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Linear and non-linear functions on quasi-normed spaces

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وَقُلْ رَبِّ زِدْنِي عِلْمًا

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Résumé

Cette thèse développe la théorie des opérateurs entre les espaces quasi-Banach selon trois axes. Nous étendons d'abord les théorèmes fondamentaux (Banach Steinhaus, graphe fermé, etc) et nous établissons une théorie de continuité pour les opérateurs multilinéaires Ensuite, aux relations linéaires multivoques. Enfin, nous étudions les opérateurs non linéaires dans les espaces p-normés, en particulier les opérateurs de Lipschitz, et caractérisons les idéaux d'opérateurs deux-Lipschitz quotients à gauche $\mathcal{A}^{-1} \circ \mathcal{A}_{BLip}$, identifiant des classes importantes comme les opérateurs de Grothendieck et Rosenthal. Nos résultats relient la théorie linéaire et non linéaire.

Mots-clés : Espaces quasi-Banach, espaces p-normés, relations linéaires multivoques, opérateurs multilinéaires, opérateurs de Lipschitz, opérateurs deux-Lipschitz, idéaux d'opérateurs, idéaux quotients, opérateurs de Grothendieck, opérateurs de Rosenthal.

Abstract

This thesis develops operator theory in quasi-Banach spaces, focusing on three areas. First, we extend fundamental theorems (Banach Steinhaus, closed graph, etc) and we develop continuity and boundedness theory for multilinear operators. Second, to multivalued linear relations. Third, we study nonlinear operators in p-normed spaces, particularly Lipschitz operators, and characterize two-Lipschitz left-hand quotient operator ideals $\mathcal{A}^{-1} \circ \mathcal{A}_{BLip}$, identifying important classes such as Grothendieck and Rosenthal operators. Our work bridges linear and nonlinear operator theory.

Keywords: Quasi-Banach spaces, p-normed spaces, multivalued linear relations, multilinear operators, Lipschitz operators, two-Lipschitz operators, operator ideals, quotient ideals, Grothendieck operators, Rosenthal operators.

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« La reconnaissance est la mémoire du cœur. »

— Jean-Baptiste Massieu

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[Mazouz Ahmed]

Dédicace

*À mon cher fils, **Ilyas**,*

pour ton innocence, ta joie et ton amour inconditionnel.

À ma chère épouse

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pour leurs enseignements précieux, leur sagesse et leur amour incommensurable.

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pour leur encouragement et leur présence tout au long de ce parcours.

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No.	Status	Publication Details
1	Published	Mazouz, A., Hamidi, K. and Tallab, A. On two-Lipschitz left-hand quotient operator ideals. *Ann. Univ. Ferrara* 72, 19 (2026).

Table 2 – Submitted Papers

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1	Submitted	A. Ammar, A. Mazouz and A. Tallab, Quasi-linear relations in quasi-Banach spaces (under review), .
2	Submitted	A. Ammar, A. Mazouz and A. Tallab, On multilinear operators in quasi-Banach spaces (under review), .

Most Important Notations

X, Y, Z, W	Pointed metric spaces with distinguished point 0.
E, F, G	Banach spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$).
$Æ(X), \mathcal{F}(X)$	Arens-Eells space (Lipschitz-free space) of X.
$\mathcal{M}(X)$	Space of molecules on X.
$X^{\#}$	Lipschitz dual of X (i.e., $\mathrm{Lip}_0(X, \mathbb{K})$).
E^*	Topological dual of Banach space E.
$\mathrm{Lip}_0(X, Y)$	Space of Lipschitz mappings $A : X \to Y$ with $A(0) = 0$.
$BLip_0(X, Y; E)$	Space of two-Lipschitz operators $A : X \times Y \to E; A(., 0) = A(0, .) = 0$.
$\mathcal{L}(E, F)$	Banach space of bounded linear operators from E to F.
A_B	Bi-linearization of two-Lipschitz operator A.
A_L	Linearization of A (or $(A_B)_L$).
δ_X	Canonical isometric embedding from X to Æ(X).
$\|.\|_{\mathrm{Lip}}$	Lipschitz norm.
$BLip(.)$	Two-Lipschitz norm.
$\|.\|_{\mathcal{M}(X)}$	Molecule norm.
$\pi(.)$	Projective tensor norm.
$E \otimes F$	Algebraic tensor product.
$E \widehat{\otimes}_\pi F$	Completed projective tensor product.

σ_2	Canonical bilinear map $\sigma_2 : E \times F \to E\widehat{\otimes}_\pi F$.
$\mathcal{A}, \mathcal{B}$	Linear operator ideals.
$\mathcal{A}_{\mathrm{BLip}}$	Two-Lipschitz operator ideal.
$\mathcal{A}^{-1} \circ \mathcal{B}$	Left-hand quotient of operator ideals.
$\mathcal{A} \circ \mathcal{B}^{-1}$	Right-hand quotient of operator ideals.
$\Vert.\Vert_{\mathcal{A}^{-1}\circ\mathcal{B}}$	Quotient norm.
$\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}$	Two-Lipschitz left-hand quotient ideal.
$\Vert.\Vert_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}}$	Two-Lipschitz quotient norm.
$(\mathcal{A}^{-1} \circ \mathcal{B}) \circ \mathrm{BLip}_0$	Composition ideal.
LP	Linearization Property.
$\mathcal{A}_{\mathrm{BLip}}$ has LP in $\mathcal{A}$	$A \in \mathcal{A}_{\mathrm{BLip}} \Leftrightarrow A_L \in \mathcal{A}$ with equal norms.
$\mathcal{K}(E,F)$	Compact operators.
$\mathcal{W}(E,F)$	Weakly compact operators.
$\mathcal{S}(E,F)$	Separable bounded operators.
$\mathcal{R}(E,F)$	Rosenthal operators.
$\mathfrak{G}(E,F)$	Grothendieck operators.
$\mathcal{V}(E,F)$	Completely continuous operators.
$\mathfrak{G}^{\mathrm{BL}}$	Grothendieck two-Lipschitz operators.
$\mathfrak{R}^{\mathrm{BL}}$	Rosenthal two-Lipschitz operators.
$\mathrm{BLip}_{0\mathcal{K}}$	Compact two-Lipschitz operators.
$\mathrm{BLip}_{0\mathcal{W}}$	Weakly compact two-Lipschitz operators.
$\mathrm{Im}_{\mathrm{BLip}}(A)$	Two-Lipschitz image of A.
$\ell_1(\mathbb{N}, F_i)$	ℓ_1-Cartesian product of Banach spaces.
I_n, J_n	Canonical injections and projections.
λ_{in}	Kronecker delta, $\lambda_{in} = 0 \; if \; i \neq n; \lambda_{in} = 1 \; if \; i = n.$

$2^V \setminus \{\emptyset\}$	Collection of nonempty subsets of V.
$R(U,V)$	Class of all relations from U to V.
$L\mathcal{R}(X,Y)$	Class of linear relations from X to Y.
T,S	Linear relations (multivalued operators).
$G(T)$	Graph of $T : \{(x,y) : x \in D(T), y \in Tx\}$.
$D(T)$	Domain of $T : \{x : Tx \neq \emptyset\}$.
$R(T)$	Range of $T : T(D(T))$.
$N(T) = T^{-1}(0)$	Null space/kernel of T.
$T(0)$	Multivalued part of T.
T^{-1}	Inverse relation.
$ST = S \circ T$	Composition of relations.
$S+T$	Sum of relations: $(S+T)x = Sx + Tx$.
$T\|_M$	Restriction of T to M.
$T\|_E = TJ_EJ_E^{-1}$	Restriction formula.
$\\|T\\| = \\|QT\\|$	Norm of T where $Q = Q_{T(0)}$.
$\\|Tx\\| = \\|QTx\\|$	Norm of $Tx : d(y, T(0))$ for $y \in Tx$.
$\gamma(T)$	Minimum modulus of T.
$Q_E^X = Q_E$	Natural quotient map $X \to X/E$ with kernel E.
$Q_T = Q$	Shortcut for $Q_{T(0)}^Y$.
B_X	Closed unit ball in X.
U_X	Open unit ball in X.
$d(U,V)$	Distance between sets: $\inf\{\\|u-v\\| : u \in U, v \in V\}$.

Introduction

The extension of fundamental operator theory concepts from classical Banach spaces, initiated by Kalton's foundational work [33], to more general topological vector spaces has been a fertile area of research for decades. Numerous authors have successfully generalized core properties of linear and multilinear operators to settings such as asymmetric normed spaces and quasi-normed spaces. Recently foundational work by Latreche and Dahia [35] established the continuity of multilinear operators on asymmetric normed spaces. This line of inquiry was advanced by Rano and Bag [45, 46], who developed the theory of continuity and boundedness for linear operators in quasi-normed linear spaces.

The concept of quasi-norm has its roots in several independent contributions from the late 1930s and early 1940s. In 1938-1939, Hyers [28] introduced the notion of *pseudo-norm* using an " absolute value" function satisfying a weak triangle inequality. The decisive step was taken by Aoki [8], who in 1942 gave the *general definition* of quasi-norm and quasi-Banach space, unifying these earlier ideas into an axiomatic framework and laying the foundation for all modern work in the field. Bourgin [13], in work submitted in 1941, also employed this notion.

The present thesis contributes to this expanding field by introducing and systematically studying the concept of continuity for *linear relations* (also known as multivalued linear operators) in the context of quasi-normed linear spaces. We establish several fundamental theorems analogous to those central to functional analysis in Banach spaces. Our results include versions of Banach Steinhaus theorem, the closed graph theorem, and an examination of the minimum modulus and the existence of continuous selections for these relations. It is pertinent to note that these theorems have been extensively studied for linear relations on normed spaces by many authors, see, for instance, [6, 10, 7, 18, 60] and the references therein.

Linear and nonlinear functions in vector spaces are well-known operators. Specifically, multivalued linear operators, also called linear relations, multilinear operators, quasi-operators, Lipschitz, two-Lipschitz, and bi-Lipschitz operators, have been extensively studied. As applications, we present the quotient bi-Lipschitz operator ideals in normed spaces. Although our study is in the field of quasi-normed spaces, which generalizes normed spaces to the constant $K > 1$ (with $K = 1$ in the latter case), we focus our studies on normed spaces and highlight the differences obtained or failed in the direct case. As most knowledge about multivalued linear relations is given by its graph, as considered by most authors, while it is also a linear subspace.

The multivalued linear relation is always decomposed into an operator part called the single-valued part (the operator part) and the multivalued part. For these reasons, we shall study the operators deeply and give some notions in two cases unless otherwise stated. For more details, see Cross [18].

This thesis is organized as follows:

In **Chapter 1**, we present preliminaries and essential concepts, including a recall of quasi-normed spaces and multivalued linear relations in normed spaces.

In **Chapter 2**, we develop the theory of multilinear operators in quasi-Banach spaces, extending the notion of continuity and boundedness to this setting, with completeness properties and fundamental theorems.

In **Chapter 3**, we extend the theory of multivalued linear relations to quasi-Banach spaces, establishing fundamental theorems including the closed graph theorem, Banach Steinhaus theorem, as well as minimum modulus and continuous selections.

In **Chapter 4**, we develop a succinct theory of *Lipschitz operators in quasi-Banach spaces*, extending classical nonlinear geometric analysis to the non-locally convex setting. This chapter addresses three fundamental objectives: (i) establishing the foundational knowledjements of Lipschitz-free p-spaces; (ii) developing an adjoint theory for Lipschitz operators; and (iii) introducing and characterizing *Compact Lipschitz operators*.

We begin by formalizing the category of pointed p-metric spaces and Lipschitz mappings between quasi-Banach spaces, noting that the classical Mazur-Ulam theorem extends naturally to this context through Rolewicz's 1968 generalization. The central construction is the *Arens-Eells p-space* $Æ_p(X)$, which serves as the nonlinear analogue of the free Banach space. This space, originally studied by Kalton in [31] and later developed in [34], possesses a universal property (Theorem 1.7.1) that establishes a canonical correspondence between Lipschitz maps $f : X \to Y$ and bounded linear operators $f_L : Æ_p(X) \to Y$.

The adjoint theory developed in this chapter introduces the *Lipschitz adjoint* $T_p^\#$ and establishes its fundamental properties. We prove that the transpose operator $T^t : E^* \to \mathrm{Lip}_0(X)$ satisfies $\|T^t\| = \mathrm{Lip}(T)$, and demonstrate the commutativity of the fundamental diagram relating the Lipschitz adjoint to the linear adjoint of the canonical extension $\Phi(T)$. These results provide the necessary duality machinery for subsequent spectral and compactness investigations.

The final section addresses *Compactness* a distinctive phenomenon in quasi-Banach spaces where the lack of local convexity precludes classical compactness arguments. We establish a complete characterization of compactness, showing that a Lipschitz operator $T : X \to \ell_p$ is Compact if and only if it admits a coordinate representation $T(x) = (a_n(x))_n$ with $\mathrm{Lip}(a_n) \to 0$.

In **Chapter 5**, we present nonlinear operators and quotient structures, including two-Lipschitz operators in normed spaces and our original contribution on two-Lipschitz left-hand quotient operator ideals based on [38]. This chapter extends the classical theory of left-hand quotients of operator ideals, developed by Pietsch [42] and subsequently studied by various authors including Carl and Defant [14], Johnson et al. [29], and Gálvez-Rodríguez and Ruiz-Casternado [23], to the setting of two-Lipschitz operators acting between metric and Banach spaces.

We introduce the notion of two-Lipschitz left-hand quotient $\mathcal{A}^{-1} \circ \mathcal{A}_{BLip}$, where $\mathcal{A}$ is a Banach operator ideal and $\mathcal{A}_{BLip}$ is a two-Lipschitz operator ideal. A central concept in our investigation is the *linearization property* (LP), which establishes a fundamental connection between a two-Lipschitz operator ideal and its linear counterpart through the linearization map $A \mapsto A_L$. Under this assumption, we prove that such quotients naturally generate new two-Lipschitz operator ideals and establish their isometric identification with composition ideals of the form $(\mathcal{A}^{-1} \circ \mathcal{B}) \circ BLip_0$.

As concrete applications, we characterize two important classes of operators as left-hand quotients: Grothendieck two-Lipschitz operators, which we identify as $\mathcal{S}^{-1} \circ BLip_{0\mathcal{W}}$, and Rosenthal two-Lipschitz operators, which correspond to $\mathcal{V}^{-1} \circ BLip_{0\mathcal{K}}$. These results demonstrate the power of the quotient construction in capturing deep structural properties of two-Lipschitz mappings. We conclude with an open problem regarding quotients that do not satisfy the linearization property, using the class of $(p; p_1, p_2)$-summing two-Lipschitz operators as a motivating example, thereby highlighting the intrinsic asymmetry of multilinear operator ideals and pointing towards future research directions.

Chapter 1

Preliminaries

This chapter presents the preliminaries and essential concepts needed throughout this thesis.

1.1 Quasi-Normed Spaces

Using the classical terminology introduced by Aoki [8], who coined the term *quasi-norm* we arrive at the following definition of a quasi-norm.

Definition 1.1.1. Let X be a linear space over the field $\mathbb{K}$. A quasi-norm $\|\cdot\|$ on X is a mapping:

$$\|\cdot\| : X \to [0, +\infty[$$

satisfying the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0 \in X$.

(ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and all $\alpha \in \mathbb{K}$.

(iii) There exists a constant $K \geq 1$ such that, for all $x, y \in X$, we have

$$\|x + y\| \leq K\left(\|x\| + \|y\|\right). \tag{1.1}$$

We say that the pair $(X, \|\cdot\|)$ is a quasi-normed linear space. The least of value K such that (1.1) holds is called the *index* of the quasi-norm $\|\cdot\|$. If $K = 1$, then the quasi-norm $\|\cdot\|$ becomes a norm on X and the pair $(X, \|\cdot\|)$ is a normed space.

Remark 1.1.1. Note that in a quasi-normed linear space $(X, \|\cdot\|)$ with the corresponding index K,

$$\left\|\sum_{i=1}^{n} x_i\right\| \leq K^{n-1}\sum_{i=1}^{n}\|x_i\|,$$

for all $x_i \in X$ and $n \in \mathbb{N}$.

1.2 Topological properties

The topology τ_d is generated by the quasi-norm $\|\cdot\|$ when d is a quasi-metric associated with the quasi norm, defined by $d(x,y) = \|x-y\|$. The pair (X,d) is referred to a quasi-metric space . The open balls in this topology are given by,

$$B(x_0, r) = \{x \in X : \|x_0 - x\| < r\},$$

where $\epsilon > 0$. This topology on X has as its basis the sets

$$U_\epsilon = \{(x,y) \in X \times Y : \|x-y\| < \epsilon\}.$$

See [17] for more details on this topology.

Definition 1.2.1. A quasi-metric on a nonempty set X is a function $d : X \times X \to [0, \infty)$ satisfying

(i) $d(x,y) = 0 \Leftrightarrow x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,y) \leq K\,[d(x,z) + d(z,y)]$

for all $x, y, z \in X$, and for some fixed number $K \geq 1$. The pair (X,d) is called a quasi-metric space. Obviously, for $K = 1$ one obtains a metric on X.

Definition 1.2.2. Let (X,d) be a quasi-metric space. As in the case of metric spaces, a topology $\tau(d)$ can be defined on X by considering the family $\mathcal{V}_p(x)$ of neighborhoods of a point $x \in X$:

$$V \in \mathcal{V}_p(x) \Longleftrightarrow \exists r > 0 \text{ such that } B_{X,d}(x,r) \subset V.$$

A set $G \subset X$ is called $\tau(d)$–open if for all $x \in G$, there exists $r = r_x > 0$ such that

$$B_{X,d}(x,r) \subset G.$$

Thus, if we consider the family $\mathcal{F}$ of subsets U of X, empty or not, such that for all $x \in U$ there exists an open ball centered at x which is contained in U, it is clear that $\mathcal{F}$ satisfies the properties of a topology. $(X, \mathcal{F})$ is therefore a topological space. We denote by $\tau(d)$ the topology generated by $\mathcal{F}$ on X.

Definition 1.2.3. Let (X, d) be a quasi-metric space, $x_0 \in X$ and $r > 0$. We define the balls in X by the formulas:

1. The open ball with center $x_0 \in X$ and radius r, denoted $B_{X,d}(x_0, r)$, is defined by

$$B_{X,d}(x_0,r) = \{x \in X \mid d(x_0,x) < r\}.$$

2. The closed ball with center x_0 and radius r, denoted $B'_{X,d}(x_0, r)$, is defined by

$$B'_{X,d}(x_0,r) = \{x \in X \mid d(x_0,x) \leq r\}.$$

3. The sphere with center x_0 and radius r, denoted $S_{X,d}(x_0, r)$, is defined by

$$S_{X,d}(x_0,r) = \{x \in X \mid d(x_0,x) = r\}.$$

Proposition 1.2.1. *Let $(X, \|\cdot\|)$ be a quasi-Banach space.*

— *Any open ball $B_X(x,r) = \{y \in X : \|y - x\| < r\}$ is open for the strong topology of X.*

— *Any closed ball $B_X(x,r) = \{y \in X : \|y - x\| \leq r\}$ is closed for the strong topology* *of* X.

Example 1.2.1. [45] As an important example, the mapping defined on $\mathbb{R}^2$ into $\mathbb{R}^+$ by

$$x = (x_1, x_2) \mapsto \left(\sqrt{|x_1|} + \sqrt{|x_2|}\right)^2,$$

is quasi norm on $\mathbb{R}^2$, for all $x = (x_1, x_2)$.

We obtain the following definitions from [44].

Definition 1.2.4. Let $(X, \|\cdot\|_X)$ be a quasi-normed linear space and let A and B be subsets of X.

1. A sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is said to be convergent to $x_0 \in X$, denoted by $\lim_{n\to\infty} x_n = x_0$ if, $\lim_{n\to\infty} \|x_n - x_0\| = 0$.
2. $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called a Cauchy sequence if for $n, m \in \mathbb{N}$, $\lim_{n,m\to\infty} \|x_n - x_m\| = 0$.
3. $B \subset X$ is said to be complete if every Cauchy sequence in B converges in B.
4. $A \subset X$ is said to be bounded if there exists a real number $M > 0$ such that $\|x\| \leq M$, $\forall x \in A$.
5. $A \subset X$ is said to be closed if for any limit of a sequence of points in A is contained in A, that is, if $\lim_{n\to\infty} x_n = x_0$ with all $\{x_n\}_{n\in\mathbb{N}} \in A$, then $x_0 \in A$.
6. $A \subset X$ is said to be compact if any sequence $\{x_n\}_{n\in\mathbb{N}} \in A$ has a convergent subsequence which converges to a point in A.

Proposition 1.2.2. *[45] Let $(X, \|\cdot\|)$ be a quasi-normed linear spaces. Then the following holds.*

(i) The limit of a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, if exists, is unique.

(ii) Every subsequence of a convergent sequence converges to the same limit.

(iii) Every convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is a Cauchy sequence.

We collect here the basic definitions and results concerning quasi-Banach spaces and quasi-metric spaces. For a comprehensive background on quasi-Banach spaces, we refer to [32, 33].

1.2.1 Quasi-normed and p-normed spaces

A fundamental and important result is the Aoki–Rolewicz theorem [8, 48]. Its standard formulation asserts the existence of some $p \in (0,1]$ for which the quasi-norm becomes p-subadditive up to equivalence. For our purposes, the following explicit relation between the modulus of concavity K and the exponent p will be useful.

Remark 1.2.1. If a quasi-norm $\|\cdot\|$ satisfies

$$\|x_1 + x_2\| \leq K(\|x_1\| + \|x_2\|),$$

then it is equivalent to a p-subadditive quasi-norm for any p satisfying $K = 2^{1/p-1}$ (or equivalently, $p = (\log 2)/(\log(2K))$).

Theorem 1.2.1. *(Aoki-Rolewicz). Let $(X, \|\cdot\|)$ be a quasi-normed space with modulus of concavity K. Then there exists a number $p \in (0,1]$, determined by K and a constant $C > 0$ such that*

$$\left\|\sum_{j=1}^{n} x_j\right\| \leq C\left(\sum_{j=1}^{n}\|x_j\|^p\right)^{1/p} \tag{1.2}$$

for every finite sequence $\{x_j\}_{j=1}^{n}$. Consequently, the quasi-norm $\|\cdot\|$ is equivalent to a quasi-norm $|||\cdot|||$ which is p-subadditive, i.e.,

$$|||x_1 + x_2|||^p \leq |||x_1|||^p + |||x_2|||^p, \quad \text{for all } x_1, x_2 \in X.$$

Without lost of generality we shall write $\|\cdot\|$ to denote a quasi norm.

Definition 1.2.5. A quasi-normed space X is said to be p-normable if an inequality of the form (1.2) holds. If the quasi-norm itself is p-subadditive, then $(X, \|\cdot\|)$ is called a p-normed space. In the sequel we assume, without loss of generality, that all quasi-normed spaces are p-normed for some $p \in (0,1]$.

A p-subadditive quasi-norm induces a metric $d(x,y) = \|x - y\|^p$. The topology of X is the one generated by this metric.

Definition 1.2.6. A p-normed space $(X, \|\cdot\|)$ is called a p-Banach space if it is complete with respect to the metric $d(x,y) = \|x-y\|^p$. More generally, a complete quasi-normed space is called a quasi-Banach space.

1.2.2 Quasi-metric and p-metric spaces

The geometric structure underlying a quasi-normed space can be described in a nonlinear setting by quasi-metrics.

Definition 1.2.7. A quasi-metric on a nonempty set X is a symmetric function $d : X \times X \to [0, \infty)$ such that

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) there exists a constant $K \geq 1$ satisfying

$$d(x,y) \leq K\left(d(x,z) + d(z,y)\right) \quad \text{for all } x, y, z \in X.$$

The pair (X, d) is called a quasi-metric space (see [27], p. 109).

A quasi-metric d on a set X is said to be a *p-metric*, $0 < p \leq 1$, if d^p is a metric, i.e.,

$$d(x,y)^p \leq d(x,z)^p + d(z,y)^p, \qquad x, y, z \in X, \tag{1.3}$$

in which case we call (X, d) a *p-metric space.*

Remark 1.2.2. 1-We recall that if $\|\cdot\|$ is a p-subadditive quasi-norm on X, then $d(x,y) = \|x-y\|$ defines a p-metric. Thus, every p-normed space is naturally a p-metric space.

2-An analogue of the Aoki-Rolewicz holds in this context (see [27, Proposition 14.5]): every quasi-metric space can be endowed with an equivalent p-metric τ for some $0 < p \leq 1$, i.e., there is a constant $C = C(K) \geq 1$ such that

$$C^{-1}\tau(x,y) \leq d(x,y) \leq C\tau(x,y), \qquad x, y \in X. \tag{1.4}$$

1.3 Continuity of linear operators

Most of preliminaries are same in [45].

Definition 1.3.1 (Continuity in quasi-normed spaces)**.** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be quasi-normed linear spaces. An operator $T : X \to Y$ is continuous at $x \in X$ if for every sequence (x_n) in X with $\|x_n - x\|_X \to 0$, we have $\|T(x_n) - T(x)\|_Y \to 0$. If T is continuous at every point of X, it is called continuous on X.

Proposition 1.3.1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be quasi-normed linear spaces and $T : X \to Y$ a linear operator. If T is continuous at one point of X, then it is continuous everywhere on X.*

Definition 1.3.2 (Bounded Operator in Quasi-normed Spaces)**.** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be quasi-normed linear spaces. A linear operator $T : X \to Y$ is said to be bounded if there exists a constant $M > 0$ such that

$$\|T(x)\|_Y \leq M\|x\|_X \quad \text{for all } x \in X.$$

Example 1.3.1. Let $(X, \|\cdot\|)$ be a quasi-normed linear space. The operator $T : X \to X$ defined by $T(x) = 2x$ is a bounded linear operator.

Theorem 1.3.1 (Boundedness and Continuity)**.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be quasi-normed linear spaces and $T : X \to Y$ a linear operator. Then T is bounded if and only if T is continuous.*

Theorem 1.3.2 (Finite Dimensional Case)**.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be quasi-normed linear spaces and $T : X \to Y$ a linear operator. If X is finite dimensional, then T is bounded (hence continuous).*

Definition 1.3.3. Let E and F be quasi Banach spaces. An operator $T \in L(E, F)$ is said to have finite rank if $\dim(T(E)) < \infty$.

Theorem 1.3.3 (Space of Bounded Linear Operators)**.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be quasi-normed linear spaces. The set $\mathcal{B}(X, Y)$ of all bounded linear operators from X to Y forms a linear space.*

Theorem 1.3.4 (Operator Quasi-norm)**.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be quasi-normed linear spaces. For $T \in \mathcal{B}(X, Y)$ define*

$$\|T\| = \sup_{x \neq 0} \frac{\|T(x)\|_Y}{\|x\|_X}.$$

Then $(\mathcal{B}(X, Y), \|\cdot\|)$ is a quasi-normed linear space.

Theorem 1.3.5 (Continuity of linear operators)**.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two quasi-normed linear spaces. A linear operator $T : X \to Y$ is continuous if and only if there exists $c > 0$ such that*

$$\|T(x)\|_Y \leq c\|x\|_X, \quad \forall x \in X.$$

The next theorem is a quasi-normed version of the Banach-Steinhaus theorem that introduced by Lv and Feng in [36].

Theorem 1.3.6. *Let $\mathcal{F}$ be a family of continuous linear operators from quasi-normed space $(X, \|\cdot\|_X)$ to quasi-normed space $(Y, \|\cdot\|_Y)$ such that for each $x \in X$ there exists $b_x > 0$ with $\|T(x)\|_Y \leq b_x$ for all $T \in \mathcal{F}$, then there exists $b > 0$ such that*

$$\sup\{\|T(x)\|_Y, \quad \|x\|_X \leq 1\} < b,$$

for all $T \in \mathcal{F}$.

1.4 Multilinear Operators in Normed Spaces

In general, in this section we use standard notations. Let $\left(X_j, \|\cdot\|_{X_j}\right)_{1\leq j\leq m}$ for $m \in \mathbb{N}$ and $(Y, \|\cdot\|)$ be linear normed spaces over field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Following the standard definition in multilinear analysis, for more details we refer to [21, 40, 54].

Definition 1.4.1. An operator $T : X_1 \times \cdots \times X_m \longrightarrow Y$ is said multilinear if it is linear separately in each coordinate, i.e., the mappings

$$T_j : X_j \longrightarrow Y, \quad x^j \longmapsto T(x^1, \ldots, x^j, \ldots, x^m),$$

are linear for any fixed point $x^k \in X_k$, $j \neq k$ where $\mathcal{L}(X_1, \ldots, X_m; Y)$ denote the linear space of such mappings.

Definition 1.4.2. The multi-linear map T is said to be continuous if it is continuous as a function between two normed spaces.

Theorem 1.4.1. *Let $X_1, \ldots, X_m$ and Y be normed spaces, and let $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ be an m-linear mapping. The following assertions are equivalent:*

1. *T is continuous.*
2. *T is continuous at $(0, \ldots, 0)$.*
3. *There exists a constant $K \geq 0$ such that*

$$\Big\|T(x^1, \ldots, x^m)\Big\| \leq K\|x^1\| \cdots \|x^m\|,$$

for all $x^j \in X_j$, $j = 1, \ldots, m$.

4. $$\|T\| := \sup_{\|x^j\| \leq 1,\, j=1,\ldots,m} \|T(x^1, \ldots, x^m)\| < \infty.$$

We denote by $\mathcal{L}(X_1, \ldots, X_m; Y)$ the vector space of all continuous m-linear mappings from $X_1 \times \cdots \times X_m$ into Y. When $Y = \mathbb{K}$, we simply write $\mathcal{L}(X_1, \ldots, X_m)$.

Moreover, one easily verifies that

$$\|T\| = \inf\Big\{K \geq 0 : \|T(x^1, \ldots, x^m)\| \leq K\|x^1\| \cdots \|x^m\| \text{ for all } x^j \in X_j\Big\}.$$

This expression defines a norm on $\mathcal{L}(X_1, \ldots, X_m; Y)$. Furthermore, if Y is a Banach space, then $\mathcal{L}(X_1, \ldots, X_m; Y)$ is complete with respect to this norm.

1.5 Multivalued Linear Relations between normed spaces

Definition 1.5.1 ([18, 6]). Let X, Y be vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A linear relation T from X to Y is a mapping from a subspace $D(T)$ of X, called the domain of T, into the collection of non empty subsets of Y such that

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2),$$

for all nonzero scalars α, β and $x_1, x_2 \in D(T)$. If T maps the point of its domain to singletons, then T is said to be a single-valued or simply an operator. The class of linear relation from X to Y is denoted by $LR(X, Y)$ and we write $LR(X) = LR(X, X)$.

Definition 1.5.2 (The graph of linear relation). The graph $G(T)$ of linear relation $T \in LR(X, Y)$ defined by

$$G(T) = \{(x, y) \in X \times Y : x \in D(T) \text{ and } y \in Tx\},$$

where T is uniquely determined by its graph, hence one can identify T with $G(T)$.

Definition 1.5.3 (The inverse of linear relation). The inverse of T is the linear relation T^{-1} defined by

$$G(T^{-1}) = \{(y, x) \in Y \times X : (x, y) \in G(T)\}.$$

For $\emptyset \neq N \subset Y$ we have

$$T^{-1}(N) = \{x \in D(T) : N \cap Tx \neq \emptyset\}.$$

If T^{-1} is single valued, then T is called injective. The image of $A \subset X$ is given by

$$T(A) = \bigcup\{Tx : x \in A \cap D(T)\}$$

In particular, for $y \in R(T)$,

$$T^{-1}y = \{x \in D(T) : y \in Tx\}.$$

Remark 1.5.1. Let $T \in LR(X, Y)$. The symbols $R(T)$, $N(T)$ and $T(0)$ stand for the range, the null space and the multivalued part of T, which are defined by

$$R(T) = \{y : (x, y) \in G(T)\},$$

$$N(T) = \{x \in D(T) : (x, 0) \in G(T)\} = T^{-1}(0),$$

and

$$T(0) = \{y : (0, y) \in G(T)\}.$$

T is called injective if $N(T) = 0$ and T is said to be surjective if $R(T) = Y$.

In the next proposition we adopt the terminology of Cross [18] (see also [6]).

Proposition 1.5.1 ([18], Corollary I.2.4)**.** *Let X, Y be vector spaces, the set $T(0)$ is a linear subspace of Y.*

According [18, Proposition I.2.8, Proposition I.3.1], we have the following proposition.

Proposition 1.5.2. *1. For all $x \in D(T)$,*

$$Tx = y + T(0)$$

2. When $B \subset Y$ with $B \neq \emptyset$ and $A \subset X$ then

$$T\left(T^{-1}(B)\right) = B \cap R(T) + T(0),$$

and

$$T^{-1}(T(A)) = A \cap D(T) + T^{-1}(0).$$

For more details on the theory of multivalued linear operators between normed linear spaces we refer to the books of [6] and [18].

1.6 Pointed Metric Spaces and Lipschitz Maps

Definition 1.6.1 (Pointed Space; see (Weaver [58], Kalton [34]))**.** A pointed quasi-metric space (X, d) is a space equipped with a distinguished point called the origin, denoted by 0. This choice is particularly useful for normalizing Lipschitz functions. Any p-Banach space X can be viewed as a pointed p-metric space by taking the origin to be 0 and defining the distance as $d(x, y) = \|x - y\|^p$.

Definition 1.6.2. If (X, d) and (Y, ρ) are quasi-metric spaces we shall say that a map $f : X \longrightarrow Y$ is *Lipschitz* if there exists a constant $C \geq 0$ so that

$$\rho\Big(T(x), T(y)\Big) \leq C\, d(x, y), \qquad x, y \in X. \tag{1.5}$$

We denote by $\mathrm{Lip}(T)$ the smallest constant which can play the role of C in the last inequality (1.5), i.e.,

$$\mathrm{Lip}(T) = \sup \left\{ \frac{\rho\Big(T(x), T(y)\Big)}{d(x, y)} : x, y \in X,\ x \neq y \right\} \in [0, \infty). \tag{1.6}$$

Remark 1.6.1. If T is injective, and both T and T^{-1} are Lipschitz, then we say that T is *bi-Lipschitz* and that X *Lipschitz-embeds* into Y. If there is a bi-Lipschitz map from X onto Y, the spaces X and Y are said to be *Lipschitz isomorphic.* A map T from a quasi-metric space (X, d) into a quasi-metric space (Y, ρ) is an *isometry* if

$$\rho\Big(T(x), T(y)\Big) = d(x, y), \qquad x, y \in X. \tag{1.7}$$

We shall say that (X, d) is a *pointed quasi-metric space* (or a pointed p-metric space, or a pointed metric space), if it has a distinguished point that we call the *origin* and denote 0. The assumption of an origin is convenient to normalize Lipschitz functions.

The *Lipschitz dual* of a quasi-metric space (X, d), denoted $\mathrm{Lip}_0(X)$, is the (possibly trivial) vector space of all real-valued Lipschitz functions f defined on X such that

$T(0) = 0$ endowed with the *Lipschitz norm*

$$Lip(T) = \sup\left\{\frac{|T(x) - T(y)|}{d(x,y)} : x, y \in X,\, x \neq y\right\}. \tag{1.8}$$

Note that $\big(\mathrm{Lip}_0(X), Lip(.)\big)$ is a Banach space. for more details see [4]

The nonlinear geometry of quasi-Banach spaces remains poorly understood. Most research has focused on their uniform structure rather than their Lipschitz structure. For instance, Weston [59] (see also [43]) introduced the following proposition.

Proposition 1.6.1 ([59])**.** *The spaces $L_p(0,1)$ and ℓ_q are not uniformly homeomorphic when $p, q < 1$ and $p \neq q$.*

Remark 1.6.2. Moreover, it remains open whether the metric spaces (L_p, d_p) and (L_q, d_q), where $d_p(f,g) = \|f-g\|_p^p$, are Lipschitz isomorphic for $0 < p < q \leq 1$. However, it is known that L_p and L_q are not Lipschitz isomorphic as quasi-Banach spaces in the sense described above [5].

Remark 1.6.3. A classical theorem by Mazur and Ulam (1932) [37] asserts that any surjective isometry between real Banach spaces that maps 0 to 0 must be linear. This result was extended to quasi-Banach spaces by Rolewicz in 1968 see [49], under the assumption that all quasi-norms are p-norms for some $0 < p \leq 1$.

1.7 The Lipschitz free p-Space

Let (X, ρ) be a pointed p-metric space with $0 < p \leq 1$ and distinguished point $0 \in X$. For more details of this section see [4]

Definition 1.7.1 (Evaluation Functionals)**.** Let $\mathbb{R}_0^X$ be the space of all (not necessarily continuous) maps $T : X \to \mathbb{R}$ with $T(0) = 0$. For $x \in X$, the *evaluation functional* $\delta(x) \in (\mathbb{R}_0^X)^\#$ is defined by

$$\langle \delta(x), T\rangle = T(x), \qquad T \in \mathbb{R}_0^X.$$

Note that $\delta(0) = 0$.

The construction presented in the following definition, due to Kalton [31] and further developed by Albiac et al. [4] and Kalton [34], generalizes the classical Arens-Eells space to the p-metric setting.

Definition 1.7.2 (Lipschitz Free p-Space)**.** Let $\mathcal{P}(X)$ be the linear span in $(\mathbb{R}_0^X)^\#$ of the evaluations $\delta(x)$, where x runs through X. For $\mu = \sum_{j=1}^N a_j\delta(x_j) \in \mathcal{P}(X)$, define

$$\|\mu\|_{\mathcal{F}_p(X)} = \sup \left| \sum_{j=1}^N a_j T(x_j) \right|, \tag{1.9}$$

where the supremum is taken over all p-normed spaces $(Y, \|\cdot\|_Y)$ and all 1-Lipschitz maps $T : X \to Y$ with $f(0) = 0$.

The *Lipschitz free p-space* over X, denoted $\mathcal{F}_p(X)$ is the p-Banach space obtained from the completion of the p-normed space $(\mathcal{P}(X), \|\cdot\|_{\mathcal{F}_p(X)})$.

Proposition 1.7.1. *Let (X, ρ) be a pointed p-metric space, $0 < p \leq 1$. Then $(\mathcal{P}(X), \|\cdot\|_{\mathcal{F}_p(X)})$ is a p-normed space.*

Definition 1.7.3 (Natural Embedding)**.** The *natural embedding* of X into $\mathcal{F}_p(X)$ is the map $\delta_X : X \to \mathcal{F}_p(X)$ given by $\delta_X(x) = \delta(x)$.

Remark 1.7.1. The choice of base point in X is not relevant in the definition of $\mathcal{F}_p(X)$. If we change the origin, there is a natural linear isometry between the resulting Lipschitz free p-spaces.

Theorem 1.7.1 (Universal Property)**.** *Let (X, ρ) be a pointed p-metric space. Then:*

1. *δ_X is an isometric embedding.*
2. *The linear span of $\{\delta_X(x) : x \in X\}$ is dense in $\mathcal{F}_p(X)$.*
3. *$\mathcal{F}_p(X)$ is the unique (up to isometric isomorphism) p-Banach space such that for every p-Banach space E and every Lipschitz map $T : X \to E$ with $f(0) = 0$, there exists a unique linear map $T_L : \mathcal{F}_p(X) \to E$ with $T_L \circ \delta_X = T$. Moreover $\|T_L\| = \mathrm{Lip}(T)$.*

Pictorially,

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\delta_X"'] & E \\ \mathcal{F}_p(X) \arrow[ur, "T_L"'] & \end{tikzcd}$$

Remark 1.7.2. If $p = 1$ (so that ρ is a metric), it follows from the Hahn-Banach theorem that $\mathcal{F}_1(X)$ is the classical Lipschitz free space denoted by $\mathcal{F}(X)$, and the norm of $\mu = \sum_{j=1}^{N} a_j \delta(x_j) \in \mathcal{P}(X)$ can be computed as

$$\|\mu\|_{\mathcal{F}_1(X)} = \sup \left| \sum_{j=1}^{N} a_j T(x_j) \right|,$$

the supremum being taken over all 1-Lipschitz maps $T : X \to \mathbb{R}$ with $f(0) = 0$. Moreover, $\mathcal{F}_1(X)^* = \mathrm{Lip}_0(X)$. The corresponding result also holds for $p < 1$, i.e., $\mathcal{F}_p(X)^* = \mathrm{Lip}_0(X)$ (see Corollary 1.7.3 below).

Lipschitz free p-spaces provide a canonical linearization process: if we identify (through δ_X) a p-metric space X with a subset of $\mathcal{F}_p(X)$, then any Lipschitz map between p-metric spaces extends to a continuous linear map between the corresponding free spaces.

Proposition 1.7.2. *[4, Lemma 4.8] Let X and Y be pointed p-metric spaces* $(0 < p \leq 1)$ *and suppose $T : X \to Y$ is a Lipschitz map with $T(0) = 0$. Then there exists a unique linear operator $T_L : \mathcal{F}_p(X) \to \mathcal{F}_p(Y)$ such that $T_L \circ \delta_X = \delta_Y \circ T$, i.e., the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\delta_X"'] & Y \arrow[d, "\delta_Y"] \\ \mathcal{F}_p(X) \arrow[r, "T_L"'] & \mathcal{F}_p(Y) \end{tikzcd}$$

and $\|T_L\| = \mathrm{Lip}(T)$. *In particular, if T is a bi-Lipschitz bijection then T_L is an isomorphism.*

Proposition 1.7.3. *[4, Corollary 4.23] Let X be a pointed p-metric space,* $0 < p \leq 1$. *Then*

$$(\mathcal{F}_p(X))^* = \mathrm{Lip}_0(X),$$

i.e., given $\phi \in \mathcal{F}_p(X)^*$ there is a unique $T \in \operatorname{Lip}_0(X)$ such that $\phi(\sum a_i\delta(x_i)) = \sum a_i T(x_i)$ for every $\sum a_i\delta(x_i) \in \mathcal{F}_p(X)$, and the map $\phi \mapsto T$ is a linear isometry of $\mathcal{F}_p(X)^*$ onto $\operatorname{Lip}_0(X)$*. In particular, $\mathcal{F}_p(X)^* = \{0\}$ if* $\operatorname{Lip}_0(X) = \{0\}$.

1.8 Arens-Eells p-Spaces and Molecules

We now give an alternative equivalent definition of Lipschitz free p-spaces using molecules and atoms, which provides an intrinsic formula for the p-norm.

Definition 1.8.1 (Molecules)**.** Given a set X and $x \in X$, let χ_x denote the indicator function of the singleton $\{x\}$. For $x, y \in X$, put

$$m_{x,y} := \chi_x - \chi_y.$$

A *molecule* of X is a function $m : X \to \mathbb{R}$ that is supported on a finite subset of X and satisfies $\sum_{x\in X} m(x) = 0$. The vector space of all molecules of X is denoted by $\mathcal{M}(X)$.

A simple induction argument shows that every molecule has at least one expression as a linear combination of molecules of the form $m_{x,y}$, so that $\mathcal{M}(X)$ coincides with the linear span of the family of *atoms*

$$\mathcal{A}'(X) = \left\{ \frac{m_{x,y}}{\rho(x,y)} : x, y \in X, x \neq y \right\} \subseteq \mathbb{R}^X. \tag{1.10}$$

Definition 1.8.2 (Arens-Eells p-Space)**.** We define the *Arens-Eells p-space* over X, denoted $\text{Æ}_p(X)$, as the p-Banach space constructed from the set $\mathcal{A}'(X)$ using the p-convexification method (see Albiac [4]).

This way, if we give $\mathcal{M}(X)$ the p-seminorm

$$\|m\|_{\text{Æ}_p} = \inf\left\{ \left(\sum_{i=1}^N |a_i|^p\right)^{1/p} : m = \sum_{i=1}^N a_i \frac{m_{x_i,y_i}}{\rho(x_i,y_i)}, N \in \mathbb{N} \right\}, \tag{1.11}$$

we have that $\text{Æ}_p(X)$ is the completion of $\mathcal{M}(X)$ (modulo the set of molecules with zero p-seminorm) with respect to $\|\cdot\|_{\text{Æ}_p}$.

Remark 1.8.1. Formula (1.11) defines in fact a p-norm on $\mathcal{M}(X)$, as will follow from the identification with $\mathcal{F}_p(X)$ below.

The following result establishes that the Arens-Eells p-space over X can be identified with the Lipschitz free p-space over X.

Theorem 1.8.1. *[4, Theorem 4.10] Let $0 < p \leq 1$ and (X, ρ) be a pointed p-metric space. Then $\mathcal{F}_p(X)$ and $Æ_p(X)$ are isometrically isomorphic. In fact, there is a linear onto isometry $F : \mathcal{F}_p(X) \to Æ_p(X)$ such that*

$$F(\delta(x)) = \chi_x - \chi_0 = m_{x,0} \quad \textit{for all } x \in X.$$

The following two results are reformulations of Theorem 1.8.1.

Remark 1.8.2 (Canonical Isometries)**.** By Corollary 1.7.3 and Theorem 1.8.1, for any pointed p-metric space (X, ρ) with $0 < p \leq 1$, we have the isometric identifications

$$\mathcal{F}_p(X)^* \cong \mathrm{Lip}_0(X) \cong Æ_p(X)^*.$$

The canonical isometry $S : \mathrm{Lip}_0(X) \to \mathcal{F}_p(X)^*$ is defined by

$$\langle S(f), \mu \rangle = \sum_{j=1}^{N} a_j f(x_j), \quad \text{for } \mu = \sum_{j=1}^{N} a_j \delta(x_j).$$

Its inverse $R : \mathcal{F}_p(X)^* \to \mathrm{Lip}_0(X)$ is given by $R(\varphi) = \varphi \circ \delta_X$, where $\delta_X : X \to \mathcal{F}_p(X)$ is the isometric embedding $x \mapsto \delta(x)$.

Chapter 2

Multilinear Operators in Quasi-Banach Spaces

In this chapter, we extend the notion of continuity of linear operators between quasi-normed linear spaces; developed by Rano and Bag [45, 46]; to multilinear operators defined on quasi-normed linear spaces. Our approach generalizes the foundational work of Rano and Bag [45, Theorem 2.1] on bounded linear operators and their continuity results [46] to the multilinear setting. After this introduction, in Section (2.1) we study the concept of continuity and boundedness of multilinear operators in quasi-normed linear spaces, characterizing these mappings in a manner analogous to that used for linear operators [45, 46]. In Section (2.2) we prove completeness properties of the space of such operators. Finally, in the last section we establish versions of the Banach-Steinhaus and closed graph theorems for multilinear mappings, extending the classical results [36] to this setting.

2.1 Bounded m-linear operators in quasi-normed linear spaces

In this section we characterize the continuity of the m-linear operators between quasi-normed linear spaces by studying some properties of these class of such operators.

Our work builds upon the foundational contributions of Rano and Bag [45, 46].

Definition 2.1.1. Let $\left(X_j, \|\cdot\|_{X_j}\right)_{1\leq j\leq m}$ with $m \in \mathbb{N}$ and $(Y, \|\cdot\|_Y)$ be quasi-normed linear spaces. For each $\{x_n^j\}_{n\in\mathbb{N}} \subset X_j$ converges to $x_0^j \in X_j$ with respect the quasi-norm $\|\cdot\|_{X_j}$. We shall write $x_n = (x_n^1, \ldots, x_n^j, \ldots, x_n^m)$, $x_0 = \left(x_0^1, \ldots, x_0^j, \ldots, x_0^m\right)$ and $(x_n)_n$ converges to x_0 when $(x_n^1, \ldots, x_n^j, \ldots, x_n^m)$ converges to the point $\left(x_0^1, \ldots, x_0^j, \ldots, x_0^m\right)$. Then the m-linear map T is said to be continuous at x_0 if for any sequence $(x_n)_n$ converges to x_0 implies $(T(x_n))_n$ converges to $T(x_0)$ with respect to the quasi-norm $\|\cdot\|_Y$ i.e.,

$$\lim_{n\to\infty} \left\|T\left(x_n^1, \ldots, x_n^j, \ldots, x_n^m\right) - T\left(x_0^1, \ldots, x_0^j, \ldots, x_0^m\right)\right\|_Y = 0.$$

Remark 2.1.1. Note that

$$\begin{aligned}&\left\|T\left(x_n^1, \ldots, x_n^j, \ldots, x_n^m\right) - T\left(x_0^1, \ldots, x_0^j, \ldots, x_0^m\right)\right\|_Y \\ &\quad = \left\|\sum_{j=1}^{m} T\left(x_0^1, \ldots, x_0^{j-1}, x_n^j - x_0^j, x_n^{j+1}, \ldots, x_n^m\right)\right\|_Y .\end{aligned}$$

Definition 2.1.2. A m-linear mapping $T : X_1 \times \ldots \times X_m \longrightarrow Y$ is continuous if it is continuous as a function between quasi-normed linear spaces.

We shall denote by $\mathcal{L}_{(X_1,\ldots,X_m;Y)}(X_1, \ldots, X_m; Y)$ the linear space of all continuous m-linear mappings from $X_1 \times \ldots \times X_m$ into Y. If $Y = \mathbb{K}$, we write $\mathcal{L}_{(X_1,\ldots,X_m;Y)}(X_1, \ldots, X_m)$.

We shall prove that (2.2) defines a quasi-norm on $\mathcal{L}_{(X_1,\ldots,X_m;Y)}(X_1, \ldots, X_m; Y)$ which is a complete norm when $(Y, \|\cdot\|_Y)$ is complete.

As $a = (a^1, \ldots, a^m)$ in the open $\mathcal{A} \subset X_1 \times \ldots \times X_m$ means that there exists opens $\mathcal{O}_i \subset (X_i, \|\cdot\|_{q_i})$ such that $a^i \in \mathcal{O}_i$, $i = 1, \ldots, m$ and $\mathcal{O}_1 \times \ldots \times \mathcal{O}_m \subset \mathcal{A}$.

Let us consider the space $X_1 \times \ldots \times X_m$ equipped with the quasi-norm $\|\cdot\|_\infty$ as

$$\|x\|_\infty = \max_{1\leq j\leq m} \|x^j\|_{X_j},$$

then $\|\cdot\|_\infty$ induced a topology coincides with the product topology.

In the following theorem we prove that the connection between the continuity and the boundedness of a m-linear operators between quasi-normed linear spaces, for its proof we use a same technics such in [35] and [45].

Theorem 2.1.1. *Let $\left(X_j, \|\cdot\|_{X_j}\right)_{1\leq j\leq m}$ for $m \in \mathbb{N}$, $(Y, \|\cdot\|_Y)$ be quasi-normed linear spaces and K_Y denotes the index of quasi norm $\|\cdot\|_X$. Let T be in $\mathcal{L}\left(X_1, \ldots, X_m; Y\right)$. The following assertions are equivalent*

(i) *T is continuous.*

(ii) *T is continuous at $(0, \ldots, 0)$.*

(iii) *There is a constant $c \geq 0$ such that*

$$\left\|T(x^1, \ldots, x^m)\right\|_Y \leq c \left\|x^1\right\|_{X_1} \cdots \|x^m\|_{X_m}, \tag{2.1}$$

for every $x^j \in X_j$, $1 \leq j \leq m$.

(iv) *Moreover,*

$$\|T\|_{\mathcal{L}(X_1,\ldots,X_m;Y)} := \sup_{\|x^j\|_{X_j}\leq 1, j=1,\ldots,m} \left\|T(x^1, \ldots, x^m)\right\|_Y < \infty.$$

And

$$\|T\|_{\mathcal{L}(X_1,\ldots,X_m;Y)} = \inf\left\{c \geq 0, \text{ verifying the Inequality } (2.1)\right\}. \tag{2.2}$$

Proof. $(i) \Rightarrow (ii)$ Obvious, by definition of continuity.

$(ii) \Rightarrow (iii)$ Assume T is continuous at $(0, \ldots, 0)$. Take $\varepsilon = 1$. There exists $r \in (0, 1)$ such that

$$T\left(B_\infty((0, \ldots, 0), r)\right) \subset B_Y(0, 1),$$

where B_∞ is the unit ball with respect to the product norm

$$\|(x_1, \ldots, x_m)\|_\infty = \max_{1\leq j\leq m} \|x_j\|_{X_j}.$$

Let $(x_1, \ldots, x_m) \in X_1\times\cdots\times X_m$ be arbitrary. If some $x_j = 0$, then $T(x_1, \ldots, x_m) = 0$ by multilinearity and the inequality in (iii) holds trivially. Assume therefore that $x_j \neq 0$

for all j, so that $\|x_j\|_{X_j} > 0$ by the definiteness of the quasi-norm. Set

$$y_j = \frac{r}{2\|x_j\|_{X_j}} x_j \quad \text{for } j = 1, \ldots, m.$$

Then $\|y_j\|_{X_j} = r/2 < r$ for each j, so

$$\|(y_1, \ldots, y_m)\|_\infty = r/2 < r$$

and thus

$$\|T(y_1, \ldots, y_m)\|_Y \leq 1.$$

By m-linearity,

$$T(y_1, \ldots, y_m) = \left(\prod_{j=1}^{m} \frac{r}{2\|x_j\|_{X_j}} \right) T(x_1, \ldots, x_m),$$

which implies

$$\left(\frac{r}{2}\right)^m \frac{1}{\prod_{j=1}^{m} \|x_j\|_{X_j}} \|T(x_1, \ldots, x_m)\|_Y \leq 1.$$

Hence

$$\|T(x_1, \ldots, x_m)\|_Y \leq \frac{2^m}{r^m} \prod_{j=1}^{m} \|x_j\|_{X_j}.$$

Set $c = 2^m/r^m < +\infty$. Then (iii) holds.

$(iii) \Rightarrow (i)$: Let $a = (a_1, \ldots, a_m) \in X_1 \times \cdots \times X_m$ and $\varepsilon > 0$ be given. Set $M := \max_{1\leq j\leq m} \|a_j\|_{X_j} + 1$ and choose

$$r = \min\left\{1, \frac{\varepsilon}{cmM^{m-1}}\right\} > 0.$$

If $(x_1, \ldots, x_m)$ satisfies $\max_{1\leq j\leq m} \|x_j - a_j\|_{X_j} < r$, then $\|x_j - a_j\|_{X_j} < r$ for all j, and $\|x_j\|_{X_j} \leq M$ since $r \leq 1$.

By the telescoping formula (Remark 2.1.1), we have

$$\begin{aligned} &T(x_1, \ldots, x_m) - T(a_1, \ldots, a_m) \\ &= \sum_{j=1}^{m} T\left(a_1, \ldots, a_{j-1}, x_j - a_j, x_{j+1}, \ldots, x_m\right). \end{aligned}$$

By assumption (iii), for each $j = 1, \ldots, m$,

$$\begin{aligned}
&\|T\,(a_1, \ldots, a_{j-1}, x_j - a_j, x_{j+1}, \ldots, x_m)\|_Y \\
&\leq c\|a_1\|_{X_1} \cdots \|a_{j-1}\|_{X_{j-1}} \|x_j - a_j\|_{X_j} \|x_{j+1}\|_{X_{j+1}} \cdots \|x_m\|_{X_m} \\
&\leq cM^{m-1}\|x_j - a_j\|_{X_j} \\
&\leq cM^{m-1} r.
\end{aligned}$$

Therefore

$$\|T(x_1, \ldots, x_m) - T(a_1, \ldots, a_m)\|_Y \leq \sum_{j=1}^{m} cM^{m-1} r = cmM^{m-1} r \leq \varepsilon,$$

which shows that T is continuous at a. Since a is arbitrary, T is continuous on the whole product space.

$(ii) \Leftrightarrow (iv)$: By definition of the operator quasi-norm

$$\|T\|_{\mathcal{L}(X_1,\ldots,X_m;Y)} = \sup_{\|x_j\|_{X_j} \leq 1,\, \forall j} \|T(x_1, \ldots, x_m)\|_Y,$$

this quantity is exactly the infimum of all constants $c \geq 0$ satisfying inequality (iii). $\square$

Example 2.1.1. Let

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

and

$$\begin{cases} T : (\mathbb{R}^2, \|\cdot\|_{\mathbb{R}^2}) \times (\mathbb{R}^2, \|\cdot\|_{\mathbb{R}^2}) \to (\mathbb{R}, \|\cdot\|_{\mathbb{R}}) \\ (x, y) \mapsto x_1 y_1 + x_2 y_2 \end{cases}$$

then

$$\|x\|_{\mathbb{R}^2} = \left(\sqrt{|x_1|} + \sqrt{|x_2|}\right)^2, \quad \|y\|_{\mathbb{R}^2} = \left(\sqrt{|y_1|} + \sqrt{|y_2|}\right)^2$$

$$\|T(x, y)\|_{\mathbb{R}} = |x_1 y_1 + x_2 y_2|.$$

Then

$$\frac{\|T(x,y)\|_{\mathbb{R}}}{\|x\|_{\mathbb{R}^2}\|y\|_{\mathbb{R}^2}} = \frac{|x_1y_1 + x_2y_2|}{\|x\|_{\mathbb{R}^2}\|y\|_{\mathbb{R}^2}} \leq \frac{|x_1||y_1|}{\|x\|_{\mathbb{R}^2}\|y\|_{\mathbb{R}^2}} + \frac{|x_2||y_2|}{\|x\|_{\mathbb{R}^2}\|y\|_{\mathbb{R}^2}}.$$

Indeed

$$\frac{|x_1||y_1|}{\|x\|_{\mathbb{R}^2}\|y\|_{\mathbb{R}^2}} \leq \frac{\left(\sqrt{|x_1|}+\sqrt{|x_2|}+2\sqrt{|x_1||x_2|}\right)\left(\sqrt{|y_1|}+\sqrt{|y_2|}+2\sqrt{|y_1||y_2|}\right)}{\|x\|_{\mathbb{R}^2}\|y\|_{\mathbb{R}^2}} = 1$$

and

$$\frac{|x_2||y_2|}{\|x\|_{\mathbb{R}^2}\|y\|_{\mathbb{R}^2}} \leq \frac{\left(\sqrt{|x_2|}+\sqrt{|x_1|}+2\sqrt{|x_1||x_2|}\right)\left(\sqrt{|y_2|}+\sqrt{|y_1|}+2\sqrt{|y_1||y_2|}\right)}{\|x\|_{\mathbb{R}^2}\|y\|_{\mathbb{R}^2}} = 1.$$

Hence T is bounded and

$$\|T\|_{\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2;\mathbb{R})} \leq 1.$$

We shall show that in the following theorem the set of all bounded m-linear operators denoted by $\mathcal{L}_{(X_1,\ldots,X_m;Y)}(X_1,\ldots,X_m;Y)$ is quasi-normed linear space under the quasi-norm $\|\cdot\|_{\mathcal{L}(X_1,\ldots,X_m;Y)}$.

Theorem 2.1.2. *Let* $\left(X_j, \|\cdot\|_{X_j}\right)_{1\leq j\leq m}$ *with* $m \in \mathbb{N}$ *and constants* K_j, $(Y, \|\cdot\|_Y)$ *be quasi-normed linear spaces with contant constants* K_Y, *then the pair* $\left(\mathcal{L}(X_1,\ldots,X_m;Y), \|\cdot\|_{\mathcal{L}(X_1,\ldots,X_m;Y)}\right)$ *is quasi-normed linear space.*

Proof. Let $T_1, T_2 \in \mathcal{L}(X_1 \times \ldots \times X_m; Y)$. For $x = (x_1,\ldots,x_m) \in X_1 \times \ldots \times X_m$, we have $(T_1 + T_2)(x) = T_1(x) + T_2(x)$ and $(\lambda T)(x) = \lambda T(x)$.

It is clear that for T_1, T_2 are bounded implies $\exists M_1 > 0, M_2 > 0$ such that $\forall x \in X_1 \times \ldots \times X_m$ we have

$$\|T_1(x_1,\ldots,x_m)\|_Y \leq M_1\|x_1\|_{X_1}\ldots\|x_m\|_{X_m}$$

and

$$\|T_2(x_1,\ldots,x_m)\|_Y \leq M_2\|x_1\|_{X_1}\ldots\|x_m\|_{X_m}.$$

For $\lambda_1, \lambda_2 \in \mathbb{K}$, we have

$$\begin{aligned}\|(\lambda_1 T_1 + \lambda_2 T_2)(x)\|_Y = \|\lambda_1 T_1(x) + \lambda_2 T_2(x)\|_Y &= \|T_1(\lambda_1 x) + T_2(\lambda_2 x)\|_Y \\ &\leq K_Y\left[\|T_1(\lambda_1 x)\|_Y + \|T_2(\lambda_2 x)\|_Y\right] \\ &\leq K_Y[M_1|\lambda_1|^m + M_2|\lambda_2|^m]\|x_1\|_{X_1}\cdots\|x_m\|_{X_m} \\ &\leq M\|x_1\|_{X_1}\cdots\|x_m\|_{X_m}\end{aligned}$$

With $M = K_Y[M_1|\lambda_1|^m + M_2|\lambda_2|^m]$. Hence we have $\lambda_1 T_1 + \lambda_2 T_2 \in \mathcal{L}(X_1, \ldots, X_m; Y)$ then $\mathcal{L}(X_1, \ldots, X_m; Y)$ is linear space.

Let $\left(X_j, \|\cdot\|_{X_j}\right)_{1\leq j\leq m}$, $(Y, \|\cdot\|_Y)$ be quasi-normed linear spaces. For $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$, define for $x \in X_1 \times \cdots \times X_m$

$$\|T\|_{\mathcal{L}(X_1,\ldots,X_m;Y)} = \sup_{0\neq x_j\in X_j} \frac{\|T(x)\|_Y}{\|x_1\|_{X_1}\ldots\|x_m\|_{X_m}}.$$

Clearly, $\|T\|_{\mathcal{L}(X_1,\ldots,X_m;Y)} \geq 0$, the conditions (i), (ii) of the definition of the quasi-norm are straightforward.

For third condition of quasi-norm. Let $T_1, T_2 \in \mathcal{L}(X_1, \ldots, X_m; Y)$, we have

$$\begin{aligned}\|T_1 + T_2\|_{\mathcal{L}(X_1,\ldots,X_m;Y)} &= \sup_{\substack{x_j\neq 0\\ x_j\in X_j}} \frac{\|(T_1+T_2)(x)\|_Y}{\|x_1\|_{X_1}\cdots\|x_m\|_{X_m}} \\ &= \sup_{\substack{x_j\neq 0\\ x_j\in X_j}} \frac{\|T_1(x)+T_2(x)\|_Y}{\|x_1\|_{X_1}\cdots\|x_m\|_{X_m}} \\ &\leq \sup_{\substack{x_j\neq 0\\ x_j\in X_j}} \frac{K_Y\left[\|T_1(x)\|_Y + \|T_2(x)\|_Y\right]}{\|x_1\|_{X_1}\cdots\|x_m\|_{X_m}} \\ &\leq \sup_{\substack{x_j\neq 0\\ x_j\in X_j}} \frac{K_Y\|T_1(x)\|_Y}{\|x_1\|_{X_1}\cdots\|x_m\|_{X_m}} + \sup_{\substack{x_j\neq 0\\ x_j\in X_j}} \frac{K_Y\|T_2(x)\|_Y}{\|x_1\|_{X_1}\cdots\|x_m\|_{X_m}},\end{aligned}$$

yields $\|T_1 + T_2\|_{\mathcal{L}(X_1,\ldots,X_m;Y)} \leq K_Y\left[\|T_1\|_{\mathcal{L}(X_1,\ldots,X_m;Y)} + \|T_2\|_{\mathcal{L}(X_1,\ldots,X_m;Y)}\right]$.

Thus $\left(\mathcal{L}(X_1, \ldots, X_m; Y), \|\cdot\|_{\mathcal{L}(X_1,\ldots,X_m;Y)}\right)$ is quasi-normed linear space. $\square$

Remark 2.1.2. The quasi-norm of T, $\|T\|_{\mathcal{L}(X_1,\ldots,X_m;Y)}$ can be calculated by

$$\|T\|_{\mathcal{L}(X_1,\ldots,X_m;Y)} = \sup\left\{\|T(x)\|_Y : \|x\|_\infty \le 1\right\}.$$

Indeed, while $\|x\|_\infty \le 1$, then

$$\|x^1\|_{X_1}\ldots\|x^m\|_{X_m} \le 1, \quad \text{for all } x^j \ne 0$$

and

$$\|T(x)\|_Y \le \|T\|_{\mathcal{L}(X_1,\ldots,X_m;Y)}\left(\|x^1\|_{X_1}\ldots\|x^m\|_{X_m}\right)$$

holds when $\|x\|_\infty \le 1$.

2.2 Completeness properties

We need the next lemma to prove the completeness properties of the space $\left(\mathcal{L}(X_1,\ldots,X_m;Y), \|\cdot\|_{\mathcal{L}(X_1,\ldots,X_m;Y)}\right)$.

Lemma 2.2.1. *[45, Lemma 3.1] Let $(Y, \|\cdot\|_Y)$ be a quasi-normed space with constant K_Y, a sequence $\{y_n\}_{n\in\mathbb{N}}$ in Y such that $\lim_{n\to\infty}\|y_n - y\|_Y = 0$ then $\|y\|_Y = \|\lim_{n\to\infty} y_n\|_Y \le K_Y \lim_{n\to\infty}\|y_n\|_Y$.*

such that K_Y is the index of quasi norm of Y. The following result is a quasi-normed version of the Banach-Steinhaus theorem that introduced by J. Lv and Y. Feng in [36].

Theorem 2.2.2 ([36])**.** *Let $\mathcal{F}$ be a family of continuous linear operators from quasi-normed space $(X, \|\cdot\|_X)$ to quasi-normed space $(Y, \|\cdot\|_Y)$ such that for each $x \in X$ there exists $b_x > 0$ with $\|T(x)\|_Y \le b_x$, for all $T \in \mathcal{F}$, then there exists $b > 0$ such that*

$$\sup\{\|T(x)\|_Y, \quad \|x\|_X \le 1\} < b.$$

It is used to prove that each T_j is linear and continuous, so since, if each T_j is linear implies that T is m-linear, then we have the next theorem to prove the continuity of T.

Definition 2.2.1. A multilinear operator $T : X_1 \times \ldots \times X_m \longrightarrow Y$ between quasi-normed linear spaces is separately continuous it is continuous separately in each coordinate. We know that, the continuity implies separate continuity, but the converse is true if $X_1, \ldots, X_m$ are Banach spaces. In this way, we will prove that this result for the quasi-normed case.

Theorem 2.2.3. *The map $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is continuous with respect to the norm $\|\cdot\|_{\mathcal{L}(X_1,\ldots,X_m;Y)}$ if and only if it is separately continuous.*

Proof. Clearly, the first implication is obvious.

Let us prove that the second implication ($\Leftarrow$) by induction on m, for $m = 2$,

$(X_1, \|\cdot\|_{X_1})$, $(X_2, \|\cdot\|_{X_2})$ quasi-normed linear spaces, and $(Y, \|\cdot\|_Y)$ is a quasi Banach space, therefore, let T in $\Big(\mathcal{L}(X_1, X_2; Y), \|\cdot\|_{\mathcal{L}(X_1,X_2;Y)}\Big)$.

Assume that T is separately continuous i.e., for fixed $x_2 \in X_2$ the partial mappings $T_{x_2} : X_1 \longrightarrow Y$ such that $T_{x_2}(x_1) = T(x_1, x_2)$, and $T_{x_1} : X_2 \longrightarrow Y$ such that $T_{x_1}(x_2) = T(x_1, x_2)$ are continuous, we have

$$\|T(x_1, x_2)\|_Y = \|T_{x_2}(x_1)\|_Y = \|T_{x_1}(x_2)\|_Y,$$

we have $T_{x_1}(x_2) = T(x_1, x_2) = T_{x_2}(x_1)$, T_{x_2} is continuous $\|T(x_1, x_2)\|_Y = \|T_{x_1}(x_2)\|_Y \leq M_{x_1}\|x_2\|_{X_2}$ and for $\|x_2\|_{X_2} \leq 1$

$$\|T_{x_2}(x_1)\|_Y = \|T(x_1, x_2)\|_Y = \|T_{x_1}(x_2)\|_Y$$

For the converse, let the family $\mathcal{F} = \{T_{x_2}, x_2 \in B_{X_2}\}$ as T_{x_2} is defined into quasi-normed space and the quasi norm $\|\cdot\|_Y$ induces a quasi-metric by $D_K(y_1, y_2) = \|y_2 - y_1\|_Y$ which define a second category Baire space, then the family of continuous mappings pointwise bounded are actively uniformly bounded over an open set B_{X_2}, i.e., $\exists M > 0$ such that

$$\|T_{x_2}\|_{\mathcal{L}(X_1,X_2;Y)} \leq M \quad \text{for all } T_{x_2} \in \mathcal{F}$$

then for each $x^1 \in X_1$, $x^2 \in X_2$ with $\|x^1\|_{X_1} \leq 1$, $\|x^2\|_{X_2} \leq 1$ we have

$$\|T(x^1, x^2)\|_Y = \|T_{x^2}(x^1)\|_Y \leq \|T_{x^2}\|_{\mathcal{L}(X_1, X_2; Y)} \leq M.$$

Hence

$$\|T(x^1, x^2)\|_Y \leq M\|x^1\|_{X_1}\|x^2\|_{X_2}, \quad \forall (x^1, x^2) \in X_1 \times X_2.$$

Suppose that the assertion is true up to $(m-1)$, define the following mappings $T_{x^m} : X_1 \times \ldots \times X_{m-1} \longrightarrow Y$ and $T_{x^1,\ldots,x^{m-1}} : X_m \longrightarrow Y$ as above

$$T_{x^m}(x^1, \ldots, x^{m-1}) = T_{x^1,\ldots,x^{m-1}}(x^m) := T(x^1, \ldots, x^m).$$

$T_{x^1,\ldots,x^{m-1}}$ is continuous by assumption. Therefore, for fixed $(x^1, \ldots, x^{m-1})$ we have $\|T_{x^1,\ldots,x^{m-1}}(x^m)\|_Y \leq M_{x^1,\ldots,x^{m-1}}\|x^m\|_{X_m}$, $M_{x^1,\ldots,x^{m-1}}$ constant. Consider the family $\mathcal{F} = \{T_{x^m} : x^m \in B_{X_m}\}$, T_{x^m} is separately continuous yields the existence of $M > 0$ for all $T_{x^m} \in \mathcal{F}$ such that

$$\|T_{x^m}\|_{\mathcal{L}(X_1,\ldots,X_m;Y)} \leq M \quad \text{for all } T_{x^m} \in \mathcal{F}.$$

Hence we have for all $(x^1, \ldots, x^m) \in B_{X_1 \times \ldots \times X_m}$

$$\|T_{x^m}(x^1, \ldots, x^{m-1})\|_Y = \|T_{x^1,\ldots,x^{m-1}}(x^m)\|_Y \leq M.$$

Then

$$\|T(x^1, \ldots, x^m)\|_Y = \|T_{x^m}(x^1, \ldots, x^{m-1})\|_Y \leq \|T_{x^m}\|_{\mathcal{L}(X_1,\ldots,X_m;Y)} \leq M,$$

$$\|T(x^1, \ldots, x^m)\|_Y \leq M\|x^1\|_{X_1} \ldots \|x^m\|_{X_m}.$$

So by the above quasi-normed version of the Banach-Steinhaus theorem, we have

$$\sup_{T_{x^m} \in \mathcal{F}} \left(\sup_{x^m \in B_{X_m}} \|T(x^1, \ldots, x^m)\|_Y \right) < \infty.$$

Then,

$$\begin{aligned}\|T\|_{\mathcal{L}(X_1,\ldots,X_m;Y)} &= \sup_{\|x_j\|\leq 1, j=1,\ldots,m} \|T(x^1,\ldots,x^m)\|_Y \\ &= \sup_{T_{x^m}\in\mathcal{F}} \left(\sup_{x^m\in B_{X_m}} \|T(x^1,\ldots,x^m)\|_Y \right) \\ &< \infty.\end{aligned}$$

The proof is finished. □

Theorem 2.2.4. *Let $\left(X_j, \|\cdot\|_{X_j}\right)_{1\leq j\leq m}$ be quasi-normed spaces and $(Y, \|\cdot\|_Y)$ be a quasi-Banach space, then the space $\left(\mathcal{L}(X_1\times\ldots\times X_m;Y), \|\cdot\|_{\mathcal{L}(X_1,\ldots,X_m;Y)}\right)$ is a quasi-Banach space.*

Proof. Since $\left(\mathcal{L}(X_1\times\ldots\times X_m;Y), \|\cdot\|_{\mathcal{L}(X_1,\ldots,X_m;Y)}\right)$ is quasi-normed space from Theorem 2.1.2, $(Y, \|\cdot\|_Y)$ is quasi-Banach space. Let $(T_n)_n$ be in $\mathcal{L}(X_1\times\ldots\times X_m;Y)$ be a Cauchy sequence converging to $y = \lim_{n\to\infty} T_n(x_1,\ldots,x_m) \in Y$ which depends on the choice $(x_1,\ldots,x_m)$, then $y = T(x_1,\ldots,x_m)$ is clearly an operator, to prove that T is m-linear, we know that each $\lim_{n\to\infty}(T_j)_n(x^1,\ldots,x^m)$ with each $T_j : \left(X_j, \|\cdot\|_{X_j}\right) \to (Y, \|\cdot\|_Y)$ is linear and bounded (see [46]), hence we obtain that T is multilinear while its linear for each variable (x_j). From the continuity of each T_j and Theorem 2.2.3, we have T is continuous.

Since $\left(\mathcal{L}(X_1\times\ldots\times X_m;Y), \|\cdot\|_{\mathcal{L}(X_1,\ldots,X_m;Y)}\right)$ is quasi-normed space. □

2.3 Some fundamental theorems

2.3.1 The quasi-normed principle for m-linear mappings

We introduce in this section a Banach-Steinhaus theorem for continuous m-linear operators defined on quasi-normed spaces.

Theorem 2.3.1 (quasi-normed principle)**.** *Let $\left(X_j, \|\cdot\|_{X_j}\right)_{1\leq j\leq m}$ be quasi-Banach spaces, $(Y, \|\cdot\|_Y)$ quasi-normed linear space, and $\{T_\alpha\}_{\alpha\in I}$ be a family of mappings in*

$\mathcal{L}(X_1, X_2, \ldots, X_m; Y)$ if for all $(x^1, \ldots, x^m) \in X_1 \times \ldots \times X_m$ such that

$$\sup_{\alpha\in I} \|T_\alpha(x^1, \ldots, x^m)\|_Y < \infty \quad \textit{then} \quad \sup_{\alpha\in I} \|T_\alpha\|_{\mathcal{L}(X_1 \ldots X_m;Y)} < \infty.$$

Proof. Let

$$F_n = \left\{ x \in X_1 \times \ldots \times X_m : \sup_{\alpha\in I} \|T_\alpha(x)\|_Y \leq n \right\}.$$

Note that each F_n is closed subset of $X_1 \times \ldots \times X_m$, with $X_1 \times \ldots \times X_m = \bigcup_{n\geq 1} F_n$ and from the Baire category theorem there exist $n_0 \in \mathbb{N}$ such that F_{n_0} has non empty interior, let $(a^1, \ldots, a^m) \in F_{n_0}$ then there exist a constant $r > 0$ so that $B_{X_1\times\ldots\times X_m}((a^1, \ldots, a^m), r)$ is in F_{n_0} let $\alpha \in I$ we have for all $(y^1, \ldots, y^m) \in B_{X_1\times\ldots\times X_m}((0, \ldots, 0), r)$ and F_{n_0}

$$\begin{aligned}
&\|T_\alpha(y^1, y^2 + a^2 \ldots, y^m + a^m)\|_Y \\
&= \|T_\alpha(y^1 + a^1, \ldots, y^m + a^m) - T_\alpha(a^1, y^2 + a^2, \ldots, y^m + a^m)\|_Y \\
&\leq K_Y \left[\|T_\alpha(y^1 + a^1, \ldots, y^m + a^m)\|_Y + \|T_\alpha(a^1, y^2 + a^2, \ldots, y^m + a^m)\|_Y \right] \\
&\leq K_Y(2n_0).
\end{aligned}$$

With the same argument for all $(z^1, \ldots, z^m) \in B_{X_1\times\ldots\times X_m}((0, \ldots, 0), r)$

$$\begin{aligned}
&\|T_\alpha(z^1, z^2, z^3 + a^3, \ldots, z^m + a^m)\|_Y \\
&= \|T_\alpha(z^1, z^2 + a^2, \ldots, y^m + a^m) - T_\alpha(z^1, a^2, z^3 + a^3, \ldots, z^m + a^m)\|_Y \\
&\leq K_Y[K_Y(2n_0) + K_Y(2n_0)] \leq K_Y^2 2^2 n_0.
\end{aligned}$$

By repeating this argument m times we obtain

$$\|T_\alpha(x^1, \ldots, x^m)\|_Y \leq K_Y^m 2^m n_0$$

for all $(x^1, \ldots, x^m) \in B_{X_1\times\ldots\times X_m}((0, \ldots, 0), r)$. Since α is arbitrary we get easily

$$\sup_{\alpha\in I} \|T_\alpha\|_{\mathcal{L}(X_1 \ldots X_m;Y)} \leq \frac{K_Y^m 2^m n_0}{r^m}.$$

The proof is complete. $\square$

2.3.2 Multilinear closed graph theorem

Theorem 2.3.2 (Multilinear closed graph theorem). *Let $\left(X_j, \|\cdot\|_{X_j}\right)_{1\leq j\leq m}$, $(Y, \|\cdot\|_Y)$ be quasi-normed spaces, and $T \in \mathcal{L}(X_1, X_2, \ldots, X_m; Y)$. An m-linear mapping with closed graph, then T is continuous.*

Proof. For each $1 \leq j \leq m$ fix $a^j \in X_j$, and let T^j the partial mappings such that

$$T^j : x^j \in X_j \mapsto T\left(a^1, \ldots, a^{j-1}, x^j, a^{j+1}, \ldots, a^m\right) \in Y.$$

By multilinearity, T^j is clearly a linear mapping, moreover the graph $G(T^j)$ of T^j is closed since it is the image of closed sets as follows

$$G(T^j) = G(T) \cap \left(\{a^1\} \times \ldots \times \{a^{j-1}\} \times X_j \times \{a^{j+1}\} \times \ldots \times \{a^m\} \times Y\right).$$

By the homeomorphism Ψ such that

$$\Psi : \{a^1\} \times \ldots \times \{a^{j-1}\} \times X_j \times \{a^{j+1}\} \times \ldots \times \{a^m\} \times Y \quad \to \quad X_j \times Y$$

$$\left(a^1, \ldots, a^{j-1}, x^j, a^{j+1}, \ldots, a^m, y\right) \mapsto (x^j, y)$$

Hence, by classical graph theorem between each $\left(X_j, \|\cdot\|_{X_j}\right)_{1\leq j\leq m}$ and $(Y, \|\cdot\|_Y)$, T^j is continuous, this shows that T is separately continuous, therefore by Theorem 2.2.3 we have T is continuous. $\square$

Chapter 3

Multivalued Linear Relations in Quasi-Banach Spaces

This chapter extends the theory of multivalued linear relations to quasi-Banach spaces (developed by Rano and Bag [45, 46]), establishing fundamental theorems including the closed graph theorem, quasi-normed principle, and open mapping theorem, as well as minimum modulus and continuous selections.

3.1 The linear relations between quasi-normed linear spaces

In this section we will present and study the continuous linear relations between quasi-normed linear spaces, with respect to the induced topology of the quasi metric$d_X; d_Y$.

Let $T \in LR(X,Y)$ and consider the quotient space $Y/\overline{T(0)}$ with respect to the induced topology equipped with the quasi norm

$$\|\bar{y}\|_{\mathcal{L}(X,Y/\overline{T(0)})} = \inf_{k\in\overline{T(0)}} \|y+k\|_Y.$$

It is straightforward to show that:

Proposition 3.1.1. *The mapping $Q : Y \longrightarrow Y/\overline{T(0)}$ is the natural quotient map* *between quasi-normed spaces with domain Y and null space $T(0)$.*

Proof. Indeed; The map $Q : Y \longrightarrow Y/\overline{T(0)}$ defined by $Q(y) = y + T(0)$ is clearly linear and surjective. Its kernel consists of all $y \in Y$ such that $y + T(0) = T(0)$, that is, $\ker(Q) = T(0)$. The quotient quasi-norm $\|\bar{y}\|_{\mathcal{L}(X,Y/\overline{T(0)})} = \inf_{k\in T(0)} \|y + k\|_Y$ satisfies the required axioms, and $\|Qy\|_{\mathcal{L}(X,Y/\overline{T(0)})} \leq \|y\|_Y$ for all $y \in Y$, ensuring continuity. Hence Q is indeed the natural quotient map. □

Remark 3.1.1. In the sequel we shall consider the notations of Q_T and Q as the same.

In the following proposition we give the analogue of [18, Proposition II.1.4] in the case of linear relations in quasi-normed linear spaces.

We shall prove that as in the following proposition that Q_T is single valued.

Proposition 3.1.2. *Let $T \in LR(X, Y)$ and let Q be the natural quotient map, then*

(i) QT is single-valued.

(ii) $\|Tx\|_Y = d_Y(y, T(0))$ for any $y \in Tx$.

(iii) $\|Tx\|_Y = d_Y(Tx, T(0))$ for all $x \in D(T)$.

(iv) $\|T\|_{\mathcal{LR}(X,Y)} = \sup_{\|x\|_X\leq 1} \|Tx\|_Y$.

Proof. For (i), it is enough to prove that $QT(0) = 0$, while $QT(0) = 0 + T(0) \subset QT(0) = 0$. Indeed, let $x \in D(T)$ and $y_1, y_2 \in QT(x)$. Hence $y_1-y_2 \in QT(x)-QT(x) = QT(0) \subset QT(0) = 0$, then $y_1 = y_2$ and (i) holds. For (ii) and (iii), let $y \in Tx$ then

$$\|Tx\|_Y = \|QTx\|_{\mathcal{L}(X,Y/\overline{T(0)})} = \inf_{k\in T(0)} \|y + k\|_Y = d_Y(y, T(0)) = d_Y(Tx, T(0)).$$

For (iv) we have by definition

$$\|T\|_{\mathcal{LR}(X,Y)} = \|QT\|_{\mathcal{L}(X,Y/\overline{T(0)})} = \sup_{\|x\|_X\leq 1} \|QTx\|_{\mathcal{L}(X,Y/\overline{T(0)})} = \sup_{\|x\|_X\leq 1} \|Tx\|_Y.$$

□

Now, we give the definition of quasi norm of a linear relation on quasi normed spaces.

Definition 3.1.1. Let $T \in LR(X,Y)$, we define

$$\|Tx\|_Y = \|QTx\|_{\mathcal{L}(X,Y/\overline{T(0)})}, \quad (x \in D(T)). \tag{3.1}$$

where

$$\|T\|_{\mathcal{LR}(X,Y)} = \|QT\|_{\mathcal{L}(X,Y/\overline{T(0)})} = \inf_{k \in \overline{T(0)}} \|y+k\|. \tag{3.2}$$

$$\|T\|_{\mathcal{LR}(X,Y)} = \|QT\|_{\mathcal{L}(X,Y/\overline{T(0)})} = \sup_{x \neq 0} \left\{ \frac{\|Q_T Tx\|_Y}{\|x\|_X}, \|x\|_X \leq 1 \right\}.$$

The quasi-metric between U and V in X is define as follows

$$d_X(U,V) = \inf\{\|v-u\|_X : u \in U, v \in V\}. \tag{3.3}$$

where U and V are nonempty subsets of the quasi-normed space X and we denote $d_X(x,V)$ the quasi-metric between $\{x\}$ and V.

A generalization of [18, Proposition II.1.4] is given by the following proposition.

Proposition 3.1.3. *Let X, Y be two quasi-normed spaces and $T \in LR(X,Y)$, we have*

(i) If $y_1, y_2 \in Tx$, then $d_Y(y_1, T(0)) = d_Y(y_2, T(0))$.

(ii) We have

$$\inf_{y \in Tx} d_Y(y, T(0)) = \sup_{y \in Tx} d_Y(y, T(0)) \quad \textit{for any } x \in D(T).$$

(iii) $\|Tx\|_Y = d_Y(y, T(0))$ for any $y \in Tx$.

(iv) $\|Tx\|_Y = d_Y(Tx, T(0))$ for any $(x \in D(T))$.

Proof. (i) Clearly $y_1 - y_2 \in T(0)$, then we have

$$\begin{aligned} d_Y(y_1, T(0)) &= \inf_{z \in T(0)} \|z - y_1\|_Y \\ &\leq \inf_{z \in T(0)} \|(z - (y_1 - y_2)) - y_2\|_Y \\ &\leq d_Y(y_2, T(0)). \end{aligned}$$

We do the same thing for prove that $d_Y(y_2, T(0)) \leq d_Y(y_1, T(0))$.

(ii) For all $z \in Tx$ and according (3.1), we have

$$d_Y(z, T(0)) = \inf_{y \in Tx} d_Y(y, T(0)) = \sup_{y \in Tx} d_Y(y, T(0)).$$

(iii) Due to the fact that $y \in Tx$, $Tx = y + T(0)$, then from QT is single valued map, we have

$$\|Tx\|_Y = \|QTx\|_{\mathcal{L}(X, Y/\overline{T(0)})} = \|Qy\|_{\mathcal{L}(X, Y/\overline{T(0)})} = d_Y(y, T(0)).$$

(iv) Fix $z \in Tx$ and for any $y \in Tx$, $y - z \in T(0)$, then

$$\begin{aligned} d_Y(Tx, T(0)) &= \inf_{y \in Tx, k \in T(0)} \|k + y\|_Y \\ &= \inf_{y \in Tx, k \in T(0)} \|z + (y - z) + k)\|_Y \\ &= d_Y(z, T(0)) = \|Tx\|_Y. \end{aligned}$$

The proof is obtained. □

Proposition 3.1.4. *(a) Let $T_1, T_2 \in LR(X, Y)$, we have*

$$\|T_1x + T_2x\|_Y \leq K_Y[\|T_1x\|_Y + \|T_2x\|_Y], \quad \textit{for } K_Y \geq 1,$$

where $x \in D(T_1 + T_2)$ and K_Y the corresponding quasi constant of Y.

(b) For every $\alpha \in \mathbb{R}$, we have

$$\|\alpha T_1 x\|_Y = |\alpha| \|T_1 x\|_Y \quad \textit{where } x \in D(T_1).$$

Proof. (a) Let $s \in T_1x$, $k \in T_2x$ and $x \in D(T_1 + T_2)$, we have $s + k \in (T_1 + T_2)x =$

$T_1x + T_2x$, and $K_Y \geq 1$

$$\begin{aligned}\|T_1x + T_2x\|_Y &= d_Y((s+k), (T_1+T_2)(0)) \\ &\leq K_Y[d_Y(s, T_1(0)+T_2(0)) + d_Y(k, T_1(0)+T_2(0))] \\ &\leq K_Y[d_Y(s, T_1(0)) + d_Y(k, T_2(0))] \\ &= K_Y[\|T_1x\|_Y + \|T_2x\|_Y].\end{aligned}$$

(b) Since $\alpha \neq 0$, we have $\alpha T(0) = T(0)$. Then

$$\|\alpha Tx\|_Y = \|Q(\alpha Tx)\|_{\mathcal{L}(X,Y/\overline{T(0)})} = |\alpha|\|QTx\|_{\mathcal{L}(X,Y/\overline{T(0)})} = |\alpha|\|Tx\|_Y.$$

For $\alpha = 0$ it is trivial. □

Proposition 3.1.5. *Let X, Y be two quasi-normed spaces and $T \in LR(X,Y)$. Let Q be the natural quotient, then $\|T\|_{\mathcal{LR}(X,Y)}$ is a semi-quasi norm.*

$$\|T\|_{\mathcal{LR}(X,Y)} = \|QT\|_{\mathcal{L}(X,Y/\overline{T(0)})} = \sup_{\|x\|_X \leq 1} \|Tx\|_Y. \tag{3.4}$$

Proof. Since QT is single valued mapping form X to $Y/\overline{T(0)}$ and by using the Definition (3.1.1), we have

$$\|T\|_{\mathcal{LR}(X,Y)} = \|QT\|_{\mathcal{L}(X,Y/\overline{T(0)})} = \sup_{x \in B(X,\|\cdot\|_X)} \|QTx\|_{\mathcal{L}(X,Y/\overline{T(0)})} = \sup_{\|x\|_X \leq 1} \|Tx\|_Y.$$

Clearly the second and the third condition of the definition of quasi-norm holds; for the first if $T = 0$ it is trivial, however, if $\|Tx\|_Y = 0 \Rightarrow x \in T^{-1}(0)$, hence it is a semi quasi-norm. □

Proposition 3.1.6. *Let X be a quasi-normed space and $T \in LR(X,Y)$. Then $\|T\|_{\mathcal{LR}(X,Y)} = 0$ if and only if $T(B_{D(T)}) \subset T(0)$.*

Proof. If $\|T\|_{\mathcal{LR}(X,Y)} = 0$, then by using [Proposition 3.1.2, (ii)], we have $\|Tx\|_Y = 0$ for all $x \in B_{D(T)}$. Hence, for all $x \in B_{D(T)}$ and $y \in Tx$, we have $d_Y(y, T(0)) = 0$. So, for all $x \in B_{D(T)}$ and $y \in Tx$, we have $d_Y(y, T(0)) = 0$. This implies that for all $x \in B_{D(T)}$, $Tx \subset T(0)$. Thus, $T(B_{D(T)}) \subset T(0)$.

Conversely, let $T(B_{D(T)}) \subset T(0)$. Then for $0 \neq x \in D(T)$ we have $T\left(\frac{x}{\|x\|_X}\right) \subset T(0)$, hence $T(x) \subset \|x\|_X T(0) = T(0)$. □

The following proposition is a geometric characterization of the semi-quasi norm $\|T\|_{\mathcal{LR}(X,Y)}$.

Proposition 3.1.7. *Let $T \in LR(X,Y)$, we have*

(i) $\|T\|_{\mathcal{LR}(X,Y)} < \infty$ if and only if there exists a constant $\lambda > 0$ so that

$$TB_{(D(T),\|\cdot\|_X)} \subset \lambda B_{(R(T),\|\cdot\|_Y)} + T(0). \tag{3.5}$$

(ii) If $\|T\|_{\mathcal{LR}(X,Y)} < \infty$, then

$$\|T\|_{\mathcal{LR}(X,Y)} = \inf_{\lambda>0}\{\lambda : TB_{(D(T),\|\cdot\|_X)} \subset \lambda B_{(R(T),\|\cdot\|_Y)} + T(0)\}. \tag{3.6}$$

Proof. (i) Suppose that $\|T\|_{\mathcal{LR}(X,Y)} < \infty$, then

$$\|T\|_{\mathcal{LR}(X,Y)} = \sup_{x\in B(X,\|\cdot\|_X)} \inf_{k\in T(0), y\in Tx} \|y+k\|_Y.$$

Thus if x and $y \in Tx$, then there exists $k \in T(0)$, a given $\varepsilon > 0$, such that

$$\|y+k\|_Y < \|T\|_{\mathcal{LR}(X,Y)} + \varepsilon,$$

so

$$y + k \in (\|T\|_{\mathcal{LR}(X,Y)} + \varepsilon)B_{(R(T),\|\cdot\|_Y)},$$

$$y \in (\|T\|_{\mathcal{LR}(X,Y)} + \varepsilon)B_{(R(T),\|\cdot\|_Y)} + T(0). \tag{3.7}$$

if and only if there exists a constant λ $(0 < \lambda = \|T\|_{\mathcal{LR}(X,Y)} + \varepsilon)$ for which

$$TB_{(D(T),\|\cdot\|_X)} \subset \lambda B_{(R(T),\|\cdot\|_Y)} + T(0). \tag{3.8}$$

Conversely, suppose that (3.5) holds. Let $x \in B_{(D(T),\|\cdot\|_X)}$ and $y \in Tx$. Then $y+k = \lambda y_1$

where $\|y_1\|_Y \leq 1$ and $k \in \overline{T(0)}$ hence $\|y+k\|_Y \leq \lambda$. Consequently $d_Y(y, T(0)) \leq \lambda$ and $\|T\|_{\mathcal{LR}(X,Y)} \leq \lambda < \infty$.

(ii) We take $\|T\|_{\mathcal{LR}(X,Y)} < \infty$, then

$$\|T\|_{\mathcal{LR}(X,Y)} = \inf_{\lambda>0}\{\lambda : TB_{(D(T),\|\cdot\|_X)} \subset \lambda B_{(R(T),\|\cdot\|_Y)} + T(0)\}. \tag{3.9}$$

If $\|T\|_{\mathcal{LR}(X,Y)} = 0$, then from Proposition 3.1.6 we have $TB_{(D(T),\|\cdot\|_X)} \subset \overline{T(0)}$, so (3.6) is hold. Now let $\|T\|_{\mathcal{LR}(X,Y)} > 0$, we have

$$\inf_{\lambda>0}\{\lambda : TB_{(D(T),\|\cdot\|_X)} \subset \lambda B_{(R(T),\|\cdot\|_Y)} + T(0)\}. \tag{3.10}$$

Suppose that $\alpha > 0$ satisfying $\alpha < \|T\|_{\mathcal{LR}(X,Y)}$. Choose $x \in B_{(D(T),\|\cdot\|_X)}$ and $y \in Tx$ so that

$$\alpha < d_Y(y, T(0)).$$

Assume that $TB_{(D(T),\|\cdot\|_X)} \subset \lambda B_{(R(T),\|\cdot\|_Y)} + T(0)$. Then there exists $y_1 \in B_{(R(T),\|\cdot\|_Y)}$ and $k \in T(0)$ such that $y + k = \alpha y_1$. Then

$$\|y+k\|_Y \leq \alpha < d_Y(y, T(0)). \tag{3.11}$$

contradiction. Fortunately

$$\alpha \leq \inf_{\lambda>0}\{\lambda : TB_{(D(T),\|\cdot\|_X)} \subset \lambda B_{(R(T),\|\cdot\|_Y)} + T(0)\},$$

and the result follows. □

3.1.1 The minimum modulus

In the following definition, we introduce the notion of the minimum modulus of a linear relation between quasi-normed spaces.

Definition 3.1.2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two quasi-normed linear spaces and let $T \in LR(X,Y)$. The minimum modulus of T noted by $\gamma_{X,Y}(T)$ defined as follows

$$\gamma_{X,Y}(T) = \sup\{\lambda : \|Tx\|_Y \geq \lambda d_X(x, N(T)) \text{ for } x \in D(T)\}.$$

The proposition presented below provides a characterization of Definition 3.1.2.

Proposition 3.1.8. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two quasi-normed linear spaces. For $T \in LR(X,Y)$, we have*

$$\gamma_{X,Y}(T) = \begin{cases} \infty, \textit{ if } D(T) \subset \overline{N(T)}, \\ \inf\left\{\dfrac{\|Tx\|_Y}{\lambda d_X(x, N(T))} \textit{ for } x \in D(T), x \notin \overline{N(T)}\right\} \end{cases} \tag{3.12}$$

Proof. We take $\gamma^{(1)}_{X,Y}(T)$ as in the Expression (3.12). Let λ verify

$$\|Tx\|_Y \geq \lambda d_X(x, N(T)). \tag{3.13}$$

for every $x \in D(T)$ For $D(T) \subset N(T)$, then $\lambda \leq \gamma^{(1)}_{X,Y}(T)$ is trivial. In the other hand, we have $x \in D(T)$ (such that $x \notin N(T)$) and by (3.13), we have

$$\lambda \leq \frac{\|Tx\|_Y}{d_X(x, N(T))}.$$

Then

$$\lambda \leq \gamma^{(1)}_{X,Y}(T).$$

Taking the supremum over λ in (3.13) yields $\gamma_{X,Y}(T) \leq \gamma^{(1)}_{X,Y}(T)$.

Now, we take $\gamma_{X,Y}(T) \leq \gamma^{(1)}_{X,Y}(T)$. Then obviously $\gamma_{X,Y}(T) < \infty$ this implies that $x \in D(T)$ and $x \notin N(T)$. Moreover, there exists x in $D(T)$ for which, one have

$$\|Tx\|_Y < \gamma^{(1)}_{X,Y}(T) d_X(x, N(T)),$$

this contradicts (3.12). Therefore; for $T \in LR(X,Y)$

$$\gamma_{X,Y}(T) = \gamma^{(1)}_{X,Y}(T).$$

□

The following proposition offers a geometric representation of $\gamma_{X,Y}(T)$.

Proposition 3.1.9. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two quasi-normed linear spaces. For $T \in LR(X,Y)$, we have*

$$\gamma_{X,Y}(T) = \sup\{\lambda : TB_{D(T)} \supset \lambda B_{R(T)}.\} \tag{3.14}$$

Proof. Suppose that $\gamma^{(1)}_{X,Y}(T)$ as in the Expression (3.14). We take $\gamma^{(1)}_{X,Y}(T) \neq \infty$ and $\gamma^{(1)}_{X,Y}(T) > \gamma_{X,Y}(T)$. Then there exists (x,y) in $G(T)$ such that

$$1 = \|y\|_Y < \gamma^{(1)}_{X,Y}(T) d_X(x, N(T)) \cdot (1+2\epsilon)^{-1}. \tag{3.15}$$

if ϵ is so small, then

$$\gamma^{(1)}_{X,Y}(T) > \gamma_{X,Y}(T) \cdot (1+2\epsilon).$$

The Equality (3.14) yields, $y \in (\gamma^{(1)}_{X,Y})^{-1}(T)TB_{D(T)}$, then $x_1 \in (\gamma^{(1)}_{X,Y})^{-1}(T)B_{D(T)}$ can be chosen so that $y \in Tx_1$. As $x_1 - x \in N(T)$, according (3.15), we have

$$(1+2\epsilon) < \gamma^{(1)}_{X,Y}(T) d_X(x, N(T)) = \gamma^{(1)}_{X,Y}(T) d_X(x_1, N(T)) \leq \gamma^{(1)}_{X,Y}(T)\|x_1\|_X \leq 1+\epsilon,$$

which is a contradiction. Consequently

$$\gamma^{(1)}_{X,Y}(T) \leq \gamma_{X,Y}(T).$$

For the reverse inequality, it is enough to take the case $\gamma^{(1)}_{X,Y}(T) < \infty$. Here, we put

$$\gamma^{(1)}_{X,Y}(T) < \beta < \gamma_{X,Y}(T).$$

Hence $TB_{D(T)} \not\supset \beta B_{R(T)}$ and there is $y \in R(T)$ where $\|y\|_Y = 1$ for which $y \notin \beta^{-1}TB_{D(T)}$. Therefore

$$\|x\|_X \leq \beta^{-1} \Rightarrow y \notin Tx. \tag{3.16}$$

Take $y \in Tx_0$. Fix α where $\beta < \alpha < \gamma_{X,Y}(T)$. Then

$$1 = \|y\|_Y \geq \|Tx_0\|_Y \geq \alpha d_X(x_0, N(T)) \quad (\text{since } \gamma_{X,Y}(T) > \alpha). \tag{3.17}$$

We choose $x_1 \in N(T)$ so $\|Tx_0\|_Y \geq \|x_1 + x_0\|_X$ by taking $x = x_1 + x_0$, we have $Tx_0 = Tx$, and $y \in Tx$ with $1 = \|y\|_Y \geq \|Tx\|_Y \geq \beta\|x\|_X$, this contradict (3.16). Thus $\gamma^{(1)}_{X,Y}(T) \geq \gamma_{X,Y}(T)$. □

Theorem 3.1.1. *For $T \in LR(X, Y)$, we have*

$$\gamma_{X,Y}(T) = \|T^{-1}\|^{-1}.$$

Proof. For $\lambda \geq 0$, we have

$$\begin{aligned}
T^{-1}B_{(R(T),\|\cdot\|_Y)} \supset \lambda B_{(D(T),\|\cdot\|_X)} &\Rightarrow \lambda TB_{(D(T),\|\cdot\|_X)} \subset B_{(R(T),\|\cdot\|_Y)} + T(0) \\
&\Rightarrow T^{-1}\left(B_{(R(T),\|\cdot\|_Y)} + T(0)\right) \supset \lambda B_{(D(T),\|\cdot\|_X)} + T^{-1}(0) \\
&\Rightarrow T^{-1}\left(B_{(R(T),\|\cdot\|_Y)}\right) + T(0) \supset \lambda B_{(D(T),\|\cdot\|_X)} + T^{-1}(0) \\
&\Rightarrow T^{-1}\left(B_{(R(T),\|\cdot\|_Y)}\right) \supset \lambda B_{(D(T),\|\cdot\|_X)}
\end{aligned}$$

Then

$$T^{-1}\left(B_{(R(T),\|\cdot\|_Y)}\right) \supset \lambda B_{(D(T),\|\cdot\|_X)}. \tag{3.18}$$

if and only if

$$\lambda T\left(B_{(D(T),\|\cdot\|_X)}\right) \subset B_{(R(T),\|\cdot\|_Y)} + T(0). \tag{3.19}$$

a) If $\|T\|_{\mathcal{LR}(X,Y)} < \infty$. By (3.6), we have

$$\begin{aligned}
\|T\|_{\mathcal{LR}(X,Y)} &= \inf_{\lambda>0}\{\lambda : \lambda^{-1}TB_{(D(T),\|\cdot\|_X)} \subset B_{(R(T),\|\cdot\|_Y)} + T(0)\} \\
&= \left(\sup_{\lambda>0}\{\lambda : \lambda TB_{(D(T),\|\cdot\|_X)} \subset B_{(R(T),\|\cdot\|_Y)} + T(0)\}\right)^{-1} \\
&= \left(\sup_{\lambda>0} T^{-1}\left(B_{(R(T),\|\cdot\|_Y)}\right) \supset \lambda B_{(D(T),\|\cdot\|_X)}\right)^{-1} = (\gamma_{X,Y}(T^{-1}))^{-1}.
\end{aligned}$$

b) If $\|T\|_{\mathcal{LR}(X,Y)} = \infty$. we have

$$TB_{(D(T),\|\cdot\|_X)} \not\subseteq \lambda B_{(R(T),\|\cdot\|_Y)} + T(0) \quad \text{for all } \lambda > 0.$$

Then

$$\lambda TB_{(D(T),\|\cdot\|_X)} \not\subseteq B_{(R(T),\|\cdot\|_Y)} + T(0) \quad \text{for all } \lambda > 0.$$

Hence for all $\lambda > 0$ and By (3.19), we have

$$T^{-1}\left(B_{(R(T),\|\cdot\|_Y)}\right) \not\supseteq \lambda B_{(D(T),\|\cdot\|_X)}.$$

Therefore by Proposition 3.1.9 $\gamma_{X,Y}(T^{-1}) = 0$ and $\gamma_{X,Y}(T^{-1}) = \|T\|^{-1}_{\mathcal{LR}(Y,X)}$. Then $(\gamma_{X,Y}(T^{-1}))^{-1} = \|T^{-1}\|^{-1}_{\mathcal{LR}(Y,X)}$. □

3.1.2 Continuous linear relations between quasi-normed spaces

In this section, it is presumed that unless specified otherwise $X, Y, Z, \ldots$ will consistently represent linear quasinormed spaces. We characterize the continuous linear relations involving quasi-normed spaces, similar to the characterization used for normed spaces. The definition of a continuous linear relation is analogous to that provided in [18, Definition II.3.1] for topological spaces. We provide the definition for the purpose of having a comprehensive understanding.

Definition 3.1.3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be quasi-normed spaces. A linear relation $T \in LR(X,Y)$ is said to be continuous if the inverse image $T^{-1}(V)$ is a neighbourhood in $D(T)$ for any neighbourhood V in $R(T)$. Also T is called open if whenever U is a neighbourhood in $D(T)$, the image $T(U)$ is a neighbourhood in $R(T)$.

Theorem 3.1.2. *(1) Let $T \in LR(X,Y)$. Then T is continuous if and only if $\|T\|_{\mathcal{LR}(X,Y)} < \infty$.*

(2) Let $T \in LR(X,Y)$. Then T is open if and only if $\gamma_{X,Y}(T) > 0$.

Proof. 1. Suppose that T is continuous. Then there exists an open neighbourhood $\mathcal{O}$

in $D(T)$ such that $\mathcal{O} \subset T^{-1}U_{R(T)}$. Hence

$$\mathcal{O} - \mathcal{O} \subset T^{-1}U_{R(T)} - T^{-1}U_{R(T)},$$

an open neighbourhood of $\{0\}$ in $D(T)$. Also $2T^{-1}U_{R(T)} = T^{-1}U_{R(T)} - T^{-1}U_{R(T)}$. Consequently there exists $\lambda > 0$ such that

$$\lambda U_{D(T)} \subset 2T^{-1}U_{R(T)}$$

and

$$\frac{1}{2}\lambda B_{(D(T),\|\cdot\|_X)} \subset T^{-1}B_{(R(T),\|\cdot\|_Y)},$$

$$\frac{1}{2}\lambda TB_{(D(T),\|\cdot\|_X)} \subset TT^{-1}B_{(R(T),\|\cdot\|_Y)} = B_{(R(T),\|\cdot\|_Y)} + T(0).$$

For the converse, assume that $\|T\|_{\mathcal{LR}(X,Y)} < \infty$. Let V be open ball in $R(T)$ with centre y. Then

$$V_0 = V - \{y\} = \alpha U_{(R(T),\|\cdot\|_Y)} \quad \text{for some } \alpha > 0.$$

We have for some $\lambda > 0$

$$TU_{(D(T),\|\cdot\|_X)} \subset \lambda U_{(R(T),\|\cdot\|_Y)} + T(0).$$

Accordingly

$$U_{(D(T),\|\cdot\|_X)} + T^{-1}(0) \subset \lambda T^{-1}U_{(R(T),\|\cdot\|_Y)} = \frac{\lambda}{\alpha}T^{-1}(V_0).$$

Thus

$$\frac{\alpha}{\lambda}U_{(D(T),\|\cdot\|_X)} + T^{-1}y \subset T^{-1}V,$$

which is a neighbourhood in $D(T)$. Therefore T is continuous.

2. It is obvious by using Theorem 3.1.1. □

The set of all continuous relations from quasi-normed space X to quasi-normed space Y denoted by $CR(X,Y)$.

For the proof of the following theorem, we use the definition of kernel of T.

Proposition 3.1.10. *Let $T \in CR(X, Y)$, then*

$$\|Tx\|_Y \leq \|T\|_{\mathcal{LR}(X,Y)} \|x\|_X, \tag{3.20}$$

for all $x \in X$, where $\|T\|_{\mathcal{LR}(X,Y)}$ is the least M satisfying $\|Tx\|_Y \leq M\|x\|_X$.

Proof. From 3.1, we have $\|T\|_{\mathcal{LR}(X,Y)} = \sup_{0 \neq \|x\|_X \leq 1} \|Tx\|_Y$, where

$$\|Tx\|_Y \leq \sup_{0 \neq \|x\|_X \leq 1} \|Tx\|_Y = \|T\|_{\mathcal{LR}(X,Y)}.$$

Timely, $\|T\left(\frac{x}{\|x\|_X}\right)\|_Y \leq \|T\|_{\mathcal{LR}(X,Y)}$ yields (3.20). The case $\|x\|_X = 0$ is trivial. $\square$

Example 3.1.1. Let $\mathbb{R}^2$ be endowed with the quasi-norm

$$\|(a, b)\|_{\mathbb{R}^2} = (\sqrt{|a|} + \sqrt{|b|})^2,$$

and let $T : (\mathbb{R}^2, \|\cdot\|_{\mathbb{R}^2}) \to (\ell_\infty, \|\cdot\|_\infty)$ be the linear relation defined by

$$T(a, b) = \left\{(x_n)_n \in \ell_\infty : \lim_{n\to\infty} x_n = a + b\right\}.$$

Then $\tilde{\Gamma}(0, 0) = c_0$, the space of null sequences, confirming that T is a multivalued linear relation.

For any $(a, b) \in \mathbb{R}^2$, we have $T(a, b) = (a + b) \cdot \mathbf{1} + c_0$, where $\mathbf{1} = (1, 1, 1, \ldots) \in \ell_\infty$. Therefore, for $y \in T(a, b)$,

$$\begin{aligned}
\|Tx\|_\infty &= \inf_{k\in c_0} \|y + k\|_\infty \\
&= \inf_{z\in c_0} \|(a + b) \cdot \mathbf{1} + z\|_\infty \\
&\leq |a + b| \qquad \text{(taking } z = 0\text{)} \\
&\leq |a| + |b| \\
&\leq (\sqrt{|a|} + \sqrt{|b|})^2 \\
&= \|(a, b)\|_{\mathbb{R}^2}, .
\end{aligned}$$

Thus T is a bounded linear relation with $\|T\| \leq 1$.

The [18, Proposition I.2.8] and the definition of kernel give us the following proposition.

Proposition 3.1.11. *Let $T \in LR(X,Y)$. Then we have*

$$N(T) \subset N(QT).$$

In particular if $\overline{T(0)}$ is relatively closed in $R(T)$, then $N(T) = N(QT)$.

Proposition 3.1.12. *Let $T \in LR(X,Y)$. Then we have*

$$\gamma_{X,Y}(T) \leq \gamma_{X,Y}(QT).$$

In particular $\gamma_{X,Y}(T) = \gamma_{X,Y}(QT)$ if $T(0)$ is relatively closed in $R(T)$.

Proof. According the Definition 3.1.2 and the Proposition 3.1.11, We have

$$\begin{aligned}\gamma_{X,Y}(T) &= \sup\{\lambda : \|Tx\|_Y \geq \lambda d_X(x, N(T)) \text{ for } x \in D(T)\}\\ &= \sup\{\lambda : \|Tx\|_Y \geq \lambda d_X(x, N(QT)) \text{ for } x \in D(T)\}\\ &= \gamma_{X,Y}(QT)\end{aligned}$$

For the equality, the Proposition 3.1.11 gives us $N(T) = N(QT)$ if $T(0)$ is relatively closed in $R(T)$, then

$$\begin{aligned}\gamma_{X,Y}(T) &= \sup\{\lambda : \|Tx\|_Y \geq \lambda d_X(x, N(T)) \text{ for } x \in D(T)\}\\ &= \sup\{\lambda : \|Tx\|_Y \geq \lambda d_X(x, N(QT)) \text{ for } x \in D(T)\}\\ &= \sup\{\lambda : \|QTx\|_{\mathcal{L}(X,Y/\overline{T(0)})} \geq \lambda d_X(x, N(T)) \text{ for } x \in D(T)\}\\ &= \gamma_{X,Y}(QT).\end{aligned}$$

□

For proving the following proposition, we use a same techniques which used in [18, Proposition II.3.10] by changing distance by quasi-metric.

Proposition 3.1.13. *Let M be a non-empty subset of $R(T)$ and let $\gamma_{X,Y}(T) < \infty$. Then for $N \subset D(T)$ we have*

$$d_Y(TN, M) \geq \gamma_{X,Y}(T) d_X(N, T^{-1}M).$$

Proof. Let $\lambda < \gamma_{X,Y}(T)$. By definition, $\|Tx\|_Y \geq \lambda d_X(x, N(T))$ for all $x \in D(T)$.

For any $y \in TN$ and $z \in M$, choose $x \in N$ with $y \in Tx$. Then:

$$d_Y(y, z) \geq d_Y(Tx, M) = \inf_{w \in M} \|Tx - w\|_Y.$$

Since $\|Tx\|_Y = d_Y(Tx, T(0))$ and $T(0) \subseteq T(T^{-1}M) = M \cap R(T) + T(0)$ by Proposition 3.1.2, we have:

$$d_Y(Tx, M) = d_Y(Tx, M \cap R(T) + T(0)) = d_Y(Tx, T(T^{-1}M)).$$

Using the quotient map $Q : Y \to Y/\overline{T(0)}$:

$$\|QTx\|_{\mathcal{L}(X,Y/\overline{T(0)})} = d_Y(y, T(0)) \geq \lambda d_X(x, N(T)) \geq \lambda d_X(x, T^{-1}M),$$

where the last inequality uses $N(T) \subseteq T^{-1}M$ when $0 \in M$.

Taking infimum over $y \in TN$ and using that QT is single-valued:

$$d_Y(TN, M) \geq \lambda d_X(N, T^{-1}M).$$

Taking supremum over $\lambda < \gamma_{X,Y}(T)$ yields the result. □

Theorem 3.1.3. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be quasi-normed spaces. Let $T \in LR(X, Y)$ and $S \in LR(Y, Z)$. Then*

$$\gamma_{X,Z}(ST) \geq \gamma_{Y,Z}\left(S|_{R(T)}\right) \gamma_{X,Y}(T) \qquad (0 \cdot \infty \ \textit{excluded}).$$

Moreover, if $S^{-1}(0) \subset R(T)$ then

$$\gamma_{X,Z}(ST) \geq \gamma_{Y,Z}(S)\gamma_{X,Y}(T).$$

Proof. Let $B_{D(T)} = \{x \in D(T) : \|x\|_X < 1\}$ and $B_{R(T)} = \{y \in R(T) : \|y\|_Y = \|Q^Y_{T(0)}y\|_{\mathcal{L}(X,Y/\overline{T(0)})} < 1\}$, then

$$B_{S(R(T))} = \{z \in S(R(T)) : \|z\|_Z = \|Q^Z_{S(0)}z\|_Z < 1\},$$

this implies that

$$\begin{aligned} T(B_{D(T)}) &\supset \gamma_{X,Y}B_{R(T)} \\ S(T(B_{D(T)})) &\supseteq \gamma_{X,Y}S(B_{R(T)}) \\ ST(B_{D(T)}) &\supseteq \gamma_{X,Y}\gamma_{Y,Z}(B_{S(R(T))}). \end{aligned}$$

$$\gamma_{X,Z}(ST) \geq \gamma_{Y,Z}\left(S|_{R(T)}\right)\gamma_{X,Y}(T).$$

□

Remark 3.1.2. The inequality $\gamma_{X,Z}(ST) \geq \gamma_{Y,Z}(S|_{R(T)}) \cdot \gamma_{X,Y}(T)$ follows directly from the geometric characterization of the minimum modulus via set inclusions (Proposition 3.1.9) and the exact homogeneity of the quasi-norm (Proposition 3.1.4(b)). Indeed, by inclusions, the result yields as in the classical normed space setting, without any additional constant arising from the quasi-norm structure.

The following proposition is quasi-normed version of quasi-normed principle theorem.

Theorem 3.1.4 (quasi-normed theorem)**.** *If $\mathcal{F}$ is a family of continuous linear relations from a quasi-normed space $(X, \|\cdot\|_X)$ to a quasi-normed space $(Y, \|\cdot\|_Y)$ such that for each $x \in X$ there exists $b_x > 0$ with $\|T(x)\|_Y \leq b_x$ for all $T \in \mathcal{F}$, then*

$$\sup_{T\in\mathcal{F}} \|T\|_{\mathcal{LR}(X,Y)} < \infty \quad \textit{for all } T.$$

Proof. For each $n \in \mathbb{N}$, define

$$A_n = \{x \in X : \sup_{T\in\mathcal{F}} \|Tx\|_Y \leq n\}.$$

Each A_n is closed: if $x_k \to x$ with $x_k \in A_n$, then for any $T \in \mathcal{F}$,

$$\|Tx\|_Y \leq K(\|Tx - Tx_k\|_Y + \|Tx_k\|_Y) \leq K(\|T\|_{\mathcal{LR}(X,Y)}\|x - x_k\|_X + n).$$

Taking $k \to \infty$ gives $\|Tx\|_Y \leq Kn$. Adjusting constants (using that K is fixed), we get $x \in A_{n'}$ for appropriate n'.

By pointwise boundedness, $X = \bigcup_{n=1}^{\infty} A_n$. By Baire category, some A_m has non-empty interior. Thus there exist $z \in X$ and $r > 0$ such that $B(z, r) \subset A_m$.

For $x \in B(0, 1)$, we have $z + rx \in B(z, r) \subset A_m$, so $\|T(z + rx)\|_Y \leq m$ for all $T \in \mathcal{F}$.

Then:

$$\begin{aligned}
\|Tx\|_Y &= \frac{1}{r}\|T(rx)\|_Y \\
&= \frac{1}{r}\|T(z + rx) - T(z)\|_Y \\
&\leq \frac{K}{r}(\|T(z + rx)\|_Y + \|Tz\|_Y) \\
&\leq \frac{K}{r}(m + m) = \frac{2Km}{r}.
\end{aligned}$$

Thus $\sup_{\|x\|_X\leq 1} \|Tx\|_Y \leq \frac{2Km}{r} < \infty$ for all $T \in \mathcal{F}$, proving $\sup_{T\in\mathcal{F}} \|T\|_{\mathcal{LR}(X,Y)} < \infty$. □

The following proposition is quasi-normed version of closed graph theorems and for The proof is a similar of such [53, Theorem 1.5] by taking the quasinorm on $X \times Y$ of the graph $\Gamma = G(T)$ as follows

$$\|(x, y)\|_\Gamma = \max(\|x\|_X, \|y\|_Y) \quad \text{for } (x, y) \in G(T).$$

Proposition 3.1.14 (Closed graph theorem)**.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be quasi-Banach spaces and $T \in LR(X, Y)$. Then the following statements are equivalent:*

1. *T and $D(T)$ are closed;*

2. *T is bounded, and $D(T)$ and $T(0)$ are closed;*

3. *T is bounded and closed.*

Proof. We prove the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): T and $D(T)$ closed $\Rightarrow$ T bounded, $D(T)$ and $T(0)$ closed.

Step 1: $T(0)$ is closed.

Since T is a linear relation, $T(0)$ is a linear subspace of Y by Proposition 3.1.1. Let $(y_n) \subset T(0)$ with $y_n \to y$ in Y. Since $(0, y_n) \in G(T)$ for all n and $G(T)$ is closed by assumption, we have $(0, y) \in G(T)$. This means $y \in T(0)$. Thus $T(0)$ is closed.

Step 2: The quotient space $Y/T(0)$.

Since $T(0)$ is closed, the quotient space $Y/T(0)$ is a quasi-Banach space with the quasi-norm

$$\|\bar{y}\|_{\mathcal{L}(X,Y/T(0))} = \inf_{k \in T(0)} \|y + k\|_Y.$$

Let $Q : Y \to Y/T(0)$ be the natural quotient map. By Proposition 3.1.1, QT is single-valued.

Step 3: QT has closed graph.

The graph of QT is $G(QT) = \{(x, \bar{y}) : x \in D(T), y \in Tx\}$. Let $(x_n, \bar{y_n}) \to (x, \bar{y})$ in $X \times Y/T(0)$ with $(x_n, \bar{y_n}) \in G(QT)$. Then there exist $y_n \in Tx_n$ and $k_n \in T(0)$ such that $y_n + k_n \to y$ in Y for some representative y.

Since $(x_n, y_n) \in G(T)$ and $(0, k_n) \in G(T)$, by linearity of the relation we have $(x_n, y_n + k_n) \in G(T)$. As $G(T)$ is closed and $(x_n, y_n + k_n) \to (x, y)$, we get $(x, y) \in G(T)$. Thus $\bar{y} = Qy = QTx$, so $(x, \bar{y}) \in G(QT)$. Hence QT has closed graph.

Step 4: QT is bounded.

Since $QT : D(T) \subseteq X \to Y/T(0)$ is a single-valued linear operator with closed graph, and $X, Y/T(0)$ are quasi-Banach spaces, we apply the Closed Graph Theorem for single-valued operators in quasi-Banach spaces (Theorem 2.3.2 with $m = 1$). Thus QT is bounded, i.e., $\|QT\|_{\mathcal{L}(X,Y/T(0))} < \infty$.

Step 5: T is bounded.

By Definition 3.1.1, $\|T\|_{\mathcal{LR}(X,Y)} = \|QT\|_{\mathcal{L}(X,Y/T(0))} < \infty$. Therefore T is bounded, and we have established (2).

(2) $\Rightarrow$ (3): T bounded, $D(T)$ and $\overline{T(0)}$ closed $\Rightarrow$ T bounded and closed.

Boundedness is given. We show T is closed. Let $(x_n, y_n) \in G(T)$ with $x_n \to x$ and $y_n \to y$. Since $D(T)$ is closed, $x \in D(T)$.

Since T is bounded, QT is continuous. We have $QTx_n = Qy_n \to Qy$ and $QTx_n \to QTx$ by continuity. Thus $Qy = QTx$, meaning $y \in Tx + T(0) = Tx$ (since $T(0)$ is the null space of Q restricted to the range). Hence $(x, y) \in G(T)$, so T is closed.

(3) $\Rightarrow$ (1): T bounded and closed $\Rightarrow$ T and $D(T)$ closed.

T is closed by assumption. We show $D(T)$ is closed. Let $x_n \in D(T)$ with $x_n \to x$ in X.

Since T is bounded, for each n there exists $y_n \in Tx_n$ with $\|y_n\|_Y \leq 2\|T\|_{\mathcal{LR}(X,Y)}\|x_n\|_X$. The sequence (y_n) is bounded in Y.

Consider the projection $\pi_X : G(T) \to X$ given by $\pi_X(x, y) = x$. This is a continuous linear operator. Since T is bounded, for each $x \in D(T)$, the fiber $\pi_X^{-1}(x) = \{(x, y) : y \in Tx\}$ satisfies that QT is single-valued and continuous.

The map π_X is surjective onto $D(T)$. Since $G(T)$ is closed (as T is closed) and X is complete, and since QT being bounded implies π_X is an open map onto its range, we conclude that $D(T)$ is closed in X.

More directly: since QT is continuous and $Y/T(0)$ is complete, the inverse image of closed sets under QT are closed. The set $\{x_n\} \cup \{x\}$ is compact, and continuity of QT ensures $D(T)$ contains its limit points.

Thus all three statements are equivalent. $\square$

Corollary 3.1.5. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be quasi-Banach spaces. If $T \in LR(X, Y)$ is a closed linear relation with closed domain, then T is continuous (bounded).*

Proof. This follows immediately from Proposition 3.1.14, implication (1) $\Rightarrow$ (2). $\square$

Remark 3.1.3. In the proof above, we use the fact that any quasi-Banach space admits an equivalent p-subadditive quasi-norm for some $p \in (0, 1]$ by the Aoki-Rolewicz theorem (Theorem 1.2.1). This ensures that the quotient space $Y/T(0)$ inherits the quasi-norm structure.

3.2 Continuous selections of linear relations

3.2.1 Continuous selections of linear relations between quasi-normed spaces

Recall that a selection is the single valued part of the multivalued linear relations for more details see [6, 18]. Note that in this section a linear relation T is established between two quasi normed linear spaces X and Y and we shall give conditions to finding a continuous selections for a given relations in quasi normed linear spaces. Also, note that .

Proposition 3.2.1. *If A is a continuous selection of T, then T is continuous. Furthermore*

$$\|T\|_{\mathcal{LR}(X,Y)} \leq \|A\|_{\mathcal{L}(X,Y)}.$$

Proof. Let A be a continuous selection of T. Then, we have

$$Tx = Ax + T(0) \quad \text{for } x \in D(T),$$

$$\|Tx\|_Y = \inf_{y \in Tx} \|y\|_Y \leq \|Ax\|_Y \leq \|A\|_{\mathcal{L}(X,Y)} \|x\|_X.$$

□

Proposition 3.2.2. *On the product quasi-normed space $X \times Y$ we shall define the quasi norm $\|(x,y)\|_\Gamma = \|x\|_X + \|y\|_Y$ on the graph $G(T)$ of T denoted by $T_G \in LR(X, X \times Y)$ with $D(T_G) = D(T)$ and for all $(x,y) \in X \times Y$,*

$$T_G x = \{(x,y) : y \in Tx\},$$

for which $\|(x,y)\|_\Gamma = \|x\|_X + \|Tx\|_Y$. As we have seen,

$$\|Tx\|_Y = \|T_G x\|_Y.$$

Proof. To prove that $\|(x,y)\|_\Gamma$ is quasinorm on T_G, the first and second conditions are straightforward since $\|\cdot\|_Y$ and $\|\cdot\|_X$ are both quasi norms. For the third condition of quasi norm let (x_1,y_1), and $(x_2,y_2)\in T_G$ then for $y_1\in Tx_1$ and $y_2\in Tx_2$, we have

$$\begin{aligned}\|(x_1,y_1)\|_\Gamma+\|(x_2,y_2)\|_\Gamma &= \|(x_1+x_2,y_1+y_2)\|_\Gamma\\ &= \|(x_1+x_2)\|_X+\|(y_1+y_2)\|_Y\\ &\le K_X[\|x_1\|_X+\|x_2\|_X]+K_Y[\|y_1\|_Y+\|y_2\|_Y]\\ &\le K_\Gamma[\|x_1\|_X+\|x_2\|_X+\|y_1\|_Y+\|y_2\|_Y]\\ &\le K_\Gamma[\|(x_1,y_1)\|_\Gamma+\|(x_2,y_2)\|_\Gamma],\end{aligned}$$

with $K_\Gamma=\max(K_X,K_Y)$. □

Proposition 3.2.3. *Let T be a linear relation having a continuous selection A. If T is continuous and $\{0\}\times T(0)$ is topologically complemented in $G(T)$ where $G(A)=PG(T)$ such that P is any bounded projection defined on $G(T)$ with*

$$\ker(P)=\{0\}\times T(0),$$

then A is continuous.

Proof. Suppose that T is continuous, we have

$$\|T_Gx\|_\Gamma=\inf_{y\in Tx}\{\|x\|_X+\|y\|_Y\}=\|x\|_X+\|Tx\|_Y\le(1+\|T\|_{\mathcal{LR}(X,Y)})\|x\|_X.$$

Since $D(T)=D(T_G)$, we have PT_Gx is single valued, hence for $x\in D(T)$

$$PT_Gx=(x,Ax).$$

As A is a single valued linear relation, hence

$$\begin{aligned}\|(x, Ax)\|_\Gamma &= \|x\|_X + \|Ax\|_Y \\ &= \|PT_G x\|_\Gamma \\ &\le \|P\|_{\mathcal{L}(X,Y)} \|T_G x\|_\Gamma = \|P\|_{\mathcal{L}(X,Y)} (\|x\|_X + \|Tx\|_Y),\end{aligned}$$

and from $\|x\|_X + \|Ax\|_Y = \|P\|_{\mathcal{L}(X,Y)}(\|x\|_X + \|Tx\|_Y)$, we have

$$\|Ax\|_Y = \|P\|_{\mathcal{L}(X,Y)}(\|x\|_X + \|Tx\|_Y) - \|x\|_X,$$

in on another hand T is bounded, then we find the continuity of A

$$\|Ax\|_Y \le [\|P\|_{\mathcal{L}(X,Y)} + \|P\|_{\mathcal{L}(X,Y)} \|T\|_{\mathcal{LR}(X,Y)} - 1] \|x\|_X.$$

$\square$

Chapter 4

Compact Lipschitz Operators between p-metric space and p-Banach space

In this chapter we extend the concept of Lipschitz operators between metric spaces and Banach spaces, introduced by Vergas et al. [56] to Lipschitz operators between p-metric spaces and p-Banach spaces.

4.1 The Lipschitz Adjoint

We now define the adjoint of a Lipschitz operator, following Sawashima [51].

Definition 4.1.1 (Lipschitz Adjoint)**.** Let X and Y be pointed p-metric spaces. For a Lipschitz operator $T \in \mathrm{Lip}_0(X, Y)$, define the *Lipschitz adjoint* $T_p^{\#} : \mathrm{Lip}_0(Y) \to \mathrm{Lip}_0(X)$ by

$$T_p^{\#}(g) = g \circ T, \quad \forall g \in \mathrm{Lip}_0(Y). \tag{4.1}$$

By the canonical isometric isomorphism (see Theorem 1.8.1 and Corollary 1.7.3), we may also view $T_p^{\#}$ as an operator $\mathcal{F}_p(Y)^* \to \mathcal{F}_p(X)^*$.

Remark 4.1.1. If $Y = E$ is a p-Banach space, then the restriction $T_p^{\#}|_{E^*}$ coincides with the usual transpose of the associated linear operator:

$$T^t(e^*)(x) = e^*(T(x)), \quad e^* \in E^*,\ x \in X. \tag{4.2}$$

Theorem 4.1.1 (Properties of the Transpose)**.** *Let $T \in \mathrm{Lip}_0(X, E)$, where E is a p-Banach space. Then the transpose operator $T^t : E^* \to \mathrm{Lip}_0(X)$ is linear and bounded with $\|T^t\| = \mathrm{Lip}(T)$.*

Proof. Linearity of T^t is clear from its definition. To prove boundedness, for any $e^* \in E^*$ and $x, x' \in X$, we have

$$\begin{aligned}
|T^t(e^*)(x) - T^t(e^*)(x')| &= |e^*(T(x)) - e^*(T(x'))| \\
&= |e^*(T(x) - T(x'))| \\
&\leq \|e^*\| \cdot \|T(x) - T(x')\| \\
&\leq \|e^*\| \cdot \mathrm{Lip}(T) \cdot d(x, x').
\end{aligned}$$

Also, $T^t(e^*)(0) = e^*(T(0)) = 0$. Hence $T^t(e^*) \in \mathrm{Lip}_0(X)$, with

$$\mathrm{Lip}(T^t(e^*)) \leq \|e^*\| \cdot \mathrm{Lip}(T).$$

Taking the supremum over $\|e^*\| \leq 1$, we get $\|T^t\| \leq \mathrm{Lip}(T)$.

To prove the reverse inequality, observe:

$$\|T(x)-T(x')\| = \sup_{\|e^*\|\leq 1} |e^*(T(x)-T(x'))| = \sup_{\|e^*\|\leq 1} |T^t(e^*)(x)-T^t(e^*)(x')| \leq \|T^t\| \cdot d(x, x').$$

Thus $\mathrm{Lip}(T) \leq \|T^t\|$, and equality follows. □

Theorem 4.1.2 (Diagram Relation)**.** *Let $T \in \mathrm{Lip}_0(X, E)$. Let S_1, R_1 and S_2, R_2 denote the canonical isometries between $\mathrm{Lip}_0(X)$, $\mathcal{F}_p(X)^*$ and $\mathrm{Lip}_0(Y)$, $\mathcal{F}_p(Y)^*$, respectively (which exist by the isometric isomorphism established in Corollary 1.7.3). Then the* *following holds:*

$$T^{\#} = R_1 \circ \Phi(T)^* \circ S_2, \quad \textit{or equivalently} \quad \Phi(T)^* = S_1 \circ T^{\#}. \tag{4.3}$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_p(Y)^* & \xrightarrow{\Phi(T)^*} & \mathcal{F}_p(X)^* \\ S_2 \uparrow & & \uparrow S_1 \\ \mathrm{Lip}_0(Y) & \xrightarrow{T^{\#}} & \mathrm{Lip}_0(X) \end{array}$$

Proof. The proof follows from the definitions and the universal property. For any $g \in \mathrm{Lip}_0(Y)$ and $\mu \in \mathcal{F}_p(X)$, we compute both sides and verify they coincide. The key identification is that $\mathcal{F}_p(X)^* \cong \mathrm{Lip}_0(X)$ via the isometry S_1, and similarly for Y, by Corollary 1.7.3. $\square$

4.2 Compact Lipschitz Operators

In this section, we study compact Lipschitz operators between p-metric spaces, which play a fundamental role in understanding operator properties in spaces like ℓ_p with $0 < p < 1$.

Definition 4.2.1 (Lipschitz Image)**.** Let X be a p-metric space and E a p-Banach space. The *Lipschitz image* of a mapping $T : X \to E$ is the set

$$\mathrm{Im}_{\mathrm{Lip}}(T) = \left\{ \frac{T(x) - T(y)}{d(x, y)} : x, y \in X, x \neq y \right\}. \tag{4.4}$$

Note that since d_X is a p-metric, the condition $\|T(x) - T(y)\| \leq C\, d_X(x, y)$ is equivalent to $\|T(x) - T(y)\| \leq C\, d_X(x, y)^p$ where $d_X(x, y)^p$ defines a metric on X (by the p-subadditivity of the quasi-norm). Thus $\mathrm{Lip}_0(X)$ coincides with the space of Lipschitz functions on (X, d_X^p).

Definition 4.2.2 (Compact Lipschitz Operator)**.** Let X be a pointed p-metric space and E a p-Banach space. We say that a base-point preserving map $T : X \to E$ is *Lipschitz compact* (respectively, *Lipschitz weakly compact*) if its Lipschitz image is relatively compact (respectively, relatively weakly compact) in E. We denote by $\mathrm{Lip}_0^{\mathcal{K}}(X, E)$ the set of all Lipschitz compact operators from X to E.

Remark 4.2.1. When $Y = \ell_p$ with $0 < p < 1$, a p-Banach space, the compactness of $T(B)$ implies that $T(B)$ is totally bounded under the metric $d_Y(y, y') = \|y - y'\|_p^p$. Unlike classical Banach spaces ($p = 1$), the unit ball in ℓ_p is not compact in infinite dimensions due to the lack of local convexity. Nevertheless, a Lipschitz operator $T : X \to \ell_p$ can still be compact if its image exhibits a form of decay or can be approximated by operators with finite-dimensional range.

Using the same techniques as that used by Vargas et al. in the proof [56, Propostion 2.1] we obtain the following result.

Proposition 4.2.1. *Let X be a pointed p-metric space, E a p-Banach space and $T \in \mathrm{Lip}_0(X, E)$. Then T is Lipschitz compact if and only if T_f is compact.*

Theorem 4.2.1 (Characterization of Compact Lipschitz Operators)**.** *Let (X, d_X) be a pointed p-metric space with $0 < p < 1$, and let $T \in \mathrm{Lip}_0(X, \ell_p)$. Then T is a compact Lipschitz operator if and only if there exists a sequence $(a_n) \subset \mathrm{Lip}_0(X)$ such that*

$$T(x) = (a_n(x))_n \quad \textit{and} \quad \mathrm{Lip}(a_n) \to 0 \textit{ as } n \to \infty. \tag{4.5}$$

Proof.

Since $T \in \mathrm{Lip}_0(X, \ell_p)$, by the universal property of the free space $\mathcal{F}_p(X)$ (Theorem 1.7.1), it uniquely extends to a bounded linear operator $T_L : \mathcal{F}_p(X) \to \ell_p$ such that $T = T_L \circ \delta_X$ and $\|T_L\| = \mathrm{Lip}(T)$.

Sufficiency. Suppose $T(x) = (a_n(x))_n$, where $a_n \in \mathrm{Lip}_0(X)$ and $\mathrm{Lip}(a_n) \to 0$. Let $b_n = (a_n)_L$ be the canonical linear extension to $\mathcal{F}_p(X)$, so that $T_L(m) = (b_n(m))_n$ defines a bounded linear operator. Then for each $x \in X$,

$$T(x) = T_L(\delta_X(x)) = (b_n(\delta_X(x)))_n = (a_n(x))_n.$$

To prove that T is compact, let $B \subset X$ be bounded. Then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\mathrm{Lip}(a_n) < \varepsilon^{1/p}$ for all $n > N$. For $x, x' \in B$,

$$|a_n(x) - a_n(x')| \leq \mathrm{Lip}(a_n) \cdot d_X(x, x')^{1/p} < \varepsilon, \quad \text{for } n > N.$$

Hence, the tail $\sum_{n=N+1}^{\infty} |a_n(x) - a_n(x')|^p$ can be made arbitrarily small uniformly in x, x'. Thus, $T(B)$ is totally bounded in ℓ_p, and since ℓ_p is complete, $\overline{T(B)}$ is precompact.

Necessity. Suppose $T \in \mathrm{Lip}_0(X, \ell_p)$ is compact. Then the extension $T_L : \mathcal{F}_p(X) \to \ell_p$ is a compact linear operator. By the theory of compact operators in p-Banach spaces, T_L admits a representation $T_L(m) = (b_n(m))_n$, where $b_n \in \mathcal{L}(\mathcal{F}_p(X), \mathbb{K})$ and $\|b_n\| \to 0$. Let $a_n \in \mathrm{Lip}_0(X)$ be the unique Lipschitz functional such that $a_n = b_n \circ \delta_X$. Then

$$T(x) = T_L(\delta_X(x)) = (b_n(\delta_X(x)))_n = (a_n(x))_n,$$

with $\mathrm{Lip}(a_n) = \|b_n\| \to 0$, completing the proof. $\square$

Example 4.2.1 (A Compact Lipschitz Operator on ℓ_p)**.** Define $T : \ell_p \to \ell_p$ by

$$T(x) = \left(\frac{x_n}{n^{1/p}}\right)_{n=1}^{\infty}, \quad \text{for } x = (x_n)_n \in \ell_p,\ 0 < p < 1. \tag{4.6}$$

1. T maps ℓ_p into itself, since

$$\|T(x)\|_p^p = \sum_{n=1}^{\infty} \frac{|x_n|^p}{n} \leq \sum_{n=1}^{\infty} |x_n|^p = \|x\|_p^p. \tag{4.7}$$

2. T is Lipschitz, as

$$\|T(x) - T(y)\|_p^p = \sum_{n=1}^{\infty} \frac{|x_n - y_n|^p}{n} \leq \sum_{n=1}^{\infty} |x_n - y_n|^p = \|x - y\|_p^p. \tag{4.8}$$

3. To show compactness, define

$$T_N(x) = \left(\frac{x_n}{n^{1/p}} \cdot \chi_{\{n \leq N\}}\right)_n, \tag{4.9}$$

a finite-rank operator. Then

$$\|T(x) - T_N(x)\|_p^p = \sum_{n=N+1}^{\infty} \frac{|x_n|^p}{n} \leq \sum_{n=N+1}^{\infty} |x_n|^p \to 0 \quad \text{as } N \to \infty. \tag{4.10}$$

Hence, T can be approximated by finite-rank operators and is compact.

4. Let $a_n(x) = x_n/n^{1/p}$, then $T(x) = (a_n(x))_n$, and

$$\mathrm{Lip}(a_n) \leq \frac{1}{n^{1/p}} \to 0, \tag{4.11}$$

so Theorem 4.2.1 applies.

Theorem 4.2.2 (Compactness of Adjoints)**.** *Let (X, d_X) be a pointed p-metric space with $0 < p \leq 1$, and let E be a p-Banach space. If $T \in \mathrm{Lip}_0(X, E)$, then T is compact if and only if T^t is a compact linear operator.*

Proof. ($\Rightarrow$) If T is compact, then $T_L : \mathcal{F}_p(X) \to E$ is compact. (Equivalently, via the isomorphism $\mathcal{F}_p(X) \cong Æ_p(X)$, the corresponding operator on is compact). Hence $T_L^* : E^* \to \mathcal{F}_p(X)^*$ is compact. Since $\tilde{T}^t = T_L^*$ (under the identification of Remark 1.8.2), T^t is compact.

($\Leftarrow$) Suppose $T^t : E^* \to \mathrm{Lip}_0(X)$ is compact. Let B_{E^*} be the unit ball of E^*. Then $T^t(B_{E^*})$ is relatively compact in $\mathrm{Lip}_0(X)$. We show that $T^t(B_{E^*})$ is equicontinuous on bounded subsets of X. Indeed, since $T^t(B_{E^*})$ is compact in $\mathrm{Lip}_0(X)$, for any $\varepsilon > 0$ and bounded $B \subset X$, there exist $g_1, \ldots, g_n \in T^t(B_{E^*})$ such that for any $g \in T^t(B_{E^*})$:

$$\sup_{x \in B} |g(x) - g_i(x)| < \varepsilon \quad \text{for some } i. \tag{4.12}$$

Each g_i is Lipschitz, hence uniformly continuous on B. The finite family $\{g_1, \ldots, g_n\}$ is equicontinuous, so by approximation, $T^t(B_{E^*})$ is equicontinuous on B. Now for $x, x' \in B$:

$$\|T(x) - T(x')\| = \sup_{\|e^*\| \leq 1} |e^*(T(x) - T(x'))| = \sup_{g \in T^t(B_{E^*})} |g(x) - g(x')|. \tag{4.13}$$

By equicontinuity, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, x') < \delta$ implies $|g(x) - g(x')| < \varepsilon$ for all $g \in T^t(B_{E^*})$. Thus $\|T(x) - T(x')\| < \varepsilon$, showing $T(\overline{B})$ is totally bounded. Hence T is compact. □

4.3 Structure of the Space of Compact Operators

Proposition 4.3.1 (Ideal Property)**.** *Let (X, d_X), (Y, d_Y), (Z, d_Z) and (W, d_W) be pointed p-metric spaces, and let E, F be p-Banach spaces. Then:*

1. *For each pair (X, E), the set $\mathrm{Lip}_0^{\mathcal{K}}(X, E)$ is a vector subspace of $\mathrm{Lip}_0(X, E)$.*
2. *If $T \in \mathrm{Lip}_0^{\mathcal{K}}(X, E)$, $A \in \mathrm{Lip}_0(E, F)$ and $B \in \mathrm{Lip}_0(W, X)$, then the composition $A \circ T \circ B \in \mathrm{Lip}_0^{\mathcal{K}}(W, F)$.*
3. *The space $(\mathrm{Lip}_0^{\mathcal{K}}(X, E), \mathrm{Lip})$ defines a Lipschitz operator ideal.*

Proof. (1) Let $T_1, T_2 \in \mathrm{Lip}_0^{\mathcal{K}}(X, E)$ and $\lambda \in \mathbb{K}$. We show that $\lambda T_1 + T_2$ is compact.

Let $B \subset X$ be bounded. Since T_1 and T_2 are compact, the sets $T_1(B)$ and $T_2(B)$ are precompact in E. Hence their closures $\overline{T_1(B)}$ and $\overline{T_2(B)}$ are compact subsets of E, because E is complete. Consider now the continuous map

$$\Phi : E \times E \longrightarrow E, \quad \Phi(u, v) = \lambda u + v. \tag{4.14}$$

The set $K := \overline{T_1(B)} \times \overline{T_2(B)}$ is compact in $E \times E$ so $\Phi(K)$ is compact in E. Clearly,

$$(\lambda T_1 + T_2)(B) \subset \Phi(K), \tag{4.15}$$

whence $(\lambda T_1 + T_2)(B)$ is precompact in E. Therefore $\lambda T_1 + T_2 \in \mathrm{Lip}_0^{\mathcal{K}}(X, E)$, and $\mathrm{Lip}_0^{\mathcal{K}}(X, E)$ is a vector subspace of $\mathrm{Lip}_0(X, E)$.

(2) Let $T \in \mathrm{Lip}_0^{\mathcal{K}}(X, E)$, $A \in \mathrm{Lip}_0(E, F)$ and $B \in \mathrm{Lip}_0(W, X)$. Let $C \subset W$ be a bounded subset. Since B is Lipschitz, $B(C)$ is bounded in X. As T is compact, $T(B(C))$ is precompact in E, thus its closure $\overline{T(B(C))}$ is compact in E.

The map $A : E \to F$ is Lipschitz, hence continuous with respect to the p-metrics.

Therefore $A(\overline{T(B(C))})$ is compact in F, and

$$(A \circ T \circ B)(C) = A\Big(T(B(C))\Big) \subset A\Big(\overline{T(B(C))}\Big), \tag{4.16}$$

so $(A \circ T \circ B)(C)$ is precompact in F. Since C was arbitrary, $A \circ T \circ B$ is compact. Moreover,

$$\mathrm{Lip}(A \circ T \circ B) \leq \mathrm{Lip}(A) \cdot \mathrm{Lip}(T) \cdot \mathrm{Lip}(B) < \infty, \tag{4.17}$$

so $A \circ T \circ B \in \mathrm{Lip}_0^{\mathcal{K}}(W, F)$.

(3) The first part shows that each $\mathrm{Lip}_0^{\mathcal{K}}(X, E)$ is a linear subspace, and the second part shows stability under composition with arbitrary Lipschitz operators acting on the left and on the right. This is precisely the defining property of a (nonlinear) Lipschitz operator ideal. $\square$

Proposition 4.3.2 (Closedness). *Let (X, d_X) be a pointed p-metric space and let E be a p-Banach space. Then $\mathrm{Lip}_0^{\mathcal{K}}(X, E)$ is a closed subspace of $\mathrm{Lip}_0(X, E)$ for the Lipschitz norm.*

Proof. Suppose $(T_n) \subset \mathrm{Lip}_0^{\mathcal{K}}(X, E)$ converges to T in $\mathrm{Lip}_0(X, E)$, i.e., $\mathrm{Lip}(T_n - T) \to 0$. We need to show that $T \in \mathrm{Lip}_0^{\mathcal{K}}(X, E)$.

Let $B \subset X$ be bounded and write

$$R := \sup\{d_X(x, y) : x, y \in B\} < \infty. \tag{4.18}$$

Fix $\varepsilon > 0$. Since $\mathrm{Lip}(T_n - T) \to 0$, there exists n_0 such that

$$\mathrm{Lip}(T_{n_0} - T) < \frac{\varepsilon}{2R}. \tag{4.19}$$

As T_{n_0} is compact, $T_{n_0}(B)$ is totally bounded in E. Hence there exist $y_1, \ldots, y_m \in E$ such that

$$T_{n_0}(B) \subset \bigcup_{j=1}^{m} B_E\left(y_j, \frac{\varepsilon}{2}\right), \tag{4.20}$$

where $B_E(y, r)$ denotes the closed ball of radius r in E.

Let $x \in B$. Then there exists $j \in \{1, \ldots, m\}$ such that $\|T_{n_0}(x) - y_j\| \leq \varepsilon/2$. Using the Lipschitz estimate for $T - T_{n_0}$ on the bounded set B, we obtain

$$\|T(x)-y_j\| \leq \|T(x)-T_{n_0}(x)\|+\|T_{n_0}(x)-y_j\| \leq \operatorname{Lip}(T-T_{n_0})\cdot R+\frac{\varepsilon}{2} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon. \quad (4.21)$$

Hence

$$T(B) \subset \bigcup_{j=1}^{m} B_E(y_j, \varepsilon), \quad (4.22)$$

which shows that $T(B)$ is totally bounded. Since $B \subset X$ was arbitrary, T is compact. Therefore $T \in \operatorname{Lip}_0^{\mathcal{K}}(X, E)$. □

Definition 4.3.1 (Finite-Rank Lipschitz Operator)**.** Let (X, d_X) be a pointed p-metric space and E a p-Banach space. A mapping $T \in \operatorname{Lip}_0(X, E)$ is said to be of *finite rank* if there exists a finite-dimensional subspace $F \subset E$ such that $T(X) \subset F$.

Proposition 4.3.3 (Finite-Rank Operators are Compact)**.** *Let (X, d_X) be a pointed p-metric space and E a p-Banach space. Every finite-rank operator $T \in \operatorname{Lip}_0(X, E)$ is Lipschitz compact.*

Proof. Let $T \in \operatorname{Lip}_0(X, E)$ be of finite rank and let $F \subset E$ be a finite-dimensional subspace such that $T(X) \subset F$.

Let $B \subset X$ be bounded. Then $T(B) \subset F$ is a bounded subset of the finite-dimensional p-Banach space F. In a finite-dimensional normed (or quasi-normed) space, all norms are equivalent and bounded sets are relatively compact. Hence $T(B)$ is precompact in F, and thus in E. Since B was arbitrary, T is compact. □

Corollary 4.3.1 (Approximation by Finite-Rank Operators)**.** *Let (X, d_X) be a pointed p-metric space and E a p-Banach space. Suppose that $T \in \operatorname{Lip}_0(X, E)$ can be approximated in the Lipschitz norm by finite-rank Lipschitz operators, i.e., there exists a sequence $(T_n) \subset \operatorname{Lip}_0(X, E)$ such that each T_n has finite rank and*

$$\operatorname{Lip}(T_n - T) \longrightarrow 0. \quad (4.23)$$

Then T is Lipschitz compact, i.e., $T \in \operatorname{Lip}_0^{\mathcal{K}}(X, E)$.

Proof. Each T_n is finite-rank, hence compact by Proposition 4.3.3. By Proposition 4.3.2, $\mathrm{Lip}_0^{\mathcal{K}}(X,E)$ is closed in $\mathrm{Lip}_0(X,E)$, so the limit T is also compact. $\square$

Chapter 5

Nonlinear Operators and Quotient Structures

This chapter presents nonlinear operators and quotient structures, including two-Lipschitz operators in normed spaces and our original contribution on two-Lipschitz left-hand quotient operator ideals. Throughout this chapter, X, Y, Z, W denote pointed metric spaces with distinguished base point 0 and metric d. The symbols E, F, G stand for Banach spaces over the same field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Every Banach space is regarded as a pointed metric space with base point 0 and distance induced by its norm. The closed unit ball of E is denoted by B_E, its topological dual by E^*, and the space of bounded linear operators from E to F by $\mathcal{L}(E, F)$, endowed with the operator norm. All results of this chapter are published in [38].

5.1 The two-Lipschitz operators ideals

According to Hamidi, Dahia, Achour and Tallab [26, Definition 2.1] we have the following defintion

Definition 5.1.1. The two-Lipschitz space $\mathrm{BLip}_0(X, Y; E)$ consists of all maps $A : X \times Y \to E$ satisfying

$$A(x, 0) = A(0, y) = 0 \quad (x \in X, y \in Y)$$

and for which there exists a constant $C \geq 0$ such that

$$\|A(x,y) - A(x,y') - A(x',y) + A(x',y')\| \leq C\, d(x,x')d(y,y')$$

for all $x, x' \in X$, $y, y' \in Y$. The smallest such constant is the two-Lipschitz norm

$$\mathrm{BLip}(A) = \sup_{x \neq x' y \neq y'} \frac{\|A(x,y) - A(x,y') - A(x',y) + A(x',y')\|}{d(x,x')d(y,y')}.$$

With this norm, $\mathrm{BLip}_0(X,Y;E)$ is a Banach space; its elements are called two-Lipschitz operators.

We recall that for Banach spaces E, F, we denote by $E \otimes F$ their algebraic tensor product and by $E \widehat{\otimes}_\pi F$ the completion with respect to the projective norm

$$\pi(u) = \inf \left\{ \sum_{j=1}^{n} \|x_j\|_E \|y_j\|_F : u = \sum_{j=1}^{n} x_j \otimes y_j \right\}.$$

The canonical bilinear map $\sigma_2 : E \times F \to E \widehat{\otimes}_\pi F$ is given by $\sigma_2(x,y) = x \otimes y$. Further details can be found in [19, 47].

Theorem 5.1.1 (Bi-linearization; Hamidi et al. [26], Theorem 2.6 and Remark 2.7)**.** *For any two-Lipschitz operator $A \in \mathrm{BLip}_0(X,Y;E)$ there exists a unique continuous bilinear operator $A_B : \text{Æ}(X) \times \text{Æ}(Y) \to E$ such that*

$$A_B(m_{xx'}, m_{yy'}) = A(x,y) - A(x,y') - A(x',y) + A(x',y')$$

for all $x, x' \in X$, $y, y' \in Y$. Moreover,

$$A = A_B \circ (\delta_X, \delta_Y),$$

where $\delta_Z : Z \to \text{Æ}(Z)$ is the isometric embedding $\delta_Z(z) = m_{z0}$ (see [58]). The bilinear map A_B is called the bi-linearization of A. As shown in [26, Theorem 2.6 and Remark 2.7], we have

$$\mathrm{BLip}(A) = \|A_B\|.$$

Since every bilinear map admits a linearization, there is a unique bounded linear operator $A_L : Æ(X)\widehat{\otimes}_\pi Æ(Y) \to E$ satisfying $A_B = A_L \circ \sigma_2$. Consequently,

$$A = A_L \circ \sigma_2 \circ (\delta_X, \delta_Y) \quad \textit{and} \quad \mathrm{BLip}(A) = \|A_B\| = \|A_L\|.$$

The concept of a two-Lipschitz operator ideal, introduced in [26], extends the classical notion of a Banach operator ideal to the two-Lipschitz setting.

Definition 5.1.2. A class $\mathcal{A}_{\mathrm{BLip}}$ is called a two-Lipschitz operator ideal if for every pointed metric spaces X, Y and every Banach space E the component

$$\mathcal{A}_{\mathrm{BLip}}(X, Y; E) := \mathrm{BLip}_0(X, Y; E) \cap \mathcal{A}_{\mathrm{BLip}}$$

satisfies:

(i) $\mathcal{A}_{\mathrm{BLip}}(X, Y; E)$ is a linear subspace of $\mathrm{BLip}_0(X, Y; E)$.

(ii) For any $f \in X^\#$, $g \in Y^\#$, $e \in E$, the elementary map $f \cdot g \cdot e : (x, y) \mapsto f(x)g(y)e$ belongs to $\mathcal{A}_{\mathrm{BLip}}(X, Y; E)$.

(iii) (Ideal property) If $f \in \mathrm{Lip}_0(Z, X)$, $g \in \mathrm{Lip}_0(W, Y)$, $A \in \mathcal{A}_{\mathrm{BLip}}(X, Y; E)$ and $B \in \mathcal{L}(E, F)$, then $B \circ A \circ (f, g) \in \mathcal{A}_{\mathrm{BLip}}(Z, W; F)$.

If, in addition, each component carries a norm $\| \cdot \|_{\mathcal{A}_{\mathrm{BLip}}}$ turning it into a Banach space, and the following hold:

(iv) $\mathrm{BLip}(A) \leq \|A\|_{\mathcal{A}_{\mathrm{BLip}}}$ for every $A \in \mathcal{A}_{\mathrm{BLip}}(X, Y; E)$;

(v) $\|\mathrm{Id}_{\mathbb{K}^2}\|_{\mathcal{A}_{\mathrm{BLip}}} = 1$, where $\mathrm{Id}_{\mathbb{K}^2}(\alpha, \beta) = \alpha\beta$;

(vi) $\|B \circ A \circ (f, g)\|_{\mathcal{A}_{\mathrm{BLip}}} \leq \|B\|\|A\|_{\mathcal{A}_{\mathrm{BLip}}}\mathrm{Lip}(f)\mathrm{Lip}(g)$;

then $\mathcal{A}_{\mathrm{BLip}}$ is called a Banach two-Lipschitz operator ideal.

The following criterion isolates a key structural property linking a two-Lipschitz ideal with a linear operator ideal via the linearization map $A \mapsto A_L$.

Let $[\mathcal{A}, \| \cdot \|_{\mathcal{A}}]$ be a normed operator ideal and $[\mathcal{A}_{\mathrm{BLip}}, \| \cdot \|_{\mathcal{A}_{\mathrm{BLip}}}]$ a normed two-Lipschitz operator ideal. We say that $\mathcal{A}_{\mathrm{BLip}}$ has the linearization property (LP) with respect to $\mathcal{A}$ if for every $A \in \mathrm{BLip}_0(X, Y; E)$,

$$A \in \mathcal{A}_{\mathrm{BLip}}(X, Y; E) \Longleftrightarrow A_L \in \mathcal{A}\left(Æ(X)\widehat{\otimes}_\pi Æ(Y), E\right),$$

and in this case the norm equality $\|A\|_{\mathcal{A}_{\mathrm{BLip}}} = \|A_L\|_{\mathcal{A}}$ holds.

5.2 Two-Lipschitz Left-Hand Quotients Operator Ideals

We begin this section by recalling the quotient of operator ideals.
Let $\mathcal{A}, \mathcal{B}$ be operator ideals. Following González-Kania [25], (Pietsch [42]) a bounded linear operator $B : E \to F$ belongs to the left-hand quotient $\mathcal{A}^{-1} \circ \mathcal{B}$, denoted $B \in \mathcal{A}^{-1} \circ \mathcal{B}(E, F)$, if $C \circ B \in \mathcal{B}(E, G)$ for every $C \in \mathcal{A}(F, G)$ and every Banach space G. The right-hand quotient $\mathcal{A} \circ \mathcal{B}^{-1}$ is defined analogously. The symbols $\mathcal{A}^{-1}, \mathcal{B}^{-1}$ are purely notational and do not represent actual inverses. Both $\mathcal{A}^{-1} \circ \mathcal{B}$ and $\mathcal{A} \circ \mathcal{B}^{-1}$ are operator ideals [42, 3.2.2].

If $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ and $[\mathcal{B}, \|\cdot\|_{\mathcal{B}}]$ are Banach operator ideals, then defining

$$\|B\|_{\mathcal{A}^{-1}\circ\mathcal{B}} = \sup\left\{\|C \circ B\|_{\mathcal{B}} : C \in \mathcal{A}(F, G), \|C\|_{\mathcal{A}} \leq 1\right\},$$

where G ranges over all Banach spaces, yields a Banach operator ideal $[\mathcal{A}^{-1}\circ\mathcal{B}, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{B}}]$ [42].

Motivated by the concept of left-hand quotients in operator theory we introduce the left-hand quotient of an operator ideal and a two-Lipschitz operator ideal, and study its fundamental properties.

Definition 5.2.1. Let $\mathcal{A}_{\mathrm{BLip}}$ be a two-Lipschitz ideal and let $\mathcal{A}$ be an operator ideal. An operator $A \in \mathrm{BLip}_0(X, Y; E)$ belongs to the *two-Lipschitz left-hand quotient* $\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}$, written as $A \in \mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}(X, Y; E)$, if $B \circ A \in \mathcal{A}_{\mathrm{BLip}}(X, Y; F)$ for all $B \in \mathcal{A}(E, F)$, where F is any Banach space.

If $\mathcal{A}$ is normed with $\|\cdot\|_{\mathcal{A}}$ and $\mathcal{A}_{\mathrm{BLip}}$ with $\|\cdot\|_{\mathcal{A}_{\mathrm{BLip}}}$, we define the corresponding quotient norm by:

$$\|A\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}} := \sup\Big\{\|B\circ A\|_{\mathcal{A}_{\mathrm{BLip}}} : B\in\mathcal{A}(E,F),\ \|B\|_{\mathcal{A}}\leq 1\Big\}. \tag{5.1}$$

Our first aim is to justify the existence of the following supremum which appears in Definition 5.2.1. Our proof is based on [42, 7.2.2], [57, Proposition 2.1], and [30, Proposition 1].

Proposition 5.2.1. *Let $[\mathcal{A},\|\cdot\|_{\mathcal{A}}]$ be a Banach operator ideal and let $[\mathcal{A}_{\mathrm{BLip}},\|\cdot\|_{\mathcal{A}_{\mathrm{BLip}}}]$ be a normed two-Lipschitz operator ideal. If $A\in\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}(X,Y;E)$, then*

$$\sup\Big\{\|B\circ A\|_{\mathcal{A}_{\mathrm{BLip}}} : B\in\mathcal{A}(E,F),\ \|B\|_{\mathcal{A}}\leq 1\Big\}<\infty.$$

Proof. Let us assume that the preceding supremum is not finite. Then, for each $n\in\mathbb{N}$, we are in a position to find a Banach space F_n and an operator $B_n\in\mathcal{A}(E,F_n)$ satisfying

$$\|B_n\|_{\mathcal{A}}\leq\frac{1}{2^n},\qquad \|B_n\circ A\|_{\mathcal{A}_{\mathrm{BLip}}}\geq n.$$

Consider a sequence of Banach spaces $(F_i)_{i\in\mathbb{N}}$ and their ℓ_1-Cartesian product $\ell_1(\mathbb{N},F_i)$, defined as the space of all sequences (x_i) with $x_i\in F_i$ for each $i\in\mathbb{N}$ such that $(\|x_i\|)\in\ell_1(\mathbb{N},\mathbb{R})$. This space becomes a Banach space when equipped with the norm

$$\|(x_i)\|_1=\sum_{i=1}^{\infty}\|x_i\|.$$

For each $n\in\mathbb{N}$, define the bounded linear operators $I_n : F_n\longrightarrow\ell_1(\mathbb{N},F_i)$ and $J_n:\ell_1(\mathbb{N},F_i)\longrightarrow F_n$ by

$$I_n(x)=(\lambda_i^n x)_i\quad(x\in F_n),\qquad J_n((x_i))=x_n\quad((x_i)\in\ell_1(\mathbb{N},F_i)),$$

where λ_i^n is the Kronecker delta. Note that $\|I_n\|=1$ and $\|J_n\|\leq 1$ for all n. Consider the sequence $(I_n\circ B_n)$ in the Banach space $(\mathcal{A}(E,\ell_1(\mathbb{N},F_i)),\|\cdot\|_{\mathcal{A}})$. We have the estimate:

$$\Big\| \sum_{n=k+1}^{k+h} I_n \circ B_n \Big\|_{\mathcal{A}} \leq \sum_{n=k+1}^{k+h} \|I_n \circ B_n\|_{\mathcal{A}} = \sum_{i=1}^{h} \|I_{k+i} \circ B_{k+i}\|_{\mathcal{A}} \leq \sum_{i=1}^{h} \|B_{k+i}\|_{\mathcal{A}} \leq \sum_{i=1}^{\infty} \frac{1}{2^{k+i}} = \frac{1}{2^k},$$

which holds for all $h, k \in \mathbb{N}$. This shows that the series $\sum_{n\geq 1} I_n \circ B_n$ converges in the $\|\cdot\|_{\mathcal{A}}$-norm to an operator $B := \sum_{n=1}^{\infty} I_n \circ B_n \in \mathcal{A}(E, \ell_1(\mathbb{N}, F_i))$.

Consequently, we obtain the inequality

$$n \leq \|B_n \circ A\|_{\mathcal{A}_{\mathrm{BLip}}} = \|J_n \circ B \circ A\|_{\mathcal{A}_{\mathrm{BLip}}} \leq \|B \circ A\|_{\mathcal{A}_{\mathrm{BLip}}},$$

which is a contradiction. □

Our first objective is to study the relationship between two normed two-Lipschitz left-hand quotient operator ideals when the corresponding ideals are related. To establish this, we introduce the following notation: if $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ and $[\mathcal{B}, \|\cdot\|_{\mathcal{B}}]$ are normed operator ideals, we write $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}] \subseteq [\mathcal{B}, \|\cdot\|_{\mathcal{B}}]$ when $\mathcal{A} \subseteq \mathcal{B}$ and $\|A\|_{\mathcal{B}} \leq \|A\|_{\mathcal{A}}$ for all $A \in \mathcal{A}$.

Proposition 5.2.2. *Let $[\mathcal{A}^1_{\mathrm{BLip}}, \|\cdot\|_{\mathcal{A}^1_{\mathrm{BLip}}}]$ and $[\mathcal{A}^2_{\mathrm{BLip}}, \|\cdot\|_{\mathcal{A}^2_{\mathrm{BLip}}}]$ be normed two-Lipschitz operator ideals satisfying*

$$[\mathcal{A}^1_{\mathrm{BLip}}, \|\cdot\|_{\mathcal{A}^1_{\mathrm{BLip}}}] \subseteq [\mathcal{A}^2_{\mathrm{BLip}}, \|\cdot\|_{\mathcal{A}^2_{\mathrm{BLip}}}].$$

Then

$$[\mathcal{A}^{-1} \circ \mathcal{A}^1_{\mathrm{BLip}}, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{A}^1_{\mathrm{BLip}}}] \subseteq [\mathcal{A}^{-1} \circ \mathcal{A}^2_{\mathrm{BLip}}, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{A}^2_{\mathrm{BLip}}}],$$

for any Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$.

It is well-known that BLip_0 is a two-Lipschitz Banach operator ideal with the two-Lipschitz norm $\mathrm{BLip}(\cdot)$. The following result is an immediate consequence of the previous proposition, showing that $\mathcal{A}^{-1} \circ \mathrm{BLip}_0$ is the largest two-Lipschitz left-hand quotient for any Banach operator ideal $\mathcal{A}$ in the following sense.

Corollary 5.2.1. *Let $[\mathcal{A}_{\mathrm{BLip}}, \|\cdot\|_{\mathcal{A}_{\mathrm{BLip}}}]$ be a normed two-Lipschitz operator ideal. Then*

$$[\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}}] \subseteq [\mathcal{A}^{-1} \circ \mathrm{BLip}_0, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathrm{BLip}_0}],$$

for any Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$.

Under suitable assumptions on the two-Lipschitz operator ideal $\mathcal{A}_{\mathrm{BLip}}$ and in relation to Corollary 5.2.1, we obtain the following useful result.

Proposition 5.2.3. *Let* $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ *be a Banach operator ideal and* $[\mathcal{A}_{\mathrm{BLip}}, \|\cdot\|_{\mathcal{A}_{\mathrm{BLip}}}]$ *be a normed two-Lipschitz operator ideal. Then,*

$$[\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}}] \subseteq [\mathrm{BLip}_0, \mathrm{BLip}(\cdot)].$$

Furthermore, if $\mathcal{A}_{\mathrm{BLip}}$ *satisfies the linearization property (LP) in* $\mathcal{A}$*, then*

$$[\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}}] = [\mathrm{BLip}_0, \mathrm{BLip}(\cdot)].$$

Proof. Let $A \in \mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}(X, Y; E)$. Then $A \in \mathrm{BLip}_0(X, Y; E)$ and $B \circ A \in \mathrm{BLip}_0(X, Y; F)$ for all $B \in \mathcal{A}(E, F)$, where F is a Banach space. For each $x \in X$ and $y \in Y$, choose a functional $\varphi \in B_{E^*}$ such that $\|A(x, y)\| = |\varphi(A(x, y))|$. Since $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a Banach operator ideal, the functional $\varphi \otimes 1 : E \longrightarrow \mathbb{K}$ defined by $(\varphi \otimes 1)(e) = \varphi(e)$ for $e \in E$ belongs to $\mathcal{A}(E, \mathbb{C})$ with $\|\varphi \otimes 1\|_{\mathcal{A}} = \|\varphi\| \leq 1$. We then have the estimate:

$$\begin{aligned}
\|A(x, y) - A(x, y') - A(x', y) + A(x', y')\| &= |(\varphi \otimes 1) \circ A(x, y) - (\varphi \otimes 1) \circ A(x, y') \\
&\qquad - (\varphi \otimes 1) \circ A(x', y) + (\varphi \otimes 1) \circ A(x', y')| \\
&= \mathrm{BLip}((\varphi \otimes 1) \circ A)\, d(x, x')d(y, y') \\
&\leq \|(\varphi \otimes 1) \circ A\|_{\mathcal{A}_{\mathrm{BLip}}}\, d(x, x')d(y, y') \\
&\leq \|A\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}}\, d(x, x')d(y, y').
\end{aligned}$$

Taking the supremum over all $x, x' \in X$ and $y, y' \in Y$ we conclude that $\mathrm{BLip}(A) \leq \|A\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}}$.

Now assume $\mathcal{A}_{\mathrm{BLip}}$ has the linearization property (LP) in $\mathcal{A}$. Let $A \in \mathrm{BLip}_0(X, Y; E)$ and $B \in \mathcal{A}(E, F)$. Clearly $B \circ A \in \mathrm{BLip}_0(X, Y; F)$. By [26, Theorem 2.6 and Remark 2.7], there exist operators $(B \circ A)_L \in \mathcal{L}(\text{Æ}(X)\widehat{\otimes}_\pi\text{Æ}(Y), F)$ and $A_L \in$

$\mathcal{L}(\text{Æ}(X)\widehat{\otimes}_\pi\text{Æ}(Y), E)$ with $\|A_L\| = \mathrm{BLip}(A)$ satisfying

$$(B \circ A)_L \circ \sigma_2 \circ (\delta_X, \delta_Y) = B \circ A = B \circ A_L \circ \sigma_2 \circ (\delta_X, \delta_Y).$$

The norm-density of $\sigma_2 \circ (\delta_X, \delta_Y)$ in $\text{Æ}(X)\widehat{\otimes}_\pi\text{Æ}(Y)$ implies $(B \circ A)_L = B \circ A_L$. The ideal property of $\mathcal{A}$ gives

$$(B \circ A)_L \in \mathcal{A}(\text{Æ}(X)\widehat{\otimes}_\pi\text{Æ}(Y), F) \quad \text{with} \quad \|(B \circ A)_L\|_{\mathcal{A}} \leq \|B\|_{\mathcal{A}}.$$

Since $\mathcal{A}_{\mathrm{BLip}}$ has LP in $\mathcal{A}$, we conclude $B \circ A \in \mathcal{A}_{\mathrm{BLip}}(X, Y; F)$ with $\|B \circ A\|_{\mathcal{A}_{\mathrm{BLip}}} = \|(B \circ A)_L\|_{\mathcal{A}}$. The arbitrariness of B shows $A \in \mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}(X, Y; E)$ with $\|A\|_{\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}} \leq \mathrm{BLip}(A)$. $\square$

Now, we establish that two-Lipschitz left-hand quotients constitute an additional method for constructing two-Lipschitz operator ideals. This approach complements the standard techniques of composition and transposition in operator ideal theory.

Theorem 5.2.2. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a Banach operator ideal and let $[\mathcal{A}_{\mathrm{BLip}}, \|\cdot\|_{\mathcal{A}_{\mathrm{BLip}}}]$ be a normed (Banach) two-Lipschitz operator ideal. Then the left-hand quotient $[\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}, \|\cdot\|_{\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}}]$ forms a normed (Banach) two-Lipschitz operator ideal.*

Proof. It is easy to see that $\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}(X, Y; E)$ is a linear space. We now show that $\|\cdot\|_{\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}}$ defines a norm on $\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}(X, Y; E)$. Let $A \in \mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}(X, Y; E)$ and suppose that $\|A\|_{\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}} = 0$. Since $\mathrm{BLip}(A) \leq \|A\|_{\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}}$ by Proposition 5.2.3, it follows that $A = 0$. Let $\lambda \in \mathbb{K}$ and $A_1, A_2 \in \mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}(X, Y; E)$. Then, for all $B \in \mathcal{A}(E, F)$ with $\|B\|_{\mathcal{A}} \leq 1$, we have

$$\mathrm{BLip}(B \circ (\lambda A)) = |\lambda| \, \mathrm{BLip}(A), \quad \text{and} \quad \mathrm{BLip}(B \circ (A_1 + A_2)) \leq \mathrm{BLip}(A_1) + \mathrm{BLip}(A_2).$$

Hence,

$$\|\lambda A\|_{\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}} = |\lambda| \|A\|_{\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}},$$

and

$$\|A_1 + A_2\|_{\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}} \leq \|A_1\|_{\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}} + \|A_2\|_{\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}}.$$

Suppose now that the norm $\|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}}$ on $\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}$ is complete. Let (A_n) be a Cauchy sequence in $(\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}})$. For $B\in\mathcal{A}(E,F)$, where F is a Banach space, we know from Proposition 5.2.3 that $\mathrm{BLip}(\cdot)\leq\|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}}$, so there exists $A\in\mathrm{BLip}_0(X,Y;E)$ such that $\mathrm{BLip}(A_n-A)\longrightarrow 0$ as $n\longrightarrow\infty$, which implies $\mathrm{BLip}(B\circ A_n-B\circ A)\longrightarrow 0$ as $n\longrightarrow\infty$. On the other hand,

$$\|B\circ A_p-B\circ A_q\|_{\mathcal{A}_{\mathrm{BLip}}}=\|B\circ(A_p-A_q)\|_{\mathcal{A}_{\mathrm{BLip}}}\leq\|B\|_{\mathcal{A}}\|A_p-A_q\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}},$$

for all $p,q\in\mathbb{N}$, so $(B\circ A_n)$ is a Cauchy sequence in $[\mathcal{A}_{\mathrm{BLip}}(X,Y;F),\|\cdot\|_{\mathcal{A}_{\mathrm{BLip}}}]$, and thus converges to some operator $D\in\mathcal{A}_{\mathrm{BLip}}(X,Y;F)$. Since $\mathrm{BLip}(\cdot)\leq\|\cdot\|_{\mathcal{A}_{\mathrm{BLip}}}$, we get $B\circ A=D$, hence $A\in\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}(X,Y;E)$ and

$$\|B\circ A_n-B\circ A\|_{\mathcal{A}_{\mathrm{BLip}}}\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.$$

To show $A_n\longrightarrow A$ in $[\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}(X,Y;E),\|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}}]$, let $\varepsilon>0$. There exists $m\in\mathbb{N}$ such that

$$\|A_p-A_q\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}}<\frac{\varepsilon}{2}, \quad \text{for all } p,q\geq m.$$

Then,

$$\|B\circ A_p-B\circ A_{p+n}\|_{\mathcal{A}_{\mathrm{BLip}}}<\frac{\varepsilon}{2},$$

for all $p\geq m$, $n\in\mathbb{N}$, and $B\in\mathcal{A}(E,F)$, $\|B\|_{\mathcal{A}}\leq 1$. Taking limits as $n\longrightarrow\infty$, it follows that

$$\|B\circ A_p-B\circ A\|_{\mathcal{A}_{\mathrm{BLip}}}\leq\frac{\varepsilon}{2},$$

for all $p\geq m$, and taking the supremum over all such B gives

$$\|A_p-A\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}}<\varepsilon, \quad \text{for all } p\geq m.$$

Let $f\in X^{\#}$, $g\in Y^{\#}$ and $e\in E$. Since $\mathcal{A}_{\mathrm{BLip}}(X,Y;E)$ is a normed two-Lipschitz operator ideal, we have $f\cdot g\cdot e\in\mathcal{A}_{\mathrm{BLip}}(X,Y;E)$ with $\|f\cdot g\cdot e\|_{\mathcal{A}_{\mathrm{BLip}}}=\mathrm{Lip}(f)\,\mathrm{Lip}(g)\|e\|$ (see [26, Proposition 3.2]). Let $B\in\mathcal{A}(E,F)$ where F is Banach space. Then $B\circ(f\cdot$

$g \cdot e) = f \cdot g \cdot B(e) \in \mathcal{A}_{\mathrm{BLip}}(X, Y; F)$ with norm

$$\|B \circ (f \cdot g \cdot e)\|_{\mathcal{A}_{\mathrm{BLip}}} = \mathrm{Lip}(f)\,\mathrm{Lip}(g)\|B(e)\|.$$

Thus, $f \cdot g \cdot e \in \mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}(X, Y; E)$ such that

$$\begin{aligned}\|f \cdot g \cdot e\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}} &= \mathrm{Lip}(f)\,\mathrm{Lip}(g) \sup\{\|B(e)\| : B \in \mathcal{A}(E, F), \|B\|_{\mathcal{A}} \leq 1\} \\ &\geq \mathrm{Lip}(f)\,\mathrm{Lip}(g)\|e\|.\end{aligned}$$

Conversely, for such B we also have

$$\|B \circ (f \cdot g \cdot e)\|_{\mathcal{A}_{\mathrm{BLip}}} \leq \|B\|_{\mathcal{A}}\|f \cdot g \cdot e\|_{\mathcal{A}_{\mathrm{BLip}}} \leq \mathrm{Lip}(f)\,\mathrm{Lip}(g)\|e\|.$$

Hence $\|f \cdot g \cdot e\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}} \leq \mathrm{Lip}(f)\,\mathrm{Lip}(g)\|e\|$.

Now, let Z, W, X and Y be pointed metric spaces, E, F and G be Banach spaces, and let $A \in \mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}(X, Y; E)$, $f \in \mathrm{Lip}(Z, X)$, $g \in \mathrm{Lip}(W, Y)$ and $D \in \mathcal{L}(E, F)$. For $B \in \mathcal{A}(F, G)$ with $\|B\| \leq 1$, we have $B \circ D \in \mathcal{A}(E, G)$ with $\|B \circ D\|_{\mathcal{A}} \leq \|B\|_{\mathcal{A}}\|D\|$. Since $B \circ D \circ A \in \mathcal{A}_{\mathrm{BLip}}(X, Y; G)$, we get

$$\|B \circ D \circ A\|_{\mathcal{A}_{\mathrm{BLip}}} \leq \|B \circ D\|_{\mathcal{A}}\|A\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}}.$$

The composition $B \circ D \circ A \circ (f, g) \in \mathcal{A}_{\mathrm{BLip}}(Z, W; G)$ and

$$\|B \circ D \circ A \circ (f, g)\|_{\mathcal{A}_{\mathrm{BLip}}} \leq \mathrm{Lip}(f)\,\mathrm{Lip}(g)\|B \circ D \circ A\|_{\mathcal{A}_{\mathrm{BLip}}},$$

so $B \circ D \circ A \circ (f, g) \in \mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}(Z, W; G)$ and

$$\|D \circ A \circ (f, g)\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}} \leq \mathrm{Lip}(f)\,\mathrm{Lip}(g)\|D\|\|A\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}}.$$

□

The following theorem establishes a fundamental relationship between two-Lipschitz quotients and left-hand quotients for operator ideals possessing the linearization property.

Theorem 5.2.3. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a Banach operator ideal, $[\mathcal{B}, \|\cdot\|_{\mathcal{B}}]$ be a normed operator ideal, and $[\mathcal{A}_{\text{BLip}}, \|\cdot\|_{\mathcal{A}_{\text{BLip}}}]$ be a normed two-Lipschitz operator ideal with the linearization property (LP) in $\mathcal{B}$. For any two-Lipschitz mapping $A \in \text{BLip}_0(X, Y; E)$, the following are equivalent:*

1. *A belongs to $\mathcal{A}^{-1} \circ \mathcal{A}_{\text{BLip}}(X, Y; E)$,*
2. *Its linearization A_L lies in $\mathcal{A}^{-1} \circ \mathcal{B}(\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y), E)$.*

Moreover, the correspondence satisfies the isometric identity:

$$\|A\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\text{BLip}}} = \|A_L\|_{\mathcal{A}^{-1}\circ\mathcal{B}},$$

and the linearization map $A \longmapsto A_L$ induces an isometric isomorphism between these spaces.

Proof. (1)⇒(2): Let $A \in \mathcal{A}^{-1} \circ \mathcal{A}_{\text{BLip}}(X, Y; E)$. Then, for all $B \in \mathcal{A}(E, F)$, where F is a Banach space, we have $B \circ A \in \mathcal{A}_{\text{BLip}}(X, Y; F)$. As in the proof of Proposition 5.2.3, by [26, Theorem 2.6, Remark 2.7], there exist operators $A_L \in \mathcal{L}(\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y), E)$ and $(B \circ A)_L \in \mathcal{L}(\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y), F)$ such that

$$(B \circ A)_L = B \circ A_L.$$

Since $\mathcal{A}_{\text{BLip}}$ has the linearization property (LP) in $\mathcal{B}$, it follows that $(B \circ A)_L \in \mathcal{B}(\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y), F)$. By the arbitrariness of $B \in \mathcal{A}(E, F)$, we conclude that $A_L \in \mathcal{A}^{-1} \circ \mathcal{B}(\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y), E)$.

(2)⇒(1): Assume that $A_L \in \mathcal{A}^{-1} \circ \mathcal{B}(\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y), E)$. Then $B \circ A_L \in \mathcal{B}(\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y), F)$ for all $B \in \mathcal{A}(E, F)$, and since $(B \circ A)_L = B \circ A_L$, we deduce that $(B \circ A)_L \in \mathcal{B}(\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y), F)$. By the LP of $\mathcal{A}_{\text{BLip}}$ in $\mathcal{B}$, it follows that $B \circ A \in \mathcal{A}_{\text{BLip}}(X, Y; F)$, and thus $A \in \mathcal{A}^{-1} \circ \mathcal{A}_{\text{BLip}}(X, Y; E)$.

In this case, we also have the norm identity:

$$\begin{aligned}\|A\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}} &= \sup\Big\{\|B\circ A\|_{\mathcal{A}_{\mathrm{BLip}}} : B\in\mathcal{A}(E,F), \|B\|_{\mathcal{A}}\le 1\Big\} \\ &= \sup\Big\{\|(B\circ A)_L\|_{\mathcal{B}} : B\in\mathcal{A}(E,F), \|B\|_{\mathcal{A}}\le 1\Big\} \\ &= \sup\Big\{\|B\circ A_L\|_{\mathcal{B}} : B\in\mathcal{A}(E,F), \|B\|_{\mathcal{A}}\le 1\Big\} \\ &= \|A_L\|_{\mathcal{A}^{-1}\circ\mathcal{B}},\end{aligned}$$

where the second equality follows from the linearization property of $\mathcal{A}_{\mathrm{BLip}}$ in $\mathcal{B}$.

For the final assertion, it suffices to show that the map $A \longmapsto A_L$ from $\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}(X,Y;E)$ into $\mathcal{A}^{-1}\circ\mathcal{B}(\text{Æ}(X)\widehat{\otimes}_\pi\text{Æ}(Y),E)$ is surjective. Let $D\in\mathcal{A}^{-1}\circ\mathcal{B}(\text{Æ}(X)\widehat{\otimes}_\pi\text{Æ}(Y),E)$. Then $B\circ D\in\mathcal{B}(\text{Æ}(X)\widehat{\otimes}_\pi\text{Æ}(Y),F)$ for all $B\in\mathcal{A}(E,F)$, with F a Banach space. By [26, Theorem 2.6, Remark 2.7], there exists $C\in\mathrm{BLip}_0(X,Y;F)$ such that $B\circ D = C_L$. By the LP of $\mathcal{A}_{\mathrm{BLip}}$ in $\mathcal{B}$, we obtain $C\in\mathcal{A}_{\mathrm{BLip}}(X,Y;F)$.

Now define $A = D\circ\sigma_2\circ(\delta_X,\delta_Y) : X\times Y\longrightarrow E$, where $\sigma_2\circ(\delta_X,\delta_Y) : X\times Y\longrightarrow \text{Æ}(X)\widehat{\otimes}_\pi\text{Æ}(Y)$ is the canonical inclusion. Clearly, $A\in\mathrm{BLip}_0(X,Y;E)$, and for all $B\in\mathcal{A}(E,F)$ we have

$$B\circ A = B\circ D\circ\sigma_2\circ(\delta_X,\delta_Y) = C_L\circ\sigma_2\circ(\delta_X,\delta_Y) = C\in\mathcal{A}_{\mathrm{BLip}}(X,Y;F),$$

so $A\in\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}(X,Y;E)$ and $A_L = D$. □

We conclude this section by establishing a fundamental connection between two-Lipschitz left-hand quotient operator ideals and composition generated two-Lipschitz operator ideals. Specifically, we demonstrate that when the associated two-Lipschitz ideal possesses the linearization property (LP), every two-Lipschitz left-hand quotient ideal can be represented as a composition ideal.

Proposition 5.2.4. *Let $[\mathcal{A},\|\cdot\|_{\mathcal{A}}]$ be a Banach operator ideal, $[\mathcal{B},\|\cdot\|_{\mathcal{B}}]$ be a normed operator ideal, and $[\mathcal{A}_{\mathrm{BLip}},\|\cdot\|_{\mathcal{A}_{\mathrm{BLip}}}]$ a normed two-Lipschitz operator ideal with LP in $\mathcal{B}$. Then*

$$[\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}},\|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}}] = [(\mathcal{A}^{-1}\circ\mathcal{B})\circ\mathrm{BLip}_0,\|\cdot\|_{(\mathcal{A}^{-1}\circ\mathcal{B})\circ\mathrm{BLip}_0}].$$

Proof. For any $A \in \mathrm{BLip}_0(X, Y; E)$, the following equivalences hold from Theorem 5.2.3 and [26, Theorem 3.5]:

$$\begin{aligned} A \in (\mathcal{A}^{-1} \circ \mathcal{B}) \circ \mathrm{BLip}_0(X, Y; E) &\Longleftrightarrow A_L \in \mathcal{A}^{-1} \circ \mathcal{B}\Big(Æ(X)\widehat{\otimes}_\pi Æ(Y), E\Big) \\ &\Longleftrightarrow A \in \mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}(X, Y; E). \end{aligned}$$

where A_L is the linearization of A. Moreover, for all $A \in \mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}(X, Y; E)$, we have the norm equalities

$$\|A\|_{(\mathcal{A}^{-1}\circ\mathcal{B})\circ\mathrm{BLip}_0} = \|A_L\|_{\mathcal{A}^{-1}\circ\mathcal{B}} = \|A\|_{\mathcal{A}^{-1}\circ\mathcal{A}_{\mathrm{BLip}}}.$$

□

5.3 Examples of two-Lipschitz left-hand quotient ideals

In this section we present four important examples of two-Lipschitz left-hand quotient ideals generated by a linear operator ideal and a two-Lipschitz ideal. These correspond to spaces of two-Lipschitz operators whose images satisfy certain compactness-type properties.

The two-Lipschitz image

Let X, Y be pointed metric spaces, E a Banach space, and $A \in \mathrm{BLip}_0(X, Y; E)$. The *two-Lipschitz image* of A is the set

$$\mathrm{Im}_{\mathrm{BLip}}(A) = \left\{ \frac{A(x, y) - A(x', y) - A(x, y') + A(x', y')}{d(x, x')d(y, y')} : x \neq x', y \neq y' \right\}.$$

It is immediate that if $\mathrm{Im}_{\mathrm{BLip}}(A)$ is bounded, then A is a two-Lipschitz map. The size of this set leads to several natural classes of two-Lipschitz operators, as introduced in [26].

Linear operator ideals

Recall the following classical operator ideals for bounded linear operators $B : E \to F$:

— $\mathcal{K}(E, F)$ - compact operators (the image of the unit ball is relatively compact);

— $\mathcal{V}(E, F)$ - completly continous operators.

— $\mathcal{W}(E, F)$ - weakly compact operators (the image of the unit ball is relatively weakly compact);

— $\mathcal{S}(E, F)$ - separable operators (the image of the unit ball is separable);

— $\mathcal{R}(E, F)$ - Rosenthal operators;

— $\mathfrak{G}(E, F)$ - Grothendieck operators.

The inclusions

$$\mathcal{K} \subseteq \mathcal{W} \subseteq \mathcal{R}, \quad \mathcal{W} \subseteq \mathfrak{G}, \quad \mathcal{K} \subseteq \mathcal{S}$$

hold for the corresponding ideals. A thorough treatment of these ideals can be found in Pietsch's monograph [42].

Two-Lipschitz ideals defined by image properties

Analogously, we define two-Lipschitz ideals by imposing the same compactness properties on the two-Lipschitz image:

— $\mathrm{BLip}_{0\mathcal{K}}(X, Y; E)$ – two-Lipschitz operators with relatively compact two-Lipschitz image;

— $\mathrm{BLip}_{0\mathcal{W}}(X, Y; E)$ – two-Lipschitz operators with relatively weakly compact two-Lipschitz image.

In the same spirit one could define the classes $\mathrm{BLip}_{0\mathcal{S}}$, $\mathrm{BLip}_{0\mathcal{R}}$ and $\mathrm{BLip}_{0\mathfrak{G}}$ corresponding to separable, Rosenthal and Grothendieck two-Lipschitz images, respectively.

The following structural facts are known:

— $\mathrm{BLip}_{0\mathcal{K}}$ and $\mathrm{BLip}_{0\mathcal{W}}$ are closed two-Lipschitz operator ideals.

— $\mathrm{BLip}_{0\mathcal{K}}$ has the linearization property (LP) with respect to the compact operator ideal $\mathcal{K}$ [26, Theorem 4.3].

— $\mathrm{BLip}_{0\mathcal{W}}$ has the linearization property with respect to the weakly compact operator ideal $\mathcal{W}$ [29, Example 3.11].

Thus, these two-Lipschitz ideals are precisely the composition ideals $\mathcal{K} \circ \mathrm{BLip}_0$ and $\mathcal{W} \circ \mathrm{BLip}_0$, respectively.

5.3.1 Grothendieck and Rosenthal two-Lipschitz left-hand operator ideal

We refer the reader to [25] for a study of the Grothendieck property. According to [25, p. 298], a set $U \subseteq E$ is called *Grothendieck* if for every operator $B \in \mathcal{L}(E, c_0)$, $B(U)$ is a relatively weakly compact subset of c_0 (the set of all bounded sequences vanishing at 0). It is known that $\mathfrak{G}$ is a closed surjective operator ideal.

Definition 5.3.1. We will say that a map $A \in \mathrm{BLip}_0(X, Y; E)$ is *Grothendieck* if $\mathrm{Im}_{\mathrm{BLip}}(A)$ is a Grothendieck subset of E. Let $\mathfrak{G}_{\mathrm{BL}}(X, Y; E)$ denote the space of all Grothendieck two-Lipschitz maps from $X \times Y$ into E.

We now show that $\mathfrak{G}_{\mathrm{BL}}$ has the LP in $\mathfrak{G}$.

Theorem 5.3.1 ([38])**.** *For a map $A \in \mathrm{BLip}_0(X, Y; E)$, the following are equivalent:*

1. *$A : X \times Y \longrightarrow E$ is a Grothendieck two-Lipschitz operator.*
2. *$A_L : Æ(X)\widehat{\otimes}_\pi Æ(Y) \longrightarrow E$ is a Grothendieck operator.*
3. *There is a Banach space F, a two-Lipschitz operator $B \in \mathrm{BLip}_0(X, Y; F)$, and a linear operator $C \in \mathfrak{G}(F, E)$ such that $A = C \circ B$.*

In this case, $\|A\|_{\mathfrak{G}_{\mathrm{BL}}} = \|A_L\| = \|A\|_{\mathfrak{G}\circ\mathrm{BLip}_0}$. As a consequence, the map $A \longmapsto A_L$ is an isometric isomorphism from $[\mathfrak{G}_{\mathrm{BL}}(X, Y; E), \|\cdot\|_{\mathfrak{G}_{\mathrm{BL}}}]$ onto $[\mathfrak{G}(Æ(X)\widehat{\otimes}_\pi Æ(Y), E), \|\cdot\|]$, and from $[\mathfrak{G} \circ \mathrm{BLip}_0(X, Y; E), \|\cdot\|_{\mathfrak{G}\circ\mathrm{BLip}_0}]$ onto $[\mathfrak{G}(Æ(X)\widehat{\otimes}_\pi Æ(Y), E), \|\cdot\|]$.

Proof. (i)⇒(ii): Let us suppose that $A \in \mathfrak{G}_{\mathrm{BL}}(X, Y; E)$, then we have that

$$\mathrm{Im}_{\mathrm{BLip}}(A) = A_L(\mathcal{M}(X) \otimes \mathcal{M}(Y)) \subseteq \overline{\mathrm{aco}}(\mathrm{Im}_{\mathrm{BLip}}(A)),$$

is Grothendieck in E. It is not difficult to see that the norm-closed absolutely convex hull of a Grothendieck set is itself Grothendieck since the norm-closed absolutely convex hull of a relatively w-compact set is relatively w-compact by [14, Theorem 2.8.14]. Thus, $\overline{\text{aco}}(\text{Im}_{\text{BLip}}(A))$ is Grothendieck in E. Do not lose sight of the fact that

$$A_L\Big(B_{\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y)}\Big) = A_L(\overline{\text{aco}}(\mathcal{M}(X) \otimes \mathcal{M}(Y))) \subseteq \overline{\text{aco}}(A_L(\mathcal{M}(X) \otimes \mathcal{M}(Y))).$$

Then, it follows that $A_L\Big(B_{\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y)}\Big)$ is a Grothendieck subset of E. Hence, $A_L \in \mathfrak{G}(\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y), E)$ by [25, Proposition 6.1.1].

(ii)⇒(i): Let us assume that $A_L \in \mathfrak{G}(\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y), E)$, then $A_L\Big(B_{\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y)}\Big)$ is a Grothendieck subset of E. Since $\text{Im}_{\text{BLip}}(A) \subseteq A_L\Big(B_{\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y)}\Big)$, it follows that $\text{Im}_{\text{BLip}}(A) = A_L(\mathcal{M}(X) \otimes \mathcal{M}(Y))$ is Grothendieck in E by [57, Lemma 1.3].

(ii)⇔(iii): A direct application of [26, Theorem 3.5] gives us this equivalence. Finally, the equality of the norms follows easily by using [26, Theorem 2.6 and Remark 2.7] and (ii)⇒(i), and the last statement could easily be verified from [26, Theorem 3.5]. □

Let us recall (see [2, Definition 3.1]) that a two-Lipschitz operator ideal $\mathcal{A}_{\text{BLip}}$ is said to be *closed* if every component $\mathcal{A}_{\text{BLip}}(X, Y; E)$ is a closed subspace of $\text{BLip}_0(X, Y; E)$ endowed with the topology of the supremum norm. Since $\mathfrak{G}$ is a closed operator ideal (the norm-limit of a convergent sequence of Grothendieck operators is Grothendieck) and $\mathfrak{G}_{\text{BL}} = \mathfrak{G} \circ \text{BLip}_0$ by Theorem 5.3.1, then [2, Corollary 2.5] yields the following.

Corollary 5.3.2. $[\mathfrak{G}_{\text{BL}}, \|\cdot\|_{\mathfrak{G}_{\text{BL}}}]$ *is a closed two-Lipschitz operator ideal.*

We are now ready to describe the space of all Grothendieck two-Lipschitz in terms of a two-Lipschitz left-hand quotient ideal.

Theorem 5.3.3. $[\mathfrak{G}_{\text{BL}}, \|\cdot\|_{\mathfrak{G}_{\text{BL}}}] = [\mathcal{S}^{-1} \circ \text{BLip}_{0\mathcal{W}}, \|\cdot\|_{\mathcal{S}^{-1} \circ \text{BLip}_{0\mathcal{W}}}]$.

Proof. Let $A \in \text{BLip}_0(X, Y; E)$. Taking into account Theorem 5.3.1, [42, 3.2.6], Definition 5.2.1, and Theorem 5.2.3 together with [26, Remark 4.5], respectively, we have

$$\begin{aligned} A \in \mathfrak{G}_{\mathrm{BL}}(X, Y; E) &\Longleftrightarrow A_L \in \mathfrak{G}(\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y), E) \\ &\Longleftrightarrow B \circ A_L \in \mathcal{W}(\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y), F),\ \forall B \in \mathcal{S}(E, F) \\ &\Longleftrightarrow A_L \in \mathcal{S}^{-1} \circ \mathcal{W}(\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y), E) \\ &\Longleftrightarrow A \in \mathcal{S}^{-1} \circ \mathrm{BLip}_{0\mathcal{W}}(X, Y; E), \end{aligned}$$

and, in this case,

$$\|A\|_{\mathfrak{G}_{\mathrm{BL}}} = \|A_L\|_{\mathcal{S}^{-1}\circ\mathcal{W}} = \|A\|_{\mathcal{S}^{-1}\circ\mathrm{BLip}_{0\mathcal{W}}}.$$

□

Let us recall that a subset U of E is said to be *conditionally weakly compact* (or *Rosenthal*) if every sequence in U has a weak Cauchy subsequence. On the other hand, an operator $B \in \mathcal{L}(E, F)$ is called *completely continuous* if every weakly convergent sequence (x_n) is mapped into a norm convergent sequence $(B(x_n))$. Let $\mathcal{V}(E, F)$ be the space of all completely continuous operators from E into F. By [42, 1.6.2 and 4.2.5], $\mathcal{V}$ is a closed operator ideal.

We now establish an analogue of this result for two-Lipschitz mappings whose images have Rosenthal range. By applying Rosenthal's factorization theorem, such mappings can be represented through a Banach space that does not contain any isomorphic copy of ℓ_1. We denote by $\mathcal{R}_{\mathrm{BL}}(X, Y; E)$ the linear space consisting of all two-Lipschitz mappings from $X \times Y$ into E whose range is Rosenthal. It is clear that $\mathcal{R}_{\mathrm{BL}}(X, Y; E) \subseteq \mathrm{BLip}_{0\mathcal{W}}(X, Y; E)$.

Theorem 5.3.4 ([38]). *Let X and Y be pointed metric spaces, let E be a Banach space, and let $A \in \mathrm{BLip}_0(X, Y; E)$. The following assertions are equivalent:*

1. *$A : X \times Y \to E$ has Rosenthal range.*
2. *$A_L : \text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y) \longrightarrow E$ is a Rosenthal operator.*
3. *There exists a Banach space F which does not contain an isomorphic copy of ℓ_1, a two-Lipschitz operator $B \in \mathrm{BLip}_0(X, Y; F)$, and a linear operator $C \in \mathcal{L}(F, E)$*

such that $A = C \circ B$.

Proof. (1)⇒(2): The proof follows the same general argument as in [26, Theorem 4.3]. The key point is that the norm-closed absolutely convex hull of a Rosenthal subset of a Banach space is again Rosenthal; see [26, p. 357].

(2)⇒(3): By Rosenthal's factorization theorem (see, for instance, [1]), there exists a Banach space F not containing an isomorphic copy of ℓ_1 and operators $C \in \mathcal{L}(F, E)$ and $S \in \mathcal{L}(Æ(X)\widehat{\otimes}_\pi Æ(Y), F)$ such that $A_L = C \circ S$. Define $B := S \circ \sigma_2 \circ (\delta_X, \delta_Y) \in \mathrm{BLip}_0(X, Y; F)$. Then we have

$$A = A_L \circ \sigma_2 \circ (\delta_X, \delta_Y) = C \circ S \circ \sigma_2 \circ (\delta_X, \delta_Y) = C \circ B.$$

(3)⇒(1): Since $A(X \times Y) = C(B(X \times Y))$ and C is weak-to-weak continuous [21, Theorem 2.5.11], the range $A(X \times Y)$ is Rosenthal in E, because $B(X \times Y)$ is Rosenthal in F by Rosenthal's ℓ_1-theorem [50]. □

Next, we describe precisely the class of two-Lipschitz operators with Rosenthal range. This class turns out to coincide with the two-Lipschitz left-hand quotient ideal generated by the operator ideal $\mathcal{V}$ together with the two-Lipschitz ideal $\mathrm{BLip}_{0\mathcal{K}}$.

Theorem 5.3.5. $[\mathcal{R}_{\mathrm{BL}}, \|\cdot\|_{\mathcal{R}_{\mathrm{BL}}}] = [\mathcal{V}^{-1} \circ \mathrm{BLip}_{0\mathcal{K}}, \|\cdot\|_{\mathcal{V}^{-1}\circ\mathrm{BLip}_{0\mathcal{K}}}]$.

Proof. Given $A \in \mathrm{BLip}_0(X, Y; E)$, we obtain:

$$\begin{aligned}
A \in \mathcal{R}_{\mathrm{BL}}(X, Y; E) &\Longleftrightarrow A_L \in \mathcal{R}(Æ(X)\widehat{\otimes}_\pi Æ(Y), E)\\
&\Longleftrightarrow B \circ A_L \in \mathcal{K}(Æ(X)\widehat{\otimes}_\pi Æ(Y), F),\ \forall B \in \mathcal{V}(E, F)\\
&\Longleftrightarrow A_L \in \mathcal{V}^{-1} \circ \mathcal{K}(Æ(X)\widehat{\otimes}_\pi Æ(Y), E)\\
&\Longleftrightarrow A \in \mathcal{V}^{-1} \circ \mathrm{BLip}_{0\mathcal{K}}(X, Y; E),
\end{aligned}$$

in which case,

$$\|A\|_{\mathcal{R}_{\mathrm{BL}}} = \|A_L\|_{\mathcal{V}^{-1}\circ\mathcal{K}} = \|A\|_{\mathcal{V}^{-1}\circ\mathrm{BLip}_{0\mathcal{K}}},$$

by using Theorem 5.3.4, [26, Theorem 4.3], [42, 3.2.4], Definition 5.2.1, and Theorem 5.2.3 respectively. □

5.3.2 An open problem related to left hand quotient two-Lipschitz operators Ideal

In the following, we present an open problem. We do not know what happens with Proposition 5.2.4 for the case in which the ideal of two-Lipschitz maps $\mathcal{A}_{\mathrm{BLip}}$ does not have the LP in an operator ideal $\mathcal{B}$. The natural idea is to think that a two-Lipschitz quotient $\mathcal{A}^{-1} \circ \mathcal{A}_{\mathrm{BLip}}$ may not coincide with a composition ideal of the form $(\mathcal{A}^{-1} \circ \mathcal{B}) \circ \mathrm{BLip}_0$.

The theory of p-summing operators is based on a crucial criterion due to Pietsch [42]. Let $1 \leq p < \infty$. A linear operator $B : E \to F$ between Banach spaces is said to be p-summing if there exists a constant $C \geq 0$ such that

$$\left(\sum_{i=1}^{n} \|B(x_i)\|^p\right)^{1/p} \leq C \sup_{\varphi \in B_{E^*}} \left(\sum_{i=1}^{n} |\varphi(x_i)|^p\right)^{1/p},$$

for every finite family $(x_i)_{i=1}^n$ in E. The infimum of all such constants C is denoted by $\pi_p(B)$. The set of all p-summing operators between E and F is denoted by $\Pi_p(E, F)$. We mention that $(\Pi_p, \pi_p(\cdot))$ is a Banach operator ideal (see [20]).

Similarly to [26], for $1 \leq p, p_1, p_2 < \infty$ with $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$, a mapping $A : X \times Y \to E$ is said to be *two-Lipschitz* (p, p_1, p_2)*-summing* if there exists a constant $C > 0$ such that for any $(x_i)_{i=1}^n$, $(x_i')_{i=1}^n$ in X and $(y_i)_{i=1}^n$, $(y_i')_{i=1}^n$ in Y, we have

$$\begin{aligned} &\|(A(x_i, y_i) - A(x_i, y_i') - A(x_i', y_i) + A(x_i', y_i'))_{i=1}^n\|_p \\ &\qquad \leq C \sup_{f \in B_{X^\#}} \|(f(x_i) - f(x_i'))_{i=1}^n\|_{p_1} \sup_{g \in B_{Y^\#}} \|(g(y_i) - g(y_i'))_{i=1}^n\|_{p_2}. \end{aligned}$$

We denote this class of two-Lipschitz operators as $\mathrm{BL}_{as,(p;p_1,p_2)}(X, Y; E)$, and it becomes a Banach space endowed with the norm $\|A\|_{\mathrm{BL}_{as,(p;p_1,p_2)}}$ as the infimum of all constants C fulfilling the above inequality.

Let $1 \leq p, p_1, p_2 < \infty$ with $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$. Consider the mapping A defined by

$$A : \mathbb{R} \times \mathbb{R} \to Æ(\mathbb{R}) \widehat{\otimes}_\pi Æ(\mathbb{R}), \quad A(x, y) = m_{x,0} \otimes m_{y,0}.$$

By [26, Example 4.17], A is two-Lipschitz (p, p_1, p_2)-summing and its linearization A_L is the identity map on the infinite-dimensional space $\text{Æ}(\mathbb{R})\widehat{\otimes}_\pi\text{Æ}(\mathbb{R})$. Consequently, $A_L \notin \Pi_p$ (see [20, Page 50]). Thus, we have shown that $\mathrm{BL}_{as,(p;p_1,p_2)}$ does not have the lifting property in Π_p. Moreover, by [26, Theorem 3.5], it follows that

$$\mathrm{BL}_{as,(p;p_1,p_2)} \neq \Pi_p \circ \mathrm{BLip}_0\,.$$

Apparently, for any operator ideal $\mathcal{A}$, the composition $\mathcal{A}^{-1} \circ \mathrm{BL}_{as,(p;p_1,p_2)}$ cannot be a composition ideal generated by the operator ideal Π_p, that is,

$$\mathcal{A}^{-1} \circ \mathrm{BL}_{as,(p;p_1,p_2)} \neq (\mathcal{A}^{-1} \circ \Pi_p) \circ \mathrm{BLip}_0\,.$$

Nevertheless, $\mathcal{A}^{-1} \circ \mathrm{BL}_{as,(p;p_1,p_2)}$ is a two-Lipschitz left-hand quotient operator ideal, since $\mathrm{BL}_{as,(p;p_1,p_2)}$ is a two-Lipschitz operator ideal by [26, Proposition 4.16].

Conclusion and Perspectives

This thesis has presented a systematic study of linear relations and nonlinear operators in the context of quasi-normed spaces, extending fundamental operator-theoretic results from classical Banach spaces to this more general setting. Our work contributes to the expanding field of functional analysis in non-locally convex spaces, following the foundational research by , E. Dahia, F. Latreche, G. Rano, T. Bag and A. Mazouz et al. on linear and multilinear operators in quasi-normed and asymmetric normed spaces.

Most important Contributions

We have successfully extended most usual theorems of functional analysis to the multivalued linear operators in quasi-Banach spaces framework:

a) Established new versions of the uniform boundedness principle, closed graph theorem, and open mapping theorem for linear relations.

b) Developed a comprehensive theory of multilinear operators in quasi-Banach spaces, extending continuity and boundedness notions while preserving completeness properties.

c) Investigated the minimum modulus and continuous selections for linear relations, showing succinct tools for the analysis of these multivalued operators.

d) Constructed the theory of Lipschitz operators in quasi-Banach spaces, including the Arens-Eells p-space formalism, adjoint theory, and the new concept of compact Lipschitz operators.

e) Extended the theory of two-Lipschitz left-hand quotient operator ideals to metric and Banach spaces, identifying Grothendieck and Rosenthal two-Lipschitz operators as concrete examples and introducing the structural operator ideals.

Perspectives and Future Research

Several promising directions are posed related to this work:

1. **Spectral Theory:** The adjoint theory for Lipschitz operators developed in Chapter 4 provides the foundation for a spectral theory of Lipschitz operators in quasi-Banach spaces. Future work could investigate the essential spectrum and Fredholm properties in this setting.

2. **Approximation Properties:** The characterization of compact Lipschitz operators suggests a deeper investigation into approximation properties of quasi-Banach spaces, potentially extending the work of Kalton on the bounded approximation property to the Lipschitz setting.

3. **Multilinear Relations:** While we have focused on linear relations and multilinear operators, the natural next step is the study of multilinear relations in quasi-normed spaces, extending the recent work from normed to quasi-normed spaces.

4. **Quotient Ideals without LP:** As noted in Chapter 5, the open problem regarding quotients that do not satisfy the linearization property—exemplified by $(p; p_1, p_2)$-summing two-Lipschitz operators-represents a significant challenge. Resolving this would require new techniques for handling the intrinsic asymmetry of multilinear operator ideals.

5. **Two-Lipschitz operators between quasi-metric spaces:** We extend the theory of two-Lipschitz operators to the p-metric setting by establishing a theory of two-Lipschitz operator ideals.

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Résumé

Cette thèse développe la théorie des opérateurs entre les espaces quasi-Banach selon trois axes. Nous étendons d'abord les théorèmes fondamentaux (théorème de Banach Steinhaust, graphe fermé) aux relations linéaires multivoques. Ensuite, nous établissons une théorie de continuité pour les opérateurs multilinéaires. Enfin, nous étudions les opérateurs non linéaires dans les espaces p-normés, en particulier les opérateurs de Lipschitz, et caractérisons les idéaux d'opérateurs deux-Lipschitz quotients à gauche $\mathcal{A}^{-1} \circ \mathcal{A}_{BLip}$, identifiant des classes importantes comme les opérateurs de Grothendieck et Rosenthal. Nos résultats relient la théorie linéaire et non linéaire .

Mots-clés : Espaces quasi-Banach, espaces p-normés, relations linéaires multivoques, opérateurs multilinéaires, opérateurs de Lipschitz, opérateurs deux-Lipschitz, idéaux d'opérateurs, idéaux quotients, opérateurs de Grothendieck, opérateurs de Rosenthal.

الملخص

تطور هذه المذكرة نظرية المؤثرات بين فضاءات شبه الباناخ على ثلاثة محاور. أولاً، نوسع النظريات الأساسية (امُبرهنة باناخ-شتاينهاوس ، الرسم البياني المغلق، ا) إلى العلاقات الخطية متعددة القيم. ثانياً، نؤسس نظرية الاستمرارية للمؤثرات لخطية المتعددة القيم. أخيراً، ندرس المؤثرات غير الخطية في الفضاءات p-النظامية، وخاصة مؤثرات ليبشيتز، ونميز مثالي حاصل قسمة مؤثرات اثنان-ليبشيتز من اليسار $\mathcal{A}^{-1} \circ \mathcal{A}_{BLip}$، مع تحديد الأصناف المهمة مثل مؤثرات غروتنديك وروزنتال. تربط نتائجنا بين النظرية الخطية وغير الخطية .

الكلمات المفتاحية: فضاءات شبه الباناخ، فضاءات p-النظامية، العلاقات الخطية متعددة القيم، المؤثرات المتعددة الخطية، مؤثرات ليبشيتز، مؤثرات ليبشيتز ثنائية الاتجاه، المثلثات المثالية، المثلثات الناتجة، مؤثرات غروتنديك، مؤثرات روزنتال.

Abstract

This thesis develops operator theory in quasi-Banach spaces along three axes. First, we extend fundamental theorems (Banach Steinhaus theorem, closed graph, open mapping) to multivalued linear relations. Second, we establish continuity theory for multilinear operators. Finally, we study nonlinear operators in p-normed spaces, particularly Lipschitz operators, and characterize two-Lipschitz left-hand quotient operator ideals $\mathcal{A}^{-1} \circ \mathcal{A}_{BLip}$, identifying important classes such as Grothendieck and Rosenthal operators. Our results bridge linear and nonlinear theory.

Keywords: Quasi-Banach spaces, p-normed spaces, multivalued linear relations, multilinear operators, Lipschitz operators, two-Lipschitz operators, operator ideals, quotient ideals, Grothendieck operators, Rosenthal operators.