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## Title

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# *On fuzzy metric spaces*

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Academic years: 2020/2021

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# ON FUZZY METRIC SPACES

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# On fuzzy metric spaces

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*Thank*

# Contents

# Introduction

The notion of metric space, introduced in 1906 by M.Fréchet and developed shorty after by F.Hausdorff, comes directly from an analysis of the main properties of usual distance. The extension to metric spaces of properties of Euclidean space which are definable from distance alone introduces a geometric language into many questions of analysis and number theory. This is how we define, from the balls, the open ones, by the natural way in which neighborhoods and the notions of limit and continuity are introduced, the study of metric spaces is an excellent introduction to general topology. In mathematics, a metric space is a set where a distance (called a metric) is defined between elements of the set. Metric space methods have been employed for decades in various applications, for example in internet search engines, image classification, or protein classification.

Today the theory of fuzzy set is very topical, which serves mathematically represented the lack of precision relating to certain classes of object and also this theory is a generalization of the theory of classical sets. Indeed, the term fuzzy set represents the theoretical basis of fuzzy logic which was established in 1965 [17] by professor L.Zadeh of the University of California, Berkeley. At that time the theory of fuzzy logic was not taken seriously, it allows to treat non-exact variables whose value can vary between 0 and 1. At the beginning, this theory was applied in non-technical fields, like commerce and medicine whose aim is to supplement expert systems and to give them decision-making skills. In 1971, L.Zadeh [18] defined the notion of fuzzy relation, in particular the partial fuzzy order. The fields of application of the theory of fuzzy sets are very numerous, it is found in automatic, to make fuzzy control and regulation, in robotics, to do trajectory planning, in image processing, to attenuate noise, of an image, to make interpolation,...,etc.

One of the main problems in the theory of fuzzy topological spaces is to obtain an appropriate and consistent notion of a fuzzy metric space. Many authors have investigated this question and several notions of a fuzzy metric space have been defined and studied. In particular, and modifying the concept of metric fuzziness introduced by Kamosil and Michalek (which is a generalization of the concept of probabilistic metric space introduced by K. Menger [8] to the fuzzy setting), George and Veeramani [4, 5], have studied a notion of fuzzy metric space. In a previous paper [7], Gregori and Romaguera proved that the class of fuzzy metric spaces, in George and Veeramani's sense, coincides with the class of metric spaces. In the light of the results obtained in [7], we think that the George and Veeramani's definition is an appropriate notion of metric fuzziness in the sense that it provides rich fuzzy topological structures which can be obtained, in many cases, from classical theorems.

The objective of this work is to study some properties and characteristics of fuzzy metric spaces. Furthermore, we study some topological properties of fuzzy metric spaces such as compact fuzzy metric space and complete fuzzy metric space.

This dissertation is organized into three chapters.

- In the first chapter, we provide a basic introduction to the binary relations, posets, t-norm. Next, we recall some basic notions of fuzzy sets and fuzzy relations.
- In the second chapter, we study properties of fuzzy metric spaces and topology generated by fuzzy metric spaces.
- In the third chapter, we study further properties in fuzzy metric spaces using the notions of sequences.

# Chapter 1

# Generalities on fuzzy sets and fuzzy relations

The purpose of this first chapter is provide a basic introduction to the binary relations, posets, t-norm. Next, we recall some basic notions of fuzzy sets and fuzzy relations. Many of the properties of these concepts will be used in next chapters.

## 1.1 Crisp sets

This section contains the basic definitions of crisp sets with several operations.

**Definition 1.1.** *A set of reference $X$ is a collection of objects, this set con be defined by*

*(i) Writing of all its elements, whose elements are $a_1, a_2, \cdots, a_n$, and we write, $X = \{a_1, a_2, \cdots, a_n\}$.*

*(ii) A property or properties are satisfied by its elements, and we write, $A = \{x \mid P(x)\}$. Where the symbol "$|$" denotes the sentence "such that" and $P(x)$ a proposition of the form "$x$" has a property.*

*(iii) A function called characteristic function $\mathcal{X}_A(x)$ which takes the value 0 for the elements that do not belong to $A$ and the value 1 for those that belong to $A$:*

$$\chi_A : X \longrightarrow \{0,1\}$$
$$x \longmapsto \begin{cases} 0 & if \quad x \notin A; \\ 1 & if \quad x \in A. \end{cases}$$

**Definition 1.2.** ***(Operations on crisp sets)*** *Let $X$ be a set, let $A$ and $B$ be two subsets on $X$.*

*(i) Inclusion: $A \subset B$ if $(x \ \in A) \Rightarrow (x \in B)$,i.e.,$\mathcal{X}_A(x) \leq \mathcal{X}_B(x)$, for any $x \ \in X$;*

*(ii)* *Equality:* $A = B$ *if* $A \subseteq B$ *and* $B \subseteq A$ *i.e.,* $(\mathcal{X}_A(x) = \mathcal{X}_B(x))$, *for any* $x \in X$;

*(iii)* *Complement:* $A^c = \{x \in X \mid x \notin A\}$ *i.e.,* $\mathcal{X}_{A^c}(x) = 1 - \mathcal{X}_A(x)$;

*(iv)* *Intersection:* $A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}$*i.e.,*$\mathcal{X}_{A\cap B}(x) = min(\mathcal{X}_A(x), \mathcal{X}_B(x))$;

*(v)* *Union:* $A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}$*i.e.,*$\mathcal{X}_{A\cup B}(x) = max(\mathcal{X}_A(x), \mathcal{X}_B(x))$;

*(vi)* *Relative complement:* $A \backslash B = A - B = A \cap B^c = \{x \in X \mid x \in A \text{ and } x \notin B\}$ *i.e.,* $\mathcal{X}_{A-B}(x) = \mathcal{X}_{A\cap B^c}(x) = min(\mathcal{X}_A(x), \mathcal{X}_{B^c}(x))$.

**Example 1.1.** *Let* $X = \{x, y, z, t, w\}$ *be a set, let* $A$ *and* $B$ *be two subsets of* $X$ *such that* $A = \{x, y, w\}$ *and* $B = \{x, y, z\}$. *Then,*

$$A^c = \{z, t\};$$

$$B^c = \{t, w\};$$

$$A \cap B = \{x, y\};$$

$$A \cup B = \{x, y, z, w\};$$

$$A \backslash B = \{w\};$$

$$B \backslash A = \{z\}.$$

**Example 1.2.** *Let* $X = [36, 42]$ *the universe of speech which expresses the degree of temperature of a human body, a person with hepatitis generally presents the following symptoms:*

1. *The person has a high fever;*
2. *His skin is yellow in color;*
3. *He has nausea.*

*We will now study the first symptom or the property having a high fever which is indicated by the degree of temperature in the classic case (see* ***Figure 1.1****)we define of temperature in the classic set* $A$ *of* $X$ *associated with the property (having a high fever) by:*

$$\mathcal{X}_A(x) = \begin{cases} 1, & if \quad x \geq 39 \\ 0, & otherwise. \end{cases}$$

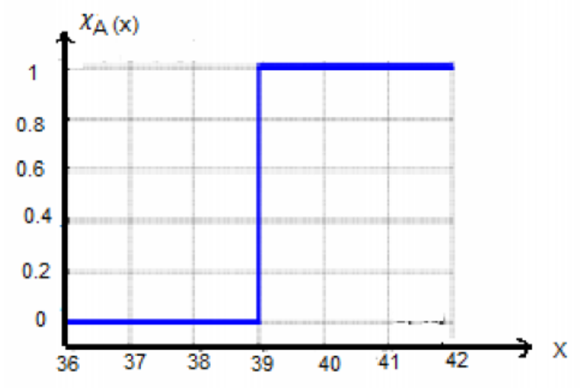

Figure 1.1

*i.e., people have a temperature over 39 are systematically have a high fever therefore they reach hepatitis, without this diagnosis being logical.*

## 1.2 Fuzzy sets

This section contains the basic definitions and properties of fuzzy sets and several operations of fuzzy sets. The notion of fuzzy set was introduced in 1965 by Lotfi A.Zadeh in the paper [17].

**Definition 1.3.** [17] *Let $X$ be a nonempty set. A fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ is characterized by a membership function $\mu_A : X \to [0,1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element $x$ in the fuzzy subset $A$ for $x \in X$.*

**Notation 1.1.** *Let $X$ be a nonempty set. The set of all fuzzy subsets of $X$ will be denoted by $F(X)$.*

**Example 1.3.** *Let $X = \{a, b, c\}$ be a set. $A_1 = \{(a, 0.3), (b, 1.0), (c, 0.7)\}$ and*
*$A_2 = \{(a, 0.0), (b, 0.9), (c, 0.7)\}$ are two fuzzy subsets on $X$.*

**Example 1.4.** *In same **Example1.2**, in this fuzzy case the (**Figure 1.2**) shows a possible diagnosis. We also define the fuzzy subset $B$ of $X$ which associated with the same property of $A$ (to have a strong fever) as*

$$\mu_B(x) = \begin{cases} 0, \quad if \quad 36 \leq x \leq 37, \\ \frac{26}{110}x - \frac{1037}{110}, \quad if \quad 37 < x < 41, \\ 1, \quad if \quad 41 < x < 42. \end{cases}$$

*i.e., a person has a temperature* $X$ *of* $[36, 42]$

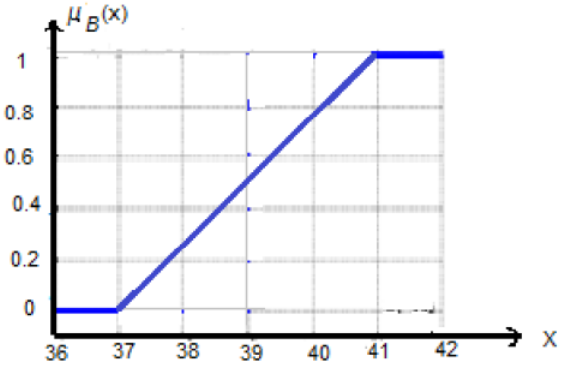

Figure 1.2: Trapezoid diagram

*If* $\mu_B(x) = 1$*, so the person with hepatitis.*

*If* $\mu_B(x) = \frac{29}{110}x - \frac{1037}{110}$*, so the person the patient with a degree of hepatitis.*

*If* $\mu_B(x) = 0$*, so the person does not have hepatitis.*

### 1.2.1 Fuzzy sets operations

For two fuzzy sets $A$ and $B$ on a set $X$, several operations are defined in the following way(see [17])

(i) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$, for any $x \in X$;

(ii) $A = B$ if $\mu_A(x) = \mu_B(x)$, for any $x \in X$;

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x) \rangle \mid x \in X\}$;

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x) \rangle \mid x \in X\}$;

(v) $\bar{A} = \{\langle x, 1 - \mu_A(x) \rangle \mid x \in X\}$.

**Example 1.5.** *If we consider the fuzzy sets* $A_1(x) = \begin{cases} 1, & if \quad 40 \leq x < 50, \\ 1 - \frac{x-50}{10}, & if \quad 50 \leq x < 60, \\ 0, & if \quad 60 \leq x \leq 100. \end{cases}$

$$A_2(x) = \begin{cases} 0, & if \quad 40 \leq x < 50, \\ \frac{x-50}{10}, & if \quad 50 \leq x < 60, \\ 1 - \frac{x-60}{10}, & if \quad 60 \leq x < 70, \\ 0, & if \quad 70 \leq x \leq 100. \end{cases}$$

*Then their union is* $(A_1 \cup A_2)(x) = \begin{cases} 1, & if \quad 40 \leq x < 50, \\ 1 - \frac{x-50}{10}, & if \quad 50 \leq x < 55, \\ \frac{x-50}{10}, & if \quad 55 \leq x \leq 60, \\ 1 - \frac{x-60}{10}, & if \quad 60 \leq x \leq 70, \\ 0, & if \quad 70 \leq x \leq 100. \end{cases}$

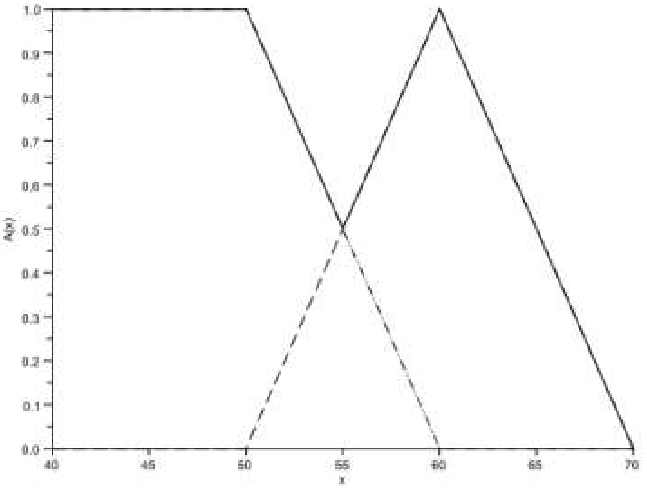

Figure 1.3: Fuzzy Union

*The intersection can be expressed as* $(A_1 \cap A_2)(x) = \begin{cases} 0, & if \quad 40 \leq x < 50, \\ \frac{x-50}{10}, & if \quad 50 \leq x < 55, \\ 1 - \frac{x-50}{10}, & if \quad 55 \leq x < 60, \\ 0, & if \quad 60 < x \leq 100. \end{cases}$

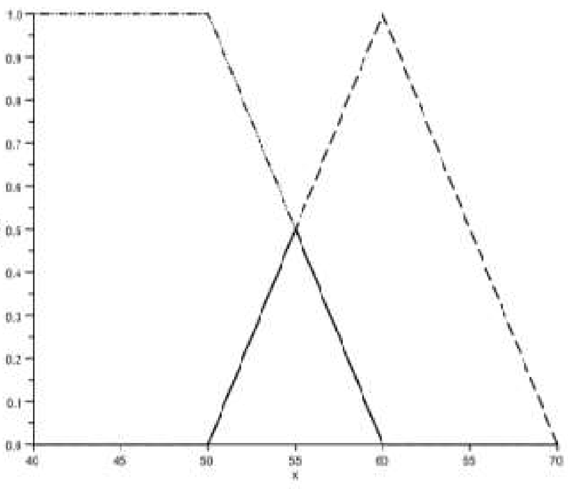

Figure 1.4: Fuzzy Intersection

*The complement of* $A_1$ *can be written* $\overline{A_1}(x) = \begin{cases} 0, & if \quad 40 \leq x < 50, \\ \frac{x-50}{10}, & if \quad 40 \leq x < 60, \\ 1, & if \quad 60 \leq x \leq 100. \end{cases}$

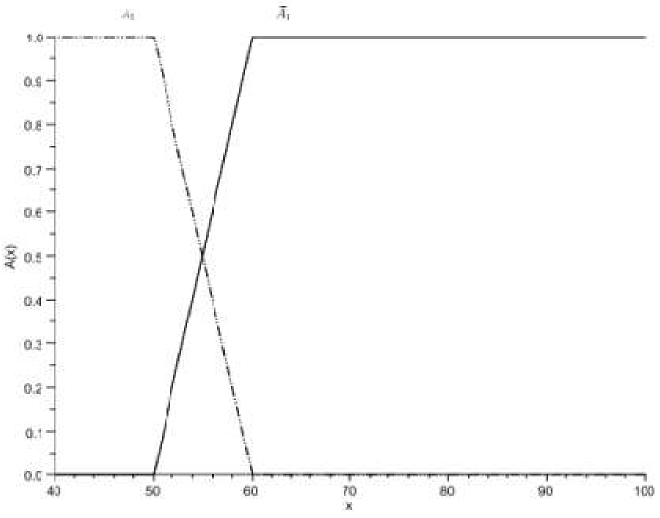

Figure 1.5: The complement of a fuzzy set

**Example 1.6.** *Let* $X = \mathbb{R}$ *and let* $A$ *be the set of reals greater than* 10 *and* $B$ *the set of reals* *close to* 1 *are characterized respectively by its membership functions*

$$\mu_A(x) = \begin{cases} 0, & if \quad x \leq 10, \\ (1+(x-10)^{-2})^{-1}, & if \quad x > 10. \end{cases}$$

*and*

$$\mu_B(x) = \begin{cases} 0, & if \quad x \leq 10, \\ (1+(x-10)^{4})^{-1}, & if \quad x > 10. \end{cases}$$

*So, we get* $A \cap B$ *set of reals greater than* 10 *and close to* 11 *given by its membership function*

$$\mu_{A\cap B}(x) = \begin{cases} 0, & if \quad x \leq 10, \\ min[(1+(x-10)^{-2})^{-1}, (1+(x-10)^{4})^{-1}], & if \quad x > 10. \end{cases}$$

*And* $A \cup B$ *the set of real numbers greater than 10 or close to 11 given by its membership function*

$$\mu_{A\cup B}(x) = max[(1+(x-10)^{-2})^{-1}, (1+(x-10)^{4})^{-1}], x \in X.$$

### 1.2.2 Characteristics of fuzzy sets

**Definition 1.4.** ***(The support of a fuzzy sets)*** [16] *Let* $A$ *be a fuzzy set on a set* $X$. *The support of* $A$ *is the crisp subset on* $X$ *given by*

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0\}.$$

**Definition 1.5.** ***(The kernel of a fuzzy sets)*** [16] *Let* $A$ *be a fuzzy set on a set* $X$. *The kernel of* $A$ *is the crisp subset on* $X$ *given by*

$$Ker(A) = \{x \in X \mid \mu_A(x) = 1\}.$$

**Definition 1.6.** ***(The highest of a fuzzy sets)*** [16] *Let* $A$ *be a fuzzy set on a set* $X$. *The height of* $A$ *is the highest value taken by its membership function given by*

$$H(A) = sup_{x\in X}\mu_A(x).$$

**Definition 1.7.** ***(The cardinality of a fuzzy sets)*** [10] *The cardinality of the fuzzy subset* $A$ *of* $X$, *noted* $\mid A \mid$, *when* $X$ *is finite, is defined by*

$$\mid A \mid = \sum_{x\in X} \mu_A(x).$$

**Example 1.7.** *Let* $X = \{a, b, c\}$ *be a set.* $A_1 = \{(a, 0.3), (b, 1.0), (c, 0.7)\}$ *and* $A_2 = \{(a, 0.0), (b, 0.9), (c, 1.0)\}$ *are two fuzzy subsets on* $X$.
*Then,* $Supp(A_1) = \{a, b, c\}$ *and* $Supp(A_2) = \{b, c\}$.
$Ker(A_1) = \{b\}$ *and* $Ker(A_2) = \{c\}$. $H(A_1) = 1$ *and* $H(A_2) = 1$. $\mid A_1 \mid = 2$ *and* $\mid A_2 \mid = 1.9$.

**Example 1.8.** *Let* $X = [0, 1]$ *with* $\alpha, \beta \in \mathbb{R}$ *and let* $a, b \in \mathbb{R}$. *We define the fuzzy set* $A$ *on* $X$ *by*

$$\mu_A(x) = \begin{cases} 0, \quad if \quad x < a - \alpha \quad or \quad b + \beta < x, \\ 1, \quad if \quad a < x < b, \\ 1 + (\frac{x-a}{\alpha}), \quad if \quad a - \alpha < x < a, \\ 1 - (\frac{b-x}{\beta}), \quad if \quad b < x < b + \beta. \end{cases}$$

*Then* $Ker(A) = [0, 1]$, $Supp(A) = [a - \alpha, b + \beta]$ *and* $H(A) = 1$.

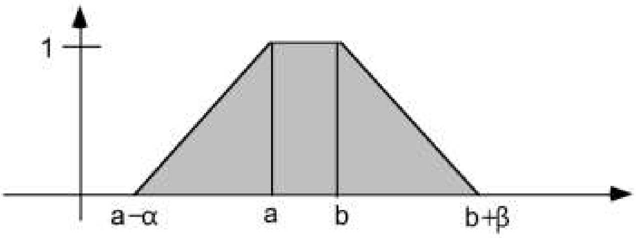

Figure 1.6

**Example 1.9.** *Let* $B$ *the fuzzy subset given by* ***Figure 1.2*** *on the set* $X = [36, 42]$. *Then,* $Supp(A) = ]37, 42]$, $H(A) = 1$, $Ker(A) = [41, 42]$, $\mid A \mid$ *is infinite.*

**Definition 1.8.** ***(The*** $\alpha$***-cut of a fuzzy set)*** [16] *Let* $A$ *be a fuzzy set on a set* $X$. *The* $\alpha$*-cut of* $A$ *is the crisp subset*

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\} \text{ where } \alpha \in [0, 1]$$

*Particular cases:*

1. *If* $\alpha = 0$*, then* $A_0 = X$
2. *If* $\alpha = 1$*, then* $A_1 = Ker(A)$.

**Example 1.10.** *Let* $X = \{1, 2, 3, ..., 10\}$*, and* $A$ *be a fuzzy subset of* $X$ *given by*
$A = \{< 1; 0.2 >, < 2; 0.5 >, < 3; 0.8 >, < 4; 1 >, < 5; 0.7 >, < 6; 0.3 >, < 7; 0 >, < 8; 0 >,$
$< 9; 0 >, < 10; 0 >\}$*. Then, the* $\alpha$*-cut of* $A$ *is given by:*
$A_0 = \{x \in X, A(x) > 0\} = X$;
$A_{0.1} = \{x \in X, A(x) \geq 0.1\} = \{1, 2, 3, 4, 5, 6\}$;
$A_{0.2} = \{x \in X, A(x) \geq 0.2\} = \{1, 2, 3, 4, 5, 6\}$;
$A_{0.3} = \{x \in X, A(x) \geq 0.3\} = \{2, 3, 4, 5, 6\}$;
$A_{0.4} = \{x \in X, A(x) \geq 0.4\} = \{2, 3, 4, 5\}$;
$A_{0.5} = \{x \in X, A(x) \geq 0.5\} = \{2, 3, 4, 5\}$;
$A_{0.6} = \{x \in X, A(x) \geq 0.6\} = \{3, 4, 5\}$;
$A_{0.7} = \{x \in X, A(x) \geq 0.7\} = \{3, 4, 5\}$;
$A_{0.8} = \{x \in X, A(x) \geq 0.8\} = \{3, 4\}$;
$A_{0.9} = \{x \in X, A(x) \geq 0.9\} = \{4\}$;
$A_1 = \{x \in X, A(x) \geq 1\} = \{4\}$.

### 1.2.3 Projection and cartesian product on fuzzy sets

**Definition 1.9.** ***(Cartesian product on fuzzy set)*** [9] *Let the fuzzy subsets* $A_1, A_2, \cdots, A_n$ *respectively defined on* $X_1, X_2, \cdots, X_n$*, we define their cartesian product* $A = A_1 \times A_2 \times \cdots \times A_n$*, as a fuzzy subset of* $X = X_1, X_2, \cdots, X_n$ *with a membership function defined for any* $x = (x_1, x_2, \cdots, x_n) \in X$ *by:*

$$\mu_A(x) = min[\mu_{A_1}(x_1), \mu_{A_2}(x_2), \cdots, \mu_{A_n}(x_n)].$$

**Example 1.11.** *Let* $X_1$ *be a set of animals* $X_1 = \{cat, cheetah, tiger\}$ *and* $X_2$ *be a set of country choices by temperature* $X_2 = \{hot, cold\}$*. The fuzzy subset* $A_1$ *represents the choices of an individual that the animals would like to own and the fuzzy subset* $A_2$ *represents its choices to the type of country in which the animal would like to live such as*
$A_1 = \{\langle cat, 0.5\rangle, \langle cheetah, 0.8\rangle, \langle tiger, 0.3\rangle\}, A_2 = \{\langle hot, 0.9\rangle, \langle cold, 0.1\rangle\}$*. We get* $A_1 \times A_2 = \{\langle(cat, hot), 0.5\rangle, \langle(cat, cold), 0.1\rangle, \langle(cheetah, hot), 0.8\rangle, \langle(cheetah, cold), 0.1\rangle, \langle(cheetah, hot), 0.8\rangle, \langle(tiger, hot), 0.3\rangle, \langle(tiger, cold), 0.1\rangle\}$*. Let be a fuzzy subset* $A$ *defined on a universe* $X_1 \times X_2$ *cartesian product of two reference sets* $X_1$ *and* $X_2$*.*

**Definition 1.10.** ***(Projection on fuzzy set)*** [9] *The projection on $X_1$ of the fuzzy set $A$ of $X_1 \times X_2$ is the fuzzy set $Proj_{X_1}(A)$ of $X_1$, whose membership function is defined by:*

$$\forall x_1 \in X_1, \mu_{Proj_{X_1}(A)}(x_1) = sup_{x_2 \in X_2} \mu_A(x_1, x_2).$$

*We defined in a similar way the projection of $A$ on $X_2$.*

**Example 1.12.** *Let $X = X_1 \times X_2$ the reference set such that $X_1$ and $X_2$ two sets which are defined in the* ***Example 1.11****, we consider $A_1 \times A_2 = A$ given by $A = \{\langle(cat, hot), 0.5\rangle, \langle(cat, cold), 0.1\rangle,$ $\langle(cheetah, hot), 0.8\rangle, \langle(cheetah, cold), 0.1\rangle, \langle(cheetah, hot), 0.8\rangle, \langle(tiger, hot), 0.3\rangle,$ $\langle(tiger, cold), 0.1\rangle\}$. Then, we get* $Proj_{X_1}(A) = \{\langle cat, max(0.5, 0.1)\rangle, \langle cheetah, max(0.8, 0.1)\rangle, \langle tiger, max(0.3, 0.1)\rangle\} = \{\langle cat, 0.5\rangle,$ $\langle cheetah, 0.8\rangle, \langle tiger, 0.3\rangle\}$. $Proj_{X_2}(A) = \{\langle hot, max(0.5, 0.8, 0.3)\rangle, \langle cold, max(0.1, 0.1, 0.1)\rangle\}$ $= \{\langle hot, 0.8\rangle, \langle cold, 0.1\rangle\}$.

In the following theorem, we study the decomposition theorem.

**Theorem 1.1.** ***(Decomposition theorem)*** [9] *Any fuzzy subset $A$ of the reference set $X$ is defined from its $\alpha$-cups for any element $x$ of $X$.*

$$\mu_A(x) = sup_{\alpha \in ]0,1]}(\alpha . \mathcal{X}_{A_\alpha}(x)).$$

*$\mathcal{X}_{A_\alpha}$ is the characteristic function of $A^\alpha$.*

*Proof.* Let the characteristic function

$$\mathcal{X}_{A_\alpha}(x) = \begin{cases} 1, & if \;\; \mu_A(x) \geq \alpha \\ 0, & if \;\; otherwise \end{cases}$$

by multiplying each member by a real number $\alpha$, we get:

$$\alpha \mathcal{X}_{A_\alpha}(x) = \begin{cases} \alpha, & if \;\; \mu_A(x) \geq \alpha \\ 0, & if \;\; otherwise. \end{cases}$$

By introducing the "sup" operator in each member, we have: $sup_{\alpha \in ]0,1]} \alpha \mathcal{X}_{A_\alpha}(x) = sup_{\alpha \in ]0,1]}\{\mu_A(x) \geq \alpha\} \Rightarrow sup_{\alpha \in ]0,1]} \alpha \mathcal{X}_{A_\alpha}(x) = sup_{\alpha \in ]0,1]}\{\alpha \leq \mu_A(x)\}$. Or (characterization of the upper bound in $\mathbb{R}$): $q = sup(A)$ if and only if $\forall q \in A, x \leq q$ ($q$ is an upper bound of $A$). Which makes it possible to establish that: $\mu_A(x) = sup_{\alpha \in ]0,1]}(\alpha . \mathcal{X}_{A_\alpha}(x))$. □

**Example 1.13.** *Let* $X = \{1, 2, \cdots, 10\}$ *and* $A = \{\langle 1, 0.2\rangle, \langle 2, 0.5\rangle, \langle 3, 0.8\rangle, \langle 4, 1\rangle, \langle 5, 0.7\rangle, \langle 6, 0.3\rangle\}$. *We have for any level* $\alpha$ *in* $[0, 1]$.

$$A_1 = \{x \in X | \mu_A(x) \geq 1\} = \{4\};$$

$$A_{0.8} = \{x \in X | \mu_A(x) \geq 0.8\} = \{3, 4\};$$

$$A_{0.7} = \{x \in X | \mu_A(x) \geq 1\} = \{3, 4, 5\};$$

$$A_{0.5} = \{x \in X | \mu_A(x) \geq 1\} = \{2, 3, 4, 5\};$$

$$A_{0.3} = \{x \in X | \mu_A(x) \geq 1\} = \{2, 3, 4, 5, 6\};$$

$$A_{0.2} = \{x \in X | \mu_A(x) \geq 1\} = \{1, 2, 3, 4, 5, 6\}.$$

*So, we get*
$\mu_A(1) = max(1 \times 0, \cdots, 0.2 \times 1, 0.1 \times 1, 0 \times 1) = 0.2;$
$\mu_A(2) = max(1 \times 0, \cdots, 0.5 \times 1, 0.4 \times 1, 0 \times 1) = 0.5;$
$\mu_A(3) = max(1 \times 0, 0.9 \times 0, 0.8 \times 1, \cdots, 0 \times 1) = 0.8;$
$\mu_A(4) = max(1 \times 1, \cdots, 0 \times 1) = 1;$
$\mu_A(5) = max(1 \times 0, \cdots, 0.7 \times 1, \cdots, 0 \times 1) = 0.7;$
$\mu_A(6) = max(1 \times 0, \cdots, 0.3 \times 1, \cdots, 0 \times 1) = 0.3.$
*Which provides the set* $A$.

## 1.3 T-norms and t-conorms

The history of triangular-norms(t-norms) started with Menger [8]. His main idea was to construct metric spaces where probability distributions are used to describe the distance between two elements.

### 1.3.1 T-norms

**Definition 1.11.** [15] *A t-norm* $T$ *on* $[0, 1]$ *is a function* $T : [0, 1]^2 \to [0, 1]$ *satisfies the following four axioms:*

*(T1) Commutativity:* $(\forall x, y \in [0, 1])(T(x, y) = T(y, x));$

*(T2) Associativity:* $(\forall x, y, z \in [0, 1])(T(x, T(y, z)) = T(T(x, y), z)));$

*(T3) Monotonicity:* $(\forall x, y, z \in [0, 1])(x \leq y \Rightarrow T(x, z) \leq T(y, z));$

*(T4) Boundary condition:* $(\forall x \in [0,1])(T(x,1) = x)$.

*Conditions* $(T4)$ *and* $(T3)$ *imply that for any t-norm* $T$ *it holds that* $T(x,y) \leq x$, $T(x,y) \leq y, T(x,y) \leq Min(x,y)$ *and* $T(x,0) = 0$.

**Example 1.14.** *The following four operations are the most common t-norms:*

*(T5) Minimum:* $T_M(x,y) = min\{x,y\}$;

*(T6) Product:* $T_P(x,y) = x \cdot y$;

*(T7) Lukasiewicz:* $T_L(x,y) = max\{x+y-1,0\}$;

*(T8) Drastic product:*

$$T_D(x,y) = \begin{cases} x, & if \quad y = 1 \\ y, & if \quad x = 1 \\ 0, & if \quad x,y < 1. \end{cases}$$

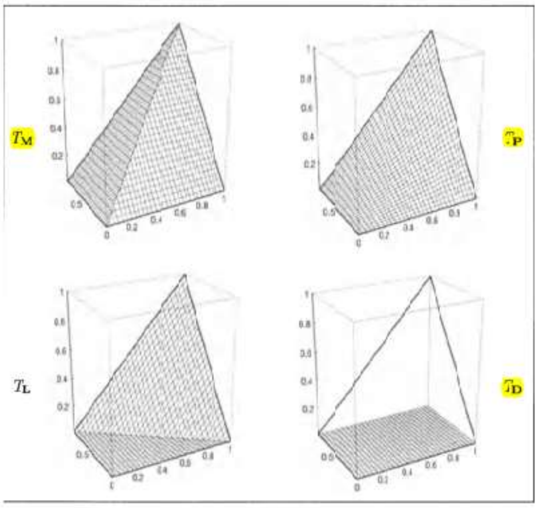

Figure 1.7: 3D plots of the four basic t-norm

Let $T$ be a t-norm on $[0,1]$. An element $\alpha \in ]0,1[$ is called a zero divisor of $T$ if there exists some $b > 0$ such that $T(a,b) = 0$. An element $\alpha \in [0,1]$ is called an idempotent element of $T$ if $T(a,a) = a$. $T$ is called Archimedean if $T(x,x) < x$, for any $x \in [0,1]$. Each $\alpha \in [a,b]$ is an idempotent element of the Minimum t-norm $T_M$(Actually $T_M$ is the only t-norm whose set of idempotent is equal $[0,1]$), $T_M$ has no zero divisor. Each $\alpha \in ]0,1[$ is a zero divisor of the

Lukasiewicz t-norm $T_L$ as well of the Drastic product t-norm $T_D$. For two t-norms $T_1$ and $T_2$ on $[0, 1]$, we define:

$$T_1 \leq T_2 \Leftrightarrow (\forall x, y \in [0, 1])(T_1(x, y) \leq T_2(x, y)).$$

Let be $T_1$ and $T_2$ two t-norms. If $T_1 \leq T_2$, then $T_1$ is called weaker than $T_2$ (or equivalently, $T_2$ is called stronger than $T_1$). Note that $T_D$ is the weakest t-norm, and $T_M$ is the strongest t-norm, i.e. for any t-norm it holds: $(T9)\ \ T_D \leq T \leq T_M$. Since, $T_L \leq T_P$, it obviously holds: $(T10)\ \ T_D \leq T_L \leq T_P \leq T_M$.

**Example 1.15.** *1.* $T_0(x, y) = \begin{cases} 0, & if \quad (x, y) \in [0, 1[^2 \\ min(x, y), & otherwise; \end{cases}$

*2.* $T_1(x, y) = max(x + y - 1, 0)$;

*3.* $T_{1.5}(x, y) = \frac{xy}{2-x-y+xy}$;

*4.* $T_2(x, y) = xy$;

*5.* $T_{2.5}(x, y) = \frac{xy}{x+y-xy}$;

*6.* $T_3(x, y) = min(x, y)$.

*We have* $T_0 \leq T_1 \leq T_{1.5} \leq T_2 \leq T_{2.5} \leq T_3$.

*Proof.* 1. $T_0(x, y) = \begin{cases} 0, & if \quad (x, y) \in [0, 1[^2; \\ min(x, y), & otherwise. \end{cases}$

If $(x, y) \in [0, 1[^2$, then $T_0 \leq T_1$.

If $(x, y) \notin [0, 1[^2$, i.e., $(x, y) \in \{1\} \times [0, 1]$ or $[0, 1] \times \{1\}$.

If $(x, y) \in \{1\} \times [0, 1] : T_0(x, y) = T_0(1, y) = y$ and $T_1(x, y) = T_1(1, y) = y$ then $T_0 \leq T_1$.

If $(x, y) \in [0, 1] \times \{1\} : T_0(x, y) = T_0(x, 1) = x$ and $T_1(x, y) = T_1(x, 1) = x$ then $T_0 \leq T_1$. So, $T_0(x, y) \leq T_1(x, y)$.

Then, $T_0 \leq T_1$

2. $T_1(x, y) = max(x + y - 1, 0)$ there are two cases:

(1) $x + y - 1 \leq 0 \Rightarrow T_1(x, y) = max(x + y - 1, 0) = 0 \leq T_{1.5}(x, y)$;

(2) $x + y - 1 > 0 \Rightarrow T_1(x, y) = max(x + y - 1, 0) = x + y - 1$.

$$\begin{aligned} T_{1.5}(x, y) - T_1(x, y) &= \frac{xy}{2 - x - y + xy} - (x + y - 1) \\ &= \frac{(xy - (x + y - 1)(2 - (x + y) + xy))}{2 - x - y + xy}. \end{aligned}$$

Then, $(2-x-y+xy)>0$, it is suffices to determine the sign of the numinator $[xy+(x+y-1)(x+y-xy-2)]$

$$\begin{aligned}(x+y-1)(x+y-xy-2)+xy &= (x+y-1)((x+y-1)-(xy+1))+xy\\ &= (x+y-1)^2-(x+y-1)(xy+1)+xy\\ &= (x+y-1)^2-x^2y-xy^2+2xy\\ &= (x+y-1)^2+(xy-x^2y)+(xy-xy^2)\geq 0.\end{aligned}$$

Therefore, $T_{1.5}(x,y)-T_1(x,y)\geq 0$.

Then, $T_1(x,y)\leq T_{1.5}(x,y)$.

(3) $T_2(x,y)=xy$

$$\begin{aligned}T_{1.5}(x,y)-T_2(x,y) &= \frac{xy}{2-x-y+xy}-xy\\ &= \frac{xy-xy(2-(x+y)+xy)}{2-x-y+xy}\end{aligned}$$

Since $2-(x+y)+xy>0$, thus it is enough to determine the sign of the numinator

$$\begin{aligned}xy-xy(2-(x+y)+xy) &= xy+xy(x+y-xy-2)\\ &= xy(x+y-xy-1)\\ &= xy((x-1)+y(1-x))\\ &= xy(x-1)(1-y)\leq 0\end{aligned}$$

Thus, $T_{1.5}(x,y)-T_2(x,y)\leq 0$. Then, $T_{1.5}(x,y)\leq T_2(x,y)$.

(4) $T_{2.5}(x,y)=\frac{xy}{x+y-xy}$

$$\begin{aligned}T_2(x,y)-T_{2.5}(x,y) &= xy-\frac{xy}{x+y-xy}\\ &= \frac{xy(x+y-xy)-xy}{x+y-xy}\\ &= \frac{xy(x+y-xy-1)}{x+y-xy}\end{aligned}$$

The denominator is positive $(x+y-xy>0)$, the numerator sign should be studied

$$\begin{aligned}xy(x+y-xy-1) &= xy(x(1-y)+(y-1))\\ &= xy((1-y)(1-x))\leq 0\end{aligned}$$

Thus, $T_2(x,y)-T_{2.5}(x,y)\leq 0$. Then, $T_2(x,y)\leq T_{2.5}(x,y)$.

3. Finally,

$$T_{2.5}(x,y) - T_3(x,y) = \begin{cases} \frac{xy}{x+y-xy} - x, & if \quad x \leq y; \\ \frac{xy}{x+y-xy} - y, & otherwise. \end{cases}$$

If $x \leq y$: $T_{2.5}(x,y) \leq min(x,y)$.

If $x > y$: $T_{2.5}(x,y) \leq min(x,y)$.

Thus, for all $(x,y) \in [0,1[^2 \;\; T_{2.5}(x,y) \leq T_3(x,y)$.

Consequently,

$$T_0(x,y) \leq T_1(x,y) \leq T_{1.5}(x,y) \leq T_2(x,y) \leq T_{2.5}(x,y) \leq T_3(x,y).$$

□

**Definition 1.12.** *A binary Operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-norm if* $([0,1], *)$, *is a topological monoid with unit* $1$ *such that* $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ $(a,b,c,d \in [0,1])$.

### 1.3.2 T-conorm

**Definition 1.13.** [15] *A t-conorm is a function* $S : [0,1]^2 \to [0,1]$ *that for any* $x, y, z \in [0,1]$ *satisfies* $(T1) - (T3)$ *and the following boundary condition* $S(x,0) = S(0,x) = x$, $S(x,1) = S(1,x) = 0$.

**Remark 1.1.** *Given a t-norm* $T$, *we find the associated dual t-conorm* $S$ *by* $S(x,y) = 1 - T(1-x, 1-y)$. *The dual t-conorms w.r.t.* $T_M$, $T_P$, $T_L$ *and* $T_D$ *are given by:*

*(S1) Maximum:* $S_M(x,y) = max\{x,y\}$;

*(S2) Probabilistic sum:* $S_P(x,y) = x + y - s \cdot y$;

*(T7) Lukasiewicz:* $S_L(x,y) = min\{x+y, 1\}$;

*(T8) Drastic sum:*

$$S_D(x,y) = \begin{cases} 1, & if \quad (x,y) \in [0,1]^2 \\ max\{x,y\}, & otherwise. \end{cases}$$

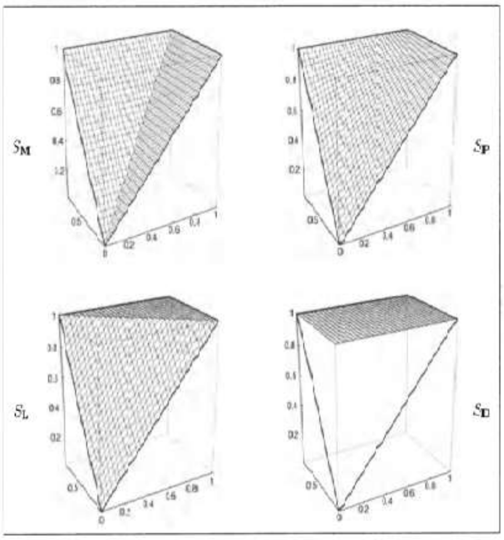

Figure 1.8: 3D plots of the four basic t-conorm

**Definition 1.14.** ***(Duality between operation)*** [9] *A t-norm $T$ and a t-conorm $S$ are said to be dual for the strict negation $n$ if they satisfy the following formulas for all $x, y \in [0, 1]$ :*
$S(x, y) = N(T(N(x), N(y)))$;
$T(x, y) = N(S(N(x), N(y)))$.

**Example 1.16.** *Let be $X = \{a, b, c\}$, let $A$ and $B$ two fuzzy subsets of $X$ such that* $A = \{\langle a, 0.2\rangle, \langle b, 0.4\rangle, \langle c, 0.8\rangle\}$, $B = \{\langle a, 0.9\rangle, \langle b, 0.1\rangle, \langle c, 0.5\rangle\}$. *We can use the operators of Lukasiewicz to define the union and the intersection by*

1. $\mu_{A\cap_T B}(x) = T(\mu_A, \mu_B) = max(\mu_A(x) + \mu_B(x) - 1, 0)$, *for any* $x \in X$;

2. $\mu_{A\cup_S B}(x) = S(\mu_A, \mu_B) = min(\mu_A(x) + \mu_B(x), 1)$, *for any* $x \in X$.

*Then, we get*

1. $A \cap_T B = \{\langle a, 0.1\rangle, \langle b, 0\rangle, \langle c, 0.3\rangle\}$;

2. $A \cup_S B = \{\langle a, 1\rangle, \langle b, 0.5\rangle, \langle c, 1\rangle\}$.

## 1.4 Fuzzy relations

In this section, we recall the basic definitions and properties of fuzzy relations and several operations on fuzzy relations. The notion of fuzzy set was introduced in 1971 by Lotfi A.Zadeh in the paper [18].

**Definition 1.15.** [18] *Let $X$ and $Y$ be two nonempty sets. A binary fuzzy relation $R$ from $X$ to $Y$, is a fuzzy subset of $X \times Y$ characterized by a membership function $R$ which associates with each pair $(x, y)$ its grade of membership $R(x, y)$ in the interval* $[0, 1]$.

**Definition 1.16.** ***(Operations on fuzzy relations)*** *Let $R$, $P$ be two fuzzy relations. $R$ is said to be contained in $P$ (or $P$ contains $R$), denoted by $R \subseteq P$, if for any $(x, y) \in X \times Y$ it holds that $R(x, y) \leq P(x, y)$. The transpose(or the inverse) $R^t$ of $R$ is the fuzzy relation from $Y$ to $X$ defined by*

$R^t = \{\langle (x, y), R^t(x, y)\rangle \mid (x, y) \in X \times Y\}$, *where* $R^t(x, y) = R(y, x)$ *for any* $(x, y) \in X \times Y$.

*The intersection of two fuzzy relations $R$ and $P$ is defined as*

$R \cap P = \{\langle (x, y), min(R(x, y), P(x, y))\rangle \mid (x, y) \in X \times Y\}$.

*The union of two fuzzy relations $R$ and $P$ is defined as*

$R \cup P = \{\langle (x, y), max(R(x, y), P(x, y))\rangle \mid (x, y) \in X \times Y\}$.

*The complement $R^c$ of $R$ is the fuzzy relation defined as*

$R^c = \{\langle (x, y), 1 - R(x, y)\rangle \mid (x, y) \in X \times Y\}$.

**Example 1.17.** *Let $R$ and $S$ be two fuzzy relations on $X \times X$ such that $x = \{x, y, z\}$, represented by the following tables*

| $R$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | 1.0 | 0.9 | 0.8 |
| $y$ | 0.9 | 1.0 | 0.8 |
| $z$ | 0.8 | 0.8 | 1.0 |

| $S$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | 0.6 | 0.2 | 0.7 |
| $y$ | 0.9 | 0.0 | 1.0 |
| $z$ | 0.1 | 0.7 | 0.6 |

*The union and intersection relations defined by*

| $R \cup S$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | 1.0 | 0.9 | 0.8 |
| $y$ | 0.9 | 1.0 | 1.0 |
| $z$ | 0.8 | 0.8 | 1.0 |

| $R \cap S$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | 0.6 | 0.2 | 0.7 |
| $y$ | 0.9 | 0.0 | 0.8 |
| $z$ | 0.1 | 0.7 | 0.6 |

*The transpose relation is given by the following table*

| $S$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | 0.6 | 0.9 | 0.1 |
| $y$ | 0.2 | 0.0 | 0.7 |
| $z$ | 0.7 | 1.0 | 0.6 |

*The complementary relation is given by the following table*

| $s$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | *0.4* | *0.8* | *0.3* |
| $y$ | *0.1* | *1.0* | *0.0* |
| $z$ | *0.9* | *0.3* | *0.4* |

**Definition 1.17.** *Let $R$, $P$ and $Q$ be three fuzzy relations from a universe $X$ to a universe $Y$.*

*(i)* *if* $R \subseteq P$*, then* $R^t \subseteq P^t$;

*(ii)* $(R \cup P)^t = R^t \cup P^t$;

*(iii)* $(R \cap P)^t = R^t \cap P^t$;

*(iv)* $(R^t)^t = R$;

*(v)* $R \cap (P \cup Q) = (R \cap P) \cup (R \cap Q)$ *and* $R \cup (P \cap Q) = (R \cup P) \cap (R \cup Q)$;

*(vi)* $R \cup P \supseteq R$, $R \cup P \supseteq P$, $R \cap P \subseteq R$, $R \cap P \subseteq P$;

*(vii)* *if* $R \supseteq P$ *and* $R \supseteq Q$*, then,* $R \supseteq P \cup Q$;

*(viii)* *if* $R \subseteq P$ *and* $R \subseteq Q$*, then,* $R \subseteq P \cap Q$.

**Definition 1.18.** [3, 2] *Let $R$ be an fuzzy relation on $X$, for short. The following properties are crucial :*

*(i)* *Reflexive: if* $R(x,x) = 1$*, for any* $x \in X$;

*(ii)* *Irreflexive: if* $R(x,x) = 0$*, for any* $x \in X$;

*(iii)* *Symmetrical: if* $R(x,y) = R(y,x)$*, for all* $x, y \in X$;

*(iv)* *Asymmetrical: if* $R(x,y) \wedge R(y,x) = 0$*, with* $x \neq y$*, for all* $x, y \in X$;

*(v)* *Antisymmetry: if* $(R(x,y) > 0) \wedge (R(y,x) > 0)$ *then,* $x = y$ *, for all* $x, y \in X$;

*(vi)* *Transitive: if* $R(x,z) \geq max_{y \in X}(min(R(x,y), R(y,z)))$*, for all* $x, y, z \in X$.

**Definition 1.19.** [3, 2] *Let $X$ be a nonempty crisp set and $R$ be a fuzzy relation on $X$. $R$ is called a fuzzy order or a partial fuzzy order if the following condition are satisfies:*

*(i)* *Reflexive: if* $R(x,x) = 1$*, for any* $x \in X$;

*(ii)* *Antisymmetry: if* $\begin{cases} R(x,y) > 0 \\ R(y,x) > 0 \end{cases} \Rightarrow x = y$

*(iii) Transitivity: if* $R(x, z) \geq max_{y \in X}(min(R(x, y), R(y, z)))$, *for all* $x, y, z \in X$.

*A nonempty set* $X$ *with a fuzzy order* $R$ *defined on it is called a fuzzy ordered set and is denoted by* $(X; R)$. *It easily follows that each partially ordered set* $(X; \leq)$ *and each fuzzy ordered set* $(X; R)$ *can be viewed as fuzzy ordered sets.*

**Example 1.18.** *Let* $X = \{a, b, c, d, e\}$. *Then, the fuzzy relation* $R$ *defined on* $X$ *by* $R = \{\langle (x, y), R(x, y) \rangle \mid x, y \in X\}$.
*where* $R$ *is given by the following tables:*

| $R(.,.)$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | 1 | 0 | 0 | 0.65 | 0.50 |
| $b$ | 0 | 1 | 0 | 0.35 | 0.45 |
| $c$ | 0 | 0 | 1 | 0 | 0.80 |
| $d$ | 0 | 0 | 0 | 1 | 0 |
| $e$ | 0 | 0 | 0 | 0 | 1 |

*is an the fuzzy order on* $X$.

**Example 1.19.** *Let* $m, n \in \mathbb{N}$. *Then, the following fuzzy relation* $R$ *on* $\mathbb{N}$ *is an fuzzy order, where*

$$R(m, n) = \begin{cases} 1, & if \quad m = n, \\ 1 - \frac{m}{n}, & if \quad m < n, \\ 0, & if \quad m > n. \end{cases}$$

*On the basis of the above definition of antisymmetry we define a complete (or total) fuzzy order as follows.*

**Definition 1.20.** [18] *A fuzzy order* $R$ *on a universe* $X$ *is called complete (or total) if for all* $x, y \in X$ *it holds that* $[R(x; y) > 0 \quad or \quad (R(x; y) = 0)] \quad or \quad [R(y; x) > 0 \quad or \quad (R(y; x) = 0)]$.

**Example 1.20.** *Let* $R$ *be a fuzzy relation on* $X = \{x, y, z\}$ *given by*

| $R$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | 1 | 0.6 | 0.6 |
| $y$ | 0 | 1 | 0.4 |
| $z$ | 0 | 0 | 1 |

$R$ *is a total fuzzy order.*

# Chapter 2

# Fuzzy metric spaces

The aim of this chapter is to study properties of fuzzy metric spaces and topology generated by fuzzy metric spaces. For more details see [4, 7].

## 2.1 Definitions

In this section, we recall the basic definitions and properties of fuzzy metric spaces. The notion of fuzzy metric spaces was introduced in 1994 by A. George and P.Veeramani in the paper [4].

**Definition 2.1.** [4] *The 3-tuple* $(X, M, *)$ *is said to be a fuzzy metric space if* $X$ *is an arbitrary set,* $*$ *is a continuous t-norm and* $M$ *is a fuzzy set on* $X^2 \times [0, +\infty[$ *satisfying the following conditions:*

*(i)* $M(x, y, t) > 0$;

*(ii)* $M(x, y, t) = 1$ *if and only if* $x = y$;

*(iii)* $M(x, y, t) = M(y, x, t)$;

*(iv)* $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

*(v)* $M(x, y, \cdot) : [0, +\infty[ \to [0, 1]$ *is continuous,* $x, y, z \in X$ *and* $t, s > 0$.

**Example 2.1.** *Let* $X = \mathbb{R}$. *Define* $a * b = ab$ *for all* $a, b \in [0.1]$ *and*

$$M(x, y, t) = \left[\exp\left(\frac{\mid x - y \mid}{t}\right)\right]^{-1},$$

*for all* $x, y \in X$ *and* $t \in [0, +\infty[$. *Then,* $(X, M, *)$ *is a fuzzy metric space. We shall show that* $M$ *is a fuzzy metric.*

*Proof.* (1) $\forall t > 0$. Assume that $x = y$. Then, this implies that $\mid x - y \mid= 0$. Then,

$$\left[\exp\left(\frac{\mid x-y \mid}{t}\right)\right]^{-1} = 1.$$

Thus, $M(x, y, t) = 1$. Conversely, assume that $M(x, y, t) = 1$. Then,

$$\left[\exp\left(\frac{\mid x-y \mid}{t}\right)\right]^{-1} = 1$$

implies that $e^{\frac{|x-y|}{t}} = e^0$. Therefore, $\frac{|x-y|}{t} = 0$, it follows that $\mid x - y \mid= 0$. Then, $x = y$. Thus, $M(x, y, t) = 1$ if and only if $x = y$.

(2) To prove $M(x, y, t) = M(y, x, t)$ we know that $\mid x-y \mid=\mid y-x \mid$ for all $x, y \in \mathbb{R}$. It follows that for all $x, y \in X$ and for all $t > 0$, $M(x, y, t) = M(y, x, t)$

(3) to prove $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$, we know that for all $x, y, z \in X$ and for all $t, s > 0$,

$$\mid x - z \mid\leq \left(\frac{t+s}{t}\right) \mid x - y \mid + \left(\frac{t+s}{s}\right) \mid y - z \mid .$$

That is $\frac{|x-z|}{t+s} \leq \frac{|x-y|}{t} + \frac{|y-z|}{s}$. Then, $e^{\frac{|x-z|}{t+s}} \leq e^{\frac{|x-y|}{t}} e^{\frac{|y-z|}{s}}$, since $e^x$ is an increasing function for $x > 0$. Then,

$$\left[\exp\left(\frac{\mid x-z \mid}{t+s}\right)\right]^{-1} \geq \left[\exp\left(\frac{\mid x-y \mid}{t}\right)\right]^{-1} * \left[\exp\left(\frac{\mid y-z \mid}{s}\right)\right]^{-1}.$$

Then, $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$.

(4) Take a sequence $\{t_n\} \in [0, +\infty[$ such that the sequence $\{t_n\}$ converges to $t \in [0, +\infty[$ where $[0, +\infty[$ is equipped with the usual metric. That is, $\lim_n \mid t_n - t \mid= 0$. Without the loss of generality, fix $x, y \in X$. Since the function $e^x$ is continuous on $\mathbb{R}$ we have $e^{\frac{|x-y|}{t_n}}$ converges to $e^{\frac{|x-y|}{t}}$ as $t_n$ converges to $t$, with respect to the usual metric. Then, $M(x, y, \bullet) : [0, +\infty[\to [0, 1]$ is continuous. Thus, $(X, M, *)$ is a fuzzy metric space.

□

We can replace $\mathbb{R}$ by any metric space $X$ and $\mid x - y \mid$ by $d(x, y)$ in the above example. Further, note that the above example holds even with the other t-norm $a * b = min(a, b)$.

**Remark 2.1.** *(i) In a fuzzy metric space* $(X, M, *)$*, whenever* $M(x, y, t) > 1 - r$ *for* $x, y$ *in* $X, t > 0,\ 0 < r < 1$*, we can find a* $t_0$*,* $0 < t_0 < t$ *such that* $M(x, y, t_0) > 1 - r$*. For any* $r_1 > r_2$*, we can find a* $r_3$ *such that* $r_1 * r_3 \geq r_2$ *and for any* $r_4$ *we can find a* $r_5$ *such that* $r_5 * r_5 \geq r_4, (r_1, r_2, r_3, r_4, r_5 \in [0, 1])$.

*(ii)* $M(x, y, *)$ *is nondecreasing for all* $x, y \in X$.

*(iii) Let* $X$ *be a non-empty set. If* $(M, T)$ *is a fuzzy metric on* $X$ *and* $T^{'}$ *is a continuous t-norm such that* $T^{'} \leq T$*, then,* $(M, T^{'})$ *is a fuzzy metric on* $X$.

Next, the following example shows that every metric induces a fuzzy metric.

**Example 2.2.** *Let* $(X, d)$ *be a metric space. Define* $a * b = ab$ *and* $M(x, y, t) = \frac{kt^n}{kt^n + m\ d(x,y)}$, $k, m, n \in \mathbb{R}^+$*. Then,* $(X, M, *)$ *is a fuzzy metric space.*

Note that the above example holds even with the t-norm $a * b = min(a, b)$. In the above example by taking $k = m = n = 1$, we get $M(x, y, t) = \frac{t}{t + d(x,y)}$. We call this fuzzy metric induced by a metric $d$ the standard fuzzy metric.

**Example 2.3.** *Let* $X = \mathbb{N}$*. Define* $a * b = ab$*, and for all* $t > 0$*, let*

$$M(x, y, t) = \begin{cases} x/y, & if \quad x \leq y, \\ y/x, & if \quad y \leq x, \quad for \quad all \quad t > 0. \end{cases}$$

*Then,* $(X, M, *)$ *is a fuzzy metric space.*

We shall show that M is a fuzzy metric.

*Proof.* (1) $\forall t > 0$. Assume that $x = y$. Thus, $\frac{x}{y} = \frac{y}{x} = 1$. Therefore, $M(x, y, t) = 1$.

Conversely,

assume that $M(x, y, t) = 1$. Hence, $\frac{x}{y} = 1$, therefore, $x = y$. Similarly if $\frac{y}{x} = 1$ then, it follows that $y = x$. Then, $M(x, y, t) = 1$ if and only if $x = y$.

(2) For all $x, y \in X$ and for all $t > 0$, clearly, $M(x, y, t) = M(y, x, t)$.

(3) To prove that $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) = 1$. We consider the following cases:

(i) $x = y = z$.

Then

$$M(x, y, t) = 1$$

$$M(y, z, s) = 1$$

$$M(x,z,t+s)=1$$

Now, $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)=1$.

It follows that $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)$ holds

(ii) $x\neq y=z$.

Without loss of generality, we may assume that $x<y$ and $y=z$. Then, $M(x,y,t)=\frac{x}{y}$. Also, we have $M(y,z,t)=1$ and $M(x,z,t+s)=\frac{x}{z}$. Now, $\frac{x}{y}*1=\frac{x}{y}$ and $\frac{x}{y}=\frac{x}{z}$. Then, $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)$ holds.

(iii) $x=y\neq z$.

Without loss of generality, we may assume that $x=y$ and $y<z$. Then, $M(x,y,t)=1$. Also, we have $M(y,z,t)=\frac{y}{z}$ and $M(x,z,t+s)=\frac{x}{z}$. Now, $1*\frac{y}{z}=\frac{y}{z}$ and $\frac{y}{z}=\frac{x}{z}$. Thus, $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)$ holds.

(iv) $x\neq y\neq z$.

With out loss of generality, we may assume that $x<y<z$. Then

$$M(x,y,t)=\frac{x}{y}$$
$$M(y,z,s)=\frac{y}{z}$$
$$M(x,z,t+s)=\frac{x}{z}.$$

Now, $z>y$ implies that $z^2>y^2$. So $\frac{1}{z^2}<\frac{1}{y^2}$. Therefore, $\frac{xy}{z^2}<\frac{xy}{y^2}$. Hence, $\frac{x}{z}*\frac{y}{z}<\frac{x}{y}$. Then

$$M(x,z,t)*M(z,y,s)<M(x,y,t+s).$$

(4) Note that $M(x,y,t)$ is independent of $t$ (that is, $M(x,y,t)$ is a constant in terms of $t$). For any $s,t>0$. We have $M(x,y,t)=M(x,y,s)$. Thus, $M(x,y,\bullet)$ is continuous. Therefore, $(X,M,*)$ is a fuzzy metric space.

□

**Remark 2.2.** *It is interesting to note that there exists no metric on* $X$ *satisfying* $M(x,y,t)=\frac{t}{t+d(x,y)}$ *where* $M(x,y,t)$ *is defined in the above example. Also, note that the above function* $M$ *is not a fuzzy metric with the t-norm defined as* $a*b=min(a,b)$.

## 2.2 Topology induced by fuzzy metric

**Definition 2.2.** [4] *Let* $(X, M, *)$ *be a fuzzy metric space. We define open ball* $B(x, r, t)$ *with centre* $x \in X$ *and radius* $r, 0 < r < 1,\ t > 0$ *as* $B(x, r, t) = \{y \in X; M(x, y, t) > 1 - r\}$.

In the following proposition, we will show that relation between open ball and open set.

**Proposition 2.1.** *Every open ball is an open set.*

*Proof.* Consider an open ball $B(x, r, t)$. Now $y \in B(x, r, t) \Rightarrow M(x, y, t) > 1 - r$. Since $M(x, y, t) > 1-r$, we can find a $t_0, 0 < t_0 < t$ such that $M(x, y, t_0) > 1-r$. Let $r_0 = M(x, y, t_0) > 1-r$. Since $r_0 > 1-r$, we can find a $s, 0 < s < 1$, such that $r_0 > 1-s > 1-r$. Now for a given $r_0$ and $s$ such that $r_0 > 1-s$ we can find $r_1, 0 < r_1 < 1$ such that $r_0 * r_1 \geq 1-s$. Now consider the ball $B(y, 1-r_1, t-t_0)$. We claim $B(y, 1-r_1, t-t_0) \subset B(x, r, t)$. Now $z \in B(y, 1-r_1, t-t_0) \Rightarrow M(y, z, t-t_0) > r_1$. Therefore, $M(x, z, t) \geq M(x, y, t_0) * M(y, z, t-t_0) \geq r_0 * r_1 \geq 1-s > 1-r$. Therefore, $z \in B(x, r, t)$ and hence, $B(y, 1-r_1, t-t_0) \subset B(x, r, t)$. □

In this theorem, we study the topology induced by fuzzy metric space.

**Theorem 2.1.** ***(Topology induced by fuzzy metric)*** *Let* $(X, M, *)$ *be a fuzzy metric space. Define* $\tau_M = \{A \subset X : x \in A$ *if and only if there exist* $t > 0$ *and* $r,\ 0 < r < 1$ *such that* $B(x, r, t) \subset A\}$. *Then,* $\tau_M$ *is a topology on* $X$.

*Proof.* (i) Clearly $\emptyset$ and $X$ belong to $\tau_M$.

(ii) Suppose that $A_1, A_2, A_3, ..., A_i \in \tau_M$, setting $U = \bigcup_{i \in I} A_i$. We shall show that $U \in \tau_M$. If $a \in U$, thus, $a \in \bigcup_{i \in I} A_i$, which implies that $a \in A_i$ for some $i \in I$. Since $A_i \in \tau_M$, there exists $0 < r < 1, t > 0$, such that $B(a, r, t) \subset A_i$. Then, $B(a, r, t) \subset A_i \subset \bigcup_{i \in I} A_i = U$. This shows that $U \in \tau_M$.

(iii) Let $A_1, A_2, A_3, ..., A_n \in \tau_M$, and $U = \bigcap_{i=I}^{n} A_i$. We shall show that $U \in \tau_M$. Let $a \in U$. thus, $a \in A_i$ for all $i \in I$. Therefore, for each $i \in I$. there exists $0 < r_i < 1, t_i > 0$ such that $B(a, r_i, t_i) \subset A_i$. Let $r = min\{r_i, i \in I\}$ and $t = max\{t_i, i \in I\}$. Hence, $r \leq r_i$ for all $i \in I$, $1 - r \geq 1 - r_i$ for $i \in I$. Also, $t > 0$. So, $B(a, r, t) \subseteq A_i$ for all $i \in I$. Then, $B(a, r, t) \subset \bigcap_{i \in I} A_i = U$. This shows that $U \in \tau_M$.

□

**Lemma 2.1.** *Let* $(X, d)$ *be a metric space and* $s, t > 0$. *The following inequality holds, for all* $n \geq 1$,

$$\frac{d(x,z)}{(t+s)^n} \leq max\left\{\frac{d(x,z)}{t^n}, \frac{d(x,z)}{s^n}\right\}$$

*Proof.* We distinguish three cases:

(1) $d(x,z) \leq d(x,y)$

(2) $d(x,z) \leq d(y,z)$

(3) $d(x,z) > d(x,y)$ and $d(x,z) > d(y,z)$

The inequality chosen is obvious in cases (1) and (2). Now, suppose (3) is satisfied and distinguish two possibilities:

(3.1) $d(x,z) = d(x,y) + d(y,z)$

(3.2) $d(x,z) < d(x,y) + d(y,z)$

Suppose (3.1) is satisfied. Put $d(x,y) = \beta d(x,y)$ with $\beta \in ]0,1[$ and hence

$$d(y,z) = (1-\beta)d(x,z).$$

Now, to show the above inequality we have to prove that

$$\frac{1}{(t+s)^n} \leq max\left\{\frac{\beta}{t^n}, \frac{1-\beta}{s^n}\right\}.$$

Then, consider the function $f(\beta) = \frac{t^n}{\beta}$ and $g(\beta) = \frac{s^n}{1-\beta}$ which are strictly decreasing and increasing, respectively. Now, the largest value of $min\left\{\frac{t^n}{\beta}, \frac{s^n}{1-\beta}\right\}$ is taken when $f(\beta) = g(\beta)$, that is, for $\beta = \frac{t^n}{t^n+s^n}$. Then,

$$(t+s)^n \geq t^n + s^n = f\left(\frac{t^n}{t^n+s^n}\right) \geq min\left\{\frac{t^n}{\beta}, \frac{s^n}{1-\beta}\right\}$$

and the chosen inequality is stated. The case (3.2) is a consequence of (3.1). □

**Example 2.4.** *Let $(X,d)$ be a metric space, and denote $B(x,r)$ the open ball centered in $x \in X$ with radius $r > 0$.*

*(i) For each $n \in \mathbb{N}$, $(X, M, T_1)$ is a fuzzy metric space where $M$ is given by*

$$M(x,y,t) = \frac{1}{e^{\frac{d(x,y)}{t^n}}} \ for\ all\ x,y \in X, t > 0,$$

*and $\tau_M = \tau(d)$. (This example when $n = 1$)*

*(ii) For each $k, m \in \mathbb{R}^+, n \geq 1, (X, M, T_1)$ is a fuzzy metric space where $M$ is given by*

$$M(x,y,t) = \frac{kt^n}{kt^n + md(x,y)} \ for\ all\ x, y \in X, t > 0,$$

*and $\tau_M = \tau(d)$.*

*Proof.* (i) It is easy to verify that $(M, T_1)$ satisfied all conditions of fuzzy metrics, in particular the triangular inequality is a consequence of the previous lemma.

Now, for $x \in X, r \in ]0,1[$ and $t > 0$ we have that

$$B_M(x,r,t) = B\Big(x, -t^n ln(1-r)\Big),$$

and

$$B(x,r) = B_M\Big(x, 1 - \frac{1}{e^{\frac{r}{t^n}}}, t\Big)$$

and then, $\tau_M = \tau(d)$.

(ii) We will only give a proof of the triangular inequality. Indeed, by the previous lemma

$$1 + \frac{md(x,z)}{k(t+s)^n} \leq max\Big\{1 + \frac{md(x,y)}{kt^n}, 1 + \frac{md(y,z)}{ks^n}\Big\}$$

hence

$$\frac{k(t+s)^n}{k(t+s)^n + md(x,z)} \geq min\Big\{\frac{kt^n}{kt^n + md(x,y)}, \frac{ks^n}{ks^n + md(y,z)}\Big\},$$

and the triangular inequality is stated. Now, for $x \in X, t > 0$ and $r \in ]0,1[$ we have that

$$B_M(x,r,t) = B\Big(x, \frac{kt^n r}{m(1-r)}\Big),$$

and

$$B(x,r) = B_M\Big(x, \frac{mr}{kt^n + mr}, t\Big),$$

and then, $\tau_M = \tau(d)$.

□

In following theorem, we will show that every fuzzy metric space implie hausdorff.

**Theorem 2.2.** *Every fuzzy metric space is Hausdorff.*

*Proof.* Let $(X, M, *)$ be the given fuzzy metric space. Let $x, y$ be two distinct points of $X$. Thus, $0 < M(x, y, t) < 1$. Let $M(x, y, t) = r$, for some $r, 0 < r < 1$. For each $r_0, r < r_0 < 1$, we can find a $r_1$ such that $r_1 * r_1 \geq r_0$. Now consider the open balls $B(x, 1 - r_1, \frac{t}{2})$ and $B(y, 1 - r_1, \frac{t}{2})$. Clearly $B(x, 1-r_1, \frac{t}{2}) \cap B(y, 1-r_1, \frac{t}{2}) = \emptyset$. For if there exists $z \in B(x, 1-r_1, \frac{t}{2}) \cap B(y, 1-r_1, \frac{t}{2})$. Then, $r = M(x, y, t) \geq M(x, z, \frac{t}{2}) * M(z, y, \frac{t}{2}) = r_1 * r_1 \geq r_0 > r$. Which is a contradiction. Hence $(X, M, *)$ is Hausdorff. □

**Corollary 2.1.** *Let $(X, d)$ be a metric space. Let $M(x, y, t) = \frac{t}{t+d(x,y)}$ be the fuzzy metric defined on $X$. Then, the topology $\tau_0$ induced by the metric $d$ and the topology $\tau$ induced by the fuzzy metric $M$ are the same.*

**Example 2.5.** *Let $X$ be the real interval $]2, +\infty[$ and consider the mapping $M$ on $X^2 \times ]0, +\infty[$ defined as follows*

$$M(a, b, t) = \begin{cases} 1, & if \quad a = b, \\ \frac{1}{a} + \frac{1}{b}, & if \quad a \neq b, t > 0. \end{cases}$$

*It is easy to verify that $(X, M, T_3)$ is a fuzzy metric space. On the other hand if we take $a = 1000, b = 3$ and $c = 10000$, then $M(a, b, t) \cdot M(b, c, s) > M(a, c, t + s)$ and so $(X, M, T_2)$ is not a fuzzy metric space. Finally, the topology $\tau_M$ is the discrete topology on $X$. Indeed, for $x \in X$, if we take $r < \frac{1}{2} - \frac{1}{x}$ then $B(x, r, t) = \{x\}$.*

**Definition 2.3.** [4] *Let $(X, M, *)$ be a fuzzy metric space. A subset $A$ of $X$ is said to be F-bounded if and only if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.*

In the following proposition, we will know $(N, *)$ is an F-bounded fuzzy metric on $X$.

**Proposition 2.2.** *Let $(X, M, *)$ be a fuzzy metric space and $k \in ]0, 1[$. Define $N(x, y, t) = max\{M(x, y, t), k\}$, for each $x, y \in X, t > 0$. Then, $(N, *)$ is an F-bounded fuzzy metric on $X$, which generates the same topology that $M$.*

*Proof.* Straightforward. □

In the following proposition, we will prove generation the same topology of $M$ result $N$ F-bounded.

**Proposition 2.3.** *Let $i \in \{1, 2, 3\}$ and $k > 0$. Suppose that $(X, M, T_i)$ is a fuzzy metric space, and define : $N(x, y, t) = \frac{k+M(x,y,t)}{1+k}$ for all $x, y \in X, t > 0$. Then, $(N, T_i)$ is an F-bounded fuzzy metric on $X$, which generates the same topology that $M$.*

*Proof.* We prove this proposition for the case $i = 2$. For seeing that $(N, T_i)$ is a fuzzy metric on $X$, we only show the triangular inequality. Now, it is an easy exercise to verify that the following relation $\frac{k+a}{1+k} \cdot \frac{k+b}{1+k} \leq \frac{k+ab}{1+k}$ holds, for all $a, b \in [0, 1]$. Then,

$$\frac{k + M(x, y, t)}{1 + k} \cdot \frac{k + M(y, z, s)}{1 + k} \leq \frac{k + M(x, y, t)M(y, z, s)}{1 + k} \leq \frac{k + M(x, z, t + s)}{1 + k}.$$

Clearly $\frac{k}{k+1}$ is a lower bound of $N(x, y, t)$, for all $x, y \in X, t > 0$. Finally, for $t > 0, r \in ]0, 1[$ it is satisfied that
$B_M(x, r, t) = B_N(x, \frac{r}{1+k}, t)$ and $B_N(x, r, t) = B_M(x, r(k + 1), t)$, and so $\tau_M = \tau_N$. The basic $i = 1, 3$ are proved analogously. □

**Definition 2.4.** [7] *A fuzzy metric space $(X, M, *)$ is called precompact if for each $r \in ]0, 1[$, and $t > 0$, there exists a finite subset $A$ of $X$ such that $X = \bigcup\{B(a, r, t) : a \in A\}$. In this case, we say that $M$ is a precompact fuzzy metric on $X$.*

**Proposition 2.4.** *Let $(X, d)$ be a metric space and let $M_d$ be the standard fuzzy metric deduced from d. Therefore, d is a precompact metric if and only if $M_d$ is a precompact fuzzy metric.*

In the following proposition, we study relation between precompact and F-bounded.

**Proposition 2.5.** *Let $(X, M, *)$ be a precompact fuzzy metric space, and suppose $a * b \neq 0$ whenever, $a, b \neq 0$. Then, $(M, *)$ is F-bounded.*

*Proof.* Let $r \in ]0, 1[$ and $t > 0$. By assumption there is a finite subset $A = \{a_1, ..., a_n\}$ of $X$ such that $X = \bigcup_{i=1}^{n} B(a_i, r, t)$. Let

$$\alpha = min\{M(a_i, a_j, t) : i, j = 1, ..., n\} > 0.$$

Let $x, y \in X$. Then $x \in B(a_i, r, t)$ and $y \in B(a_j, r, t)$ for some $i, j \in \{1, ..., n\}$.
Therefore, $M(x, a_i, t) > 1 - r$ and $M(y, a_j, t) > 1 - r$. Now,

$$M(x, y, 3t) \geq M(x, a_i, t) * M(a_i, a_j, t) * M(a_j, y, t) \geq (1 - r) * a * (1 - r) > 1 - s$$

for some $s \in ]0, 1[$ by the assumption on $*$, and so $M$ is F-bounded. □

In this same manner, wr get the following theorem, which provides every compact of fuzzy metric space implie F-bounded.

**Theorem 2.3.** *Every compact subset $A$ of a fuzzy metric space $X$ is F-bounded.*

*Proof.* Given $A$ is a compact subset of $X$. Fix $t > 0$ and $0 < r < 1$. Consider an open cover $\{B(x,r,t) : x \in A\}$ of $A$. Since, $A$ is compact, there exist $x_1, x_2, ..., x_n \in A$ such that $A \subseteq \bigcup B(x_i,r,t)$. Let $x, y \in A$. Then, $x \in B(x_i,r,t)$ and $y \in B(x_j,r,t)$ for some $i, j$. Therefore, $M(x,x_i,t) > 1-r$ and $M(y,x_j,t) > 1-r$. Now, let $\alpha = min\{M(x_i,x_j,t) : 1 \leq i,j \leq n\}$. Then, $\alpha > 0$. Now, $M(x,y,3t) \geq M(x,x_i,t) * M(x_i,x_j,t) * M(x_j,y,t) \geq (1-r) * (1-r) * \alpha$. Taking $t' = 3t$ and $(1-r)*(1-r)*\alpha > 1-s,\ 0 < s < 1$, we have $M(x,y,t') > 1-s$ for all $x, y \in A$. Hence, $A$ is F-bounded. □

**Remark 2.3.** *In a fuzzy metric space every compact set is closed and bounded.*

# Chapter 3

# Further properties on fuzzy metric spaces

The aim of based paper is the study further properties in fuzzy metric spaces using the notions of sequences. For more details see [5, 4].

## 3.1 Sequences on fuzzy metric spaces

The notion of sequence in fuzzy metric space is introduced by A. George and P.Veeramani [5].

**Definition 3.1.** [5] *Let* $(X, M, *)$ *be a fuzzy metric space and* $(X, M)$ *be sequences in* $X$*. We say that* $x_n \to x$ *(*$x_n$ *converges to* $x$ *if and only if* $M(x_n, x, t)$ *converges to* $1$ *as* $n$ *tends to* $\infty$*).*

**Definition 3.2.** [4] *A sequence* $\{x_n\}$ *in a fuzzy metric space* $(X, M, *)$ *is a cauchy sequence if for each* $\epsilon > 0, t > 0$ *there exists* $n_0 \in N$ *such that* $M(x_n, x_m, t) > 1 - \epsilon$ *for all* $n, m \geq n_0$.

**Remark 3.1.** *Let* $(X, M, *)$ *be fuzzy metric space and* $d$ *be a metric and* $(X, M_d)$ *be sequences on* $M$*.*

**Definition 3.3.** [5] *A fuzzy metric space in which every cauchy sequence is convergent is called a* ***complete fuzzy metric space****.*

**Note 3.1.** *We note that with the above definition, even* $\mathbb{R}$ *fails to be complete.*

**Example 3.1.** *Consider* $S_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ *in* $(\mathbb{R}, M, \cdot)$ *where*

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

*such that (*$d$ *is a metric on* $\mathbb{R}$*).*
*Then*

$$M(S_{n+p}, S_n, t) = \frac{t}{t + \mid S_{n+p} - S_n \mid} = \frac{t}{t + (\frac{1}{n+1}) + (\frac{1}{n+2}) + ... + (\frac{1}{n+p})}.$$

*Thus*

$$lim_{n\to+\infty}M(S_{n+p},S_n,t)=1.$$

*Therefore, $\{S_n\}$ is a Cauchy sequence in the fuzzy metric space $\mathbb{R}$. If $\mathbb{R}$ is fuzzy complete then there exists $x\in\mathbb{R}$ such that $M(S_n,x,t)$ converges to $1$ as $n$ tends to $+\infty$.*
*From this it follows that $\frac{t}{t+|S_n-x|}$ converges to $1$ as $n$ tends to $+\infty$.*
*Further, $\mid S_n-x\mid$ converges to $0$ as $n$ tends to $+\infty$, and so $S_n$ converges to $x$ in $\mathbb{R}$ which is not true.*

## 3.2 Properties on fuzzy metric spaces using sequences

In the following theorem, we study equivalence between the sequence $\{x_n\}$ and $M(x_n,x,t)$.

**Theorem 3.1.** *Let $(X,M,*)$ be a fuzzy metric space and $\tau_M$ be the topology induced by the fuzzy metric. Then, for a sequence $\{x_n\}$ in $X$, the sequence $\{x_n\}$ converges to $x$ if and only if $M(x_n,x,t)$ converges to 1 as $n$ tends to $\infty$.*

*Proof.* Let $t>0$. Suppose that $\{x_n\}$ converges to $x$. Thus, for $0<r<1$, there exists $n_0\in\mathbb{N}$ such that $x_n\in B(x,r,t)$ for all $n\geq n_0$. It follows that $M(x_n,x,t)>1-r$ and therefore, $1-M(x_n,x,t)<r$. Hance, $M(x_n,x,t)$ converges to 1 as $n$ tends to $\infty$. Conversely, if for each $t>0, M(x_n,x,t)$ converges to 1 as $n$ tends to $\infty$ thus for $0<r<1$, there exists $n_0\in\mathbb{N}$ such that $1-M(x_n,x,t)<r$ for all $n\geq n_0$. It follows that $M(x_n,x,t)>1-r$ for all $n\geq n_0$. Hence, $x_n\in B(x,r,t)$ for all $n\geq n_0$, and then the sequence $\{x_n\}$ converges to $x$. □

**Remark 3.2.** *Let $(X,d)$ be a metric space such that $d$ be a metric and $\{x_n\}$ be a sequence in $X$. Then, $lim_n d(x_n,x)=0$ if and only if $lim_n M_d(x_n,x,t)=1$ for all $t>0$ and $x\in X$. Then, $\{x_n\}$ is a cauchy sequence in $(X,d)$ if and only if it is cauchy sequence in $(X,M_d,*)$.*

**Proposition 3.1.** *If $(X,d)$ is a metric space and $d$ is a metric and $(M_d,*)$ is the fuzzy metric induced by $d$, then:*

*(i) The topology $\tau_d$ on $X$ generated by $d$ coincides with topology $\tau_{M_d}$ generated by the standard fuzzy metric $M_d$.*

*(ii) $\{x_n\}$ is a $d$-Cauchy sequence (i.e., a Cauchy sequence in $(X,d)$ if and only if it is a Cauchy sequence in $(X,M_d,*)$.*

*(iii)* $A \subset X$ *is bounded in* $(X, d)$ *if and only if it is F-bounded in* $(X, M_d, *)$.

**Definition 3.4.** [5]***(Closed ball)*** *Let* $(X, M, *)$ *be a fuzzy metric space. Then, we define a* ***closed ball*** *with the center* $x \in X$ *and the radius* $r$, $0 < r < 1, t > 0$, *as*
$B[x, r, t] = \{y \in X : M(x, y, t) \geq 1 - r\}$.

**Lemma 3.1.** *Every closed ball in a fuzzy metric space* $(X, M, *)$ *is a closed set.*

*Proof.* Let $y \in \overline{B[x, r, t]}$. Since $X$ is first countable, there exits a sequence $\{y_n\}$ in $B[x, r, t]$ such that the sequence $\{y_n\}$ converges to $y$. Therefore $M(y_n, y, t)$ converges to 1 for all $t$. For a given $\epsilon > 0$, $M(x, y, t + \epsilon) \geq M(x, y_n, t) * M(y_n, y, \epsilon)$. Then,
$M(x, y, t + \epsilon) \geq lim_n M(x, y_n, t) * lim_n M(y_n, y, \epsilon) \geq (1 - r) * 1 = 1 - r$. (If $M(x, y_n, t)$ is bounded, the sequence $\{y_n\}$ has a subsequence, which we again denote by $\{y_n\}$ for which $lim_n M(x, y_n, t)$ exists). In particular for $n \in N$, take $\epsilon = \frac{1}{n}$. Hence, $M(x, y, t + \frac{1}{n}) \geq 1 - r$. Then, $M(x, y, t) = lim_n M(x, y, t + \frac{1}{n}) \geq 1 - r$. Thus, $y \in B[x, r, t]$. Hence, $B[x, r, t]$ is a closed set. □

**Example 3.2.** *The subset* $[a; b]$ *of* $\mathbb{R}$ *equipped with the usual metric is closed since its complement* $\mathbb{R} - [a, b]$ $= ]\infty-, a[ \cup ]b, +\infty[$, *the union of two open infinite intervals is open. Similarly* $[a, +\infty[$ *is closed, because its complement* $]\infty-, a]$ *is open.*

In the following theorem, we will show that the intersection of dense open sets.

**Theorem 3.2.** *Let* $(X, M, *)$ *be a complete fuzzy metric space. Hence, the intersection of a countable number of dense open sets is dense.*

*Proof.* Suppose that $X$ be the given complete fuzzy metric space. Let $B_0$ be a nonempty open set. Let $D_1, D_2, D_3, ...$ be dense open sets in $X$. Since $D_1$ is dense in $X$, $B_0 \cap D_1 \neq \emptyset$. Let $x_1 \in B_0 \cap D_1$. Since $B_0 \cap D_1$ is open, there exists $0 < r_1 < 1, t > 0$, such that $B(x_1, r_1, t_1) \subset B_0 \cap D_1$. Choose $r_1^{'} < r_1$ and $t^{'} = min\{t_1, 1\}$ such that $B(x_1, r_1^{'}, t_1^{'}) \subset B_0 \cap D_1$. Let $B_1 = B(x_1, r_1^{'}, t_1^{'})$. Since $D_2$ is dense in $X$, $B_1 \cap D_2 \neq \emptyset$. Let $x_2 \in B_1 \cap D_2$. Since $B_1 \cap D_2$ is open, there exists $0 < r_2 < \frac{1}{2}$ and $t_2 > 0$ such that $B(x_2, r_2, t_2) \subset B_1 \cap D_2$. Choose $r_2^{'} < r_2$ and $t_2^{'} = min\{t_2, \frac{1}{2}\}$ such that $B[x_2, r_2^{'}, t_2^{'}] \subset B_1 \cap D_2$. Let $B_2 = B(x_2, r_2^{'}, t_2^{'})$. Similarly proceeding by induction we can find an $x_n \in B_{n-1} \cap D_n$. Since $B_{n-1} \cap D_n$ is open, there exists $0 < r_n < \frac{1}{n}$ and $t_n > 0$ such that $B(x_n, r_n, t_n) \subset B_{n-1} \cap D_n$. Choose $r_n^{'} < r_n, t_n^{'} = min\{t_n, \frac{1}{n}\}$ such that $B[x_n, r_n^{'}, t_n^{'}] \subset B_{n-1} \cap D_n$. Let $B_n = B(x_n, r_n^{'}, t_n^{'})$. Now we claim that $\{x_n\}$ is a cauchy sequence. For a given $t > 0, \epsilon > 0$ choose $n_0$ such that $\frac{1}{n_0} < t$ and $\frac{1}{n_0} < \epsilon$. Therefore, for $n \geq n_0, m \geq n$. $M(x_n, x_m, t) \geq M(x_n, x_m, \frac{1}{n}) \geq 1 - (\frac{1}{n}) \geq 1 - \epsilon$. Then, $\{x_n\}$ is a cauchy sequence. Since, $X$ is complete, the sequence $\{x_n\}$ converges to $x$ in $X$. But $x_k \in B[x_n, r_n^{'}, t_n^{'}]$ for all $k \geq n$ and by the previous result $B[x_n, r_n^{'}, t_n^{'}]$ is a closed set. Therefore, $x \in B[x_n, r_n^{'}, t_n^{'}] \subset B_{n-1} \cap D_n$ for all $n$. Then, $B_0 \cap (\bigcap_{n-1}^{\infty} D_n) \neq \emptyset$. Therefore, $\bigcap_{n-1}^{\infty} D_n$ is dense in $X$. □

**Example 3.3.** *The set* $\mathbb{R}$ *of real numbers with the usual metric is the completion of the set* $\mathbb{Q}$ *of rational numbers, since* $\mathbb{R}$ *is complete and* $\mathbb{Q}$ *is a dense subset of* $\mathbb{R}$.

In the following proposition, we study product of two fuzzy metric speces.

**Proposition 3.2.** *Let* $M(X_1, M_1, *)$ *and* $M(X_2, M_2, *)$ *be fuzzy metric spaces. For* $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2, t > 0$. *Let* $M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t)$. *Then,* $M$ *is a* ***fuzzy metric*** *on* $X_1 \times X_2$.

*Proof.*

1. Since $M_1(x_1, y_1, t) > 0$ and $M_2(x_2, y_2, t) > 0$ this implies that $M_1(x_1, y_1, t) * M_2(x_2, y_2, t) > 0$. Thus, $M((x_1, x_2), (y_1, y_2), t) > 0$

2. Suppose that for all $t > 0$. $(x_1, x_2, t) = (y_1, y_2, t)$. This implies that $x_1 = y_1$ and $x_2 = y_2$, for all $t > 0$. Therefore, $M_1(x_1, y_1, t) = 1$ and $M_2(x_2, y_2, t) = 1$. It follows that $M(x, y, t) = 1$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Conversely, suppose that $M(x, y, t) = 1$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. This implies that $M_1(x_1, y_1, t) * M_2(x_2, y_2, t) = 1$. Since, $0 < M_1(x_1, y_1, t) \leq 1$ and $0 < M_2(x_2, y_2, t) \leq 1$, it follows that $M_1(x_1, y_1, t) = 1$ and $M_2(x_2, y_2, t) = 1$. Then, $x_1 = y_1$ and $x_2 = y_2$. Then, $x = y$

3. To prove that $M(x, y, t) = M(y, x, t)$. We observe that $M_1(x_1, y_1, t) = M_1(y_1, x_1, t)$ and $M_2(x_2, y_2, t) = M_2(y_2, x_2, t)$. It follows that for all $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$ and $t > 0$, $M((x_1, x_2), (y_1, y_2), t) = M((y_1, y_2), (x_1, x_2), t)$.

4. Since, $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are fuzzy metric spaces we have that $M_1(x_1, z_1, t + s) \geq M_1(x_1, y_1, t) * M_1(y_1, z_1, s)$ and $M_2(x_2, z_2, t + s) \geq M_2(x_2, y_2, t) * M_2(y_2, z_2, s)$, for all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X_1 \times X_2$ and $s, t > 0$. Then, $M((x_1, x_2), (z_1, z_2), t + s) = M_1(x_1, z_1, t + s) * M_2(x_2, z_2, t + s)$ $M((x_1, x_2), (z_1, z_2), t + s) \geq M_1(x_1, y_1, t) * M_1(y_1, z_1, s) * M_2(x_2, y_2, t) * M_2(y_2, z_2, s) \geq M_1(x_1, y_1, t) * M_2(x_2, y_2, t) * M_1(y_1, z_1, s) * M_2(y_2, z_2, s) \geq M((x_1, x_2), (y_1, y_2), t) * M((y_1, y_2), (z_1, z_2), s)$.

5. Note that $M_1(x_1, y_1, t)$ and $M_2(x_2, y_2, t)$ are continuous with respect to $t$ and $*$ is continuous. It follows that $M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t)$, is also continuous.

□

In the same manner, we get the following proposition which provides a product of two complete fuzzy metric spaces.

**Proposition 3.3.** *Let* $M(X_1, M_1, *)$ *and* $M(X_2, M_2, *)$ *be two fuzzy metric spaces. We define* $M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t)$*. Therefore,* $M$ *is a* ***complete fuzzy metric*** *on* $X_1 \times X_2$ *if and only if* $(X_1, M_1, *)$ *and* $(X_2, M_2, *)$ *are complete.*

*Proof.* Suppose that $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete fuzzy metric spaces. Let $\{a_n\}$ be a cauchy sequence in $X_1 \times X_2$. Note that $a_n = \{x_1^n, x_2^n\}$ and $a_m = \{x_1^m, x_2^m\}$. Thus, $M(a_n, a_m, t)$ converges to 1. This implies that $M((x_1^n, x_2^n), (x_1^m, x_2^m), t)$ converges to 1 for each $t > 0$. It follows that $M_1(x_1^n, x_1^n, t) * M_2(x_2^n, x_2^m, t)$ converges to 1 for all $t > 0$. Thus, $M_1(x_1^n, x_1^n, t)$ converges to 1 and also $M_2(x_2^n, x_2^m, t)$ converges to 1. Then, $\{x_1^n\}$ is a cauchy sequence in $(X_1, M_1, *)$ and $\{x_2^n\}$ is a cauchy sequence in $(X_2, M_2, *)$. Since, $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete fuzzy metric spaces, there exists $x_1 \in X_1$ and $x_2 \in X_2$ such that $M_1(x_1^n, x_1, t)$ converges to 1 and $M_2(x_2^n, x_2, t)$ converges to 1 for all $t > 0$. Let $a = (x_1, x_2)$. Then, $a \in X_1 \times X_2$. It follows that $M(a_n, a, t)$ converges to 1 for all $t > 0$. This shows that $(X, M, *)$ is complete. Conversely, suppose that $(X, M, *)$ is complete. We shall show that $(X, M_1, *)$ and $(X, M_2, *)$ are complete. Let $\{x_1^n\}$ and $\{x_2^n\}$ be cauchy sequence in $(X, M_1, *)$ and $(X, M_2, *)$ respectively. Therefore, $M_1(x_1^n, x_1^m, t)$ converges to 1 and $M_2(x_2^n, x_2^m, t)$ converges to 1 for all $t > 0$. It follows that $M(x_1^n, x_2^m, t) = M_1(x_1^n, x_1^m, t) * M_2(x_2^n, x_2^m, t)$ converges to 1. Let $x^n = (x_1^n, x_2^m)$ in $X_1 \times X_2$ for $n \geq 1$. Hence, $\{x^n\}$ is a cauchy sequence in $X$. Since, $(X, M, *)$ is complete, there exists $x \in X_1 \times X_2 = X$ such that $M(x_1^n, x, t)$ converges to 1. Since, $x \in X_1 \times X_2$, we may put $x = (x_1, x_2)$, $x_1 \in X_1$ and $x_2 \in X_2$. Clearly, $M_1(x_1^n, x_1, t)$ converges to 1 and $M_2(x_2^n, x_2, t)$ converges to 1. Hence, $(X, M_1, *)$ and $(X, M_2, *)$ are complete. This completes the proof. □

# Conclusion

In this work. First, we have studies the notion of fuzzy metric space and some of thier properties. Furthermore, we have studies the notions of compact fuzzy metric space and complete fuzzy metric space as a specific types of fuzzy metric spaces. In future work, we plain to study other properties on fuzzy metric spaces for example connexity and convexity.

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# ملخص

في هذا العمل, قمنا بدراسة الفضاءات المترية الفازية وبعض الخواص الاساسية ايضا, قمنا بدراسة نوع معين من هذه الفضاءات المترية وهي الفضاءات المترية المتراصة الفازية وكذا الفضاءات المترية التامة الفازية.

**الكلمات المفتاحية :**

مجموعة فازية, فضاء متري, فضاء متري متراص, فضاء متري تام.

# Abstract

In this work. First, we study the notion of fuzzy metric space and some of thier properties. Furthermore, we treat the notions of compact fuzzy metric space and complete fuzzy metric space as a specific type of fuzzy metric spaces.

**Key words :**

Fuzzy set, metric space, compact metric space, complete metric space.

# Résumé:

Dans ce travail, nous étudions les espaces métriques flous ainsi que certaines propriétés de base. Nous avons également étudié deux types spécifiques de ces espaces métriques, les espaces métriques flous compacts et les espaces métriques flous complets.

**Mots clés:**

Ensemble flou, espace métrique, espace métrique compact, espace métrique complet.