-\alpha)}\right] \\ &+\frac{c\alpha^2b^4}{l^4}\sin\left(\frac{bx}{l}\right)\left[\frac{3t^2}{2}-\frac{2t^{3-\alpha}}{\Gamma(4-\alpha)}\right]-\frac{c\alpha^3b^6}{l^6}\sin\left(\frac{bx}{l}\right)\frac{t^3}{6} \end{aligned} \tag{3.4}$$

When $\alpha = 1$ Eq. (3.4) takes the form:

$$u(x,t) = c\sin\left(\frac{bx}{l}\right)\left[1+\left(\frac{-\alpha b^2t}{l^2}\right)+\frac{1}{2!}\left(\frac{-\alpha b^2t}{l^2}\right)^2+\frac{1}{3!}\left(\frac{-\alpha b^2t}{l^2}\right)^3\right] \tag{3.5}$$

which is first four terms of the series of the exact solution $u(x,t) = c\sin\left(\frac{bx}{l}\right)\exp^{(\frac{-\alpha b^2 t}{l^2})}$ Table 3.1 shows the approximate solution for Prob. (3.1) obtained for different values of α at $c = \alpha = l = b = 1$. It is clear that the approximations obtained by the method are in high agreement with those obtained using the exact solution. It is evident that the efficiency of the method can be enhanced by computing further terms of $u(x,t)$.

t	x	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	u_{Exact}
0.25	0.25	0.120391	0.141808	0.168363	0.19264	0.192678
	0.5	0.233297	0.274799	0.326257	0.373303	0.373377
	0.75	0.331697	0.390705	0.463867	0.530755	0.530861
	1.0	0.409475	0.482318	0.572635	0.655208	0.655338
0.5	0.25	0.071327	0.104075	0.132745	0.149473	0.150058
	0.5	0.138219	0.201678	0.257236	0.289653	0.290786
	0.75	0.196517	0.286743	0.365734	0.411823	0.413435
	1.0	0.242597	0.353979	0.451492	0.508389	0.510378
0.75	0.25	0.074051	0.102631	0.116828	0.114038	0.116865
	0.5	0.143498	0.198882	0.226393	0.220985	0.226465
	0.75	0.204022	0.282767	0.321882	0.314193	0.321983
	1.0	0.251862	0.34907	0.397357	0.387866	0.397483
1.0	0.25	0.110219	0.120805	0.109939	0.082468	0.091015
	0.5	0.213586	0.234098	0.213042	0.159809	0.176371
	0.75	0.303673	0.332837	0.302899	0.227213	0.250761
	1.0	0.374879	0.410881	0.373923	0.280490	0.309560

Table 3.1: Numerical values when $\alpha = 0.25, 0.5, 0.75$ and 1 for Eq. (3.1)

Example 3.2 *In this Example we apply the 2^{nd} Modification.*

We consider the one dimensional linear inhomogenous fractional Klein-Gordon equation *given by :*

$$D_t^{\alpha}u - \frac{\partial^2 u}{\partial x^2} + u = 6x^3\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} + (x^3 - 6x)t^3, \quad t > 0, x \in \mathbb{R}, 1 < \alpha \leqslant 2 \tag{3.6}$$

subject to the initial conditions:

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0. \tag{3.7}$$

by the MHPM,in view of (2.43),the homotopy for (3.6)and(3.7) becomes

$$D_t^{\alpha}u - 6x^3\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} = p\left[(x^3 - 6x)t^3 + \frac{\partial^2 u}{\partial x^2} - u\right], \tag{3.8}$$

where $f_0 = 6x^3(t^{3-\alpha}/\Gamma(4-\alpha), f_1 = (x^3 - 6x)t^3$.
Substituting (2.44) and the initial condition (3.7)into (3.8)and equating the terms with equal powers of p,we obtain the following set of equations:

$$\begin{aligned}
&p^0 : D_t^{\alpha}u_0 = 6x^3\frac{t^{3-\alpha}}{\Gamma(4-\alpha)}, \quad u_0(x,0) = 0, \frac{\partial u_0(x,0)}{\partial t} = 0.\\
&p^1 : D_t^{\alpha}u_1 = (x^3 - 6x)t^3 + \frac{\partial^2 u_0}{\partial x^2} - u_0, \quad u_1(x,0) = 0, \frac{\partial u_1(x,0)}{\partial t} = 0.\\
&p^2 : D_t^{\alpha}u_2 = \frac{\partial^2 u_1}{\partial x^2} - u_1, \quad u_2(x,0) = 0, \frac{\partial u_2(x,0)}{\partial t} = 0.\\
&\vdots
\end{aligned}$$

by solving the obove set of equations ,the first few components of the modified homotopy perturbation solution for(3.6)and(3.7) are

$$u_0(x,t) = x^3t^3, \quad u_j(x,t) = 0, \quad j \geqslant 1.$$

the exact solution $u(x,t) = x^3t^3$ *follows immediately*

Example 3.3 *In this Example we apply both Modifications on the same equation.*
Consider the following one-dimensional linear in-homogeneous fractional wave equation:

$$D_t^{\alpha}u + \frac{\partial u}{\partial x} = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin x + t\cos x, \quad t > 0, \quad x \in \mathbb{R}, \quad 0 < \alpha \leqslant 1 \tag{3.9}$$

subject to the initial condition:

$$u(x,0) = 0. \tag{3.10}$$

The 1^{st} Modification MHPM1 :

According to the MHPM1, in view of (2.43), the homotopy for (3.9) and (3.10) can be constructed as:

$$\frac{\partial u}{\partial t} - \left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin x + t \cos x \right) = p \left[\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} - D_t^{\alpha} u \right] \tag{3.11}$$

According to the homotopy (3.11), we obtain the following set of linear partial differential equations:

$$\begin{aligned}
\frac{\partial u_0}{\partial t} &= \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin x + t \cos x, \quad u_0(x,0) = 0, \\
\frac{\partial u_1}{\partial t} &= \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial x} - D_t^{\alpha} u_0, \quad u_1(x,0) = 0, \\
\frac{\partial u_2}{\partial t} &= \frac{\partial u_1}{\partial t} - \frac{\partial u_1}{\partial x} - D_t^{\alpha} u_1, \quad u_2(x,0) = 0,
\end{aligned}$$

Consequently, solving the above equations for u_0, u_1, u_2 the first few components of homotopy perturbation solution for Equation (3.9) are derived as follows

$$\begin{aligned}
u_0 &= \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} \sin x + \frac{t^2}{2} \cos x, \\
u_1 &= \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} \sin x + \frac{t^2}{2} \cos x - \frac{t^{3-\alpha}}{\Gamma(4-\alpha)} \cos x + \frac{t^3}{6} \cos x - \frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} \sin x \\
&\vdots
\end{aligned}$$

Hence the series of the solution is

$$\begin{aligned}
u(t,x) &= u_0(t,x) + u_2(t,x) + u_2(t,x) + \ldots \\
&= \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} \sin x + t^2 \cos x - \frac{t^{3-\alpha}}{\Gamma(4-\alpha)} \cos x + \frac{t^3}{6} \cos x - \frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} \sin x
\end{aligned}$$

Hence, if we eliminating the noise terms in the above series of solution the result is the exact solution of (3.9) when $\alpha = 1$ which is

$$u(x,t) = t \sin x$$

The 2^{nd} Modification MHPM2 :

According to the MHPM2, in view of (2.43), the homotopy for (3.9) and (3.10) can be *constructed as:*

$$D_t^{\alpha}u - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin x = p\left[t\cos x - \frac{\partial u}{\partial x}\right], \tag{3.12}$$

where $f_0 = t^{1-\alpha}/\Gamma(2-\alpha)\sin x$, $f_1 = t\cos x$. *Substituting (2.44) and the initial condition (3.10) into (3.12) and equating the terms with identical powers of* p , *we obtain the following set of equations:*

$$\begin{aligned} p^0 &: D_t^{\alpha}u_0 = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin x, \quad u_0(x,0) = 0 \\ p^1 &: D_t^{\alpha}u_1 = t\cos x - \frac{\partial u_0}{\partial x}, \quad u_1(x,0) = 0 \\ p^2 &: D_t^{\alpha}u_2 = -\frac{\partial u_1}{\partial x}, \quad u_2(x,0) = 0 \\ &\vdots \end{aligned} \tag{3.13}$$

Now applying the operator I^{α} *to both sides of Equation (3.13) and using initial condition yields*

$$\begin{aligned} u_0 &= t\sin x \\ u_1 &= 0 \\ u_2 &= 0 \\ &\vdots \end{aligned}$$

Consequently, the first few components of the modified homotopy perturbation solution for(3.9) and (3.10)are derived as follows:

$$u_0(x,t) = t\sin x, \quad u_j(x,t) = 0, \quad j \geqslant 1.$$

The exact solution $u(x,t) = t\sin x$ *follows immediately.*

The success of obtaining the exact solution by one iteration only is a result of the proper selection of f_0 *and* f_1,

3.2 Nonlinear Problems

For nonlinear problems, there exists no method that yields exact solutions and therefore only approximate solutions can be derived. In this section the (MHPM) is used to obtain the exact solution for four nonlinear problems.

Example 3.4 *In this Example we apply the* 1^{st}*Modification consider nonlinear fractional Differential equation:*

$$D_t^{\alpha}u - u(t) + u^2(t) = 2t, \quad t > 0. \tag{3.14}$$

with the initial condition : $u(0) = 0$
Based on Modified Homotopy Perturbation Method, homotopy equation (2.21) became:

$$\frac{\partial u}{\partial t} - 2t = p\left[\frac{\partial u}{\partial t} + u - u^2 - D_t^{\alpha}u\right]$$

So, the solution of this equation is

$$u = u_0 + pu_1 + p^2u_2 + \ldots,$$

By substituting the basic assumptions and initial conditions for the homotopy equation obtained

$$\begin{aligned}
\frac{\partial u_0}{\partial t} &= 2t, \quad u_0(0) = 0 \\
\frac{\partial u_1}{\partial t} &= \frac{\partial u_0}{\partial t} + u_0 - u_0^2 - D_t^{\alpha}u_0, \quad u_1(0) = 0 \\
\frac{\partial u_2}{\partial t} &= \frac{\partial u_1}{\partial t} + u_1 - 2u_0u_1 - D_t^{\alpha}u_1, \quad u_2(0) = 0 \\
\frac{\partial u_3}{\partial t} &= \frac{\partial u_2}{\partial t} + u_2 - 2u_0u_2 - u_1^2 - D_t^{\alpha}u_2, \quad u_3(0) = 0 \\
&\vdots
\end{aligned}$$

Then, it is integrated so that it can be obtained

$$\begin{aligned}
u_0 &= t^2,\\
u_1 &= t^2 + \frac{t^3}{3} - \frac{t^5}{5} - \frac{2t^{3-\alpha}}{\Gamma(4-\alpha)}\\
u_2 &= t^2 + \frac{2t^3}{3} - \frac{3t^5}{5} - \frac{13t^6}{90} + \frac{t^8}{20} - \frac{4t^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{2t^{4-\alpha}}{\Gamma(5-\alpha)} + \frac{24t^{6-\alpha}}{\Gamma(7-\alpha)} + \frac{2t^{4-2\alpha}}{\Gamma(5-2\alpha)} + \frac{4t^{7-2\alpha}}{(7-2\alpha)\Gamma(4-\alpha)}\\
&\vdots
\end{aligned}$$

Therefore, solution function of equation (3.14):

$$\begin{aligned}
u(t) &= u_0(t) + u_1(t) + u_2(t) + \dots\\
&= 3t^2 + t^3 + \frac{t^4}{12} - \frac{4t^5}{5} - \frac{13t^6}{90} + \frac{t^8}{20} - \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{2t^{4-\alpha}}{\Gamma(5-\alpha)} + \frac{24t^{6-\alpha}}{\Gamma(7-\alpha)} + \frac{2t^{4-2\alpha}}{\Gamma(5-2\alpha)}\\
&+ \frac{4t^{7-2\alpha}}{(7-2\alpha)\Gamma(4-\alpha)} + \dots
\end{aligned}$$

Example 3.5 *in this Example we apply the* 2^{nd}*Modification,*
Now consider the following nonlinear fractional initial value prolem FIVP:

$$D^\alpha u = \frac{40320}{\Gamma(9-\alpha)}t^{8-\alpha} - 3\frac{\Gamma(5+\alpha/2)}{\Gamma(5-\alpha/2)}t^{4-\alpha/2} + \frac{9}{4}\Gamma(\alpha+1) + \left(\frac{3}{2}t^{\alpha/2} - t^4\right)^3 - u^{3/2}, \quad 0 < \alpha \leqslant 2 \tag{3.15}$$

subject to the initial conditions

$$u(0) = 0, \quad \frac{\partial u(0)}{\partial t} = 0 \quad for \quad \alpha > 1$$

the nonlinear term $u^{3/2}$ *in (3.15) is expanded using the Taylor series as follow*

$$u^{3/2} \approx 1 + \frac{3}{2}(u-1) + \frac{3}{8}(u-1)^2 = -\frac{1}{8} + \frac{3}{4}u + \frac{3}{8}u^2. \tag{3.16}$$

According to (2.35), we can construct the following homotopy

$$D^\alpha u + p\left[-\frac{1}{8} + \frac{3}{4}u + \frac{3}{8}u^2\right] = \sum_{n=0}^{\infty} p^n f_n(t). \tag{3.17}$$

where we take $f_n(t)$ to be given by

$$\begin{aligned}
f_0(t) &= \frac{40320}{\Gamma(9-\alpha)} t^{8-\alpha} - 3\frac{\Gamma(5+\alpha/2)}{\Gamma(5-\alpha/2)} t^{4-\alpha/2} + \frac{9}{4}\Gamma(\alpha+1)\\
f_1(t) &= \left(\frac{3}{2}t^{\alpha/2} - t^4\right)^3\\
f_n(t) &= 0, \quad n \geqslant 2
\end{aligned}$$

Substituting (2.13) into (3.17) and equating the terms with the same power of p, we obtain

$$\begin{aligned}
&p^0 : D^\alpha u_0 = f_0(t),\\
&p^1 : D^\alpha u_1 = f_1(t) - \left[-\frac{1}{8} + \frac{3}{4}u_0 + \frac{3}{8}u_0^2\right],\\
&p^2 : D^\alpha u_2 = -\left[\frac{3}{4}u_1 - \frac{3}{8}(2u_0 u_1)\right],\\
&\vdots
\end{aligned} \tag{3.18}$$

In (3.16) we have taken the first three terms of the Taylor expansion series of the nonlinear term $u^{3/2}$ in order to show that the computation of u_n, $n \geqslant 2$, depends heavily on u_0 and u_1, but if we use the whole terms of the Taylor expansion series, i.e.

$$u^{3/2} = \omega(u) = \sum_{k=0}^{\infty} \omega^{(k)}(u)(u-1)^k,$$

then the first two linear equations can be given by

$$\begin{aligned}
&p^0 : D^\alpha u_0 = f_0(t),\\
&p^1 : D^\alpha u_1 = f_1(t) - \sum_{k=0}^{\infty} \omega^{(k)}(u_0)(u_0-1)^k = f_1(t) - u_0^{3/2},\\
&p^2 : D^\alpha u_2 = -\left[\frac{3}{4}u_1 - \frac{3}{8}(2u_0 u_1)\right],\\
&\vdots
\end{aligned}$$

Applying the operator I^α, which is the inverse operator of D^α, we obtain

$$\begin{aligned}
u_0 &= u(0) + \frac{\partial u(0)}{\partial t}t + I^\alpha[f_0] \\
&= I^\alpha\left[\frac{40320}{\Gamma(9-\alpha)}t^{8-\alpha} - 3\frac{\Gamma(5+\alpha/2)}{\Gamma(5-\alpha/2)}t^{4-\alpha/2} + \frac{9}{4}\Gamma(\alpha+1)\right] \\
&= \left(\frac{3}{2}t^{\alpha/2} - t^4\right)^2 \\
u_1 &= I^\alpha[f_1] - I^\alpha[u_0^{3/2}] \\
&= I^\alpha\left[\left(\frac{3}{2}t^{\alpha/2} - t^4\right)^3\right] - I^\alpha\left[\left(\frac{3}{2}t^{\alpha/2} - t^4\right)^3\right] = 0
\end{aligned}$$

According to (3.18), it is clear that $u_m = 0 \quad, m \geqslant 2$. Hence, the exact solution

$$u = \left(\frac{3}{2}t^{\alpha/2} - t^4\right)^2$$

Example 3.6 *in this Example we apply both Modifications on the same equation Consider the nonlinear time-fractional advection partial differential equation:*

$$D_t^\alpha u + u\frac{\partial u}{\partial x} = x + xt^2, \quad t > 0, x \in \mathbb{R}, 0 < \alpha \leqslant 1, \tag{3.19}$$

subject to the initial condition

$$u(x,0) = 0 \tag{3.20}$$

The 1st Modification MHPM1:

in view of MHPM1 (2.22),the homotopy for Eq(3.19)be constructed as

$$\frac{\partial u}{\partial t} - x - xt^2 = p\left[\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} - D_t^\alpha u\right] \tag{3.21}$$

Substituting (2.44) and the initial condition (3.20) into (3.21) and equating the terms with identical powers of p, we obtain the following set of linear partial differential equations

$$\begin{aligned}
\frac{\partial u_0}{\partial t} &= x + xt^2, \quad u_0(x,0) = 0, \\
\frac{\partial u_1}{\partial t} &= \frac{\partial u_0}{\partial t} - u_0\frac{\partial u_0}{\partial x} - D_t^\alpha u_0, \quad u_1(x,0) = 0, \\
\frac{\partial u_2}{\partial t} &= \frac{\partial u_1}{\partial t} - u_0\frac{\partial u_1}{\partial x} - u_1\frac{\partial u_0}{\partial x} - D_t^\alpha u_1, \quad u_2(x,0) = 0, \\
&\vdots
\end{aligned} \tag{3.22}$$

Consequently, the first few components of the homotopy perturbation solution for Eq. (3.19) are derived as follows

$$
\begin{aligned}
u_0(x,t) &= x\left[t+\frac{t^3}{3}\right],\\
u_1(x,t) &= x\left[t-\frac{2t^5}{15}-\frac{t^7}{63}-\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}-\frac{2t^{4-\alpha}}{\Gamma(5-\alpha)}\right],\\
u_2(x,t) &= x\left[t-\frac{t^3}{3}-\frac{2t^5}{15}+\frac{t^7}{45}+\frac{2t^9}{567}-\frac{4t^{11}}{2475}-\frac{4t^{13}}{12285}-\frac{t^{15}}{59535}-\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}+\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)}\right.\\
&\quad+\left(\frac{2}{\Gamma(3-\alpha)}-\frac{2}{\Gamma(4-\alpha)}\right)\frac{t^{4-\alpha}}{(4-\alpha)}+\left(\frac{4}{\Gamma(5-\alpha)}+\frac{16}{\Gamma(6-\alpha)}\right)\frac{t^{6-\alpha}}{(6-\alpha)}\\
&\quad+\left(\frac{80}{\Gamma(8-\alpha)}-\frac{4}{15\Gamma(3-\alpha)}\right)\frac{t^{8-\alpha}}{(8-\alpha)}-\left(\frac{8}{15\Gamma(5-\alpha)}+\frac{2}{63\Gamma(3-\alpha)}\right)\frac{t^{10-\alpha}}{(10-\alpha)}\\
&\quad-\frac{4t^{12-\alpha}}{63(12-\alpha)\Gamma(5-\alpha)}+\left(\frac{2}{\Gamma(5-2\alpha)}-\frac{1}{\Gamma(3-\alpha)^2}\right)\frac{t^{5-2\alpha}}{(5-2\alpha)}-\frac{4t^{7-2\alpha}}{(7-2\alpha)\Gamma(3-\alpha)\Gamma(5-\alpha)}\\
&\quad\left.-\frac{4t^{9-2\alpha}}{(9-2\alpha)\Gamma(5-\alpha)^2}\right],\\
&\ \ \vdots
\end{aligned}
$$

and so on, in the same manner the rest of components can be obtained using the Mathematical package. The third-order term approximate solution for Eq. (3.19) is given by

$$
\begin{aligned}
u(x,t) &= x\left[3t-\frac{4t^5}{15}+\frac{2t^7}{315}+\frac{2t^9}{567}-\frac{4t^{11}}{2475}-\frac{4t^{13}}{12285}-\frac{t^{15}}{59535}-\frac{3t^{2-\alpha}}{\Gamma(3-\alpha)}+\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)}\right.\\
&\quad+\left(\frac{2}{\Gamma(3-\alpha)}-\frac{4}{\Gamma(4-\alpha)}\right)\frac{t^{4-\alpha}}{(4-\alpha)}+\left(\frac{4}{\Gamma(5-\alpha)}+\frac{16}{\Gamma(6-\alpha)}\right)\frac{t^{6-\alpha}}{(6-\alpha)}\\
&\quad+\left(\frac{80}{\Gamma(8-\alpha)}-\frac{4}{15\Gamma(3-\alpha)}\right)\frac{t^{8-\alpha}}{(8-\alpha)}-\left(\frac{8}{15\Gamma(5-\alpha)}+\frac{2}{63\Gamma(3-\alpha)}\right)\frac{t^{10-\alpha}}{(10-\alpha)}\\
&\quad-\frac{4t^{12-\alpha}}{63(12-\alpha)\Gamma(5-\alpha)}+\left(\frac{2}{\Gamma(5-2\alpha)}-\frac{1}{\Gamma(3-\alpha)^2}\right)\frac{t^{5-2\alpha}}{(5-2\alpha)}-\frac{4t^{7-2\alpha}}{(7-2\alpha)\Gamma(3-\alpha)\Gamma(5-\alpha)}\\
&\quad\left.-\frac{4t^{9-2\alpha}}{(9-2\alpha)\Gamma(5-\alpha)^2}\right],
\end{aligned}
$$

The 2^{nd} Modification MHPM2 :

in view of MHPM2 the homotopy for (3.19)and (3.20)

$$D_t^\alpha u - x = p\left[xt^2 - u\frac{\partial u}{\partial x}\right]. \tag{3.23}$$

substituting (2.44) and the initial condition of(3.20)into (3.23) as above ,we have

$$\begin{aligned}
p^0 &= D_t^\alpha u_0 = x, \quad u_0(x,t) = 0,\\
p^1 &= D_t^\alpha u_1 = xt^2 - u_0\frac{\partial u_0}{\partial x}, \quad u_1(x,t) = 0,\\
p^2 &= D_t^\alpha u_2 = -u_1\frac{\partial u_0}{\partial x} - u_0\frac{\partial u_1}{\partial x}, \quad u_2(x,t) = 0,\\
&\vdots
\end{aligned}$$

solving the above set of equations,we have the folloxing first few components of the modified homotopy perturbation solution for (3.19)and (3.20)

$$u_0(x,t) = xt, \quad u_j(x,t) = 0, \quad j \geqslant 1.$$

the exact solution follows $u(x,t) = xt$ immediatly

Example 3.7 *in this example we apply the standard HPM and the 1^{st} Modification on the same equation :*

we consider nonlinear time-fractional advection partial differential equation:

$$D_t^\alpha u + u\frac{\partial u}{\partial x} = 2t + x + t^3 + xt^2, \quad x > 0, \quad 0 < \alpha \leqslant 2 \tag{3.24}$$

subject to the initial condition:

$$u(x,0) = 0, \tag{3.25}$$

The exact solution for the special case $\alpha = 1$ is given by:

$$u(x,t) = t^2 + xt \tag{3.26}$$

The standard HPM :

First we use standard HPM for solve the Equation (3.24), according to homotopy, we have

$$D_t^\alpha u = p(-u\frac{\partial u}{\partial x} + 2t + x + t^3 + xt^2) \tag{3.27}$$

Substituting Equation (2.44) in to (3.27) equating the terms with having identical power of p, we obtain the following series of equations

$$\begin{aligned}
&p^0 : D_t^\alpha u_0 = 0, \quad u_0(x,t) = 0,\\
&p^1 : D_t^\alpha u_1 = -u_0\frac{\partial u_0}{\partial x} + 2t + x + t^3 + xt^2\\
&p^2 : D_t^\alpha u_2 = -u_0\frac{\partial u_1}{\partial x} - u_1\frac{\partial u_0}{\partial x},\\
&\vdots
\end{aligned} \tag{3.28}$$

Now applying the operator I^α to both sides of Equation (3.28) and using initial condition yields

$$\begin{aligned}
u_0 &= \frac{t^\alpha}{\Gamma(\alpha+1)},\\
u_1 &= \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{xt^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)},\\
u_2 &= -\frac{t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - \frac{t^{3\alpha+2}\Gamma(2\alpha+3)}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)}\\
&\vdots
\end{aligned}$$

Hence the series of the solution is:

$$\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + ...\\
&= \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{xt^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\\
&- \frac{t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - \frac{t^{3\alpha+2}\Gamma(2\alpha+3)}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} + ...
\end{aligned}$$

The 1^{st} Modification MHPM1 :

by the MHPM1, the homotopy for (3.24) and (3.25) can be written as:

$$\frac{\partial u}{\partial t} - (2t + x + t^3 + xt^2) = p\left[\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} - D_t^{\alpha} u\right] \tag{3.29}$$

According to homotopy (3.29) we obtain the following set of partial differential equations

$$\begin{aligned}
\frac{\partial u_0}{\partial t} &= 2t + x + t^3 + xt^2, \quad u_0(x,0) = 0,\\
\frac{\partial u_1}{\partial t} &= \frac{\partial u_0}{\partial t} - u_0\frac{\partial u_0}{\partial x} - D_t^{\alpha} u_0, \quad u_1(x,0) = 0\\
\frac{\partial u_2}{\partial t} &= \frac{\partial u_1}{\partial t} - u_0\frac{\partial u_1}{\partial x} - u_1\frac{\partial u_0}{\partial x} - D_t^{\alpha} u_1, \quad u_2(x,0) = 0\\
&\vdots
\end{aligned}$$

Consequently, solving the above equations for u_0, u_1, u_2 the first few components of homotopy perturbation solution for Equation (3.24) are derived as follows

$$\begin{aligned}
u_0(x,t) &= t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3},\\
u_1(x,t) &= t^2 + xt + \frac{7t^6}{72} + \frac{t^8}{96} - \frac{2t^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{xt^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{6t^{5-\alpha}}{\Gamma(6-\alpha)} - \frac{xt^{4-\alpha}}{\Gamma(5-\alpha)},\\
u_2(x,t) &= t^2 - \frac{t^4}{4} - \frac{14t^6}{72} - \frac{143xt^8}{2880} + \frac{2783t^{10}}{302400} + \frac{5t^{12}}{8064} - \frac{3t^{\alpha-3}}{\Gamma(4-\alpha)} - t^{5-\alpha}\\
&\left[-\frac{12}{\Gamma(6-\alpha)} + \frac{3}{(5-\alpha)\Gamma(4-\alpha)}\right] + \frac{t^{7-\alpha}}{\Gamma(7-\alpha)}\left[\frac{1}{\Gamma(3-\alpha)} + \frac{2}{3\Gamma(4-\alpha)}\right.\\
&\left.+ \frac{7}{72\Gamma(7-\alpha) + \dfrac{2}{\Gamma(5-\alpha)}}\right] + t^{9-\alpha}\left[\frac{2}{\Gamma(6-\alpha)} + \frac{1}{\Gamma(5-\alpha)} + \frac{1}{(96\Gamma(9-\alpha)}\right]\\
&+ \frac{2t^{4-2\alpha}}{\Gamma(5-2\alpha)} + \frac{6t}{\Gamma(7-2\alpha)} + x\left[t + \frac{2t^3}{3} - \frac{4t^5}{15} - \frac{38t^7}{28835} - \frac{2t^{11}}{2079} - \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\right]\\
&+ 2t^{4-\alpha}\left[\frac{1}{\Gamma(5-\alpha)} + \frac{1}{(4-\alpha)\Gamma(3-\alpha)}\right] + \frac{2t^{6-\alpha}}{\Gamma(6-\alpha)}\left[\frac{1}{\Gamma(5-\alpha)} + \frac{1}{15\Gamma(6-\alpha)}\right]\\
&+ \frac{t^{8-\alpha}}{\Gamma(8-\alpha)}\left[\frac{8}{3\Gamma(5-\alpha)} + \frac{1}{63\Gamma(8-\alpha)} + \frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)}\right]\\
&\vdots
\end{aligned}$$

Hence the series of the solution is:

$$\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + ... \\
&= 3t^2 + 3xt + \frac{t^4}{4} + \frac{xt^3}{3} - \frac{7t^6}{72} - \frac{2xt^5}{15} - \frac{xt^7}{63} - \frac{t^8}{96} - \frac{2t^{3-\alpha}}{\Gamma(4-\alpha)} \\
&- \frac{xt^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{6t^{5-\alpha}}{\Gamma(6-\alpha)} - \frac{xt^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{14t^6}{72} - \frac{143xt^8}{2880} + \frac{2783t^{10}}{302400} \\
&+ \frac{5t^{12}}{8064} - \frac{3t^{\alpha-3}}{\Gamma(4-\alpha)} - t^{5-\alpha}\left[-\frac{12}{\Gamma(6-\alpha)} + \frac{3}{(5-\alpha)\Gamma(4-\alpha)} \right] \\
&+ \frac{t^{7-\alpha}}{\Gamma(7-\alpha)}\left[\frac{1}{\Gamma(3-\alpha)} + \frac{2}{3\Gamma(4-\alpha)} + \frac{7}{72\Gamma(7-\alpha) + \frac{2}{\Gamma(5-\alpha)}} \right] \\
&+ t^{9-\alpha}\left[\frac{2}{\Gamma(6-\alpha)} + \frac{1}{\Gamma(5-\alpha)} + \frac{1}{(96\Gamma(9-\alpha)} \right] + \frac{2t^{4-2\alpha}}{\Gamma(5-2\alpha)} + \frac{6t}{\Gamma(7-2\alpha)} \\
&+ x\Bigg[t + \frac{2t^3}{3} - \frac{4t^5}{15} - \frac{38t^7}{28835} - \frac{2t^{11}}{2079} - \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2t^{4-\alpha}[\frac{1}{\Gamma(5-\alpha)} \\
&+ \frac{1}{(4-\alpha)\Gamma(3-\alpha)} \Bigg] + \frac{2t^{6-\alpha}}{\Gamma(6-\alpha)}\left[\frac{1}{\Gamma(5-\alpha)} + \frac{1}{15\Gamma(6-\alpha)} \right] + \frac{t^{8-\alpha}}{\Gamma(8-\alpha)} \\
&\left[\frac{8}{3\Gamma(5-\alpha)} + \frac{1}{63\Gamma(8-\alpha)} \right] + \frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} \Bigg] + \dots
\end{aligned}$$

Note that that if we take $\alpha = 1$ then the first few components the solution of equation (3.24) is given

$$\begin{aligned}
u_0(x,t) &= t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3}, \\
u_1(x,t) &= -\frac{7t^6}{72} - \frac{2t^5}{15} - \frac{xt^7}{63} - \frac{t^8}{96} - \frac{t^4}{4} - \frac{2xt^3}{3} \\
u_2(x,t) &= \frac{2t^5}{15} - \frac{7t^6}{12} - \frac{22xt^7}{315} + \frac{143xt^8}{2880} + \frac{38xt^9}{2835} \\
&+ \frac{2783t^{10}}{302400} + \frac{2t^{11}}{2079} + \frac{5t^{12}}{8064}
\end{aligned}$$

we can cancel the noise term $-\frac{t^4}{4} - \frac{xt^3}{3}$ between u_0 and u_1 and remaining term of u_0 still satisfy the equation, and so on. Hence we get in special case

$$u(x,t) = t^2 + xt \tag{3.30}$$

which is exactly the exact solution

t	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1$		Exact solution
	HPM	MHPM1	HPM	MHPM1	HPM	MHPM1	HPM	MHPM1	
0.1	0.65584	0.05707	0.43738	0.05342	0.24625	0.04573	0.12833	0.03003	0.03000
0.2	0.81658	0.14802	0.63668	0.13656	0.42824	0.11602	0.26672	0.08026	0.08000
0.3	0.97727	0.27175	0.80800	0.24860	0.59233	0.21080	0.40525	0.15083	0.15000
0.4	1.15164	0.42749	0.97218	0.38875	0.74332	0.32958	0.53406	0.24173	0.24000
0.5	1.34334	0.61449	1.13773	0.55624	0.88264	0.47176	0.64323	0.35261	0.35000
0.6	1.55367	0.83174	1.30923	0.74997	1.01071	0.63627	0.72281	0.48249	0.48000
0.7	1.78326	1.07752	1.48946	0.96806	1.12753	0.82120	0.76274	0.62940	0.63000
0.8	2.03243	1.34874	1.68024	1.20728	1.23283	1.02318	0.75281	0.78983	0.80000
0.9	2.30143	1.64033	1.88280	1.46233	1.32609	1.23674	0.68265	0.95815	0.99000
1.0	2.59049	1.94428	2.09795	1.72495	1.40660	1.45343	0.54167	1.12569	1.20000

Table 3.2: Comparison of HPM and MHPM1 with exact solution of equation (3.24) for different values of α and $x = 0.2$

Table 3.2 shows the approximate solution for Equation (3.24) obtained for different values of α using homotopy perturbation method and modified homotopy perturbation method. The value $\alpha = 1$ is the only case for which we know exact solution $u(x,t) = t^2 + xt$ and our approximate solution using modified homotopy perturbation method is more accurate than homotopy perturbation method. It is be noted that the second-order term series and third-order term series was used in evaluating the approximate solutions by modified homotopy perturbation method and homotopy perturbation method respectively for Table3.2.

Conclusion

In this work, we carefully proposed an efficient two modifications of the Homotopy Perturbation Method for calculating analytical approximate and exact solutions for some linear (nonlinear) ordinary (partial) differential equations of fractional order by using only a minimal number of iterations and minimize the already reduced volume of calculations.

The results of **Example 3.3** and **Example 3.6** show us that the 2^{nd} Modification MHPM2 is remarkably very effective, very simple, and can obtain the exact solutions by using only one iteration and requires less computational work when compared with the 1^{st} Modification MHPM1,on the other hand the **Example 3.7** show us that the 1^{st} Modification MHPM1 has a very high accuracy comparing with standerd HPM,so we can say that these two kind of modifications are full advantage of the Homotopy perturbation method HPM.

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ملخص

في هاذا العمل طبقنا تعديلين على طريقة اضطراب التماثل لهي اللذان اقترحهما اوديبات و موماني للحصول على حلول للمعدلات الخطية (غير الخطية) و العادية (الجزئية) ذات الترتيب الكسري، المشتق الكسري يعطى بدلالة كابوتو.
تم اعطاء بعض الامثلة التوضيحية للكشف عن فعالية وملاءمة هاذين التعديلين .

كلمات مفتاحية : حساب التفاضل والتكامل الكسري، المشتقات الكسرية، المعادلات التفاضلية الكسرية، طريقة اضطراب التماثل ، طريقة اضطراب التماثل المعدل .

Abstract

In this work, we applied two modifications of He's homotopy perturbation method HPM suggested by Momani and Odibat to obtaining solutions of linear (nonlinear) ordinary (partial) differential equations of fractional order. The fractional derivative is described in the Caputo sense.
Some illustrative examples are given to reveal the effectiveness and convenience of these modifications.

Keywords : Fractional calculus, fractional derivatives, Fractional Differential Equations, Homotopy Perturbation Method, Modified Homotopy Perturbation Method.

Résumé

Dans ce travail, nous avons appliqué deux modifications de la méthode de perturbation par homotopie de He HPM suggéré par Momani et Odibat pour obtenir des solutions de linéaires (non linéaires) ordinaires (partielles) équations différentielles d'ordre fractionnaire. La dérivée fractionnaire est décrite dans le Caputo sens.
Quelques exemples illustratifs sont donnés pour révéler l'efficacité et la commodité de ces modifications.

Mot-clés : Calcul Fractionnaire, dérivées fractionnaires, équations différentielles fractionnaires, méthode d'homotopie perturbatrice, méthode d'homotopie perturbatrice modifiée