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The Number of Fuzzy Clopen Sets in Fuzzy Topological Spaces

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Contents

List of Symbols

Symbol	**Meaning**
(X, τ)	Topological space
τ	Topology (collection of open sets)
$\mathbb{R}, \mathbb{N}, \mathbb{Q}$	Real, natural, and rational numbers
A^c, $X \setminus A$	Complement of A
$\bar{A}$	Closure of A
A^0, $\text{int}(A)$	Interior of A
$B(x, \varepsilon)$	Open ball centered at x with radius ε
(a, b), $[a, b)$, $[a, b]$	Intervals in $\mathbb{R}$
$\mathcal{B}$	Basis for a topology
T_0, T_1, T_2	Separation axioms (Kolmogorov, Fréchet, Hausdorff)
$\lim x_n = x$	Sequence convergence
$f : X \to Y$	Continuous map from X to Y
$f^{-1}(U)$	Preimage of U under f
χ_A	Characteristic function of A
$\tau_Y = \{U \cap Y : U \in \tau\}$	Subspace topology on Y
$X \times Y$	Product of topological spaces
$\bigcup, \bigcap$	Union, intersection
$\mathcal{F}, \mathcal{C}$	Fuzzy open and closed sets
$\mu_A(x)$	Membership of x in fuzzy set A
$p(x)$	Fuzzy point at x
cA, A^c	Complement of fuzzy set A
$f^{-1}[A]$	Preimage of fuzzy set A
$f[A]$	Image of fuzzy set A
sup, inf	Supremum, infimum
Open cover	$X \subseteq \bigcup U_\alpha$
nbhd	neighborhood

Introduction

Topology is a fundamental branch of mathematics with numerous applications in limited sets, discrete sets, and geometric spaces. This work is structured around the classical and fuzzy aspects of topology as follows:

In chapter 1, we explore the classical theory of topology, beginning with the basic concepts of topological spaces. Various types of topologies are discussed, including the discrete topology, the indiscrete topology, and the complement finite (cofinite) topology. Key notions such as interior points, open sets, closed sets, closure sets, and important properties including the countability axioms, sequence convergence, continuity, homeomorphisms, subspace topology, and finite product topologies are also covered.

chapter 2 introduces the theory of fuzzy topological spaces—an extension of classical topology that allows for modeling uncertainty and imprecision. The foundational definitions and conceptual framework of fuzzy topology are presented to bridge the gap between classical topological ideas and more generalized fuzzy systems.

Finally,In Chapter 3, we examine the number of fuzzy clopen sets within fuzzy topological spaces. The classification and properties of clopen sets are discussed in detail, including observations about their structure

his work thus provides a comprehensive study—from classical topological concepts to advanced fuzzy topologies—highlighting both theoretical development and applications.

Chapter 1

Topology Basics

In this chapter, I discuss the foundational concepts of topology, focusing on bases of topology, closed sets, and closures. Additionally, I explore key topics such as countability, sequences, weak T-axioms, first-countability, continuity, homeomorphisms, and subspaces, providing a deeper understanding of their roles in topology.

Definition 1.1. [4] Topology is a branch of mathematics that describes mathematical spaces, in particular the properties that stem from a space's shape. Many of the shapes topologists deal with are incredibly strange, so much so that practically all everyday objects such as bowls, pets, and trees make up a small minority. The word "topology" derives from the Greek words for place (*topos*) and study (*-logy*).

Topology is important as a guide in several areas of study:

- Theoretical physics (in particular the successors of quantum mechanics such as quantum field theory and string theory)
- Cosmology (for determining the shape of the universe)
- Biology (for the tangling of DNA and predicting the growth of organs and other body parts)
- Computer science (for determining the large-scale structure of data sets)
- Robotics (where a robot arm's motions are planned based on the shape of a space with a number of dimensions equal to the number of arm joints)

1.1 Topological spaces

Definition 1.2. [6] A *topology* on a nonempty set X is a collection of subsets of X, called *open sets*, such that:

(a) the empty set $\emptyset$ and the set X are open;

(b) the union of an arbitrary collection of open sets is open;

(c) the intersection of a finite number of open sets is open.

A subset A of X is a *closed set* if and only if its complement, $A^c = X \setminus A$, is open.

More formally, a collection $\mathcal{T}$ of subsets of X is a topology on X if:

(a) $\emptyset, X \in \mathcal{T}$;

(b) if $G_\alpha \in \mathcal{T}$ for $\alpha \in A$, then $\bigcup_{\alpha \in A} G_\alpha \in \mathcal{T}$;

(c) if $G_i \in \mathcal{T}$ for $i = 1, 2, \ldots, n$, then $\bigcap_{i=1}^{n} G_i \in \mathcal{T}$.

We call the pair $(X, \mathcal{T})$ a *topological space*; if $\mathcal{T}$ is clear from the context, then we often refer to X as a topological space.

Example 1.1. Let X be a nonempty set. The collection $\{\emptyset, X\}$, consisting of the empty set and the whole set, is a topology on X, called the *trivial topology* or

indiscrete topology. The power set $\mathcal{P}(X)$ of X, consisting of all subsets of X, is a topology on X, called the *discrete topology.*

Example 1.2. Let (X, d) be a metric space. Then the set of all open sets is a topology on X, called the *metric topology. For instance*, a subset G of $\mathbb{R}$ is open with respect to the standard, metric topology on $\mathbb{R}$ if and only if for every $x \in G$ there is an open interval I such that $x \in I$ and $I \subset G$.

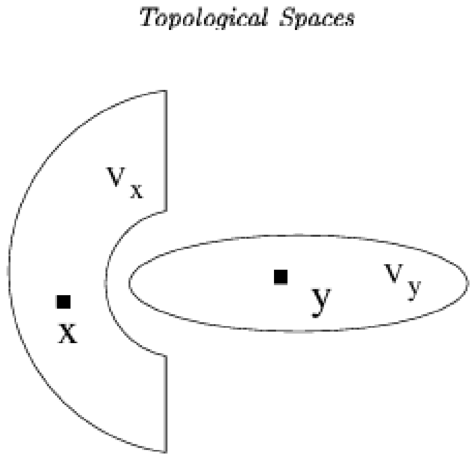

Figure 1.1: The Hausdorff property

1.1.1 Discrete Topology

[8] Let X be a set, and let $\mathcal{P}(X)$ be the power set of X. Then $\mathcal{P}(X)$ is a topology on X, which is called the *discrete topology* and is often denoted by $\mathcal{T}_d$.

Example: Let $X = \{a, b\}$. Then:

$$\mathcal{T}_d = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

1.1.2 Indiscrete Topology

[8] Let X be a set. Then $\mathcal{T} = \{\emptyset, X\}$ is a topology on X, which is called the *indiscrete topology* (also known as the trivial topology) and is often denoted by $\mathcal{T}_i$.

This topology is the smallest (coarsest) topology that can be defined on X.

Example: Let $X = \{a, b\}$. Then the collection

$$\mathcal{T}_i = \{\emptyset, X\} = \{\emptyset, \{a, b\}\}$$

1.1.3 Complement Finite Topology (Cofinite Topology)

[8] Let X be an infinite set then:

$$\tau_{co} = \{U \subseteq X \mid U^c \text{ is finite}\} \cup \{\emptyset\}$$

is a topology on X which is called the cofinite topology and denoted by τ_{co}.

1.1.4 Interior Points and Open Sets

Let (X, d) be a metric space and $A \subset X$ a subset.

1. A point $a \in A$ is called **interior** if there exists an open ball $B_r(a) \subset A$ for some $r > 0$. The set of all interior points is called the **interior** of A.
2. The subset A is called **open** if every $a \in A$ is an interior point.

1.2 Closed sets and closures

1.2.1 Closed sets

[7] A subset A of a topological space X is said to be closed if the set $X - A$ is open.

Example 1.3. The subset $[a, b] \subset \mathbb{R}$ is closed because its complement

$$\mathbb{R} - [a, b] = (-\infty, a) \cup (b, +\infty)$$

is open. Similarly, $[a, +\infty)$ is closed, because its complement $(-\infty, a)$ is open. These facts justify our use of the terms "closed interval" and "closed ray." The subset $[a, b)$ of $\mathbb{R}$ is neither open nor closed.

Example 1.4. In the plane $\mathbb{R}^2$, the set

$$\{(x, y) \mid x \geq 0 \text{ and } y \geq 0\}$$

is closed because its complement is the union of the two sets

$$(-\infty, 0) \times \mathbb{R} \quad \text{and} \quad \mathbb{R} \times (-\infty, 0),$$

each of which is a product of open sets in $\mathbb{R}$ and is, therefore, open in $\mathbb{R}^2$.

Example 1.5. In the finite complement topology on a set X, the closed sets consist of X itself and all finite subsets of X.

Example 1.6. In the discrete topology on the set X, every set is open; it follows that every set is closed as well.

1.2.2 Closure sets

Definition 1.3. [6] **L**et $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. We define the *closure* of A in $(X, \mathcal{T})$, which we denote with $\bar{A}$, by:

$$x \in \bar{A} \text{ if and only if for every open set } U \text{ containing } x, U \cap A \neq \emptyset.$$

Or, in symbols:

$$\bar{A} = \{x \in X : \forall U \in \mathcal{T} \text{ such that } x \in U, U \cap A \neq \emptyset\}.$$

When there can be no confusion about the topological space with respect to which a closure is being considered, we will simply write $\bar{A}$ without specifying it in $(X, \mathcal{T})$.

In words, $x \in \bar{A}$ if and only if any open set containing x also contains an element of A.

After we define what it means for a set to be closed, we will be able to present an alternate way of defining the closure of a set.

Before going any further with examples, we examine some elementary properties of closures, the proofs of which use only the definition of closure and absolutely no cleverness or new ideas.

Proposition 1.1. [6] **L**et $(X, \mathcal{T})$ be a topological space and let $A, B \subseteq X$. Then:

1. $A \subseteq \bar{A}$.
2. $\bar{\bar{A}} = \bar{A}$. (A mathematician interested in using fancy words would say that taking closures is an *idempotent operation.*)
3. $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
4. If $X \setminus A$ is open, then $\bar{A} = A$.

5. Trivially, $\bar{\emptyset} = \emptyset$ and $\bar{X} = X$.

Proof. Let A be a subset of a topological space, and let $\bar{A}$ denote its closure. We want to show that $\bar{A} = A$.

1. Let $x \in \bar{A}$. Then, by definition of closure, any open set U containing x intersects A, meaning there exists at least one point $y \in U \cap A$. Hence, $x \in A$.
2. By (1), we have $\bar{A} \subseteq A$. On the other hand, let $x \in A$, and let U be an open set containing x. We need to show that $U \cap A \neq \emptyset$. By assumption, $U \cap A \neq \emptyset$, so there is at least one point y in this intersection. Since U is an open set containing an element y in the closure of A, we conclude that $U \cap A \neq \emptyset$ by definition of $\bar{A}$.

Thus, we have shown that $A \subseteq \bar{A}$ and $\bar{A} \subseteq A$, proving that $\bar{A} = A$. □

Example 1.7. Let $A = (0, 1) \subset \mathbb{R}$. The closure of A is:

$$\overline{A} = [0, 1]$$

Here, 0 and 1 are limit points of A and must be included in the closur

Example 1.8. let $X = \{a, b, c\}$. For $A = \{a\}$:

$$\overline{A} = \{a\}$$

Since A is already closed, its closure is itself.

Example 1.9. Let $X = \mathbb{N}$ (natural numbers) with the co-finite topology (closed sets are finite sets or X). For $A = \{1\}$:

$$\overline{A} = \{1\}$$

For $B = \{2, 4, 6, \dots\}$ (even numbers), the closure is:

$$\overline{B} = \mathbb{N}$$

Because B is infinite, and the only closed superset is $\mathbb{N}$.

1.3 The Countability Axioms

In this section, we restate this definition and introduce a new “countability axiom.” Both of the countability axioms involve countable (versus uncountable) bases of topologies.

Definition 1.4. [7] A topological space X has a **countable basis at a point** x if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points satisfies the **First Countability Axiom**, or is **first-countable**.

Theorem 1.1. [7] Let X be a topological space.

1. Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse holds if X is first-countable.

2. Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first-countable.

Definition 1.5. [7] If a topological space X has a countable basis for its topology, then X satisfies the **Second Countability Axiom**, or is **second-countable**.

Example 1.10. The real line $\mathbb{R}$ (under the standard topology) is first-countable; consider $\mathcal{B} = \{(a, b) \mid a < b,\ a, b \in \mathbb{Q}\}$.

Similarly, $\mathbb{R}^n$ is first-countable; let $\mathcal{B}$ be all products of open intervals with rational endpoints.

Even $\mathbb{R}^\omega$ (with the product topology) is second-countable; let $\mathcal{B}$ be all products $\prod_{n\in\mathbb{N}} U_n$ where finitely many U_n are open intervals with rational endpoints and the remaining $U_n = \mathbb{R}$.

1.4 Sequences, weak T-axioms, and first countability

1.4.1 Convergence

Definition 1.6. [6] Let (X, τ) be a topological space. A sequence $\{x_n\}_{n=1}^{\infty}$ is said to converge to a point $x \in X$ if for every open set U containing x, there exists an $N \in \mathbb{N}$

such that $x_n \in U$ for all $n > N$.

In this case, we write

$$\lim_{n\to\infty} x_n = x,$$

or more commonly, we simply write $x_n \to x$.

Recalling the language of tails, the way I would actually say this is that x_n converges to x if there is a tail of the sequence in any open set containing x.

1.4.2 When is sequence convergence not weird?

In this section, we will formalize some properties of topological spaces called *separation axioms* that will ensure things are less weird. One of these, the *Hausdorff property*, is particularly important. We will approach it through some weaker properties first.

Definition 1.7. [6] A topological space (X, τ) is said to be T_0 (or, much less commonly, a *Kolmogorov space*) if for any pair of distinct points $x, y \in X$, there exists an open set U that contains one of them and not the other.

One would usually express this by saying that (X, τ) is T_0 provided that given a pair of distinct points, one of them has an open neighborhood not containing the other.

This is a very weak property that we will not discuss much, as it is rarely useful to ask for a space to be just T_0; we will usually ask for more. We introduce this property because it is the weakest of the T-axioms, of which we will see several throughout the course. The only space of any consequence we have seen that fails to be T_0 is the *indiscrete space*.

Next, a property that we foreshadowed while discussing closed sets, though the definition may not seem familiar at first.

Definition 1.8. [6] A topological space (X, τ) is said to be T_1 (or, much less commonly, a *Fréchet space*) if for any pair of distinct points $x, y \in X$, there exist open sets U and V such that U contains x but not y, and V contains y but not x.

This is a very slight strengthening of T_0.

Example 1.11. Let X be an infinite set. Then the space $(X, \tau_{\text{co-finite}})$ is T_1. Indeed, given distinct points $x, y \in X$, the open sets $U = X \setminus \{y\}$ and $V = X \setminus \{x\}$ witness this. However, as we saw in Example 2.3.8, sequences in this space can converge to many different points.

For example, in $(\mathbb{R}, \tau_{\text{co-finite}})$, the sequence $1, 2, 3, 4, \ldots$ converges to all points, since any co-finite subset of X contains a tail of the sequence.

This should make you a little bit uncomfortable. Your intuition from the usual topology on $\mathbb{R}^n$ says that a convergent sequence should converge to only one point. The next definition is what guarantees this. It is by far the most important of the weaker separation axioms.

Definition 1.9. [6] A topological space (X, τ) is said to be T_2, or more commonly, a *Hausdorff space*, if for every pair of distinct points $x, y \in X$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem 1.2. [6] Let (X, τ) be a Hausdorff space. Then every sequence in X converges to at most one point.

Proof. Suppose X is Hausdorff, and let $\{x_n\}$ be a sequence in X. Suppose $x_n \to x$ and $x_n \to y$ with $x \neq y$.

Then, since X is Hausdorff, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$. By the definition of convergence, some tail of the sequence $\{x_n\}$ is contained in U. But then that tail (and therefore all subsequent terms of the sequence) is disjoint from V, meaning $x_n \not\to y$, which contradicts our assumption.

Thus, a sequence in a Hausdorff space can converge to at most one point. □

Example 1.12.

1. $\mathbb{R}$ with the usual topology, the Sorgenfrey line, and $(X, \tau_{\text{discrete}})$ for any set X are all Hausdorff.
2. The Furstenberg topology on $\mathbb{Z}$ discussed in the Big List is Hausdorff (you proved this, in fact, though I spelled out the property in the question rather than giving its name).

3. $(X, \tau_{\text{indiscrete}})$ is almost always not Hausdorff. Under what circumstances is it Hausdorff?

4. The cofinite topology on an infinite set and the cocountable topology on an uncountable set are both not Hausdorff.

5. $(\mathbb{R}, \tau_{\text{ray}})$ is not Hausdorff, since any two nonempty open sets intersect.

1.4.3 Sequences and closures

Proposition 1.2. [6] Let (X, τ) be a topological space, let $A \subseteq X$, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements of A. If $a_n \to a$, then $a \in A$.

Proof. If $a \notin A$ (i.e., if a is close to A but not in A), it feels like we should be able to find a sequence of points from A that converges to a. Unfortunately, this is not always the case.

Example 1.13. Consider the co-countable topology on $\mathbb{R}$, where a set is open if its complement is at most countable. Let $A = \mathbb{R} \setminus \{7\}$. We will show that 7 is in the closure of A, but no sequence in A can converge to 7.

Why is 7 in the closure of A?

Any open set containing 7 must have a complement that is at most countable. This means the open set includes almost all real numbers, so it must contain some points from A. Since every open neighborhood of 7 intersects A, we conclude that 7 is in the closure of A.

Why can no sequence in A converge to 7?

In this topology, a sequence can only converge if it is eventually constant (meaning it stays the same after some point). But a sequence in A can never be eventually constant at 7 because $7 \notin A$. So, no sequence in A can converge to 7.

1.4.4 First countability

Definition 1.10. [6] Let (X, τ) be a topological space and let $x \in X$. A *local basis* at x is a collection of open sets $\mathcal{B}_x \subseteq \tau$ satisfying the following two properties:

1. $x \in B$ for all $B \in \mathcal{B}_x$.
2. For any open set U containing x, there exists $B \in \mathcal{B}_x$ such that $B \subseteq U$.

Example 1.14.

1. Let x be a point in $\mathbb{R}$ with the usual topology. Then the collection

   $$\{(a,b) \subset \mathbb{R} \mid a < x < b\}$$

   is a local basis at x.

   In fact, we can refine this by finding a countable local basis in the familiar way:

   $$\{(a,b) \subset \mathbb{R} \mid a < x < b \text{ and } a, b \in \mathbb{Q}\}.$$

   An even better choice is:

   $$\mathcal{B}_x = \left\{ \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \mid n \in \mathbb{N} \right\}.$$

   These are the local bases we will usually consider in $\mathbb{R}$ with the usual topology.

2. More generally, in $\mathbb{R}^n$ with the usual topology, the following collections are all local bases at x:

   $$\{B(x,\epsilon) \mid \epsilon > 0\}$$

   $$\{B(x,\epsilon) \mid \epsilon > 0 \text{ and } \epsilon \in \mathbb{Q}\}$$

   $$\mathcal{B}_x = \{B(x, 1/n) \mid n \in \mathbb{N}\}.$$

   Note that the latter two are countable while the first one is not.

3. In the Sorgenfrey line, a point x. Then both $[x,b) \subset \mathbb{R} : b > x$ and

   $$B_x = \{[x, x + \frac{1}{n}) : n \in \mathbb{N}\}$$

   are local bases at x. (Show this.) Again, the latter is countable while the former is not.

4. In $(X, \tau_{\text{discrete}})$, for any $x \in X$, the singleton $\{x\}$ is an open set and therefore must be an element of any local base at x. In fact, it turns out that

$$\mathcal{B}_x = \{\{x\}\}$$

is a local base at x.

5. More generally, it should be easy to see that if $\{x\}$ is open in a topological space, then $\{x\}$ is a local base at x. As a result, for example, $\{p\}$ is a local base at p in (X, τ_p). This is not true for other points in that space, though. If $x \neq p$ in this space, then the smallest local base at x would be $\{x, p\}$.

Definition 1.11. [6] A topological space (X, T) is said to be first countable if every point in X has a countable local basis.

This is the key property that allows sequences to capture all the information we feel they should capture. We make note of some examples. We have already shown most of what there is to show here.

Example 1.15. 1. $\mathbb{R}^n$ with the usual topology and the Sorgenfrey line are first countable, as is every discrete space.

2. Spaces that are not first countable are relatively rare. The only ones we have encountered so far are $(X, T_{\text{co-finite}})$ and $(X, T_{\text{co-countable}})$, provided that X is uncountable.

Before we use first countability to prove the results we were hoping for, we establish an equivalent definition which looks like a slight strengthening, but actually is not.

1.5 Continuity and homeomorphisms

Definition 1.12. [13] Let $f : X \to Y$ be a function, $x_o \in X$ and T, U be the topologies on X, Y respectively. Then f is said to be continuous at $x_o \in X$ if given any U-open set H containing $f(x_o)$, there exists a T-open set G containing x_o such that $f(G) \subset H$.

Definition 1.13. [13] Let $f : X \to Y$ be a function and T, U be topologies on X, Y respectively. Then f is said to be continuous (or $T - U$ continuous) if it is continuous at each point of X.

Theorem 1.3. [13] The function $f : (X, T) \to (Y, U)$ is continuous if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

Proof. Let $f : (X, T) \to (Y, U)$ be a map.

Suppose f is continuous. Let G be an open subset of Y. To prove that $f^{-1}(G)$ is open in X:

If $f^{-1}(G) = \emptyset$ then $f^{-1}(G) \in T$.

If $f^{-1}(G) \neq \emptyset$ then there exists $x \in f^{-1}(G)$ so that $f(x) \in G$.

Since f is continuous at x, there exists $H \in T$ such that $x \in H$ and $f(H) \subset G$.

Thus $x \in H \subset f^{-1}(G), H \in T$.

Hence $f^{-1}(G)$ is a neighbourhood of each of its points and so $f^{-1}(G)$ is T-open.

Conversely, suppose that $f : (X, T) \to (Y, U)$ is a map such that $f^{-1}(V)$ is open in X for each open set $V \subset Y$. To prove that f is continuous, let $V \in U$ be arbitrary. Then by assumption, $f^{-1}(V)$ is open in X.

Taking $U = f^{-1}(V)$ so that $U \in T$, we get $f(U) = f(f^{-1}(V)) \subset V$, which proves that f is a continuous map.

Theorem 1.4. [13] Let $f : (X, T) \to (Y, U)$ be a map. Let $\mathcal{S}$ be a subbase for the topology U on Y. Then f is continuous if and only if $f^{-1}(S)$ is open in X whenever $S \in \mathcal{S}$.

Theorem 1.5. [13] A map $f : (X, T) \to (Y, U)$ is closed if and only if $\overline{f(A)} \subset f(\overline{A})$ for every $A \subset X$.

Theorem 1.6. [13] A homeomorphic image of a second countable space is second countable.

Theorem 1.7. [13] The first axiom of countability is a topological property.

Example 1.16. Give an example of a function which is continuous and closed but not open.

Solution. Consider a function $f : (\mathbb{R}, U) \to (\mathbb{R}, U)$ such that $f(x) = 1$ for all $x \in \mathbb{R}$. Then f is evidently a constant map. Let $G \subset U$ be an arbitrary open set. Then:

$$f^{-1}(G) = \{x \in \mathbb{R} \mid f(x) \in G\} = \{x \in \mathbb{R} \mid 1 \in G\}.$$

Thus,

$$f^{-1}(G) = \begin{cases} \mathbb{R}, & \text{if } 1 \in G, \\ \emptyset, & \text{if } 1 \notin G. \end{cases}$$

Since $\mathbb{R}$ and $\emptyset$ are both open sets in $\mathbb{R}$, it follows that $f^{-1}(G)$ is open, proving that f is continuous.

Let $F \subset \mathbb{R}$ be a closed set. Then:

$$f(F) = \{f(x) \mid x \in F\} = \{1\},$$

which is a finite subset of $\mathbb{R}$, and hence closed. Thus, f is a closed map.

Let $A \subset \mathbb{R}$ be open. Then:

$$f(A) = \{f(x) \mid x \in A\} = \{1\},$$

which is not open.

Therefore, A being open implies that $f(A)$ is not open. Thus, f is not an open map.

Finally, f is continuous, closed, but not open.

1.6 Subspace Topology

Definition 1.14. [10] Let (X, T) be a topological space with topology T. If Y is a subset of X, the collection

$$T_Y = \{U \cap Y \mid U \in T\}$$

is a topology on Y, called the *subspace topology.* With this topology, Y is called a *subspace* of X; its open sets consist of all intersections of open sets of X with Y.

Check that T_Y is a topology: 1. It contains $\emptyset$ and Y because

$$\emptyset = Y \cap \emptyset \quad \text{and} \quad Y = Y \cap X$$

where $\emptyset$ and X are elements of T.

2. It is closed under finite intersections:

$$(U_1 \cap Y) \cap \cdots \cap (U_n \cap Y) = (U_1 \cap \cdots \cap U_n) \cap Y$$

Since T is a topology, $U_1 \cap \cdots \cap U_n \in T$, so the result belongs to T_Y.

3. It is closed under arbitrary unions:

$$\bigcup_{\alpha \in J} (U_\alpha \cap Y) = \left(\bigcup_{\alpha \in J} U_\alpha \right) \cap Y$$

Since T is a topology, $\bigcup_{\alpha \in J} U_\alpha \in T$, so the result belongs to T_Y.

Thus, T_Y satisfies the axioms of a topology.

Lemma 1.1. [10]

Let B be a basis for the topology of X. Then the collection

$$B_Y = \{B \cap Y \mid B \in B\}$$

is a basis for the subspace topology on Y.

Definition 1.15. [10] If Y is a subspace of X, we say that a set U is *open in* Y if $U \in T_Y$; this implies in particular that it is a subset of Y. We say that U is *open in* X if $U \in T_X$.

Lemma 1.2. [10]

Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof. Since U is open in Y, we have $U = Y \cap V$ for some set V open in X. Since Y and V are both open in X, so is $Y \cap V$.

Theorem 1.8. [10] If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Proof. The set $U \times V$ is the general basis element for $X \times Y$, where U is open in X and V is open in Y. Therefore,

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B).$$

Since $U \cap A$ and $V \cap B$ are the general open sets for the subspace topologies on A and B, respectively, the set $(U \cap A) \times (V \cap B)$ is the general basis element for the product topology on $A \times B$. Hence, the bases for the subspace topology on $A \times B$ and for the product topology on $A \times B$ are the same, implying that the topologies are the same.

Example 1.17. Consider the subset $Y = [0, 1] \subset \mathbb{R}$, in the subspace topology. The subspace topology has as basis

$$B = (a, b) \cap Y$$

where (a, b) is an open interval in $\mathbb{R}$.

Such a set is of one of the following types:

$$(a, b) \cap Y = \begin{cases} (a, b) & \text{if } a, b \in Y \\ [a, b) & \text{if } a \in Y, b \notin Y \\ (a, 1] & \text{if } b \in Y, a \notin Y \\ Y \text{ or } \emptyset & \text{if } a, b \notin Y \end{cases}$$

By definition, each of these sets is open in Y. But sets of the second and third types are not open in the larger space $\mathbb{R}$.

Note: These sets form a basis for the order topology on Y. Thus, we see that in the case of the set $Y = [0, 1]$, its subspace topology and its order topology are the same.

Example 1.18. Let Y be the subset $[0, 1) \subset \mathbb{R}$. In the subspace topology on Y, the one-point set $\{2\}$ is open, because

$$\{2\} = \left(\frac{3}{2}, \frac{5}{2}\right) \cap Y.$$

However, in the order topology on Y, the set $\{2\}$ is not open.

Any basis element for the order topology on Y that contains 2 is of the form

$$(x, x') \cap Y \quad \text{where } a < x' = 2$$

for some $a \in Y$. Such a set necessarily contains points of Y less than 2, showing that $\{2\}$ is not open in the order topology.

1.7 Finite product topologies

Definition 1.16. [15] Let (X, T) and (Y, U) be topological spaces. The **product topology** on $X \times Y$ is the topology generated by the basis:

$$\{U \times V : U \in T, V \in U\}.$$

More generally, if $(X_1, T_1), \ldots, (X_n, T_n)$ are topological spaces, the product topology on

$$\prod_{i=1}^{n} X_i = X_1 \times \cdots \times X_n$$

is the topology generated by the basis:

$$\{U_1 \times U_2 \times \cdots \times U_n : U_i \in T_i \text{ for all } i = 1, \ldots, n\}.$$

Proposition 1.3. [15] Let (X, T) and (Y, U) be topological spaces, and let B_X and B_Y be bases on X and Y that generate T and U, respectively. Then

$$B = \{U \times V : U \in B_X, V \in B_Y\}$$

is a basis for the product topology on $X \times Y$.

Proof. Since $B_X \subset T$ and $B_Y \subset U$, every element of B is open in the product topology. Now fix an open set U in the product topology and some point $(x, y) \in U$. By definition of the product topology, there must be some $U_X \in T$ and $U_Y \in U$ such that $(x, y) \in U_X \times U_Y \subset U$. Using the fact that B_X and B_Y are bases, find sets $B_X \in B_X$ and $B_Y \in B_Y$

such that $x \in B_X \subset U_X$ and $y \in B_Y \subset U_Y$. But then we have:

$$(x, y) \in B_X \times B_Y \subset U_X \times U_Y \subset U,$$

so $B_X \times B_Y$ is the set we were looking for.

□

Remark 2.3. we will usually write X^2 instead of $X \times X$, X^3 instead of $X \times X \times X$, and so on. This agrees with the usual notation for $\mathbb{R}^n$.

Example 1.19.

1. A product of discrete spaces is discrete, and a product of indiscrete spaces is indiscrete.
2. $(\mathbb{R}_{\text{usual}})^2 = \mathbb{R}^2_{\text{usual}}$.
3. $(\mathbb{R}_{\text{Sorgenfrey}})^2$ is an interesting space. This is the space generated by the basis of rectangles with their left and bottom edges closed. You will explore this space more through some Big List problems.

Chapter 2

Fuzzy Topological Spaces

In this chapter, we examine the concept of fuzzy topology and We discuss fundamental notions such as neighborhoods, closed sets, interior, and more, while exploring the essential properties of fuzzy topology.

2.1 Definitions and Examples

In this section, we recall the notion of fuzzy topology.

Definition 2.1. [2] A **fuzzy topological space** is a pair (X, τ) consisting of a non-empty set X and a family τ of fuzzy sets in X satisfying the following conditions:

(1) $\emptyset, X \in \tau$.

(2) If $\{A_i : i \in \Omega\}$ is an arbitrary family of fuzzy sets in τ, then $\bigcup_{i\in\Omega} A_i \in \tau$.

(3) If $A, B \in \tau$, then $A \cap B \in \tau$.

The family τ is called a **fuzzy topology** on X, and the pair (X, τ) is termed a **fuzzy topological space** (FTS). Members of τ are called τ-open fuzzy sets (OFS). A fuzzy set is τ-closed (CFS) if its complement is τ-open. Henceforth, we refer to τ-open (resp. τ-closed) sets simply as **open** (resp. **closed**). Analogous to classical topologies, the **indiscrete fuzzy topology** contains only $\emptyset$ and X, while the **discrete fuzzy topology** includes all fuzzy sets. A fuzzy topology T is **coarser** than τ if $T \subset \tau$.

Definition 2.2. [12] Let X be a non empty set and $\tau : I^X \to I$ be a mapping satisfies the following condition:

1. $\tau(\emptyset) = \tau(X) = 1$,
2. for every $A, B \in I^X$, then $\tau(A) \wedge \tau(B) \leq \tau(A \cap B)$,
3. for any $A_i \in I^X$, $\forall i \in I$, then $\wedge_i \tau(A_i) \leq \tau(\sqcup_i A_i)$.

(X, τ) is called a fuzzy topological space in Šostak's sense, $\tau(A)$ is called degree of openness of the fuzzy set A.

Definition 2.3. Let (X, τ) be a fuzzy topological space.

(1) For each element of (X, τ) is called open.

(2) A fuzzy set is closed if and only if its complement is a fuzzy open set.

Example 2.1. (1) Let $X = \{a, b\}$ and A be a fuzzy subset on X given by $A = \{(a, 0.5), \langle b, 0.8\rangle\}$. Then $\tau = \{\emptyset, A, X\}$ is a fuzzy topology and (X, τ) is a fuzzy topological space. Here, A is a fuzzy open set, while $\emptyset$ and X are both open and closed fuzzy sets.

Example 2.2. Let $X = [0, 1]$ and let $k \in (0, 1]$. Consider the function:

$$f_k(x, y) = \begin{cases} 3kx & \text{if } x \in [0, \frac{1}{2}]; \\ 3k(1 - x) & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

The family $\tau = \{f_k : 0 < k \leq 1\} \cup \{\emptyset, X\}$ forms a fuzzy topology, making (X, τ) a fuzzy topological space.

Definition 2.4. [3] A fuzzy set U in a fuzzy topological space (X, τ) is called a neighborhood of a fuzzy set A if and only if there exists an open fuzzy set O such that $A \subset O \subset U$. This definition focuses on neighborhoods of fuzzy sets rather than individual points.

Example 2.3. (1) Let $X = \{1, 2, 3, 4\}$ with $\tau = \{\emptyset, X, \{1\}, \{3\}, \{1, 3\}, \{2, 3\}\}$. Then:

$$N(1) = \{\{1\}, \{1, 3\}, \{1, 4\}, X\}; \quad N(3) = \{\{3\}, \{1, 3\}, \{2, 3\}, X\}.$$

(2) Let $X = \{p, q, r, s\}$ with $\tau = \{\emptyset, X, \{p\}, \{p, r\}, \{p, s\}, \{p, r, s\}\}$. Then:

$$N(p) = \{\{p\}, \{p, q\}, \{p, r\}, \{p, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, X\};$$

$$N(q) = \{X\};$$

$$N(r) = \{\{p, r\}, \{p, r, q\}, \{p, r, s\}\};$$

$$N(s) = \{\{p, s\}, \{p, s, q\}, \{p, s, r\}, X\}.$$

Theorem 2.1. [3] A fuzzy set A is open if and only if it is a neighborhood of every fuzzy set B contained within it.

($\Leftarrow$) Since $A \subset A$, there exists an open fuzzy set O such that $A \subset O \subset A$. Hence, $A = O$ and A is open.

The neighborhood system of a fuzzy set is the family of all neighborhoods of the fuzzy set.

Theorem 2.2. [3] If U is the neighborhood system of a fuzzy set, then the finite intersections of members of U belong to U, and each fuzzy set which contains a member of U belongs to U.

Proof. Let R and S be neighborhoods of a fuzzy set A. By definition, there exist open neighborhoods R_0 and S_0 such that $R_0 \subseteq R$ and $S_0 \subseteq S$. Since the intersection of two open sets is also open, $R_0 \cap S_0$ is an open neighborhood of A. Furthermore, since $R_0 \cap S_0 \subseteq R \cap S$, it follows that $R \cap S$ is a neighborhood of A.

Thus, the intersection of two (and, by induction, any finite number of) neighborhoods of A is also a neighborhood of A. Consequently, if a fuzzy set R contains a neighborhood of A, it must also contain an open neighborhood of A, implying that R itself is a neighborhood of A. □

Definition 2.5. [3] Let A and B be fuzzy sets in a FTS (X, τ), and let $B \subset A$. Then B is called an interior fuzzy set of A if and only if A is a neighborhood of B. The union of all interior fuzzy sets of A is called the interior of A and is denoted by A^0.

Definition 2.6. [9] Let (X, τ) be a fuzzy topological space. A is a fuzzy subset of X.

The closure of A is a fuzzy set $\overline{A}$ defined by:

$$\overline{A} = \left\{ \left\langle x, \max_{x \in X} \mu_A(x) \right\rangle \mid x \in X \right\}.$$

Example 2.4. [9] Let A and B be two fuzzy sets on $X = \mathbb{R}$:

$$A(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{3}; \\ 3x - 1 & \text{if } \frac{1}{3} \leq x \leq 1. \end{cases}$$

Let $B(x)$ be a fuzzy set on $X = \mathbb{R}$ defined as:

$$B(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{5}; \\ -5x + 2 & \text{if } \frac{1}{5} \leq x \leq \frac{2}{5}; \\ 0 & \text{if } \frac{2}{5} \leq x \leq 1. \end{cases}$$

And the curve of $A \cup B$,Then, the collection $\tau = \{\emptyset, A, B, A \cup B, X\}$ forms a fuzzy topology on X, with the following properties:

$$\overline{A} = B^c, \quad \overline{B} = A^c, \quad (A^c)^0 = B, \quad (B^c)^0 = A, \quad \text{and} \quad ((A \cup B)^c)^0 = \emptyset.$$

Theorem 2.3. [3] Let A be a fuzzy set in a fuzzy topological space (FTS) (X, τ). Then A^0 is open and is the largest open fuzzy set contained on A. The fuzzy set A is open if and only if $A = A^0$.

Proof. By Definition 1.4, A^0 is itself an interior fuzzy set of A. Hence, there exists an open fuzzy set O such that $A^0 \subset O \subset A$. But O is an interior fuzzy set of A. Hence, $O \subset A^0$. Therefore $A^0 = O$. Thus, A^0 is open and is the largest open fuzzy set contained in A. If A is open, then $A \subset A^0$, for A is an interior fuzzy set of A. Hence, $A = A^0$. The converse is obviously true. □

Definition 2.7. [11] Let (X, τ) be a fuzzy topological space. Then a subfamily B of τ is called a *base* for τ if every member of τ can be written as a union of members of B.

Definition 2.8. [11] Let (X, τ) be a fuzzy topological space. Then a subfamily S of τ is called a *subbase* for τ if the family of finite intersections of its members forms a base for

τ.

Definition 2.9. [14] A fuzzy topology τ is said to be *generated* by a subfamily S of fuzzy sets on X if every member of τ is a union of finite intersections of members of S.

Example 2.5.

1. Let $X = \mathbb{R}$. Then (X, τ) is a fuzzy topological space, and $B = \{(a, b] : a, b \in \mathbb{R} \text{ and } a < b\}$. Therefore, there exists a unique fuzzy topology τ for which B is a base.

2. Let $X = \mathbb{R}$ have the usual fuzzy topology. Then $S = \{(-\infty, b] : b \in \mathbb{R}\} \cup \{[a, \infty) : a \in \mathbb{R}\}$ is a subbase but not a base.

2.2 Fundamental properties of fuzzy topological spaces

In this section, we explore the fundamental properties of fuzzy topological spaces.

Definition 2.10. [3] Let f be a function from X to Y. Let B be a fuzzy set on Y with membership function $\mu_B(y)$. Then the inverse of B, written as $f^{-1}[B]$, is a fuzzy set on X whose membership function is defined by

$$\mu_{f^{-1}[B]}(x) = \mu_B(f(x)) \quad \text{for all } x \in X.$$

Conversely, let A be a fuzzy set in X with membership function μ_A. The image of A, written as $f[A]$, is a fuzzy set on Y whose membership function is given by

$$\mu_{f[A]}(y) = \begin{cases} \sup\{\mu_A(z) \mid z \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for all $y \in Y$, where $f^{-1}(y) = \{x \mid f(x) = y\}$.

Theorem 2.4. [3] Let X and Y be two fuzzy topological spaces, and let $f : X \to Y$ be a function. Let A, A' be two fuzzy subsets of X, and let B, B' be two fuzzy subsets of Y. Then:

1. $f^{-1}(B') = (f^{-1}(B))'$ for any fuzzy set B on Y.
2. $(f(A))' \subseteq f(A')$ for any fuzzy set A on X.
3. If $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$, where B_1, B_2 are fuzzy sets on Y.
4. If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$, where A_1, A_2 are fuzzy sets on X.
5. $f(f^{-1}(B)) \subseteq B$ for any fuzzy set B on Y.
6. $A \subseteq f^{-1}(f(A))$ for any fuzzy set A on X.
7. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then

$$(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$$

for any fuzzy set C on Z, where $g \circ f$ is the composition of g and f.

Proof of (1). For any $x \in X$,

$$\begin{aligned}\mu_{f^{-1}[B']}(x) &= \mu_{B'}(f(x)) \\ &= 1 - \mu_B(f(x)) \\ &= 1 - \mu_{f^{-1}[B]}(x) \\ &= \mu_{(f^{-1}[B])'}(x).\end{aligned}$$

(2) For any $y \in Y$, if $f^{-1}[y]$ is not empty, then

$$\mu_{f[A']}(y) = \sup_{z \in f^{-1}[y]} \{\mu_{A'}(z)\} = \sup_{z \in f^{-1}[y]} \{1 - \mu_A(z)\} = 1 - \inf_{z \in f^{-1}[y]} \{\mu_A(z)\},$$

and

$$\mu_{[f[A]]'}(y) = 1 - \mu_{f[A']}(y) = 1 - \sup_{z \in f^{-1}[y]} \{\mu_A(z)\}.$$

Hence,

$$\mu_{f[A']}(y) \geq \mu_{[f[A]]'}(y).$$

(3) For any $x \in X$,

$$\mu_{f^{-1}[B_1]}(x) = \mu_{B_1}(f(x))$$

and

$$\mu_{f^{-1}[B_2]}(x) = \mu_{B_2}(f(x)) \quad \text{for any } x \in X.$$

Since $B_1 \subset B_2$,

$$\mu_{f^{-1}[B_1]}(x) \leq \mu_{f^{-1}[B_2]}(x) \quad \text{for any } x \in X.$$

Hence,

$$f^{-1}[B_1] \subset f^{-1}[B_2].$$

(4)

$$\mu_{f[A_1]}(y) = \sup_{z \in f^{-1}[y]} \{\mu_{A_1}(z)\} \quad \text{and} \quad \mu_{f[A_2]}(y) = \sup_{z \in f^{-1}[y]} \{\mu_{A_2}(z)\}.$$

Therefore, $A_1 \subset A_2$ implies

$$\mu_{f[A_1]}(y) \leq \mu_{f[A_2]}(y) \quad \text{for any } y \in Y.$$

Since

$$f[A_1] \subset f[A_2].$$

(5) If $f^{-1}[y]$ is not empty,

$$\mu_{f[f^{-1}[B]]}(y) = \sup_{z \in f^{-1}[y]} \{\mu_{f^{-1}[B]}(z)\} = \sup_{z \in f^{-1}[y]} \{\mu_B(f(z))\} = \mu_B(y).$$

If $f^{-1}[y]$ is empty,

$$\mu_{f[f^{-1}[B]]}(y) = 0.$$

Therefore,

$$\mu_{f[f^{-1}[B]]}(y) \leq \mu_B(y) \quad \text{for any } y \in Y.$$

(6) For any $x \in X$,

$$\mu_{f^{-1}[f[A]]}(x) = \mu_{f[A]}[f(x)] = \sup_{z \in f^{-1}[f(x)]} \{\mu_A(z)\} \geq \mu_A(x).$$

(7) For any $x \in X$,

$$\mu_{(g \circ f)^{-1}[C]}(x) = \mu_C[g \circ f(x)] = \mu_C[g[f(x)]] = \mu_{g^{-1}[C]}[f(x)] = \mu_{f^{-1}[g^{-1}[C]]}(x).$$

□

Definition 2.11. [3] A function f from a FTS (X, τ_1) to a FTS (Y, τ_2) is *fuzzy continuous* if and only if the inverse of each τ_2-open fuzzy set is τ_1-open set.

Clearly, if f is a fuzzy continuous function from X to Y and g is a fuzzy continuous function from Y to Z, then the composition $g \circ f$ is a fuzzy continuous function from X to Z, for $(g \circ f)^{-1}[V] = f^{-1}[g^{-1}[V]]$ for each fuzzy set V on Z. Using the fuzzy continuity of g and f, it follows that if V is open, so is $(g \circ f)^{-1}[V]$.

Theorem 2.5. [3] If X and Y are two fuzzy topological spaces, and f is a function from

X to Y, then the conditions below are related as follows: (a) and (b) are equivalent; (c) and (d) are equivalent; (a) implies (c); and (d) implies (e).

(a) The function f is fuzzy continuous.

(b) The inverse of every closed fuzzy set is closed.

(c) For each fuzzy set A on X, the inverse of every nbhd of $f[A]$ is a nbhd of A.

(d) For each fuzzy set A on X and each nbhd V of $f[A]$, there is a nbhd W of A such that $f[W] \subset V$.

(e) For each sequence of fuzzy sets $\{A_n, n = 1, 2, \ldots\}$ on X which converges to a fuzzy set A in X, the sequence $\{f[A_n], n = 1, 2, \ldots\}$ converges to $f[A]$.

Proof. (a) $\Leftrightarrow$ (b): This is an immediate consequence of the fact that $f^{-1}[B'] = \{f^{-1}[B]\}'$ for every fuzzy set B on Y.

(a) $\Rightarrow$ (c): If f is fuzzy continuous, A is a fuzzy set on X, and V is a nbhd of $f[A]$, then V contains an open nbhd W of $f[A]$. Since $f[A] \subset W \subset V$, we have $f^{-1}[f[A]] \subset f^{-1}[W] \subset f^{-1}[V]$. But $A \subset f^{-1}[f[A]]$ and $f^{-1}[W]$ is open. Thus, $f^{-1}[V]$ is a nbhd of A.

(c) $\Rightarrow$ (d): Since $f^{-1}[V]$ is a nbhd of A, we have $f[W] = f[f^{-1}[V]] \subset V$, wherever $W = f^{-1}[V]$.

(d) $\Rightarrow$ (c): Suppose V be a nbhd of $f[A]$. Then there is a nbhd W of A such that $f[W] \subset V$. Hence, $f^{-1}[f[W]] \subset f^{-1}[V]$. Moreover, since $W \subset f^{-1}[f[W]]$, $f^{-1}[V]$ is a nbhd of A.

(d) $\Rightarrow$ (e) If V is a nbhd of $f[A]$, there is a nbhd W of A such that $f[W] \subset V$. Since $\{A_n, n = 1, 2, \ldots\}$ is eventually contained in W, i.e., there is an m such that for $n \geq m$, $A_n \subset W$, we have $f[A_n] \subset f[W] \subset V$ for $n \geq m$. Therefore $\{f[A_n], n = 1, 2, \ldots\}$ converges to $f[A]$.

A fuzzy homeomorphism is a fuzzy continuous one-to-one map of a FTS X onto a FTS Y such that the inverse of the map is also fuzzy continuous. If there exists a fuzzy homeomorphism of one fuzzy space onto another, the two fuzzy spaces are said to be F-homeomorphic and each is a fuzzy homeomorph of the other. Two FTS's are topologically fuzzy equivalent if and only if they are F-homeomorphic. □

2.3 Compact fuzzy topological space

In this section, we study the compactness properties for fuzzy topological spaces.

Definition 2.12. [2, 3]

A family $\mathcal{A}$ of fuzzy sets is a cover of a fuzzy set B if and only if $B \subset \bigcup\{A \mid A \in \mathcal{A}\}$. It is an open cover if and only if each member of $\mathcal{A}$ is an open fuzzy set. A subcover of $\mathcal{A}$ is a subfamily of $\mathcal{A}$ which is also a cover.

Example 2.6. Let $C = \{(-n, n) \mid n \in \mathbb{N}\}$ and $C' = \{(-3n, 3n) \mid n \in \mathbb{N}\}$. Both are open covers, but C' is a subcover of C.

Definition 2.13. [3] A FTS (X, τ) is compact if and only if each open cover has a finite subcover.

Example 2.7. (i) Let B_x be a fuzzy set with continuous membership function μ_{B_x} such that

$$\mu_{B_x}(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the family $\{B_x\}$, $x \in X$ forms an open cover of (X, τ). Clearly, it contains no finite subcover if X is not a finite set, hence (X, τ) is not compact.

(ii) Let $\{V_n\}$, $n = 1, 2, \ldots$, be a countable family of disjoint subsets on X such that their union is X. Consider the family of fuzzy sets $\{B_n\}$, $n = 1, 2, \ldots$, with continuous membership functions such that

$$\mu_{B_n}(y) = \begin{cases} 1 & \text{if } y \in V_n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\{B_n\}$, $n = 1, 2, \ldots$, is a countable open cover of (X, τ). If all V_n's are nonempty, then $\{B_n\}$ has no finite subcover.

Definition 2.14. [3] A family $\mathcal{A}$ of fuzzy sets has the finite intersection property if and only if the intersection of the members of each finite subfamily of $\mathcal{A}$ is nonempty.

Theorem 2.6. [3] A FTS is compact if and only if each family of closed fuzzy sets which has the finite intersection property has a nonempty intersection.

Proof. If $\mathcal{A}$ is a family of fuzzy sets in a FTS (X, τ), then $\mathcal{A}$ is a cover of X if and only if $\bigcup\{A \mid A \in \mathcal{A}\} = X$, or if and only if $(\bigcup\{A \mid A \in \mathcal{A}\})' = X' = \emptyset$, or if and only if $\cap\{A' \mid A \in \mathcal{A}\} = \emptyset$ by the De Morgan's laws. Therefore, the fuzzy space X is compact if and only if each family of open fuzzy sets on X such that no finite subfamily covers X, fails to be a cover, and this is true if and only if each family of closed fuzzy sets which possesses the finite intersection property has a nonempty intersection. $\square$

Theorem 2.7. [3] Let f be a fuzzy continuous function carrying the compact FTS (X) onto the FTS (Y). Then Y is compact.

Proof. Let $\mathcal{B}$ be an open cover on Y. Then,

$$\mu_{\underset{B\in\mathcal{B}}{\cup} f^{-1}[B]}(x) = \sup_{B\in\mathcal{B}}\{\mu_{f^{-1}[B]}(x)\} = \sup_{B\in\mathcal{B}}\{\mu_B(f(x))\} = 1 \quad \text{for any } x \in X,$$

the family of all fuzzy sets of the form $f^{-1}[B]$, for B in $\mathcal{B}$, is an open cover of X which has a finite subcover. However, if f is onto, then it is easily seen that $f[f^{-1}[B]] = B$ for any fuzzy set B on Y. Thus, the family of images of members of the subcover is a finite subfamily of $\mathcal{B}$ which covers Y and consequently Y is compact. $\square$

Chapter 3

Number of Fuzzy Clopen Sets in Fuzzy Topological Spaces

In this chapter, I discuss clopen sets in both topological and fuzzy topological spaces. I begin with clopen sets in classical topology and their role in understanding space structure. Then, I explore open sets in fuzzy topology and how they differ from classical open sets. Finally, I examine clopen sets in fuzzy topology and their significance in fuzzy connectedness.

3.1 Number of Clopen Topological Spaces

3.1.1 Observations

[1] In a topological space (M), all elements of M are defined to be open sets, and their complements $V = M \setminus V$ are defined to be closed sets (V^c). A natural question is whether there exists a set that is both open and closed. An example of such sets can be seen in Figure 3.1. The aim of this report is to investigate these kinds of sets. Firstly, for the sake of brevity, let us call sets that are both open and closed *clopen sets*. In other words, a set $V \subseteq M$ is clopen if V is open and V is closed. Observe that p_1 and $p_2 \cup p_3$ are clopen sets in Figure 1. Only by considering the definitions and some simple cases, one can make the following observations.

Observation 1 For a topological space $(M, \mathcal{T})$:

1. $\emptyset$ and M are always clopen. This follows from the fact that, by definition, $\emptyset$ and M

are open sets, and they are the complement of each other (so they are closed sets too, which satisfies the definition of a clopen set).

2. Every subset of M is clopen if and only if $\mathcal{T}$ contains all subsets of M.

   Proof: If every subset of M is contained in $\mathcal{T}$, then every subset of M is open (by definition). Also, since the complement of any subset of M is another subset of M, the complement is also in $\mathcal{T}$, hence open. Therefore, every subset is also closed. Thus, if every subset of M is in $\mathcal{T}$, then every subset of M is clopen.

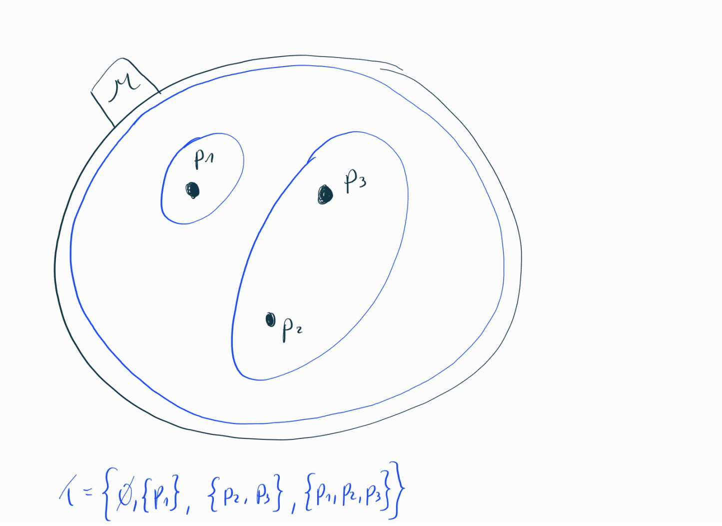

Figure 3.1: Open and closed sets. In this figure, p_1 is an open set (since it is in $\mathcal{T}$), and its complement $p_2 \cup p_3$ is a closed set. Observe that since $p_2 \cup p_3$ is also contained in $\mathcal{T}$, it is open, and therefore p_1 is also closed. Thus, both p_1 and $p_2 \cup p_3$ are open and closed; i.e., they are clopen sets.

Now suppose that all subsets of M are clopen, but not all subsets of M are contained in $\mathcal{T}$. This means that there exists $V \subseteq M$ such that $V \notin \mathcal{T}$. But that would mean V is not an open set, thus V cannot be clopen. Hence, if $\mathcal{T}$ contains only clopen sets, then all subsets of M must be contained in $\mathcal{T}$ (proof by contradiction).

3. If V is clopen, then $\mathcal{C} = \{V, V^c\}$ is an open cover of M. This follows from the fact that any element of M must lie either in V or in its complement V^c (by definition), and that both V and V^c are open sets. This is precisely the meaning of V being clopen.

3.1.2 Hausdorff property

[1] Next, let us investigate the relationship between clopen sets and the Hausdorff property. Recall that a topological space $(M, \mathcal{T})$ is *Hausdorff* if for any $p_1, p_2 \in M$ with $p_1 \neq p_2$, there exist $V_1, V_2 \in \mathcal{T}$ such that $p_1 \in V_1$, $p_2 \in V_2$, and $V_1 \cap V_2 = \emptyset$.

	contains clopen sets (other than $\emptyset$ and $\mathcal{M}$)	does not contain clopen sets (other than $\emptyset$ and $\mathcal{M}$)
Hausdorff	$\mathcal{M} = \{p_1, p_2\}$ $\tau = \{\mathcal{M}, \emptyset, \{p_1\}, \{p_2\}\}$	?
not Hausdorff	$\mathcal{M} = \{p_1, p_2, p_3\}$ $\tau = \{\mathcal{M}, \emptyset, \{p_1\}, \{p_2, p_3\}\}$	$\mathcal{M} = \{p_1, p_2, p_3\}$ $\tau = \{\mathcal{M}, \emptyset, \{p_1\}, \{p_2\}, \{p_1, p_2\}\}$

Table 3.1: Clopen sets and Hausdorff property.

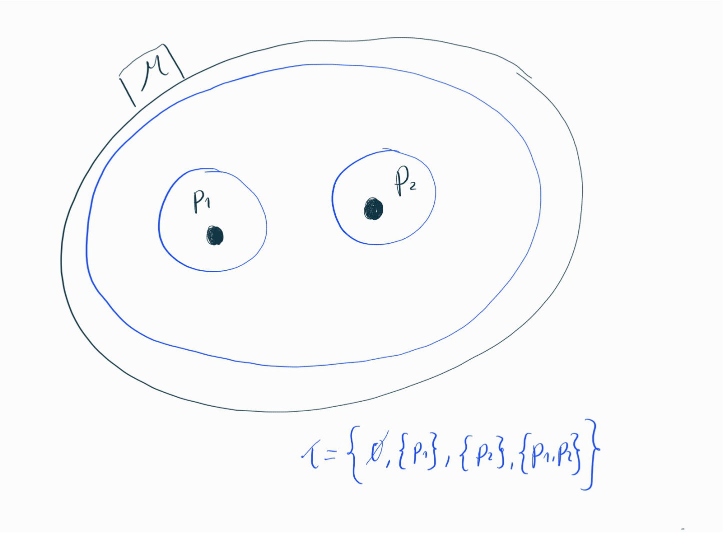

Figure 3.2: A Hausdorff topological space containing clopen sets (other than $\emptyset$ and $\mathcal{M}$). Observe that it is possible to separate p_1 and p_2 using open sets (so it is Hausdorff). Also, observe that $\{p_1\}$ and $\{p_2\}$ are clopen.

Table 1 provides one-to-one examples for a Hausdorff topological space containing clopen sets (other than $\emptyset$ and M), a non-Hausdorff topological space containing clopen sets (other than $\emptyset$ and M), and a non-Hausdorff topological space not containing clopen sets (other than $\emptyset$ and M). An explanation for each case can be seen in Figures 2, 3, and 4. However, one can notice that for the entry of a Hausdorff topological.

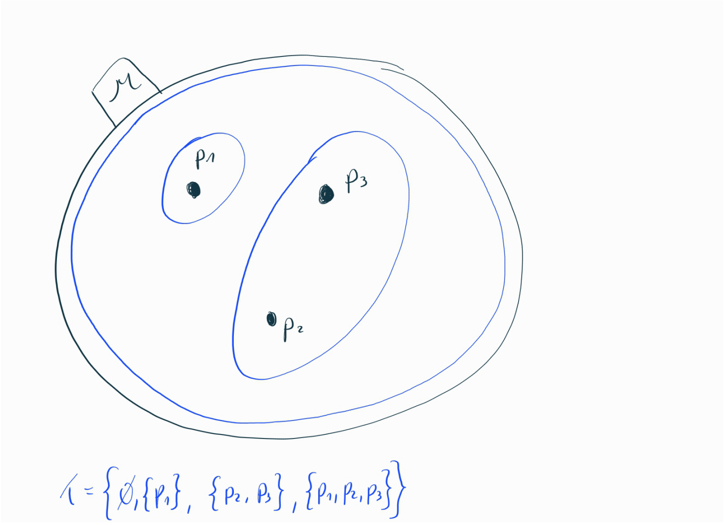

Figure 3.3: A non-Hausdorff topological space containing clopen sets (other than $\emptyset$ and M). Observe that p_2 and p_3 cannot be separated using open sets (so it is not Hausdorff). Also, observe that p_1, p_2, and p_3 are clopen.

A space not containing clopen sets (other than $\emptyset$ and M) is not straightforward to provide an (easy) example. In fact, it turns out that if $M \neq \emptyset$, then no such set exists, which will follow from a following, stronger proposition. But, to prove that proposition, we need to prove the following lemma first.

Lemma 3.1. [1] Let (M) be a Hausdorff topological space. For any element $p \in M$, if the subset containing only this element is not an open set, then any finite subset of M containing p is not an open set either (i.e., for $p \in M$, if $p \in V \subset M$ with $V \neq \emptyset$, then V is not open).

Proof: I am going to use proof by mathematical induction. Let us assume that (M) is Hausdorff, and that for a $p \in M$, p is not an open set.

Next, let us check the subsets of M containing p. Let us start with the ones containing the least number of elements (the most basic cases):

- When the subset has one element: there is only one subset of M that has one element and contains p, namely $\{p\}$, which is not an open set by our basic assumption. No subset of M containing one element is open if it contains p.

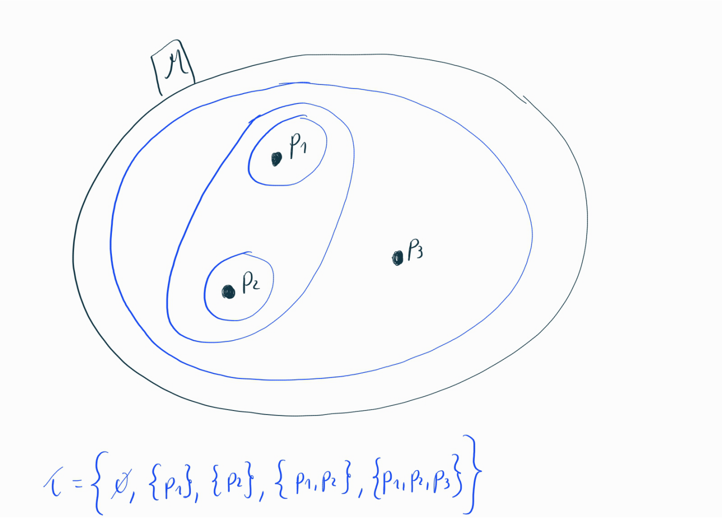

Figure 3.4: A non-Hausdorff topological space not containing clopen sets (other than $\emptyset$ and M). Observe that p_1 and p_3 cannot be separated using open sets (so it is not Hausdorff). Also, observe that there are no clopen sets in this topological space (except for $\emptyset$ and M): neither of the other open sets (p_1, p_2, $p_1 \cup p_2$) are closed, since their complements ($p_2 \cup p_3$, $p_1 \cup p_3$, p_3) are not open sets, thus they are not clopen.

When the subset has two elements: this is a set containing p and another random element of M, that is a set $\{p, p'\}$ with $p \in M$ and $p \neq p'$. Now let us assume that this can be an open set, that is $\{p, p'\}$ is open in M for which $p \in \{p, p'\}$. However, the topological space is Hausdorff with $p \in V$, $p' \in V'$, and $V \cap V' = \emptyset$. Then $p \in V \implies \{p, p'\} \cap V = \{p\}$, which is a contradiction, thus $\{p, p'\}$ cannot be open for any $p \in M$. No subset of M containing two elements is open if it contains p.

We have seen that the statement we want to prove works for subsets containing one or two elements. Next let us assume that it also holds for subsets containing no more than k elements, and see if it holds for any set containing $k+1$ elements. For this let us assume that the set $\{p, p_1, p_2, \dots, p_k\}$ is open and contains distinct elements (in other words, this is a set containing $k+1$ distinct elements that also contains p), but any proper subset of it containing p is not open. Since the topological space is Hausdorff, for $1 \leq i \leq k$, there exist open sets V, V_i with $p \in V$, $p_i \in V_i$ (where $p_i \neq p$), and $V \cap V_i = \emptyset$. The intersection

$$(\{p, p_1, p_2, \dots, p_k\} \setminus V) \cap \{p, p_1, p_2, \dots, p_{i-1}, p_{i+1}, \dots, p_k\}$$

is a proper subset of $\{p, p_1, p_2, \dots, p_k\}$. This implies $\{p, p_1, p_2, \dots, p_{i-1}, p_{i+1}, \dots, p_k\}$ is an open set, which is a contradiction, since any subset of $\{p, p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_k\}$ is a proper subset of $\{p, p_1, p_2, \ldots, p_k\}$ containing p (and no proper subset of $\{p, p_1, p_2, \ldots, p_k\}$ containing p is supposed to be open).

From these, by mathematical induction, one gets that if $p \in \bigcap \mathcal{V}$, then any finite subset of M containing p is not open. However, since p is an arbitrary element of M, this proof can be applied to any element of M, thus we have proven the statement we wanted to prove (i.e., for $p \in M$, if $p \in \bigcap \mathcal{V}$ and $\mathcal{V} \subset M$ with $|\mathcal{V}| < \infty$, then $\mathcal{V} \notin \mathcal{T}$).

Remark 1. The reason why we need the set $V \subset M$ to be finite in Lemma 3.1 is that the recursion (by which we prove that V is not an open set) can only be performed a finite number of times. This means that V can have an arbitrarily large number of elements, but not an infinite number of elements, in order for the proof to still work.

3.2 Number of Fuzzy Clopen Sets in Fuzzy Topological Spaces

Definition 3.1. [5] A *fuzzy point* p in a set X is a fuzzy set in X given by:

$$p(x) = \begin{cases} t & \text{for } x = x_p \ (0 < t < 1) \\ 0 & \text{for } x \neq x_p \end{cases}$$

where x_p is called the *support* of p and $p(x_p) = t$ is called the *value* of p. A *fuzzy crisp point* q in X is a fuzzy set in X given by:

$$q(x) = \begin{cases} 1 & \text{for } x = x_q \\ 0 & \text{for } x \neq x_q \end{cases}$$

remark 3.1. p is a fuzzy singleton if and only if p is either a fuzzy point or a fuzzy crisp point.

A fuzzy set λ is called *fuzzy clopen* if it is fuzzy open and fuzzy closed simultaneously. A *fuzzy crisp set* is a fuzzy set on X whose range is a subset of $\{0, 1\}$ (i.e., a characteristic function).

Recall that a topology T on a nonempty set X is a subset of $\mathcal{P}(X)$ that contains $\emptyset$ and X, and is closed under arbitrary unions and finite intersections. A topology on X is a sublattice of $\mathcal{P}(X)$ with maximum element X (denoted by 1) and minimum element $\emptyset$ (denoted by 0).

Definition 3.2. [5] A set $H \subseteq X$ is called *open* in (X, T) if $H \in T$. Complements of open sets are called *closed sets.* A set M is *clopen* in (X, T) if $\{M, M^c\} \subseteq T$. The collection of all (fuzzy) clopen sets in (X, T) is denoted by $\mathrm{Co}(X, T)$ or $\mathrm{Co}(X)$.

The cardinality of a set X is denoted by $\mathrm{card}(X)$. $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ denote the sets of natural, rational, and real numbers respectively. T_{ind}, T_{dis}, T_{Sor} denote the indiscrete, discrete, and Sorgenfrey topologies respectively.

We follow for the definitions of c_λ, $\mathrm{Co}\, M_\lambda = c_\lambda \cap \chi_M$ (where χ_M denotes the characteristic function of M), and fuzzy subspace topology.

Definition 3.3. [5] A fuzzy point p in X is said to *belong to* a fuzzy set λ in X (notation: $p \in \lambda$) if $p(x_p) < \lambda(x_p)$. Two fuzzy points p and q are *distinct* if their supports are distinct (i.e., $x_p \neq x_q$).

For an arbitrary fuzzy set λ on X, the ordinary set

$$\{\mu : \mu \text{ is a fuzzy set on } X \text{ and } \mu \subseteq \lambda\}$$

is called the *fuzzy power class* of λ, denoted by $\mathcal{P}_f(\lambda)$.

A fuzzy set λ on X is a function from X to $[0, 1]$. The *complement* c_λ is defined by $c_\lambda(x) = 1 - \lambda(x)$. For $r \in [0, 1]$, r denotes the constant fuzzy set $r(x) = r$ for all $x \in X$.

Definition 3.4. [5] A *fuzzy topology* τ on X is a collection of fuzzy sets containing 0, 1, and closed under finite intersections and arbitrary unions. Elements of τ are called *fuzzy open sets*; their complements are *fuzzy closed sets*.

In any topological space (X, T), the complement of a clopen set is clopen. Thus the number of clopen sets must be even if finite. However, this number cannot be 6. Fora proved the following results:

Theorem 3.1. [5] Let (X, T) be a topological space such that $CO(X, T)$ is finite. Then

$$\text{card } CO(X, T) = 2^k \text{ for some } k \in \mathbb{N}.$$

Theorem 3.2. [5] Given a nonempty set X and $n \geq 1$ is any cardinal number. If $|X| \geq n$, then there exists a topology T on X such that

$$\text{card } CO(X, T) = 2^n.$$

One application of (fuzzy) clopen sets is that they can be used to describe (fuzzy) connectedness. In particular, a (fuzzy) topological space (X, T) is (fuzzy) connected if and only if

$$\text{card } CO(X, T) = 2.$$

It is clear that in general $CO(X,T)$ need not be a topological space even if (X,T) is 0-dimensional (i.e., T has a base consisting of clopen sets). Indeed, $CO(\mathbb{R},T_{\text{Sorg}})$ is not a topology on $\mathbb{R}$ because $(0,1)$ is not a clopen set in $(\mathbb{R},T_{\text{Sorg}})$ although

$$(0,1)=\bigcup_{n\in\mathbb{N}}\left[\frac{1}{n},1\right).$$

Several researchers have enumerated the topologies on a finite set.Others have studied the number of open sets of finite topologies . Fora has studied the number of clopen sets of arbitrary topological space. He used an algebraic approach for his goal.

Enumeration of $\mathrm{CO}(X,T)$ for Fuzzy Topological Space (X,T)

The following result shows that the number of fuzzy clopen sets in a fuzzy topological space may be any natural number (odd as well as even) greater than 1.

Proposition 3.1. [5] For any ordinary nonempty set X and any natural number n, let $T=\left\{\frac{j}{n}:j=0,1,2,\ldots,n-1,n\right\}$. Then (X,T) is a fuzzy topological space with $\mathrm{CO}(X,T)=T$ of cardinality $n+1\geq 2$, which may be even as well as odd.

It is clear that the fuzzy constant set $\lambda=\frac{j}{n}$ is clopen in (X,T) because $\lambda^c=\frac{n-j}{n}\in T$ for $0\leq j\leq n$.

Proposition 3.2. [5] If (X,T) is a fuzzy disconnected space, then there exist nontrivial (i.e., $0,1\neq$) fuzzy clopen sets λ, η on X such that $0.5\leq\eta\leq\lambda$.

Proof: Since (X,T) is a fuzzy disconnected space, there exists a nontrivial fuzzy clopen set μ in X. Now taking $\eta=\mu\cap\mu^c$ and $\lambda=\mu\cup\mu^c$ will complete the proof.

In the above Proposition 2.2, there is no guarantee that λ,η are nonconstant. Indeed, taking $X=[0,1]$ and

$$T=\{0,1,\mu,\mu^c,0.25,0.75\}$$

where

$$\mu(x)=\begin{cases}0.25, & \text{for } x\in\mathbb{Q}\cap X,\\ 0.75, & \text{for } x\in X\setminus\mathbb{Q},\end{cases}$$

then $\eta = \mu \cap \mu^c = 0.25$ and $\lambda = \mu \cup \mu^c = 0.75$. Notice that (X, T) has six fuzzy clopen sets, two of which are nonconstant.

The following result shows that $\mathrm{CO}(X, T)$ can be even denumerable as well as having cardinality the continuum c.

Proposition 3.3. [5] For any ordinary nonempty set X, let

(i) $T = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$,

(ii) $\tau = \{c : 0 \leq c \leq 1\}$.

Then (X, T) and (X, τ) are fuzzy topological spaces satisfying the conclusion, i.e., $\mathrm{CO}(X, T) = T$ is denumerable and $\mathrm{card}(\mathrm{CO}(X, \tau)) = \mathrm{card}(\tau) = c$.

The following result gives an upper bound for $\mathrm{card}\,\mathrm{CO}(X, T)$ for fuzzy topological space (X, T). Of course, its lower bound is 2. Its value is exactly 2 for connected fuzzy topological space (X, T).

Proposition 3.4. [5] Let X be an infinite set and T be a fuzzy topology on X. Then

$$2 \leq \mathrm{card}(\mathrm{CO}(X, T)) \leq 2^{|X|}.$$

Notice that we have used the fact that $2^{\aleph_0} = 2^{|X|} = c$ for X an infinite set.

The following result shows that the number of crisp fuzzy clopen sets obeys the order obtained by Fora [5]. Namely, we have the following convergence result with ordinary topological spaces.

Theorem 3.3. [5] Let (X, T) be a fuzzy topological space such that $\mathrm{CO}(X, T)$ is finite. Then the collection of all crisp fuzzy clopen sets in (X, T) has cardinality 2^n for some natural number n.

Proof. Denote the collection of all crisp fuzzy clopen sets in (X, T) by $K(X, T)$. Then

$$K(X, T) = \{\mu : \mu \in \mathrm{CO}(X, T) \text{ and } \mu = \chi_A \text{ for some } A \subseteq X\}.$$

For $\alpha, \beta \in K(X,T)$, define the binary operation $*$ by

$$\alpha * \beta = \chi_{A\Delta B}$$

where $A = \alpha^{-1}(\{1\})$, $B = \beta^{-1}(\{1\})$, and $A\Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference.

This operation makes $(K(X,T), *)$ an abelian group with the identity element 0. Moreover, for any $\alpha \in K(X,T)$, we have $\alpha * \alpha = 0$, i.e., each element is its own inverse. Thus applying the same technique as in Fora [5], we get

$$(K(X,T), *) \cong C_1 \times C_2 \times \cdots \times C_n$$

where $C_i \cong \mathbb{Z}_2$ for $i = 1, 2, \ldots, n$. Hence,

$$\operatorname{card} K(X,T) = \operatorname{card}(C_1 \times C_2 \times \cdots \times C_n) = 2^n.$$

Another topological proof. Denote the collection of all crisp fuzzy clopen sets in (X,T) by $K(X,T)$. Then

$$K(X,T) = \{\mu : \mu \in \mathrm{CO}(X,T) \text{ and } \mu = \chi_A \text{ for some } A \subseteq X\}.$$

Define

$$\tau = \left\{\mu^{-1}(\{1\}) : \mu \in K(X,T)\right\}.$$

Then τ is indeed a topology on X because $\mathrm{CO}(X,T)$ is finite. Moreover, $\mathrm{CO}(X,\tau)$ is finite. Hence by Theorem 1.1 (see Fora [5]),

$$\operatorname{card}(\mathrm{CO}(X,\tau)) = 2^n \text{ for some } n \in \mathbb{N}.$$

Henceforth,

$$\operatorname{card}(K(X,T)) = \operatorname{card}(\mathrm{CO}(X,\tau)) = 2^n.$$

Example 3.1 (Indiscrete Fuzzy Topology)**.** Let X be any set, and let $\tau = \{0_X, 1_X\}$. The only clopen sets are 0_X and 1_X. Thus, there are exactly **2** fuzzy clopen sets.

Example 3.2 (Discrete Fuzzy Topology)**.** Let $\tau = [0,1]^X$. Every fuzzy set is open, so every set is also closed (since A^c is open). Hence, **all** fuzzy subsets are clopen, resulting in uncountably many clopen sets.

Example 3.3 (Sierpiński-Type Fuzzy Topology)**.** Let $X = \{x\}$ and $\tau = \{0_X, 1_X, A\}$, where $A(x) = 0.5$. Here, $A^c = 1 - A = A$, so A is clopen. Thus, τ has **3** clopen sets: 0_X, 1_X, and A.

Example 3.4 (Clopen Sets from Generators)**.** Let $X = \{x, y\}$ and define:

$$A(x) = 0.8, \quad A(y) = 0.6; \quad B = A^c \quad (B(x) = 0.2,\, B(y) = 0.4).$$

Let τ be the topology generated by A and B. Then:

- $A \wedge B = \min(A, B)$ is clopen (with values 0.2 at x, 0.4 at y),
- $A \vee B = \max(A, B)$ is clopen (equal to A),
- 0_X and 1_X are clopen.

This topology contains **at least 4** distinct fuzzy clopen sets.

Example 3.5 (Counting Fuzzy Clopen Sets in Finite Fuzzy Topological Spaces)**.** Let $X = \{x, y\}$ and $\tau = \{0_X, 1_X, A, A^c\}$, where $A(x) = 0.3$, $A(y) = 0.7$. Here, A and A^c are clopen. Thus, there are **4** clopen sets: $0_X, 1_X, A, A^c$.

Example 3.6 (Discrete vs. Indiscrete Fuzzy Topologies)**.** Let $X = \{x\}$ and $\tau = \{0_X, 1_X, A\}$ where $A(x) = 0.5$. The 0.5-level set $A_{0.5} = \{x\}$, which is crisp clopen. This aligns with A being fuzzy clopen.

Example 3.7 (Number of Clopen Sets Under Different t-Norms)**.** Consider $X = \{x\}$ with τ closed under product t-norm. Let $A(x) = 0.5$. Then $A * A^c = 0.5 \times 0.5 = 0.25 \neq 0_X$. Thus, A is *not* clopen here, unlike in the minimum t-norm case.

Example 3.8 (Characterization of Spaces with Finite Clopen Sets)**.** Let $X = \{x, y\}$ and τ be generated by clopen sets A and A^c, where $A(x) = 0.4$, $A(y) = 0.6$. The clopen sets are $0_X, 1_X, A, A^c$ (finite).

Conclusion

This thesis presents a comprehensive exploration of topological structures, beginning with the foundational concepts of classical topology and advancing toward the nuanced and flexible framework of fuzzy topology. In Chapter 1, we established the groundwork by discussing key topological concepts such as open and closed sets, interior and closure, continuity, homeomorphisms, and different types of topologies. These classical ideas form the basis for understanding more complex and abstract structures.

Building upon this foundation, Chapter 2 introduced fuzzy topological spaces, extending classical topology to accommodate uncertainty and gradation. By allowing degrees of membership, fuzzy topology offers a powerful framework for modeling situations where classical binary logic proves insufficient. This chapter underscored the theoretical importance and practical relevance of fuzzy systems in addressing imprecision and ambiguity in various real-world contexts.

In Chapter 3, we focused on the concept of fuzzy clopen sets—sets that are simultaneously fuzzy open and fuzzy closed. We examined the number and structural properties of such sets.

Overall, this thesis demonstrates the progression from classical to fuzzy topology, highlighting both the theoretical advancements and the broader applicability of topological thinking. By bridging these domains, the study not only deepens our understanding of topological structures but also opens new avenues for research and application in areas involving uncertainty, such as decision theory, artificial intelligence, and systems analysis.

Bibliography

[1] Adam, ”Open and closed sets in topological spaces – Groups and their representations”, Report MATEFY. [2023]

[2] C.L. Chang, ”Fuzzy topological spaces”, *Journal of Mathematical Analysis and Applications*, 24, 182–190 (1968).

[3] C.L. Chang, ”Division of computer research and technology”, National Institutes of Health, Bethesda, Maryland, 24, 182–190 (1968).

[4] R. Coolman, ”What Is Topology?”, *Live Science*, June 23, 2015.

[5] A.A.A. Fora, ”The Number of Fuzzy Clopen Sets in Fuzzy Topological Spaces”, *Journal of Mathematical Sciences and Applications*, 5(1), 24–26 (2017).

[6] I. Khatchatourian, ”Closed sets and limit points”, MAT327 Lecture Notes, University of Toronto. [2019]

[7] J.R. Munkres, *Topology* (2nd ed.), Pearson (2000).

[8] Dr.MohammedJabbarHussein ,”Chapter One: Topological Spaces”, Lecture Notes, University of Mustansiriyah, January 9, 2019.

[9] N. Palaniappan, *Fuzzy Topology*, Narosa Publications, Harrow (2002).

[10] P. Parimal, ”Lecture 15: The subspace topology, closed sets”, Retrieved from https://ece.iisc.ac.in/ parimal/2015/proofs/lecture-15.pdf (2015).

[11] P. Pao-Ming, L. Ying-Ming, ”Fuzzy topology I. Neighborhood structure of a fuzzy point and Moore-Smith convergence”, *Journal of Mathematical Analysis and Applications*, 76, 571–599 (1980).

[12] A. Šostak, ”On a General Theory of Fuzzy Topological Spaces”, Proceedings of the 13th Winter School on Abstract Analysis, 379–390, 1985.

[13] J. Srivastava, ”Continuity and Homeomorphism”, Lecture material, Department of Mathematics and Statistics, DDU Gorakhpur University, India, [2020].

[14] R. Srivastava, S.N. Lal, A.K. Srivastava, ”Fuzzy T1-topological spaces”, *Journal of Mathematical Analysis and Applications*, 102(2), 442–448 (1984).

[15] I.Y. Vierya, ”Lecture 8: Product topology”, MAT327 Course Notes, University of Toronto. [2019]