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Theme

On gap between closed operators

Presented by:
M^r BELHOUT Sami

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In front of the jury composed of:

Heraiz Toufik	M.C.B ,	University of M'sila	**Chairperson**
TALLAB Abdelhamid	M.C.A ,	University of M'sila	**Supervisor**
MAZOUZ Ahmed	M.A.A,	University of M'sila	**Co-Supervisor**
HAMIDI Khaled	M.C.B ,	University of M'sila	**Examiner**

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Dedication

To my remarkable family,

On this momentous occasion of my graduation, I am overwhelmed with profound gratitude for the unwavering support, boundless love, and endless encouragement you have bestowed upon me throughout my university journey. Today is not just a celebration of my individual achievements but a testament to the remarkable strength and unity that defines our family, I dedicate my graduation to all of you.

Acknowledgments

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Last but not least, I would like to express my gratitude to all my teachers and everyone who has directly or indirectly contributed to the realization of this project.

ملخص:

الغرض من هذه الأطروحة هو تقديم الفكرة بين مؤثرين خطيين مغلقين في فضاءات باناخ.
في هذه الأطروحة ، نقدم مفهوم الفجوة بين مؤثرين مغلقين في مساحات باناخ التي قدمها كاتو ونعطي خصائصها. لقد أثبت أنه إذا كان T و S مؤثرين خطيين مغلقين من X إلى Y ، فسيتم قياس المسافةْ بين T و S من حيث الفجوةْ بين الفضاءات الجزئية الخطية (الرسوم البيانية لـ S و T ، $G(T)$ و $G(S)$ للمنتج المعياري للمساحة $X \times Y$.

الكلمات المفتاحية: (المؤثرالخطي، الفجوة، الرسوم البيانية.)

Abstract:

The goal of this memory is to introduce the notion between two closed operators on Banach spaces.
In this memory we present the concept of gap between two closed operators on Banach spaces introduced by Kato and give its proprieties, He proved that if T and S are two closed linear operators from X into Y, the "distance" between T and S will be measured in terms of the "gap" between the linear subspaces (graphs of S and T) $G(T)$ and $G(S)$ of the product normed space $X \times Y$.

Keywords : (Linear operator, Gap, Graph.)

Resumé:

Le but de ce mémoire est d'introduire la notion entre deux opérateurs fermés sur les espaces de Banach.
Dans ce mémoire, nous présentons la notion d'écart entre deux opérateurs fermés sur des espaces de Banach introduite par Kato et donnons ses propriétés. Il a prouvé que si T et S sont deux opérateurs linéaires fermés de X dans Y, la "distance" entre T et S sera mesurée en termes d'"écart" entre les sous-espaces linéaires (graphes de S et T) $G(T)$ et $G(S)$ du produit normé de l'espace $X \times Y$.

Mots-clés: (Opérateur linéaire, Ecart, Graph.)

List of Symbols

Notation	Name
$\mathbb{K}$	the field of reel or complex numbers
$L(X,Y)$	the collection of all linear operators from X into Y
$\mathcal{L}(X,Y)$	the collection of all continuous linear operators from X into Y.
$(\mathcal{L}(X), \Vert.\Vert)$	the space of continuous linear operators from X into X.
$(\mathcal{L}(X,Y), \Vert.\Vert)$	the space of continuous linear operators from X into Y.
T linear operator, T^{-1}	an inverse linear operator of T define from Y into X.
T linear operator, T^*	the adjoint linear operator of T .
T linear operator, $D(T)$	the domain of T .
T linear operator, $N(T)$	the kernel of T .
T linear operator, T^*	the range of T.
T linear operator, $\alpha(T)$	the dimenion of $N(T)$.
T linear operator, $\beta(T)$	the dimenion of $R(T)$.
$\mathcal{U}(X,Y)$	the collection of all unbounded linear operators from X into Y.
$G(T)$	the graph of linear operator T.
$\mathcal{C}(X,Y)$	the collection of all closed linear operators from X into Y.
$M^{\perp}$	is the annihilator of M.
$\hat{\delta}(M,N)$	is the gap topology between two subspaces M and N.
$\hat{\delta}(S,T)$	is the gap topology between two linear operators S and T.

Introduction

Before starting the analysis, we need to introduce the notion of the Gap between linear subspaces and between linear operators. This notion was introduced by M. G. Kreinet M. A. Krasnoselski in [9] at 1940 and T.Kato [8] at 1976, they showed that the notion of Gap between the two closed subspaces of a Banach space is expected to be very useful for a number of studies concerning the perturbation of closed semifredholm operators, in particular for solving the spectral stability problem. Let $\mathcal{H}$ be a Hilbert space and let T and S be two closed linear operators on $\mathcal{H}$. The gap between T and S is a measure of how far apart these two operators are in terms of their respective domains, ranges, and spectra. The gap between T and S is defined as the infimum of all positive real numbers δ such that $\|Tx\| \geq \delta\|Sx\|$ for all $x \in \mathcal{D}(S)$. In other words, the gap between T and S is the smallest positive number δ such that T is δ-dominated by S on its common domain. Geometrically, the gap between T and S measures the size of the largest ball around the origin in $\mathcal{H}$ that is entirely contained in the domain of S and is mapped by T into a ball of radius less than δ in the range of T. Intuitively, the gap between T and S is small when T and S are very similar as linear mappings. The gap between two closed linear operators is an important concept in functional analysis and has many applications in areas such as operator theory, differential equations, and mathematical physics. It allows us to compare and analyze different linear operators on a common space and is often used to establish various types of convergence and stability results in these fields This memory is divided into three chapters. In the first chapter we give some ingredients which used in this work. For the second chapter, we introduce in details, the gap between two closed subspaces, also between two closed and closable operators by using some properties. Finally in the last chapter, we present some application of the gap in the convergence of operators and spectrum.

Chapter 1

Preliminaries

1.1 Continuity of linear operators

Definition 1.1.1. ***(Normed space)***

Let X be a vector space on Field $\mathbb{K}$, a norm on X is a mapping

$$\|\bullet\| : X \longrightarrow \mathbb{K}$$

verifying the following properties:

1. $\|x\| = 0 \Leftrightarrow x = 0, \forall x \in X.$

2.$\|\lambda x\| = |\lambda|.\|x\|, \forall x \in X, \forall \lambda \in \mathbb{K}$.

3.$\|x + y\| \leqslant \|x\| + \|y\|, \forall x, y \in X.$ *and the pair* $(X, \|.\|)$ *is said to be a normed vector space.*

Definition 1.1.2. ***(Complete space)***

A normed vector space $(X, \|.\|)$ *is said to be* ***complete or Banach*** *if all Cauchy sequence is convergent in X for its norm* $\|.\|$.

Lemma 1.1.1. *Any normed vector space* $(X, \|.\|)$ *of finite dimensional is Banach .*

Example 1.1.1. *:* $(C([0, 1], \mathbb{R}), \| \,.\, \|_\infty)$ *is a Banach .*

Definition 1.1.3. ***(Inner product)***

Let X be a $\mathbb{K}$-linear space, An inner product or scalar product on X is a mapping, $< .,. >$: $X \times X \to \mathbb{C}$ *for any pair of vectors x and y in X which* $(x, y) \to < x, y >$ *satisfies the following conditions:*

1) $< x, x > \geqslant 0, \forall x \in X.$

2) $< x, x >= 0 \Leftrightarrow x = 0, \ \forall x \in X.$

3) $< \lambda x, y >= \lambda < x, y >, \forall x, y \in X, \forall \lambda \in \mathbb{C}.$

4) $< x + y, z >=< x, z > + < y, z >, \forall x, y, z \in X.$

5) $\overline{< x, y >} =< y, x >, \forall x, y \in X.$

Definition 1.1.4. ***(Hilbert space)***

A vector space $\mathcal{H}$ over $\mathbb{C}$ endowed with an inner product is called a pre-Hilbert space.

If $(\mathcal{H}, < .,. >)$ *is a pre-Hilbert space, then we define a norm $\|.\|$ on $\mathcal{H}$ by setting:*

$$\|x\| = \sqrt{< x, x >}$$

for all $x \in \mathcal{H}$ In this event, we say that the norm $\|.\|$ is deduced from the inner product $< .,. >$. A pre-Hilbert space $\mathcal{H}$ which is complete is called a Hilbert space.

Definition 1.1.5. ***(Operators)***
Let X and Y be two normed vector spaces. We say that an application T defined on a subset $X_0 \subset X$ in Y is an operator if X_0 is linear subspace of X.

Definition 1.1.6. ***(Linear operator)***
Let $(X, \| \,.\, \|_1)$ and $(Y, \| \,.\, \|_2)$ be two Banach spaces over the same field $\mathbb{K}$. A mapping $T : X \longrightarrow Y$ satisfying $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in D(T)$ and $\alpha, \beta \in \mathbb{K}$, is called a linear operator or a linear transformation. The set of all linear operators from X into Y will be denoted by $L(X, Y)$, when $X = Y$, this is simply denoted by $L(X)$.

Example 1.1.2. *:*
Let :

$$\begin{array}{rccc} T: & C([0,1]) & \longrightarrow & \mathbb{R} \\ & f & \longmapsto & T(f) = \int_0^1 f(x)dx. \end{array}$$

Then, T is a linear operator.

1.2 Bounded linear operators

1.2.1 Bounded linear operator

Definition 1.2.1. ***(Continuous operator)***
Let X and Y be two Banach spaces, T a linear operator defined on subset $D(T) \subset X$ in Y is said to be continuous at point x_0 of $D(T)$ if for any sequence (x_n) of $D(T)$ converges to x_0, the sequence $(T(x_n))_n$ converges to $T(x_0)$ That is :

$$\lim_{n \longrightarrow \infty} T(x_n) = T(\lim_{n \longrightarrow \infty} x_n) = T(x_0)$$

Remark 1.2.1. *The linear operator T is said to be continuous on X if it is continuous at each point of the set X.*

Proof. Let $(x_n)_n \longrightarrow x$,this sequence is written as follows:

$$\begin{aligned} x_n &= [x_0 + (x_n - x)] + (x - x_0) \\ &= y_n + (x - x_0) \end{aligned}$$

and the sequence y_n converge to x_0; and T is a linear operator, then

$$\begin{aligned} T(x_n) &= T([x_0 + (x_n - x)] + (x - x_0)) \\ &= T(y_n) + T(x - x_0) \end{aligned}$$

The operator T is continuous at point x_0; then :

$$\begin{aligned}
\lim_{n\longrightarrow\infty} T(x_n) &= \lim_{n\longrightarrow\infty}[T([x_0+(x_n-x)]+(x-x_0))] \\
&= \lim_{n\longrightarrow\infty} T(y_n)+T(x)-T(x_0) \\
&= T(\lim_{n\longrightarrow\infty} y_n)+T(x)-T(x_0) \\
&= T(x_0)+T(x)-T(x_0) \\
&= T(x)
\end{aligned}$$

then T is continuous on X_0 and $\lim_{n\longrightarrow\infty} T(x_n)=T(x)$. □

Corollary 1.2.1.

Let X, Y, Z be Banach spaces and let $T\in\mathcal{L}(X,Y)$ and $S\in\mathcal{L}(Y,Z)$. Then, $S\circ T\in\mathcal{L}(X,Z)$.

Definition 1.2.2. *Let X be a normed space. Then, the space of all the continuous linear functional from X into $\mathbb{K}$ is called the dual of X and denoted $X^*=\mathcal{L}(X,\mathbb{K})$. Hence X^* is the set of linear forms from X into $\mathbb{K}$.*

Definition 1.2.3.

A linear operator $T: X\longrightarrow Y$ satisfying the following property:there exists $C\geqslant 0$ such that

$$\| T(x) \|\leqslant C \| x \| \tag{1.1}$$

for all $x\in X$, T is called a bounded (or continuous) linear operator. The collection of all bounded linear operators from X into Y will be denoted by $\mathcal{L}(X,Y)$
with $\mathcal{L}(X,X):=\mathcal{L}(X)$.

Example 1.2.1.

1) Let T a Voltera integral operator defined by:

$$\begin{aligned}
T:\ & C([0,1]) \longrightarrow C([0,1]) \\
& f \longrightarrow Tf(x)=\int_0^x f(t)dt
\end{aligned}$$

it is clear that T is lineair and continuous for the norm $\|f\|_\infty=\max_{t\in[0,1]}|f(t)|$
Moreover

$$\|Tf(x)\|\leqslant\|f\|_\infty, \|T\|=1$$

2) Let S a linear operator define from $L^2(0,1)$ into $L^2(0,1)$ by :

$$Sf(x)=xf(x)$$

S is continuous and

$$\|Sf(x)\|_2\leqslant\|f\|_2, \|S\|=1$$

Proposition 1.2.1.

If $T, S \in \mathcal{L}(X,Y)$ and $\alpha \in \mathbb{K}$, then the following properties are hold:
1) $T+S \in \mathcal{L}(X,Y)$;
2) $\alpha T \in \mathcal{L}(X,Y)$.

Proof.
1) Using the linearity of both T and S it follows that :

$$\begin{aligned} (T+S)(\alpha x+\beta y) &= T(\alpha x+\beta y)+S(\alpha x+\beta y) \\ &= \alpha T(x)+\beta T(y)+\alpha S(x)+\beta S(y) \\ &= \alpha(T+S)(x)+\beta(T+S)(y) \end{aligned}$$

Then $T+S \in \mathcal{L}(X,Y)$.
Similarly, using the triangle inequality and the continuity of both T and S,we obtain

$$\begin{aligned} \| (T+S)(x) \| &= \| T(x)+S(x) \| \\ &\leqslant \| T(x) \| + \| S(x) \| \\ &\leqslant C_1 \| x \| + C_2 \| x \| \\ &= C \| x \| \end{aligned}$$

which yields $T+S$ is continuous
2) The proof is obvious and hence is omitted .
by (1) and (2) , $\mathcal{L}(X,Y)$ is a vector space.
3) We will prove it for TS as the proof for TS is quite similar. Using the linearity of both T and S and (2) it follows that

$$\begin{aligned} (TS)(\alpha x+\beta y) &= T[S(\alpha x+\beta y)] \\ &= T[\alpha S(x)+\beta S(y)] \\ &= T[\alpha S(x)]+T[\beta S(y)] \\ &= \alpha(TS)(x)+\beta(TS)(y) \end{aligned}$$

and hence TS is linear. Similarly, using the continuity of both T and S, we obtain that :

$$\begin{aligned} \| (TS)(x) \| &= \| T[S(x)] \| \\ &\leqslant \| C_1 \| . \| S(x) \| \\ &\leqslant C_1.C_2 \| x \| \\ &= C \| x \| \end{aligned}$$

which yields TS is bounded (continuous) and $\| TS \| \leqslant C$.
Moreover, if $T \in \mathcal{L}(X,Y)$, then we define :

$$\| T \| = \sup_{x \in X, x \neq 0} \frac{\| T(x) \|_2}{\| x \|_1} \tag{1.2}$$

Further,by 1.2 it can be shown that :

$$\begin{aligned} \| T \| &= \sup_{x \in X}\{\frac{\| T(x) \|_2}{\| x \|_1}, \| x \|_1 \neq 0\} \\ &= \sup_{x \in X}\{\| T(x) \|_2, \| x \|_1 \leqslant 1\} \\ &= \sup_{x \in X}\{\| T(x) \|_2, \| x \|_1 = 1\} \end{aligned}$$

also , $\| T(x) \|_2 \leqslant \| T \| . \| x \|_2$ for all $x \in X$

Moreover , if $T, S \in \mathcal{L}(X, Y)$:

$$\| TS \| \leqslant \| T \| . \| S \| .$$

And the proof is ended. □

Proposition 1.2.2. *[6] Let X and Y be two normed vector spaces:*

1) $\|.\|$ is a norm on $\mathcal{L}(X;Y)$.

2) If Y is a Banach space then $\mathcal{L}(X;Y)$ is a Banach space.

Theorem 1.2.1. *:*

If $T : X \longrightarrow Y$ is a linear operator, then the following statements are equivalent:

a) T is continuous .

b) T is continuous at 0_X .

c) There exists a constant $C > 0$ such that

$$\| T(x) \|_2 \leqslant C. \| x \|_1$$

for each $x \in X$.

1.2.2 The inverse operator

Definition 1.2.4.

An operator $T \in \mathcal{L}(X)$ is called invertible if there exists $S \in \mathcal{L}(X)$ such that:

$$TS = ST = I$$

S is called the inverse of T and is denoted by T^{-1}, We denote $\mathcal{GL}(X)$ the set of $\mathcal{L}(X)$ inverse operators.

Remark 1.2.2. *The inverse of a bounded operator is not always bounded.*

Proposition 1.2.3. *Let $S \in \mathcal{L}(X, Y)$ and $R \in \mathcal{L}(Y, Z)$*

If S and R are invertible then the operator $T = RS$ is invertible and we have :

$$T^{-1} = S^{-1}R^{-1}.$$

Definition 1.2.5.
Let X and Y be two Banach spaces. If $T \in \mathcal{L}(X,Y)$, then its kernel $N(T)$ and range $R(T)$ are, respectively, defined by,

$$N(T) = \{x \in X : T(x) = 0\}.$$

and

$$R(T) = \{T(x) : x \in X\}.$$

It is clear that $N(T) \subset X$ while $R(T) \subset Y$.

Definition 1.2.6. *Let T be a linear operator, we call,*
(i) The dimension of the kernel of T is denoted by:

$$\alpha(T) := \dim \mathcal{N}(T).$$

(ii) The co-dimension of the image of T is denoted by:

$$\beta(T) := codim \mathcal{R}(T) := \dim Y/\mathcal{R}(T).$$

Remark 1.2.3. *For a linear operator T, we have:*
(i) $\mathcal{D}(T)$ is linear subspace of X.
(ii) $\mathcal{N}(T)$ is linear subspace of X.
(iii) $\mathcal{R}(T)$ is linear subspace of Y.
(iv) $G(T)$ is linear subspace of $X \times Y$.

Remark 1.2.4.
If $T \in \mathcal{L}(X,Y)$, then

- *T is said to be one-to-one, if $N(T) = \{0\}$.*
- *T is said to be onto, if $R(T) = Y$.*
- *T is said to be invertible, if it is both one-to-one and onto.*

1.3 Unbounded linear operators

Definition 1.3.1.
Let $T : D(T) \subset X \longrightarrow Y$ be a linear operator, the set $D(T)$ which is called the domain of T. If T is not continuous, then T is called an unbounded operator, the set of all unbounded linear operators from X into Y will be denoted $\mathcal{U}(X,Y)$, with $\mathcal{U}(X,X) = \mathcal{U}(X)$.

Example 1.3.1. *Let $X = Y = C([0,1])$, an operator T defined from $D(T) = C^1([0,1])$ into $C([0,1])$ by :*

$$Tf(x) = \frac{df}{dx}$$

T is linear but not continuous. Indeed, we consider $f_n(x) = x^n$, $n \in \mathbb{N}$, then

$$\|f_n\|_\infty = \max_{x\in([0,1]}|f_n(x)| = \max_{x\in([0,1]}|x^n| = 1$$

$$\|Tf_n(x)\| = \max_{x\in([0,1]}|nx^{n-1}| = n \underset{n\to\infty}{\longrightarrow} = \infty$$

so, $\|Tf_n(x)\|_\infty \nleq C\|f_n\|_\infty$ then the operator T is not bounded.

Definition 1.3.2.
Let T, S be in $\mathcal{U}(X,Y)$, then S is said to be an extension of T if $T(x) = S(x)$ for all $x \in D(T)$ and $D(T) \subset D(S)$.
Moreover, S is said to be an extension of T if and only if $D(T) \subset D(S)$.

1.4 Closed and closable operators

Definition 1.4.1.
If $T : D(X) \subset X \longrightarrow Y$ is a linear operator, then its graph is defined by :

$$G(T) = \{(x, Tx) \in X \times Y : x \in D(T)\}.$$

Definition 1.4.2.
A linear operator $T : T(X) \subset X \longrightarrow Y$ is said to be closed if its graph $G(T)$ as a subset of $X \times Y$ is closed. The collection of closed linear operators from X into Y is denoted by $\mathcal{C}(X,Y)$. When $X = Y$, this is simply denoted by $\mathcal{C}(X)$.

Proposition 1.4.1.
A linear operator T on Banach space X is closed if for every sequence (x_n) in $D(T)$ such that $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$, we have

$$x \in D(T) \quad and \quad Tx = y.$$

Theorem 1.4.1.
Let X, Y to be Banach spaces, and T is a closed linear operator from X into Y, with $D(T) = X$, then T is bounded ($T \in \mathcal{L}(X,Y)$).

Remark 1.4.1.
Let X, Y to be Banach spaces, and T is a linear operator from X into Y ,

- *If T is closed. Then $N(T)$ is closed.*

- *Let T be an unbounded linear operator in Banach space X with domain $D(T)$, if T is closed, its bounded on $D(T)$ if and only if $D(T)$ is a closed subspace of X.*

Proposition 1.4.2. *Let X, Y be two Banach spaces and T in $\mathcal{L}(X,Y)$, $G(T)$ its graph. If T is invertible, then*

$$G(T) = G(T^{-1}).$$

We denote the graph of T^{-1} by $(G(T))^{-1}$ such that $(G(T))^{-1} = G(T^{-1})$.

Proof. For all $x \in D(T)$, we have $T(x) = y$ such that $T^{-1}(y) = x$ and $y \in Y$, so

$$\begin{aligned} G(T^{-1}) &= \{(y, T^{-1}y), y \in Y\} \\ &= \{(Tx, T^{-1}y), x \in X\} \\ &= \{(Tx, x), x \in X\} \\ &= G(T). \end{aligned}$$

Then $G(T) = G(T^{-1})$. □

Theorem 1.4.2. ***(Closed graph theorem)***
Let T a linear operator X from Y into which X, Y are two Banach spaces, It is assumed that the graph of T, $G(T)$ is closed in $X \times Y$ Then T is continuous.

Definition 1.4.3. ***(Closable operator)***
A linear operator $T : T(X) \subset X \longrightarrow Y$ is said to be closable if the closure $\overline{G(T)}$ of $G(T)$ is a graph .

Proposition 1.4.3. *An operator $T \in \mathcal{L}(X)$ is closable if for every sequence (x_n) of elements in $D(T)$such that $x_n \longrightarrow 0$ and $Tx_n \longrightarrow y$, for some $y \in X$, we have $y = 0$.*

Proposition 1.4.4. *[5]*
Let T and S be two unbounded linear operators acting from X into Y such that $D(T) \subseteq D(S)$ and $T(x) = S(x)$ for all $x \in D(T)$, then S is called an extension of T.

Definition 1.4.4.
An unbounded linear operator $T : D(T) \subset X \longrightarrow Y$ is said to be closable, if it has a closed extension.

Remark 1.4.2.
When T is closable, there is a closed operator $\overline{T}$ with $G(\overline{T}) = \overline{G(T)}$. It follows immediately that $\overline{T}$ is the smallest closed extension of T .

Proposition 1.4.5.

Let X and Y be two Banach spaces, we assume that $S \in \mathcal{L}(X,Y)$ and $T : X \longrightarrow Y$ is an unbounded linear operator.

1) For $S+T$ to be closed it is necessary and sufficient that T is closed.

2) For $S+T$ to be closable it is necessary and sufficient that T is closable and $\overline{S+T} = S + \overline{T}$.

Remark 1.4.3. *Let X and Y be two Banach spaces and $T : X \longrightarrow Y$ an unbounded operator then : T is closable if and only if :*

$$\forall (f_n)_n \subset D(T), f_n \underset{n\longrightarrow\infty}{\longrightarrow} 0 \quad if \quad Tf_n \quad converges \Longrightarrow Tf_n \longrightarrow 0.$$

Example 1.4.1. *Let $T : C([0,1]) \longrightarrow \mathbb{R}$ and $D(T) = C^1([0,1])$ with $Tf = f'(0)$ so T is not closable because, if $f_n(x) = \frac{1}{n}\sin nx \in C^1([0,1])$ then $\|x_n\|_\infty = \frac{1}{n} \longrightarrow 0$ and $\|Tx_n\|_\infty = 1$.*

Remark 1.4.4.

Any closed operator is closable but the opposite is false.

Remark 1.4.5.

i) If T is one to one and closed then T^{-1} is closed and the closed graph theorem implies that T^{-1} is bounded.

ii) if T is closed, $\mathcal{N}(T)$, the null space of T is closed.

iii) If T is bounded on $D(T)$, it is closable, where $D(\overline{T})$ is the closure of $D(T)$ in $\mathcal{H}$ and $\overline{T}$ is bounded in $D(\overline{T})$ with $\|\overline{T}\| = \|T\|$.

Example 1.4.2. *Let $T_k u = u''$, $k = 1,2,3,4$ be differential operators in $L^2([0,1])$ with domains*

$D(T_1) = \{u \in C^2([0,1]), u(0) = u(1) = 0\}$

$D(T_2) = C^2([0,1])$

$D(T_3) = \{u \in H^2([0,1]),\ u(0) = u(1) = 0\}$

$D(T_4) = L^2([0,1])$.

So we have $D(T_k) \subset L^2([0,1])$, $k = 1,2,3,4$ then T_k, $k = 1,2,3,4$ is an unbounded operator on $L^2([0,1])$, and we have $D(T_1) \subset D(T_2) \subset D(T_4)$, then we note $T_1 \subset T_2 \subset T_4$ (on sense of extension), and also, $T_3 \subset T_4$.

In the other hand, we have T_1 and T_2 are not closed, because we can may choose sequence of functions $u_n \in C^2([0,1])$ such that $u_n \to u$ and $u_n'' \to v$ in $L^2([0,1])$, where v is not continuous, hence u is not C^2

1.5 The adjoint of linear operator

Definition 1.5.1. *Let X, Y be two Banach spaces and let T a linear operator with dense domain. We call the adjoint of T the operator $T^* : \mathcal{D}(T^*) \subset Y^* \to X^*$ where*

$$\mathcal{D}(T^*) = \{y^* \in Y^* : \exists\ x^* \in X^* : (Tx, y^*)_Y = (x, x^*)_X, \forall\ x \in \mathcal{D}(T)\},$$

such that $T^*y^* = x^*$.

Proposition 1.5.1. [8]
Let $T : \mathcal{D}(T) \subset X \to Y$ a linear operator such that $\mathcal{D}(T)$ dense in X then
1) $G(-T)^0 = (G(T^*))^{-1}$.
2) T^* is closed operator.

Proof. (1) Let $T^* : \mathcal{D}(T^*) \subset Y^* \to X^*$ the adjoint of T then for all $y^* \in \mathcal{D}(T^*)$ there exists $x^* \in X^*$ such that $(Tx, y^*) = (x, x^*) = (x, T^*y^*)$ then $(x, T^*y^*) + (-Tx, y^*) = 0$ hence $\Big((x, -Tx), (T^*y^*, y^*)\Big) = 0$. this implies that for all $(x, -Tx) \in G(-T)$ we have $G(-T)^0 = (G(T^*))^{-1}$.
(2) We have $G(-T)^0$ is closed linear subspace, then $(G(T^*))^{-1}$ is closed, hence $G(T^*)$ is closed and T^* is closed operator. □

Theorem 1.5.1. [11, Theorem 3.3] *Let $T \in \mathcal{L}(X, Y)$, then the adjoint of T verified:*

$$\|T\| = \|T^*\|.$$

Proof. We have $|T^*y^*x| = |y^*(Tx)| \leq \|y^*\|\ \|Tx\| \leq \|y^*\|\ \|T\|\ \|x\|$ then,

$$\|T^*y^*\| = \sup_{x \neq 0} \frac{|T^*y^*x|}{\|x\|} \leq \|y^*\|\ \|T\|.$$

That implies

$$\|T^*\| \leq \|T\|.$$

In other hand, we have

$$\begin{aligned} |y^*(Tx)| &\leq \|T^*y^*\|\ \|x\| \\ &\leq \|y^*\|\|T^*\|\|x\|. \end{aligned}$$

Also $\|Tx\| = \sup_{y^* \neq 0} \frac{|y^*(Tx)|}{\|y^*\|} \leq \|x\|\ \|T^*\|$ for all $x \in X$, then

$$\|T\| \leq \|T^*\|$$

. □

Now, we give a definition of adjoint of operator defined on Hilbert spaces and some properties.

Definition 1.5.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, the unique linear operator $T^* \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that for all $x \in \mathcal{H}_1$; $y \in \mathcal{H}_2$ we have, $\langle Tx, y\rangle = \langle x, T^*y\rangle$ is called the adjoint of T.*

Proposition 1.5.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. then there exists a unique $T^* \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that, for all $x \in \mathcal{H}_1$; $y \in \mathcal{H}_2$, we have ,*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle .$$

Moreover, $\|T^*\| = \|T\|$, The adjoint of an operator will usually be used for operators in $\mathcal{L}(\mathcal{H})$,rather than $\mathcal{L}(\mathcal{H}, \mathcal{H})$. There is one notable exception.

Proposition 1.5.3. *If $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ then T is an isomorphism if and only if T is invertible and $T^{-1} = T^*$.*

Theorem 1.5.2. *Let $\mathcal{H}$ be a complex Hilbert space and let $T \in \mathcal{L}(\mathcal{H})$ be invertible, then T is inverter and we have :*

$$(T^{-1})^* = (T^*)^{-1}$$

1.6 Self-adjoint operators

Definition 1.6.1. ***(Auto-adjoint operator)***
An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be Self-adjoint (or sometimes Hermitian) operator if, whatever x and y in $\mathcal{H}$; we have :

$$\langle Tx, y\rangle = \langle x, Ty\rangle$$

Definition 1.6.2. ***(Normal operator)***
An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if, $T^*T = TT^*$.

Proposition 1.6.1. *[3] if $\mathcal{H}$ is a $\mathbb{C}$-Hilbert space and $T \in \mathcal{L}(\mathcal{H})$, then T is hermitian if and only if $\langle Tx, y\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$.*

Proposition 1.6.2. *Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space and $T \in \mathcal{L}(\mathcal{H})$, if $\langle Tx, x\rangle = 0$ for all $x \in H$ then $T \equiv 0$.*

Theorem 1.6.1.
if $T \in \mathcal{L}(\mathcal{H})$ is Self-adjoint operator then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

Chapter 2

Gap between two closed operators

In this chapter, we give the definition of the gap between closed operators.

2.1 Gap between two closed linear subspaces

2.1.1 Gap between two linear subspaces at normed space

Definition 2.1.1.
Let M be a subspace of the Banach space X, let $x \in X$. The distance between x and the subspace M is denoted by $d(y, M)$ and is defined by

$$d(x, M) = \inf_{y \in M} \{d(x, y)\}.$$

Such that d is a distance on X.

Definition 2.1.2. *Let M and N be two subspaces of the Banach space X. The gap between M and N is denoted by $\overset{\wedge}{\delta}(M, N)$ and is defined by*

$$\overset{\wedge}{\delta}(M, N) = \max\{\delta(M, N), \delta(N, M)\}.$$

The number $\delta(M, N)$ is defined by

$$\delta(M, N) = \begin{cases} \sup\limits_{x \in S_M} \{d(x, N)\} & , if M \neq \{0\}. \\ 0, & otherwise. \end{cases}$$

While S_M the unit sphere of M, it means that $\forall x \in S_M, \|x\| = 1$.
Evidently, $\delta(M, N) \leqslant 1$, because there exists $C > 0$, $d(x, N) \leqslant C \parallel x \parallel$ for all $x \in M$.

Remark 2.1.1.
It is easy to see that
1) $\delta(M, N) = \sup\limits_{x \in M,\ \|x\| \leq 1} d(x, N)$.
2) $\sup\limits_{x \in M,\ \|x\| \leq c} d(x, N) = C\ \delta(M, N)$, *for all* $C > 0$.

Proposition 2.1.1.

We have the following assertions:

1) $\delta(M, 0) = 1$ *if* $M \neq \varnothing$.

2) $\delta(M, N) = 0$ *if and only if* $M \subset N$.

3) $\overset{\wedge}{\delta}(M, N) = 0$ *if and only if* $M = N$.

4) $\overset{\wedge}{\delta}(M, N) = \overset{\wedge}{\delta}(N, M)$.

5) $0 \leqslant \delta(M, N) \leqslant 1$ *and* $0 \leqslant \overset{\wedge}{\delta}(M, N) \leqslant 1$.

6) $\delta(M, N) = \delta(\overline{M}, \overline{N})$.

Proof. According the definition, we have :

1) If $M \neq \varnothing$, then $\exists y \in M$ and $x = \frac{y}{\|y\|} \in M$ which $\|x\| = 1$ so

$$\begin{aligned} \delta(M, 0) &= \sup_{x \in S_M} d(x, 0) \\ &= \sup_{x \in S_M} \|x\| \\ &= 1. \end{aligned}$$

2) If $\delta(M, N) = 0$, then $\sup\limits_{x \in S_M} \{d(x, N)\} = 0$, so $d(x, N) = 0$, i.e.,

$$x \in N \quad and \quad M \subset N.$$

For $M \subset N$ it is evident to see that $\delta(M, N) = 0$.

3) For $\hat{\delta}(M, N) = 0$ then, $\max\{\delta(M, N), \delta(N, M)\} = 0$, it means that

$$\delta(M, N) = 0 \quad and \quad \delta(N, M) = 0.$$

so that by (2), we get

$$M \subset N \quad and \quad N \subset M.$$

then $M = N$.

4) We have,

$$\begin{aligned} \overset{\wedge}{\delta}(M, N) &= \max\{\delta(M, N), \delta(N, M)\} \\ &= \max\{\delta(N, M), \delta(M, N)\} \\ &= \overset{\wedge}{\delta}(N, M). \end{aligned}$$

5) a) We have two cases

- If $M = \{0\}$ then we assume by definition $\delta(M,N) = 0$ and $0 \leqslant \delta(M,N) \leqslant 1$.

- If $M \neq \{0\}$ and $0 \in N$, $\forall x \in S_M$, then

$$\begin{aligned} 0 \leqslant d(x,N) &= \inf_{y \in N} \{d(x,y)\} \\ &\leqslant d(x,0) = 1. \end{aligned}$$

So that $\forall x \in S_M$: $0 \leqslant d(x,N) \leqslant 1$ and for that

$$0 \leqslant \sup_{x \in S_M} \{d(x,N)\} \leqslant 1.$$

Then

$$0 \leqslant \delta(M,N) \leqslant 1$$

b) Moreover,

$$0 \leqslant \delta(M,N) \leqslant 1 \quad and \quad 0 \leqslant \delta(N,M) \leqslant 1.$$

So that

$$0 \leqslant \max\{\delta(M,N), \delta(N,M)\} \leqslant 1.$$

It means that $0 \leqslant \hat{\delta}(M,N) \leqslant 1$.

6) We have $d(x,N) = d(x,\overline{N})$, then

$$\begin{aligned} \delta(M,N) &= \sup_{x \in S_M} d(x,N) \\ &= \sup_{x \in S_{\overline{M}}} d(x,N) \\ &= \sup_{x \in S_{\overline{M}}} d(x,\overline{N}) \\ &= \delta(\overline{M},\overline{N}). \end{aligned}$$

□

Remark 2.1.2. • *$\hat{\delta}(M,N)$ is defined the **Gap maximum**, denoted by $\delta_{max}(M,N)$, with*

$$\hat{\delta}(M,N) = \delta_{max}(M,N).$$

- *$\delta_{min}(M,N)$ is the **Gap minimum** defined by*

$$\delta_{min}(M,N) = \min\{\delta(M,N), \delta(N,M)\}.$$

- *The spherical gap between M and N is denoted by $\hat{\delta}_0(M,N)$ and is defined by :*

$$\hat{\delta}_0(M,N) = \max\{\delta_0(M,N), \delta_0(N,M)\}.$$

and

$$\delta_0(M,N) = \sup_{x \in S_M} \{d(x,\mathcal{S}_N)\}.$$

Where $\mathcal{S}_N$ is the closed unit sphere of subspace N.

2.1.2 Gap between two subspaces at Hilbert spaces

Lemma 2.1.1. *[7] Let X be a Hilbert space and M, N be closed subspaces of X, then*

$$\delta(M,N) = \|(1-P_N)P_M\|.$$

where P_M and P_N are the orthogonal projections on M and N respectively. The orthogonal projection on M is denoted by P_M defined by :

$$P_M x = \begin{cases} x & if \quad x \in M. \\ 0 & \quad otherwise. \end{cases}$$

Proof. For all $x \in X$ $d(x,N) = \|x - P_N x\|$ then:

$$\begin{aligned}
\delta(M,N) &= \sup_{x\in M,\ \|x\|\le 1} d(x,N) \\
&= \sup_{x\in M,\ \|x\|\le 1} \|x - P_N x\| \\
&= \sup_{x\in M,\ \|x\|\le 1} \|P_M x - P_N P_M x\| \\
&= \sup_{x\in M,\ \|x\|\le 1} \|(1-P_N)P_M x\| \\
&= \sup_{x\in X,\ \|x\|= 1} \|(1-P_N)P_M x\| \\
&= \|(1-P_N)P_M\|.
\end{aligned}$$

□

Theorem 2.1.1. [7, Theorem 1.2 , p.4] *Let $\mathcal{H}$ be a Hilbert space and M, N be closed subspaces of $\mathcal{H}$, then*

$$\hat{\delta}(M,N) = \|P_M - P_N\|.$$

Proof. We have $\hat{\delta}(M,N) = \max\big[\delta(M,N);\delta(N,M)\big]$ then after lemma 2.1.1 we have

$$\hat{\delta}(M,N) = \max\Big[\|(1-P_N)P_M\|;\|(1-P_M)P_N\|\Big].$$

Or we have $P_M^2 = P_M \circ P_M = P_M$ and $(1-P_M)^2 = (1-P_M)$ then

$$\begin{aligned}
\big\|(P_M - P_N)x\big\|^2 &= \big\|P_M x - P_N P_M x + P_N P_M x - P_N x\big\|^2 \\
&= \big\|(1-P_N)P_M x - P_N(1-P_M)x\big\|^2 \\
&\le \|(1-P_N)P_M x\|^2 + \big\|P_N(1-P_M)x\big\|^2 \\
&\le \big\|(1-P_N)P_M\big\|^2\big\|P_M x\big\|^2 + \big\|P_N(1-P_M)\big\|^2\big\|(1-P_M)x\big\|^2.
\end{aligned}$$

At other way,

$$\begin{aligned}
\|P_N(1-P_M)\| &= \|(P_N(1-P_M))^*\| \\
&= \|(1-P_M)^* P_N^*\| \\
&= \|(1-P_M)P_N\| \\
&\le C.
\end{aligned}$$

It implies that $\|(P_M - P_N)x\|^2 \leq C^2\|P_M x\|^2 + C^2\|(1-P_M)x\|^2 = C^2\|x\|^2$.
hence

$$\|(P_M - P_N)x\|^2 \leq C^2\|x\|^2.$$

then $\max\left[\|(1-P_N)P_M x\|; \|P_N(1-P_M)x\|\right] \leq \|(P_M - P_N)x\| \leq C\|x\|$ for all $x \in X$
hence $\sup_{x\in X,\ \|x\|\leq 1}\left[\max\left(\|(1-P_N)P_M x\|; \|P_N(1-P_M)x\|\right)\right] \leq \sup_{x\in X,\ \|x\|\leq 1}\|(P_M - P_N)x\|$ that implies $\max\left[\|(1-P_N)P_M\|; \|P_N(1-P_M)\|\right] \leq \|P_M - P_N\| \leq C$.
On the other hand, we have $\|(1-P_N)P_M\| = C$ and $\|P_N(1-P_M)\| = \|(1-P_M)P_N\| = C$.
then $C \leq \|P_M - P_N\| \leq C$ hence $\|P_M - P_N\| = C$. □

2.2 The gap and the dimension

The following lemma is basic in the study of the gaps between closed linear subspaces.

Lemma 2.2.1. *Let M, N be linear subspaces in a Banach space X. If $\dim M \geqslant \dim N$, there exists $x \in M$ such that*

$$d(x, N) = \|x\| > 0.$$

Remark 2.2.1. *[8]Let M, $\tilde{M}$ be linear subspaces in a Banach space X. If $\tilde{M} \cup M = X$ and $\tilde{M} \cap M = \phi$ then $\tilde{M}$ is called the quotient space of M and it can be written*

$$\|\tilde{x}\| = \|x\| > 0.$$

Note that M is closed since $\dim M < \infty$ by hypothesis.

Corollary 2.2.1. *Let M, N to be closed linear subspaces. If $\hat{\delta}(M,N) < 1$ and $\dim M \leqslant \dim N$, then $\dim M = \dim N$.*

2.3 The gap and duality

There is a simple relationship between the gap function in a Banach space X and that in the dual space X^*.

Definition 2.3.1. *For any subset $M \subset X$, the annihilator of M which denoted $M^\perp$ is the closed linear subspace of X^* and*

$$M^\perp = \{x^* \in X^* : x^*(x) = 0, \forall x \in M\}.$$

Proposition 2.3.1. *[8] Let M and N two closed linear sub-spaces then*

$$(M+N)^\perp = M^\perp \cap N^\perp.$$

Evermore $(M+N)^\perp$ is closed linear subspace but $M^\perp \cap N^\perp$ need not be closed, however, that it is true if and only if $M+N$ is closed.

Lemma 2.3.1. *[8] Let M be a closed linear subspace of X, $M \neq 0$ and $M \neq X$, then*
1) $d(x^*, M^\perp) = \sup\limits_{x \in S_M} |\langle x, x^* \rangle|, x^* \in X^*$.
2) $d(x, M) = \sup\limits_{x \in S_{M^\perp}} |\langle x, x^* \rangle|, x \in X$.

Theorem 2.3.1. *[8] For closed linear subspaces M, N of X, we have:*

$$\delta(M, N) = \delta(N^\perp, M^\perp).$$

and

$$\hat{\delta}(M, N) = \hat{\delta}(M^\perp, N^\perp).$$

Proof. 1. For closed linear subspaces M, N of X, we have

$$\begin{aligned}
\delta(M, N) &= \mathop{sup}_{x \in S_M} \{d(x, N)\} \\
&= \sup_{x \in S_M} \sup_{y^* \in S_{N^\perp}} |\langle x, y^* \rangle| \\
&= \sup_{y^* \in S_{N^\perp}} \sup_{x \in S_M} |\langle x, y^* \rangle| \\
&= \mathop{sup}_{y^* \in S_{N^\perp}} d(y^*, M^\perp) \\
&= \delta(N^\perp, M^\perp).
\end{aligned}$$

The above proof applies to the case where $M \neq \varnothing$ and $N \neq X$. If $M = \varnothing$, then $M^\perp = X^*$ so that $\delta(M, N) = \delta(N^\perp, M^\perp) = 0$.
If $N = X$, then $N^\perp = 0$ so that $\delta(M, N) = \delta(N^\perp, M^\perp) = 0$.

2. For closed linear subspaces M, N of X, we have

$$\begin{aligned}
\hat{\delta}(M, N) &= \max\{\delta(M, N), \delta(N, M)\} \\
&= \max\left\{\delta(N^\perp, M^\perp), \delta(M^\perp, N^\perp)\right\} \\
&= \max\left\{\delta(M^\perp, N^\perp), \delta(N^\perp, M^\perp)\right\} \\
&= \hat{\delta}(M^\perp, N^\perp).
\end{aligned}$$

□

2.4 Gap between closed operators

2.4.1 Gap between two closed operators

Definition 2.4.1. *For $T, S \in \mathcal{C}(X,Y)$, their graphs $G(T)$, $G(S)$ are closed linear subspaces of the product space $X \times Y$. We set*

$$\delta(T,S) = \delta(G(T), G(S))$$

and

$$\begin{aligned} \hat{\delta}(T,S) &= \hat{\delta}(G(T), G(S)) \\ &= \max\{\delta(T,S), \delta(S,T)\} \end{aligned}$$

and $\hat{\delta}(T,S)$will be called the gap between T and S.

Remark 2.4.1.
The spherical gap between T and S is denoted by $\overset{\wedge}{\delta}_0(T,S)$ and is defined by

$$\overset{\wedge}{\delta}_0(T,S) = \max\{\delta_0(T,S), \delta_0(S,T)\}$$

Where $\mathcal{S}_{G(S)}$ is the closed unit sphere of subspace $G(S)$ and

$$\delta_0(T,S) = \sup_{M \in \mathcal{S}_{G(T)}} \{d(M, \mathcal{S}_{G(S)})\}$$

which $M \in \mathcal{S}_{G(T)}$ means $M(x,Tx) \in G(T)$: $\|x\|^2 + \|Tx\|^2 = 1$.

Theorem 2.4.1. [4, Theorem 2.2] *Let X, Y be two Banach spaces and $T \in \mathcal{L}(X,Y)$ then*

$$\delta(T,0) = \frac{\|T\|}{\sqrt{1+\|T\|^2}}.$$

Proof. Let $x \in X$ such that

$$\|x\|_X^2 + \|Tx\|_Y^2 = 1. \tag{2.1}$$

We have

$$\begin{aligned} \delta(T,0) = \delta(G(T), G(0)) &= \sup_{x\in X, \|x\|_X^2+\|Tx\|_Y^2=1} \left[\inf_{y\in X} (\|x-y\|_X^2 + \|Tx\|_Y^2)^{\frac{1}{2}}\right] \\ &= \sup_{x\in X, \|x\|_X^2+\|Tx\|_Y^2=1} \|Tx\|. \end{aligned}$$

Let $z \in X$ such that $\|z\| = 1$. We pose $x = cz$ with $c = \dfrac{1}{(\|z\|^2 + \|Tz\|^2)^{\frac{1}{2}}}$. It is evident that x it satisfy to (2.1), hence

$$\begin{aligned} \delta(T,0) &= \sup_{z\in X} \frac{\|Tz\|}{(\|z\|^2+\|Tz\|^2)^{\frac{1}{2}}} \\ &= \sup_{z\in X, \|z\|=1} \frac{\|Tz\|}{(1+\|Tz\|^2)^{\frac{1}{2}}} \\ &= \frac{\|T\|}{(1+\|T\|^2)^{\frac{1}{2}}}. \end{aligned}$$

□

Theorem 2.4.2. [4, Theorem 2.3] *Let X, Y be two Banach spaces $T \in \mathcal{L}(X,Y)$, Then*

$$\hat{\delta}(T,0) = \frac{\|T\|}{\sqrt{1+\|T\|^2}}.$$

Proof. For show that $\hat{\delta}(T,0) = \frac{\|T\|}{\sqrt{1+\|T\|^2}}$ we must show $\delta(0,T) \leq \delta(T,0)$, we have

$$\begin{aligned}
\delta(0,T) = \delta(G(0),G(T)) &= \sup_{x\in\mathcal{S}(G(0))} dist\big((x,0),G(T)\big) \\
&= \sup_{x\in X,\|x\|=1} \inf_{y\in X} \big(\|(x,0)-(y,Ty)\|\big) \\
&= \sup_{x\in X,\|x\|=1} \inf_{y\in X} \big(\|x-y\|^2+\|Ty\|^2\big)^{\frac{1}{2}}.
\end{aligned}$$

We pose

$$f(x,y) = \big(\|x-y\|^2+\|Ty\|^2\big)^{\frac{1}{2}}.$$

Let $x \in \mathcal{S}(X)$ and $y = \dfrac{x}{(1+\|Tx\|^2)^{\frac{1}{2}}}$. We have

$$dist\big((x,0),G(T)\big) \leq f(x,y) = \frac{\|Tx\|}{(1+\|Tx\|^2)^{\frac{1}{2}}}.$$

Because the function $\Psi(t) = \dfrac{t}{1+t}$ is increasing then

$$\begin{aligned}
\delta(0,T) &= \sup_{x\in X,\|x\|=1} dist\big((x,0),G(T)\big) \\
&\leq \sup_{\|x\|=1} \frac{\|Tx\|}{(1+\|Tx\|^2)^{\frac{1}{2}}} = \frac{\|T\|}{(1+\|T\|^2)^{\frac{1}{2}}} = \delta(T,0).
\end{aligned}$$

Now, we have $\delta(0,T) \leqslant \delta(T,0)$, i.e.,

$$\begin{aligned}
\hat{\delta}(T,0) &= \max\{\delta(0,T),\delta(T,0)\} \\
&= \delta(T,0).
\end{aligned}$$

Then and after the Theorem 2.4.1

$$\hat{\delta}(T,0) = \frac{\|T\|}{\sqrt{1+\|T\|^2}}.$$

The proof is complete. □

Theorem 2.4.3. [7, Theorem 1.5] *Let X, Y be two normed spaces and S, T be two linear operators, and let A in $\mathcal{L}(X,Y)$, then*

$$\delta(T+A,S+A) \leq (2+\|A\|^2)\delta(T,S).$$

And

$$\hat{\delta}(T+A,S+A) \leq (2+\|A\|^2)\hat{\delta}(T,S).$$

Proof. Wehave $A \in \mathcal{L}(X,Y)$ then $\mathcal{D}(A) = X$ so that, $\mathcal{D}(S+A) = \mathcal{D}(S)$ et $\mathcal{D}(T+A) = \mathcal{D}(T)$. For $x \in \mathcal{D}(T)$ and

$$\|x\|_X^2 + \|(T+A)x\|_Y^2 = 1.$$

Then

$$\|Tx\|_Y \leq \|(T+A)x\|_Y + \|Ax\|_Y \leq \|(T+A)x\|_Y + \|A\|\|x\|_X,$$

then after Cauchy-Schwarz inequality we have

$$\|Tx\|_Y^2 \leq (\|x\|_X^2 + \|(T+A)x\|_Y^2)(1+\|A\|^2) = 1 + \|A\|^2.$$

And $\|x\|_X^2 \leq 1$ then $\|x\|_X^2 + \|Tx\|_Y^2 \leq 2 + \|A\|^2$. So that

$$(\|x\|_X^2 + \|Tx\|_Y^2)^{\frac{1}{2}} \leq (2+\|A\|^2)^{\frac{1}{2}}.$$

also, for all $y \in \mathcal{D}(S)$ we have

$$\begin{aligned} \|(T+A)x - (S+A)y\|_Y &= \|Tx - Sy + A(x-y)\|_Y \\ &\leq \|Tx - Sy\|_X + \|A\|\|x-y\|_X. \end{aligned}$$

Therefore,

$$\|(T+A)x - (S+A)y\|_Y^2 \leq (\|x-y\|_X^2 + \|Tx - Sy\|_Y^2)(1+\|A\|^2)$$

and

$$(\|x-y\|_X^2 + \|(T+A)x - (S+A)y\|_Y^2)^{\frac{1}{2}} \leq (2+\|A\|^2)^{\frac{1}{2}}(\|x-y\|_X^2 + \|Tx - Sy\|_Y^2)^{\frac{1}{2}}.$$

Then

$$\begin{aligned} &\delta(T+A, S+A) \\ = &\sup_{x\in\mathcal{D}(T), \|x\|_X^2 + \|(T+A)x\|_Y^2 = 1} \left[\inf_{y\in\mathcal{D}(S)} (\|x-y\|_X^2 + \|(T+A)x - (S+A)y\|_Y^2)^{\frac{1}{2}}\right] \\ \leq &(2+\|A\|^2)^{\frac{1}{2}} \sup_{x\in\mathcal{D}(T), (\|x\|_X^2 + \|Tx\|_Y^2)^{\frac{1}{2}} \leq (2+\|A\|^2)^{\frac{1}{2}}} \left[\inf_{y\in\mathcal{D}(S)} (\|x-y\|_X^2 + \|Tx - Sy\|_Y^2)^{\frac{1}{2}}\right] \\ = &(2+\|A\|^2)\delta(T,S). \end{aligned}$$

Then

$$\delta(T+A, S+A) \leq (2+\|A\|^2)\delta(T,S).$$

And

$$\begin{aligned} \hat{\delta}(T+A, S+A) &= max\left\{\delta(T+A, S+A), \delta(S+A, T+A)\right\} \\ &\leqslant \left\{(2+\|A\|^2)\delta(T,S), (2+\|A\|^2)\delta(S,T)\right\} \\ &\leqslant (2+\|A\|^2)\max\left\{\delta(T,S), \delta(S,T)\right\} \\ &\leqslant (2+\|A\|^2)\hat{\delta}(T,S). \end{aligned}$$

So, $\hat{\delta}(T+A, S+A) \leqslant (2+\|A\|^2)\hat{\delta}(T,S)$. □

Corollary 2.4.1. *[8] Let X, Y two Banach spaces and $T, S \in \mathcal{C}(X,Y)$ and $A \in \mathcal{B}(X,Y)$. Then*

$$\hat{\delta}(S+A, T+A) \leqslant 2(1+\|A\|^2)\hat{\delta}(T,S).$$

Theorem 2.4.4. *[10] Let X, Y be two normed spaces and T, S be two invertible linear operators from X into Y, then*

$$\delta(T,S) = \delta(T^{-1}, S^{-1}),$$

and

$$\hat{\delta}(T,S) = \hat{\delta}(T^{-1}, S^{-1}).$$

Proof. 1. We have by proposition 1.4.2 and $\delta(G(T), G(S)) = \delta\big((G(T))^{-1}, (G(S))^{-1}\big)$ then

$$\begin{aligned} \delta(T,S) &= \delta(G(T), G(S)) \\ &= \delta\big((G(T))^{-1}, (G(S))^{-1}\big) \\ &= \delta(G(T^{-1}), G(S^{-1})) \\ &= \delta(T^{-1}, S^{-1}). \end{aligned}$$

2. We have also

$$\begin{aligned} \hat{\delta}(T,S) &= \max\{\delta(T,S); \delta(S,T)\} \\ &= \max\{\delta(T^{-1}, S^{-1}); \delta(S^{-1}, T^{-1})\} \\ &= \hat{\delta}(T^{-1}, S^{-1}). \end{aligned}$$

And we end the proof. □

Lemma 2.4.1. *Let X, Y be two Banach spaces and $T \in \mathcal{L}(X,Y)$. If $S \in \mathcal{C}(X,Y)$ and $\delta(S,T) < (1+\|T\|^2)^{-1/2}$, then $S \in \mathcal{L}(X,Y)$ (so that $D(S)$ is closed).*

Proof. Let q be an element of the unit sphere of $G(S)$: $q = (x, Sx) \in G(S)$ and

$$\|x\|^2 + \|Sx\|^2 = \|q\|^2 = 1.$$

Let C be any number such that $\delta(S,T) < C < (1+\|T\|^2)^{-1/2}$, then q has distance smaller than C from $G(T)$, so that there exists $p = \{y, Ty\} \in G(T)$ such that $\|q-p\| < C$ and

$$\|x-y\|^2 + \|Sx - Ty\|^2 = \|q-p\|^2 < C^2.$$

Set $A = S - T$, we have

$$\begin{aligned} \|Ax\|^2 &= \|S(x) - T(y) - T(x-y)\|^2 \\ &\leqslant (\|S(x) - T(y)\| + \|T\|.\|x-y\|)^2 \\ &\leqslant C^2\left(1+\|T\|^2\right). \end{aligned}$$

Since,

$$1 = \|x\|^2 + \|Tx + Ax\|^2 \leqslant (1 + \|T\|^2).\|x\|^2 + 2\|T\|.\|x\|.\|Ax\| + \|Ax\|^2$$

by property of spherical gap, we have

$$\|Ax\|^2 \leqslant \; C^2.(1 + \|T\|^2).[(1 + \|T\|^2).\|x\|^2 + 2\|T\|.\|x\|.\|Ax\| + \|Ax\|^2].$$

Solving this inequality for $\|Ax\|$, we obtain

$$\begin{aligned} \|Ax\| &\leqslant \frac{C.(1 + \|T\|)[(1 - C^2)^{1/2} + C\|T\|]}{1 - C^2.(1 + \|T\|^2)} \|x\| \\ &\leqslant \frac{C.(1 + \|T\|)}{1 - C.(1 + \|T\|^2)^{1/2}} \|x\| \\ &\leqslant C^{'} \|x\|. \end{aligned}$$

With $C' = \frac{C.(1+\|T\|)[(1-C^2)^{1/2}+C\|T\|]}{1-C^2.(1+\|T\|^2)}$, note that the denominators are positive.
It is homogeneous in x, it is true for every $x \in D(S)$ without any normalization, both $A, T \in \mathcal{L}(X, Y)$, so $S = T + A$ is bounded. □

Lemma 2.4.2. *Let X,Y be two normed spaces and* $T \in \mathcal{L}(X, Y)$*. If* $S \in \mathcal{C}(X, Y)$ *and* $\delta(T, S) < (1 + \|T\|^2)^{-1/2}$*, then S is densely defined.*

Proof. Let (x, Tx) is an element $G(T)$, where $\|x\|^2 + \|Tx\|^2 = 1$.
Let $C > 0$ be such that $\delta(T, S) < C < (1 + \|T\|^2)^{-1/2}$. Then there is a (y, Sy) satisfying,

$$\|x - y\|^2 + \|Sx - Ty\|^2 < C^2.$$

Hence $\|x-y\| < C$, then $d(x, \overline{D(S)}) < C$, and because $(1+\|T\|^2)^{1/2}\|x\| \leqslant 1$, then $d(x, \overline{D(S)}) < C(1 + \|T\|^2)^{1/2}\|x\|$. The last inequality is homogeneous and therefore true for every $x \in X$.
Since $C < (1 + \|T\|^2)^{-1/2}$, then $C(1 + \|T\|^2)^{1/2} < 1$, We pose that $M \neq x$, it exists $x_0 \in X$ such that $\|x_0\| = 1$ and $d(x_0, M) > C(1 + \|T\|^2)^{1/2}\|x_0\|$, its absurd.
Then $\overline{D(S)} = X$, then S is densely defined. □

Theorem 2.4.5. *[7] Let X, Y be two normed spaces and* $T \in \mathcal{L}(X, Y)$*. If* $S \in \mathcal{C}(X, Y)$ *is so close to T that* $\delta(S, T) < (1 + \|T\|^2)^{-1/2}$*, then* $S \in \mathcal{L}(X, Y)$ *and*

$$\|S - T\| \leqslant \frac{(1 + \|T\|^2).\delta(S, T)}{1 - (1 + \|T\|^2)^{1/2}.\delta(S, T)}.$$

Proof. First step: If $x \in \mathcal{D}(S)$ then for all $y \in \mathcal{D}(T)$ we have

$$\begin{aligned} \|(S - T)x\|_Y &= \|Sx - Ty - T(x - y)\|_Y \\ &\leq \|Sx - Ty\|_Y + \|T\|\|x - y\|_X. \end{aligned}$$

Then after Cauchy-Schwarz inequality we have

$$\|(S - T)x\|_Y \leq \left(\|x - y\|_X^2 + \|Sx - Ty\|_X^2\right)^{\frac{1}{2}} \left(1 + \|T\|^2\right)^{\frac{1}{2}}$$

and

$$\begin{aligned}\|(S-T)x\|_Y &\leq \inf_{y\in\mathcal{D}(T)}\left(\|x-y\|_X^2+\|Sx-Ty\|_X^2\right)^{\frac{1}{2}}\left(1+\|T\|^2\right)^{\frac{1}{2}}\\ &\leq \delta(S,T)\left(1+\|T\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

second step: We assume that

$$\|x\|_X^2+\|Sx\|_Y^2=1 \tag{2.2}$$

Then

$$\begin{aligned}1 &\leq \left(\|x\|_X^2+\|Tx+(S-T)x\|_Y^2\right)^{\frac{1}{2}}\\ &\leq \left(\|x\|_X^2+\left(\|T\|\|x\|_X+\|(S-T)x\|_Y\right)^2\right)^{\frac{1}{2}}\\ &\leq \left(\|x\|_X^2+\|T\|^2\|x\|_X^2+2\|T\|\|x\|\|(S-T)x\|_X+\|(S-T)x\|_Y^2\right)^{\frac{1}{2}}\\ &\leq \left((1+\|T\|^2)\|x\|_X^2+2(1+\|T\|^2)^{\frac{1}{2}}\|x\|_X\|Sx-Tx\|_X+\|Sx-Tx\|_Y^2\right)^{\frac{1}{2}}.\end{aligned}$$

We have also $\left((1+\|T\|^2)\|x\|_X^2+2(1+\|T\|^2)^{\frac{1}{2}}\|x\|_X\|Sx-Tx\|_X+\|Sx-Tx\|_Y^2\right)^{\frac{1}{2}}=$ $\left(\left((1+\|T\|^2)^{\frac{1}{2}}\|x\|_X+\|Sx-Tx\|_Y\right)^2\right)^{\frac{1}{2}}=(1+\|T\|^2)^{\frac{1}{2}}\|x\|_X+\|Sx-Tx\|_Y$. In other hand,

$$1\leq(1+\|T\|^2)^{\frac{1}{2}}\|x\|_X+\|Sx-Tx\|_Y$$

or

$$\|(S-T)x\|_Y\leq\delta(S,T)\left(1+\|T\|^2\right)^{\frac{1}{2}}.$$

Where

$$\|(S-T)x\|_Y\leq(1+\|T\|^2)\delta(S,T)\|x\|_X+\left(1+\|T\|^2\right)^{\frac{1}{2}}\delta(S,T)\|Sx-Tx\|_Y.$$

Therefore, for all $x\in\mathcal{D}(S)$ and after (2.2)

$$\|(S-T)x\|_Y-\left(1+\|T\|^2\right)^{\frac{1}{2}}\delta(S,T)\|Sx-Tx\|_Y\leq(1+\|T\|^2)\delta(S,T)\|x\|_X,$$

then

$$\left(1-(1+\|T\|^2)^{\frac{1}{2}}\delta(S,T)\right)\|(S-T)x\|_Y\leq(1+\|T\|^2)\delta(S,T)\|x\|_X,$$

where

$$\|(S-T)x\|_Y\leq\frac{1+\|T\|^2}{1-(1+\|T\|^2)^{\frac{1}{2}}\delta(S,T)}\delta(S,T)\|x\|_X.$$

As this inequality is true for all $x\in\mathcal{D}(S)$ then $S-T\in\widetilde{\mathcal{L}}(X,Y)$.
Therefore, $S\in\widetilde{\mathcal{L}}(X,Y)$ et $\|S-T\|_{\mathcal{D}(S)}\leq\dfrac{1+\|T\|^2}{1-\sqrt{1+\|T\|^2}\delta(S,T)}\delta(S,T)$ □

Theorem 2.4.6. *[8] Let X, Y be two Banach spaces and $T \in \mathcal{L}(X,Y)$. If $S \in \mathcal{C}(X,Y)$ is so close to T that $\delta(S,T) < (1+\|T\|^2)^{-1/2}$, then $S \in \mathcal{L}(X,Y)$ and*

$$\|S-T\| \leqslant \frac{(1+\|T\|^2).\delta(S,T)}{1-(1+\|T\|^2)^{1/2}.\delta(S,T)}.$$

Proof. we have after Lemma 2.4.2 $S \in \mathcal{L}(X,Y)$ and

$$\|S-T\|_{\mathcal{D}(S)} \leq \frac{1+\|T\|^2}{1-\sqrt{1+\|T\|^2}\delta(S,T)}\delta(S,T).$$

As we have $S \in \mathcal{C}(X,Y)$ then $D(S)$ is closed, i.e., $\mathcal{D}(S) = \overline{\mathcal{D}(S)} = X$.
Where $S \in \mathcal{L}(X,Y)$ and

$$\|S-T\| \leq \frac{1+\|T\|^2}{1-\sqrt{1+\|T\|^2}\delta(S,T)}\delta(S,T).$$

□

Theorem 2.4.7. ***(The gap between two closable operators)*** *[10]*
Let X, Y be two normed spaces and T, S be two closable linear operators from X into Y. If $T, S \in \mathcal{C}(X,Y)$, then

$$\delta(T,S) = \delta(\overline{T},\overline{S}).$$

And

$$\hat{\delta}(T,S) = \hat{\delta}(\overline{T},\overline{S}).$$

Proof. (i) We have $G(T)$ is linear subspace of $X \times Y$ then after theorem 1.4.2 we have $\delta(G(T),G(S)) = \delta(\overline{G(T)},\overline{G(S)})$ and T et S are two closable operators then $\overline{G(T)} = G(\overline{T})$ et $\overline{G(S)} = G(\overline{S})$. there implies that

$$\begin{aligned}
\delta(T,S) &= \delta(G(T),G(S)) \\
&= \delta(\overline{G(T)},\overline{G(S)}) \\
&= \delta(G(\overline{T}),G(\overline{S})) \\
&= \delta(\overline{T},\overline{S}).
\end{aligned}$$

(ii) We have

$$\begin{aligned}
\hat{\delta}(T,S) &= \max\{\delta(T,S);\delta(S,T)\} \\
&= \max\{\delta(\overline{T},\overline{S});\delta(\overline{S},\overline{T})\} \\
&= \hat{\delta}(\overline{T},\overline{S}).
\end{aligned}$$

□

Corollary 2.4.2. *Let X and Y be two banach spaces. We assume that $T \in \mathcal{L}(X,Y)$, $S : X \longrightarrow Y$ a closable linear operator and $\delta(T,S) < \dfrac{1}{\sqrt{1+\|T\|^2}}$.*
Then $S \in \mathcal{L}(X,Y)$ and

$$\|S-T\| \leq \frac{1+\|T\|^2}{1-\sqrt{1+\|T\|^2}\delta(T,S)}\delta(T,S).$$

Proof. we have after lemma 2.4.2 S is a dense-domain linear operator then S^* exists. By replacing in Theorem2.4.5, S by S^* and T by T^* so we obtain $S^* \in \mathcal{L}(Y^*,X^*)$ and

$$\|S^*-T^*\| \leq \frac{1+\|T^*\|^2}{1-\sqrt{1+\|T^*\|^2}\delta(S^*,T^*)}\delta(S^*,T^*).$$

After Theorem 1.5.1, which $\|T^*\| = \|T\|$ then

$$\|S-T\| \leq \frac{1+\|T\|^2}{1-\sqrt{1+\|T\|^2}\delta(S,T)}\delta(S,T).$$

□

Proposition 2.4.1. *Let $T,S \in \mathcal{C}(X,Y)$, Similarly we can define the distance $\hat{d}(T,S)$ between T and S as equal to $d(G(T),G(S))$. Under this distance function $T_n \in \mathcal{C}(X,Y)$ becomes a metric space. In this space the convergence of a sequence to a $T \in \mathcal{C}(X,Y)$.*
But we have $\hat{\delta}(T,S) \leqslant \overset{\wedge}{\delta_0}(T,S) \leqslant 2\hat{\delta}(T,S)$, this is true if and only if $\hat{\delta}(T_n,T) \longrightarrow 0$. . In this case we shall also say that the operator T_n converges to T.
It should be remarked that earlier we defined the convergence of operators only for operators of the class $\mathcal{B}(X,Y)$. Actually we introduced several different notions of convergence: convergence in norm, strong and weak convergence. We shall show in a moment that the notion of generalized convergence introduced above for closed operators is a generalization of convergence in norm for operators of $\mathcal{L}(X,Y)$.

Remark 2.4.2. *When T varies over $\mathcal{C}(X,Y)$, $G(T)$ varies over a proper subset of the set of all closed linear subspaces of $X \times Y$. This subset is not closed and consequently, $\mathcal{C}(X,Y)$ is not a complete metric space (assuming, of course, that $\dim X \geqslant 1$, $\dim Y \geqslant 1$). It is not trivial to see this in general, but it is easily seen if $Y = X$. Consider the sequence nI where I is the identity operator in X. $G(nI)$ is the subset of $X \times X$ consisting of all elements n^1x, x, $x \in X$, and it is readily seen that $\lim G(nI)$ exists and is equal to the set of all elements of the form $0, x$,$x \in X$. But this set is not a graph.*
Thus nI is a Cauchy sequence in $\mathcal{L}(X)$ without a limit.

Theorem 2.4.8. *[4] Let X, Y be two Banach spaces and $X \times Y$ is normed space by the norm $\|(x,y)\| = \left(\|x\|^2+\|y\|^2\right)^{\frac{1}{2}}$ for all $x \in X$ and $y \in Y$. if T, $S \in \mathcal{L}(X,Y)$ then*

$$\delta(T,S) \leq \Big(2+\min\{\|T\|^2,\|S\|^2\}\Big)\frac{\|T-S\|}{(1+\|T-S\|^2)^{\frac{1}{2}}}.$$

Proof. We have $\delta(T,S) = \delta((T-S)+S, 0+S)$ and $\delta(T,S) = \delta(T+0,(S-T)+T)$. and by theorem 2.4.3 we have

$$\delta((T-S)+S,0+S) \le 2(1+\|S\|^2)\,\delta(T-S,0) \ \text{ and } \ \delta(0+T,(S-T)+T) \le 2(1+\|T\|^2)\,\delta(0,S-T).$$

and after theorem 2.4.1,
We get $\delta(T-S,0) = \dfrac{\|T-S\|}{(1+\|T-S\|^2)^{\frac{1}{2}}}$ and $\delta(0,S-T) \le \dfrac{\|T-S\|}{(1+\|T-S\|^2)^{\frac{1}{2}}}$ then

$$\delta(T,S) \le (2+\|S\|^2)\frac{\|T-S\|}{(1+\|T-S\|^2)^{\frac{1}{2}}},$$

and

$$\delta(T,S) \le (2+\|T\|^2)\frac{\|T-S\|}{(1+\|T-S\|^2)^{\frac{1}{2}}}.$$

it implies that

$$\delta(T,S) \le \Big(2+\min\{\|T\|^2,\|S\|^2\}\Big)\frac{\|T-S\|}{(1+\|T-S\|^2)^{\frac{1}{2}}}.$$

□

Theorem 2.4.9. *[8] Let $T \in \mathcal{C}(X,Y)$ and let A be T-bounded with relative bound less than 1, so that we have the inequality (1.1) with $b<1$. Then $S = T+A \in \mathcal{C}(X,Y)$ and*

$$\hat{\delta}(S,T) \leqslant (1-b)^{-1}(a^2+b^2)^{1/2}.$$

In Particular if $A \in \mathcal{L}(X,Y)$, then

$$\hat{\delta}(T+A,T) \leqslant \|A\|.$$

Remark 2.4.3. *Theorem 2.16 shows that $\mathcal{L}(X,Y)$ is an open subset of $\mathcal{C}(X,Y)$. by theorem (2.4.5) and theorem (2.4.9) show that within this open subset $\mathcal{L}(X,Y)$, the topology defined by the distance function $\hat{\delta}$ (or, equivalently, by the gap function $\hat{\delta}_0$) is identical with the norm topology.*

Theorem 2.4.10. [7, Théorème 1.9] *Let X, Y be two normed spaces and $T,S \in L(X,Y)$, we assume that T is invertible such that $T^{-1} \in \mathcal{L}(Y,X)$.*
(1) *If $\delta(S,T) < \dfrac{1}{\sqrt{1+\|T^{-1}\|^2}}$ then S is invertible, $S^{-1} \in \widetilde{\mathcal{L}}(Y,X)$ and*

$$\|S^{-1}-T^{-1}\|_{\mathcal{R}(S)} \le \frac{1+\|T^{-1}\|^2}{1-\sqrt{1+\|T^{-1}\|^2}\delta(S,T)}\delta(S,T).$$

(2) *If S est un opérateur à domaine dense et $\delta_{\max}(T,S) < \dfrac{1}{\sqrt{1+\|T^{-1}\|^2}}$.*
Then S is invertible, $S^{-1} \in \mathcal{L}(Y,X)$ and

$$\|S^{-1}-T^{-1}\|_{\mathcal{R}(S)} \le \frac{1+\|T^{-1}\|^2}{1-\sqrt{1+\|T^{-1}\|^2}\delta(T,S)}\delta(T,S).$$

And what's more, $\overline{\mathcal{R}(S)} = Y$.

Proposition 2.4.2. *Si X et Y be two Banach spaces and $S \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$ then S^{-1} is closed and $S^{-1} \in \widetilde{\mathcal{L}}(Y, X)$ then $\mathcal{D}(S^{-1})$ is closed.*
Or we have $\overline{\mathcal{R}(S)} = Y$ where $\mathcal{D}(S^{-1}) = \overline{\mathcal{D}(S^{-1})} = Y$ then $S^{-1} \in \mathcal{L}(X, Y)$ and

$$\|S^{-1} - T^{-1}\| \leq \frac{1 + \|T^{-1}\|^2}{1 - \sqrt{1 + \|T^{-1}\|^2}\delta(T, S)}\delta(T, S).$$

Theorem 2.4.11. *Let $T, S \in \mathcal{C}(X, Y)$ be densely defined. Then*

$$\delta(S, T) = \delta(T^*, S^*)$$

and

$$\hat{\delta}(S, T) = \hat{\delta}(S^*, T^*)$$

Proof. We have

$$\begin{aligned}
\delta(T^*, S^*) &= \delta(G(T^*), G(S^*)) \\
&= \delta(G'(T^*), G'(S^*)) \\
&= \delta(G(-T)^{\perp}, G(-S)^{\perp}) \\
&= \delta(G(S), G(T)) \\
&= \delta(S, T).
\end{aligned}$$

where $G(T) \subset X \times Y$ is the graph of T and $G'(T^*) \subset X^* \times Y^*$ is the inverse graph of $T^* \in \mathcal{C}(Y^*, X^*)$, note that $G'(T^*) = G(-T)^{\perp}$, and $\delta(G(S^*), G(T^*)) = \delta(G'(S^*), G'(T^*))$ is due to the special choice of the norm in the product space. □

Chapter 3

Applications

3.1 Resolvent set and spectrum of a linear operator

Definition 3.1.1. *Let T be a closable linear operator in a Banach space X. The resolvent set and the spectrum of T are, respectively, defined as*

$$\rho(T) = \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - T \text{ is injective and } (\lambda - T)^{-1} \in \mathcal{L}(X)\right\} \ , \sigma(T) = \mathbb{C} \backslash \rho(T)$$

and the point spectrum, continuous spectrum, and the residual spectrum are defined as

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \text{ such that } \ T \text{ is not injective}\},$$

$$\sigma_c(T) = \left\{\lambda \in \mathbb{C} \text{ such that } T \text{ is injective, } \overline{R(\lambda - T)} = X, R(\lambda - T) \neq X\right\}$$

$$\sigma_r(T) = \left\{\lambda \in \mathbb{C} \text{ such that } T \text{ is injective, } \overline{R(\lambda - T)} \neq X,\right\}$$

Remark 3.1.1. *Note that, if $\rho(T) \neq \emptyset$ then T is closed. In fact, if $\lambda \in \rho(T)$ then $(\lambda - T)^{-1}$is closed, which is also valid for $\lambda - T$ then, according to the closed graph theorem we deduce that*

$$\rho(T) = \{\lambda \in \mathbb{C} \text{ sucht hat } \lambda - T \text{ is bijective } \}$$

and hence

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

Proposition 3.1.1. *Let X be a Banach space. Let $T \in \mathcal{L}(X)$.*
(i) $\rho(T)$ is open in $\mathbb{C}$.
(ii) $\sigma(T)$ is non empty compact in $\mathbb{C}$, include in $B(0, \|T\|)$
$\Big($ where $\mathcal{B}(0, \|T\|)$: the closed ball $B(0, \|T\|)$.

Proposition 3.1.2. *Let X a Banach space and $T : \mathcal{D}(T) \subset X \longrightarrow X$ a linear operator. If $\rho(T) \neq \emptyset$ then T is closed.*

3.1.1 Essential Spectrum of linear operator

Definition 3.1.2. *Let X be a Banach space and $T \in \mathcal{C}(X)$. We define the essential spectrum of Weyl of operator T by:*

$$\sigma_w(T) := \bigcap_{K \in \ \mathcal{K}(X)} \sigma(T + K).$$

Proposition 3.1.3. [11, Theorem 7.27] *Let X a Banach space and $T \in \mathcal{C}(X)$ then we have $\lambda \notin \sigma_w(T)$ if and only if $\lambda - T \in \Phi(X)$ and $i(\lambda - T) = 0$.*
So, we can define the essential spectrum of Weyl of T by

$$\sigma_w(T) = \mathbb{C}\backslash\{\lambda \in \mathbb{C} : (\lambda - T) \in \Phi(X) \ et \ i(\lambda - T) = 0\}.$$

Proposition 3.1.4. *Let $(T, D(T))$ be a closed, densely defined, and linear operator with a nonempty resolvent set .$\rho(T)$, for each $\lambda_0 \in \rho(T)$ we have*

$$\sigma\left((\lambda_0 - T)^{-1}\right) = (\lambda_0 - \sigma(T))^{-1}$$

3.2 Convergence

Definition 3.2.1. [1, Definition] *Let (T_n) be a sequence of bounded linear operators and T be a bounded linear operator in Banach space X.*
(i) We call convergence by norm and we note it by $T_n \xrightarrow{n} T$ if

$$\|T_n - T\| \longrightarrow 0 \quad when \quad n \to \infty.$$

(ii) We call pointwise convergence par point et on le note $T_n \xrightarrow{p} T$ if

$$\|T_n x - Tx\| \to 0 \quad when \quad n \to \infty, \quad for\ all \quad x \in X.$$

(iii) We call convergence collectively compact and note it $T_n \xrightarrow{c.c} T$ if $T_n \xrightarrow{p} T$ and for a certain $n_0 \in \mathbb{N}$ we have

$$\bigcup_{n \geq n_0} \left\{(T_n - T)x : \ x \in X, \ \|x\| \leq 1\right\}$$

is a relatively compact set in X.

Proposition 3.2.1. *[10, Remark 4.0.1] (i) Let T is compact linear operator, then we say $T_n \xrightarrow{c.c} T$, if $T_n \xrightarrow{p} T$ and for a certain $n_0 \in \mathbb{N}$ we have $\bigcup_{n \geq n_0} \left\{T_n x : \ x \in X, \ \|x\| \leq 1\right\}$ is a relatively compact set in X.*
(ii) Si $T_n \xrightarrow{n} T$ ou $T_n \xrightarrow{c.C} T$, then $T_n \xrightarrow{p} T$.
(iii) the reverse of (ii) isn't true. In fact, we have the following example
Let $X := l^p, \ 1 \leq p < \infty$. For $n = 1, 2, \ldots$ let

$$\begin{array}{rcl} I : X & \longrightarrow & X \\ x & \longmapsto & x \end{array}$$

such that $x = \sum_{k=1}^{+\infty} x(k)e_k$ and $T_n x = \sum_{k=1}^{n} x(k)e_k$. This implies that T_n is bounded operator with finite range X et $T_n \xrightarrow{p} I$, but $T_n \not\xrightarrow{n} I$ because $\|T_n - I\| = 1$ for all n, and $T_n \not\xrightarrow{c.c} I$ because for all $n_0 \in \mathbb{N}$ we have

$$e_k \in \left\{(I - T_n)x : \ x \in X, \ \|x\| \leq 1\right\} \quad for \quad k = n_0 + 1, n_0 + 2, \ldots$$

but (e_k) *admits no convergent sub-sequence. Means that if* $T_n \xrightarrow{p} I$*, then* $T_n \overset{n}{\not\longrightarrow} I$ *and* $T_n \overset{c.c}{\not\longrightarrow} I$.

Definition 3.2.2. [?, Definition 1.5] *Let* (T_n) *a sequence of closable linear operators of* X *into* Y *and* $T : \mathcal{D}(T) \subset X \longrightarrow Y$ *a closable linear operator. We say that* (T_n) *converge au generally sens toward* T *if* $\delta_{\max}(T_n, T) \to 0$ *when* $n \to +\infty$.

Theorem 3.2.1. *Let* $T \in \mathcal{L}(X)$. $T_n \xrightarrow{g} T$ *if, and only if,* $T_n \in \mathcal{L}(X)$ *for sufficiently larger* n *and* T_n *converges to* T.

Proof. Let us assume that T_n converges in the generalized sense to T and that $T \in \mathcal{L}(X)$. Let $n_0 \in \mathbb{N}$ such that $\widehat{\delta}(T, T_n) < \frac{1}{\sqrt{1+\|T\|^2}}$ holds for all $n \geq n_0$. Suppose that $x \in \mathcal{D}(\overline{T_n})$ where $n \geq n_0$ is fixed. First, we show that

$$\|(T - \overline{T_n})x\| \leq \delta(\overline{T_n}, T)\left(1 + \|T^2\|\right)^{\frac{1}{2}}. \tag{3.1}$$

Indeed,

$$\|(T - \overline{T_n})x\| \leq \|Ty - \overline{T_n}x\| + \|T\|\|x - y\|, \ \forall y \in X.$$

By using the Cauchy-Schwarz inequality, we deduce that

$$\|(T - \overline{T_n})x\| \leq (\|x - y\|^2 + \|Ty - \overline{T_n}x\|^2)^{\frac{1}{2}}\left(1 + \|T\|^2\right)^{\frac{1}{2}}.$$

So, we have

$$\left\|(T - \overline{T_n})x\right\| \leq \left[\inf_{y \in Y}\left(\|x - y\|^2 + \ |Ty - \overline{T_n}x\|^2\right)^{\frac{1}{2}}\right]\left(1 + \|T\|^2\right)^{\frac{1}{2}}. \tag{3.2}$$

Hence, the inequality (3.1) follows immediately by using (3.2). Now, let $x \in \mathcal{D}(\overline{T_n})$, such that $\|x\|^2 + \|\overline{T_n}x\|^2 = 1$. So, it is easy to prove that

$$1 \leq \left(1 + \|T\|^2\right)^{\frac{1}{2}}\left\|x\right\| + \left\|(T - \overline{T_n})x\right\|. \tag{3.3}$$

Let $x \in \mathcal{D}(\overline{T}_n)$ such that $\|x\| \leq 1$. Then we have

$$\left\|(T - \overline{T_n})x\right\| \leq \left(\frac{1 + \|T\|^2}{1 - \sqrt{1 + \|T\|^2\delta(\overline{T_n}, T)}}\delta(\overline{T_n}, T)\right)\left\|x\right\|. \tag{3.4}$$

Since this inequality is homogeneous in x, then (3.4) is also true for any $x \in \mathcal{D}(\overline{T_n})$. The fact that $\delta(T_n, T) = \delta(\overline{T_n}, \overline{T}) = \delta(\overline{T_n}, T)$ and $\mathcal{D}(T_n) \subset \mathcal{D}(\overline{T_n})$, allows us to conclude that

$$\left\|(T - T_n)x\right\| \leq \left(\frac{1 + \|T\|^2}{1 - \sqrt{1 + \|T\|^2\delta(T_n, T)}}\delta(T_n, T)\right)\left\|x\right\|, \forall x \in \mathcal{D}(T_n). \tag{3.5}$$

By virtue of (3.5), the operator T_n is bounded on $\mathcal{D}(T_n)$. This implies that $\mathcal{D}(T_n)$ is closed. Hence, $\overline{\mathcal{D}(T_n)} = \mathcal{D}(T_n) = X$, $T_n \in \mathcal{L}(X, Y)$, and

$$\left\|T_n - T\right\| \leq \left(\frac{1 + \|T\|^2}{1 - \sqrt{1 + \|T\|^2\delta(T_n, T)}}\right)\delta(T_n, T), \ \forall n \geq n_0. \tag{3.6}$$

The relation (3.6) enables us to conclude that T_n converges to T. Conversely, we suppose that T_n converges to T. So, T is bounded. Now, we can write $\widehat{\delta}(T_n, T) = \widehat{\delta}\Big((T_n - T) + T, 0 + T\Big)$. This implies that

$$\widehat{\delta}(T_n, T) \leq 2\Big(1 + \|T\|^2\Big)\widehat{\delta}(T_n - T, 0). \tag{3.7}$$

Theorem 2.4.2 leads to the following inequality $\widehat{\delta}(T_n, T) \leq 2\Big(1 + \|T\|^2\Big)\frac{\|T_n - T\|}{\sqrt{1 + \|T_n - T\|^2}}$. As a result, T_n converges in the generalized sense to T. □

Theorem 3.2.2. [2, Theorem 2.3] *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of closable linear operators on X and let T be a closable linear operator on X.*

$T_n \xrightarrow{g} T$, if, and only if, $T_n + S \xrightarrow{g} T + S$, for all $S \in \mathcal{L}(X)$.

Proof. Let $S \in \mathcal{L}(X, Y)$, then

$$\begin{aligned} \widehat{\delta}(T_n + S, T + S) &= \widehat{\delta}(\overline{T_n + S}, \overline{T + S}) \\ &= \widehat{\delta}(\overline{T_n} + S, \overline{T} + S). \end{aligned}$$

By using Theorem 2.4.3, we have $\widehat{\delta}(\overline{T_n} + S, \overline{T} + S) \leq 2\,(1 + \|S\|^2)\,\widehat{\delta}(\overline{T_n}, \overline{T})$. Hence,

$$\widehat{\delta}(T_n + S, T + S) \leq 2(1 + \|S\|^2)\widehat{\delta}(T_n, T). \tag{3.8}$$

In other terms, $\widehat{\delta}(\overline{T_n}, \overline{T}) = \widehat{\delta}\left(\overline{T_n + S} - S, \overline{T + S} - S\right)$, hence,

$$\widehat{\delta}(T_n, T) \leq 2(1 + \|S\|^2)\widehat{\delta}(T_n + S, T + S). \tag{3.9}$$

If T_n converges in the generalized sense to T then $\widehat{\delta}(T_n, T) \to 0$. So, by using (3.8) we have, $\widehat{\delta}(T_n + S, T + S) \to 0$. Then, $T_n + S$ converges in the generalized sense to $T + S$. Conversely, if $T_n + S$ converges in the generalized sense to $T + S$ and according to (3.9) we have, $\widehat{\delta}(T_n + S, T + S) \to 0$ as $n \to \infty$. Hence, T_n converges in the generalized sense to T. □

Theorem 3.2.3. *Let $T_n \xrightarrow{g} T$. Then, T^{-1} exists and $T^{-1} \in \mathcal{L}(Y)$, if, and only if, T_n^{-1} exists and $T_n^{-1} \in \mathcal{L}(X)$ for sufficiently large n and T_n^{-1} converges to T^{-1}.*

Proof. Now, let us argue by contradiction. We assume that there exists $x \in \mathcal{D}(T_n)$ such that $\|x\| = 1$ and $T_n x = 0$, for all $n \in \mathbb{N}$. In other words, T_n converges in the generalized sense to T then there exists $N \in \mathbb{N}$ such that $\widehat{\delta}(T_n, T) < \frac{1}{\sqrt{1+\|T^{-1}\|^2}}$, $\forall n \geq N$. Then there exists $\delta > 0$ such that

$$\widehat{\delta}(T_n, T) < \delta < \frac{1}{\sqrt{1 + \|T^{-1}\|^2}}.$$

Since $(x, 0) \in G(T_n)$, then there exists $y \in \mathcal{D}(T)$ such that

$$\|x - y\|^2 + \|\overline{T_n}x - Ty\|^2 < \delta^2.$$

Hence, we have

$$1 = \|x\|^2 \leq \Big(\|x-y\| + \|y\|\Big)^2 \leq \Big(\|x-y\| + \|T^{-1}\|\|Ty\|\Big)^2, \tag{3.10}$$

and, by using both the Schwarz inequality and (3.10) we infer that

$$1 \leq \delta^2\Big(1 + \|T^{-1}\|^2\Big) < 1$$

is a contradiction. So, T_n^{-1} exists for sufficiently larger n and by virtue of (3.6), we deduce that $T_n^{-1} \in \mathcal{L}(Y, X)$, and

$$\|T^{-1} - T_n^{-1}\| \leq \left(\frac{1 + \|T^{-1}\|^2}{1 - \sqrt{1 + \|T^{-1}\|^2}\delta(T_n^{-1}, T^{-1})}\right)\delta(T_n^{-1}, T^{-1}). \tag{3.11}$$

By using Theorem 2.4.4 , and also the estimation (3.11) we infer that

$$\|T^{-1} - T_n^{-1}\| \leq \left(\frac{1 + \|T^{-1}\|^2}{1 - \sqrt{1 + \|T^{-1}\|^2}\delta(T_n, T)}\right)\delta(T_n, T). \tag{3.12}$$

According to (3.12), we also emphasize that T_n^{-1} converges to T^{-1}. Conversely, for the reasoning of the proof of theorem 2.4.2, it is sufficient to just replace T and T_n by T^{-1} and T_n^{-1} respectively.

□

Conclusion

In this memory, we have study some results on gap between two closed operators

- some properties of closed and closable linear operators between two Banach spaces.
- Gap between two subspaces, its properties and gap between two subspaces at Hilbert space, gap between two bounded linear operators and the gap between two closed linear operators, we introduce some theorems and properties.
- If X and Y are two Banach spaces, T a linear operator from X into Y, then we can study the convergence of (T_n) in several cases.
- investigated for future research (open problem).

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