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The n-Normed Space of p-Summable sequences

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Dedication

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To my parents, whose unwavering support and encouragement have been my guiding light throughout this journey. Your sacrifices and love have made this achievement possible.

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Lebid Alla

Contents

List of Symbols

Symbol	Notation
$\mathbb{K}$	The field of real $\mathbb{R}$ or complex numbers $\mathbb{C}$.
$\|\cdot,\ldots,\cdot\|$	n-norm.
$(X, \|\cdot,\ldots,\cdot\|)$	n-normed space.
$\mathscr{L}(X,Y)$	The vector space of linear operators of X in Y.
$\mathscr{L}(X)$	The collection of all linear operators from X into X.
$B(X,Y)$	The space of all bounded linear operators from X to Y.
$B(X^n,Y)$	The space of all bounded n-linear operators from X^n into Y.
$B_n(X,X)$	The space of all n-bounded linear operators from X into X^n.
$B_n^n(X,X)$	The space of all n-bounded of type-I operators from X^n into X^n.
$C^1[0,1]$	The space of continuous functions on $[0,1]$.
$\langle .,. \rangle$	An inner product is a function that satisfies the properties(Conjugate symmetry,Linearity,Positive definiteness).
$\mathrm{sign}(\sigma)$	is a function that extracts the sign of a real number. It tells you whether a number is positive, negative or zero.
$\ell_p(X)$	is the set of all sequences for which the series of the p p-th power of the norm converges.
$Lp(X)$	the space of all measurable functions (applications) for which the p-the power of the norm value is integrable.
$F_k(z)$	represents the n-norm considered as a function of its k-the argument while all other arguments are kept fixed.

Introduction

Functional analysis has evolved significantly since its inception, providing mathematicians with powerful tools to study infinite-dimensional spaces. Among these developments, the concept of n-normed spaces stands as a natural generalization of classical normed spaces. While a norm measures the length of a single vector, an n-norm measures the volume of the parallelepiped spanned by n vectors, thereby extending geometric intuition to higher dimensions.

The theory of n-normed spaces originated with the work of Gähler [36] in the 1960s, contrary to some historical accounts. Gähler introduced the notion of 2-normed spaces in 1963 and later generalized it to n-normed spaces. Subsequently, Gunawan and Mashadi [19, 21] systematically developed the theory, providing fundamental results and establishing connections with classical functional analysis. These spaces have found applications in various areas, including the study of nonlinear partial differential equations, approximation theory, and fixed point theory [17].

One of the distinctive features of n-normed spaces is that an n-norm is not necessarily homogeneous in each argument individually; rather, it satisfies a multilinearity condition. More precisely, an n-norm $\|\cdot,\ldots,\cdot\|$ on a vector space X is a function from X^n to $\mathbb{R}_{\geq 0}$ that is:

- Positive definite: $\|x_1,\ldots,x_n\| = 0$ if and only if $x_1,\ldots,x_n$ are linearly dependent,
- Multilinear: linear in each argument,
- Alternating: changes sign under permutation of arguments,
- Positive: $\|x_1,\ldots,x_n\| \geq 0$.

The theory of n-normed spaces has attracted considerable attention from many researchers. For instance, Gunawan and Mashadi [19, 21] extensively studied the structure of n-normed spaces and their finite-dimensional properties. In 2014, Kir and Kiziltunc [17] established fixed point theorems for contraction mappings in n-normed spaces. More recently, Batkunde, Gunawan, and Pangalela [1] investigated bounded linear functionals on the n-normed space of p-summable sequences, while Batkunde and Rumlawang [2] extended these results to bounded 2-linear functionals.

The study of n-dual spaces, which is the central theme of this dissertation, generalizes the classical concept of dual spaces to the n-linear setting. While the dual space of a normed space consists of all bounded linear functionals, the n-dual space consists of all bounded n-linear functionals. This generalization, initiated by Gunawan [20] and further developed by Gozali, Gunawan, and Neswan [17], reveals deep connections between the summability properties of sequences and the geometry of n-normed spaces.

Several authors have contributed to the understanding of n-dual spaces. Chen and Song [4] characterized isometries in linear n-normed spaces, while Chu, Lee, and Park [5] investigated the Aleksandrov problem in this setting. Freese and Cho [10] provided a comprehensive geometric treatment of linear 2-normed spaces. Meitei and Singh [35] studied bounded n-linear operators, and Srivastava and Singh [35] analyzed Cauchy sequences in ℓ^p viewed as an n-normed space.

In this dissertation, we focus on the n-dual spaces of ℓ^p, the space of p-summable sequences, when equipped with two distinct n-norms: Gähler's n-norm and Gunawan's n-norm. Our main objectives are:

1. To identify the n-dual space of $(\ell^p, \|\cdot\|_p)$ as a normed space,
2. To determine the n-dual space of $(\ell^p, \|\cdot,\dots,\cdot\|_p^G)$, where $\|\cdot,\dots,\cdot\|_p^G$ is Gähler's n-norm,
3. To determine the n-dual space of $(\ell^p, \|\cdot,\dots,\cdot\|_p^H)$, where $\|\cdot,\dots,\cdot\|_p^H$ is Gunawan's n-norm,
4. To establish the equivalence between these two n-dual spaces.

This dissertation is divided into four chapters. Chapter 1 presents preliminary concepts and essential results on n-normed spaces, n-Banach spaces, linear and n-linear operators, as well as continuity and boundedness properties. This chapter serves as the foundation for the subsequent analysis.

Chapter 2 develops the theory of n-normed and n-Banach spaces in greater depth. We study continuous bounded n-linear operators and characterize the space of all bounded n-linear operators from X^n into Y, with particular attention to Lebesgue spaces.

Chapter 3 concerns the study of weak p-summable spaces and their relationship with n-dual spaces. We establish the identification of n-dual spaces for ℓ^p equipped with both Gähler's and Gunawan's n-norms.

Chapter 1

On the n-normed spaces and their properties

1.1 The n-normed spaces

Gähler was the first to develop theories of n-normed spaces in 1960s [10, 19, 20] ,and later, Misiak [21] developed the theory more extensively. Notion of boundedness in 2-normed space was then introduced by White [6].

Definition 1.1.1. *[21]*
Let X be a real vector space with $\dim X \geq n$ where n is a positive integer. A real valued function $\|\cdot,\ldots,\|: X^n \to \mathbb{R}$ is called an n-norm on X if the following conditions hold:

1. $\|x_1,\ldots,x_n\| = 0$ *iff* $x_1,\ldots,x_n$ *are linearly dependent.*
2. $\|x_1,\ldots,x_n\|$ *remains invariant under permutations of* $x_1,\ldots,x_n$.
3. $\|\alpha x_1, x_2,\ldots,x_n\| = |\alpha| \|x_1,\ldots,x_n\| \quad \forall x_1,\ldots,x_n \in X$ *and* $\alpha \in \mathbb{R}$.
4. $\|x_0 + x_1, x_2,\ldots,x_n\| \leq \|x_0,\ldots,x_n\| + \|x_1,\ldots,x_n\|$ *for all* $x_0, x_1,\ldots,x_n \in X$.

Remark 1.1.2. *[10] The concept of 2-normed spaces was initially investigated and developed by Gähler in 1960s and has been extensively developed by Diminnie, Gähler, White and many others [4, 5, 35]. Let X be a real linear space of dimension greater than 1 and $\|\cdot\|$ be a real valued function on $X \times X$ satisfying the following conditions:*

(2N_1) $\|x,y\|=0$ *if and only if* x *and* y *are linearly dependent.*

(2N_2) $\|x,y\|=\|y,x\|$.

(2N_3) $\|\alpha x,y\|=|\alpha|\|x,y\| \quad \forall x,y\in X$ *and* $\alpha\in\mathbb{R}$.

(2N_4) $\|x+y,z\|\leq\|x,z\|+\|y,z\| \quad \forall x,y,z\in X$.

Then, $\|\cdot,\|$ *is called a* 2*-norm on* X *and* $(X,\|\cdot,\|)$ *is called a linear 2-normed space.* 2*-norms are non-negative and* $\|x,y+\alpha x\|=\|x,y\|$ *for every* $x,y\in X$ *and* $\alpha\in\mathbb{R}$.

Example 1.1.3. *Let* X *be a real vector space with* $\dim X\geq n$, n *is a positive integer and be equipped with an inner product* $\langle\ldots\rangle$. *Then the standard* n*-norm on* X *is given by:*

$$\|x_1,\ldots,x_n\|^{\mathrm{S}}=\sqrt{\det[\langle x_i,x_j\rangle]}. \tag{1.1}$$

A trivial example of an n*-normed space is* $X=\mathbb{R}^n$ *equipped with the Euclidean* n*-norm:*

$$\|x_1,\ldots,x_n\|_E=|\det(x_{ij})|,$$

where $x_i=(x_{i1},\ldots,x_{in})\in\mathbb{R}^n$ *for each* $i=1,2,\ldots,n$.

Indeed, the Equality 1.1 verifies the four axioms of an n*-norm,then*

1. *Positivity : The Gram matrix* $G=(g_{ij})$ *with* $g_{ij}=\langle x_i,x_j\rangle$ *is positive semidefinite. Hence* $\det(G)\geq 0$. *From linear algebra,* $\det(G)=0$ *if and only if the vectors* $x_1,\ldots,x_n$ *are linearly dependent. Therefore*

$$\|x_1,\ldots,x_n\|^{\mathrm{S}}=\sqrt{\det(G)}=0 \iff x_1,\ldots,x_n \text{ are linearly dependent.}$$

Thus N1 holds.

2. *Symmetry: Let* σ *be a permutation of* $\{1,\ldots,n\}$. *The Gram matrix of* $(x_{\sigma(1)},\ldots,x_{\sigma(n)})$ *is* $G'=PGP^T$, *where* P *is the permutation matrix corresponding to* σ. *Since* $\det(P)=\pm1$ *and* $\det(P^T)=\det(P)$, *we have*

$$\det(G')=\det(PGP^T)=\det(P)\det(G)\det(P^T)=(\det P)^2\det(G)=\det(G).$$

Hence

$$\|x_{\sigma(1)},\ldots,x_{\sigma(n)}\|^{\mathrm{S}}=\|x_1,\ldots,x_n\|^{\mathrm{S}}.$$

If $x_i=x_j$ *for some* $i\neq j$, *then* G *has two identical rows, so* $\det(G)=0$ *and the norm is zero.*

3. *Homogeneity: Fix* $k\in\{1,\ldots,n\}$ *and let* $\alpha\in\mathbb{R}$. *Replace* x_k *by* αx_k. *For any* i,j:

   - *If* $i=k$ *and* $j\neq k$: $\langle\alpha x_k,x_j\rangle=\alpha\langle x_k,x_j\rangle$.
   - *If* $i\neq k$ *and* $j=k$: $\langle x_i,\alpha x_k\rangle=\alpha\langle x_i,x_k\rangle$.
   - *If* $i=j=k$: $\langle\alpha x_k,\alpha x_k\rangle=\alpha^2\langle x_k,x_k\rangle$.

Thus the new Gram matrix G' differs from G by multiplying the k-th row by α and the k-th column by α. Consequently,

$$\det(G') = \alpha^2 \det(G).$$

Therefore

$$\|x_1, \ldots, \alpha x_k, \ldots, x_n\|^{\mathrm{S}} = \sqrt{\alpha^2 \det(G)} = |\alpha| \sqrt{\det(G)} = |\alpha| \|x_1, \ldots, x_n\|^{\mathrm{S}}.$$

4. *Triangle inequality:*

   Fix $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in X$. Define the map $f : X \to \mathbb{R}_{\geq 0}$ by

$$f(y) = \|x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_n\|^{\mathrm{S}}.$$

   We need to show $f(y+z) \leq f(y) + f(z)$ for all $y, z \in X$.

   Consider the exterior product $\bigwedge^n X$. The wedge product is multilinear and alternating. Define

$$u(y) = x_1 \wedge \cdots \wedge x_{k-1} \wedge y \wedge x_{k+1} \wedge \cdots \wedge x_n \in \bigwedge^n X.$$

   The inner product on $\bigwedge^n X$ induces a norm $\|\cdot\|_\wedge$ satisfying

$$\|u(y)\|_\wedge = \|x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_n\|^{\mathrm{S}} = f(y).$$

   By multilinearity,

$$u(y+z) = x_1 \wedge \cdots \wedge (y+z) \wedge \cdots \wedge x_n = u(y) + u(z).$$

   Since $\|\cdot\|_\wedge$ is a norm, it satisfies the triangle inequality:

$$f(y+z) = \|u(y+z)\|_\wedge = \|u(y) + u(z)\|_\wedge \leq \|u(y)\|_\wedge + \|u(z)\|_\wedge = f(y) + f(z).$$

Remark 1.1.4. *Note that the value of $\|x_1, \ldots, x_n\|^{\mathrm{S}}$ represents the volume of the n-dimensional parallelepiped spanned by $x_1, \ldots, x_n$.*

Indeed, the determinant on the right-hand side of equation (1) is known as Gram's determinant, and its value is always nonnegative. Geometrically, this determinant equals the square of the volume of the parallelepiped spanned by $x_1, \ldots, x_n$, which confirms the geometric interpretation stated in the remark above. Moreover, if $(X, \|\cdot, \ldots, \cdot\|)$ is an n-normed space, then

$$\|x_1, \ldots, x_n\| = \|x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n,\ x_2, \ldots, x_n\|,$$

for all $x_1, \ldots, x_n \in X$ and $\alpha_2, \ldots, \alpha_n \in \mathbb{R}$ [10].

Now, we present the construction of quotient spaces of the n-normed spaces with respect to a linearly independent set. Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space and $Y = \{y_1, \ldots, y_n\}$ be a linearly independent set in X. For a $j \in \{1, \ldots, n\}$ we consider $Y \setminus \{y_j\}$. We define a subspace of X, namely

$$Y_j^0 := \operatorname{span} Y \setminus \{y_j\} := \left\{ \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i y_i : \alpha_i \in \mathbb{R} \right\}.$$

For any $u \in X$, the corresponding coset in X is

$$\bar{u} := \left\{ u + \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i y_i : \alpha_i \in \mathbb{R} \right\}.$$

Then we have

$$\bar{0} = \operatorname{span} Y \setminus \{y_j\} = Y_j^0$$

and if $\bar{u} = \bar{v}$ then $u - v \in \operatorname{span} Y \setminus \{y_j\}$.

We define quotient spaces of X as

$$X_j^* := X / Y_j^0 := \{\bar{u} : u \in X\}.$$

1.2 The n-bounded and n-continuous linear operators

Gozali et al.in [17], also introduced the notion of bounded n-linear functionals on n-normed spaces. Zofia Lewandowska introduced the notion of 2-linear operators on 2-normed sets in [2]. Subsequently, Soenjaya introduced the notions of continuity and boundedness of n-linear operators in [7, 24].

Let $(X, \|.\|)$ and $(X, \|, \ldots, \|)$ be respectively a normed space and an n-normed space.

Definition 1.2.1. *[17, 23]*

1. *Let $(X, \|\dot{\|})$ be a normed space and $(X, \|, \ldots, \|)$ be n-normed spaces. An operator $T : (X, \|.\|) \to (X, \|, \ldots, \|)$ is **n-continuous of type-A** at $x \in X$ if for all $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\|Tx_1 - Tx, x_2 - x, \ldots, x_n - x\| + \|x_1 - x, Tx_2 - Tx, \ldots, x_n - x\| + \\ \cdots + \|x_1 - x, x_2 - x, \ldots, Tx_n - Tx\| < \varepsilon$$

whenever $\|x_1 - x\| \|x_2 - x\| \ldots \|x_n - x\| < \delta$, where $x_1, \ldots, x_n \in X$.

T is n-continuous of type-A if it is n-continuous of type-A at each $x \in X$.

2. *Let $(X,\|,\dots,\|)$ and $(Y,\|,\dots,\|)$ be n-normed spaces. Let $T: X \to Y$ be an operator. T is **n-continuous of type-B** at $x \in X$ if for $\varepsilon > 0$, there is a $\delta > 0$ such that*
$$\|Tx_1 - Tx, Tx_2 - Tx, \dots, Tx_n - Tx\| < \varepsilon$$
whenever
$$\|x_1 - x, x_2 - x, \dots, x_n - x\| < \delta.$$
T is n-continuous of type-B on X if it is n-continuous of type-B at each $x \in X$.
When $n = 1$, it is reduced to usual notion of continuity in normed space

3. *An operator $T : (X, \|\cdot\|, \dots, \|\cdot\|) \to (X, \|\cdot\|, \dots, \|\cdot\|)$ is n-continuous of type C at $x \in X$ if for all $\varepsilon > 0$, there is a $\delta > 0$ such that*
$$\|Tx_1 - Tx, x_2 - x, \dots, x_n - x\| + \|x_1 - x, Tx_2 - Tx, \dots, x_n - x\| + \dots + \|x_1 - x, x_2 - x, \dots, Tx_n - Tx\| < \varepsilon$$
whenever $\|x_1 - x, x_2 - x, \dots, x_n - x\| < \delta$, where $x_1, x_2, \dots, x_n \in X$. T is n-continuous of type-C if it is n-continuous of type-C at each $x \in X$.

Definition 1.2.2. *[17]*

1. *An operator $T : (X, \|.\|) \to (X, \|,\dots,\|)$ is **n-bounded of type-A** if there is a constant K such that for all $x_1, x_2, \dots, x_n \in X$*
$$\|Tx_1, x_2, \dots, x_n\| + \|x_1, Tx_2, \dots, x_n\| + \dots + \|x_1, x_2, \dots, Tx_n\| \le K\|x_1\| \dots \|x_n\|.$$

2. *An operator $T : (X, \|,\dots,\|) \to (Y, \|,\dots,\|)$ is **n-bounded of type-B** if there is a constant K such that for all $x_1, \dots, x_n \in X$*
$$\|Tx_1, \dots, Tx_n\| \le K\|x_1, \dots, x_n\|.$$

3. *An operator $T : (X, \|,\dots,\|) \to (X, \|,\dots,\|)$ is **n-bounded of type-C** if there is a constant K such that for all $x_1, x_2, \dots, x_n \in X$*
$$\|Tx_1, x_2, \dots, x_n\| + \|x_1, Tx_2, \dots, x_n\| + \dots + \|x_1, x_2, \dots, Tx_n\| \le K\|x_1, \dots, x_n\|.$$

Proposition 1.2.3. 1. *For all $x_1, x_2, \dots, x_n \in X$. If T is an n-bounded operator of type-A, define $\|T\|_n^A$ by*
$$\|T\|_n^A = \sup_{\|x_1,\dots,x_n\| \neq 0} \left\{ \frac{\|Tx_1, x_2, \dots, x_n\| + \|x_1, Tx_2, \dots, x_n\| + \dots + \|x_1, x_2, \dots, Tx_n\|}{\|x_1\| \cdots \|x_n\|} \right\}.$$

2. *For all $x_1, x_2, \dots, x_n \in X$. If T is an n-bounded operator of type-B, define $\|T\|_n^B$ by*
$$\|T\|_n^B = \sup_{\|x_1,\dots,x_n\| \neq 0} \frac{\|Tx_1, \dots, Tx_n\|}{\|x_1, \dots, x_n\|}.$$

3. *For all $x_1, x_2, \ldots, x_n \in X$. If T is an n-bounded operator of type C, define $\|T\|_n^C$ by*

$$\|T\|_n^C = \sup_{\|x_1,\ldots,x_n\|\neq 0} \left\{ \frac{\|Tx_1, x_2, \ldots, x_n\| + \|x_1, Tx_2, \ldots, x_n\| + \cdots + \|x_1, x_2, \ldots, Tx_n\|}{\|x_1, \ldots, x_n\|} \right\}.$$

Remark 1.2.4. *When $n = 1$, it is reduced to usual notion of continuity in normed space.*

We now present another crucial type of n-continuity and n-boundedness, which we shall refer to as *D-type*. This notion differs from the previously introduced concepts in that it imposes a direct algebraic condition on the images of linearly independent sets, rather than relying solely on the behavior of the n-norm.

Definition 1.2.5. *[23]*

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|, \ldots, \|\cdot\|)$ be respectively a normed space and a n-normed space, and $T : X \to Y$ be an operator. T is n-continuous of type-D at $x \in X$ if for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|Tx_1 - Tx, Tx_2 - Tx, \cdots, Tx_n - Tx\| < \varepsilon$$

whenever $\|x_1 - x\|\|x_2 - x\| \cdots \|x_n - x\| < \delta$, where $x_1, \cdots, x_n \in X$.

T is n-continuous of type-D if it is n-continuous at each $x \in X$.

When $n = 1$, this notion of n-continuity of type-D becomes the notion of continuity in a normed space.

Definition 1.2.6. *An operator $T : (X, \|\cdot\|) \to (Y, \|\cdot\|, \ldots, \|\cdot\|)$ is n-bounded of type-D if there is a constant K such that for all $x_1, \ldots, x_n \in X$,*

$$\|Tx_1, \ldots, Tx_n\| \leq K\|x_1\| \cdots \|x_n\|.$$

Define $\|T\|_n^D$ by

$$\|T\|_n^D = \sup_{x_i \in X,\, \|x_i\| \neq 0} \frac{\|Tx_1, \ldots, Tx_n\|}{\|x_1\| \cdots \|x_n\|}.$$

Example 1.2.7.

1. *Let X be an inner product space equipped with standard n-norm $\|\cdot\|, \ldots, \|\cdot\|^S$ and $T : (X, \|\cdot\|) \to (X, \|\cdot\|, \ldots, \|\cdot\|^S)$ be an operator such that $Tx = cx$ for all $x \in X$ and $c \in \mathbb{R}$. Then T is n-bounded of type-D.*

2. *Let $X = \mathbb{R}^2$ be a normed space equipped with Euclidean 2-norm $\|\cdot\|^E$ and $T : (X, \|\cdot\|) \to (X, \|\cdot\|^E)$ be an operator such that $Tx_i = (x_{i1}, x_{i1})$, where $x_i = (x_{i1}, x_{i2}) \in \mathbb{R}^2$ for $i = 1, 2, \ldots$ and $\|x_i\| = \sqrt{x_{i1}^2 + x_{i2}^2}$. Then, T is 2-bounded of type-D. The operator T in example 3.4 is 2-continuous of type-D. Indeed,*

1. *Let X be an inner product space with standard n-norm $\|\cdot, \ldots, \cdot\|^S$. Define $T : (X, \|\cdot\|) \to (X, \|\cdot, \ldots, \cdot\|^S)$ by $Tx = cx$ for all $x \in X$, where $c \in \mathbb{R}$.*

   We show that T is n-bounded of type-D.

by Definition 1.2.6. An operator T is n-bounded of type-D if here exists a constant $M \geq 0$ such that for all $x_1, \ldots, x_n \in X$,

$$\|Tx_1, \ldots, Tx_n\|^{\mathrm{S}} \leq M \prod_{k=1}^{n} \|x_k\|.$$

For any $x_1, \ldots, x_n \in X$, we have

$$\|Tx_1, \ldots, Tx_n\|^{\mathrm{S}} = \|cx_1, \ldots, cx_n\|^{\mathrm{S}}.$$

Using the homogeneity property of the n-norm,

$$\|cx_1, \ldots, cx_n\|^{\mathrm{S}} = |c|^n \|x_1, \ldots, x_n\|^{\mathrm{S}}.$$

From Hadamard's inequality (or the fundamental property of the Gram determinant),

$$\|x_1, \ldots, x_n\|^{\mathrm{S}} \leq \prod_{k=1}^{n} \|x_k\|.$$

Therefore,

$$\|Tx_1, \ldots, Tx_n\|^{\mathrm{S}} \leq |c|^n \prod_{k=1}^{n} \|x_k\|.$$

Taking $M = |c|^n$, we obtain

$$\|Tx_1, \ldots, Tx_n\|^{\mathrm{S}} \leq M \prod_{k=1}^{n} \|x_k\|.$$

Hence T is n-bounded of type-D with $\|T\|_{n\text{-}D} \leq |c|^n$.

Since every bounded linear operator is continuous, T is also n-continuous of type-D.

2. *Let $X = \mathbb{R}^2$ with the Euclidean norm $\|x\| = \sqrt{x_1^2 + x_2^2}$ for $x = (x_1, x_2) \in \mathbb{R}^2$. The Euclidean 2-norm is defined by*

$$\|x, y\|^E = |\det(x, y)| = \left|\det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\right| = |x_1 y_2 - x_2 y_1|.$$

Define $T : (X, \|\cdot\|) \to (X, \|\cdot\|^E)$ by

$$Tx = (x_1, x_1), \quad \textit{where } x = (x_1, x_2) \in \mathbb{R}^2.$$

For two vectors $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$ *in* $\mathbb{R}^2$,

$$Tx = (x_1, x_1), \qquad Ty = (y_1, y_1).$$

Now compute the Euclidean 2*-norm of* (Tx, Ty)*:*

$$\|Tx, Ty\|^E = |\det(Tx, Ty)| = \left|\det\begin{pmatrix} x_1 & y_1 \\ x_1 & y_1 \end{pmatrix}\right| = |x_1 y_1 - x_1 y_1| = 0.$$

Thus, for all $x, y \in \mathbb{R}^2$,

$$\|Tx, Ty\|^E = 0.$$

We need to check the 2*-boundedness condition: there exists* $M \geq 0$ *such that for all* $x, y \in \mathbb{R}^2$,

$$\|Tx, Ty\|^E \leq M\|x\|\|y\|.$$

Since $\|Tx, Ty\|^E = 0$ *for all* x, y*, the inequality becomes*

$$0 \leq M\|x\|\|y\|,$$

which is true for any $M \geq 0$*. We can choose* $M = 0$*. Therefore,* T *is* 2*-bounded of type-D with* $\|T\|_{2\text{-}D} = 0$*.*

Consequently, T *is also* 2*-continuous of type-D.*

Theorem 1.2.8. *[1, Theorem 3.8] Let* $T : (X, \|\cdot\|) \to (Y, \|\cdot\|, \ldots, \|\cdot\|)$ *be a linear operator. Then, the following statements are equivalent.*

1. T *is* n*-continuous of type-*D*.*
2. T *is* n*-continuous of type-*D *at* $0 \in X$*.*
3. T *is* n*-bounded of type-*D*.*

Proof. It is obvious that (1) implies (2).
(2) $\Longrightarrow$ (3): Suppose T is n-continuous of type-D at $0 \in X$. By definition, there is a $\delta > 0$ such that

$$\|Tu_1, \ldots, Tu_2\| < 1$$

whenever

$$\|u_1\|\|u_2\| \cdots \|u_n\| < \delta.$$

Let $(x_1, \ldots, x_n) \in X^n$.

If $\|x_1\|\|x_2\|\cdots\|x_n\| = 0$, at least one of $x_1,\ldots,x_n$ is 0. Then by linearity of T, at least one of $Tx_1, Tx_2,\ldots,Tx_n$ is 0. It implies that $Tx_1, Tx_2,\ldots,Tx_n$ are linearly dependent. Hence, $\|Tx_1,\ldots,Tx_n\| = 0$.

If $\|x_1\|\|x_2\|\cdots\|x_n\| \neq 0$, let

$$u_i = \left(\frac{\delta}{4}\right)^{\frac{1}{n}} \frac{x_i}{\|x_i\|}, \quad i = 1,2,\ldots,n.$$

$$\|u_1\|\,\|u_2\|\cdots\|u_n\| = \frac{\delta}{4} < \delta.$$

Then, we have

$$\begin{aligned}\|Tu_1, Tu_2,\ldots,Tu_n\| &= \left\| T\left(\frac{\delta}{4}\right)^{\frac{1}{n}} \frac{x_1}{\|x_1\|}, T\left(\frac{\delta}{4}\right)^{\frac{1}{n}} \frac{x_2}{\|x_2\|},\ldots, T\left(\frac{\delta}{4}\right)^{\frac{1}{n}} \frac{x_n}{\|x_n\|} \right\| \\ &= \frac{\delta}{4}\cdot\frac{1}{\|x_1\|\cdots\|x_n\|}\|Tx_1, Tx_2,\ldots,Tx_n\| \\ \Rightarrow \|Tx_1, Tx_2,\ldots,Tx_n\| &= \frac{4}{\delta}\|x_1\|\cdots\|x_n\|\,\|Tu_1, Tu_2,\ldots,Tu_n\| \\ &< \frac{4}{\delta}\|x_1\|\cdots\|x_n\|\cdot 1\end{aligned}$$

then, T is n-bounded of type-D.

$(3) \Rightarrow (1):$ Suppose T is n-bounded of type-D.

Then for $x \in X$,

$$\|Tx_1 - Tx, Tx_2 - Tx,\ldots,Tx_n - Tx\| \leq \|T\|_n^D \|x_1 - x\|\cdots\|x_n - x\|.$$

Let $\varepsilon > 0$ be given.

Let $\delta = \frac{\varepsilon}{1+|T|_n^D}$ with $\|x_1 - x\|\|x_2 - x\|\cdots\|x_n - x\| < \delta$.

Then,

$$\begin{aligned}\|Tx_1 - Tx, Tx_2 - Tx,\ldots,Tx_n - Tx\| &\leq \|T\|_n^D \|x_1 - x\|\cdots\|x_n - x\| \\ &< \|T\|_n^D\cdot\delta \\ &= \|T\|_n^D\cdot\frac{\varepsilon}{1+\|T\|_n^D} \\ &< \varepsilon.\end{aligned}$$

Thus, for given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|Tx_1 - Tx, Tx_2 - Tx,\ldots,Tx_n - Tx\| < \varepsilon$$

whenever $\|x_1 - x\|\|x_2 - x\|\cdots\|x_n - x\| < \delta$, where $x_1,\ldots,x_n \in X$.

Therefore, T is n-continuous of type-D. This completes the proof.

□

Proposition 1.2.9. *[1] Let X be a real vector space with $\dim(X) \geq n$, where n is a positive integer, equipped with a norm $\|\cdot\|$ and an n-norm $\|\cdot,\ldots,\cdot\|$. Let $(Y, \|\cdot,\ldots,\cdot\|)$ be an n-normed space, and let $T : X \to Y$ be a linear operator.*

If T is n-bounded of both types B and D, then

$$\|T\|_n^B = \|T\|_n^D.$$

Proof. If T is n-bounded of type B,

$$\|T\|_n^B = \sup_{\|x_1\|,\ldots,\|x_n\| \neq 0} \frac{\|Tx_1, \cdots, Tx_n\|}{\|x_1, \ldots, x_n\|}.$$

If T is n-bounded of type D,

$$\|T\|_n^D = \sup_{x_i \in X, \|x_i\| \neq 0} \frac{\|Tx_1, \cdots, Tx_n\|}{\|x_1, \ldots, x_n\|}.$$

Let $x_i \in X$ with $\|x_i\| \neq 0$ for $i = 1, 2, \ldots, n$.
We Define,

$$x_i = \frac{\|x_i\| y_i}{\sqrt{\|y_1\|, \ldots, \|y_n\|}}, \quad y_i \in X \text{ and } \|y_1, \ldots, y_n\| \neq 0$$

$$= \frac{\|x_i\| y_i}{\alpha}, \quad \alpha = \sqrt{\|y_1\|, \ldots, \|y_n\|}.$$

Now,

$$\|Tx_1, \cdots, Tx_n\| = \|T(x_1, \cdots, x_n)\|.$$

$$\|Tx_1, \ldots, Tx_n\| = \|T\left(\frac{|x_1| y_1}{\alpha}, \ldots, T\left(\frac{|x_n| y_n}{\alpha}\right)\right)\|$$

$$= \left\| \frac{|x_1| \cdots |x_n|}{\alpha^n} \|Ty_1, \ldots, Ty_n\| \right\|$$

$$\Rightarrow \left\| \frac{|Tx_1, \ldots, Tx_n|}{\|x_1\| \cdots \|x_n\|} \right\| = \left\| \frac{|Ty_1, \ldots, Ty_n|}{\|y_1, \ldots, y_n\|} \right\| \tag{3.9.1}$$

Taking supremum of the right side of (3.9.1) over $\{y_1, \ldots, y_n\} \in X^n : \|y_1, \ldots, y_n\| \neq 0\}$, we have

$$\frac{\|Tx_1, \ldots, Tx_n\|}{\|x_1\| \cdots \|x_n\|} \leq \|T\|_n^B.$$

It is true for all $(x_1, \ldots, x_n) \in X^n$ and each $x_i \neq 0$.
Therefore,

$$\sup_{x_i \in X, \|x_i\| \neq 0} \frac{\|Tx_1, \ldots, Tx_n\|}{\|x_1\| \cdots \|x_n\|} \leq \|T\|_n^B$$

$$\Rightarrow \|T\|_n^D \leq \|T\|_n^B.$$

Again, Taking supremum of the left side of (3.9.1) over $\{(x_1,\ldots,x_n) \in X^n : \|x_i\| \neq 0, i = 1,2,\ldots,n\}$, we have

$$\|T\|_n^D \geq \frac{\|Ty_1,\ldots,Ty_n\|}{\|y_1,\ldots,y_n\|}$$

It is true for all $(y_1,\ldots,y_n) \in X^n$ and $\|y_1,\ldots,y_n\| \neq 0$. Therefore,

$$\|T\|_n^D \geq \sup_{x_1 \in X, \|x_1\| \neq 0} \frac{\|Ty_1,\ldots,Ty_n\|}{\|y_1,\ldots,y_n\|}$$

$$\Rightarrow \|T\|_n^D \geq \|T\|_n^B.$$

This completes the proof. □

Proposition 1.2.10. *[1] Let X be an inner product space equipped with the standard n-norm $\|\cdot,\ldots,\cdot\|^S$, and let $(Y, \|\cdot,\ldots,\cdot\|)$ be an n-normed space. If $T : X \to Y$ is n-bounded of type B, then T is n-bounded of type D.*

Proof. Since T is n-bounded of type-B,

$$\|Tx_1,\ldots,Tx_n\| \leq K\|x_1,\ldots,x_n\|^S.$$

Moreover, by Hadamard's determinant theorem,

$$\|x_1,\ldots,x_n\|^S = \sqrt{\det(\langle x_i, x_j\rangle)} \leq \sqrt{\|x_1\|^2\|x_2\|^2\cdots\|x_n\|^2} = \|x_1\|\|x_2\|\cdots\|x_n\|.$$

Therefore,

$$\|Tx_1,\ldots,Tx_n\| \leq K\|x_1\|\|x_2\|\cdots\|x_n\|.$$

Hence, T is n-bounded of type D. This completes the proof. □

Proposition 1.2.11. *Let X be a real vector space equipped with a norm $\|\cdot\|$ and an n-norm $\|\cdot,\ldots,\cdot\|^S$. Let $T : X \to X$ be a linear operator. If T is n-bounded of both type-A and type-C, then*

$$\|T\|_n^A = \|T\|_n^C.$$

Proof. $T : X \to X$ is n-bounded of type-C. It implies that there exists a constant K such that for all $x_1, x_2, \ldots, x_n \in X$,

$$\|Tx_1, x_2,\ldots,x_n\|^S + \|x_1, Tx_2,\ldots,x_n\|^S + \cdots + \|x_1, x_2,\ldots,Tx_n\|^S \leq K\|x_1,\ldots,x_n\|^S.$$

But,

$$\|x_1, \ldots, x_n\|^S = \sqrt{\det(\langle x_i, x_j \rangle)}.$$

Applying Hadamard inequality,

$$\|x_1, \ldots, x_n\|^S \leq \|x_1\| \|x_2\| \cdots \|x_n\|.$$

Therefore, for all $x_1, x_2, \ldots, x_n \in X$,

$$\|Tx_1, x_2, \ldots, x_n\|^S + \|x_1, Tx_2, \ldots, x_n\|^S + \cdots + \|x_1, x_2, \ldots, Tx_n\|^S \leq K\|x_1\| \cdots \|x_n\|.$$

It implies T is n-bounded of type-A. This completes the proof. □

Chapter 2

The space ℓ_p and its natural n-norm

Definition 2.0.1 (The sequence space $\ell^p(X)$). *Let $X = (X, \|\cdot\|_X)$ be a normed space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $1 \le p < \infty$. We define the space of p-summable sequences in X by*

$$\ell^p(X) := \left\{ (x_n)_{n=1}^{\infty} \subset X \,:\, \sum_{n=1}^{\infty} \|x_n\|_X^p < \infty \right\}.$$

For $(x_n) \in \ell^p(X)$, the ℓ^p-norm is given by

$$\|(x_n)\|_{\ell^p(X)} := \left(\sum_{n=1}^{\infty} \|x_n\|_X^p \right)^{1/p}.$$

For the case $p = \infty$, we define the space of bounded sequences in X by

$$\ell^{\infty}(X) := \left\{ (x_n)_{n=1}^{\infty} \subset X \,:\, \sup_{n \in \mathbb{N}} \|x_n\|_X < \infty \right\},$$

equipped with the norm

$$\|(x_n)\|_{\ell^{\infty}(X)} := \sup_{n \in \mathbb{N}} \|x_n\|_X.$$

If X is a Banach space, then $\ell^p(X)$ is a Banach space for every $1 \le p \le \infty$.

2.1 ℓ^p and its natural 2-norm

Definition 2.1.1. *[16][2-Norm on ℓ^p] Let $X = (X, \|\cdot,\cdot\|_X)$ be a 2-normed space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, For $1 \le p < \infty$ and $x, y \in \ell^p$, define*

$$\|x, y\|_p := \left[\frac{1}{2} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left| \det \begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} \right|^p \right]^{1/p}.$$

For $p = \infty$, define

$$\|x, y\|_\infty := \sup_{j,k} \left| \det \begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} \right|.$$

Theorem 2.1.2 (Equivalence of the p-norm definition to the standard 2-norm for $p = 2$)**.** *[26] For $p = 2$, the above definition reduces to the standard 2-norm induced by the inner product on ℓ^2:*

$$\|x, y\|_2 = \left[\det \begin{pmatrix} \sum_j x_j^2 & \sum_j x_j y_j \\ \sum_j x_j y_j & \sum_j y_j^2 \end{pmatrix} \right]^{1/2}.$$

Proof. Start from the determinant identity for two vectors in ℓ^2:

$$\det \begin{pmatrix} \sum_j x_j^2 & \sum_j x_j y_j \\ \sum_j x_j y_j & \sum_j y_j^2 \end{pmatrix} = \sum_{j,k} x_j y_k \det \begin{pmatrix} x_j & y_j \\ x_k & y_k \end{pmatrix}.$$

Averaging with the analogous expression obtained by swapping roles yields

$$2 \det(x, y) = \sum_{j,k} \left| \det \begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} \right|^2.$$

Hence

$$\|x, y\|_2 = \left[\frac{1}{2} \sum_{j,k} \left| \det \begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} \right|^2 \right]^{1/2} = \left[\det \begin{pmatrix} \sum_j x_j^2 & \sum_j x_j y_j \\ \sum_j x_j y_j & \sum_j y_j^2 \end{pmatrix} \right]^{1/2}.$$

□

Proposition 2.1.3. *[37, Fact 2.1][Basic Inequality for $\|\cdot,\cdot\|_p$] Let $1 \le p \le \infty$. Then for all $x, y \in \ell^p$,*

$$\|x, y\|_p \le 2^{1-1/p} \|x\|_p \|y\|_p,$$

with the understanding that $2^{1-1/\infty} = 2$.

Proof. 1. **Case $1 \le p < \infty$:** By the triangle inequality,

$$|x_j y_k - x_k y_j| \le |x_j||y_k| + |x_k||y_j|.$$

Raise to power p, sum over j, k, and use Minkowski's inequality for double series:

$$\left(\sum_{j,k} |x_j y_k - x_k y_j|^p \right)^{1/p} \le \left(\sum_{j,k} (|x_j||y_k| + |x_k||y_j|)^p \right)^{1/p}.$$

Minkowski gives

$$\le \left(\sum_{j,k} |x_j|^p |y_k|^p \right)^{1/p} + \left(\sum_{j,k} |x_k|^p |y_j|^p \right)^{1/p} = 2\|x\|_p \|y\|_p.$$

Then

$$\|x, y\|_p = \left(\frac{1}{2} \sum_{j,k} |\cdot|^p \right)^{1/p} \le 2^{-1/p} \cdot 2\|x\|_p \|y\|_p = 2^{1-1/p} \|x\|_p \|y\|_p.$$

2. **Case $p = \infty$:**

$$|x_j y_k - x_k y_j| \le |x_j||y_k| + |x_k||y_j| \le 2\|x\|_\infty \|y\|_\infty,$$

so the supremum over j, k is $\le 2\|x\|_\infty \|y\|_\infty$. Thus $\|x, y\|_\infty \le 2\|x\|_\infty \|y\|_\infty$, which matches $2^{1-1/\infty} = 2$.

The proof is achieved. □

Proposition 2.1.4. *[37, Fact 2.2] [Well-Definedness of the 2-Norm] For each $1 \le p \le \infty$, the function $\|\cdot,\cdot\|_p$ satisfies the four axioms of a* ***2-norm****. i.e., it satisfies Remark 1.1.2.*

Proof. Axioms (2) and (3) are immediate from the determinant properties.
Axiom (1): If x and y are linearly dependent, every 2×2 determinant vanishes, so $\|x, y\|_p = 0$. Conversely, if $\|x, y\|_p = 0$, then for all j, k, $\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} = 0$, which forces x and y to be linearly dependent (otherwise there exist two indices with a non-zero determinant).
Axiom (4): The determinant satisfies

$$\left|\det\begin{pmatrix} x_j + z_j & x_k + z_k \\ y_j & y_k \end{pmatrix}\right| \le \left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right| + \left|\det\begin{pmatrix} z_j & z_k \\ y_j & y_k \end{pmatrix}\right|.$$

Raising to power p, summing over j, k, applying Minkowski's inequality to the double series, and incorporating the factor $\frac{1}{2}$ yields (4). □

Corollary 2.1.5. *For each $1 \le p \le \infty$, the space ℓ^p equipped with $\|\cdot,\cdot\|_p$ is a* ***2-normed space****.*

Remark 2.1.6. ***1.*** *For $p = 2$, the factor $2^{1-1/2} = \sqrt{2}$ is not optimal; Hadamard's inequality gives $\|x, y\|_2 \le \|x\|_2 \|y\|_2$ (a special case with a better constant). The given factor is sufficient for general p.*
2. *The definition originates from rewriting the standard 2-norm on ℓ^2 in terms of pairwise determinants, generalizing naturally to ℓ^p by replacing the square with the p-th power.*
3. *For any two vectors, $\|x, y\|_p$ measures the "p-area" of the parallelogram spanned by them, summed over all coordinate pairs in a Minkowski-type sense.*
4. *The text does not prove it here, but ℓ^p with this 2-norm is a* ***2-Banach space*** *(i.e., complete in the sense of 2-norms) because the usual ℓ^p norm controls the 2-norm via the inequality above.*

2.1.1 Completeness

Cauchy sequences in 2-normed spaces

In the following definition, we introduce the definition of a Cauchy sequence. Note that there are two definitions of Cauchy sequences in 2-normed spaces.

Definition 2.1.7. *[35] Let $\{y,z\}$ be a linearly independent set on a 2-normed space $(X,\|\cdot,\cdot\|)$. A sequence (x_k) in X is called a Cauchy with respect to the set $\{y,z\}$ if*

$$\lim_{k,l\to\infty} \|x_k - x_l, y\| = 0$$

and

$$\lim_{k,l\to\infty} \|x_k - x_l, z\| = 0$$

Definition 2.1.8. *[25] A sequence (x_k) in a 2-normed space $(X,\|\cdot,\cdot\|)$ is called a Cauchy sequence with respect to the $\|\cdot,\cdot\|$ if*

$$\lim_{k,l\to\infty} \|x_k - x_l, z\| = 0$$

for every nonzero $z \in X$.

Remark 2.1.9. *Definition 2.1.26 is clearly stronger than Definition 2.1.7.*

Definition 2.1.10. *[36] A sequence (x_k) in a 2-normed space X is called a convergent sequence, if there is an x in X such that*

$$\lim_{k\to\infty} \|x_k - x, z\| = 0$$

for every nonzero z in X .

Remark 2.1.11. *In the light of the Definition 2.1.7, we can give another definition of convergent sequence in 2-normed space, clearly weaker than the Definition 2.1.10.*

Definition 2.1.12. *[36] Let $\{y,z\}$ be a linearly independent set on a 2-normed space $(X,\|\cdot,\cdot\|)$. A sequence (x_k) in X is called convergent to some x in X with respect to the set $\{y,z\}$ if*

$$\lim_{k\to\infty} \|x_k - x, y\| = 0$$

and

$$\lim_{k\to\infty} \|x_k - x, z\| = 0.$$

Since ℓ^p is complete with respect to the usual norm $\|\cdot\|_p$ (see [35, pp. 91–92]), it remains to investigate its completeness with respect to the natural 2-norm $\|\cdot,\cdot\|_p$. The next lemma provides a key ingredient for establishing this result.

Lemma 2.1.13. *A sequence in ℓ^p is convergent in the 2-norm $\|\cdot,\cdot\|_p$ if and only if it is convergent in the usual norm $\|\cdot\|_p$. Similarly, a sequence in ℓ^p is Cauchy with respect to$\|\cdot,\cdot\|_p$ if and only if it is Cauchy with respect to $\|\cdot\|_p$.*

In general, let $\|\cdot\|_p$ be a 2-normed space of dimension at least 2. Let a_1, a_2 be a linearly independent subset of X. We define the derived norm $\|\cdot\|_p^*$, by:

$$\|x\|_p^* := \left(\|x, a_1\|^p + \|x, a_2\|^p\right)^{1/p}, \qquad 1 \le p < \infty,$$

or, For $p = \infty$.

$$\|x\|_\infty^* := \sup\{\|x, a_1\|, \|x, a_2\|\},$$

For our 2-normed space ℓ^p, we choose

$$[a_1 = (1,0,0,\ldots), \qquad a_2 = (0,1,0,\ldots),]$$

and define $\|\cdot\|_q^*$ with respect to $\{a_1, a_2\}$.

Proposition 2.1.14. *[37, Fact 2.5] The derived norm $\|\cdot\|_p^*$ is equivalent to the usual norm $\|\cdot\|_p$ on ℓ^p. More precisely,*

$$\|x\|_p \le \|x\|_p^* \le 2^{1/p}\|x\|_p,$$

for all $x \in \ell^p$. In particular,

$$\|\cdot\|_\infty^* = \|\cdot\|_\infty.$$

Proof. Let $1 \le p < \infty$. For every $x \in \ell^p$, we compute

$$\|x, a_1\|_p^p = \sum_{j \ne 1} |x_j|^p$$

and

$$\|x, a_2\|_p^p = \sum_{j \ne 2} |x_j|^p,$$

whence

$$\|x\|_p^* = \left[|x_1|^p + |x_2|^p + 2\sum_{j>3} |x_j|^p\right]^{1/p}.$$

We therefore see that

$$\|x\|_p \le \|x\|_p^* \le 2^{1/p}\|x\|_p,$$

that is, $\|\cdot\|_p^*$ and $\|\cdot\|_p$ are equivalent. The proof for $p = \infty$ is similar. □

Proposition 2.1.14 tells us in particular that $\|\cdot\|_p$ is dominated by $\|\cdot\|_p^*$; As we shall see below, this is what we actually need to prove Lemma 2.1.13.

Proof. Suppose that $x(m)$ converges to some $x \in \ell^p$ in the 2-norm $\|\cdot,\cdot\|_p$. With respect to $a_1 = (1,0,0,\ldots)$ and $a_2 = (0,1,0,\ldots)$, define $\|\cdot\|_p^*$ as before. Then, since

$$\lim_{m\to\infty} \|x(m) - x, a_1\|_p = 0$$

and

$$\lim_{m\to\infty} \|x(m) - x, a_2\|_p = 0,$$

we have

$$\lim_{m\to\infty} \|x(m) - x\|_p^* = 0,$$

that is, $x(m)$ converges to x in $\|\cdot\|_p^*$. But $\|\cdot\|_p$ is dominated by $\|\cdot\|_p^*$ and so we conclude that $x(m)$ also converges to x in $\|\cdot\|_p$. □

Theorem 2.1.15. *[36] The space ℓ^p is complete with respect to the 2-norm $\|\cdot,\cdot\|_p$.*

Proof. Let $x(m)$ be Cauchy in ℓ^p with respect to $\|\cdot,\cdot\|_p$. Then, by Lemma 2.1.13, $x(m)$ is Cauchy with respect to the usual norm $\|\cdot\|_p$. But we know that ℓ^p is complete with respect to $\|\cdot\|_p$, and so $x(m)$ must converge to some $x \in X$ in $\|\cdot\|_p$. By another application of Lemma 2.1.13, $x(m)$ also converges to x in $\|\cdot,\cdot\|_p$. This shows that ℓ^p is complete with respect to the 2-norm $\|\cdot,\cdot\|_p$. □

3. ℓ^p and its natural n-norm

By using properties of determinants and a limiting argument as in the ℓ^2 case, the standard n-norm on ℓ^2 can be written as

$$\|x_1,\ldots,x_n\|_2 := \left[\frac{1}{n!}\sum_{j_1=1}^{\infty}\cdots\sum_{j_n=1}^{\infty}\left|\det(x_{ij})\right|^2\right]^{1/2},$$

where x_{ij} denotes the j-th coordinate of x_i.

For $1 \le p < \infty$, the above definition can be extended as follows..

Definition 2.1.16. *For $1 \le p < \infty$ and $x_1,\ldots,x_n \in \ell^p$, define*

$$\|x_1,\ldots,x_n\|_p := \left[\frac{1}{n!}\sum_{j_1=1}^{\infty}\cdots\sum_{j_n=1}^{\infty}\left|\det(x_{ij})\right|^p\right]^{1/p}.$$

For $p = \infty$, define

$$\|x_1,\ldots,x_n\|_\infty := \sup_{j_1,\ldots,j_n}\left|\det(x_{ij})\right|,$$

as in [24].

Proposition 2.1.17. *[27] For $1 \le p \le \infty$ and $x_1, \dots, x_n \in \ell^p$, the following inequality holds:*

$$\|x_1, \dots, x_n\|_p \le (n!)^{1-\frac{1}{p}} \prod_{i=1}^{n} \|x_i\|_p,$$

with the understanding that $(n!)^{1-\frac{1}{\infty}} = 1$ when $p = \infty$.

Proof. We prove the case $1 \le p < \infty$; the case $p = \infty$ follows by taking the supremum over $j_1, \dots, j_n$ in the finite-dimensional bound $|\det| \le \prod_{i=1}^{n} \|x_i\|_\infty$.

For $p < \infty$, write the determinant as a sum over permutations:

$$\det(x_{ij}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} x_{i,\, j_{\sigma(i)}}.$$

By the triangle inequality for the p-norm in the j-indices,

$$\left| \sum_{\sigma} a_\sigma \right|^p \le (n!)^{p-1} \sum_{\sigma} |a_\sigma|^p.$$

Applying this to $a_\sigma = \operatorname{sgn}(\sigma) \prod_{i=1}^{n} x_{i, j_{\sigma(i)}}$ gives

$$\left| \det(x_{ij}) \right|^p \le (n!)^{p-1} \sum_{\sigma \in S_n} \prod_{i=1}^{n} \left| x_{i, j_{\sigma(i)}} \right|^p.$$

Summing over $j_1, \dots, j_n$ and using that each term $\prod_i \left| x_{i, j_{\sigma(i)}} \right|^p$ yields $\prod_i \|x_i\|_p^p$ independently of σ, we obtain

$$\sum_{j_1, \dots, j_n} \left| \det(x_{ij}) \right|^p \le (n!)^{p-1} \cdot n! \cdot \prod_{i=1}^{n} \|x_i\|_p^p = (n!)^p \prod_{i=1}^{n} \|x_i\|_p^p.$$

Dividing by $n!$ and taking the $1/p$-power yields the claimed inequality. □

Proposition 2.1.18. *For every $1 \le p \le \infty$, the function $\|\cdot, \dots, \cdot\|_p$ defines an n-norm on ℓ^p.*

Proof. The first three properties follow directly from the multilinearity and alternating nature of the determinant, together with the homogeneity of the p-norm in the j-summation.

For the triangle inequality, fix all arguments except the k-th. Write

$$F_k(z) = \left[\frac{1}{n!} \sum_{j_1, \dots, j_n} \left| \det(\dots, z_{j_k}, \dots) \right|^p \right]^{1/p}.$$

As a function of z, F_k is the ℓ^p-norm of the vector

$$\left(\sqrt[n!]{\frac{1}{n!}}\ \det(\dots, z_{j_k}, \dots) \right)_{j_1, \dots, j_n}.$$

Since the map $z \mapsto \mathbf{det}(\ldots, z_{j_k}, \ldots)$ is linear in z, the Minkowski inequality for the ℓ^p-norm implies

$$F_k(x_k + y_k) \leq F_k(x_k) + F_k(y_k).$$

Thus property (iv) holds. □

Corollary 2.1.19 $((\ell^p, \|\cdot, \ldots, \cdot\|_p))$. *For every $1 \leq p \leq \infty$, the space ℓ^p equipped with $\|\cdot, \ldots, \cdot\|_p$ is an n-normed space.*

Remark 2.1.20. *The factor $n!$ appearing in the definition is essential for the $p = 2$ case to coincide with the usual n-norm derived from the Gram determinant, and it also makes the constant in Proposition 3.1 scale as $(n!)^{1-1/p}$ rather than $(n!)^1$.*

Theorem 2.1.21 (Minkowski's Inequality). *Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space over $\mathbb{R}$ or $\mathbb{C}$. For $p \geq 1$ and vectors $x_{k,i}, y_{k,i} \in X$ with ($k = 1, \ldots, n$ and $i = 1, \ldots, m$,), we obtain*

$$\left(\sum_{i=1}^{m} \left|x_{1,i} + y_{1,i}, x_{2,i} + y_{2,i}, \ldots, x_{n,i} + y_{n,i}\right|^p\right)^{1/p} \leq \left(\sum_{i=1}^{m} |x_{1,i}, x_{2,i}, \ldots, x_{n,i}|^p\right)^{1/p} + \left(\sum_{i=1}^{m} |y_{1,i}, y_{2,i}, \ldots, y_{n,i}|^p\right)^{1/p}.$$

Proof. For each fixed index i $(1 \leq i \leq m)$, define

$$a_i := \|x_{1,i}, x_{2,i}, \ldots, x_{n,i}\|, \qquad b_i := \|y_{1,i}, y_{2,i}, \ldots, y_{n,i}\|,$$

$$c_i := \|x_{1,i} + y_{1,i},\ x_{2,i} + y_{2,i},\ \ldots,\ x_{n,i} + y_{n,i}\|.$$

Using property 4 of the n-norm (triangle inequality in the last argument) repeatedly, we obtain

$$c_i \leq a_i + b_i \qquad \text{for all } i.$$

Now apply the classical Minkowski inequality for ℓ^p sequences to the nonnegative numbers (a_i) and (b_i):

$$\left(\sum_{i=1}^{m} (a_i + b_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^{m} a_i^p\right)^{1/p} + \left(\sum_{i=1}^{m} b_i^p\right)^{1/p}.$$

Since $c_i \leq a_i + b_i$ implies $c_i^p \leq (a_i + b_i)^p$ for $p \geq 0$, we have

$$\left(\sum_{i=1}^{m} c_i^p\right)^{1/p} \leq \left(\sum_{i=1}^{m} (a_i + b_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^{m} a_i^p\right)^{1/p} + \left(\sum_{i=1}^{m} b_i^p\right)^{1/p}.$$

Substituting the definitions of a_i, b_i, c_i yields exactly the desired inequality. □

Proposition 2.1.22 (Hölder's Inequality). *Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space. For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and vectors $x_{k,i} \in X$ ($k = 1, \ldots, n$, $i = 1, \ldots, m$), set*

$$a_i := \|x_{1,i}, x_{2,i}, \ldots, x_{n,i}\| \geq 0.$$

Then

$$\sum_{i=1}^{m} a_i \leq \left(\sum_{i=1}^{m} a_i^p\right)^{1/p} \left(\sum_{i=1}^{m} a_i^q\right)^{1/q}.$$

Equivalently,

$$\sum_{i=1}^{m} \|x_{1,i},\ldots,x_{n,i}\| \leq \left(\sum_{i=1}^{m} \|x_{1,i},\ldots,x_{n,i}\|^p\right)^{1/p} \left(\sum_{i=1}^{m} \|x_{1,i},\ldots,x_{n,i}\|^q\right)^{1/q}.$$

Proof. Observe that $a_i \geq 0$ for all i. We can write

$$a_i = (a_i^p)^{1/p} \cdot (a_i^q)^{1/q}.$$

Applying the classical Hölder inequality for two sequences of nonnegative numbers (with exponents p and q such that $1/p + 1/q = 1$), we obtain

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} (a_i^p)^{1/p} (a_i^q)^{1/q} \leq \left(\sum_{i=1}^{m} a_i^p\right)^{1/p} \left(\sum_{i=1}^{m} a_i^q\right)^{1/q}.$$

□

Completeness

Cauchy sequences in n-normed spaces

As in 2-normed spaces, we can also give two definitions for Cauchy sequences in n-normed spaces. It is known that in some cases, especially in the finite dimensional case and the standard case, two definitions are equivalent. What is not clear is in the infinite dimensional case. In this paper, we will prove that these two definitions are still equivalent in ℓ^p space. For more detail (see [35]).

Definition 2.1.23. *[35, 16]*

1. *A sequence (x_k) in an n-normed space $(X, \|\cdot,\ldots,\cdot\|)$ is said to be convergent to some $x \in X$ in the n-norm if for each $\varepsilon > 0$ there exists a positive integer $n_0 = n_0(\varepsilon)$ such that*

   $$\|x_k - x, y_2, \ldots, y_n\| < \varepsilon$$

   for all $k \geq n_0$ and for every $y_2, \ldots, y_n \in X$.

2. *A sequence (x_k) in an n-normed space $(X, \|\cdot,\ldots,\cdot\|)$ is said to be a Cauchy with respect to the n-norm if*

   $$\lim_{k,l \to \infty} \|x_k - x_l, y_2, \ldots, y_n\| = 0,$$

   for every $y_2, \ldots, y_n \in X$.

[16]

As in the case of 2-normed spaces, we can give another definition of Cauchy sequences for n-normed spaces.

Definition 2.1.24 (Convergence and Cauchy in 2-normed spaces)**.** *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space.*

- *A sequence (x_n) in X is said to* ***converge*** *to some $x \in X$ in the 2-norm if*

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

  for every $y \in X$.

- *(x_n) is said to be* ***Cauchy*** *with respect to the 2-norm if*

$$\lim_{l,m\to\infty} \|x_l - x_m, y\| = 0$$

  for every $y \in X$.

- *X is said to be* ***complete*** *with respect to the 2-norm if every Cauchy sequence in X converges to some $x \in X$.*

Definition 2.1.25 (Convergence and Cauchy in n-normed spaces)**.** *[25, 35] Let $(X, \|\cdot,\dots,\cdot\|)$ be an n-normed space.*

- *A sequence (x_m) in X is said to* ***converge*** *to some $x \in X$ in the n-norm if*

$$\lim_{m\to\infty} \|x_m - x, x_2, \dots, x_n\| = 0$$

  for every $x_2, \dots, x_n \in X$.

- *(x_m) is said to be* ***Cauchy*** *with respect to the n-norm if*

$$\lim_{l,m\to\infty} \|x_l - x_m, x_2, \dots, x_n\| = 0$$

  for every $x_2, \dots, x_n \in X$.

- *X is said to be* ***complete*** *with respect to the n-norm if every Cauchy sequence in X converges to some $x \in X$.*

The following is a generalisation of 2.1.13.

Definition 2.1.26. *Let $A := \{a_1, \dots, a_n\}$ be a linearly independent set in an n-normed space $(X, \|\cdot,\dots,\cdot\|)$. A sequence (x_k) in X is said to be* ***Cauchy with respect to the set A*** *if*

$$\lim_{k,l\to\infty} \|x_k - x_l, a_{i_2}, \dots, a_{i_n}\| = 0,$$

for every choice of indices $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$.

Remark 2.1.27. *The definition 2.1.7 is clearly stronger than Definition 2.1.26. However, from the results in [21, 22], we understand that the two definitions are still equivalent in ℓ^p and L^p (the space of all p-integrable functions) spaces. For other spaces, for example, for $C[a,b]$ (the space of all continuous functions defined on a closed interval $[a,b]$), we still do not know whether they are equivalent as in ℓ^p and L^p.*

In the space $C[a,b]$, the equivalence is obtained only for some specific vectors; we do not know the equivalence for arbitrary vectors. This is an open problem worth exploring. For details, we recommend readers to review the references [33, 11].

E. Junaedi [33], in her thesis, used tent functions (functions whose graphs look like a tent, i.e., triangular shape) $a_1,\dots,a_n$ supported on the closed intervals

$$[a,a+h],[a+h,a+2h],\dots,[a+(n-1)h,b],$$

where $h=(b-a)/n$, and showed that if a sequence satisfies the Cauchy condition in the n-norm with respect to the set $\{a_1,\dots,a_n\}$, then it also satisfies the Cauchy condition in the usual supremum norm on $C[a,b]$.

We will show the equivalence of Definition 2.1.7 and Definition 2.1.26 for ℓ^p, which can be done similarly for L^p.

Lemma 2.1.28. *[14, 3, 37] For every $x_1,\dots,x_n\in\ell^p$ with $1\le p\le\infty$, the following inequality holds:*

$$\|x_1,\dots,x_n\|_p\le(n!)^{1-\frac{1}{p}}\|x_1\|_p\cdots\|x_n\|_p$$

Lemma 2.1.29. *Let $\{a_1,\dots,a_n\}$ be a linearly independent set on ℓ^p. Then the following function*

$$\|x\|_p^*=\left[\sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}}\|x,a_{i_2},\dots,a_{i_n}\|_p^p\right]^{\frac{1}{p}}$$

defines a norm on ℓ^p.

Lemma 2.1.30. *Let $\{a_1,\dots,a_n\}$ be a linearly independent set on ℓ^p. Then the norm $\|\cdot\|_p^*$ is equivalent to the usual norm $\|\cdot\|_p$ on ℓ^p. Precisely, for every $x\in\ell^p$ we have*

$$\frac{n\|a_1,\dots,a_n\|_p}{(2n-1)\big(\|a_1\|_p+\cdots+\|a_n\|_p\big)}\|x\|_p\le\|x\|_p^*\le(n!)^{1-\frac{1}{p}}\left[\sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}}\|a_{i_2}\|_p^p\cdots\|a_{i_n}\|_p^p\right]^{\frac{1}{p}}\|x\|_p.$$

Theorem 2.1.31. *[35, Fact 3.4]* The sequence (x_k) is a Cauchy sequence in ℓ^p according to Definition 2.1.23 if and only if there exists a linearly independent set $A=\{a_1,\dots,a_n\}$ such that the sequence (x_k) is a Cauchy sequence in ℓ^p with respect to the set A.

Proof. Assume that (x_k) is a Cauchy sequence in ℓ^p according to Definition 2.1.26. Then

$$\lim_{k,l\to\infty} \|x_k - x_l, y_2, \dots, y_n\|_p = 0,$$

for every $y_2, \dots, y_n \in \ell^p$. Hence, there exists a linearly independent set $A = \{a_1, \dots, a_n\}$ on ℓ^p,

$$\lim_{k,l\to\infty} \|x_k - x_l, a_{i_2}, \dots, a_{i_n}\|_p = 0,$$

for any $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$. Thus, we obtain Definition 2.4.

Now, suppose that (x_k) is a Cauchy sequence in ℓ^p according to Definition 2.1.25. Then for $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$ we have

$$\lim_{k,l\to\infty} \|x_k - x_l, a_{i_2}, \dots, a_{i_n}\|_p = 0.$$

Hence we obtain

$$\begin{aligned}
\lim_{k,l\to\infty} \|x_k - x_l\|_p^* &= \lim_{k,l\to\infty} \left[\sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \|x_k - x_l, a_{i_2}, \dots, a_{i_n}\|_p^p \right]^{\frac{1}{p}} \\
&= \left[\sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \lim_{k,l\to\infty} \|x_k - x_l, a_{i_2}, \dots, a_{i_n}\|_p^p \right]^{\frac{1}{p}} \\
&= 0.
\end{aligned}$$

By Lemma 2.1.30, we then conclude that

$$\lim_{k,l\to\infty} \|x_k - x_l\|_p = 0.$$

Hence, for every $y_2, \dots, y_n \in \ell^p$, we have by Lemma 2.1;

$$\|x_k - x_l, y_2, \dots, y_n\|_p \leq (n!)^{1-\frac{1}{p}} \|x_k - x_l\|_p \|y_2\|_p \cdots \|y_n\|_p.$$

Thus, we obtain

$$\lim_{k,l\to\infty} \|x_k - x_l, y_2, \dots, y_n\|_p = 0$$

for every $y_2, \dots, y_n \in \ell^p$. Hence, Definition 2.3 is obtained. This completes the proof. □ □

Corollary 2.1.32. Let $A := \{a_1, \dots, a_n\}$ and $B := \{b_1, \dots, b_n\}$ be linearly independent sets on ℓ^p. The sequence (x_k) is a Cauchy sequence with respect to the set A if and only if the sequence (x_k) is a Cauchy sequence with respect to the set B.

Proof. Let (x_k) be a Cauchy sequence in ℓ^p with respect to the set A. Then from Theorem 3.1 (x_k) is a Cauchy sequence in ℓ^p according to Definition 2.1.26. Thus, we have from Theorem 2.1.31 that there exists a linearly independent set, i.e., say, $B = \{b_1, \ldots, b_n\}$ such that (x_k) is a Cauchy sequence in ℓ^p with respect to the set B.

For the converse, change the position of A and B. Hence, we have the result. □

Lemma 2.1.33. *A sequence in ℓ^p is convergent in the n-norm $\|\cdot, \ldots, \cdot\|_p$ if and only if it is convergent in the usual norm $\|\cdot\|_p$. Similarly, a sequence in ℓ^p is Cauchy with respect to $\|\cdot, \ldots, \cdot\|_p$ if and only if it is Cauchy with respect to $\|\cdot\|_p$. As before, the if' parts of Lemma 2.1.29 are obvious and the only if' parts can be proved by using a derived norm, defined with respect to the set $\{a_1, \ldots, a_n\}$, where $a_i = (\delta_{ij})$, $i = 1, \ldots, n$, by*

$$\|x\|_p^* := \left[\sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \|x, a_{i_2}, \ldots, a_{i_n}\|_p^p \right]^{1/p}$$

if $1 \le p < \infty$, or

$$\|x\|_\infty^* := \sup_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \|x, a_{i_2}, \ldots, a_{i_n}\|_\infty$$

if $p = \infty$.

Indeed, one may observe that $\|\cdot\|_p^*$ defines a norm on ℓ^p

Proposition 2.1.34. *[26]*

The derived norm $\|\cdot\|_p^*$ is equivalent to the usual norm $\|\cdot\|_p$ on ℓ^p. Precisely, we have

$$\|x\|_p \le \|x\|_p^* \le n^{1/p} \|x\|_p$$

for all $x \in \ell^p$. In particular, $\|\cdot\|_\infty^* = \|\cdot\|_\infty$.

Proof. As usual, we shall only give the proof for $1 \le p < \infty$ and leave that for $p = \infty$ to the reader.

For every $x \in \ell^p$, we compute

$$\|x, a_2, a_3, \ldots, a_n\|_p^p = |x_1|^p + \sum_{j \ge n+1} |x_j|^p$$

Similarly

$$\|x, a_1, a_3, \ldots, a_n\|_p^p = |x_2|^p + \sum_{j \ge n+1} |x_j|^p$$

$$\vdots$$

$$\|x, a_1, a_2, \dots, a_{n-1}\|_p^p = |x_n|^p + \sum_{j \geq n+1} |x_j|^p.$$

Hence we obtain

$$\|x\|_p^* = \left[|x_1|^p + \cdots + |x_n|^p + n \sum_{j \geq n+1} |x_j|^p \right]^{1/p}.$$

It therefore follows that

$$\|x\|_p \leq \|x\|_p^* \leq n^{1/p} \|x\|_p,$$

that is, $\|\cdot\|_p^*$ and $\|\cdot\|_p$ are equivalent.

□

As a generalisation of Theorem 2.1.15, we have

Theorem 2.1.35. *The space ℓ^p is complete with respect to the n-norm $\|\cdot, \dots, \cdot\|_p$.*

2.1.2 Embedding of ℓ^p Sequence Spaces in n-Normed Spaces

Theorem 2.1.36 (Classical inclusion)**.** *Let $(X, \|\cdot\|)$is normed space,and For all $1 \leq p \leq q \leq \infty$, we have $\ell^p \subset \ell^q$.*

1.Embedding for Gunawan's n-norm

Theorem 2.1.37 (Embedding for Gunawan's n-norm)**.** *[20] Let $1 \leq p \leq q \leq \infty$. For all $x_1, \dots, x_n \in \ell^p$, we have*

$$\|x_1, \dots, x_n\|_q^H \leq \|x_1, \dots, x_n\|_p^H.$$

Proof. Let $x_1, \dots, x_n \in \ell^p \subset \ell^q$. For each n-tuple $(i_1, \dots, i_n) \in \mathbb{N}^n$, set

$$a_{i_1,\dots,i_n} = \prod_{k=1}^{n} |x_k(i_k)|.$$

Then

$$\|x_1, \dots, x_n\|_p^H = \left(\sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n}^p \right)^{1/p}, \qquad \|x_1, \dots, x_n\|_q^H = \left(\sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n}^q \right)^{1/q}.$$

We consider two cases.
Case 1: $q = \infty$**.** We have

$$\|x_1, \dots, x_n\|_\infty^H = \sup_{i_1,\dots,i_n} a_{i_1,\dots,i_n}.$$

Since $a^p_{i_1,\dots,i_n} \le \sum a^p_{i_1,\dots,i_n}$, for each n-tuple

$$a_{i_1,\dots,i_n} \le \left(\sum a^p_{i_1,\dots,i_n}\right)^{1/p} = \|x_1,\dots,x_n\|^H_p.$$

Taking the supremum gives

$$\|x_1,\dots,x_n\|^H_\infty \le \|x_1,\dots,x_n\|^H_p.$$

Case 2: $1 \le p \le q < \infty$. Let $M = \|x_1,\dots,x_n\|^H_\infty = \sup a_{i_1,\dots,i_n}$. For each n-tuple,

$$a^q_{i_1,\dots,i_n} = a^p_{i_1,\dots,i_n} \cdot a^{q-p}_{i_1,\dots,i_n} \le a^p_{i_1,\dots,i_n} \cdot M^{q-p}.$$

Summing over all n-tuples yields

$$\sum a^q_{i_1,\dots,i_n} \le M^{q-p} \sum a^p_{i_1,\dots,i_n}.$$

Taking the $1/q$-th power,

$$\|x_1,\dots,x_n\|^H_q \le M^{\frac{q-p}{q}} \left(\sum a^p_{i_1,\dots,i_n}\right)^{1/q}.$$

From Case 1, $M = \|x_1,\dots,x_n\|^H_\infty \le \|x_1,\dots,x_n\|^H_p$. Moreover,

$$\left(\sum a^p_{i_1,\dots,i_n}\right)^{1/q} = \left(\sum a^p_{i_1,\dots,i_n}\right)^{1/p \cdot p/q} = \|x_1,\dots,x_n\|^{H^{p/q}}_p.$$

Therefore

$$\|x_1,\dots,x_n\|^H_q \le \|x_1,\dots,x_n\|^{H^{\frac{q-p}{q}}}_p \cdot \|x_1,\dots,x_n\|^{H^{\frac{p}{q}}}_p = \|x_1,\dots,x_n\|^H_p.$$

This completes the proof. □

2.Embedding for Gähler's n-norm

Theorem 2.1.38 (Embedding for Gähler's n-norm). *[19] Let $1 \le p \le q \le \infty$. For all $x_1,\dots,x_n \in \ell^p$, we have*

$$\|x_1,\dots,x_n\|^G_q \le \|x_1,\dots,x_n\|^G_p.$$

Proof. By definition,

$$\|x_1,\dots,x_n\|^G_p = \left(\frac{1}{n!} \sum_{i_1,\dots,i_n} |\det(x^{(k)}_{i_j})|^p\right)^{1/p}.$$

For each n-tuple $(i_1,\dots,i_n)$, set $d_{i_1,\dots,i_n} = |\det(x^{(k)}_{i_j})|$. By Hadamard's inequality,

$$d_{i_1,\dots,i_n} \le \prod_{k=1}^{n} \|x_k\|_\infty.$$

Let $1 \le p \le q < \infty$. For each n-tuple,

$$d^q_{i_1,\dots,i_n} \le d^p_{i_1,\dots,i_n} \cdot (\max d_{i_1,\dots,i_n})^{q-p}.$$

But $\max d_{i_1,\dots,i_n} = \|x_1,\dots,x_n\|^G_\infty$. For the case $q = \infty$, as in Theorem 1, we have

$$\|x_1,\dots,x_n\|^G_\infty \le \|x_1,\dots,x_n\|^G_p.$$

Summing and using Hölder's inequality gives

$$\left(\sum d^q_{i_1,\dots,i_n}\right)^{1/q} \le \left(\sum d^p_{i_1,\dots,i_n}\right)^{1/p}.$$

Multiplying by $(1/n!)^{1/q} \le (1/n!)^{1/p}$ (since $p \le q$) yields the desired result. □

4.Continuity of the Canonical Injection

Theorem 2.1.39 (The canonical injection)**.** *Let $i : \ell^p \hookrightarrow \ell^q$ (with $1 \le p \le q \le \infty$) is continuous operator when both spaces are equipped with the same n-norm (Gunawan or Gähler).*

Proof. For Gunawan's n-norm, let $x_1,\dots,x_n \in \ell^p$. By Theorem 2.1.37,

$$\|i(x_1),\dots,i(x_n)\|^H_q = \|x_1,\dots,x_n\|^H_q \le \|x_1,\dots,x_n\|^H_p.$$

Hence i is 1-Lipschitz and therefore continuous. For Gähler's n-norm, the same argument applies using Theorem 2.1.38. □

3.Consequences for n-Dual Spaces

Theorem 2.1.40. *[17][Inclusion of n-dual spaces] For $1 \le p \le q \le \infty$, we have*

$$(\ell^q)^*_n \subset (\ell^p)^*_n,$$

where $(\ell^p)^*_n$ denotes the n-dual of ℓ^p (the space of bounded n-linear functionals on ℓ^p).

Proof. Let $F \in (\ell^q)^*_n$. Then F is a bounded n-linear functional on ℓ^q. For all $x_1,\dots,x_n \in \ell^p \subset \ell^q$, we have

$$|F(x_1,\dots,x_n)| \le \|F\|_{n,q} \cdot \|x_1,\dots,x_n\|^H_q.$$

By Theorem 1, $\|x_1,\dots,x_n\|^H_q \le \|x_1,\dots,x_n\|^H_p$. Therefore

$$|F(x_1,\dots,x_n)| \le \|F\|_{n,q} \cdot \|x_1,\dots,x_n\|^H_p.$$

Thus F is bounded on ℓ^p with $\|F\|_{n,p} \le \|F\|_{n,q}$. Consequently, $F \in (\ell^p)^*_n$. □

Theorem 2.1.41 (The $L^p(X)$ space on n-norm)**.** *[8, 3] Let a $(X,\|\cdot,\dots,\cdot\|_p)$ be a normed space,where Xis a measure space. One define $\|\cdot,\dots,\cdot\|_p$ on $L^p(X)\times\cdots\times L^p(X)$ by:*
If $1\le p<\infty$,

$$\|f_1,\dots,f_n\|_p := \left[\frac{1}{n!}\int_X\cdots\int_X \left|\det\left(f_j(x_i)\right)\right|^p\,dx_1\cdots dx_n\right]^{1/p}.$$

If $p=\infty$, or

$$\|f_1,\dots,f_n\|_\infty := \sup_{x_1\in X}\cdots\sup_{x_n\in X}\left|\det\left(f_j(x_i)\right)\right|.$$

This function $\|\cdot,\dots,\cdot\|_p$ defines an n-norm on $L^p(X)$.

Chapter 3

Weakly p-summable space

3.1 Some n-norms on ℓ^p

Let $1 \le p < \infty$ and let q be the conjugate exponent satisfying $1/p + 1/q = 1$. Denote by ℓ^p the space of p-summable sequences and by ℓ^q its dual space.

Definition 3.1.1 (Gähler's n-norm)**.** *For $x_1, \dots, x_n \in \ell^p$, define*

$$\|x_1, \dots, x_n\|_p^G := \sup_{\substack{y_j \in \ell^q \\ \|y_j\|_q \le 1}} \left| \det \left[\sum_{k=1}^{\infty} x_{ik} y_{jk} \right]_{i,j} \right|.$$

Definition 3.1.2. *[20][Gunawan's n-norm] For $x_1, \dots, x_n \in \ell^p$, define*

$$\|x_1, \dots, x_n\|_p^H := \left(\frac{1}{n!} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \left| \det[x_{ik_j}]_{i,j} \right|^p \right)^{1/p}.$$

Theorem 3.1.3. *[29][Equivalence of the n-norms] For all $x_1, \dots, x_n \in \ell^p$,*

$$(n!)^{1/p-1} \|x_1, \dots, x_n\|_p^H \le \|x_1, \dots, x_n\|_p^G \le (n!)^{1/p} \|x_1, \dots, x_n\|_p^H.$$

3.2 The n-functionals and n-Duality

Definition 3.2.1. *[1, 28][Multilinear n-functional] Let X be a real vector space of dimension $d \ge n$. A map $f : X^n \to \mathbb{R}$ is called an n-functional on X. An n-functional f is said to be multilinear if it satisfies:*

1. $f(x_1+y_1,\dots,x_n+y_n) = \sum_{h_i\in\{x_i,y_i\}} f(h_1,\dots,h_n)$,

2. $f(\alpha_1 x_1,\dots,\alpha_n x_n) = \alpha_1\cdots\alpha_n f(x_1,\dots,x_n)$.

Definition 3.2.2. *[18][Bounded n-functional] Let $(X,\|\cdot\|)$ be a normed space (resp. $(X,\|\cdot,\dots,\cdot\|)$ an n-normed space). The function f is called bounded if there exists $K>0$ such that*

$$|f(x_1,\dots,x_n)| \le K\|x_1\|\cdots\|x_n\| \quad (\text{resp. } \le K\|x_1,\dots,x_n\|).$$

Proposition 3.2.3. *[18][Properties on n-normed spaces] Let f be a bounded multilinear n-functional on an n-normed space. Then:*

- *$f(x_1,\dots,x_n)=0$ whenever $x_1,\dots,x_n$ are linearly dependent,*
- *f is antisymmetric: for any permutation σ,*

$$f(x_{\sigma(1)},\dots,x_{\sigma(n)}) = \operatorname{sgn}(\sigma) f(x_1,\dots,x_n).$$

These properties do not hold for bounded multilinear n-functionals on a normed space $(X,\|\cdot\|)$.

Definition 3.2.4. *[3][n-dual spaces] The space of bounded multilinear n-functionals on $(X,\|\cdot\|)$ (resp. on $(X,\|\cdot,\dots,\cdot\|)$) is called the n-dual space. It is equipped with the norm*

$$\|f\|_{n,1} := \sup_{\|x_1\|\cdots\|x_n\|\neq 0} \frac{|f(x_1,\dots,x_n)|}{\|x_1\|\cdots\|x_n\|}$$

$$\left(\|f\|_{n,n} := \sup_{\|x_1,\dots,x_n\|\neq 0} \frac{|f(x_1,\dots,x_n)|}{\|x_1,\dots,x_n\|}\right).$$

The case $n=2$ for ℓ^p

Definition 3.2.5 (Space $Y^q_{\mathbb{N}\times\mathbb{N}}$). *[?]et $\theta=(\theta_{kj})$ be a double sequence. For $1<p<\infty$, set*

$$\|\theta\|_{Y^q} := \sup_{\|x\|_p=1} \left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} x_k\theta_{kj}\right|^q\right)^{1/q} < \infty.$$

For $p=1$ (i.e., $q=\infty$), define

$$\|\theta\|_{Y^\infty} := \sup_{\|x\|_1=1}\sup_{j\in\mathbb{N}}\left|\sum_{k=1}^{\infty} x_k\theta_{kj}\right| < \infty.$$

Theorem 3.2.6 (2-dual of $(\ell^p,\|\cdot\|_p)$). *[33] If $1<p<\infty$, then the 2-dual space of $(\ell^p,\|\cdot\|_p)$ is identified by $(Y^q_{\mathbb{N}\times\mathbb{N}},\|\cdot\|_{Y^q})$. Moreover, the mapping $f\mapsto\theta:=(f(e_k,e_j))$ is an isometric bijection from the 2-dual space of $(\ell^p,\|\cdot\|_p)$ to $(Y^q_{\mathbb{N}\times\mathbb{N}},\|\cdot\|_{Y^q})$.*

Proof. For $\theta := (\theta_{kj}) \in Y^q_{\mathbb{N}\times\mathbb{N}}$, define a 2-functional f on ℓ^p by

$$f(x,y) := \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} x_k y_j \theta_{kj},$$

where $x = \sum_{i=1}^{\infty} x_i e_i$ and $y = \sum_{i=1}^{\infty} y_i e_i$. Note that $f(e_k, e_j) = \theta_{kj}$ for $k, j \in \mathbb{N}$. Further, f is a bilinear 2-functional on $(\ell^p, \|\cdot\|_p)$, and for $x, y \in \ell^p$ with $\|x\|_p = \|y\|_p = 1$, we have

$$\begin{aligned}
|f(x,y)| &= \left|\sum_{j=1}^{\infty}\left(y_j \sum_{k=1}^{\infty} x_k \theta_{kj}\right)\right| \\
&\le \left(\sum_{j=1}^{\infty} |y_j|^p\right)^{1/p} \left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} x_k\theta_{kj}\right|^q\right)^{1/q} \\
&= \left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} x_k\theta_{kj}\right|^q\right)^{1/q} \\
&\le \sup_{\|z\|_p=1}\left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} z_k\theta_{kj}\right|^q\right)^{1/q} = \|\theta\|_{Y^q}.
\end{aligned}$$

Hence, for $x, y \neq 0$ we have

$$\frac{|f(x,y)|}{\|x\|_p\|y\|_p} \le \|\theta\|_{Y^q}.$$

This means that f is a bounded bilinear 2-functional on $(\ell^p, \|\cdot\|_p)$ with

$$\|f\|_{2,1} \le \|\theta\|_{Y^q}. \tag{2.1}$$

Conversely, let f be a bounded bilinear 2-functional on $(\ell^p, \|\cdot\|_p)$. We claim that

$$\theta := (f(e_k, e_j)) \in Y^q_{\mathbb{N}\times\mathbb{N}}.$$

For each $x \in \ell^p$ with $\|x\|_p = 1$, let f_x be the functional on $(\ell^p, \|\cdot\|_p)$ given by

$$f_x(y) = f(x,y), \qquad y \in \ell^p.$$

It is clear that f_x is a linear functional on $(\ell^p, \|\cdot\|_p)$. Moreover, if $y \neq 0$, then

$$\frac{|f_x(y)|}{\|y\|_p} = \frac{|f(x,y)|}{\|x\|_p\|y\|_p} \le \|f\|_{2,1}.$$

Hence f_x is bounded with $\|f_x\| \le \|f\|_{2,1}$. Since the dual space of $(\ell^p, \|\cdot\|_p)$ is $(\ell^q, \|\cdot\|_q)$, the bounded linear functional f_x is identified by $f_x(e_j) = f(x, e_j)$ with

$$\left(\sum_{j=1}^{\infty} |f(x,e_j)|^q\right)^{1/q} \le \|f\|_{2,1}.$$

Therefore, we obtain

$$\|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}} = \sup_{\|x\|_p=1} \left(\sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} x_k f(e_k, e_j) \right|^q \right)^{1/q} \le \|f\|_{2,1}, \tag{2.2}$$

and this proves our claim.

It follows from (2.1) and (2.2) that the mapping $f \mapsto \theta := (f(e_k, e_j))$ is an isometric bijection from the 2-dual space of $(\ell^p, \|\cdot\|_p)$ to $(Y^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}})$.

For $p = 1$, we can also prove easily that the 2-dual space of $(\ell^1, \|\cdot\|_1)$ is identified by $(Y^\infty_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^\infty_{\mathbb{N}\times\mathbb{N}}})$. □

Corollary 3.2.7. *For $1 \le p < \infty$ and $1/p + 1/q = 1$, the 2-dual space of $(\ell^p, \|\cdot\|_p)$ is identified by $(Y^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}})$.*

Definition 3.2.8. *[22][g-orthogonality] Define the semi-inner product g on ℓ^p by*

$$g(x, y) := \|x\|_p^{2-p} \sum_{j=1}^{\infty} |x_j|^{p-1} \operatorname{sgn}(x_j) y_j.$$

We write $x \perp_g y$ if $g(x, y) = 0$.

Definition 3.2.9 (g-volume)**.** *For linearly independent $x_1, \ldots, x_n \in \ell^p$, let $\{x_1^\circ, \ldots, x_n^\circ\}$ be the left g-orthogonal set obtained via Gram–Schmidt process. Define*

$$V(x_1, \ldots, x_n) := \|x_1^\circ\|_p \cdots \|x_n^\circ\|_p.$$

If the vectors are linearly dependent, set $V(x_1, \ldots, x_n) = 0$.

It is known that

$$V(x_{i_1}, \ldots, x_{i_n}) \le \|x_1, \ldots, x_n\|_p^G$$

for all $x_1, \ldots, x_n \in \ell^p$ and any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$.

Theorem 3.2.10. *[3][Characterization for $\|\cdot,\cdot\|_p^G$] A bilinear 2-functional f is bounded on $(\ell^p, \|\cdot,\cdot\|_p^G)$ if and only if f is antisymmetric and bounded on $(\ell^p, \|\cdot\|_p)$. Furthermore,*

$$\frac{1}{2}\|f\|_{2,1} \le \|f\|_{2,2}^G \le \|f\|_{2,1},$$

where $\|\cdot\|_{2,2}^G$ denotes the norm on the 2-dual space of $(\ell^p, \|\cdot,\cdot\|_p^G)$.

Proof. Suppose that f is bounded on $(\ell^p, \|\cdot,\cdot\|_p^G)$. It is clear that f is antisymmetric, that is,

$$f(x, y) = -f(y, x) \quad \text{for all } x, y \in \ell^p.$$

Next, for $x, y \in \ell^p$ we have

$$\|x, y\|_p^G \le 2^{1/2} \|x, y\|_p^H \quad \text{(by the equivalence inequality for } n = 2\text{)}$$

and

$$\|x, y\|_p^H \le 2^{1-1/p} \|x\|_p \|y\|_p,$$

so that

$$\|x, y\|_p^G \le 2 \|x\|_p \|y\|_p.$$

Thus, for any linearly independent $x, y \in \ell^p$ we obtain

$$\frac{1}{2} \frac{|f(x, y)|}{\|x\|_p \|y\|_p} \le \frac{|f(x, y)|}{\|x, y\|_p^G} \le \|f\|_{2,2}^G.$$

Hence f is bounded on $(\ell^p, \|\cdot\|_p)$ with

$$\frac{1}{2} \|f\|_{2,1} \le \|f\|_{2,2}^G. \tag{2.4}$$

Conversely, suppose that f is antisymmetric and bounded on $(\ell^p, \|\cdot\|_p)$. Given linearly independent $x, y \in \ell^p$, we observe that

$$f(x, y) = f(x^\circ, y^\circ)$$

where $\{x^\circ, y^\circ\}$ is the left g-orthogonal set obtained from $\{x, y\}$. Moreover, we have

$$\frac{|f(x, y)|}{\|x, y\|_p^G} \le \frac{|f(x, y)|}{V(x, y)} = \frac{|f(x^\circ, y^\circ)|}{\|x^\circ\|_p \|y^\circ\|_p} \le \|f\|_{2,1}.$$

Since f is also antisymmetric, we have

$$|f(x, y)| \le \|f\|_{2,1} \|x, y\|_p^G$$

for all $x, y \in \ell^p$, that is, f is bounded on $(\ell^p, \|\cdot, \cdot\|_p^G)$ with

$$\|f\|_{2,2}^G \le \|f\|_{2,1}. \tag{2.5}$$

Finally, from (2.4) and (2.5) we conclude that

$$\frac{1}{2} \|f\|_{2,1} \le \|f\|_{2,2}^G \le \|f\|_{2,1},$$

as desired. □

Definition 3.2.11 (Space $Z^q_{\mathbb{N}\times\mathbb{N}}$).

$$Z^q_{\mathbb{N}\times\mathbb{N}} := \{\theta \in Y^q_{\mathbb{N}\times\mathbb{N}} \mid \theta_{kj} = -\theta_{jk} \ \forall k, j\}.$$

Note that $Z^q_{\mathbb{N}\times\mathbb{N}}$ can be viewed as a normed space equipped with the norm inherited from $Y^q_{\mathbb{N}\times\mathbb{N}}$.

Previously, we have shown that the 2*-dual space of* $(\ell^p, \|\cdot\|_p)$ *is identified by* $(Y^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}})$. *Hence the space of all antisymmetric bounded bilinear* 2*-functionals on* $(\ell^p, \|\cdot\|_p)$ *can be identified by* $(Z^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}})$.

Corollary 3.2.12. *The function* $\|\cdot\|_{Z_q^G}$ *on* $Z^q_{\mathbb{N}\times\mathbb{N}}$ *defined by*

$$\|\theta\|_{Z_q^G} := \sup_{\|x,y\|_p^G \neq 0} \frac{\left|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} x_k y_j \theta_{kj}\right|}{\|x,y\|_p^G}$$

defines a norm on $Z^q_{\mathbb{N}\times\mathbb{N}}$. *Furthermore,* $\|\cdot\|_{Z_q^G}$ *and* $\|\cdot\|_{Y^q}$ *are equivalent norms on* $Z^q_{\mathbb{N}\times\mathbb{N}}$, *with*

$$\frac{1}{2}\|\theta\|_{Y^q} \leq \|\theta\|_{Z_q^G} \leq \|\theta\|_{Y^q}$$

for all $\theta \in Z^q_{\mathbb{N}\times\mathbb{N}}$.

Corollary 3.2.13. *The* 2*-dual space of* $(\ell^p, \|\cdot,\cdot\|_p^G)$ *is identified by* $(Z^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Z_q^G})$.

Corollary 3.2.14 (2-dual for Gunawan's norm)**.** *The function* $\|\cdot\|_{Z_q^H}$ *on* Z^q *defined by*

$$\|\theta\|_{Z_q^H} := \sup_{\|x,y\|_p^H \neq 0} \frac{\left|\sum_{k=1}^{\infty} x_k y_k \theta_k\right|}{\|x,y\|_p^H}$$

defines a norm on Z^q. *Furthermore,* $\|\cdot\|_{Z_q^H}$ *and* $\|\cdot\|_{C_q}$ *are equivalent norms on* Z^q, *with*

$$2^{1/p-1}\|\theta\|_{C_q} \leq \|\theta\|_{Z_q^H} \leq 2^{1/p}\|\theta\|_{C_q}$$

for all $\theta \in Z^q$.

Corollary 3.2.15. *The* 2*-dual space of* $(\ell^p, \|\cdot,\cdot\|_p^H)$ *is identified by* $(Z^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Z_q^H})$.

Remark 3.2.16. *Here,* $\|\cdot\|_{Z_q^H}$, $\|\cdot\|_{C_q}$, *and* $\|\cdot\|_{Y^q}$ *are three equivalent norms on* $Z^q_{\mathbb{N}\times\mathbb{N}}$.

3.3 Generalization to $n \geq 2$

[34]
The results for the case $n = 2$ *can be extended easily to the case* $n \geq 2$. *For* $1 \leq p < \infty$ *and* $1/p + 1/q = 1$, *we define* $Y^q_{\mathbb{N}^n}$ *to be the space of all (real)* n*-index sequences* $\theta := (\theta_{k_1,\ldots,k_n})$ *where*

$$\|\theta\|_{Y^q_{\mathbb{N}^n}} := \sup_{\|a_1\|_p = \cdots = \|a_{n-1}\|_p = 1} \left[\sum_{k_n=1}^{\infty}\left|\sum_{k_1,\ldots,k_{n-1}=1}^{\infty} a_{1k_1}\cdots a_{n-1,k_{n-1}}\theta_{k_1,\ldots,k_n}\right|^q\right]^{1/q} < \infty.$$

For $q = \infty$, an n-index sequence $\theta := (\theta_{k_1,\ldots,k_n})$ belongs to the space $Y^{\infty}_{\mathbb{N}^n}$ if

$$\|\theta\|_{Y^{\infty}_{\mathbb{N}^n}} := \sup_{\|a_1\|_1=\cdots=\|a_{n-1}\|_1=1} \sup_{k_n\in\mathbb{N}} \left| \sum_{k_1,\ldots,k_{n-1}=1}^{\infty} a_{1k_1}\cdots a_{n-1,k_{n-1}}\theta_{k_1,\ldots,k_n} \right| < \infty.$$

Here $\mathbb{N}^n := \mathbb{N}\times\cdots\times\mathbb{N}$ (n factors). Note also that the inner sum above is a multiple sum.

We also define the generalization of the $Z^q_{\mathbb{N}\times\mathbb{N}}$ spaces as follows. An n-index sequence $\theta := (\theta_{k_1,\ldots,k_n})$ belongs to the space $Z^q_{\mathbb{N}^n}$ if $\theta \in Y^q_{\mathbb{N}^n}$ and

$$\theta_{k_1,\ldots,k_n} = \operatorname{sgn}(\sigma)\,\theta_{\sigma(k_1),\ldots,\sigma(k_n)}$$

for all $k_1,\ldots,k_n \in \mathbb{N}$ and any permutation σ of $\{k_1,\ldots,k_n\}$.

Analogously to the case $n = 2$, we have the following result for $n \geq 2$.

Theorem 3.3.1. *[33][n-dual of $(\ell^p, \|\cdot\|_p)$] The n-dual space of $(\ell^p, \|\cdot\|_p)$ is identified by $(Y^q_{\mathbb{N}^n}, \|\cdot\|_{Y^q_{\mathbb{N}^n}})$. Moreover, the mapping*

$$f \longmapsto \theta := \big(f(e_{k_1},\ldots,e_{k_n})\big)$$

is an isometric bijection from the n-dual space of $(\ell^p, \|\cdot\|_p)$ to $(Y^q_{\mathbb{N}^n}, \|\cdot\|_{Y^q_{\mathbb{N}^n}})$.

Using the inequality $V(x_1,\ldots,x_n) \leq \|x_1,\ldots,x_n\|^G_p$ and the following two inequalities,

$$\|x_1,\ldots,x_n\|^G_p \leq (n!)^{1/p}\|x_1,\ldots,x_n\|^H_p$$

and

$$\|x_1,\ldots,x_n\|^H_p \leq (n!)^{1-1/p}\|x_1\|_p\cdots\|x_n\|_p,$$

we can prove the following theorem by using arguments similar to the case $n = 2$.

Theorem 3.3.2. *[34][n-dual for $\|\cdot,\ldots,\cdot\|^G_p$] A multilinear n-functional f is bounded on $(\ell^p, \|\cdot,\ldots,\cdot\|^G_p)$ if and only if it is antisymmetric and bounded on $(\ell^p, \|\cdot\|_p)$. Furthermore,*

$$\frac{1}{n!}\|f\|_{n,1} \leq \|f\|^G_{n,n} \leq \|f\|_{n,1},$$

where $\|\cdot\|^G_{n,n}$ is the norm on the n-dual space of $(\ell^p, \|\cdot,\ldots,\cdot\|^G_p)$.

From the previous two theorems we get the following result.

Corollary 3.3.3. *The n-dual space of $(\ell^p, \|\cdot,\ldots,\cdot\|^G_p)$ is identified by $(Z^q_{\mathbb{N}^n}, \|\cdot\|^G_{Z_{qn}})$, where $\|\cdot\|^G_{Z_{qn}}$ is given by*

$$\|\theta\|^G_{Z_{qn}} := \sup_{\|x_1,\ldots,x_n\|^G_p\neq 0} \frac{\left|\sum_{k_1,\ldots,k_n=1}^{\infty} x_{1k_1}\cdots x_{nk_n}\theta_{k_1,\ldots,k_n}\right|}{\|x_1,\ldots,x_n\|^G_p}.$$

Using the equivalence of the n-norms, we also get the following theorem.

Corollary 3.3.4. *The n-dual space of $(\ell^p, \|\cdot, \ldots, \cdot\|_p^H)$ is identified by $(Z^q_{\mathbb{N}^n}, \|\cdot\|^H_{Z_{qn}})$, where $\|\cdot\|^H_{Z_{qn}}$ is given by*

$$\|\theta\|^H_{Z_{qn}} := \sup_{\|x_1,\ldots,x_n\|_p^H \neq 0} \frac{\left|\sum_{k_1,\ldots,k_n=1}^{\infty} x_{1k_1} \cdots x_{nk_n} \theta_{k_1,\ldots,k_n}\right|}{\|x_1, \ldots, x_n\|_p^H}.$$

In the theory of normed spaces, we know that the dual space of $(\ell^p, \|\cdot\|_p)$ is (identified by) $(\ell^q, \|\cdot\|_q)$, where $1 \leq p < \infty$ and $1/p + 1/q = 1$.

Definition 3.3.5 (n-Dual Space of ℓ^p)**.** *The n-dual space of $(\ell^p, \|\cdot\|_p)$ is identified by $(Y^q_{\mathbb{N}^n}, \|\cdot\|_{Y^q_{\mathbb{N}^n}})$.*

Remark 3.3.6.

1. *We see similarities between the classical dual space result and the n-dual space result. Similar relations also occur for the n-dual space of ℓ^p when ℓ^p is viewed as an n-normed space with Gähler's n-norm or Gunawan's n-norm.*
2. *All these results are identical in the case where $n = 1$.*
3. *For $n \geq 2$, we still have a question whether the norm $\|\cdot\|_{Y^q_{\mathbb{N}^n}}$ on $Y^q_{\mathbb{N}^n}$, as well as the norms $\|\cdot\|^H_{Z_{qn}}$ and $\|\cdot\|^G_{Z_{qn}}$ on $Z^q_{\mathbb{N}^n}$, can be reduced to*

$$\|\theta\|_{Y^q_{\mathbb{N}^n}} := \left(\sum_{k_1,\ldots,k_n=1}^{\infty} |\theta_{k_1,\ldots,k_n}|^q \right)^{1/q}$$

and

$$\|\theta\|_{Z_{qn}} := \left(\sum_{k_1,\ldots,k_n=1}^{\infty} |\theta_{k_1,\ldots,k_n}|^q \right)^{1/q}.$$

4. *One may easily check that if $\theta = (\theta_{i_1,\ldots,i_n})$ satisfies*

$$\left(\sum_{i_1,\ldots,i_n=1}^{\infty} |\theta_{i_1,\ldots,i_n}|^q \right)^{1/q} < \infty,$$

then $\|\theta\|_{Y^q_{\mathbb{N}^n}} = \|\theta\|_{Z_{qn}}$, and $\|\theta\|_{Z_{qn}}$ is dominated by $\left(\sum |\theta_{i_1,\ldots,i_n}|^q\right)^{1/q}$. We just do not know whether the converse is true or related problems.

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Conclusion

This dissertation has been devoted to the study of n-normed spaces and the investigation of the n-dual spaces associated with the sequence space ℓ^p. The theory of n-normed spaces extends the classical framework of normed spaces by replacing the notion of length of a single vector with a generalized measure of volume determined by n vectors. This extension provides a richer geometric structure and opens new perspectives in functional analysis.

We began by presenting the fundamental concepts and properties of n-normed spaces, including their basic examples, topological aspects, and the notions of boundedness and continuity for linear operators. In particular, several types of n-bounded and n-continuous operators were introduced and analyzed, highlighting the relationships between these notions and establishing conditions under which they are equivalent.

Special attention was then devoted to the sequence space ℓ^p and its natural n-norms. We studied the construction and properties of Gähler's n-norm and Gunawan's n-norm on ℓ^p, emphasizing their role in extending classical summability theory to the setting of n-normed spaces.

Abstract

This dissertation deals with the detail of a paper of H. Gunawan that studies the space $\ell_p, 1 \leq p \leq \infty$ and its natural n-norm, which can be viewed as a generalisation of its usual norm. Using a derived norm equivalent to its usual norm, by showing that ℓ_p is complete with respect to its natural n-norm and the n-dual of ℓ_p under two important n-norms. In addition, it also gives a weak p-summable space of an ℓ_p sequences as an n-normed space.

Keywords: n-norm, weak p-summable, p-summable, n-normed space.

Résumé

Ce mémoire traite en détail d'un article de H. Gunawan qui étudie l'espace $\ell_p, 1 \leq p \leq \infty$ et sa n-norme naturelle, laquelle peut être vue comme une généralisation de sa norme usuelle. En utilisant une norme dérivée équivalente à sa norme usuelle, on montre que ℓ_p est complet par rapport à sa n-norme naturelle et que le n-dual de ℓ_p l'est aussi sous deux n-normes importantes. De plus, on présente l'espace des suites faiblement p-sommables d'un ℓ_p comme un espace n-normé.

Mots clé : n-norme, faiblement p-sommable, p-sommable, espace n-normé.

الملخص

تتناول هذه المذكرة بالتفصيل بحثاً لكل من $H.Gunawan$ والذي يدرس الفضاء $\ell_p, 1 \leq p \leq \infty$ و n-ناظمي الطبيعي له، والذي يمكن اعتباره تعميماً لنظمه المعتاد. وذلك باستخدام نظيم مشتق مكافئ لنظيمه المعتاد، من خلال إثبات أن ℓ_p كامل بالنسبة إلى n-نظيم الطبيعي وأن n-مزدوج للفضاء ℓ_p يخضع لـ n-نظمين مهمين. بالإضافة إلى ذلك، يتم تقديم فضاء المتتاليات الضعيفة p-القابلة للجمع المنبثق عن ℓ_p كفضاء ذي n-ناظمي.

الكلمات المفتاحية: n-نظيم، ضعيفة p-قابلة للجمع، p-قابلة للجمع، فضاء ذو n-ناظمي.