REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE MINISTERE DE L'ENSEIGNEMENT SUPERIEUR ET DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE MOHAMED BOUDIAF - M'SILA

FACULTE DE SCIENCE ETTECHNOLOGIE

DEPARTEMENT: GENIE ELECTRIQUE

FILIERE: AUTOMATIQUE

OPTION : ROBOTIQUE

Mémoire Présenté pour L'obtention
du Diplôme de Master Professional

Par : **TOUFIK SACI**

Intitulé :

Approximation methods of fractional order system

Soutenu devant le jury composé de :

Dr. Idir Abdelhakim	**Président**	**Univ. De M'sila**
Pr. KHETTAB Khatir	**Promoteur**	**Univ. De M'sila**
Dr. BOUKHALFA Abdelouaheb	**Co- Promoteur**	**Univ. De M'sila**
Dr. Roubache Touik	**Examinateur**	**Univ. De M'sila**

Année universitaire : 2022/2023

Thanks

*In the **name of** الله, the All-Merciful, the Most-Merciful*

Praise be to الله the only one and peace and salvation on him who has no messenger after him and on his family, his companions and all those who follow his way until the day of resurrection

*First of all, this work would not be as rich and would not have been possible without the help and supervision of **Mr Khatir Khatab and Mr, abdelwahab Boukhalfa,** we thank them for the quality of his exceptional supervision, for thier patience, hier rigor and availability during our preparation of this dissertation.*

also address my sincere thanks to the members of the juries for having kindly examined and judged this work.

We would also like to thank all the members of our family, our brothers and our dear sisters.

Finally, we would like to express our gratitude to all our friends and colleagues for the moral and material support.

Summary

List of images

General Introduction

Linear fractional systems (LFS) have a long history in the field of control systems. The use of LFS in control systems can be traced back to the early 1960s when researchers began to study the synthesis of linear controllers for systems with fractional-order dynamics.

In the early days, the study of LFS was focused on stability analysis and controller synthesis for fractional-order systems using classical control techniques such as the root locus and Nyquist methods. However, the analysis and synthesis of LFS using classical techniques were limited by the complexity of the transfer function and the presence of poles and zeros at infinity.

In the 1980s, researchers began to develop modern control techniques for LFS using the theory of linear matrix inequalities (LMIs) and convex optimization. The LMI-based techniques enabled the analysis and synthesis of LFS for large-scale systems with multiple inputs and outputs and provided a unified framework for the design of robust and optimal controllers.

The 1990s saw the emergence of model-based control techniques for LFS, such as the H-infinity control and the mu-synthesis, which enabled the design of controllers that guarantee performance and robustness under model uncertainty and disturbances.

In recent years, the use of LFS in control systems has expanded to include data-driven and machine learning-based methods for system identification and control. These methods use data to learn a low-dimensional model of the system, which can then be used for analysis and control.

In summary, the use of LFS in control systems has a long and rich history. From the early days of stability analysis and controller synthesis to the modern techniques of LMI-based control and model-based control, LFS has been a fundamental concept in control theory and has enabled the design of robust and optimal controllers for a wide range of systems.

Linear fractional systems (LFS) are a class of dynamical systems that arise in various fields of science and engineering, including control theory, signal processing, and communication systems. An LFS is a linear time-invariant (LTI) system where the input and output signals are related through a rational transfer function of the form H(s) = N(s)/D(s), where N(s) and D(s) are polynomial functions of the Laplace variable s. LFS are also known as linear systems with rational transfer functions.

The behavior of LFS is characterized by the poles and zeros of the transfer function, which determine the stability and frequency response of the system. However, exact analysis and synthesis of LFS can be challenging due to the complexity of the transfer function and the presence of poles and zeros at infinity.

To overcome these challenges, several approximation techniques have been developed to simplify the transfer function of LFS while preserving their essential properties.

One common approach to approximating linear fractional systems is based on the use of Padé approximants. Padé approximants are rational functions that can be used to approximate a given function or system. The basic idea behind Padé approximants is to construct a rational function that has the same Taylor series expansion as the given function up to a certain order. In the context of linear fractional systems, Padé approximants can be used to approximate the transfer function of the system.

Another approach to approximating linear fractional systems is based on the use of model reduction techniques. Model reduction techniques are used to reduce the complexity of a given system by eliminating unnecessary degrees of freedom. In the context of linear fractional systems, model reduction techniques can be used to reduce the order of the transfer function of the system. One popular model reduction technique is called balanced truncation. Balanced truncation is a method that can be used to reduce the order of a linear system while preserving its key properties.

A third approach to approximating linear fractional systems is based on the use of frequency domain methods. Frequency domain methods are used to analyze and control linear systems in the frequency domain. In the context of linear fractional systems, frequency domain methods can be used to approximate the transfer function of the system by fitting a simpler model to the frequency response data of the system.

In recent years, the approximation of LFS has gained significant attention due to the emergence of data-driven and machine learning-based methods for system identification and control. These methods use data to learn a low-dimensional model of the system, which can then be used for analysis and control.

In summary, the approximation of LFS is a fundamental problem in system theory and has applications in several fields of science and engineering. The goal of this thesis is to investigate and compare different approximation techniques for LFS and their applications in control and signal processing.

CHAPTER I

Fractional Order Systems

I.1 : Introduction

In the 1960s, customers took a great interest in fractional ordering systems. These systems are generally described by fractional differential equations, since it may help solve many problems and bridge the gaps that other systems suffer from. In the frequency domain, they are represented by irrational transfer functions. Because of these irrational functions, the fractal order system is rarely studied. Since there are no exact analytical solutions, numerical and approximate techniques are widely used for their analysis, analysis, and implementation. In this chapter, we will introduce the definition of fractional calculus and the coefficients of fractions, some key properties, and the Laplace transform of fractional derivatives. We will introduce some methods of operator approximation and fractional order transfer, in particular

I.2 : Brief of history

The fractional calculus (FC) may be considered an old and yet novel topic. It is an old topic because, starting from some speculations of G.W. Leibniz (1695, 1697) and L. Euler (1730), it has been developed progressively up to now. However, it may be considered a novel topic as well.

Only since the Seventies, the FC has been the object of specialized conferences and treatises. For the first conference the merit is due to B. Ross who, shortly after his Ph.D. dissertation on fractional calculus, organized the First Conference on Frac- tional Calculus and its Applications at the University of New Haven in June 1974, and edited its proceedings.

For the first monograph, the merit is ascribed to K.B. Oldham and J. Spanier, who, after a joint collaboration begun in 1968, published a book devoted to fractional calculus in 1974. This collaboration between a chemist (Oldham) and a mathemati- cian (Spanier) in treating problems of mass and heat transfer in terms of the so-called semi-derivatives and semi-integrals, clearly manifested the origin of new era for FC based both on physical intuition and mathematical versatility. In 1987, the huge book of S. Samko, A. Kilbas and O. Marichev, referred to now as “encyclopedia” of FC, appeared first in Russian, and later with an English edition in 1993. Nowadays, the series of books, journals and texts devoted to fractional calculus and its applications includes several dozens of titles and this list is expected to grow up yet more, in the forthcoming years. [1]

I.3 : Usage and applications

The subject of fractional calculus has applications in diverse and widespread fields of engineering and science such as electromagnetics, viscoelasticity, fluid mechanics, electrochemistry, biological population models, optics, and signals processing. It has been used to model physical and engineering processes that are found to be best described by fractional differential equations. The fractional derivative models are used for accurate modelling of those systems that require accurate modelling of damping. In these fields, various analytical and numerical methods including their applications to new problems have been proposed in recent years. This special issue on “Fractional Calculus and its Applications in Applied Mathematics and Other Sciences” is devoted to study the recent works in the above fields of fractional calculus done by the leading researchers. The papers for this special issue were selected after a careful and studious peer-review process. [2]

I.4 : Fractional Order Derivation

Fractional differentiation is the generalization of integer differentiation to any non-integer orders. This generalization can be obtained from the fractional integration of the thus giving the Riemann-Liouville definition and the Caputo definition. Another generalization, based on the usual definition of integer differentiation is proposed by Grunwald-Letnikov.

I.4.1 : Riemann–Liouville definition

In general, one can define fractional-order derivatives of $\boldsymbol{f}$ as well by

$$\frac{d^{\alpha}}{dx^{\alpha}}f \overset{\text{def}}{=} \frac{d^{\lceil\alpha\rceil}}{dx^{\lceil\alpha\rceil}} I^{\lceil\alpha\rceil-\alpha}f \tag{I.1}$$

where ⌈·⌉ denotes the ceiling function. One also obtains a differintegral interpolating between differentiation and integration by defining [3]

$$D_x^{\alpha} f(x) = \begin{cases} \frac{d^{\lceil\alpha}}{dx^{\lceil\alpha\rceil}} I^{\lceil\alpha\rceil-\alpha} f(x) & \alpha > 0 \\ f(x) & \alpha = 0 \ \text{ with } \alpha \text{ is Intiger} \\ I^{-\alpha} f(x) & \alpha < 0 \end{cases} \tag{I.2}$$

An alternative fractional derivative was introduced by Caputo in **1967**, and produces a derivative that has different properties: it produces zero from constant functions and, more importantly, the initial value terms of the Laplace Transform are expressed by means of the values of that function and of its derivative of integer order rather than the derivatives of fractional order as in the Riemann-Liouville derivative. The Caputo fractional derivative with base point $\boldsymbol{x}$, is then:

$$D_x^{\alpha} f(y) = \frac{1}{\Gamma(1-\alpha)} \int_x^y f'(y-u)(u-x)^{-\alpha} du. \tag{I.3}$$

Another representation is:

$${}_a\tilde{D}_x^{\alpha} f(x) = I^{\lceil\alpha\rceil-\alpha} \left(\frac{d^{\lceil\alpha\rceil} f}{dx^{\lceil\alpha\rceil}} \right) \tag{I.4}$$

Now, we are going to take a basic function and see the result.

The first derivative is as usual

$$f'(x) = \frac{d}{dx} f(x) = kx^{k-1}. \tag{I.5}$$

Repeating this gives the more general result that

$$\frac{d^a}{dx^a}x^k = \frac{k!}{(k-a)!}x^{k-a} \tag{I.6}$$

which, after replacing the factorials with the gamma function, leads to

$$\frac{d^a}{dx^a}x^k = \frac{\Gamma(k+1)}{\Gamma(k-a+1)}x^{k-a},\ k > 0. \tag{I.7}$$

For $\boldsymbol{k = 1}$ and $\boldsymbol{a = \frac{1}{2}}$, we obtain the half-derivative of the function $\boldsymbol{x \mapsto x}$ as

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x = \frac{\Gamma(1+1)}{\Gamma\left(1-\frac{1}{2}+1\right)}x^{1-\frac{1}{2}} = \frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)}x^{\frac{1}{2}} = \frac{1}{\frac{\sqrt{\pi}}{2}}x^{\frac{1}{2}} \tag{I.8}$$

To demonstrate that this is, in fact, the "half derivative" (where $\boldsymbol{H^2 f(x) = Df(x)}$), we repeat the process to get:

$$\boldsymbol{\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\frac{2x^{\frac{1}{2}}}{\sqrt{\pi}} = \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(1+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}+1\right)}x^{\frac{1}{2}-\frac{1}{2}} = \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)}x^0 = \frac{2\frac{\sqrt{\pi}}{2}x^0}{\sqrt{\pi}} = 1,} \tag{I.9}$$

because $\boldsymbol{\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}}$ and $\boldsymbol{\Gamma(1) = 1}$

which is indeed the expected result of

$$\left(\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\right)x = \frac{d}{dx}x = 1 \tag{I.10}$$

For negative integer power $\boldsymbol{k}$, $\boldsymbol{1/\Gamma}$ is 0 , so it is convenient to use the following relation:

$$\frac{d^a}{dx^a}x^{-k} = (-1)^a\frac{\Gamma(k+a)}{\Gamma(k)}x^{-(k+a)}\ \text{ for } k \geq 0. \tag{I.11}$$

This extension of the above differential operator need not be constrained only to real powers; it also applies for complex powers. For example, the $\boldsymbol{(1+i)}$-th derivative of the $\boldsymbol{(1-i)}$-th derivative yields the second derivative. Also setting negative values for $\boldsymbol{a}$ yields integrals.

For a general function $f(x)$ and $0 < \alpha < 1$, the complete fractional derivative is

$$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^{\alpha}} dt \tag{I.12}$$

For arbitrary α, since the gamma function is infinite for negative (real) integers, it is necessary to apply the fractional derivative after the integer derivative has been performed.

I.4.2 : Definition of Caputo

Another option for computing fractional derivatives is the Caputo fractional derivative. It was introduced by Michele Caputo in his 1967 paper [4].In contrast to the Riemann-Liouville fractional derivative, when solving differential equations using Caputo's definition, it is not necessary to define the fractional order initial conditions. Caputo's definition is illustrated as follows, where again $n = \lceil \alpha \rceil$.

$${}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \tag{I.13}$$

There is the Caputo fractional derivative defined as:

$$D^{\nu} f(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-u)^{(n-\nu-1)} f^{(n)}(u) du \quad (n-1) < \nu < n \tag{I.14}$$

which has the advantage that is zero when $f(t)$ is constant and its Laplace Transform is expressed by means of the initial values of the function and its derivative. Moreover, there is the Caputo fractional derivative of distributed order defined as

$${}_a^b D^{\nu} f(t) = \int_a^b \phi(\nu) \left[D^{(\nu)} f(t) \right] d\nu = \int_a^b \left[\frac{\phi(\nu)}{\Gamma(1-\nu)} \int_0^t (t-u)^{-\nu} f'(u) du \right] d\nu \tag{I.15}$$

where $\varphi(v)$ is a weight function and which is used to represent mathematically the presence of multiple memory formalisms.

I.4.3 : Caputo-Fabrizio fractional derivative

In a paper of 2015, M. Caputo and M. Fabrizio presented a definition of fractional derivative with a non singular kernel, for a function $\boldsymbol{f(t)}$ of $\boldsymbol{C^1}$ given by [5]:

$$ {}_{a}^{CF}D_{t}^{\alpha}f(t) = \frac{1}{1-\alpha}\int_{a}^{t} f'(\tau)e^{\left(-\alpha\frac{t-\tau}{1-\alpha}\right)}d\tau \tag{I.16} $$

where $\mathbf{a < 0, \alpha \in 0, 1}$

I.4.4 : Atangana-Baleanu fractional derivative

In 2016, Atangana and Baleanu suggested differential operators based on the generalized Mittag-Leffler function. The aim was to introduce fractional differential operators with non-singular nonlocal kernel. Their fractional differential operators are given below in Riemann-Liouville sense and Caputo sense respectively. For a function $\boldsymbol{f(t)}$ of $\boldsymbol{C^1}$ given by [6]:

$$ {}_{a}^{ABC}D_{t}^{\alpha}f(t) = \frac{AB(\alpha)}{1-\alpha}\int_{a}^{t} f'(\tau)E_{\alpha}\left(-\alpha\frac{(t-\tau)^{\alpha}}{1-\alpha}\right)d\tau \tag{1.17} $$

If the function is continuous, the Atangana-Baleanu derivative in Riemann-Liouville sense is given by:

$$ {}_{a}^{ABC}D_{t}^{\alpha}f(t) = \frac{AB(\alpha)}{1-\alpha}\frac{d}{dt}\int_{a}^{t} f(\tau)E_{\alpha}\left(-\alpha\frac{(t-\tau)^{\alpha}}{1-\alpha}\right)d\tau, \tag{I.18} $$

The kernel used in Atangana-Baleanu fractional derivative has some properties of a cumulative distribution function. For example, for all $\boldsymbol{\alpha \in (0, 1]}$, the function $\boldsymbol{E_{\alpha}}$ is increasing on the real line, converges to 0 in $-\infty$, and $\boldsymbol{E_{\alpha}(0) = 1}$. Therefore, we have that, the function $\boldsymbol{x \mapsto 1 - E_{\alpha}(-x^{\alpha})}$ is the cumulative distribution function of a probability measure on the positive real numbers. The distribution is therefore defined, and any of its multiples, is called a Mittag-Leffler distribution of order $\boldsymbol{\alpha}$. It is also very well-known that, all these probability distributions are absolutely continuous. In particular, the function Mittag-Leffler has a particular case $\boldsymbol{E_1}$, which is the exponential function, the Mittag-Leffler distribution of order 1 is therefore an exponential distribution. However, for $\boldsymbol{\alpha \in (0, 1)}$, the Mittag-Leffler distributions are heavy-tailed. Their Laplace transform is given by:

$$ \mathbb{E}\left(e^{-\lambda X_{\alpha}}\right) = \frac{1}{1+\lambda^{\alpha}} \tag{I.19} $$

This directly implies that, for $\boldsymbol{\alpha} \in (\mathbf{0}, \mathbf{1})$, the expectation is infinite. In addition, these distributions are geometric stable distributions.

I.5 : Properties of Fractional Order Integration and Derivation

The properties of fractional order integration and derivation are similar to those of integer-order integration and derivation, but with some important differences. Here are some of the key properties:

Linearity: Fractional order integration and derivation are linear operations. That is, if f(t) and g(t) are functions and a and b are constants, then:

$$\mathbf{D^{\alpha}[af(t) + bg(t)] = aD^{\alpha}[f(t)] + bD^{\alpha}[g(t)] \text{ and } \int [0,t]\big(af(\tau) + bg(\tau)\big)d\tau = a\int [0,t]f(\tau)d\tau + b\int [0,t]g(\tau)d\tau} \tag{I.20}$$

Commutativity: Fractional order integration and derivation do not commute with each other. That is, in general:

$$\mathbf{D^{\alpha}[\int [0,t]\, f(\tau)\, d\tau] \neq \int [0,t]D^{\alpha}[f(\tau)]\, d\tau} \tag{I.21}$$

Fractional order integration and derivation are inverse operations: Applying a fractional order integration followed by a fractional order derivation (or vice versa) results in the original function. That is:

$$\mathbf{D^{\alpha}[\int [0,t]\, f(\tau)\, d\tau] = f(t) \text{ and } \int [0,t]D^{\alpha}[f(\tau)]\, d\tau = f(t)} \tag{I.22}$$

Fractional order derivation of a constant is zero. That is:

D^α[C] = 0 (where C is a constant)

Fractional order integration of a constant is a linear function of time. That is:

$$\mathbf{\int [0,t]C\, d\tau = C * t} \tag{I.23}$$

These are some of the main properties of fractional order integration and derivation. It is worth noting that, unlike integer-order differentiation and integration, there is no simple physical interpretation of fractional order differentiation and integration. Rather, these operators are mathematical constructs that have been found to be useful in modeling complex systems.

I.6 : Fractional Order Operators

A fractional-order operator is a mathematical operator that generalizes the concept of differentiation and integration to non-integer orders. In other words, it allows for fractional differentiation and integration.

The most common type of fractional-order operator is the fractional-order derivative, which is defined as follows:

$$\boldsymbol{D^{\alpha[f(t)]} = \frac{1}{\Gamma(n-\alpha)} * \int (0\ to\ t)\left(\frac{f(\tau)}{}(t-\tau)^{\alpha+1-n}\right) d\tau} \tag{I.24}$$

where α is the order of the derivative, f(t) is a function of time, n is an integer greater than α, and Γ is the gamma function.

The fractional-order derivative can be interpreted as a generalization of the ordinary derivative, where α=1 corresponds to the first-order derivative, α=2 corresponds to the second-order derivative, and so on.

Fractional-order operators have found applications in many fields, including control theory, signal processing, and image processing. They offer a more flexible way of modeling complex systems that exhibit non-integer order dynamics [7].

I.7 : Responses of basic fractional order function with the basic input signals

In this part, we will take a basic fractional-order transfer function when:

$$\mathbf{G(s) = \frac{1}{1+\tau_0 s^{\alpha}}} \tag{I.25}$$

$$\boldsymbol{\alpha = [1.1\ ; 1.2; 1.5; 1.7; 1.9]} \tag{I.26}$$

$$\tau_0 = 1 \tag{I.27}$$

I.7.1 : Response in frequency domain

The next diagram is plotted using MATLAB directly with the build-in function (bode)

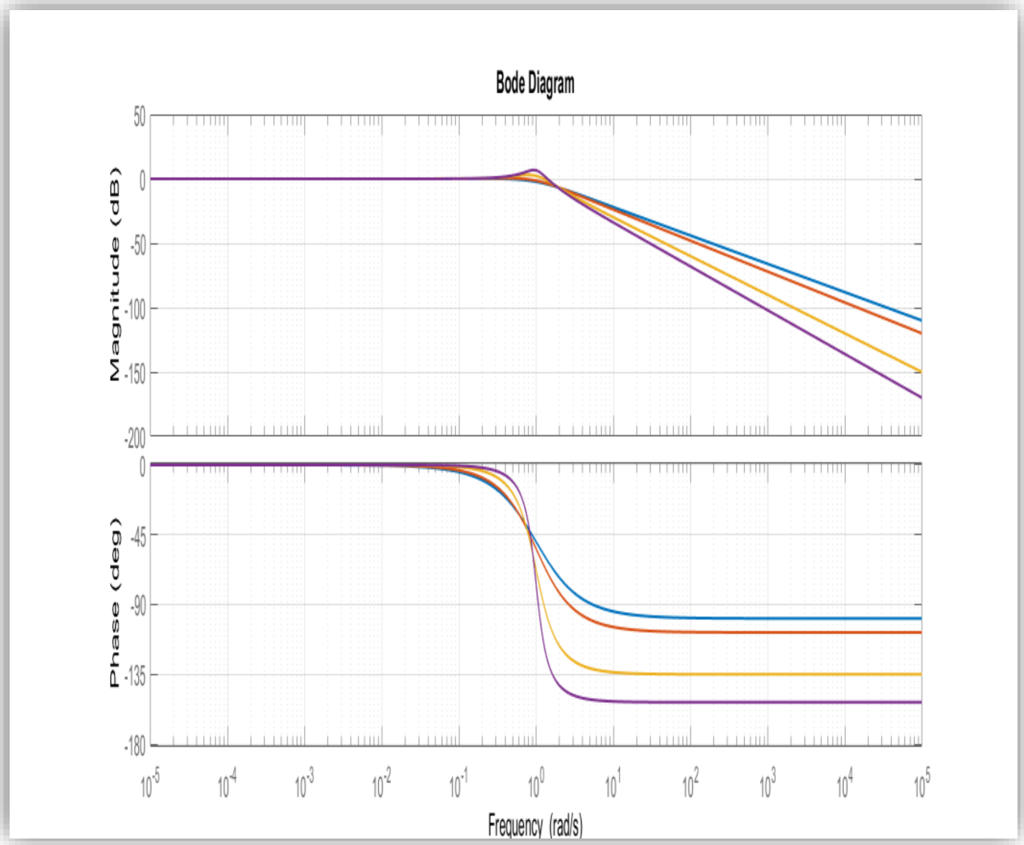

Figure I.1 : Bode diagram of G(s)

I.7.2 : Response in Time domain

I.7.2.1 : Step response

$$\mathrm{E(S)} = \frac{1}{\mathrm{S}} \qquad \text{(I.28)}$$

$$\mathbf{G(s)} = \frac{\mathbf{1}}{\mathbf{1+\tau_0 s^{\alpha}}} \qquad \text{(I.29)}$$

$$\mathbf{Y(S) = E(S) * G(S)} \qquad \text{(I.30)}$$

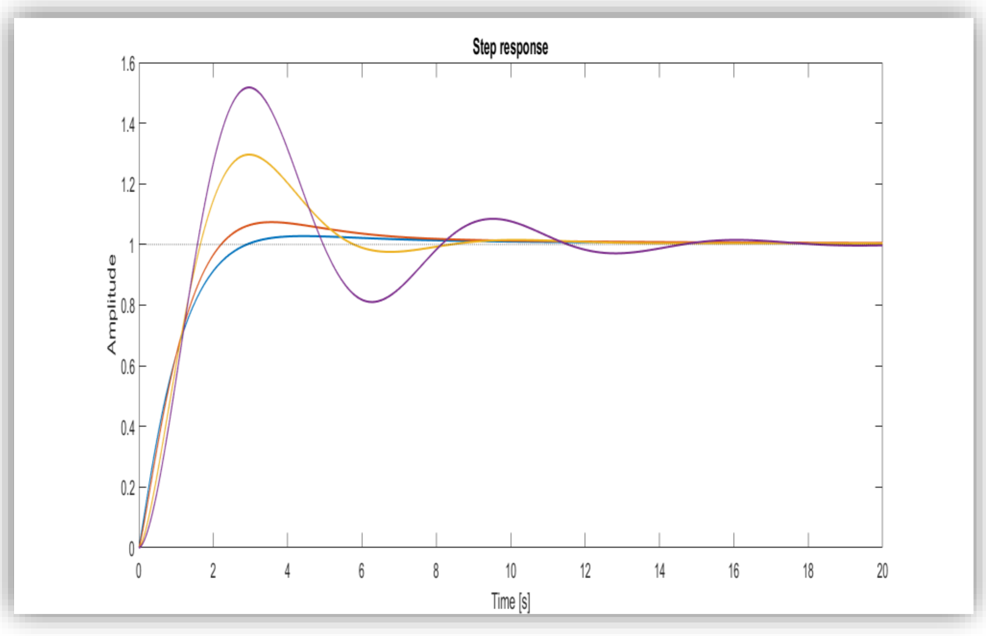

Figure I.1 : Step response of G(s),$\alpha = [\mathbf{1.1\ 1.2\ 1.5\ 1.7\ 1.9}]$

I.7.2.2 : Impulse response

$$E(S) = L\ \{\delta(t)\} \tag{I.31}$$

$$Y(S) = E(S) * G(S) \tag{I.32}$$

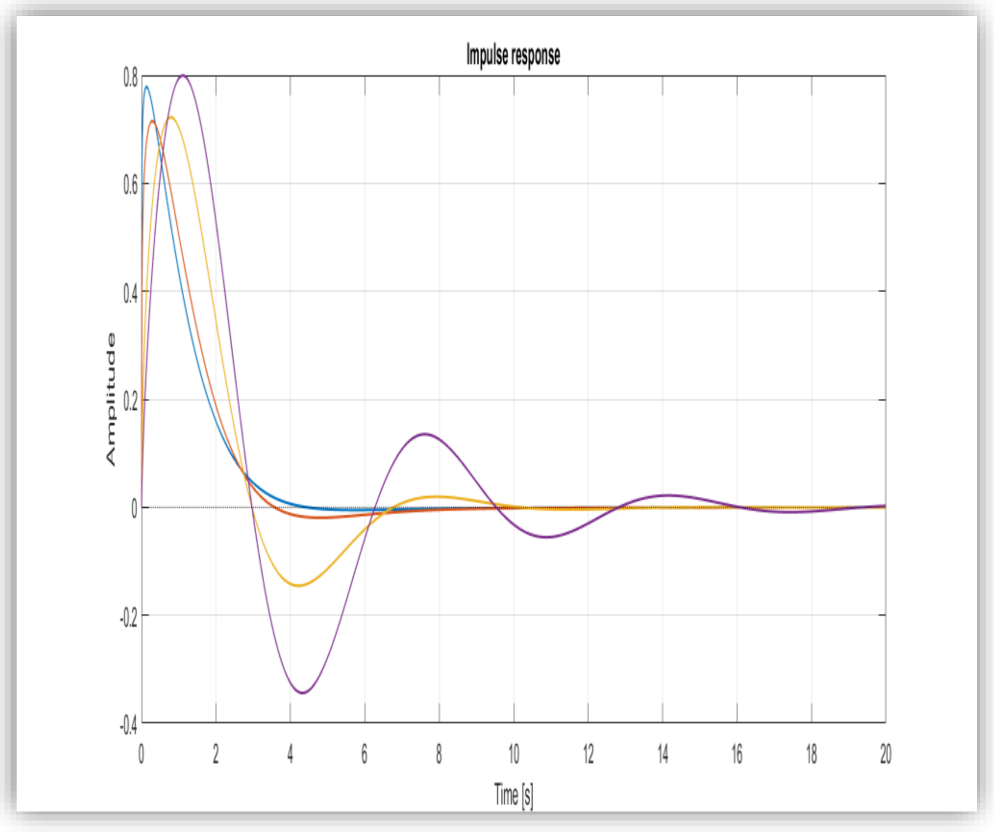

Figure I.2 : Impulse response of G(s)

I.8 : Conclusion

In this chapter we provide a brief history of linear fractional ordering systems. We also discussed the basic definitions of linear ordering systems and several different definitions of differentiation. We also presented a simple function response representation of a linear fractional ordering system.

CHAPTER II

Fractional Order Systems Approximation

II.1 : Introduction

In this chapter, we explore various methods of approximating fractional order systems, including modern numerical and analog approximation methods. We also discuss in detail the method of Charf, which is an analog approximation method for fractional order differentiators.

We first introduce the concept of fractional order systems and the challenges associated with their analysis and control. We then discuss the various numerical approximation methods, including the fractional Taylor series method, nonlinear least squares fitting method, fractional finite element method, fractional Adams-Bashforth method, and Grunwald-Letnikov fractional finite difference method. Next, we introduce the various analog approximation methods, including metamaterials, fractional order RC networks, fractional order PID controllers, fractional order oscillators, and fractional order operational amplifiers.

We then focus on the method of *Charf*, which is a widely used analog approximation method for fractional order differentiators. We discuss the circuit topology and design principles of the method of *Charf* and compare it with other analog approximation methods. We also present simulation results that demonstrate the effectiveness of the method of *Charf* for approximating fractional order differentiators.

II.2 : Numeric approximation to the fractional order system

Through many years of study, many Numeric methods have emerged to approximate systems of fractional order, and these methods are different in many ways, so it is difficult to classify them according to a specific criterion, but to simplify the study, we can say that there are three strategies [8]:

- Calculation of the analytical expression of the output of the system: This method involves calculating the exact analytical expression of the output of the fractional order system. This may be feasible for simple systems, but can be challenging for more complex systems.

- Approximation of the non-integer model by a discrete rational model: This method involves approximating the non-integer model using a discrete rational model, such as a transfer function or a difference equation. This can be an efficient and accurate method for numerical approximation of fractional order systems.
- Approximation of the non-integer model by a continuous rational model then the discretization of the latter: This method involves approximating the non-integer model using a continuous rational model, such as a Laplace transform or a differential equation, and then discretizing the latter to obtain a discrete approximation. This can also be an effective method for numerical approximation of fractional order systems.

II.2.1 : Fractional Taylor Series Method

This method based on Caputo fractional derivatives that we mentioned above as initial point. This method can give a proper result till the fourth order [10]

$a \in [0,1]$

${}^{C}D_t^{\alpha} f(t)$ is continuous

$$\Delta_k = H^{k\alpha} - L^{k\alpha} - \Gamma(k\alpha+1)\sum_{j=1}^{k-1} \frac{L^{j\alpha}\Delta_{k-j}}{\Gamma(j\alpha+1)\Gamma((k-j)\alpha+1)} \tag{II.1}$$

$$f(x) = \sum_{j=0}^{n} {}_{c}D_a^{j\alpha} f(a_1)\frac{\Delta_j}{\Gamma(j\alpha+1)} + R_n(x, a_1, a) \tag{II.2}$$

II.2.2 : Fractional Adams-Bashforth Method

In recent decades, the fractional calculus and fractional differential equations have attracted much attention and increasing interest due to their potential applications in science and. Here we study a fractional differential equation in the following form [9]:

$$D_*^{\alpha} y(t) = f(t, y(t)),\ y^{(k)}(0) = y_0^{(k)},\ k = 0,1,\dots,n-1 \tag{II.3}$$

where $\alpha > 0$ and $n := \lceil \alpha \rceil$ is the first integer not less than α. $D_*^{\alpha} y(t)$ is the α th-order (always fractional) derivative of $y(t)$ in the Caputo sense, which is defined by

$$D_*^{\alpha} z(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} z^{(n)}(\tau) d\tau,\ n-1 < \alpha < n \in Z^+, \tag{II.4}$$

where $z^{(n)}$ denotes the derivative of integer nth order of z. If we require the function f to be continuous and satisfy a Lipschitz condition with respect to the second argument y with Lipschitz constant L on a suitable set G, then the initial value problem determines a unique solution on some interval $[0, T]$, we always assume that f fulfils the above condition, so Eqhas one and only one solution defined on $[0, T]$. This solution solves the following Volterra integral equation:

$$y(t) = \sum_{k=0}^{n-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u, y(u)) du,\ t \le T. \tag{II.5}$$

Now numerical integration of differential the equation is transformed into numerical quadrature of an associated integral equation

The fractional Adams method for solving Eqs was first studied by Diethelm, Ford and Freed. They worked on a uniform grid $\{t_j = jh : j = 0,1, \dots, N\}$ with some integer N and step length $h = T/N$, and let $y_j \approx y(t_j)$. In detail, their derived computation scheme is as follows:

$$\begin{cases} y_{k+1}^P &= \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j), \\ y_{k+1} &= \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}^P) \right), \end{cases} \tag{II.6}$$

Were

$$a_{j,k+1} = \frac{h^{\alpha}}{\alpha(\alpha+1)} \cdot \begin{cases} (k^{\alpha+1} - (k-\alpha)(k+1)^{\alpha}) & \text{if } j = 0 \\ ((k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}) & \text{if } 1 \le j \le k \\ 1 & \text{if } j = k+1 \end{cases}$$

and

$$b_{j,k+1} = \frac{h^{\alpha}}{\alpha} ((k+1-j)^{\alpha} - (k-j)^{\alpha}),\ j = 0,1,2, \dots, k \tag{II.7}$$

II.2.3 : Grunwald-Letnikov Fractional Finite Difference Method

The Grunwald-Letnikov fractional finite difference method is a numerical method for approximating fractional derivatives of a function. It is based on the definition of fractional derivatives in terms of fractional difference operators, which are discrete analogs of fractional differentiation [12].

The Grunwald-Letnikov method uses a finite difference formula to approximate the fractional derivative of a function f(x) at a point x = xn, with respect to the independent variable x. The fractional derivative is defined using the Caputo derivative.

II.3 : Analog approximation to the fractional order system

II.3.1 : Fractional Order RC Networks

Fractional order RC networks are a type of analog approximation method for fractional order systems. They involve using fractional order capacitors and/or resistors to approximate the behavior of a given fractional order system. The basic idea is to replace the fractional order system with an equivalent circuit consisting of a network of fractional order RC elements.

The fractional order RC network can be designed using a variety of methods, including the method of characteristics, the method of transfer functions, and the method of impedance matching. The choice of method depends on the specific problem being solved and the desired level of accuracy.

One advantage of using fractional order RC networks is that they can be easily implemented using analog circuit components, such as operational amplifiers and capacitors. This allows for the design of low-power, low-cost circuits for fractional order systems.

However, there are some limitations to using fractional order RC networks. One limitation is that they may not be able to accurately model systems with highly nonlinear behavior. Additionally, the accuracy of the approximation may be limited by the availability of fractional order capacitors and resistors.

Overall, fractional order RC networks are a useful tool for approximating fractional order systems using analog circuits, but their effectiveness depends on the specific problem and the level of accuracy required [11].

II.3.2 : Fractional Order PID Controllers

The fractional order proportional-integral-derivative (FOPID) controller has the tunable integral and differential orders, creating the possibility to provide better control performance . However, the design of the FOPID controller is also more difficult.

To simplify the process, will represent the FOPID as :

$$c(s) = k_P\left(1 + \frac{k_i}{s^{\lambda}} + k_d s^{\mu}\right) \tag{II.8}$$

where Kp, Ki, and Kd represent the gains of the proportional, integral, and derivative components, respectively; λ and μ are the real number orders with 0<λ<1 and 0<μ<1.

The typical unit negative feedback control system can be represented as eq 001, where G(s) and C(s) are the plant and controller, respectively, and nr and n are the reference and output signals, respectively. The classic frequency-domain method depends on three specifications, i.e., the gain crossover frequency ω_c, the phase margin φ_m, and the slope of the phase at ω_c, yielding, [13]

$$|C(j\omega_c)G(j\omega_c)| = 1, \tag{II.9}$$

$$\boldsymbol{Arg}[\boldsymbol{C}(\boldsymbol{j}\boldsymbol{\omega}_{\boldsymbol{c}})] + \boldsymbol{Arg}[\boldsymbol{G}(\boldsymbol{j}\boldsymbol{\omega}_{\boldsymbol{c}})] = -\boldsymbol{\pi} + \boldsymbol{\varphi}_{\boldsymbol{m}}, \tag{II.10}$$

$$d\big[Arg[C(j\omega)G(j\omega)]\big]d\omega \;|||\; \omega = \omega_c = 0 \tag{II.11}$$

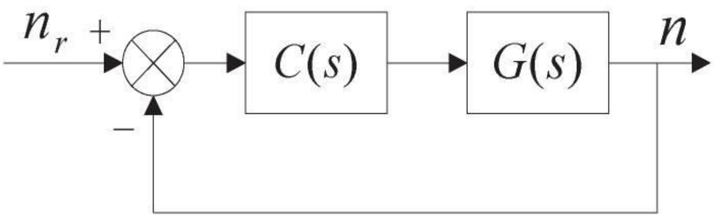

Figure II.1 : Closed loop fraction order system

Then implement the PID with any type of approximation transfer function like Padé approximation or Youla-Kucera parametrization [24]

II.3.3 : Method of Carlson

The method is derived from Newton's regular process used for the iterative approximation of order α. The method is based on the following relationship [14]:

$$\left(H(s)^{\frac{1}{\alpha}} - G(s)\right) = 0 \Rightarrow H(s) = (G(s))^{\alpha} \qquad (II.12)$$

By setting α=1q and m=q2 in each iteration, starting from H0(s)=1 as initial value, the approximation is obtained by the following formula:

$$H_i(s) = H_{i-1}(s)\frac{(q-m)(H_{i-1}(s))^2 + (q+m)G(s)}{(q+m)(H_{i-1}(s))^2 + (q-m)G(s)}$$

II.3.4 : Fractional Order Operational Amplifiers

Fractional order systems can be approximated using operational amplifier-based circuits. One method for approximating fractional order systems using operational amplifiers is the frequency-dependent negative impedance converter (FDNIC) technique.

The FDNIC technique involves the use of operational amplifiers and passive elements (resistors and capacitors) to realize a fractional order system. The fractional order system is approximated by designing a circuit that has a frequency-dependent negative impedance, which can be achieved by using a feedback loop with a frequency-dependent gain.

The FDNIC technique can be used to realize both integer and non-integer order systems, including fractional order systems. The method involves the use of a combination of passive elements and operational amplifiers to create a transfer function that approximates the fractional order system transfer function.

The FDNIC-based circuits have the advantage of being easily implemented using standard operational amplifiers and passive elements, and they can be easily tuned to approximate a desired fractional order system. However, they may require the use of high-precision components and may be sensitive to component tolerances and environmental changes.

Other operational amplifier-based techniques for approximating fractional order systems include the use of fractional order capacitors and active inductors. These techniques involve the use of specialized components to create fractional order elements that can be used to realize a fractional order system. The choice of method depends on the specific application and the design requirements.

II.3.5 : Method of Outstalop

The method is based on the recursive pole-and-zero approximation of the non-integer derivation operator. The objective of this method is to approach the non-integer integration or derivation operator by an integer model bounded in frequency of finite dimension. This approximation requires two steps [15]:

- The frequency truncation of the integration or derivation operator.
- The approximation of the non-integer operator bounded in frequency by an integer model.

II.3.5.1 : Generalized differentiator

The generalized differentiator is defined by the following transfer function:

$$D(s) = \left(\frac{s}{\omega_c}\right)^m \tag{II.13}$$

Where $m \in \mathcal{R}$ and ω_c is called the transitional frequency. Thus, for strictly positive orders the relation (eq number) defines a non-integer order derivative and for strictly negative orders an integrator of non-integer order.

First, we perform a truncation, it consists in limiting the differential behavior of $\left(\frac{s}{\omega_c}\right)$, on the interval $[\omega_A \ \omega_B]$ centered on ω_c . In reality, the truncation will be performed on the wider interval $[\omega_l \ \omega_h]$, for more precision.

With:

$$\omega_l \ll \omega_A \text{ and } \omega_h \gg \omega_B$$

We then introduce the fractional order derivative bounded in frequency proposed by Oustaloup.

$$D_{bf}(s) = \left(C_0 \frac{1+\frac{s}{\omega_l}}{1+\frac{s}{\omega_h}}\right)^m , \ -1 < m < 1 \tag{II.14}$$

Or

$$C_0 = \frac{\omega_l}{\omega_c} = \frac{\omega_c}{\omega_h} \Rightarrow \omega_c = \sqrt{\omega_h \omega_l} \tag{II.15}$$

To ensure unity gain at the frequency ω_c

In a second phase, the approach consists in approximating the bounded derivative in frequency (eq number), by a recursive distribution of real poles and zeros. This gives an entire order transfer.

$$D_{bf}(s) = \lim_{N \to \infty} D_N(s) \tag{II.16}$$

Or

$$D_N(s) = \left(\frac{\omega_c}{\omega_h}\right)^m \prod_{i=-N}^{N} \frac{1+\frac{s}{z_i}}{1+\frac{s}{P_i}} \tag{II.17}$$

Z_i and P_i represent zeros and poles of rank i; these N+1 zero-pole pairs are determined by the following recursive relations:

$$\frac{P_i}{Z_i} = a > 0 \qquad \text{(II.18)}$$

$$\frac{P_i}{z_i} = \alpha > 0 \qquad \text{(II.19)}$$

$$\frac{z_{i+1}}{z_i} = \frac{P_{i+1}}{P_i} = \alpha\eta > 1 \qquad \text{(II.20)}$$

The factors α and η are called recursive factors. And they give a constant ratio $\alpha\eta$ between two poles or two consecutive zeros. And N and generally set such that $\alpha\eta$ is approximately equal to 5.

The following figure represents the asymptotic Bode diagram of $DN(s)$, the latter is smoothed to arrive at the diagram of $D_{bf}(s)$

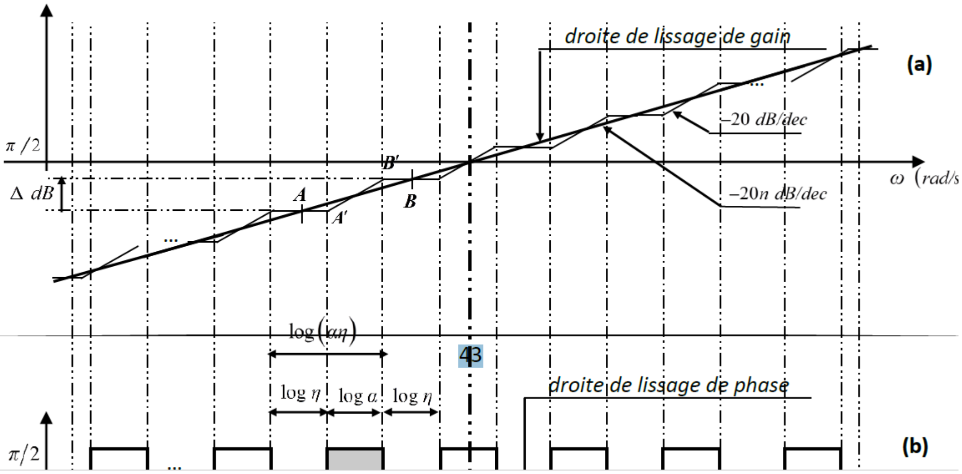

Figure I.2 : asymptotic Bode diagram of $DN(s)$

From the slopes of the two segments AB et $A'B'$ which are given respectively by:

$$20\mathrm{mdB/dec} = \frac{\Delta \mathrm{dB}}{\log(\alpha)+\log(\eta)} \tag{II.21}$$

$$\frac{20\mathrm{dB}}{\mathrm{dec}} = \frac{\Delta \mathrm{Db}}{\mathrm{LOG(a)}} \tag{II.22}$$

We can deduce, by taking their ratio, the relation between the order m and the recursive factors α and η:

$$\mathrm{m} = \frac{\log(\mathrm{a})}{\log(\mathrm{a}\eta)} \tag{II.23}$$

We can also, from the previous diagram, deduce the following formulas:

$$\frac{\omega_h}{\omega_l} = (\alpha\eta)^{2N+1} \Rightarrow (\alpha\eta) = \left(\frac{\omega_h}{\omega_l}\right)^{1/(2N+1)} \tag{II.24}$$

$$\alpha = (\alpha\eta)^m \ \text{ et } \ \eta = (\alpha\eta)^{1-m}$$

Les pôles et zéros de rang *i* peuvent être, aussi, écrits sous la forme suivante :

$$\begin{cases} Z_i = \left(\frac{\omega_h}{\omega_l}\right)^{\frac{i+N+1/(2-n/2)}{2N+1}} \omega_l \\ P_i = \left(\frac{\omega_h}{\omega_l}\right)^{\frac{i+N+1/(2+n/2)}{2N+1}} \omega_l \end{cases} \tag{II.25}$$

The advantage of this approximation method lies in its simplicity of implementation. If the order m>1 or m<−1, only the non-integer part is approximated by an integer model.

II.4 : Method Of CHAREF

Charef's method of approximation involves approximating a given fractional order system using a rational function of lower order. The method is particularly well-suited for approximating standard fractional operators (such as integrators and differentiators) of rational functions, and can be used to implement them as correctors in feedback control systems.

The basic idea of the method is to approximate the fractional power pole (FPP) of the system using a rational function. The FPP is a function that describes the behavior of the system in the frequency domain, and is given by [16]:

II.4.1 : Approximation Of First Order Fractional Systems

The Charef's method can be used to approximate first-order fractional systems of the form [15]:

$$\mathbf{G(S)} = \frac{\mathbf{K}}{(\frac{\mathbf{1}}{\mathbf{P_t}}\mathbf{S+1})^{\boldsymbol{\alpha}}} \tag{II.26}$$

Where:

K is the gain,

$\frac{\mathbf{1}}{\mathbf{P_t}}$ is the time constant,

α is the fractional order.

First, we fine bode diagram in order to study the frequency domain(figure bode).

In the Bode diagram of $H(s)$ the line with a slope of $-20m\ dB/dec$ is approximated by a number of interconnected zig-zags, with alternating slopes between 0 dB/dec and -20 db/dec as shown in (figure bode) The properties at high and low frequencies, of the transfer function of the fractional 1st order model, show that the first and last singularities of the approximation must be poles. We can therefore rewrite as follows:

$$H(s) = \frac{1}{\left(1+\frac{s}{P_T}\right)^m} = \lim_{N\to\infty} \frac{\prod_{i=0}^{N-1}\left(1+\frac{s}{Z_i}\right)}{\prod_{i=0}^{N}\left(1+\frac{s}{p_i}\right)} \tag{II.27}$$

With $N + 1$ is the total number of singularities which is determined by the frequency band of the system.

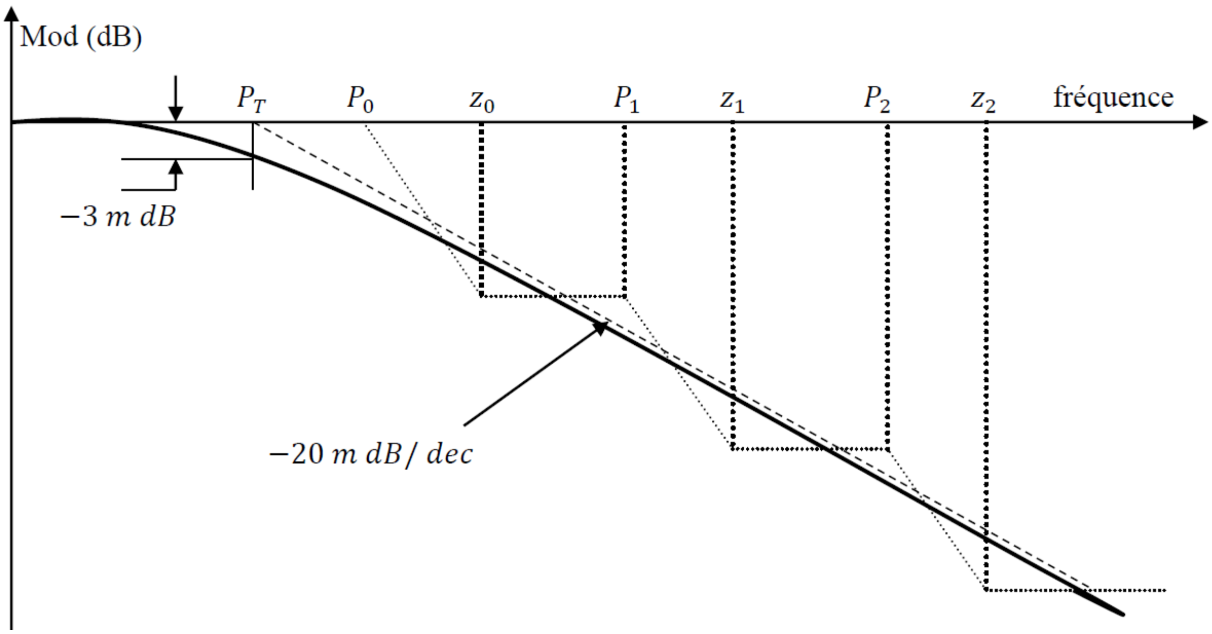

Figure I Erreur ! Il n'y a pas de texte répondant à ce style dans ce document.**3 :** **Method of Charef**

Assuming that the error between the crisscross and the desired line is $y\ (dB)$ then the poles and zeros of the singularity function can be found as follows:

First pole : $P_0 = P_t 10^{\left[\frac{y}{20m}\right]}$

First zero : $z_0 = P_0 10^{\left[\frac{y}{10(1-m)}\right]}$

Second pole : $P_1 = z_0 10^{\left[\frac{y}{10m}\right]}$

Second zero : $z_1 = P_1 10^{\left[\frac{y}{10(1-m)}\right]}$

N^{th} zero : $z_{n-1} = P_{N-1} 10^{\left[\frac{y}{10(1-m)}\right]}$

N^{th} zero: $P_N = z_{N-1} 10^{\left[\frac{y}{20m}\right]}$

Where P_T is the frequency corresponding to $-3m\ dB$ as in (BODE DIAGRAM), P_0 is the first singularity which is determined by $y\ (dB)$ and P_N is the last singularity, determined by N.

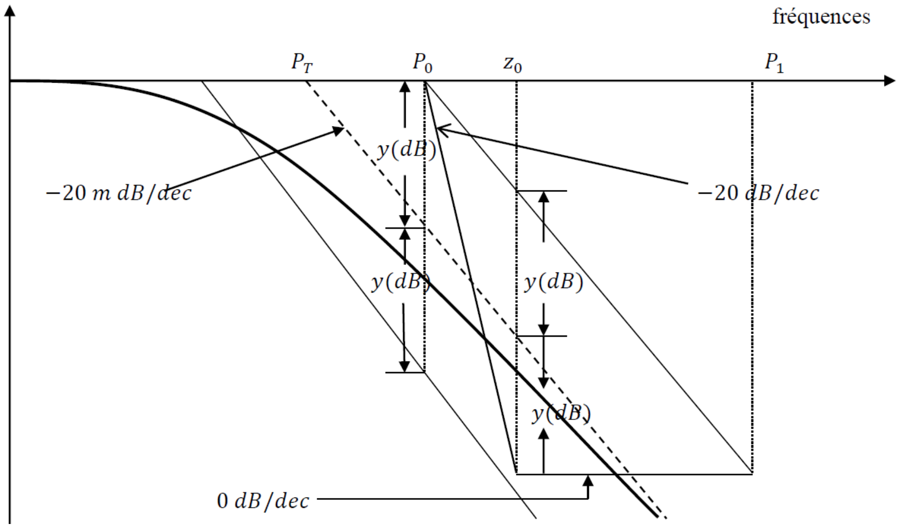

Figure I.4 : Charef approximation

$$a = 10^{\left[\frac{y}{10(1-m)}\right]} \qquad \text{(II.28)}$$
$$b = 10^{\left[\frac{y}{10m}\right]}$$

a and b are the position ratios.

$$ab = 10^{\left[\frac{y}{10m(1-m)}\right]} \qquad \text{(II.29)}$$

From there we can deduce the values of the poles and zeros of the singularity function, which are defined by the following relations:

$$P_i = (ab)^i P_0 \tag{II.30}$$

$$Z_i = (ab)^i aP_0 \tag{II.31}$$

For *i*=1,2,3, …

So, the approximate rational function is written in the form:

$$H(s) = \frac{1}{\left(1+\frac{s}{P_T}\right)^m} \approx \frac{\prod_{i=0}^{N-1}\left(1+\frac{s}{Z_i}\right)}{\prod_{i=0}^{N-1}\left(1+\frac{s}{p_i}\right)} = \frac{\prod_{i=0}^{N-1}\left(1+\frac{s}{(ab)^i aP_0}\right)}{\prod_{i=0}^{N-1}\left(1+\frac{s}{(ab)^i P_0}\right)} \tag{II.32}$$

With:

$$N = \text{integer}\left(\frac{\log\left(\frac{\omega_{max}}{P_0}\right)}{\log(ab)}\right) + 1$$

Where ω_{max} is the maximum frequency of the specified working frequency band, and the last two poles must satisfy the following condition:

$$P_{N-1} < \omega_{max} < P_N$$

II.4.2 : Approximation Of Multiple Order Fractional Systems

The Charef method can also be used to approximate multiple order fractional systems, which are systems with more than one fractional order [16].

A multiple-fractal system can be modeled in the frequency domain by the multiple PPF as follows [17]:

$$H(s) = \frac{1}{\prod_{i=1}^{p}\left(1+\frac{s}{P_{T_i}}\right)^{m_i}} \tag{II.33}$$

where $P_{T_i} < P_{T_{i+1}}$ and $0 < m_i < 1$ for $i = 1,2,3 \ldots\ldots, p$ are the cutoff frequinces.

As for the fractal 1st order system, each section of the multiple system can be approximated by a number of singularities. So the function (exact last one h(s)) can be approximated as follows:

$$H(s) = \frac{\prod_{i_1=0}^{N_1}\left(1+\frac{s}{Z_{i_1}}\right)}{\prod_{i_1=0}^{N_1}\left(1+\frac{s}{P_{i_1}}\right)} \frac{\prod_{i_2=0}^{N_2}\left(1+\frac{s}{Z_{i_2}}\right)}{\prod_{i_2=0}^{N_2}\left(1+\frac{s}{P_{i_2}}\right)} \cdots \frac{\prod_{i_p=0}^{N_p}\left(1+\frac{s}{Z_{i_p}}\right)}{\prod_{i_p=0}^{N_p}\left(1+\frac{s}{P_{i_p}}\right)} \tag{II.34}$$

The singularities are chosen so as to have an alternation between the poles and the zeros, to ensure that the residuals are real and positive. We start by choosing the poles for each section.

$P_n = (ab)^n P_0 \quad n = 1,2,3 \ldots\ldots, M$

$M = N_1 = N_2 + \ldots\ldots + N_p + (p-1)$

The zeros are chosen according to the two poles which will be adjacent to them. Indeed, since the process starts with a pole $P0$, each section must end with a zero for the section succeeding it to start with a pole. The zeros are therefore generated by the following functions:

1st section: $z_{i_1} = (ab)^{i_1} a_1 P_0, \qquad i_1 = 0,1,2,\ldots, N_1$

2nd section: $z_{i_2} = (ab)^{(N_1+1+i_2)} a_2 P_0, \qquad i_2 = 0,1,2,\ldots, N_2$

Pth section: $z_{i_2} = (ab)^{(N_1+N_2+\cdots+N_{p-1}+(p+1)+i_p)} a_p P_0, \qquad i_p = 0,1,2,\ldots, N_p$

Where $N_1, N_2, \ldots\ldots, N_p$ are determined in the same way as for the fractional 1st order system, i.e., by fixing the frequency band and the deviation y (dB) desired. And with (ab) constant for the whole system. We therefore have the following approximation of the system:

$$H(s) \approx \sum_{n=0}^{M} \frac{k_n}{\left(1+\frac{s}{(ab)^n p_0}\right)} \tag{II.35}$$

With

$$k_n = \frac{\prod_{i_1=0}^{N_1}\left[1-\frac{(ab)^{n-i_1}}{a_1}\right]\prod_{i_2=0}^{2}\left[1-\frac{(ab)^{n}-(N_1+1+i_2)}{a_2}\right]\ldots\prod_{i_p=0}^{N_p+1}\left[1-\frac{(ab)^{n-(N_1+\cdots+N_{p-1}+(p-1)+i_p)}}{a_p}\right]}{\prod_{i=0}^{M}[1-(ab)^{n-1}]} \tag{II.36}$$

II.4.3 : Approximation of the fractional order integrator:

This method is efficient and easy to use, the goal of which is to approximate by a rational function the non-integer integrator. First, the operator is modeled by a PPF function in a useful frequency band. Then, the latter is approximated by a rational function using Charef's method exposed in the previous section. So with this method one can achieve any desired accuracy on any frequency band [18].

For a given frequency band of practical utility $[\omega l \ \omega h]$, the integrating operator presented by formula (II.1) can be modeled by the following transfer function PPF:

$$G_l(s) = \frac{K_l}{\left(1+\left(\frac{s}{\omega_c}\right)\right)^{m|}} \tag{II.37}$$

Assuming that: $\omega \in [\omega_l \ \ \omega_h]$, $\omega \gg \omega_c$:

$$G(s) = \frac{K_l}{\left(\frac{s}{\omega_c}\right)^m} = \frac{K_t\omega_c^m}{s^m} = \frac{1}{s^m} = G_l(s) \tag{II.38}$$

With $K_i = \frac{1}{\omega_c^m}$ and ω_c is the cutoff frequency obtained in the Bode diagram at $-3\,m\ dB$ which is calculated by: $\omega_c = \sqrt{10^{\left[\frac{\varepsilon}{10m}\right]} - 1}$,where ε is the maximum allowed error between slopes of the fractional order integrator and its PPF function in a given frequency band.

To represent the function by a time-invariant model, we will approximate it by the singularity function method seen in the previous section The method consists of approximating the slope at $-20m\ dB/dec$, in the Bode diagram, by alternating slopes at $-20\ dB/dec$ and $0\ dB/dec$. The latter corresponds to an alternation of poles and zeros on the negative real axis of the complex plane S.

$$P_0 < Z_0 < P_1 < Z_1 < \cdots < Z_{N-1} < P_N$$

We get the following approximation:

$$G_l(s) = \frac{K_l}{\left(1+\left(\frac{s}{\omega_c}\right)\right)^{m|}} \cong K_t \frac{\prod_{i=0}^{N-1}\left(1+\frac{s}{Z_0(ab)^2}\right)}{\prod_{i=0}^{N}\left(1+\frac{s}{P_0(ab)^2}\right)} \tag{II.39}$$

Obtained by choosing y andω_{max} , which can be set to $100\omega_h$, and with:

$$P_0 = \omega_c 10^{\frac{y}{20m}} \text{ and } Z_0 = aP_0 \tag{II.40}$$

We have seen that the poles and zeros P_i and Z_i can be written as in (eq number) and (eq number), respectively. From this, one can deduce the approximation of the non-integer integrator in a given frequency band.

$$G(s) = \frac{K_l}{\left(1+\left(\frac{s}{\omega_c}\right)\right)^{m}} \cong K_l \frac{\prod_{i=0}^{N-1}\left(1+\frac{s}{Z_i}\right)}{\prod_{i=0}^{N}\left(1+\frac{s}{P_i}\right)} \tag{II.41}$$

To see the contribution of each pole, we decompose the function (eq nuber) into a sum of simple fractions (or residues). We obtain :

$$G_t(s) = K_t \frac{\prod_{i=0}^{N-1}\left(1+\frac{s}{z_0(ab)^i}\right)}{\prod_{i=0}^{N}\left(1+\frac{s}{P_0(ab)'}\right)} = \sum_{i=0}^{N} \frac{h_i}{\left(1+\frac{s}{P_0(ab)^2}\right)} \tag{II.42}$$

With:

$$h_i = K_I \frac{\prod_{j=0}^{N-1}\left(1-\frac{(ab)^{i-j}}{a}\right)}{\prod_{j=0.j\neq 1}^{N}(1-(ab)^{i-j})}$$

II.4.4 : Approximation of the fractional order derivative:

In order to implement fractional order correctors, the integrator approximation method has been extended to the fractional order differentiator, and likewise the PPF has been extended to the ZPF: Fractional Power Zero (FPZ : Fractional Power Zero).

For a given frequency band of practical utility, the fractional order differentiator, can be modeled by the following ZPF function:

$$\mathbf{G(S) = K_D \left(1+\frac{s}{\omega_c}\right)^m} \tag{II.43}$$

Assuming that $\boldsymbol{\omega} \in [\boldsymbol{\omega_l}\ \boldsymbol{\omega_h}], \boldsymbol{when}\ \boldsymbol{\omega} \gg \boldsymbol{\omega_c}. \boldsymbol{So}$:

$$\boldsymbol{G(s) = K_D \left(\frac{s}{\omega_c}\right)^m = \frac{K_D s^m}{\omega_c^m} = s^m = G_D(s)} \tag{II.44}$$

With $\boldsymbol{K_D} = \boldsymbol{\omega_c}$ and $\boldsymbol{\omega_c}$ is the cutoff frequency obtained in the Bode diagram at $-3\ m\ dB$ which is calculated by: $\boldsymbol{\omega_c} = \boldsymbol{\omega_l}\sqrt{\mathbf{10}^{[\frac{\varepsilon}{10m}]} - \mathbf{1}}$, where $\boldsymbol{\varepsilon}$ is error max of the fractional order differentiator (eq number) and its function ZPF (eq number) in a given frequency band.

To represent the function (eq number) by a time-invariant model, we will approximate it by the method of singularities seen previously. It consists in this case, in the approximation of the slope at 20 $m\ dB/dec$ of the Bode diagram of the ZPF, by a succession of slopes at 20 dB/dec and 0 dB/dec. They correspond to an alternation of zeros and poles on the negative real axis of the complex plane S. As following:

$$\mathbf{Z_0 < Z_1 < P_1 < \cdots < Z_N < P_N}$$

We get the following approximation:

$$\boldsymbol{G(s) = K_D\left(1+\frac{s}{\omega_c}\right)^m = K_D\frac{\prod_{i=0}^{N}\left(1+\frac{s}{Z_i}\right)}{\prod_{i=0}^{N}\left(1+\frac{s}{P_i}\right)}} \tag{II.45}$$

As with the fractional order integrator, the zeros and poles of the approximation are in geometric progression. By specifying y and$\boldsymbol{\omega_{max}}$, can be set to 100 $\boldsymbol{\omega_h}$. The following parameters are determined:

$$\mathbf{Z_0 = \omega_c 10^{(\frac{y}{20m})}\ \ and\ P_0 = aZ_0}$$

a,b and N remain the same as in (eq number), (eq number) and (eq number), respectively.

By replacing $\boldsymbol{P_i}$ ***and*** $\boldsymbol{Z_i}$ by their respective formulas, (eq number) and (eq number), we obtain the following approximation:

$$\mathbf{G_D(s) = s^m = K_D\left(1+\frac{s}{\omega_c}\right)^m \cong K_D\,\frac{\prod_{i=0}^{N}\left(1+\frac{s}{z_0(ab)^i}\right)}{\prod_{i=0}^{N}\left(1+\frac{s}{P_0(ab)^i}\right)}} \qquad \textbf{(II.46)}$$

The calculation of the residues of the function (II.41), gives the following formula:

$$G_D(s) = G_0 + \sum_{i=0}^{N}\frac{k_i s}{\left(1+\frac{s}{(ab)^i P_0}\right)} \qquad (II.47)$$

With $G_0 = K_D$ and :

$$k_i = -\frac{K_D}{P_0(ab)^i}\,\frac{\prod_{j=0}^{N}(1-a(ab)^{i-j}}{\prod_{j=0,j\neq 1}^{N}(1-(ab)^{i-j}} \qquad (II.48)$$

II.5 : Conclusion

In this chapter, we have discussed various methods of approximating fractional order systems, including modern numerical and analog approximation methods. We have shown that these methods can provide accurate and efficient approximations for fractional order systems of various orders and complexities.

We have also focused on the method of Charf, which is a widely used analog approximation method for fractional order differentiators. We have shown that the method of Charf can provide accurate and stable approximations for low to moderate order fractional differentiators, and we have presented simulation results that demonstrate the effectiveness of the method of Charf.

Overall, this chapter highlights the importance of approximating fractional order systems and presents various methods to achieve accurate and efficient approximations. The methods discussed in this chapter can be applied to a wide range of engineering and scientific problems where fractional order systems arise.

CHAPTER III

Simulation and compilation

III.1 : Introduction

This chapter represents the final stage of this study where we take a transformation function and apply various approximation methods to it in order to be able to compare the accuracy and performance of some of the methods that were mentioned earlier in this work, such as Charef method, Oustaloop and identification method.

III.2 : Application

For comparison, two different systems with different properties were selected in order to make a more comprehensive judgment about the efficiency of each approximation method.

III.2.1 : First system

Put:

$$G_1(S) = \frac{1}{s^{\{2.3\}}+3s^{\{1.6\}}+3} \qquad \text{(III.1)}$$

$G_1(S)$ is the transfer function of the first system.

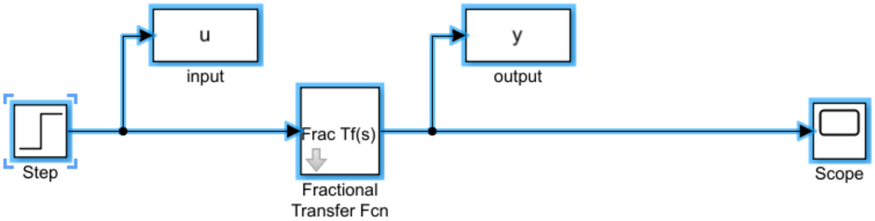

Figure III.1 : simulation bloc for sys1

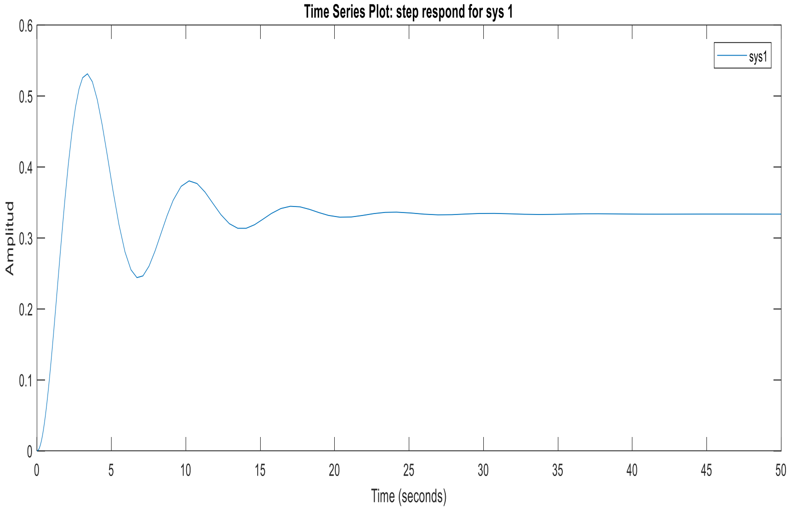

Figure III.2 : step respond for sys1

III.2.1.1 : Identification method

Can use the approximation using MATLAB system identification tool based on the input matrix u and the output matrix y as explained in figure []

First open system identification toolbox from APP the import the Time-Domain data and choice Estimate Transfer Function Model to get the approximation.

$$G_{1_{Iden}}(s) = \frac{0.05274\,s + 0.2742}{s^2 + 0.3826\,s + 0.8199} \qquad \text{(III.2)}$$

Using MATLAB stepinfo after system dentification to get the table:

Identification	**D**	**Tp**	**Tm**	**Tr**
Sysytem	**0.2**	**3.3706**	**18.6695**	**1.3192**

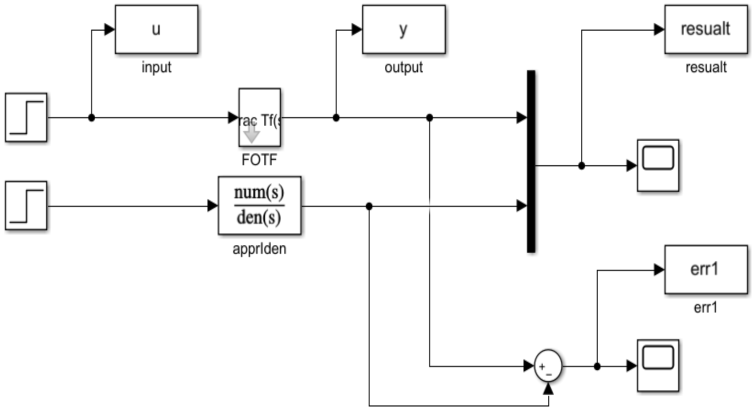

Figure III.3 : Simulation bloc for sys1 and iden1

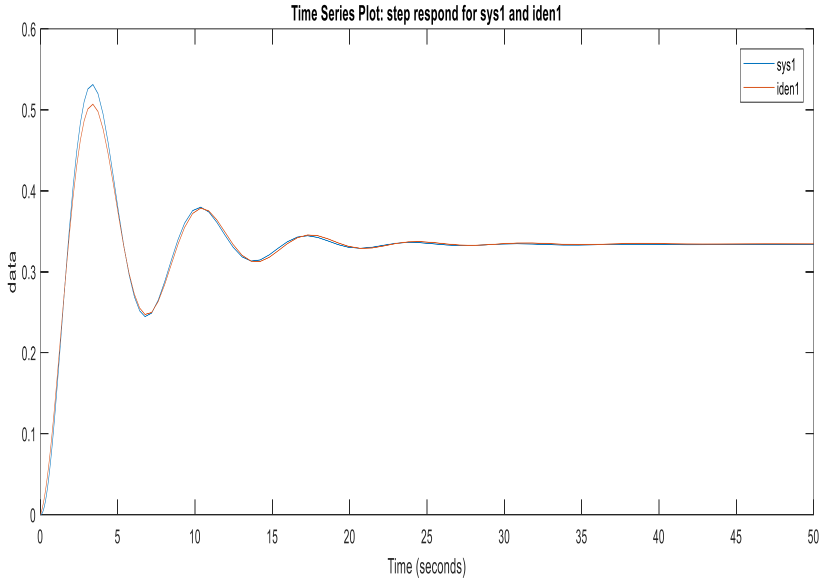

Figure III.4 : step respond for sys1 and iden1

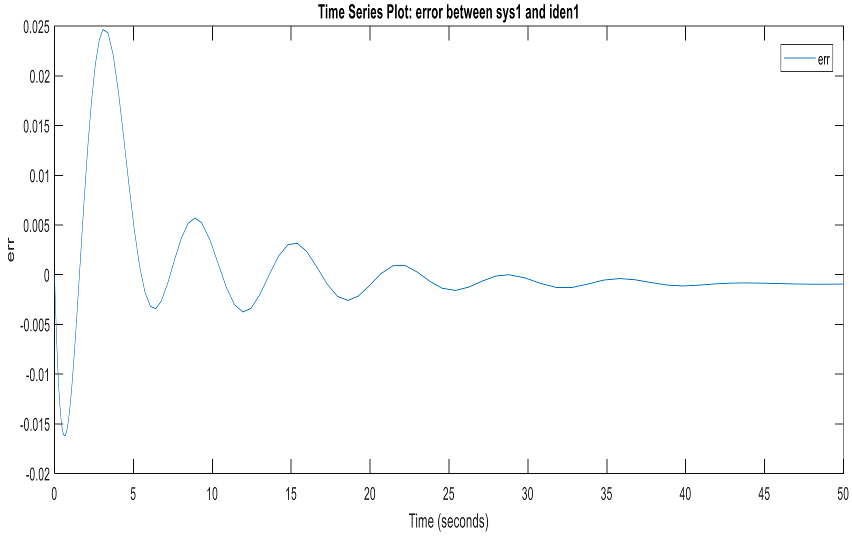

Figure III.5 : Error between sys1 and iden1

See that the err seems to be small and acceptable.

III.2.1.2 : Oustalop method

Using the following parameter : r=0.3 ; r=0.5 ; N=5 ; wb=0.001 ; wh=1000, we get:

$$G_{1_{\mathrm{Ous}}}(s)=\frac{\begin{array}{c}0.063096\ (s+7154)(s+5565)(s+1341)(s+1043)(s+251.2)\\(s+195.4)(s+47.07)(s+36.61)(s+8.82)(s+6.861)(s+1.653)(s+1.286)(s+0.3097)\\(s+0.2409)(s+0.05803)(s+0.04514)(s+0.01087)(s+0.008458)(s+0.002037)\\(s+0.001585)(s+0.0003818)(s+0.000297)\end{array}}{\begin{array}{c}(s+7168)(s+3375)(s+1347)(s+635.2)(s+255)(s+120.9)\\(s+49.56)(s+23.9)(s+10.57)(s+5.33)(s+2.402)(s+1.21)(s+0.3353)\\(s+0.2398)(s+0.05834)(s+0.04513)(s+0.01088)(s+0.008458)(s+0.002038)\\(s+0.001585)(s+0.0003818)(s+0.000297)(s^2+\ 0.402s\ +\ 0.8534)\end{array}}$$

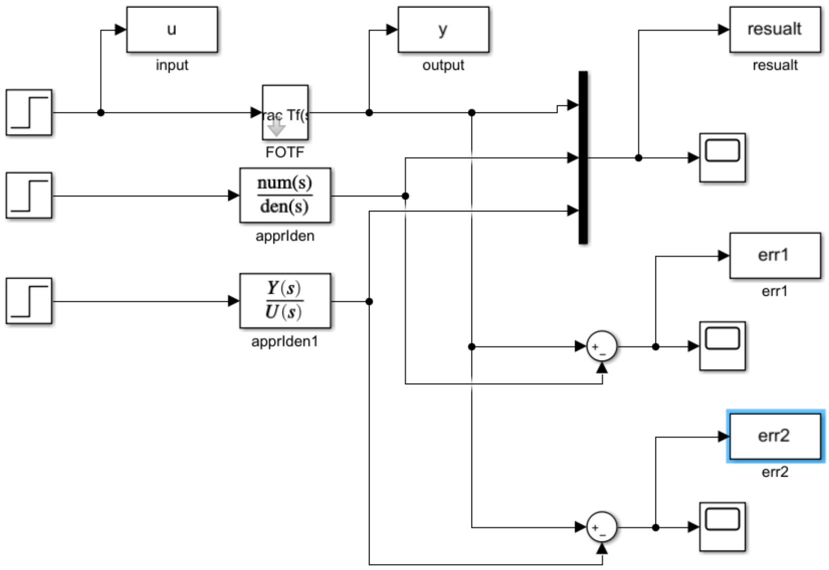

Figure III.6 : Simulation bloc for sys1 and iden1 and ous1

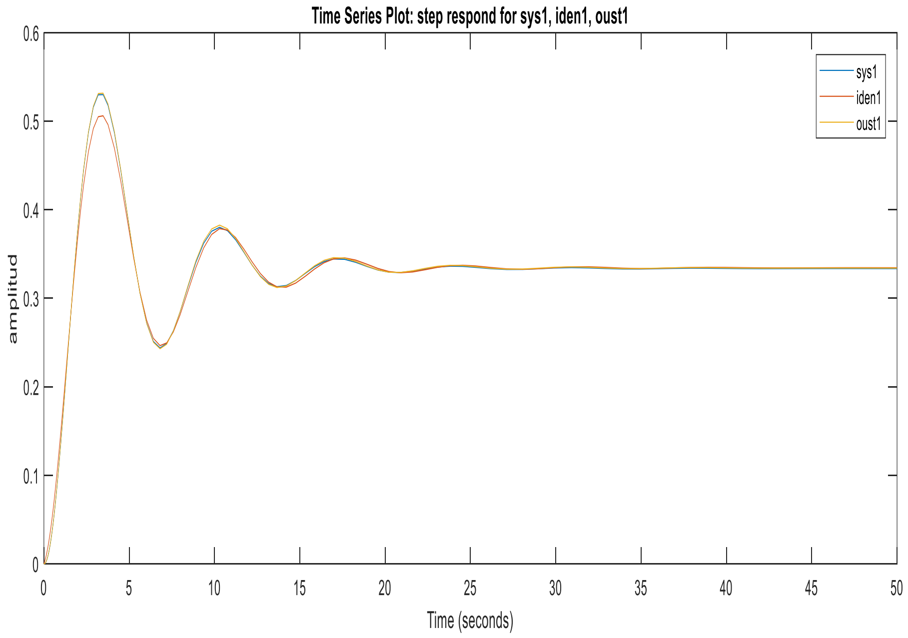

Figure III.7 : step respond for sys1, iden1, oust1

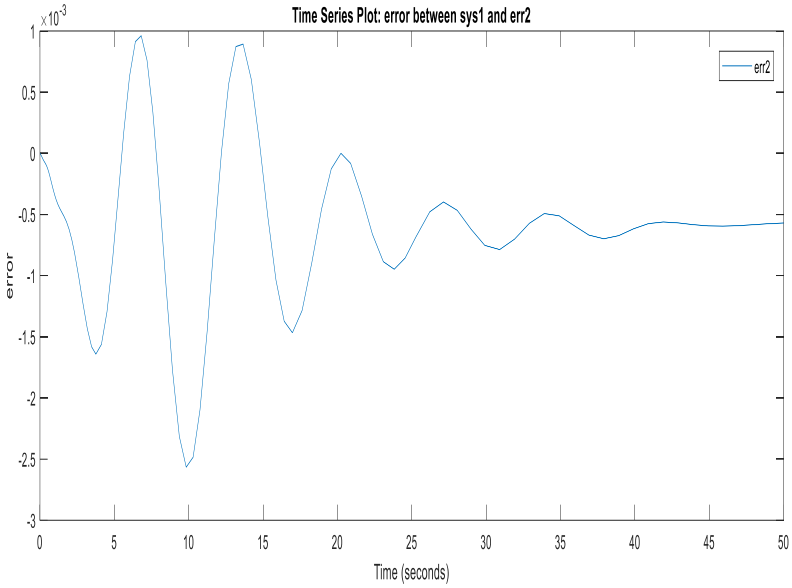

Figure III.8 : error between sys1 and ous1

The error is more wavy but its value is ten times less than identification method.

III.2.1.3 : Charef method

We apply the Charef algorithm to get the following approximation:

$$G_{1_{\mathrm{Char}}}(s)=\frac{\begin{array}{c}0.04559\,s^{26}+\ 1266\,s^{25}+\ 723.4\,s^{24}+\ 326.1\,s^{23}\\ +\,253.7\,s^{22}+\ 61.13\,s^{21}+\ 47.55\,s^{20}+\ 11.45\,s^{19}+\ 8.907\,s^{18}+\ 2.146\,s^{17}\\ +\,1.669\,s^{16}+\ 0.4021\,s^{15}+\ 0.3128\,s^{14}+\ 0.07535\,s^{13}+\ 0.05862\,s^{12}\\ +\,0.01412\,s^{11}+\ 0.01098\,s^{10}+\ 0.002645\,s^{9}+\ 0.002058\,s^{8}+\ 0.0004955\,s^{7}\\ +\,0.0003856\,s^{6}+\ 9.286e-05\,s^{5}+\ 7.225e-05\,s^{4}+\ 1.74e-05\,s^{3}\\ +\,1.354e-05\,s^{2}+\ 2.462e-06\,s\ +\ 1.231e-06\end{array}}{\begin{array}{c}s^{29}+\ 27820\,s^{28}+\ 11110\,s^{27}+\ 7164\,s^{26}+\ 3374\,s^{25}+\ 1346\,s^{24}\\ +\,635\,s^{23}+\ 254.9\,s^{22}+\ 120.8\,s^{21}+\ 49.47\,s^{20}+\ 23.84\,s^{19}+\ 10.51\,s^{18}\\ +\,5.304\,s^{17}+\ 2.376\,s^{16}+\ 1.209\,s^{15}+\ 0.3336\,s^{14}+\ 0.2399\,s^{13}+\ 0.05832\,s^{12}\\ +\,0.04513\,s^{11}+\ 0.01088\,s^{10}+\ 0.008458\,s^{9}+\ 0.002038\,s^{8}+\ 0.001585\,s^{7}\\ +\,0.0003818\,s^{6}+\ 0.000297\,s^{5}+\ 5.4e-05\,s^{4}+\ 2.7e-05\,s^{3}+\ s^{2}+\ 0.4013\,s\\ +\,0.9107\end{array}}$$

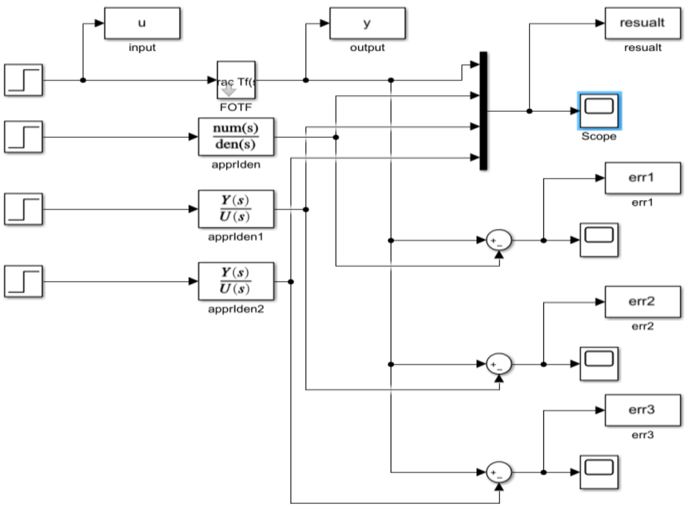

Figure III.9 : Simulation bloc for all three methods

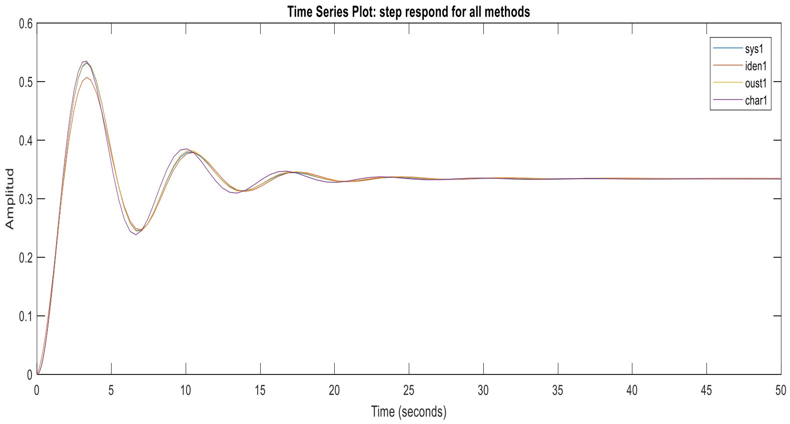

Figure III.10 : step respond for all methods

Figure III.11 : error between sys1 and char1

III.2.2 : Second system

$$G_2 = \frac{1}{s^{\{1.31\}}+s^{\{0.97\}}+1.69} \tag{III.3}$$

Using the same steps and methods as th

e First system, get the simulation in figure{}:

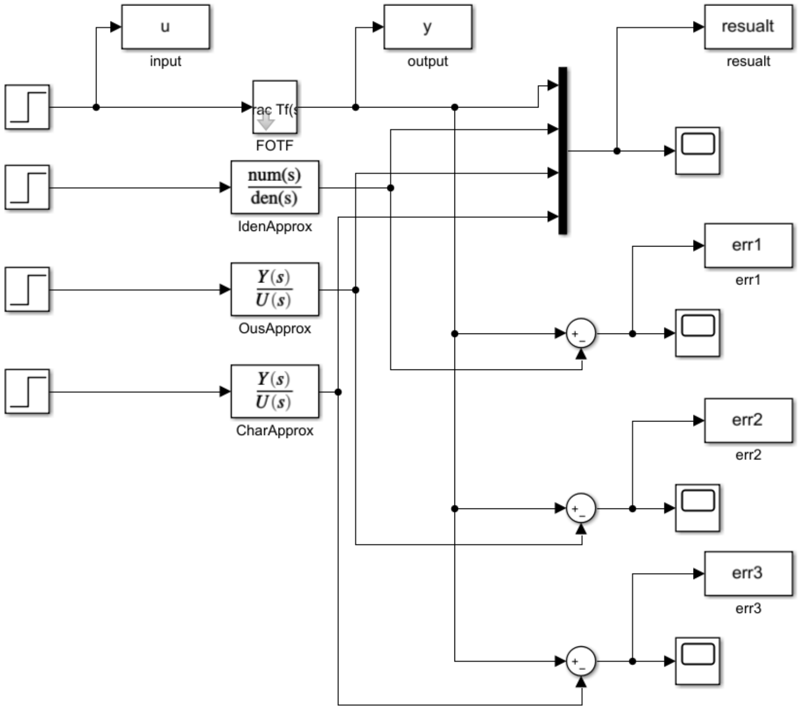

Figure III.12 : simulation bloc for G2 with all three methods

III.2.2.1 : Identification method

$$G_{2_{\text{Iden}}}(s) = \frac{0.4352\,s + 0.3619}{s^2 + 1.303\,s + 0.6109} \tag{III.4}$$

III.2.2.2 : Oustaloop method

$$G_{2_{\mathrm{Iden}}} = \frac{\begin{gathered} 0.057544s^{\{22\}} + 1088s^{\{21\}} + 6.5101e + 06s^{\{20\}} + 1.4766e + 10s^{\{19\}} \\ +1.402e + 13s^{\{18\}} + 5.5839e + 15s^{\{17\}} + 9.6574e \\ +17s^{\{16\}} + 7.1228e + 19s^{\{15\}} \\ +2.2963e + 21s^{\{14\}} + 3.1666e + 22s^{\{13\}} + 1.911e + 23s^{\{12\}} + 4.9358e \\ +23s^{\{11\}} + 5.58e + 23s^{\{10\}} + 2.6999e + 23s^{\{9\}} + 5.7171e \\ +22s^{\{8\}} + 5.1785e + 21s^{\{7\}} + 2.0503e + 20s^{\{6\}} \\ +3.4619e + 18s^{\{5\}} + 2.5384e + 16s^{\{4\}} + 7.8077e \\ +13s^{\{3\}} + 1.0052e + 11s^{\{2\}} + 4.9063e + 07s + 7578.8 \end{gathered}}{\begin{gathered} s^{\{23\}} + 16546s^{\{22\}} + 7.8749e + 07s^{\{21\}} + 1.4432e \\ +11s^{\{20\}} + 1.0817e + 14s^{\{19\}} + 3.4849e + 16s^{\{18\}} + 4.828e \\ +18s^{\{17\}} + 2.9412e + 20s^{\{16\}} + 7.856e + 21s^{\{15\}} + 9.4171e + 22s^{\{14\}} + \\ 5.1627e + 23s^{\{13\}} + 1.3715e + 24s^{\{12\}} + 1.8827e + 24s^{\{11\}} + 1.406e + 24s^{\{10\}} \\ +5.4674e + 23s^{\{9\}} + 1.044e + 23s^{\{8\}} + 9.0479e + 21s^{\{7\}} + 3.5146e + 20s^{\{6\}} \\ +5.8875e + 18s^{\{5\}} + 4.3022e + 16s^{\{4\}} + 1.3214e \\ +14s^{\{3\}} + 1.7002e + 11s^{\{2\}} + 8.2962e + 07s + 12814 \end{gathered}}$$

III.2.2.3 : Charef method

$$G_{2_{\mathrm{Char}}} = \frac{\begin{gathered} 0.041024s^{\{26\}} + 16631s^{\{25\}} + 5.4907e + 08s^{\{24\}} + 6.4298e + 12s^{\{23\}} \\ +3.1756e + 16s^{\{22\}} + 6.6646e + 19s^{\{21\}} + 6.1146e + 22s^{\{20\}} \\ +2.4006e + 25s^{\{19\}} + 4.1254e + 27s^{\{18\}} + 3.0346e + 29s^{\{17\}} + 9.7714e \\ +30s^{\{16\}} + 1.3468e + 32s^{\{15\}} + 8.126e + 32s^{\{14\}} + 2.0987e + 33s^{\{13\}} + 2.3726e \\ +33s^{\{12\}} + 1.1481e + 33s^{\{11\}} + 2.4319e + 32s^{\{10\}} + 2.2044e + 31s^{\{9\}} + 8.7423e + 29s^{\{8\}} \\ +1.4819e + 28s^{\{7\}} + 1.0962e + 26s^{\{6\}} + 3.4441e + 23s^{\{5\}} + 4.6587e + 20s^{\{4\}} + 2.5863e + \\ 17s^{\{3\}} + 5.7291e + 13s^{\{2\}} + 4.2198e + 09s + 78479 \end{gathered}}{\begin{gathered} s^{\{27\}} + 4.0695e + 05s^{\{26\}} + 1.0676e + 10s^{\{25\}} + 9.9183e + 13s^{\{24\}} \\ +3.8045e + 17s^{\{23\}} + 6.3439e + 20s^{\{22\}} + 4.5615e \\ +23s^{\{21\}} + 1.4408e + 26s^{\{20\}} + 1.9758e + 28s^{\{19\}} + 1.1962e + 30s^{\{18\}} + 3.18e + 31s^{\{17\}} \\ +3.7999e + 32s^{\{16\}} + 2.0812e + 33s^{\{15\}} + 5.5529e + 33s^{\{14\}} + 7.7043e + 33s^{\{13\}} \\ +5.8358e + 33s^{\{12\}} + 2.295e + 33s^{\{11\}} + 4.4129e + 32s^{\{10\}} + 3.8401e + 31s^{\{9\}} + 1.4966e \\ +30s^{\{8\}} + 2.5189e + 28s^{\{7\}} + 1.8576e + 26s^{\{6\}} + 5.8294e + 23s^{\{5\}} + 7.881e + 20s^{\{4\}} \\ +4.3741e + 17s^{\{3\}} + 9.6885e + 13s^{\{2\}} + 7.136e + 09s + 1.3271e + 05 \end{gathered}}$$

Finally get the following resualt:

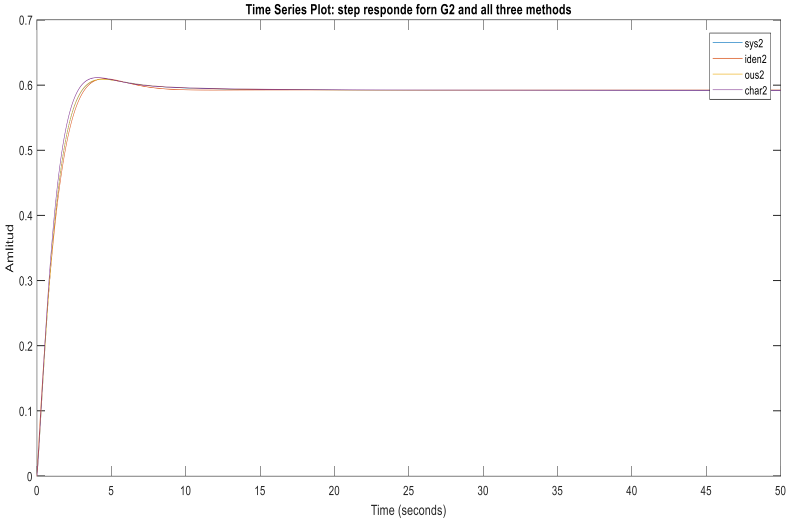

Figure III.12 : step respond for G2 with all methods

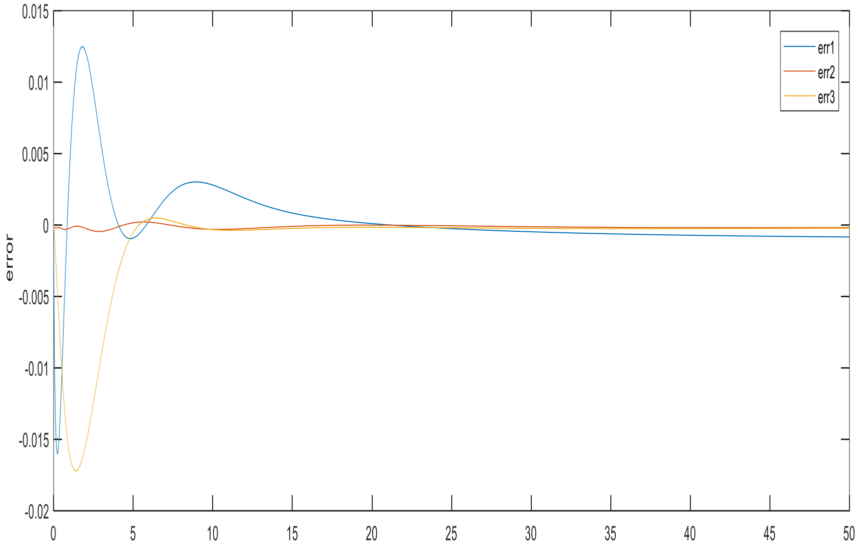

Figure III.13 : error for G2 and all three methods

III.3 : Comparation results

After completing the simulations on the two systems, we can say that all three approximation methods provide acceptable results, as the error rate does not exceed 2.5% and this is less than the acceptable error rate estimated at 5% at most.

However, when comparing the three methods, it can be noted the convergence of the results between the Charef method and the identification method as both of them have an error rate of up to 2.3% while oustaloop method's error is just 0.035% wich is much better.

III.4 : Conclusion

In this chapter, we simulate two fractional order systems, apply the three selected approximation methods, and compare the results in order to determine the method with the least error.

General Conclusion

In this graduation note, we have studied fractional order systems, the most important definitions like Caputo definition, formula, basic calculation methods, and approximation methods.

Between digital approximation methods and analog approximation methods, we choose to work with the analog approximation methods like Charef method, identificaton methods and Oustalop methods.

We choose these three methods because of its pupillarity so we wanted to know if its really useful, knowing that using fractional order system in actual is very difficult and challenging.

To do this, first we studied the three methods model and algorithm for various systems and applied it.

Then we simulated these methods on two different systems and showed the results on MATLAB 2017b using *FOMCON* toolbox.

After that, we compared the results obtained from the simulations by calculating the percentage deviation of the approximated functions from the original function.

All methods gave acceptable and scientifically applicable results, with a convergence of results between Identification method and Charef method the two methods and a clear superiority of Oustalop method.

In view of the results obtained during this graduation study, it can be said that these three analog methods can be applied practically in the industrial field, given that the error rate is within the acceptable range, but there is a clear defect, which is the necessity of having processors with high computational capacity in order to use smaller time samples and thus obtain on a lot of samples in order to get more accurate results.

REERENCES

[1]: J. Tenreiro Machado, Virginia Kiryakova, Francesco Mainard, Recent history of fractional calculus, CORE.AC.UK

[2]: Santanu Saha Ray, Abdon Atangana, S. C. Oukouomi Noutchie, Muhammet Kurulay, Necdet Bildik, and Adem Kilicman. Fractional Calculus and Its Applications in Applied Mathematics and Other Sciences, Hindawi Publishing Corporation, Volume 2014, Article ID 849395, 2 pages.

[3]: P.I. Lizorkin, « Intégrale de Riemann-Liouville », dans Michiel Hazewinkel, Encyclopædia of Mathematics, Springer, 2002

[4]:Caputo, Michele (1967). "Linear model of dissipation whose Q is almost frequency independent. II". Geophysical Journal International. 13 (5): 529–539. Bibcode:1967GeoJ...13..529C. doi:10.1111/j.1365-246x.1967.tb02303.x..

[5]:Caputo, Michele; Fabrizio, Mauro (2015). "A new Definition of Fractional Derivative without Singular Kernel". Progress in Fractional Differentiation and Applications. 1 (2): 73–85. Retrieved 7 August 2020.

[6]:Ramakrishnan Ganesh, , Nazim I. Mahmudov. APPROXIMATE CONTROLLABILITY OF FRACTIONAL FUNCTIONAL EQUATIONS WITH INFINITE DELAY. Juliusz Schauder Centre for Nonlinear Studies Volume 43, No. 2, 2014, 345–364

[7]:Salah Eddine Bouakkaz • Ben Djaballah Tahar. Réduction de modèles d'un système d'ordre fractionnaire. Université Mohamed Khider de Biskra Master 2022

[8]: Li, Y., Chen, Y., & Podlubny, I. (2010). Digital Approximation Methods for Fractional-Order Systems: A Review. IEEE Transactions on Circuits and Systems I: Regular Papers, 57(11), 2526-2534. DOI: 10.1109/TCSI.2010.2048563.

[9]: Changpin Li, Chunxing Tao.On the fractional Adams methodI. Computers and Mathematics with Applications 58 (2009) 15731588

[10]: David Usero, Fractional Taylor Series for Caputo Fractional Derivatives. Construction of Numerical Schemes. Dpto. de Matem¶atica Aplicada. Facultad de CC. Qu¶³micas. Universidad Complutense de Madrid. Spain.

[11]: Singh, S. K., Maheshwari, S., Sharma, P., & Saxena, S. (2017). Approximation of Fractional-Order RC Networks Using Analog Signal Processing Circuits. International Journal of Electronics and Communication Engineering & Technology, 8(5), 389-400.

[12]:Rudolf Scherer, Shyam L. Kalla, Yifa Tangc, Jianfei Huang. The Grünwald–Letnikov method for fractional differential equations. Computers and Mathematics with Applications 62 (2011) 902–917

[13]:link: https://www.ncbi.nlm.nih.gov/pmc/articles/PMC7908999/ . access at 10/6/2023

[14]: Carlson, B. C. (1963). A Simplified Approach to the Analysis of Complex Systems. IEEE Transactions on Circuit Theory, 10(2), 149-156.

[15]: BENZAHIA Houd, BELOUADAH Brahim. Utilisation des méthodes d'identification pour l'approximation des systèmes linéaires d'ordre fractionnaire. univ-msila Master 2022

[16]: A. Charef, A. Djouambi, and A.V.Besancon, 'Optimal approximation, Simulation and analog realization of the fundamental fractional order transfer function ', Int. J. Appl. Math. Comput. , Vol. 17, No. 4, 455–462 DOI: 10.2478/v10006-007-0037-9, 2007.

[17]:Charef A., Sun. H.H., Tsao .Y.Y., Onaral. B. «Fractal system as represented bysingularity function»; IEEE Trans. On Automatic Control, Vol. 37, No. 9, 1992.

ملخص

في هذا العمل سنتطرق إلى طرق تقريب أنظمة الدرجة الكسرية وبالتحديد الطرق التناظرية مثل طريقة شارف حيث سنجري محاكاة باستعمال برنامج ماتلاب لاثنين من أنظمة الدرجة الكسرية كمثال ثم نطلق طرق التقريب المختارة على هذين النظامين وذلك من أجل مقارنة النتائج والتحديد الطرقة ذات أقل نسبة خطأ بين الطرق المختارة

Abstract

In this work, we will look at the methods of approximating fractional order systems, specifically analog methods such as the Scharf method, where we will conduct a simulation using the MATLAB program for two fractional order systems as an example, and then launch the selected approximation methods on these two systems in order to compare the results and determine the method with the lowest error rate among the chosen methods

résumé

Dans ce travail, nous discuterons des méthodes d'approximation des systèmes de degrés fractionnaires, en particulier des méthodes analogiques telles que la méthode de Scharf, où nous effectuerons une simulation en utilisant le programme MATLAB pour deux systèmes de order fractionnaires à titre d'exemple, puis lancerons les méthodes d'approximation sélectionnées. sur ces deux systèmes afin de comparer les résultats et de déterminer la méthode avec le taux d'erreur le plus faible parmi les méthodes choisies