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Chebyshev Polynomials For Linear Ordinary Differential Equations

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Dedication

I dedicate this thesis to:

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***My whole big family.**

***All my friends.**

***To all departement teachers of mathematical and numerical analysis, to all Master students of 2022.**

ملخص:

إن الهدف الرئيسي من هذا العمل هو تقديم طريقة عددية لإيجاد حلول تقريبية للمعادلات التفاضلية العادية من الدرجة الثانية. وذلك باستخدام كثيرات حدود تشيبيشيف. ثم تطبيق هذه الطريقة على بعض الأمثلة لمقارنة النتائج مع الحل الحقيقي و النتائج المحصل عليها باستخدام طرق عددية أخرى لرؤية مدى دقة و كفاءة هذه الطريقة.

الكلمات المفتاحية: فضاء هيلبرت, معادلات تفاضلية خطية, كثيرات حدود تشيبيشيف, طريقة التقريب.

Abstract:

The main objective of this work is to present a numerical method for finding approximate certain solutions of second-order ordinary differential equations. using Chebyshev Polynomials. Then, applying this method on some examples for comparing the results with exact solution and the obtainable results using other numerical methods to show the accuracy and the effectiveness of this method.

Keywords: Hilbert space, Linear ordinary differential equations, Chebyshev polynomials, approximation method.

Résumé:

L'objectif principal de ce travail est de présenter une méthode numérique pour trouver certaines solutions approximatives d'équations différentielles ordinaires du second ordre. En utilisant les polynômes de Tchebychev. Il suffit donc d'appliquer cette méthode sur un échantillon d'exemples pour comparer les résultats réalisés avec la solution exacte et les résultats obtenus par d'autres méthodes numériques et ce pour s'assurer de l'efficacité et la fiabilité de cette méthode.

Mots-clés: Espace de Hilbert, équations différentielles ordinaires linéaires, les polynômes de Tchebychev, Méthode d'approximation.

Contents

Introduction	v
1 Basic Concepts	1
1.1 Vector spaces	1
1.2 Normed spaces	2
1.3 Inner product spaces	2
1.4 Complete spaces	3
1.5 Hilbert spaces	3
1.6 Orthogonality	3
2 Ordinary Differential Equations (ODEs)	5
2.1 Introduction	5
2.2 Linear Equations	6
2.3 Homogeneous Linear Equations	6
2.4 General Solution	6
2.5 The classification of Ordinary Differential Equations	7
2.5.1 First-order Ordinary Differential Equations	7
2.5.2 Second-Order Ordinary Differential Equations	8
3 Chebyshev Polynomials	11
3.1 Approximation using Chebyshev Polynomials	11
3.2 The first-kind polynomial $T_n(x)$	11
3.3 Derivatives of $T_n(x)$	13
3.4 Properties of the chebyshev polynomials	14
3.4.1 Rodrigues Formula	14
3.4.2 The generating function	14
3.4.3 Recurrence Relations	14

3.4.4	Orthogonality	15
4	Applications	17
4.1	Diferential equations	17
4.2	Particular Solution of an Ordinary Differential Equation . . .	18
4.3	Illustrations of Chebyshev Polynomial Approximation	19
4.4	An Approximate Particular Solution	22
4.5	Examples of Approximation	25
5	Conclusion	31

Introduction

While, historically, finite difference methods have been and remain the standard numerical technique for solving ordinary differential equations, newer alternative methods can be more effective in certain contexts. In particular we consider here methods founded on orthogonal expansions the so-called spectral and pseudospectral methods with special reference to methods based on expansions in Chebyshev polynomials.

In this thesis, we develop a method for finding approximate particular solutions for second order ordinary differential equations. We use Chebyshev polynomials to approximate the source function and the particular solution of an ordinary differential equation. The derivatives of each Chebyshev polynomial will be represented by linear combinations of Chebyshev polynomials, and hence the derivatives will be reduced and differential equations will become a system of algebraic equations. Another advantage of the method is that it does not need the expansion of Chebyshev polynomials. This method is also compared with an alternative approach for particular solutions. Examples including approximation, particular solution, a class of variable coefficient equation, and initial value problem are given to demonstrate the use and effectiveness of these methods.

The first chapter, entitled to defined Basic Concepts. We present it some the definitions and some properties such as the vector spaces, Normed spaces, inner product spaces ,hilbert spaces and Orthogonality.

In the second chapter we overview of Ordinary Differential Equation (ODE), Then classification of ordinary differential equations first and second order, and we present some properties for them.

In the third chapter, we present the Chebyshev polynomials , in which we give definitions and some properties of Chebyshev polynomials such as the generating formula, the recursive relation and Rodrigues formula, as well as orthogonality.

In the fourth chapter, We shall present the approximate particular solutions for second order ordinary differential equations. by Chebyshev polynomial approximation and the reduction of order for the derivatives of Chebyshev polynomials.

Finally, In the fifth chapter we present a Conclusion.

Chapter 1

Basic Concepts

1.1 Vector spaces

Definition 1 *By a vector space we mean a nonempty set E with two operations:*

$(x, y) \mapsto x + y$ from $E \times E$ into E called addition,

$(\lambda, x) \mapsto \lambda x$ from $F \times E$ into E called multiplication by scalars,

such that the following conditions are satisfied for all $x, y, z \in E$ and $\alpha, \beta \in F$:

(a) $x + y = y + x$;

(b) $(x + y) + z = x + (y + z)$;

(c) For every $x, y \in E$ there exists a $z \in E$ such that $x + z = y$;

(d) $\alpha(\beta x) = (\alpha\beta)x$;

(e) $(\alpha + \beta)x = \alpha x + \beta x$;

(f) $\alpha(x + y) = \alpha x + \alpha y$;

(g) $1x = x$.

Elements of E are called vectors. If $F = \mathbb{R}$, then E is called a real vector space, and if $F = \mathbb{C}$, E is called a complex vector space.

1.2 Normed spaces

Definition 2 (Norm)

A function $x \mapsto \|x\|$ from a vector space E into $\mathbb{R}$ is called a norm if it satisfies the following conditions:

- (a) $\|x\| = 0$ implies $x = 0$;
- (b) $\|\lambda x\| = |\lambda| \|x\|$ for every $x \in E$ and $\lambda \in F$;
- (c) $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in E$.

Definition 3 (Normed spaces)

A vector space with a norm is called a normed space.

A normed space is sometimes written $\{E, \|\cdot\|\}$, which emphasizes well the two-part nature of a normed space.

Definition 4 (Convergence in a normed space)

Let $(E, \|\cdot\|)$ be a normed space. We say that a sequence (x_n) of elements of E converges to some $x \in E$,

if for every $\varepsilon > 0$ there exists a number M such that for every $n \geq M$ we have $\|x_n - x\| < \varepsilon$. In such a case we write $\lim_{n \rightarrow \infty} x_n = x$ or simply $x_n = x$.

1.3 Inner product spaces

Definition 5 Let E be a complex vector space. A mapping $\langle \cdot, \cdot \rangle : E \times E \longrightarrow \mathbb{C}$ is called an inner product in E if for any $x, y, z \in E$ and $\alpha, \beta \in \mathbb{C}$ the following conditions are satisfied:

- (a) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (the bar denotes the complex conjugate);
- (b) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$;
- (c) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ implies $x = 0$

Definition 6 A vector space with an inner product is called an **inner product space**.

1.4 Complete spaces

Definition 7 (Cauchy sequence) Let (X, d) be a metric space. We say that a sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ is a Cauchy sequence if for all $\epsilon > 0$, there exists an $N_\epsilon \in \mathbb{N}$ such that for all $n, m \geq N_\epsilon$, $d(x_n, x_m) < \epsilon$ or in short (and not completely precisely) if

$$\lim_{n, m \rightarrow +\infty} d(x_n, x_m) = 0$$

Remark 8 Every convergent sequence is Cauchy, but the converse is not true.

Definition 9 We say that a metric space (X, d) is complete if every Cauchy sequence in X has a limit in X , i.e., every Cauchy sequence is convergent.

1.5 Hilbert spaces

Definition 10 A complete inner product space is called a Hilbert space by completeness of an inner product space E , we mean completeness of E as a normed space with the norm defined by inner product.

Example 11 Since $\mathbb{C}$ is complete, it is a Hilbert space, and so is $\mathbb{C}^N$.

1.6 Orthogonality

Definition 12 (Orthogonal and orthonormal systems)

A family S of nonzero vectors in an inner product space E is called an orthogonal system if $x \perp y$ for any two distinct elements of S . If, in addition, $\|x\| = 1$ for all $x \in S$, then S is called an orthonormal system.

Every orthogonal set of nonzero vectors can be normalized: If S is an orthogonal system, then $S_1 = \left\{ \frac{x}{\|x\|} \mid x \in S \right\}$ is an orthonormal system.

Definition 13 (Orthogonality) If x, y are vectors in a Hilbert space H , then we say that x and y are orthogonal, written

$$x \perp y, \text{ if } \langle x, y \rangle = 0.$$

We say that subsets A and B are orthogonal, written $A \perp B$, if $x \perp y$ for every $x \in A$ and $y \in B$. The orthogonal complement $A^\perp$ of a subset A is the set of vectors orthogonal to A ,

$$A^\perp = \{x \in H \mid x \perp y \text{ for all } y \in A\}$$

Theorem 14 *The orthogonal complement of a subset of a hilbert space is a closed linear subspace.*

Chapter 2

Ordinary Differential Equations (ODEs)

2.1 Introduction

An equation involving the derivatives of an unknown function y of a single variable x over an interval $x \in (I)$. More clearly and precisely speaking, a well defined ODE must the following features:

It can be written in the form:

$$F [x, y, y', y'', \dots, y^n] = 0; \quad (2.1)$$

where the mathematical expression on the right hand side contains

1. variable x .
2. function y of x .
3. some derivatives of y with respect to x .

The values of variables x, y must be specified in a certain number field, such as $\mathbb{N}$, $\mathbb{R}$, or $\mathbb{C}$;

The variation region of variable x of Eq2.1. must be specified, such as $x \in (a, b)$.

2.2 Linear Equations

A differential equation of the order n is linear if and only if on the form,

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + \cdots + a_n(x)y^{(n)} = f(x).$$

2.3 Homogeneous Linear Equations

The linear differential equation is homogeneous, if and only if $f(x) \equiv 0$. Linear homogeneous equations have the important property that linear combinations of solutions are also solutions. In other words, if $y_1, y_2, \dots, y_m$ are solutions and $c_1, c_2, \dots, c_m$ are constants then

$$y = c_1y_1 + c_2y_2 + \dots + c_my_m \tag{2.2}$$

is also a solution.

2.4 General Solution

It represents the set of all solutions, i.e., the set of all functions which satisfy the equation in the interval (I) .

For example, given the differential equation

$$y' = 3x^2$$

Its general solution is

$$y = x^3 + C.$$

where C is an arbitrary constant. To select a specific solution, one needs to determine the constant C with some additional conditions. For instance, the constant C can be determined by the value of y at $x = 0$. This condition is called the initial condition, which completely determines the solution. More generally, it will be shown in the future that given a, b there is a unique solution y of the differential equation with the initial condition $y(a) = b$. Geometrically, this means that the one-parameter family of curves $y = x^3 + C$ do not intersect one another and they fill up the plane $\mathbb{R}^2$.

2.5 The classification of Ordinary Differential Equations

2.5.1 First-order Ordinary Differential Equations

Recall that the general form of an ODE of order n is

$$f(x, y, y', y'', \dots, y^n) = 0, \quad x \in [a, b], \quad (2.3)$$

If we choose $n = 1$, then the equation

$$f(x, y, y') = 0, \quad x \in [a, b], \quad (2.4)$$

is a first-order ordinary differential equation. Since (2.4) is of order one, we may guess that the solution (if it exists) contains a constant c . That is, the family of solutions will be of the form

$$F(x, y, c) = 0. \quad (2.5)$$

Let us write (eq2.4) as

$$M(x, y)dx + N(x, y)dy = 0 \quad (2.6)$$

1- Linear Differential Equations

A first-order linear differential equation has the form:

$$a(x)y' + b(x)y = f(x) \quad (2.7)$$

where $a(x)$, $b(x)$, $f(x)$ are continuous functions of x on some interval (I) .

To bring it to normal form $y' = f(x, y)$ we have to divide both sides of the equation by $a(x)$. This is possible only for those x where $a(x) \neq 0$. After possibly shrinking (I) we assume that $a(x) \neq 0$ on (I) . So our equation has the form (standard form)

$$y' + p(x)y = q(x)$$

with

$$p(x) = b(x)/a(x), \quad q(x) = f(x)/a(x),$$

both continuous on (I) . Solving for y' we get the normal form for a linear first order ODE, namely

$$y' = q(x) - p(x)y.$$

2- Separable Equations

A separable first order ODE has the form

$$M(x)dx + N(y)dy = 0 \quad (2.8)$$

Then we say the variables are separable, and the solution can be obtained by integrating both sides with respect to x . Thus,

$$\int M(x)dx + \int N(y(x))\frac{dy}{dx}dx = c,$$

or for simplicity one can write,

$$\int M(x)dx + \int N(y)dy = c,$$

3- Linear Homogeneous Equations

The linear Differential Equations is homogeneous, if and only if $f(x) = 0$

$$a(x)y' + b(x)y = 0$$

2.5.2 Second-Order Ordinary Differential Equations

Second-order ODEs explicitly contain a second derivative term, but no higher derivatives. These equations are of the form

$$f(x, y, y', y'') = 0$$

The quantities x, y, y' may not appear explicitly in a second-order ODE such as in the equation, $y'' = 0$. A function satisfying the equation $f(x, y, y', y'') = 0$ is called a solution. It can be verified by substituting the solution into the ODE.

A second-order differential equation. For the first-order ODEs, we found that their solutions contain one arbitrary constant, but for the second-order ODEs, the solutions must contain two arbitrary constants because two integrations are required to obtain these solutions. In general, finding the family of solutions for $f(x, y, y', y'') = 0$ is difficult and in most cases impossible, in particular when the equation is nonlinear. For this reason and the fact that linear equations are very often used in engineering and in the applied sciences, in this chapter we will deal with linear ODEs.

2.5. THE CLASSIFICATION OF ORDINARY DIFFERENTIAL EQUATIONS 9

1-Linear Differential Equations

A second-order ODE is called linear if it is written

$$a(x)y'' + b(x)y' + c(x)y = f(x) \quad (2.9)$$

This equation is linear in y and its derivatives.

Here, a, b, c and f are any given functions of x on some interval (I) .

2- Linear Differential Equations with Constant Coefficients

We have emphasized that there are no general methods for solving second order linear differential equations. However, there are some special cases for which solution methods do exist. In this and the following sections we consider such a case, linear equations with constant coefficients.

Linear differential equations of the 2ⁿd order with coefficients constants is a differential equation of the form

$$ay'' + by' + cy = f(x). \quad (2.10)$$

where $f(x)$ is a continuous function, a, b and c are given constants, and $a \neq 0$.

3- Homogeneous Equations

1) Consider a second-order homogeneous linear equation in the general form:

$$a(x)y'' + b(x)y' + c(x)y = 0. \quad (2.11)$$

where a, b, c , and f are any given functions of x on some interval (I) .

The trivial solution, $y = 0$, is a particular solution of the homogeneous linear equation.

Let $y_1(x), y_2(x)$ be a fundamental system of solutions (nontrivial linearly independent particular solutions) of equation (2.12). Then the general solution is given by:

$$y = C_1y_1(x) + C_2y_2(x). \quad (2.12)$$

where C_1 and C_2 are arbitrary constants.

2) Let $y_1 = y_2(x)$ be any nontrivial particular solution of equation (2.9). Then its general solution can be represented as:

$$y = y_1 \left(C_1 + C_2 \int \frac{e^{-F}}{y_1^2} dx \right), \text{ where } F = \int \frac{b(x)}{a(x)} dx. \quad (2.13)$$

4- Non homogeneous Equations

A second-order nonhomogeneous linear equation has the form

$$a(x) y'' + b(x) y' + c(x) y = f(x)$$

where a, b, c , and f are any given functions of x on some interval (I) .

The objectives of this section are to determine the “structure” of the set of solutions

of (1).

As we shall see, there is a close connection between equation (2.9) and

$$a(x) y'' + b(x) y' + c(x) y = 0$$

In this context, equation (2.11) is called the reduced equation of equation (2.9).

Remark 15 *This result illustrates why the emphasis is on linear homogeneous equations. To find the general solution of the nonhomogeneous equation (2.9) we need a fundamental set of solutions of the reduced equation (2.11) and one particular solution of 2.9.*

Chapter 3

Chebyshev Polynomials

3.1 Approximation using Chebyshev Polynomials

The Chebyshev polynomials of the first and second kind are denoted by $T_n(x)$ and $U_n(x)$ respectively. The subscript n is the degree of these polynomials.

The Chebyshev polynomials of the first and second kind are closely related. For example, a Chebyshev polynomial of first kind can be represented as a linear combination of two Chebyshev polynomials of second kind ,

$$T_n(x) = \frac{1}{2} (U_n(x) - U_{n-2}(x)) ,$$

and the derivative of a Chebyshev polynomial of first kind can be written in terms of a Chebyshev polynomial of second kind.

$$T'_n(x) = nU_{n-1}(x), \quad n = 1, 2, \dots$$

In this chapter, we direct our attention to the Chebyshev polynomials of first kind.

3.2 The first-kind polynomial $T_n(x)$

Definition 16 *The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n , defined by the relation*

$$T_n(x) = \cos n\theta \text{ when } x = \cos \theta \tag{3.1}$$

If the range of the variable x is the interval $[-1, 1]$, then the range of the corresponding variable θ can be taken as $[0, \pi]$. These ranges are traversed in opposite directions, since $x = -1$ corresponds to $\theta = \pi$ and $x = 1$ corresponds to $\theta = 0$.

It is well known (as a consequence of de Moivre's Theorem) that $\cos n\theta$ is a polynomial of degree n in $\cos \theta$, and indeed we are familiar with the elementary formulae

$$\begin{aligned}\cos 0\theta &= 1, \\ \cos 1\theta &= \cos \theta. \\ \cos 2\theta &= 2 \cos^2 \theta - 1, \\ \cos 3\theta &= 4 \cos^3 \theta - 3 \cos \theta, \\ \cos 4\theta &= 8 \cos^4 \theta - 8 \cos^2 \theta + 1, \dots\end{aligned}$$

We may immediately deduce from 3.1, that the first few Chebyshev polynomials are

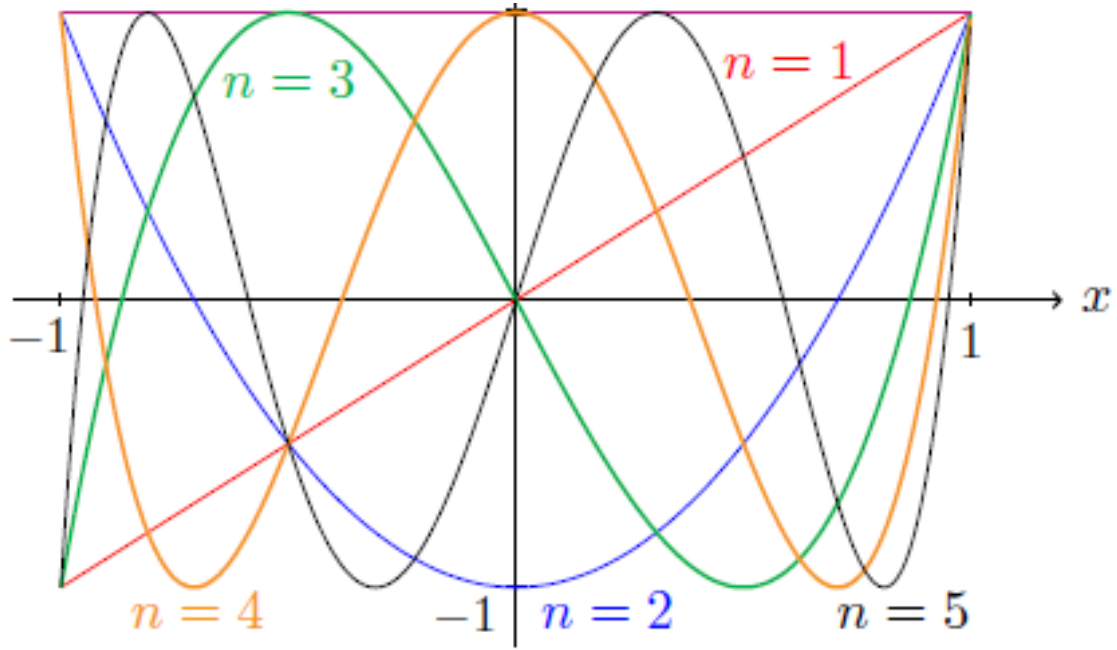
$$\begin{aligned}T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1, \dots\end{aligned}$$

In practice it is neither convenient nor efficient to work out each $T_n(x)$ from first principles.

Rather by combining the trigonometric identity

$$\cos n\theta + \cos (n-2)\theta = 2 \cos \theta \cos (n-1)\theta$$

Graphs of $T_n(x)$ for $n = 1$ to $n = 5$



Plot of Chebyshev polynomials $T_n(x)$ for $n = 0, 1, 2, \dots, 5$.

3.3 Derivatives of $T_n(x)$

The formula for the derivative of $T_n(x)$ in terms of first-kind polynomials is not quite as simple but There is relatively simple formula which we can obtain as follows:

$$\begin{aligned}
 \frac{d}{dx} T_n(x) &= \frac{d}{d\theta} (\cos n\theta) / \frac{d}{d\theta} \cos \theta \\
 &= \frac{n \sin(n\theta)}{\sin \theta} \\
 &= \frac{\frac{1}{2}n (\cos(n-1)\theta - \cos(n+1)\theta)}{\sin^2 \theta} \\
 &= \frac{\frac{1}{2}n (T_{n-1}(x) - T_{n+1}(x))}{1 - x^2}
 \end{aligned}$$

Thus, for $|x| \neq 1$,

$$\frac{d}{dx}T_n(x) = \frac{n}{2} \frac{T_{n-1}(x) - T_{n+1}(x)}{1-x^2}$$

Higher derivatives may be obtained by similar formulae.

3.4 Properties of the chebyshev polynomials

3.4.1 Rodrigues Formula

The Chebyshev polynomials $T_n(x)$ can be expressed as Rodrigues Formula in terms of derivatives

$$T_n(x) = \frac{(-2)^n n! \sqrt{1-x^2}}{(2n)!} \frac{d^n}{dx^n} (1-x^2)^{n-\frac{1}{2}}$$

with $n = 0, 1, 2, 3, 4, \dots$

3.4.2 The generating function

The generating function for Chebyshev polynomials is given as follows

$$\sum_{n=0}^{\infty} t^n T_n(x) = \frac{1-xt}{1-2xt+t^2}$$

The proof consists of letting $x = \cos\theta$ and taking the real part of both sides of the geometric series

$$\sum_{n=0}^{\infty} t^n e^{in\theta} = \frac{1}{1-te^{i\theta}}.$$

3.4.3 Recurrence Relations

There is a simple recurrence relation between the Chebyshev polynomials which is most easily derived using the trigonometric identity

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)$$

A more general recurrence relation isnce
of the power series will be 1. On substituting this power series into the differential equation, we find

$$(1 - x^2) \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} n a_n x^{n-1} + \lambda \sum_{n=0}^{\infty} a_n x^n = 0.$$

If we equate the coefficients of x_n to zero we get

$$(n+2)(n+1) a_{n+2} + (\lambda - n^2) a_n = 0.$$

$$T_{n+m}(x) + T_{n-m}(x) = 2T_n(x) T_m(x).$$

3.4.4 Orthogonality

The Chebyshev polynomials $\{T_n(x)\}$ form a complete orthogonal set on the interval $[a, b] = [-1, 1]$ with a weighted function $w(x) = (1 - x^2)^{-\frac{1}{2}}$.

This means

$$\int_{-1}^1 (1 - x^2)^{-\frac{1}{2}} T_n(x) T_m(x) dx = \begin{cases} 0, & n \neq m \\ \pi, & n = m = 0 \\ \frac{\pi}{2}, & n = m = 1, 2, 3, \dots \end{cases}$$

Using this orthogonality property, a piecewise continuous function on interval $[-1, 1]$ can be expressed in terms of the Chebyshev polynomials

$$\sum_{n=0}^{\infty} C_n T_n(x) = \begin{cases} f(x), & \text{for continuous } f(x) \\ \frac{1}{2} [f(x^-) + f(x^+)], & \text{at discontinuous points} \end{cases}$$

where

$C_0 = \frac{1}{\pi} \int_{-1}^1 (1 - x^2)^{-\frac{1}{2}} f(x) dx$ and $C_n = \frac{2}{\pi} \int_{-1}^1 (1 - x^2)^{-\frac{1}{2}} f(x) T_n(x) dx$
for $n \in \mathbb{N}$.

Chapter 4

Applications

The main idea of this chapter is the weierstrass theorem.

Theorem 17 (weierstrass) *for all continuous functions on $[a, b]$ there exists a polynomials $p_n(x)$ such that :*

$$\forall \varepsilon > 0 \quad |p_n(x) - f(x)| < \varepsilon \quad , \quad \forall x \in [a, b]$$

we consider a linear second order ordinary differential equation (ODE),

$$ay'' + by' + cy = f(x) , \tag{4.1}$$

where $f(x)$ is a continuous function, a, b and c are given constants, and $a \neq 0$.

We are interested to find a particular solution y_p for 4.1

4.1 Differential equations

Chebyshev polynomials of the first kind satisfy the ordinary differential equation:

$$(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + \lambda y = 0$$

The point $x = 0$ is an ordinary point. This means that we can express the

solution in the form of a power series $y = \sum_{n=0}^{\infty} a_n x^n$. The singularities of the differential equation are at $x = \pm 1$ and so the radius of converge

This leads to the recurrence relation:

$$\frac{a_{n+1}}{a_n} = \frac{n^2 - \lambda}{(n+2)(n+1)}$$

We see that there are two power series solutions, one series containing only even powers of x and the other an odd power series.

4.2 Particular Solution of an Ordinary Differential Equation

We hope to be able to find a particular solution for 4.1 that is nonhomogeneous. A particular solution of 4.1 is a function that satisfies 4.1.

The particular solution to an ordinary differential equation can be obtained by assigning numerical values to the parameters in the general solution [3]. We note that there are many possible answers for a particular solution.

A particular solution y_p will allow us to reduce, for example, an initial value problem.

$$\begin{cases} ay''(x) + by'(x) + cy(x) = f(x), \\ y(x_0) = y_0, y'(x_0) = y_1, \end{cases}$$

to a homogeneous equation that is subject to a different initial data.

$$\begin{cases} ay_h''(x) + by_h'(x) + cy_h(x) = f(x), \\ y_h(x_0) = y_0 - y_p(x_0), y_h'(x_0) = y_1 - y_p'(x_0), \end{cases}$$

with $y = y - y_p$. The method of particular of solutions (MPS) allows us to split the solution y into a particular solution y_p and a homogeneous solution y_h .

In this thesis, we use Chebyshev polynomials for approximating equations and their particular solutions. Chebyshev polynomials [1] of the first kind are solutions to the Chebyshev differential equations

$$(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + n^2 y = 0 \text{ for } |x| < 1$$

If the right hand side function $f(x)$ in 4.1 is a polynomial of degree n .

$$\begin{aligned} f(x) &= p_n(x) \\ &= d_n x^n + d_{n-1} x^{n-1} + \dots + d_1 x + d_0 \end{aligned}$$

where $d_n \neq 0$, we can find a particular solution of 4.1 that is a polynomial.

If $f(x)$ is not a polynomial, we approximate it using Chebyshev polynomials.

We then look for a particular solution that is expressed as a linear combination of Chebyshev polynomials.

4.3 Illustrations of Chebyshev Polynomial Approximation

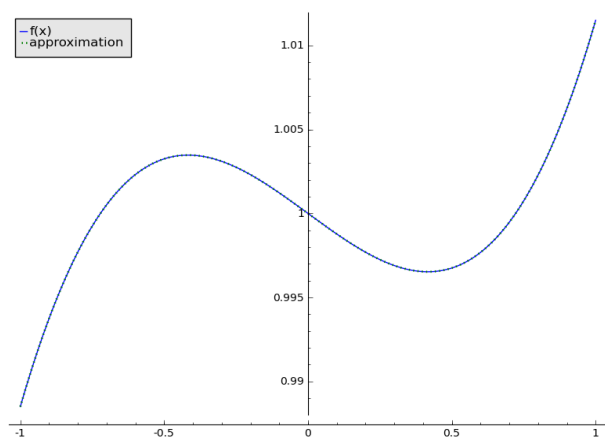
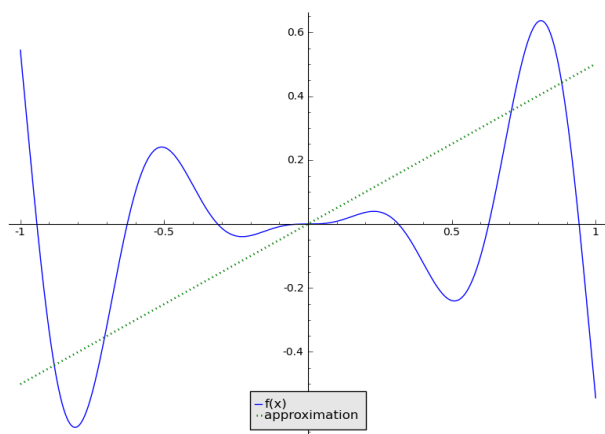
Let P_n be the n^{th} degree polynomial approximation to the function $f(x)$.

Example 18 let $f(x) = \frac{3}{125}x^3 - \frac{x}{80} + 1$

the function $f(x)$ and its Chebyshev polynomial approximation P_3 are shown in Figure 2. The approximation polynomial is identical to the function $f(x)$ as we expected.

Example 19 let $f(x) = x^2 \sin(10x)$

Its Chebyshev polynomial approximation P_n with $n = 1, 4, 9, 16$ are shown in Figures 3-6. As n increases, the error of the approximation polynomial decreases.

Figure 2: Plot of P_3 Figure 3: Plot of $f(x)$ with T_1

4.3. ILLUSTRATIONS OF CHEBYSHEV POLYNOMIAL APPROXIMATION21

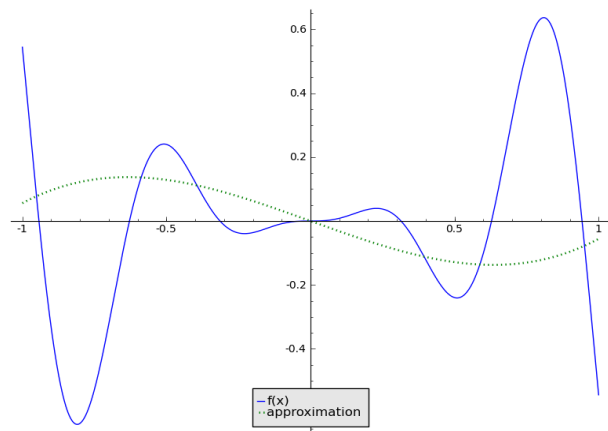


Figure 4: the plot of $f(x)$ with T_4

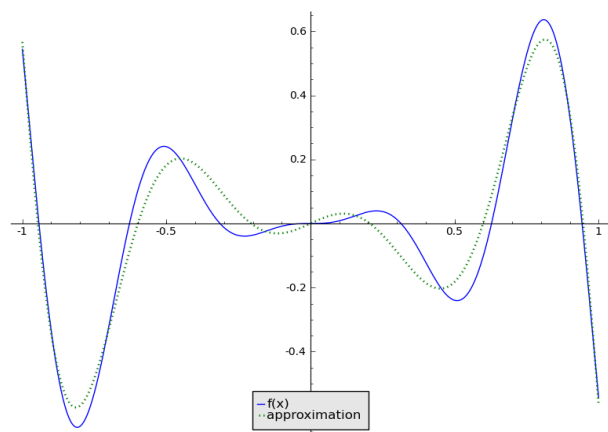


Figure 5: the plot of $f(x)$ with T_9

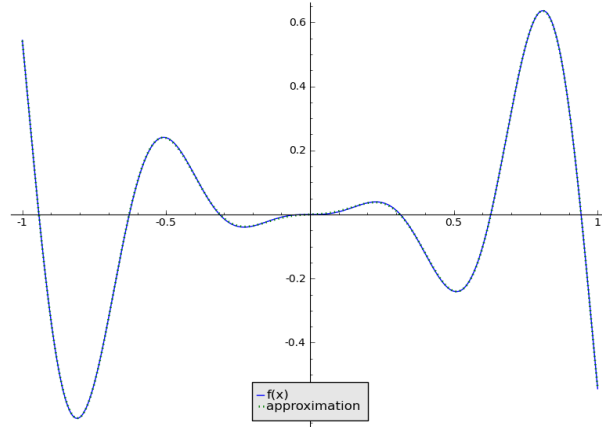


Figure 6: the plot of $f(x)$ with T_{16}

4.4 An Approximate Particular Solution

in the equation

$$ay'' + py'(x) + qy(x) = f(x)$$

We first approximate $f(x)$ by $P_n(x)$ using Chebyshev polynomials. Then we find a particular solution y_p of the equation

$$y'' + py'(x) + qy(x) = p_n(x) \quad (4.2)$$

We let the particular solution be in the form of

$$y_p = \sum_{j=0}^m q_j T_j(x). \quad (4.3)$$

The coefficients q_j for (4.3) are to be determined. We let $m=n$ if $c \neq 0$ in (equ 4.2) $m=n+1$ if $c=0$, $b \neq 0$, $m = n + 1$ if $c=0$, $b=0$.

Substituting (4.3) into (4.2)

$$\sum_{j=0}^m q_j T_j''(x) + \sum_{j=0}^m q_j T_j'(x) + \sum_{j=0}^m q_j T_j(x) = p_n(x)$$

Our next goal is to use linear combinations of Chebyshev polynomials to represent $T'_j(x)$ and $T''_j(x)$ for each j . That is the first and second order derivatives of Chebyshev polynomials $T'_j(x)$ and $T''_j(x)$ are represented in terms of $T_k(x)$, $k=0, \dots, j$. Then we arrive at a system of algebraic equations of q_j by equating coefficients to find the solution y_p given by (10).

According to [8] and its tables for representation coefficients, we reduce the first and second order derivatives as follows

$$T'_j(x) = \sum_{k=0}^{j-1} b_k T_k(x) \quad (4.4)$$

where

$$b_{2l} = 0, \text{ for } l = 0, 1, \dots, \frac{j}{2} - 1,$$

$$b_{2l+1} = 2j, \text{ for } l = 0, 1, \dots, \frac{j}{2} - 1,$$

for even j , and

$$\begin{aligned} b_0 &= j, \\ b_{2l} &= 2j, \text{ for } l = 0, 1, \dots, \frac{j-1}{2}, \\ b_{2l+1} &= 0, \text{ for } l = 0, 1, \dots, \frac{j-3}{2}, \end{aligned}$$

for odd j .

Next, we expand the second order derivative $T''_j(x)$ in terms of Chebyshev polynomials $T_i(x)$, $i = 0, \dots, j-2$. With c_i being the representation coefficients for $T''_j(x)$ that is

$$T''_j(x) = \sum_{i=0}^{j-2} c_i T_i(x). \quad (4.5)$$

$$\begin{aligned}
T_j''(x) &= \left(T_j'(x) \right)' \\
&\quad \left(\sum_{k=0}^{j-1} b_k T_k(x) \right)' \\
&\quad \sum_{k=0}^{j-1} b_k T_k'(x) \\
&\quad \sum_{k=0}^{j-1} b_k \left(\sum_{i=0}^{k-1} b_i T_i(x) \right) \\
&\quad \sum_{k=0}^{j-1} \sum_{i=0}^{k-1} b_k b_i T_i(x)
\end{aligned} \tag{4.6}$$

4.5 Examples of Approximation

Example 20 (1) *we consider the equation*

$$y''(x) + 2y(x) = P_4(x)$$

with $P_4(x) = 2T_1(x) + 5T_4(x)$. We look for a particular solution $y_p = \sum_{j=0}^4 q_j T_j(x)$.

By (4.4)-(4.6), we list the coefficients for $T'_j(x)$, $j = 0, 1, \dots, 4$ in Table 1, which is equivalent to the following,

$$\begin{aligned} (T_0)' &= 0, \\ (T_1)' &= T_0(x), \\ (T_2)' &= 4T_1(x), \\ (T_3)' &= 3T_0(x) + 6T_2(x), \\ (T_4)' &= 8T_1(x) + 8T_3(x). \end{aligned}$$

$j = 0$
$j = 1 \quad b_0 = 1$
$j = 2 \quad b_0 = 0 \quad b_1 = 4$
$j = 3 \quad b_0 = 3 \quad b_1 = 0 \quad b_2 = 6$
$j = 4 \quad b_0 = 0 \quad b_1 = 8 \quad b_2 = 0 \quad b_3 = 8$

Table 1: Coefficients for
 $T'_j(x)$, $j = 0, 1, \dots, 4$

$j = 0$	
$j = 1$	$b_0 = 1,$ $b_0 = 0$
$j = 2$	$b_1 = 4 \cdot b_1 \cdot b_0 = 4 \cdot 1$ $b = 3,$
$j = 3$	$b_1 = 0, b_1 \cdot b_0 = 0 \cdot 1$ $b_2 = 6, b_2 \cdot b_0 = 6 \cdot 0, b_2 \cdot b_1 = 6 \cdot 4$ $b = 0,$ $b_1 = 8, b_1 \cdot b_0 = 8 \cdot 1$
$j = 4$	$b_2 = 0, b_2 \cdot b_0 = 0 \cdot 0, b_2 \cdot b_1 = 0 \cdot 4$ $b_2 = 8, b_3 \cdot b_0 = 8 \cdot 3, b_3 \cdot b_1 = 8 \cdot 0, b_3 \cdot b_2 = 8 \cdot 6$

Table 2: Coefficients for $T_j(x), j = 0, 1, \dots, 4$

Thus,

$$y_p = (q_1 + 3q_3) T_0(x) + (4q_2 + 8q_4) T_1(x) + 6q_3 T_2(x) + 8q_4 T_3(x). \quad (4.7)$$

The coefficients for second order derivatives $T_j''(x), j = 0, 1, \dots, 4$ are listed in Table 2 below,

Now we obtain the second order derivatives for $T_j, j = 0, 1, \dots, 4$. We list the results as follows:

$$\begin{aligned} (T_0)'' &= 0, \\ (T_1)'' &= 0, \\ (T_2)'' &= 4T_0(x), \\ (T_3)'' &= 24T_1(x), \\ (T_4)'' &= 32T_0(x) + 48T_2(x). \end{aligned}$$

We notice, for example, $c_0 = 32, c_1 = 0$, and $c_2 = 48$ for (4.5) when $j = 4$. Therefore,

$$\begin{aligned} y_p'' &= 4q_2 T_0(x) + 24q_3 T_1(x) + 32q_4 T_0(x) + 48q_4 T_2(x) \\ &= (4q_2 + 32q_4) T_0(x) + 24q_3 T_1(x) + 48q_4 T_2(x) \end{aligned}$$

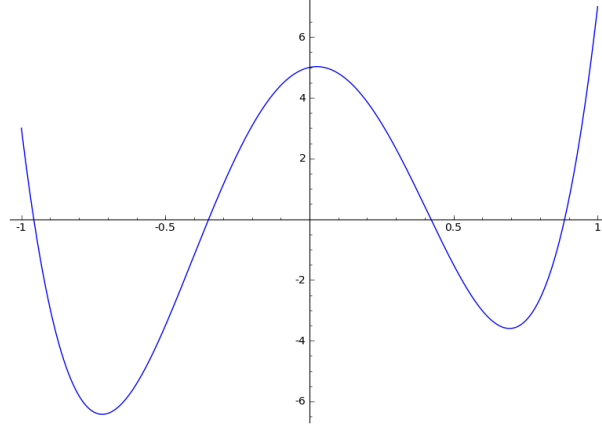


Figure7:Plot of the right hand side function
in Example 1

We obtain the following linear system of equations by comparing coefficients of T_j , $j = 0, 1, \dots, 4$.

$$\begin{cases} 4q_0 + 4q_2 + 32q_4 = 0 \\ 2q_1 + 24q_3 = 2 \\ 2q_2 + 48q_4 = 0 \\ 2q_3 = 0 \\ 2q_4 = 5. \end{cases} \quad (4.8)$$

The solution for (4.8) is $q_4 = \frac{5}{2}$, $q_3 = 0$, $q_2 = -60$, $q_0 = 80$.

The plots of the right hand side function and the particular solution are shown in Figures 7-8.

Assume we use P_n with $n = 4$ to approximate the function $f(x)$ in Eq: $ay'' + py'(x) + qy(x) = f(x)$ Without loss of generality, we assume $c \neq 0$ in (1). Using our method of reduction of order, the coefficients q_j , $j = 0, 1, \dots, 4$ of the particular solution satisfy the following system of equations,

$$cq_0 + bq_1 + 4aq_2 + 3bq_3 + 32aq_4 = \frac{1}{2}C_0, \quad (4.9)$$

$$cq_1 + 4bq_2 + 24aq_3 + 8bq_4 = C_1,$$

$$cq_2 + 6bq_3 + 48aq_4 = C_2,$$

$$cq_3 + 8bq_4 = C_3,$$

$$cq_4 = C_4.$$

Example 21 (2) *In this example, we consider*
 $y''(x) + y'(x) + 1.25y(x) = 3x^3 + x^2 + 2x + 7.$

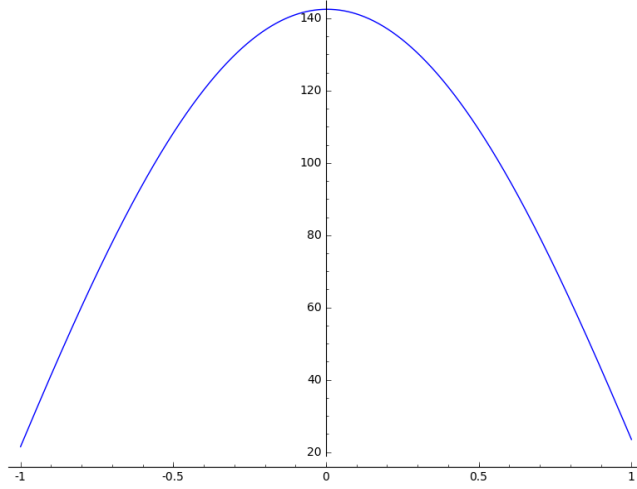


Figure 8: Plot of the particular solution for Example 2.

We notice here the function $f(x) = 3x^3 + x^2 + 2x + 7$. We can still look for a particular solution in the form of $y_p = \sum_{j=0}^4 q_j T_j(x)$, but q_4 will be 0 as expected. The Chebyshev polynomial representation for $f(x)$ is

$$f(x) = \sum_{j=0}^4 p_j T_j(x)$$

with $p_0 = \frac{15}{2}, p_1 = \frac{17}{4}, p_2 = \frac{1}{2}, p_3 = \frac{3}{4}$. According to the system of equations ((4.9), we obtain the following system for $q_j, j = 0, \dots, 4$,

$$1.25q_0 + q_1 + 4q_2 + 3q_3 + 32q_4 = \frac{15}{2},$$

$$1.25q_1 + 4q_2 + 24q_3 + 8q_4 = \frac{17}{4},$$

$$1.25q_2 + 6q_3 + 48q_4 = \frac{1}{2},$$

$$1.25q_3 + 8q_4 = \frac{3}{4},$$

$$1.25q_4 = 0.$$

Using backward substitution, we get $q_4 = 0$, $q_3 = \frac{3}{5}$, $q_2 = \frac{-62}{25}$, $q_1 = \frac{-23}{125}$, $q_0 = \frac{7902}{625}$.

Chapter 5

Conclusion

The Chebyshev polynomials have been in existence for over a hundred years and they have been used for solving many different problems. In this thesis, we have a new strategy using chebyshev polynomials for a common problem. We obtain satisfactory results because of the excellent convergence rate of Chebyshev approximation, which is very close to the minimax polynomial which minimizes the maximum error in approximation. We find approximate particular solutions by Chebyshev polynomial approximation and the reduction of order for the derivatives of Chebyshev polynomials.

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