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Décomposition d'entiers illimités et des termes d'ordre illimités des suites récurrentes linéaires

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Dedication

I dedicate this modest work to my dear parents who gave me the efforts and encouragements to continue my study and my research, to my family my sisters and my brothers, also I dedicate my work to my dear wife and my dear children.

To my uncles and my ants, and without forget my friends and my colleagues.

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Introduction

Nonstandard analysis is a mathematical tool among others, used today by a significant number of mathematicians. We use in this work the version which was developed thereafter by E. Nelson[10], [21].

Writing a "large" integer N as a product of distinct prime factors was always an interesting problem in the number theory literature. But since, this question is very delicate when N is an any given integer, A. Boudaoud proposed to factorize, instead of N, an integer in its small neighborhood[6]. Hence the natural framework of this idea is the nonstandard mathematics, because in such a language that we can use the words: small, large, etc. This idea of factorization is expressed by the model

$$N = (\text{standard}) + (\text{unlimited}) \times (\text{unlimited}). \tag{1}$$

In this sense, let $U_n = U_n(P,Q)$ and $V_n = V_n(P,Q)$ be the sequences

$$\begin{cases} U_0 = 0,\, U_1 = 1,\, U_n = PU_{n-1} - Q\cdot\, U_{n-2}, \text{ for } n \geq 2 \\ V_0 = 2,\, V_1 = P,\, V_n = PV_{n-1} - QV_{n-2}, \text{ for } n \geq 2 \end{cases} \tag{2}$$

where P, Q are non-zero integers and $D = P^2 - 4Q \neq 0$ is the discriminant of the characteristic polynomial of $X^2 - PX + Q$. The sequence (2) is called " Lucas sequences associated to the pair (P,Q)". A. Boudaoud [6] proved that if $D > 0$, then for any unlimited n, U_n and V_n differs by a limited integer from a product of two unlimited integers.. That is, U_n and V_n are representable according to the model (1). The problem of representing of any unlimited integer N, which is not an element of (2), as a sum of a standard integer s and of a product two unlimited integers ω_1 and ω_2 is still open.

Actually, the notion of representation in number theory is very vague and contains several kinds, we quote for example:

a) An integer $n > 1$ may be, by the fundamental theorem of arithmetic, presented as a product of prime numbers.

b) A positive integer n can be, under suitable conditions, represented as a sum of two squares, three squares, four squares, five squares [25], [26].

c) A positive integer n can be represented as a sum of positive integers (partition theory) [15], [12], [8]. And so on.

The aim of this thesis is to see what we have in the mathematical literature concerning the problem of representation (chapter 3). In addition, is to determine in some famous sequences of integers, the unlimited terms which can be written, that is, presented, according to the model described in (1).

Thus, the first chapter is devoted to basic notions forming environment and language of this thesis. That is, it recalls the necessary tools from nonstandard analysis and number theory.

The second chapter has as object the study of some famous sequences of integers such as Fibonacci sequence, generalized Fibonacci sequence, Horadam sequence, ... This study prepares the ground for chapter 4 since in which we study the representation of unlimited terms of theses sequences according to (1).

In the third chapter, we present some methods for writing an integer N like Cantor's development of an integer [23]. Also we recall the representation of a positive integer as a sum of distinct Fibonacci numbers [39], of distinct Lucas numbers[7], of two squares [23], ... Among our contribution, the representation of a positive integer as a sum of distinct tribonacci numbers[4].

In the last chapter and based on properties of the sequences cited in Chapter 2, we give several families of integers whose unlimited terms can be represented according to the model (1).

List of symbols

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denoted respectively the set of positive integers (or natural numbers), the set of integers, the set of rational numbers , the set of reels numbers and the set of complex numbers .

- We denote the integers by a, b, c, d, i, j, k, l, m, n, r, and s.
- The letters p and q usually denoted prime numbers.
- The sequence of prime numbers 2, 3, 5, 7,....will be denoted by p_1, p_2, p_3,....Given an integer $n \geq 2$, we write often $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_r^{\alpha_r}$ to denote its canonical representation as a product of primes $q_1 < q_2 < q_3 ... < q_r$, the exponents α_1, α_2, α_3,...α_r, are positive integers.

In the IST[(1)] theory we give the following notations

- $x \cong y$ for x is equivalent to y when $x - y$ is infinitesimal.
- $x \ll y$ for x is inferior and not equivalent to y .
- $|x| \cong +\infty$ (resp $x \cong +\infty$) for x is unlimited (resp x is unlimited and positive).
- $|x| \ll \infty$ (resp $x \ll +\infty$) for x is limited (resp x is limited or negative).
- $x \sim y$ for x and y are asymptotic i.e. $\dfrac{x}{y} \sim 1$.
- $x \gtrapprox y$ if $x > y$ and $x \cong y$.
- $x \gnapprox y$ if $x > y$ and $x \ncong y$.
- $x \lessapprox y$ if $x < y$ and $x \cong y$.
- $x \lnapprox y$ if $x < y$ and $x \ncong y$.

(1) Which is an abreviation of Internal Set Theory

Chapter 1

General concepts on number theory and nonstandard analysis

1.1 Overview on nonstandard analysis

Historically one finds that Leibniz, Euler and Cauchy are the first who began the use of infinitely small quantities. The absence of rigor surrounding the use of this notion found solution after an approach due to Abraham Robinson in 1961. In 1977 Edward Nelson provided another presentation of the non-standard analysis - called IST (Internal Set Theory) - based on ZFC to which is added a new unary predicate called "standard". The use of this predicate is governed by the following three axioms[21],[27],[10], where we use the abbreviation $\forall^{st}xF(x)$ for $[st(x) \implies F(x)]$ and $\exists^{st}xF(x)$ for $[st(x) \wedge F(x)]$.

1.1.1 Transfer principle (general)

For all standard formula $F(x, t_1, t_2, t_3, t, ...t_n)$ having no free variables other than x,t_1,t_2,t_3,t,...t_n, the following utterance is an axiom.

$$\forall^{st}\ t_1, t_2, t_3, t, ...t_n,\ \forall^{st}xF(x, t_1, t_2, t_3, t, ...t_n) \Rightarrow \forall x\ F(x, t_1, t_2, t_3, t, ...t_n) \qquad (T)$$

Similarly (equivalent formula)

$$\forall^{st}\ t_1, t_2, t_3, t, ...t_n,\ \exists x\ F(x, t_1, t_2, t_3, t, ...t_n) \Rightarrow \exists^{st}xF(x, t_1, t_2, t_3, t, ...t_n) \qquad (T)$$

1.1.2 Idealization principle

For any internal formula B, containing at least two free variables x and y the following statement is an axiom

$$\left(\forall^{st\ fini} Z,\ \exists X,\ \forall Y \in Z,\ B\left(X, Y\right)\right) \Leftrightarrow \left(\exists X_0, \forall^{st} y B\left(X_0, Y\right)\right). \qquad (I)$$

From the axiom of transfer T, it follows that 0 is standard and the successor of a standard integer is standard. Thus 1 is standard, 2 is standard, ...

By the axiom of idealization I, it follows

$$\exists \omega \in \mathbb{N} : \forall^{st} n \in \mathbb{N} : n \leq \omega$$

Such an integer ω which is greater than all the standard integers, will be called an unlimited integer. Any integer greater than an unlimited integer is an unlimited.

1.1.3 Standardization principle

For any formula $F(z)$ (internal or external), the following utterance is an axiom

$$\forall^{st} x\ \exists^{st} y\ \forall^{st} z (z \in y \Leftrightarrow z \in x \text{ et } F(z)) \qquad (S)$$

y is the standardized of the set $\{z \in x \mid F(z)\}$, and we put $y =^s \{z \in x \mid F(z)\}$.

Example 1.1.1 *Let ω be unlimited , and δ be an infinitesimal real. Then*

1. ${}^{\mathbf{s}}\{z \in \mathbb{N} \ / z > \omega\} = \phi$.
2. ${}^{\mathbf{s}}B\left(0, \delta\right) = \{0\}$, and ${}^{\mathbf{s}}B\left(\sqrt{\delta}, \delta\right) = \phi$.
3. ${}^{\mathbf{s}}\{z \in \mathbb{N} \ / z < \omega\} = \mathbb{N}$.

One of the most important results of this axiom is the following theorem.

Theorem 1.1.1 *(Theorem of the standard part of real) For any limited real x, there exists a unique standard real denoted ${}^\circ x$ (standard part) of x, such that $x \simeq {}^\circ x$.*

Example 1.1.2 *Let a be a standard number, and δ be an infinitesimal real. Then*

1. $^\circ(a+\delta)=a$
2. $^\circ\sqrt{3}=\sqrt{3}$.

Rules of calculation

The nonstandard numbers satisfy all the expected rules of calculation. Let a and b be limited reals, $a\ncong 0$, ε and η be infinitesimal ($\varepsilon\cong 0,\eta\cong 0$), and ω be unlimited real ($\omega\cong\infty$). Then

1. $a+\omega\cong\infty$.
2. $\varepsilon\times\eta\cong 0$.
3. $a\times\omega\cong\infty$.
4. $a\times\varepsilon\cong 0$.
5. $|a\ +b|\ll\infty$.
6. $\frac{a}{\omega}\cong 0$.
7. $|a\ \times b|\ll\infty$.
8. $\varepsilon+\eta\cong 0$.
9. $a+\varepsilon\cong a$.

Let us show, for example, that the sum of two infinitesimals is infinitesimal. Let ε and η be two infinitesimals, and let $r>0$ be standard, since $\frac{r}{2}$ is also standard, it majors $|\varepsilon|$ and $|\eta|$, and therefore $r=\frac{r}{2}+\frac{r}{2}$ majors $|\varepsilon+\eta|$.

We have said above that there are nonstandard integers and we have seen that this existence is obtained from the idealization principle. Now we give the different types of reals.

Definition 1.1.1 *We give some definitions*

1. An integer is unlimited if its absolute value is greater than any standard integer.
2. A real is unlimited if its absolute value is greater than any standard integer.
3. A real is infinitesimal if its absolute value is less than any standard real.
4. A real is appreciable if it is not unlimited and not infinitesimal.
5. Two reals are equivalent if their difference is infinitesimal.

1.2 Preliminaries on number theory

We will review some important notions of number theory, which we need for our work.

Euclidean division in $\mathbb{Z}$

Let a, $b \in \mathbb{Z}$, $a \neq 0$.we say that a divides b if there exists an integer c such that $b = ac$, in which case we write $a \mid b$. If a does not divide b, we write $a \nmid b$. In the case $a \mid b$ and $1 \leq a < b$, we say that a is a proper divisor of b. We write $p^{\alpha} \parallel n$ if $p^{\alpha} \mid n$ but $p^{\alpha+1}$ does not divide n.

Theorem 1.2.1 *(fundamental theorem of arithmetic) Every integer $n \geq 2$ can be written as a product of primes, and this representation is unique, except for the order in which the prime factors are disposed.*

Remark 1.2.1 *In particular, n can be written uniquely as $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_r^{\alpha_r}$,where $(q_i)_{1 \leq i \leq r}$ are distinct primes, and $(\alpha_i)_{1 \leq i \leq r}$ are positive integers. If $q_1, q_2, ...q_r$ are primes and if $a = \prod_{i=1}^{r} q_i^{\alpha_i}$ and $b = \prod_{i=1}^{r} q_i^{\beta_i}$ with $\alpha_i \geq 0$ and $\beta_i \geq 0$ for $i = 1, 2..., r$, then*

$$(a, b) = \prod_{i=1}^{r} q_i^{\min(\alpha_i,\, \beta_i)},$$

and

$$[a, b] = \prod_{i=1}^{r} q_i^{\max(\alpha_i,\, \beta_i)}$$

In the same way if $a = \prod_{i=1}^{r} q_i^{\alpha_i}$, $b = \prod_{i=1}^{r} q_i^{\beta_i}$ *and* $c = \prod_{i=1}^{r} q_i^{\gamma_i}$ *with* $\alpha_i \geq 0$, $\beta_i \geq 0$ *and* $\gamma_i \geq 0$ *for* $i = 1, 2..., r$, *then*

$$(a, b, c) = \prod_{i=1}^{r} q_i^{\min\{\alpha_i,\, \beta_i,\, \gamma_i\}} \text{ and } [a, b, c] = \prod_{i=1}^{r} q_i^{\max\{\alpha_i,\, \beta_i,\, \gamma_i\}}.$$

Theorem 1.2.2 *(theorem of prime numbers) Let* $\pi(x)$ *the number of prime numbers* $\leq x$. *Then*

$$\lim_{x \longrightarrow \infty} \frac{\pi(x)}{(x \diagup \log x)} = 1$$

Theorem 1.2.3 *(Euclid's theorem) There are infinitely many prime numbers.*

Proof. Suppose that we have only a finite number of primes $p_1, p_2, ..., p_n$. Consider the integer

$$N = p_1 p_2 \cdots p_n + 1.$$

Since $N > 1$, it follows from the fundamental theorem of arithmetic that N is divisible by some prime p. If $p = p_i$ for some $i = 1, ..., n$, then p divides $N - p_1 p_2 \cdots p_n = 1$, which is absurd. Therefore, $p \neq p_i$ for all $i = 1, ..., n$. Which is also contradiction. ■

1.2.1 Congruences

Let $a,\ b,\ m \in \mathbb{Z},\ m \neq 0$. We say that a is congruent to b modulo m, and we write $a \equiv b \pmod{m}$, if $m \mid (a - b)$; otherwise, we write $a \not\equiv b \pmod{m}$.

Let a, b, c, d and $m \in \mathbb{Z}$, $m > 0$. Then

1. $a \equiv a \pmod{m}$
2. $a \equiv b \pmod{m}$, if and only if $b \equiv a \pmod{m}$;
3. If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$;
4. If $a \equiv b \pmod{m}$, and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$ and $ax + cy \equiv bx + dy \pmod{m}$, for every $x,\ y \in \mathbb{Z}$.
5. $a \equiv b \pmod{m}$, and $d \mid m$, $d > 0$, then $a \equiv b \pmod{d}$

Let $m_1, m_2, ..., m_r \in \mathbb{N}$ and $x,\ y \in \mathbb{Z}$. Then

(i) $ax \equiv ay \,(\operatorname{mod} m)$ if and only if $x \equiv y \,(\operatorname{mod} m \diagup (a, m))$;

(ii) If $ax \equiv ay \,(\operatorname{mod} m)$ and $(a, m) = 1$, then $x \equiv y \,(\operatorname{mod} m)$;

(iii) $x \equiv y \,(\operatorname{mod} m_i)$ for $i = 1, 2, ..., r$ if and only if $x \equiv y \,(\operatorname{mod} [m_1, m_2, ..., m_r])$.

Definition 1.2.1 *(Residue modulo m)Let x , $y \in \mathbb{Z}$. If $x \equiv y \,(\operatorname{mod} m)$, then y is called a residue of x modulo m. A set $\{y_1, y_2, ..., y_m\}$ is called a complete system of residues modulo m if for every integer x there exists one and only one y_i such that $x \equiv y_i \,(\operatorname{mod} m)$.*

Definition 1.2.2 *(Reduced system of residues) Areduced system modulo m is a set of integers r_i such that $(r_i, m) = 1$, $r_i \not\equiv r_j \,(\operatorname{mod} m)$ if $i \neq j$, and such that every x prime to m is congruent modulo m to some member r_i of the set.*

Definition 1.2.3 *(Euler function)The Euler function ϕ is defined by*

$$\begin{aligned} \phi &: \mathbb{N}^* \to \mathbb{N}, \text{ such that} \\ \forall n &\in \mathbb{N},\ \phi(n) = \sharp \{0 < m \leq n \text{ with } (n, m) = 1\}. \end{aligned}$$

Now we list some famous theorems of this part.

Theorem 1.2.4 *(Fermat's theorem) Let p be a prime number and a be a positive integer such that p does not divide a. Then $a^{p-1} \equiv 1 \,(\operatorname{mod} p)$. In addition, for every integer a, we have the congruence $a^p \equiv a \,(\operatorname{mod} p)$.*

Theorem 1.2.5 *(Euler's theorem) Let $m \in \mathbb{N}$ and $a \in \mathbb{Z}$ such that $(a,\ m) = 1$. Then $a^{\phi(m)} \equiv 1 \,(\operatorname{mod} m)$.*

Proof. Let $\{r_1, ..., r_{\phi(m)}\}$ be a reduced set of residues modulo m. Since$(a, m) = 1$, we have $(a_{r_i}, m) = 1$ for $i = 1, ..., \phi(m)$. Consequently, for every $i \in \{1, ..., \phi(m)\}$ there exists $\sigma(i) \in \{1, ..., \phi(m)\}$ such that

$$ar_i \equiv r_{\sigma(i)} (\operatorname{mod} m).$$

Moreover, $ar_i \equiv ar_j (\bmod m)$ if and only if $i = j$, and so σ is a permutation of the set $\{1, ..., \phi(m)\}$ and $\{ar_1, ..., ar_{\phi(m)}\}$ is also a reduced set of residues modulo m. It follows that

$$\begin{aligned} a^{\phi(m)} r_1 r_2 \cdots r_{\phi(m)} &\equiv (ar_1)(ar_2)\cdots(ar_{\phi(m)})(\bmod m) \\ &\equiv r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(\phi(m))})(\bmod m) \\ &\equiv r_1 r_2 \cdots r_{\phi(m)} (\bmod m). \end{aligned}$$

Dividing by $r_1 r_2 \cdots r_{\phi(m)}$, we obtain

$$a^{\phi(m)} \equiv 1 (\bmod m).$$

This completes the proof. ■

Note that the latter theorem implies Fermat's little theorem, since $\phi(m) = m - 1$, when m is prime.

Corollary 1.2.1 *Let $m \in \mathbb{N}$, $a, b \in \mathbb{Z}$ and $(a, m) = 1$. Then the congruence $ax \equiv b \, (\bmod m)$ has a solution given by $x_0 = a^{\phi(n)-1} b$.*

Theorem 1.2.6 *(Wilson) Let m be a positive integer. Then m is a prime if and only if*

$$(m-1)! \equiv -1 \, (\bmod m) \, .$$

Proof. This is true for $p = 2$ and $p = 3$, since $1! \equiv -1 (\bmod 2)$ and $2! \equiv -1 (\bmod 3)$. Let $p \geq 5$. For every integer $a \in \{1, 2, ..., p-1\}$ there exists a unique integer $a^{-1} \in \{1, 2, ..., p-1\}$ such that

$$aa^{-1} \equiv 1 (\bmod p).$$

We have $a = a^{-1}$ if and only if $a = 1$ or $a = p - 1$. Therefore, we can partition the $p - 3$ numbers in the set $\{2, 3, ..., p - 2\}$ into $(p - 3)/2$ pairs of integers $\left\{a_i, a_i^{-1}\right\}$ such that $a_i . a_i^{-1} = 1 \, (\bmod p)$, for $i = 1, ..., (p - 3)/2$. Then

$$\begin{aligned} (p-1)! &= 1 \cdot 2 \cdot 3 \cdots (p-2)(p-1) \\ &\equiv (p-1) \prod_{i=1}^{(p-3)/2} a_i . a_i^{-1} \\ &\equiv p - 1 \\ &\equiv -1 \, (\bmod p) \, . \end{aligned}$$

This completes the proof. ■

Definition 1.2.4 *(pseudo prime) Let a be an integer, $a \geq 2$. A composite number n is called pseudo prime in base a if $a^{n-1} \equiv 1 \pmod{n}$.*

1.3 Quadratic residues

Definition 1.3.1 *For all a such that $(a, m) = 1$, a is called a quadratic residue modulo m if the equation $x^2 \equiv a \bmod m$ has a solution. If it has no solution, then a is called a quadratic nonresidue modulo m.*

Definition 1.3.2 *If p be an odd prime. Then*

$$\left(\frac{a}{p}\right) = \begin{cases} +1; & \text{if } a \text{ is a quadratic residue modulo } p \\ 0; & \text{if } p \mid a \\ -1; & \text{if } a \text{ is a quadratic nonresidue modulo } p. \end{cases}$$

$\left(\frac{a}{p}\right)$ *is called the Legendre symbol.*

Theorem 1.3.1 *Let p be an odd prime. Then*

1. $\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}$,
2. $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$,
3. $a \equiv b \pmod{p}$ implies that $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$,
4. $\left(\frac{1}{p}\right) = 1$, $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$.

Theorem 1.3.2 *(Quadratic reciprocity) If p and q are distinct odd primes, then*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\{(p-1)/2\}\{(q-1)/2\}}.$$

Example 1.3.1 *We will again use the rules set out above to calculate* $\left(\dfrac{541}{2011}\right)$

$$\begin{aligned}
\left(\frac{541}{2011}\right) &= \left(\frac{388}{541}\right), \ car\ 2011 \equiv 388\,[541] \\
&= \left(\frac{2}{541}\right)^2 \left(\frac{97}{541}\right), \ car\ 388 = 2^2 \times 97 \\
&= \left(\frac{97}{541}\right) \\
&= \left(\frac{541}{97}\right) \\
&= \left(\frac{56}{97}\right), \ car\ 541 \equiv 56\,[97] \\
&= \left(\frac{7}{97}\right)\left(\frac{2}{97}\right)^3, \ car\ 56 = 7 \times 2^3 \\
&= \left(\frac{97}{7}\right)\left(\frac{2}{97}\right)^3 \\
&= \left(\frac{-1}{7}\right)\left(\frac{2}{97}\right)^3, \ car\ 97 \equiv 1\,[7] \\
&= (-1)\,(1)^3 \\
&= -1.
\end{aligned}$$

1.3.1 Arithmetic functions

Definition 1.3.3 *An arithmetic function is a map from* $\mathbb{N}^*$ *into* $\mathbb{C}$.

- An arithmetic function is called multiplicative if $f(1) = 1$ and if $f(mn) = f(n)\,f(m)$ when $(n, m) = 1$ for all m, n in $\mathbb{N}^*$.
- An arithmetic function is completely multiplicative if $f(1) = 1$ and if $f(mn) = f(n)\,f(m)$ for all positive integers m and n.

Some arithmetic functions

Here are some arithmetic functions;

1. $\tau(n)$: The number of (positive) divisors of n;
2. $\sigma(n)$: The sum of divisors of n; for every reel number r, $\sigma_r(n) = \sum\limits_{d|n} d^r$;

3. $\omega(n)$: The total number of distinct prime factors of n, or in other terms $\omega(n) = \sum_{p|n} 1$ and $\omega(1) = 0$;

4. $\phi(n)$: Euler function, such that $\phi(n) = \sharp\{0 < m \leq n \text{ with } (n, m) = 1\}$;

Since each integer $n \geq 2$ can be written in the canonical form $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_r^{\alpha_r}$,We have the following theorem

Theorem 1.3.3 *Let n be a positive integer. Then*

$$\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_r + 1),$$

and

$$\sigma(n) = \prod_{i=1}^{r} \frac{q_i^{\alpha_i+1} - 1}{q_i - 1}.$$

Proposition 1.3.1 *[23]Let n be a positive integer, the equation $\phi(\tau(n)) = \tau(\phi(n))$ possesses infinity many solutions.*

Proof. It seems natural to consider the numbers n of the form $n = 2^k$, where k is a positive integer. Such a number is a solution of $\phi(\tau(n)) = \tau(\phi(n))$, if $\phi(k+1) = \tau(2^{k-1})$, which means $\phi(k+1) = k$ which happens exactly when $k+1$ is prime, that is $k = p - 1$, with p prime. The result follows. ■

Proposition 1.3.2 *[23]Let n be a positive integer, the equation $\gamma(\sigma(n)) = n$ possesses two solutions.*

Proof. We have $\gamma(\sigma(1)) = \gamma(1) = 1$, then $n = 1$ is a solution. Let $n > 1$ is a solution, if $n = p^\alpha$, then $\gamma(\sigma(n)) = n$ implies $\sigma(p) = p^\alpha$, let $p + 1 = p^\alpha$, which is impossible. then n has at least two distinct prime factors. Writing n in the canonical form

$$n = q^{\alpha_1} q^{\alpha_2} ... q^{\alpha_r}, \text{ where } (r \geq 2),$$

then

$$(q_1 + 1)(q_2 + 1)...(q_r + 1) = q_1^{\alpha_1} q_2^{\alpha_2} ... q_r^{\alpha_r}.$$

If $q_1 \geq 3$, then the number $(q_1+1)(q_2+1)\dots(q_r+1)$ is even, so the number $q_1^{\alpha_1}q_2^{\alpha_2}\dots q_r^{\alpha_r}$ is odd, which is impossible, hence $q_1 = 2$, for that

$$(q_1+1)(q_2+1)\dots(q_r+1) = q_1^{\alpha_1}q_2^{\alpha_2}\dots q_r^{\alpha_r} \Leftrightarrow 3(q_2+1)\dots(q_r+1) = 2^{\alpha_1}q_2^{\alpha_2}\dots q_r^{\alpha_r}.$$

Since $q_r|\, 3(q_2+1)\dots(q_r+1)$, it must divides 3 or one of the factors q_i+1 for such $2 \leq i \leq r$, since this last alternative is impossible, we have $q_r \mid 3$ then $q_r = q_2 = 3$. So the equation $3(q_2+1)\dots(q_r+1) = 2^{\alpha_1}q_2^{\alpha_2}\dots q_r^{\alpha_r}$ becomes

$$(2+1)(3+1) = 2^2.3 = 2^{\alpha_1}3^{\alpha_2}.$$

Thus by the uniqueness of the factorization, we conclude that $\alpha_1 = 2$ and $\alpha_2 = 1$, which gives rise to the solution $n = 12$, It is easily to see that the only two solutions of $\gamma(\sigma(n)) = n$ are $n = 1$, and $n = 6$. ■

Proposition 1.3.3 *[23]Let n be a positive integer, we have $\frac{\sigma(n)}{n} \geq \frac{\sigma(d)}{d}$, for all divisor d of n*

Proof. The result is certainly true for $n = 1$. For $n \geq 2$, we have $n = q_1^{\alpha_1}q_2^{\alpha_2}\dots q_r^{\alpha_r}$. If $d \mid n$, then $d = q_1^{\beta_1}q_2^{\beta_2}\dots q_r^{\beta_r}$, when $0 \leq \beta_i \leq \alpha_i$, for $i = 1, 2, \dots, r$, so we have

$$\frac{\sigma(n)}{n} = \prod_{i=1}^{r}\left(1+\frac{1}{q_i}+\dots+\frac{1}{q_i^{\alpha_i}}\right) \geq \prod_{i=1}^{r}\left(1+\frac{1}{q_i}+\dots+\frac{1}{q_i^{\beta_i}}\right) = \frac{\sigma(d)}{d}$$

■

Amicable numbers

An amicable pair of numbers are two numbers, each of which equal the sum of the proper divisors of the other. the smallest such a pair is 220 and 284, we have

$$\begin{aligned} 220 &= 2^2.5.11 \\ 284 &= 2^2.71 \end{aligned}$$

The proper divisors of 220 are therefore 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110, where

$$1+2+4+5+10+11+20+22+44+55+110 = 284$$

and the proper divisors of 284 are 1, 2, 4, 71, 142, where

$$1+2+4+71+142 = 220$$

We see therefore that we can also define an amicable pair of numbers as a pair m and n such that

$$\sigma(m) = \sigma(n) = m + n$$

The first amicable pairs and their sum and factors are depicted in the follows table

m	Factors	n	Factors	$m+n$	Factors
220	$2^2.5.11$	284	$2^2.71$	504	$2^3.3^2.7$
1184	$2^5.37$	1210	$2.5.11^2$	2394	$2.3^2.7.19$
2620	$2^2.5.131$	2924	$2^2.17.43$	5544	$2^3.3^2.7.11$
5020	$2^2.5.51$	5564	$2^2.13.107$	10584	$2^3.3^3.7^2$
6232	$2^3.19.41$	6368	$2^5.199$	12600	$2^3.3^2.5^2.7$
10744	$2^3.17.79$	10856	$2^3.23.59$	21600	$2^5.3^3.5^2$

We will see in the following theorem that was discovered by the Arabic mathematician (Thabit Ben Korra), a sufficient conditions to obtain a couplets (M, N) of amicable numbers.

Theorem 1.3.4 *(Thabit Ben Korrah 826-901) If the integers $p = 3.2^{k-1} - 1$, $q = 3.2^k - 1$ and $r = 9.2^{2k-1} - 1$ are prime for certain positive integer k, then the numbers $M = 2^k pq$ and $N = 2^k r$ form a pair of amicable numbers.*

Proof. We must show that $\sum\limits_{d|M,\ d<M} d = N$, and $\sum\limits_{d|N,\ d<N} d = M$

We have

$$\begin{aligned}
\sum_{d|M,\ d<M} &= \sigma(M) - M = \left(2^{k+1} - 1\right)(p+1)(q+1) - 2^k pq \\
&= \left(2^{k+1} - 1\right).3.2^{k-1}.3.2^k - 2^k\left(3.2^{k-1} - 1\right)\left(3.2^k - 1\right) \\
&= 3^2.2^{3k-1} - 2^k \\
&= N
\end{aligned}$$

Similarly we can show that $\sum\limits_{d|N,\ d<N} d = M$. ■

Remark 1.3.1 *We can define an amicable triple (l, m, n) to be a sequence of three numbers l, m, and n such that*

$$\sigma(l) = \sigma(m) = \sigma(n) = l + m + n$$

There are many examples of such triplets such as

n	m	l	$n+m+l$
1980	2016	2556	6552
9180	9504	11556	30240
21168	22200	2712	46080

Some special numbers

1. A positive integer n is called **triangular** if there exists a positive integer m such that

$$\begin{aligned} n &= 1+2+3+...m \\ &= \frac{m(m+1)}{2} \end{aligned}$$

2. A positive integer n is **perfect** if $\sigma(n) = 2n$. More generally, n is called k-perfect $(k \geq 2)$, if $\sigma(n) = kn$.

3. The numbers $F_n = 2^{2^n} - 1$ are called **Fermat** numbers.

4. The numbers $M_p = 2^p - 1,\ 0,$ when p is prime, are called **Mersenne** numbers.

5. A positive integer n is called **Carmichael** number if $b^{n-1} \equiv 1 \pmod{n}$ for every positive b such that $(b, n) = 1$.

6. The positive integer $n \geq 2$ is called **powerful** number if $p \mid n$ implies $p^2 \mid n$.

Chapter 2

Study of some famous linear recurrent sequences

We call a recurrent linear sequence $(S_n)_{n\in\mathbb{N}}$ of order p any sequence of values in a commutative field $\Bbbk$ (for example $\mathbb{R}$ or $\mathbb{C}$) defined for all $n \geq n_0$ by a relation of linear recurrence of the form

$$\forall n \geq n_0 : \ S_{n+p} = a_0 S_n + a_1 S_{n+1} + ... + a_{p-1} S_{n+p-1} \qquad (*)$$

where $a_0, a_1, ..., a_{p-1}$ are p fixed elements in $\Bbbk$ with a_0 non-zero.

2.1 Solutions of linear recurrent sequences

Such a sequence is entirely determined by its p first terms and by the recurrence relation. The linear recurrent sequences of order 1 are the geometric sequences. The study of higher order linear recurrent sequences is reduced to a problem of linear algebra. The characteristic polynomial associated to $(*)$ is defined by

$$P(X) = X^p - \sum_{i=0}^{p-1} a_i X^i = X^p - a_{p-1}X^{p-1} - a_{p-2}X^{p-2} - ... - a_1 X - a_0.$$

The degree of $P(X)$ is thus equal to the order of the recurrent relation. In particular, in the case of second order sequences, the polynomial is of degree 2 and can therefore be factorized by the usual process [32].

We study these sequences to see in the third chapter, the possibility of written an integer n as a sum of some terms of a given sequences $(S_n)_{n\in\mathbb{N}}$, namely, the representation of n by a sum of elements of $(S_n)_{n\in\mathbb{N}}$. In this sense, we will have the representation by using the terms of Fibonacci[29], tribonacci[4], Lucas[7], ... Also, in the fourth chapter, we will study the possibility of written the unlimited terms of a certain sequences according the model $s+\omega_1\omega_2$, where s is a standard integer and ω_1, ω_2 are unlimited integers.

Linear recurrent sequence of order 1

The linear recurrent sequences of order 1 are the geometric sequences

$$\begin{aligned} & U_{n_0} \text{ given, } n_0 \in \mathbb{N} \\ U_{n+1} &= qU_n,\ n \geq n_0 \end{aligned}$$

where q is a real called the raison of the sequence. The general term is given by

$$U_n = U_{n_0} q^{n-n_0},\ n \geq n_0$$

Linear recurring sequence of order 2

Let a and b be two fixed scalars of $\mathbb{k}$ with b non-zero. The recurrent relation is

$$\begin{aligned} & U_0,\ U_1 \text{ given} \\ U_{n+2} &= aU_{n+1} + bU_n,\ n \geq 0 \end{aligned}$$

The general term is given as following

- $\lambda r_1^n + \mu r_2^n$ if r_1 and r_2 are two distinct roots in $\mathbb{k}$ of polynomial $X^2 - aX - b$,
- $(\lambda + \mu n)\, r_0^n$ if r_0 is multiple root of the polynomial $X^2 - aX - b$, with λ, μ are parameters in $\mathbb{k}$ determined by the first two values of the sequence.
- If the two roots r_1, r_2 of the polynomial $X^2 - aX - b$ are two conjugate complexes $\rho e^{i\theta}$ and $\rho e^{-i\theta}$, then the general term of the sequence is written as $\rho^n\left(A\cos(n\theta) + B\sin(n\theta)\right)$.

2.2 Fibonacci and Lucas sequences and its generalizations

Definition 2.2.1 *The following recursive relation defines the nth Fibonacci number, F_n:*

$$\begin{aligned} F_1 &= F_2 = 1 \\ F_{n+2} &= F_{n+1} + F_n;\ n \geq 1 \end{aligned}$$

The first terms of Fibonacci are: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,...

Fibonacci sums We expose some sums of Fibonacci numbers

Corollary 2.2.1 *Let F_n be the nth Fibonacci number, we have*

$$\begin{aligned} \sum_{i=1}^{n} F_i &= F_{n+2} - 1 \\ \sum_{i=1}^{n} F_{2i-1} &= F_{2n} \\ \sum_{i=1}^{n} F_{2i} &= F_{2n+1} - 1 \end{aligned}$$

In the fourth chapter, we will use these identities to prove that these sums, for n is unlimited, are decomposable according to the model $s + \omega_1\omega_2$, where s is a standard integer and ω_1, ω_2 are unlimited integers.

2.2.1 Generalized Fibonacci Sequence (GFS)

We consider the sequence $(G_n)_{n\geq 1}$, where

$$\begin{aligned} G_1 &= a,\ G_2 = b, \text{ and} \\ G_n &= G_{n-1} + G_{n-2}, n \geq 3. \end{aligned}$$

The first terms are: a, b, $a+b$, $a+2b$, $2a+3b$, $3a+5b$,... . It is called the Generalized Fibonacci Sequence (GFS)[19].

Theorem 2.2.1 *Let G_n denotes the nth term of the (GFS). Then*

$$G_n = aF_{n-2} + bF_{n-1},\ n \geq 3$$

which follows easily by induction.

Proof. Since $G_3 = aF_1 + bF_2$, the statement is true when $n = 3$. Let k an arbitrary integer ≥ 3. Assume that the statement is true for all integers i,

$$G_i = G_{i-1} + G_{i-2}, i \geq 3.$$

where $3 \leq i \leq k$. Then:

$$\begin{aligned} G_{k+1} &= G_k + G_{k-1} \\ &= (aF_{k-2} + bF_{k-1}) + (aF_{k-3} + bF_{k-2}) \\ &= a(F_{k-2} + F_{k-3}) + b(F_{k-1} + F_{k-2}) \\ &= aF_{k-1} + bF_k. \end{aligned}$$

Thus, by induction the formula holds for every integer $n \geq 3$. ■

Now we study some properties in the following theorem.

Theorem 2.2.2 *Let G_n denote the nth term of the (GFS), we have*

$$\sum_{i=1}^{n} G_{k+i} = G_{n+k+2} - G_{k+2}$$

Proof. By precedent theorem

$$\begin{aligned} \sum_{i=1}^{n} G_{k+i} &= a\sum_{i=1}^{n} F_{k+i-2} + b\sum_{i=1}^{n} F_{k+i-1} \\ &= a(F_{n+k} - F_k) + b(F_{n+k+1} - F_{k+1}) \\ &= (aF_{n+k} + bF_{n+k+1}) - (aF_k + bF_{k+1}) \\ &= G_{n+k+2} - G_{k+2} \end{aligned}$$

■

Lucas sequences

Let P, Q be non-zero integers. For each $n \geq 0$, let us define $U_n = U_n(P, Q)$ and $V_n = V_n(P, Q)$ as follows

$$\begin{cases} U_0 = 0,\ U_1 = 1,\ U_n = PU_{n-1} - QU_{n-2},\ n \geq 2 \\ V_0 = 2,\ V_1 = P,\ V_n = PV_{n-1} - QV_{n-2},\ n \geq 2 \end{cases}$$

The sequences $U = (U_n(P,Q))_{n\geq 0}$ and $V = (V_n(P,Q))_{n\geq 0}$ are called the first and second respectively Lucas sequences with parameters P, Q. The sequence $(V_n(P,Q))_{n\geq 0}$ is also called the companion Lucas sequence with parameters P, Q[24], and the polynomial $X^2 - PX + Q$ is called the characteristic polynomial of $(U_n(P,Q))_{n\geq 0}$, and $(V_n(P,Q))_{n\geq 0}$. Let $D = P^2 - 4Q$ be its discriminant, and assume that $D \neq 0$ (to exclude a degenerate case). We have two roots $\alpha = \dfrac{P+\sqrt{D}}{2}$, $\beta = \dfrac{P-\sqrt{D}}{2}$. Thus, $\alpha \neq \beta$, $\alpha + \beta = P$, $\alpha\beta = Q$, and $(\alpha-\beta)^2 = D$. For any non-zero integers P and Q, we have

$$\begin{aligned} \frac{X}{1-PX+QX^2} &= \sum_{n=0}^{\infty} U_n X^n \\ \frac{2-PX}{1-PX+QX^2} &= \sum_{n=0}^{\infty} V_n X^n \end{aligned}$$

Now we give the link between the Fibonacci terms and the special Lucas terms L_n defined by $L_n = V_n(1,-1)$, $n \geq 0$[7].

$$\begin{aligned} L_0 &= 2,\ L_1 = 1 \\ L_{n+2} &= L_{n+1} + L_n,\ n \geq 0. \end{aligned}$$

Then

$$L_n = F_{n+1} + F_{n-1},\ n \geq 0$$

where

$$F_{-1} = 1,\ F_0 = 0$$

and

$$F_n = F_{n-1} + F_{n-2},\ n \geq 1$$

Theorem 2.2.3 *(Dudly and Tucker 1971)[20] Let F_n be the nth Fibonacci number and L_n be the nth Lucas number. We have*

$$\begin{array}{ll} 1) F_{4n}+1 = F_{2n-1}L_{2n+1} & 2) F_{4n}-1 = F_{2n+1}L_{2n-1} \\ 3) F_{4n+1}+1 = F_{2n+1}L_{2n} & 4) F_{4n+1}-1 = F_{2n}L_{2n+1} \\ 5) F_{4n+2}+1 = F_{2n+2}L_{2n} & 6) F_{4n+2}-1 = F_{2n}L_{2n+2} \\ 7) F_{4n+3}+1 = F_{2n+1}L_{2n+2} & 8) F_{4n+3}-1 = F_{2n+2}L_{2n+1} \end{array}$$

Proof. To prove this theorem we need the following property.

$$\begin{aligned} F_{m+n}+F_{m-n} &= \begin{cases} F_nL_m \text{ if } n \text{ is odd} \\ F_mL_n \text{ otherwise} \end{cases} \\ F_{m+n}-F_{m-n} &= \begin{cases} F_mL_n \text{ if } n \text{ is odd} \\ F_nL_m \text{ otherwise} \end{cases} \end{aligned}$$

Let us prove the formulas 1) and 3).

$$\begin{aligned} F_{4n}+1 &= F_{4n}+F_2 = F_{(2n+1)+(2n-1)} + F_{(2n+1)-(2n-1)} \\ &= F_{2n-1}L_{2n+1} \end{aligned}$$

and

$$\begin{aligned} F_{4n}+1 &= F_{4n}+F_1 = F_{(2n+1)+(2n)} + F_{(2n+1)-(2n)} \\ &= F_{2n+1}L_{2n}. \end{aligned}$$

The other formulas are proved by a similar way. ■

Theorem 2.2.4 *Let G_k denote the kth generalized Fibonacci number*

$$1. G_{m+n}+G_{m-n} = \begin{cases} G_mL_n \textit{ if } n \textit{ is even} \\ (G_{m+1}+G_{m-1})F_n \textit{ otherwise (Koshy, 1998)} \end{cases}$$

$$2. G_{m+n}+G_{m-n} = \begin{cases} (G_{m+1}+G_{m-1})F_n, \textit{ if } n \textit{ is even} \\ G_mL_n \textit{ otherwise (Koshy, 1998).} \end{cases}$$

Theorem 2.2.5 *(Koshy,1999) Let G_k denote the kth generalized Fibonacci number with $G_0 = a$ and $G_1 = b$. Then*

1. $G_{4m} + b = (G_{2m} + G_{2m-1}) F_{2m-1}$
2. $G_{4m+1} + a = G_{2m+1} L_{2m}$
3. $G_{4m+2} + b = G_{2m+2} L_{2m}$
4. $G_{4m+3} + a = (G_{2m+3} + G_{2m+1}) F_{2n+1}$
5. $G_{4m} - b = G_{2m+1} L_{2m-1}$
6. $G_{4m+1} - a = (G_{2m+2} + G_{2m}) F_{2m}$
7. $G_{4m+2} - b = (G_{2m+3} + G_{2m+1}) F_{2m}$
8. $G_{4m+3} - a = G_{4m+2} L_{2m+1}$

2.2.2 Horadam's sequence

Fibonacci sequence was generalized in many ways. One of the most widely used extension is the one given by A. F. Horadam. He defined the generalized sequence of numbers as follows[28].

Let $(W_n)_{n\in\mathbb{N}}$ be a sequence defined by

$$\begin{aligned} W_0 &= a, W_1 = b \\ W_n &= W_n(a, b; p, q) = pW_{n-1} - qW_{n-2}, \; n \geq 2 \end{aligned}$$

where a, b, p, and q are integers.

Remark 2.2.1 *We can see that the Lucas term $U_n = W_n(0, 1, ; p, q)$ and Fibonacci term $F_n = W_n(0, 1; 1, -1)$.*

We now define a new extension of the sequence $(W_n)_{n\in\mathbb{N}}$. The generalized Pseudo Fibonacci (GPF) sequence $(G'_n)_{n\in\mathbb{N}}$ is defined as the sequence satisfying the following non-homogeneous recurrent relation

$$\begin{aligned} G'_0 &= a, G'_1 = b. \\ G'_{n+2} &= pG'_{n+1} - qG'_{n-1} + At^n, \; n \geq 0 \end{aligned}$$

where A is non-zero integer and $t \neq 0$.

Here a, b, p, q are integers and α, β are distinct roots of characteristic equation $x^2 - px + q = 0$ of corresponding the homogeneous equation.

First few GPF numbers are:

$$\begin{aligned}
G'_0 &= a,\, G'_1 = b,\\
G'_2 &= (pb - qa) + A,\\
G'_3 &= \left(p^2b - pqa - qb\right) + pA + At,\\
G'_4 &= \left(p^3b - p^2qa - 2pqb + q^2a\right) + \left(p^2 - q\right) A + pAt + At^2,\\
G'_5 &= \left(p^4b - p^3qa - 3p^2qb + 2q^2a + q^2b\right) + \left(p^3 - 2pq\right) A + \left(p^2 - q\right) pAt + pAt^2 + At^3.
\end{aligned}$$

Observing that each GPF number G'_n $n \geq 2$ consists of two parts. The first part is an expression in p, q, a and b, while the second is a polynomial in t whose coefficient are A times terms in p and q. This is shown in the following tables:

- Table 01 of first part of G'_n, $n \geq 2$:

n	Expression in p, q, a, b
2	$pb - qa$
3	$p^2b - pqa - qb$
4	$p^3b - p^2qa - 2pqb + q^2a$
5	$p^4b - p^3qa - 3p^2qb + 2q^2a + q^2b$
6	$p^5b - 4p^3qb - p^4qa + 3q^2a + 2q^2$

Table 01

- Table 02 of second part of G'_n, $n \geq 2$:

n	A	At	At^2	At^3
2	1			
3	p	1		
4	$p^2 - q$	p	1	
5	$p^3 - 2pq$	$p^2 - q$	p	1

Table 02

From above tables, we have the following relation between GPF and horadam numbers W_n.

Theorem 2.2.6 *[28]For $m \geq 2$ we have*

$$G'_m = W_m(a, b; p, q) + A\sum_{k=1}^{m-1} W_k(0, 1; p, q)\, t^{m-k-1}$$

The term G'_n of sequence $\{G'_m\}$ satisfy the non-homgeneous recurrence relation

$$G'_{m+2} = pG'_{m+1} - qG'_m + At^m.$$

This sequences have strong and robust properties that can be used in the representation of integers for producing more efficient and safer results in the field of cryptography.

2.3 Lehmer sequence

In 1930, Lehmer considered the following sequence

$$L_n(P,Q) = \begin{cases} \dfrac{\alpha^n - \beta^n}{\alpha - \beta}, \text{ if } n \text{ is odd,} \\ \dfrac{\alpha^n - \beta^n}{\alpha^2 - \beta^2}, \text{ if} n \text{ is even.} \end{cases}$$

where P, Q are non-zero integers, α, β are the roots of the polynomial $X^2 - \sqrt{P}X + Q$, such that $\Delta = P - 4Q > 0$, $(\alpha + \beta)^2$ and $\alpha\beta$ are coprime positive integers, $\dfrac{\alpha}{\beta} \neq \pm 1$. All these properties ensure that the two denominators $(\alpha - \beta)$ and $\left(\alpha^2 - \beta^2\right)$ are not equal to zero, and the terms $(L_n(P,Q))_{n\geq 0}$ are integers[31], [1]. The sequence $L = (L_n(P,Q))_{n\geq 0}$ is called Lehmer sequence with integer parameters P, Q.

Lemma 2.3.1 *[1]Let $L = (L_n(P,Q))_{n\geq 0}$ be Lehmer sequence. We have*

$$L_n(P,Q) = \begin{cases} \dfrac{(\alpha^n - \beta^n)}{(\alpha - \beta)} = \left(\alpha^{n-1} + \alpha^{n-2}\beta + \alpha^{n-3}\beta^2 + ... + \beta^{n-1}\right) \in \mathbb{Z}^*, \text{ if } n \text{ is odd} \\ \dfrac{(\alpha^n - \beta^n)}{\left(\alpha^2 - \beta^2\right)} = \left(\alpha^{\mathbf{n-2}} + \alpha^{\mathbf{n-4}}\beta^2 + \alpha^{\mathbf{n-6}}\beta^4 + ... + \beta^{\mathbf{n-2}}\right) \in \mathbb{Z}^*, \text{ if } n \text{ is even.} \end{cases}$$

Those last formulas are proved by using symmetric and elementary functions of the variable α and β.

Remark 2.3.1 *For each $U_n(\alpha,\beta)$ the nth term of Lucas, and $L_n(\alpha,\beta)$ the nth term of Lehmer we have*

$$U_n(\alpha,\beta) = \begin{cases} L_n(\alpha,\beta)\,; \text{ if } n \text{ is odd} \\ (\alpha+\beta)\,L_n(\alpha,\beta)\,; \text{ if } n \text{ is even} \end{cases}$$

Cyclotomic polynomials [1]

We define the cyclotomic polynomials by

$$\Phi_n(X,Y) = \prod_{1\le k<n,\ (k,n)=1} \left(X - e^{\mathbf{2ik\pi/n}}Y\right)$$

however for each $n \in \mathbb{N}^*$, we have:

$$\alpha^n - \beta^n = \prod_{\mathbf{d|n}} \Phi_{\mathbf{d}}(\alpha,\beta)$$

$$U_n(\alpha,\beta) = \prod_{\mathbf{d|n,\ d\neq 1}} \Phi_{\mathbf{d}}(\alpha,\beta)\,,\ L_n(\alpha,\beta) = \prod_{\mathbf{d|n,\ d>2}} \Phi_{\mathbf{d}}(\alpha,\beta)\,.$$

In particular, for each $n \in \mathbb{N}^*$, $\Phi_n(\alpha,\beta)$ divides $U_n(\alpha,\beta)$ and $L_n(\alpha,\beta)$. This will permit, after a short study of the arithmetic of Lucas and Lehmer sequences, to give a cyclotomic criterion before an important restriction, in terms of pairs of Lucas and Lehmer defectives.

Example 2.3.1 *For $(P,Q) = (1,-3)$ we have $\alpha = \dfrac{1-\sqrt{13}}{2}$, $\beta = \dfrac{1+\sqrt{13}}{2}$, $(\alpha+\beta)^2 = 1$, $\alpha\beta = -3$. Then*

$$\begin{aligned}
L_3(\alpha,\beta) &= \frac{\alpha^3-\beta^3}{\alpha-\beta} = \alpha^2+\alpha\beta+\beta^2 = (\alpha+\beta)^2 - \alpha\beta = 1-(-3) = 4. \\
L_3(\alpha,\beta) &= \left(\frac{1-\sqrt{13}}{2} - e^{\frac{2\pi i}{3}}\frac{1+\sqrt{13}}{2}\right)\left(\frac{1-\sqrt{13}}{2} - e^{\frac{4\pi i}{3}}\frac{1+\sqrt{13}}{2}\right) = 4. \\
L_5(\alpha,\beta) &= \frac{\alpha^5-\beta^5}{\alpha-\beta} = \alpha^4+\alpha\beta+\alpha\beta+\alpha\beta+\beta^4 = 19. \\
L_5(\alpha,\beta) &= \phi_5(\alpha,\beta) = \prod_{1\le k\le 4,\ (k,5)=1} \left(\alpha - e^{\mathbf{2ik\pi/n}}\beta\right) \\
&= \left(\frac{1-\sqrt{13}}{2} - e^{\frac{2\pi i}{5}}\frac{1+\sqrt{13}}{2}\right)\left(\frac{1-\sqrt{13}}{2} - e^{\frac{4\pi i}{5}}\frac{1+\sqrt{13}}{2}\right)\times \\
&= \left(\frac{1-\sqrt{13}}{2} - e^{\frac{6\pi i}{5}}\frac{1+\sqrt{13}}{2}\right)\left(\frac{1-\sqrt{13}}{2} - e^{\frac{8\pi i}{5}}\frac{1+\sqrt{13}}{2}\right) \\
&= 19.
\end{aligned}$$

Remark 2.3.2 *In [1] we find many properties of Lehmer sequences $L_{\mathbf{n}}(\alpha\ ,\beta)_{\mathbf{n}\in\mathbb{N}^*}$, which can give more informations about the representation of positive integers as a sum of distinct Lehmer numbers and the decomposition of unlimited terms of Lehmer according to the required model. Then we will give in chapter 04 some applications concerning this remark.*

2.4 Pell, modified Pell and Pell-Lucas numbers

The Pell sequence $(P_n)_{n\in\mathbb{N}}$ is as important as the Fibonacci sequence, is defined by the following recurrent relation [9],

$$\begin{cases} P_0 = 0,\ P_1 = 1 \\ P_n = 2P_{n-1} + P_{n-2},\ n \geq 2, \end{cases}$$

the Pell-Lucas sequence $(Q_n)_{n\in\mathbb{N}}$ is defined by

$$\begin{cases} Q_0 = 2,\ Q_1 = 2 \\ Q_n = 2Q_{n-1} + Q_{n-2},\ n \geq 2 \end{cases}$$

and modified Pell sequence $(q_n)_{n\in\mathbb{N}}$ is defined by

$$Q_n = 2q_n.$$

Also, for more informations, we can find these sequences in[20], with initial values $P_1 = 1$, $P_2 = 2$ and $P_n = 2P_{n-1} + P_{n-2}$, $n \geq 3$.

In the table below, we give some elements of Pell, modified Pell and Pell-Lucas numbers.

n	0	1	2	3	4	5	6	7	8	9	10	...
P_n	0	1	2	5	12	29	70	169	408	985	2378	...
Q_n	2	2	6	14	34	82	198	478	1154	2786	6726	...
q_n	1	1	3	7	17	42	99	239	577	1393	3363	....

Horadam gave some results as follows:

$$\begin{aligned} q_n &= P_{n+1} - P_n \\ q_{n+1} &= P_{n+1} + P_n \\ P_{n+1} &= \frac{q_{n+1} + q_n}{2} \end{aligned}$$

Binet's form of P_n, Q_n and q_n are

$$\begin{cases} P_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta} \\ Q_n = \alpha^n + \beta^n \\ q_n = \dfrac{\alpha^n + \beta^n}{\alpha + \beta} \end{cases},$$

where α and β are the roots of the characteristic polynomial $x^2 - 2x - 1 = 0$ of the sequence $(P_n)_{n\in\mathbb{N}}$.

Remark 2.4.1 *Putting*

$$N = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \begin{pmatrix} q_{n+2} \\ q_{n+1} \end{pmatrix} &= N \begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix} \\ \begin{pmatrix} P_{n+2} \\ P_{n+1} \end{pmatrix} &= \frac{1}{2} N \begin{pmatrix} q_{n+1} \\ q_n \end{pmatrix} \end{aligned}$$

Theorem 2.4.1 *[9]Let n and m be integers. The following determinantal equalities are held.*

a) $N^n = \begin{cases} 2^{\frac{n}{2}} \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix}; & \text{if } n \text{ is even} \\ 2^{\frac{n-1}{2}} \begin{pmatrix} q_{n+1} & q_n \\ q_n & q_{n-1} \end{pmatrix}; & \text{if } n \text{ is odd} \end{cases}$

b) $\det(N^n) = 2^n$

c) $N^{n+m} = N^n + N^m$

d) $N^{n-m} = N^n + N^{-m}$.

Proof. By induction. ■

Theorem 2.4.2 *[9] For any integers n and m . We have*

1. $\begin{cases} P_n^2 - P_{n+1}P_{n-1} = (-1)^n\,; & \text{if } n \text{ is even} \\ q_{n+1}q_{n-1} - q_n^2 = 2\,(-1)^{n+1}\,; & \text{if } n \text{ is odd} \end{cases}$,

2. $P_{n+1}P_{n-1} + P_n^2 = \dfrac{1}{2}\,(q_{2n} + (-1)^n)$,

3. $q_{n+1}q_{n-1} + q_n^2 = q_{2n} + (-1)^{n+1}$,

4. $P_{2n} = 2P_n q_n$,

5. $P_{m+n} = P_m P_{n+1} + P_{m-1}P_n$,

6. $2P_{m+n} = q_m q_{n+1} + q_{m-1}q_n$,

7. $q_{m+n} = P_m q_{n+1} + P_{m-1}q_n$,

8. $(-1)^{m+1}P_{n-m} = P_n q_{m-1} - q_{n-1}P_m$,

9. $(-1)^{m+1}q_{n-m} = P_n q_{m+1} - P_{n+1}q_m$,

10. $2(-1)^{m+1}P_{n-m} = q_n q_{m+1} - q_{n+1}q_m$,

11. $Q_{m+n} = 2Q_{m+n} - (-1)^n\,Q_{m-n}$, which is discovered by Prasad and Rao in (2002).

Proof. We have

$$N^n = \begin{cases} 2^{\frac{n}{2}} \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix}; & \text{if } n \text{ is even} \\ 2^{\frac{n-1}{2}} \begin{pmatrix} q_{n+1} & q_n \\ q_n & q_{n-1} \end{pmatrix}; & \text{if } n \text{ is odd} \end{cases},$$

then

$$\det(N^n) = \begin{cases} \det\left(2^{\frac{n}{2}} \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix}\right); & \text{if } n \text{ is even} \\ \det\left(2^{\frac{n-1}{2}} \begin{pmatrix} q_{n+1} & q_n \\ q_n & q_{n-1} \end{pmatrix}\right); & \text{if } n \text{ is odd} \end{cases},$$

i.e.

$$\det(N^n) = 2^n = \begin{cases} \left(2^{\frac{n}{2}}.2^{\frac{n}{2}}P_{n+1}P_{n-1} - 2^{\frac{n}{2}}.2^{\frac{n}{2}}P_n^2\right); & \text{if } n \text{ is even} \\ \left(2^{\frac{n-1}{2}}.2^{\frac{n-1}{2}}q_{n+1}q_{n-1} - 2^{\frac{n-1}{2}}.2^{\frac{n-1}{2}}q_n^2\right); & \text{if } n \text{ is odd} \end{cases}$$

which is implies

$$\det(N^n) = 2^n = \begin{cases} \left(-2^{\frac{n}{2}}\right)2^{\frac{n}{2}}\left(P_n^2 - P_{n+1}P_{n-1}\right); & \text{if } n \text{ is even} \\ \left(-2^{\frac{n-1}{2}}\right)2^{\frac{n-1}{2}}\left(q_n^2 - q_{n+1}q_{n-1}\right); & \text{if } n \text{ is odd} \end{cases}$$

Hence

$$\begin{cases} \left(P_n^2P_{n+1} - P_{n-1}\right) = \frac{2^n}{(-2)^n} = (-1)^n \\ \left(q_n^2 - q_{n+1}q_{n-1}\right) = \frac{.2^n}{(-2)^{n-1}} = -2(-1)^n = 2(-1)^{n+1} \end{cases}.$$

Thus we have proved the formula1) and 2) at the same time. Similarly we prove the other formulas. ■

2.5 Pascal's triangle

Recall that the binomial coefficients are coefficients occurring in the binomial expansion of an expression of the form $(x+y)^n$ as following[19]

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k$$

where $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

Pascal's triangle is represented as follows

$$\begin{array}{ccccccccc} & & & & \binom{0}{0} & & & & \\ & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \end{array}$$

That is

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

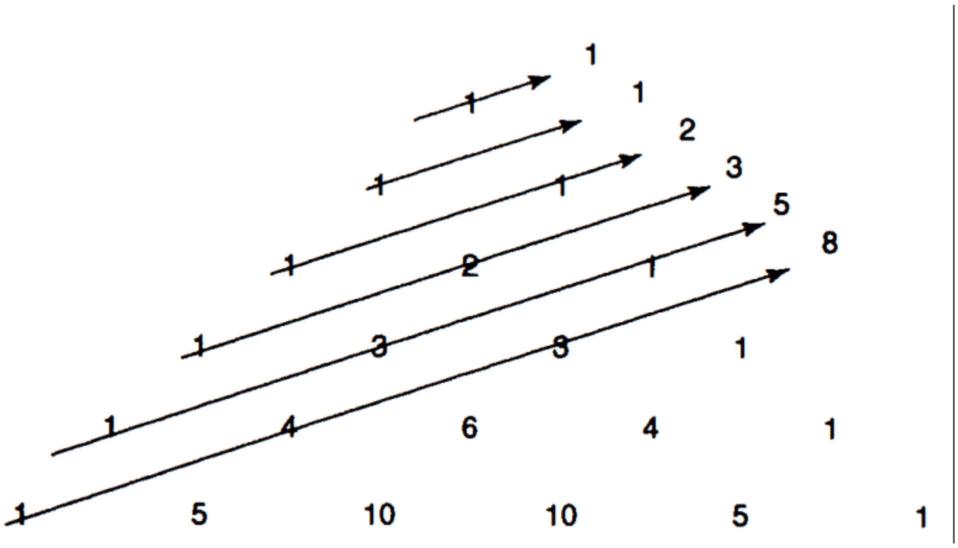
Pascal's triangle and Fibonacci numbers[19]

(Lucas Formula 1876) Let F_n be the nth Fibonacci number, the link between Pascal numbers and Fibonacci numbers is

$$F_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}; \ n \geq 0.$$

The figure 2.5.1 illustrates these sums.

01



1.png Figure 2.5.1 : Pascal triangle anf Fibonacci numbers

Proof. By induction ■

Example 2.5.1 *We have*

1. $F_6 = \sum_{k=0}^{\lfloor 5/2 \rfloor} \binom{5-k}{k} = \sum_{k=0}^{2} \binom{5-k}{k} = \binom{5}{0} + \binom{4}{1} + \binom{3}{2} = 1 + 4 + 3 = 8.$

Properties More generally, recall that $(L_n = F_{n+1} + F_{n-1})$, and we will have the following formula

a) $F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k;\ n \geq 0.$ A similar argument yields yet another identity by Lucas numbers

b) $L_{2n} = \sum_{k=0}^{n} \binom{n}{k} L_k;\ n \geq 0$, for example

$$\begin{aligned}
\sum_{k=0}^{4} \binom{4}{k} L_k &= \binom{4}{0} L_0 + \binom{4}{1} L_1 + \binom{4}{2} L_2 + \binom{4}{3} L_3 + \binom{4}{4} L_4 \\
&= 2 + 4 + 18 + 16 + 7 \\
&= 47 \\
&= L_8
\end{aligned}$$

c) $(-1)^{n-1} F_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k F_k$; $n \geq 0$, for example

$$\begin{aligned}\sum_{k=0}^{5} \binom{5}{k} (-1)^k F_k &= \binom{5}{0} F_0 - \binom{5}{1} F_1 + \binom{5}{2} F_2 - \binom{5}{3} F_3 + \binom{5}{4} F_4 - \binom{5}{5} F_5 \\ &= 0 - 5 + 10 - 20 + 15 - 5 \\ &= (-1)^5 F_5.\end{aligned}$$

Similarly we can be shown that

d) $(-1)^{n-1} L_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k L_k$; $n \geq 0$, for example

$$\begin{aligned}\sum_{k=0}^{4} \binom{4}{k} (-1)^k L_k &= \binom{4}{0} L_0 - \binom{4}{1} L_1 + \binom{4}{2} L_2 - \binom{4}{3} L_3 + \binom{4}{4} L_4 \\ &= 2 - 4 + 18 - 16 + 7 \\ &= 7 \\ &= (-1)^4 L_4.\end{aligned}$$

Pascal's triangle and Pell-Lucas numbers[19]

We can extract Pell-Lucas numbers from Pascal triangle as following theorem

$$\begin{aligned}Q_n &= \sum_{k=0}^{\lfloor n/2 \rfloor} \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] 2^{n-2k-1} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} 2^{n-2k-1}.\end{aligned}$$

For example

$$\begin{aligned}
Q_7 &= \sum_{k=0}^{3}\left[\binom{7-k}{k}+\binom{6-k}{k-1}\right]2^{6-2k}\\
&= \left[\binom{7}{0}+\binom{6}{-1}\right]2^6+\left[\binom{6}{1}+\binom{5}{0}\right]2^4+\\
&\quad \left[\binom{5}{2}+\binom{4}{1}\right]2^2+\left[\binom{4}{3}+\binom{3}{2}\right]2^0\\
&= (1+0)\,2^6+(6+1)\,2^4+(10+4)\,2^2+(4+3)\,2^0\\
&= 239.
\end{aligned}$$

Pascal's triangle and Pell numbers

Theorem 2.5.1 *Now we give the link between Pascal numbers and Fibonacci numbers as follows*

$$\sum_{i=0}^{n}\binom{n}{i}P_i=\begin{cases}2^{n/2}P_n & \text{if } n \text{ is even}\\ 2^{(n-1)/2}Q_n & \text{otherwise}\end{cases}$$

Corollary 2.5.1

$$\sum_{i=0}^{n}(-1^i)\binom{n}{i}P_i=\begin{cases}0 & \text{if } n \text{ is even}\\ -2^{(n-1)/2} & \text{otherwise.}\end{cases}$$

$$\sum_{i=0}^{n}(-1^i)\binom{n}{i}Q_i=\begin{cases}2^{n/2} & \text{if } n \text{ is even}\\ 0 & \text{otherwise.}\end{cases}$$

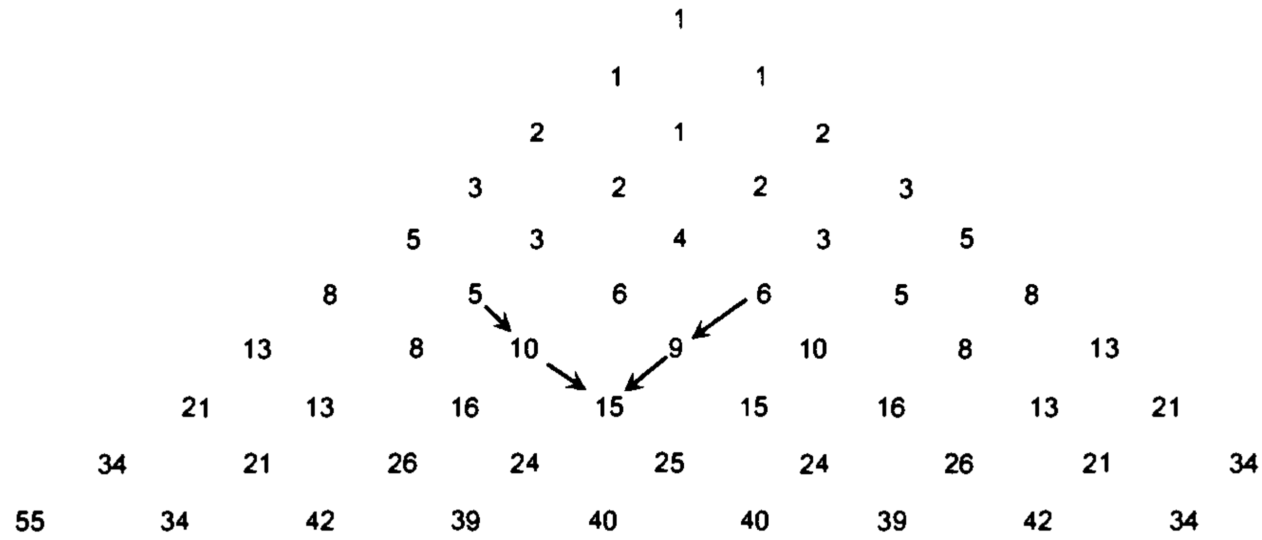
Theorem 2.5.2

$$\sum_{i=0}^{n}\binom{n}{i}P_i^2=\begin{cases}2^{(3n-4/2)}Q_n & \text{if } n \text{ is even}\\ 2^{(3n-4)/2}P_n & \text{otherwise.}\end{cases}$$

2.6 Hosoya's triangle

In 1976 H. Hosoya introduced the triangular array in the following Figure, which is closely linked to Fibonacci numbers. We call it Hosoya's triangle. Besides the array being symmetric about the vertical line through the middle, the top two northeast and southeast diagonals consist of Fibonacci numbers. Every interior number can be obtained by adding the two previous numbers, on its diagonal; for example, $15 = 10 + 5 = 6 + 9$[19].

02



2.png

Hosoya's triangle

Definition 2.6.1 *Every entry $H(n, j)$ of the array, we can define recursively:*

$$\begin{aligned} H(0,0) &= H(1,0) = H(1,1) = H(2,1) = 1 \\ H(n,j) &= H(n-1,j) + H(n-2,j) \\ &= H(n-1,j-1) + H(n-2,j-2), \end{aligned}$$

where $n \geq j \geq 0$ and $n \geq 2$.

Link between $H(n, j)$ and Fibonacci numbers[19]

Since

$$H(n,0) = H(n-1,0) + H(n-2,0)$$

where $H(0,0) = 1 = F_1$ and $H(1,0) = 1 = F_2$, it follows that

$$H(n,0) = F_{n+1}.$$

Likewise since

$$H(n,n) = H(n-1,n) + H(n-2,n)$$

it follows that

$$H(n,n) = F_{n+1}.$$

We can show that

$$H(n,1) = H(n,n-1) = F_n.$$

Successively, we have

$$\begin{aligned}
H(n,j) &= H(n-1,j) + H(n-2,j) \\
&= [H(n-2,j) + H(n-3,j)] + H(n-2,j) \\
&= 2H(n-2,j) + H(n-3,j) \\
&= 2[H(n-3,j) + H(n-4,j)] + H(n-3,j) \\
&= 3H(n-3,j) + 2H(n-4,j).
\end{aligned}$$

Continuing , we get a close link between $H(n,j)$ and Fibonacci numbers

$$H(n,j) = F_{k+1}H(n-k,j) + F_k H(n-k-1,j),$$

where $1 \leq k \leq n-j-1$. In particular, if $k = n-j-1$, then

$$\begin{aligned}
H(n,j) &= F_{n-j}H(j+1,j) + F_{n-j-1}H(j,j) \\
&= F_{n-j}F_{j+1} + F_{n-j-1}F_{j+1} \\
&= F_{j+1}(F_{n-j} + F_{n-j-1}) \\
&= F_{j+1}F_{n-j+1}.
\end{aligned}$$

For example

$$\begin{aligned}
H(7,3) &= 15 = 3.5 = F_4.F_5 \\
H(9,6) &= 39 = 3.13 = F_4.F_7.
\end{aligned}$$

2.7 Tribonacci sequence

The tribonacci numbers T_n are defined by the recurrent linear relation[19]

$$\begin{aligned} T_1 &= T_2 = 1, T_3 = 2 \\ T_n &= T_{n-1} + T_{n-2} + T_{n-3},\ n \geq 4 \end{aligned}$$

The first tribonacci numbers are 1, 1, 2, 4, 7, 13, 24, 44, 81, ...

2.7.1 Generalized Tribonacci Sequence (GTS).

To this end, consider the sequence $(T'_n)_{n\in\mathbb{N}}$ given by

$$\begin{aligned} T'_1 &= a, T'_2 = b, T'_3 = c \\ T'_n &= T'_{n-1} + T'_{n-2} + T'_{n-3},\ n \geq 4, \end{aligned}$$

where a, b, c are integers.

The first tribonacci numbers are a, b, c, $a+b+c$, $a+2b+2c$, $2a+3b+4c$,

Definition 2.7.1 *The sequence $(T'_n)_{n\in\mathbb{N}}$ is called the generalized tribonacci sequence (GTS).*

Theorem 2.7.1 *Let T'_n denote the nth term of the GTS. Then*

$$\begin{aligned} T'_1 &= a, T'_2 = b, T'_3 = c \\ T'_n &= aT_{n-3} + b\left(T_{n-4} + T_{n-3}\right) + cT_{n-2},\ n \geq 5. \end{aligned}$$

Proof. By induction, since

$$T'_5 = aT_2 + b\left(T_1 + T_2\right) + cT_3 = a + 2b + 2c,$$

the statement is true when $n = 5$. Assume that the given statement is true for all integer n, when $n \geq 5$

$$T'_n = aT_{n-3} + b\left(T_{n-4} + T_{n-3}\right) + cT_{n-2}.$$

Then we have

$$
\begin{aligned}
T'_{n+1} &= T'_n + T'_{n-1} + T'_{n-2} \\
&= (aT_{n-3} + b\,(T_{n-4} + T_{n-3}) + cT_{n-2}) + (aT_{n-4} + b\,(T_{n-5} + T_{n-4}) + cT_{n-3}) \\
&\quad + (aT_{n-5} + b\,(T_{n-6} + T_{n-5}) + cT_{n-4}) \\
&= a\,(T_{n-3} + T_{n-4} + T_{n-5}) + b\,((T_{n-4} + T_{n-5} + T_{n-6}) + (T_{n-3} + T_{n-4} + T_{n-5})) + \\
&\quad c\,(T_{n-2} + T_{n-3} + T_{n-4}) \\
&= aT_{n-2} + b\,(T_{n-3} + T_{n-2}) + cT_{n-1}.
\end{aligned}
$$

Thus, by induction, the formula holds for every integer $n \geq 5$. ■

Chapter 3

Representation of integers

Every positive integer may be presented by many forms like a product of factors, difference of squares, a polynomial form, sum of square numbers [32], [34],...

- **Product of factors**

The fundamental theorem of arithmetic

Theorem 3.0.2 *(fundamental theorem of arithmetic) Every positive integer can be written uniquely (up to order) as the product of prime numbers.*

Example 3.0.1 *We have* $60 = 2^2.3.5$.

- **As a sum of distinct factorials** [23]

Theorem 3.0.3 *Let r be a positive integer, and let n be a positive integer that can be written as a sum of r distinct factorials, i.e for which there exist a positive integers $d_1 < d_2 < ... < d_r$, such that*

$$n = d_1! + d_2! + ... + d_r!.$$

Then this representation is unique.

Proof. We assume that two representations exist, i.e there exist a positive integers $e_1 < e_2 < ... < e_r$ such that

$$n = d_1! + d_2! + ... + d_r! = e_1! + e_2! + ... + e_r!$$

We proceed by induction. If $r = 1$, then it's clear that $d_1 = e_1$, in which case the result is proved. Assume that the result is true for $r - 1$, and show that it is true for r. Without losing the generality we can assume that $d_r > e_r$. in which case $d_r \geq e_r + 1$, then we have

$$(e_r + 1)! \leq d_r! < d_1! + d_2! + ...d_r! = e_1! + e_2! + ... + e_r! \leq re_r!$$

so that $e_r + 1 < r$ et $e_r \leq r - 2$. But the e_i are distinct, we must have $e_r \geq r$, a contradiction , or uniqueness. ■

- **As a sum of primes**

Conjecture 3.0.1 *(Goldbach)[25] Every even natural number $n \geq 4$ is the sum of two primes.*

3.1 Cantor representation

Theorem 3.1.1 *Each positive integer n can be represented as*

$$n = a_m m! + a_{m-1} (m - 1)! + ... + a_2 2! + a_1 1!,$$

where a_j , $1 \leq j \leq m$, are integers satisfying $0 \leq a_j \leq j$. This representation, called Cantor's representation, is unique [23].

Proof. Let m be the greatest positive integer such that $m! \leq n$, and let a_m the greatest positive integer such that $a_m.m! \leq n$. It is clear that $0 < a_m \leq m$; otherwise this would contradicts the maximal choice of m. If $a_m.m! = n$, then the cantor development of n is $n = a_m.m!$. Otherwise $a_m.m! < n$, in this case we put $d_1 = n - a_m m! > 0$. Let m_1 be the greatest positive integer such that $m_1! \leq d_1$ and let a_{m_1} the greatest positive integer such that $a_{m_1}.m_1! \leq d_1$. As above, we have $0 < a_{m_1} \leq m_1$. If $a_{m_1}.m_1! = d_1$, then the Cantor development of n is $n = a_m.m! + a_{m_1}.m_1!$, where $0 < a_{m_1} \leq m_1 < m$. If $a_{m_1}.m_1! < d_1$, then we put $d_2 = d_1 - a_{m_1} m_1!$ and we chose m_2 as the greatest positive integer such that $m_2! \leq d_2$. And so on,..., we construct a sequence of positive integers $m > m_1 > m_2 > ...$, with the corresponding integers $0 < a_{m_i} \leq m_i$. Since the sequence $(m_i)_{i \geq 1}$ is discresing, it must be finite.

Now we show the uniqueness of this representation. Assuming that for $0 \leq a_j, b_j \leq j$, we have

$$n = a_m m! + ... + a_1 1! = b_m m! + ...b_1 1!.$$

So

$$(a_m - b_m)\, m! + ... + (a_1 - b_1)\, 1! = 0.$$

If the two representations are different, then there exist the least integer j such that $1 \leq j < m$ and $a_j \neq b_j$. Then

$$j! \left((a_m - b_m) \frac{m!}{j!} + ... + (a_{j+1} - b_{j+1})\,(j+1) + (a_j - b_j) \right) = 0$$

and thus

$$\begin{aligned} b_j - a_j &= (a_m - b_m) \frac{m!}{j!} + ... + (a_{j+1} - b_{j+1})\,(j+1) \\ &= (j+1) \left((a_m - b_m) \frac{m!}{(j+1)!} + ... + (a_{j+1} - b_{j+1}) \right), \end{aligned}$$

which implies

$$(j+1) \mid (b_j - a_j)\,.$$

Because

$$0 \leq a_j, b_j \leq j,$$

then

$$a_j = b_j,$$

contradiction, which completes the proof. ■

Example 3.1.1 *We have*

1) $23 = 3.3! + 2.2! + 1.1!$,

2) $719 = 5.5! + 4.4! + 3.3! + 2.2! + 1.1!$.

3.2 Unique representation of integers as sum of distinct Fibonacci and Lucas numbers

Theorem 3.2.1 *Let $\{F_n \colon n \in \mathbb{N}\}$ be the set of Fibonacci numbers defined previously. Then every positive integer can be written as a sum of distinct Fibonacci numbers [23].*

Proof. Let N be an arbitrary positive integer. And let F_{i_1} such that $F_{i_1} \leq N < F_{i_1+1}$. We take $\Delta_1 = N - F_{i_1}$. If $\Delta_1 = 0$ we have finished, because $N = F_{i_1}$. Otherwise, Let F_{i_2} with $i_2 < i_1$ such that $F_{i_2} \leq \Delta_1 < F_{i_2+1}$. If $\Delta_2 = \Delta_1 - F_{i_2} = 0$ we have finished, since in the case $N = \Delta_1 + F_{i_1} = \Delta_2 + F_{i_2} + F_{i_1} = F_{i_2} + F_{i_1}$, and so on. The process has an end because the sequence $(\Delta_i)_{i \geq 1}$of positive integers is decreasing, so that eventually we give $\Delta_r = 0$ for certain positive integer r in which case we have$N = F_{i_1} + F_{i_2} + ...F_{i_r}$. ■

Remark 3.2.1 *This representation is not unique, for example*

$$\begin{aligned} 25 &= 13 + 8 + 2 + 1 + 1 &&= F_7 + F_6 + F_3 + F_2 + F_1 \\ &= 21 + 2 + 1 + 1 &&= F_8 + F_3 + F_2 + F_1. \end{aligned}$$

In the following theorem, Zeckendorf proved that every positive integer can be written, in unique way, as a sum of not consecutive Fibonacci numbers.

Theorem 3.2.2 *(Zeckendorf's theorem) [**39**]Let N be a strictly positive integer. There is a unique set of strictly positive integers c_0, c_1, $...c_k$ such that*

- For every integer $i \in [0; k-1]$, $c_{i+1} > c_i$.
- F_{c_i} is the c_ith Fibonacci number.
- The integers c_i and the Fibonacci numbers also are not consecutive.
- $N = \sum\limits_{0}^{k} F_{c_i}$.

The sum can be reduced to single term; for example $5 = F_5$.

Also, in this sense, we have in[39] that every positive integer can be written as a sum of distinct Lucas numbers, where in general such representation is not unique; for example

$$\begin{aligned} 50 &= 47+3 = L_9 + L_3 \\ 50 &= 29+11+7+3 = L_8 + L_6 + L_5 + L_3 \end{aligned}$$

Theorem 3.2.3 *[7]Every positive integer n has a unique representation of the form*

$$n = \sum_{0}^{\infty} \alpha_m L_m$$

Where $\alpha_m = \alpha_m(n)$ is a binary digit $(0,$ or $1)$, for $m \geq 0$, α_m satisfy the following constraints

$$\begin{aligned} \alpha_m \alpha_{m+1} &= 0, \text{ for } m \geq 0 \\ \alpha_0 \alpha_2 &= 0 \end{aligned}$$

One want exclude every existence of two consecutive Lucas numbers in the representation, which is the same requirement that gives unique representation in Zeckendorf's theorem[27].

Proof. We begin by the following lemmas, which will be useful in this theorem. ■

Lemma 3.2.1 *Let n a positive integer. Then*

$$L_n - 1 = L_{n-1} + L_{n-3} + ... + L_{1,2}(n), \; n \geq 2,$$

where

$$L_{1,2}(n) = \begin{cases} 2L_1 \text{ if } n \text{ is even} \\ L_2 \text{ if } n \text{ is odd.} \end{cases}$$

(i. e)

$$\begin{aligned} L_{2n+1} - 1 &= L_{2n} + L_{2n-2} + ... + L_4 + L_2, \; n \geq 1, \\ L_{2n} - 1 &= L_{2n-1} + L_{2n-3} + ... + L_3 + 2L_1, \; n \geq 1. \end{aligned}$$

Proof. By induction. ■

Lemma 3.2.2 *Let n a positive integer. Then*

$$L_{n+2} = 1 + \sum_{i=0}^{n} L_i, \text{ for } n \geq 0$$

Proof. By induction ■

Lemma 3.2.3 *Let $n = \sum_{0}^{\infty} \alpha_m L_m$, where $\alpha_m = \alpha_m(n)$ is a binary digit $(0$, or $1)$ such that*

$$\begin{aligned} \alpha_m \alpha_{m+1} &= 0 \text{ for } m \geq 0 \\ \alpha_0 \alpha_2 &= 0. \end{aligned}$$

Then for every positive integer n this representation is unique.

Lemma 3.2.4 *Let $n = \sum_{0}^{k} \beta_m L_m, k \geq 2$, where β_i is a binary digit such that*

$$\begin{aligned} \beta_m + \beta_{m+1} &\neq 0, \text{ for } 0 \leq m \leq k-2 \\ \beta_0 + \beta_2 &\neq 0 \\ \beta_k &= 1. \end{aligned}$$

Such representation for n is unique.

In the next section we will give, by using a techniques that resemble to those used in theorem 3.2.1 and theorem 3.2.3, a representation for every positive integer by means of the elements of tribonacci sequence.

3.3 Unique representation of integers as sum of distinct tribonacci numbers

Theorem 3.3.1 *Let $(T_n)_{n\in\mathbb{N}}$ be the sequence of tribonacci defined previously. Any positive integer n can be written as the sum of distinct tibonacci numbers[4].*

Proof. Let $N > 0$ be an integer.

First, we prove that the sequence $(T_i)_{i\geq 1}$ verifies

$$T_{i+1} < 2T_i \ , \forall i \geq 1.$$

We easily check this formula for $i = 1,2,3,4$. For $i \geq 5$, we have $T_{i+2} = T_{i+1} + T_i + T_{i-1} < T_{i+1} + (T_i + T_{i-1} + T_{i-2}) = 2T_{i+1}$.

Now, put $\triangle_0 = N$. Let i_1 be the largest integer such that

$$T_{i_1} \leq \triangle_0 < T_{i_1+1}.$$

Put $\triangle_1 = \triangle_0 - T_{i_1}$. If $\triangle_1 = 0$, then $N = T_{i_1}$ and this completes the proof. Otherwise, from the fact that $\triangle_1 = \triangle_0 - T_{i_1} < T_{i_1}$, we choose $i_2 < i_1$ such that

$$T_{i_2} \leq \triangle_1 < T_{i_2+1}.$$

Put $\triangle_2 = \triangle_1 - T_{i_2}$. If $\triangle_2 = 0$, then $\triangle_1 - T_{i_2} = 0$, then $\triangle_1 = T_{i_2}$ and consequently $\triangle_1 = T_{i_2} = N - T_{i_1}$; since $\triangle_0 = N$. Hence, $N = T_{i_1} + T_{i_2}$ and this ends the proof. Otherwise (i.e. $\triangle_2 > 0$), the same reasons like before allows us to choose $i_3 < i_2$ such that

$$T_{i_3} \leq \triangle_2 < T_{i_3+1}.$$

And so on.

Since the sequence $i_1 > i_2 > i_3 > ... > ...$ is decreasing, it must have an end i_k. Hence $N = T_{i_1} + T_{i_2} + ... + T_{i_k}$ and consequently

$$N = \sum_{i=1}^{i=i_1} \alpha_i T_i.$$

■

Remark 3.3.1 *The representation in the latter theorem is not unique for example*

$$\begin{aligned} 12 &= 7 + 4 + 1 = T_5 + T_4 + T_1 t. \\ 21 &= 13 + 7 + 1 = T_6 + T_5 + T_2. \end{aligned}$$

Lemma 3.3.1 ***1.*** *For every integer $n \geq 5$, we have*

$$T_n = 3 + T_{n-2} + 2\sum_{i=2}^{i=n-3} T_i$$

Proof. From the definition of $(T_n)_{n\geq 1}$ we have successively

$$\begin{array}{rcl} T_n &=& T_{n-1} + \underline{T_{n-2}} + T_{n-3} \\ T_{n-1} &=& T_{n-2} + T_{n-3} + T_{n-4} \\ T_{n-2} &=& T_{n-3} + T_{n-4} + T_{n-5} \\ T_{n-3} &=& T_{n-4} + T_{n-5} + T_{n-6} \\ && \vdots \quad \vdots \quad \vdots \quad \vdots \\ T_5 &=& T_4 + T_3 + T_2 \\ T_4 &=& \underline{T_3} + T_2 + \underline{T_1} \ . \end{array}$$

After assuming side by side and simplifying the right side, we get

$$T_n = T_1 + T_3 + T_{n-2} + 2\sum_{i=2}^{i=n-3} T_i$$

Then

$$T_n = 3 + T_{n-2} + 2\sum_{i=2}^{i=n-3} T_i.$$

We can also easily obtain this result by induction. Indeed, the statement is true when $n = 5$. because $7 = T_5 = 3 + 2 + 2\,(1) = 7$. Assume the given statement is true for all integer n, when $n \geq 5$

$$\begin{array}{rcl} T_{n+1} &=& T_n + T_{n-1} + T_{n-2} \\ &=& 3 + T_{n-2} + 2\sum_{i=2}^{i=n-3} T_i + T_{n-1} + T_{n-2} \\ &=& 3 + T_{n-1} + 2T_{n-2} + 2\sum_{i=2}^{i=n-3} T_i \\ &=& 3 + T_{n-1} + 2\sum_{i=2}^{i=n-2} T_i. \end{array}$$

Which confirms the theorem. ■

Theorem 3.3.2 *Let* $N = \sum_{1}^{\infty} \alpha_m T_m$, *if*

$$\left\{ \begin{array}{l} \alpha_m \alpha_{m+1} = 0 \\ \alpha_m \alpha_{m+2} = 0 \end{array} \right. \quad m \geq 1.$$

where α_m *(*$m \geq 1$*) is the binary digit (zero or one), then this representation is unique.*

To prove this theorem we need the following lemma

Lemma 3.3.2 *We have*

$$\sum_{m=1}^{m=k} \alpha_m T_m < 2 \sum_{m=2}^{m=k-2} T_m + T_{k-2},\ k \geq 6$$

Proof. We put

$$\begin{aligned} L &= 2 \sum_{m=2}^{m=k-2} T_m - \sum_{m=1}^{m=k} \alpha_m T_m + T_{k-2} \\ &= \sum_{m=2}^{m=k-2} (2 - \alpha_m) T_m - \alpha_1 - \alpha_{k-1} T_{k-1} - \alpha_k T_k + T_{k-2} \end{aligned}$$

And we train to get $L > 0$. We distinguish two cases

A) $\alpha_k = 0$

$$L = \sum_{m=2}^{m=k-3} (2 - \alpha_m) T_m - \alpha_1 - \alpha_{k-1} T_{k-1} + T_{k-2} + (2 - \alpha_{k-2}) T_{k-2}$$

In this case

$$L > \sum_{m=2}^{m=k-3} (2 - \alpha_m) T_m + 2T_{k-2} - T_{k-1} - 1$$

We know that

$$\sum_{m=2}^{m=k-3} (2 - \alpha_m) T_m > 0,$$

and

$$2T_{k-2} - T_{k-1} - 1 > 0,\ \forall k > 1$$

So

$$L > 0$$

B) $\alpha_k = 1$ (which implies that $\alpha_{k-1} = \alpha_{k-2} = 0$) So

$$\begin{aligned} L &= \sum_{m=2}^{m=k-2} (2-\alpha_m) T_m - \alpha_1 - T_k + T_{k-2} \\ &= \sum_{m=2}^{m=k-3} (2-\alpha_m) T_m + (2-\alpha_{k-2}) T_{k-2} - \alpha_1 - T_k + T_{k-2} \\ &= \sum_{m=2}^{m=k-3} (2-\alpha_m) T_m + 3T_{k-2} - T_k - \alpha_1 \\ &= \sum_{m=2}^{m=k-4} (2-\alpha_m) T_m + (2-\alpha_{k-3}) T_{k-3} + 3T_{k-2} - T_k + -\alpha_1 \\ &> \sum_{m=2}^{m=k-4} (2-\alpha_m) T_m + T_{k-3} + 3T_{k-2} - T_k - 1 \end{aligned}$$

Since

$$T_k = T_{k-1} + T_{k-2} + T_{k-3}$$

So

$$L > \sum_{m=2}^{m=k-4} (2-\alpha_m) T_m + T_{k-3} + 3T_{k-2} - T_{k-1} - T_{k-2} - T_{k-3} - 1$$

(i.e)

$$L > \sum_{m=2}^{m=k-4} (2-\alpha_m) T_m + 2T_{k-2} - T_{k-1} - 1$$

Observe that we sum up to $k-4$, so $k-4 \geq 2$ i.e, $k \geq 6$.

Also

$$2T_{k-2} - T_{k-1} - 1 > 0 \text{ if } k-2 \geq 6$$

So we take $k \geq 8$.

Now we prove by induction that

$$2T_{k-2} - T_{k-1} - 1 > 0 \text{ if } k \geq 8$$

- For $k = 8$, we have $2T_6 - T_7 - 1 = 2(13) - 24 - 1 = 1 > 0$.
  - Assume that $2T_{k-2} - T_{k-1} - 1 > 0$ for $k \geq 8$.

- We prove that $2T_{k-2} - T_{k-1} - 1 > 0$ for $k+1$ i.e $2T_{k-1} - T_k - 1 > 0$.

$$\begin{aligned} 2T_{k-1} - T_k - 1 &= 2T_{k-1} - T_{k-1} - T_{k-2} - T_{k-3} - 1 \\ &= T_{k-1} - T_{k-2} - T_{k-3} - 1 \\ &= (T_{k-2} - T_{k-3} - T_{k-4}) - T_{k-2} - T_{k-3} - 1 \\ &= T_{k-4} - 1 > 0,\ k-4 \geq 4. \end{aligned}$$

So

$$L > \sum_{m=2}^{m=k-4} (2-\alpha_m)\, T_m + 2T_{k-2} - T_{k-1} - 1 > 0$$

Finally

$$L = 2\sum_{m=2}^{m=k-2} T_m - \sum_{m=1}^{m=k} \alpha_m T_m + T_{k-2} > 0$$

which implies

$$\sum_{m=1}^{m=k} \alpha_m T_m < 2\sum_{m=2}^{m=k-2} T_m + T_{k-2},\ \forall k > 6$$

Which finished the proof of lemma.

Let us return to the proof of theorem. Indeed, assume that N has a competing representation,

$$N = \sum_{m=1}^{\infty} \beta_m T_m$$

Where β_m is the binary digit (zero or one) such that:

$$\begin{cases} \beta_m \beta_{m+1} = 0 \\ \beta_m \beta_{m+2} = 0 \end{cases},\ m \geq 1$$

Assume, for a proof by contradiction, that the two representations are not identical, that is,

$$\sum_{1}^{\infty} |\alpha_m - \beta_m| \neq 0.$$

Then, let k be the largest value of m such that $\alpha_k \neq \beta_k$, Clearly $k \geq 4$, and since $\alpha_k \neq \beta_k$, we may assume without loss of generality that $\alpha_k = 1$, $\beta_k = 0$. It follows that,

$$l = \sum_{m=1}^{m=k} \alpha_m T_m = \sum_{m=1}^{m=k} \beta_m T_m = \sum_{m=1}^{m=k-1} \beta_m T_m, \text{ because } \beta_k = 0,\ l \leq n$$

with $\alpha_k = 1$. Then

$$l = \sum_{m=1}^{m=k} \alpha_m T_m = \sum_{m=1}^{m=k-1} \alpha_m T_m + T_k \geq T_k.$$

According the previous lemma we have

$$l = \sum_{m=1}^{m=k} \alpha_m T_m < 2\sum_{m=2}^{m-2} T_m + T_{k-2}.$$

$$l = \sum_{m=1}^{m=k} \alpha_m T_m < T_{k-2}.$$

According the previous proposition we have

$$l = \sum_{m=1}^{m=k} \alpha_m T_m < T_k$$

Finally we obtain

$$T_k \leq l = \sum_{m=1}^{k} \alpha_m T_m < T_k$$

Which is a contradiction. Hence the representation is unique[4]. ■

Chapter 4

Integers N may be represented according to the Model

$s + \omega_1.\omega_2$

In the chapter 2 and 3, we have studied some famous linear recurrent sequences, as well as their representation for positive integers. In the current chapter we are interested in the representation of the terms of unlimited order of some linear recurrent sequences according to the model: standard + unlimited x unlimited.

Also we will study the representation of some special unlimited positive integers according to the same model.

4.1 Representation of Lucas and Lehmer numbers as $s + \omega_1.\omega_2$

Let P, Q be nonzero integers. Consider the polynomial $X^2 - PX + Q$; its discriminant is $D = P^2 - 4Q$ and its roots are

$$\begin{cases} \alpha = \dfrac{P + \sqrt{D}}{2} \\ \beta = \dfrac{P - \sqrt{D}}{2} \end{cases}.$$

So

$$\begin{cases} \alpha+\beta=P \\ \alpha\beta=Q \\ \alpha-\beta=\sqrt{Q} \end{cases}.$$

For $D\neq 0$ we define the sequences of Lucas numbers

$$U_n(\alpha,\beta)=\frac{\alpha^n-\beta^n}{\alpha-\beta} \text{ and } V_n(\alpha,\beta)=\alpha^n+\beta^n,\ n\geq 0.$$

Particularly $U_0(\alpha,\beta)=0$, $U_1(\alpha,\beta)=1$, while $V_0(\alpha,\beta)=2$, $V_1(\alpha,\beta)=P$.

The sequences $(U_n(\alpha,\beta))_{n\geq 0}$ and $(V_n(\alpha,\beta))_{n\geq 0}$ are called also the Lucas sequences associated to the pair (α,β), and

$$U_n(\alpha,\beta)=\begin{cases} L_n(\alpha,\beta) & \text{if } n \text{ is odd} \\ (\alpha+\beta)L_n(\alpha,\beta) & \text{if } n \text{ is even} \end{cases},$$

where $(L_n(\alpha,\beta))_{n\geq 0}$ are the Lehmer sequences associated to the pair (α,β) as previously defined. The fellow theorem proves that any term of Lucas sequences whose the index is unlimited may be represented by the following formula

$$\text{standard } + \text{ unlimited} \times \text{unlimited}.$$

Theorem 4.1.1 *[6] Let P,Q be nonzero integers such that $D>0$. Then, for any unlimited n , the two sequences U_n and V_n differs by a limited integer from a product of two unlimited integers.*

Corollary 4.1.1 *If $V_1(\alpha,\beta)=P$ is standard, then for any unlimited n the term $L_n(\alpha,\beta)$ different of product of two unlimited integers by a standard integer.*

Proof. We distinguish two cases

a) If n is odd, then, for all P is standard, the corollary result from the fact that $U_n(\alpha,\beta)=L_n(\alpha,\beta)$.

b) If $n=2k$ i.e. n is even, then

$$\begin{aligned} U_{2k}(\alpha,\beta) &= (\alpha+\beta)L_{2k}(\alpha,\beta) \\ U_{2k}(\alpha,\beta) &= U_k(\alpha,\beta)V_k(\alpha,\beta) \end{aligned}$$

we have

$$L_n(\alpha,\beta) = \frac{U_n(\alpha,\beta)}{P} = \frac{U_{2k}(\alpha,\beta)}{P} = \frac{U_k(\alpha,\beta)\,V_k(\alpha,\beta)}{P}.$$

Then

$$\begin{cases} P \mid U_k(\alpha,\beta) \\ \quad\text{or} \\ P \mid V_k(\alpha,\beta) \end{cases}.$$

Since $U_k(\alpha,\beta)$ and $V_k(\alpha,\beta)$ are unlimited because k is unlimited and P is standard, then

$$L_n(\alpha,\beta) = 0 + \omega_1\omega_2.$$

■

Illustration We will expose some terms of Lehmer which are decomposable as

standard+unlimited × unlimited.

We take $P = -1$, $Q = -3$. Then $D = 13$ which implies $\alpha = \dfrac{1-\sqrt{13}}{2}$, $\beta = \dfrac{1+\sqrt{13}}{2}$.
We have different cases

- n **prime.** Let us take for example $n = 797$.

$$\begin{aligned} L_{797}(\alpha,\beta) &= \frac{\left(\frac{1-\sqrt{13}}{2}\right)^{797} - \left(\frac{1+\sqrt{13}}{2}\right)^{797}}{\left(\frac{1-\sqrt{13}}{2}\right) - \left(\frac{1+\sqrt{13}}{2}\right)} \\ &= 1437\,389\,236\,495\,279\,528\,483\,327\,958\,104\,298\,156\,075\,884\,219\,751\,704\,703 \\ &\quad 898\,018\,544\,501\,140\,545\,826\,818\,578\,159\,771\,134\,891\,807\,904\,564\,903\,116 \\ &\quad 114\,425\,632\,618\,494\,347\,460\,490\,598\,172\,372\,736\,799\,223\,403\,601\,407\,316 \\ &\quad 637\,860\,657\,195\,040\,986\,365\,241\,164\,425\,086\,718\,712\,220\,331\,139\,381\,846 \\ &\quad 010\,665\,941\,907\,132\,525\,732\,000\,053\,216\,169\,129\,991\,751\,432\,002\,856\,195 \\ &\quad 521\,661\,554\,560\,874\,419. \end{aligned}$$

In this case we have $L_n = U_n = \left(\frac{D}{n}\right) \bmod (n)$.

$$
\begin{aligned}
L_n &= \left(\frac{13}{797}\right) + 797 \times N \\
&= 1 + 797 \times 1803\,499\,669\,379\,271\,679\,401\,917\,136\,893\,724\,160\,697\,470 \\
&\quad 790\,152\,703\,518\,065\,267\,935\,384\,116\,117\,724\,991\,942\,484\,029\,027\,467 \\
&\quad 763\,995\,689\,966\,268\,650\,471\,308\,178\,788\,390\,791\,079\,796\,954\,043\,584 \\
&\quad 440\,681\,811\,294\,112\,066\,044\,994\,551\,060\,277\,272\,729\,286\,279\,077\,900 \\
&\quad 525\,360\,376\,827\,025\,573\,207\,039\,731\,420\,209\,702\,039\,814\,303\,705\,415 \\
&\quad 519611\,031\,055\,749\,062\,554\,824\,995\,811235\,333\,594.
\end{aligned}
$$

where $\left(\frac{13}{797}\right)$ is symbol of Legendre, and in this case $\left(\frac{13}{797}\right) = 1$.

- n **even**. Let us take for example $n = 620 = 2 \times 310$.

$$
\begin{aligned}
L_{620}(\alpha,\beta) &= \frac{\left(\frac{1-\sqrt{13}}{2}\right)^{620} - \left(\frac{1+\sqrt{13}}{2}\right)^{620}}{\left(\frac{1-\sqrt{13}}{2}\right)^{2} - \left(\frac{1+\sqrt{13}}{2}\right)^{2}} \\
&= 109\,404\,844\,896\,334\,076\,171\,340\,342\,385\,530\,648\,690\,326\,652\,319\,032\,842 \\
&\quad 254\,356\,321\,908\,025\,627\,172\,899\,700\,641\,065\,448\,676\,929\,386\,861\,696\,701 \\
&\quad 227\,873\,322\,643\,184\,793\,961\,039\,096\,288\,863\,688\,555\,600\,828\,263\,671\,565 \\
&\quad 175\,321\,806\,658\,018\,374\,131\,492\,986\,703\,199\,271\,726\,022\,164\,897\,982\,717 \\
&\quad 882\,759\,913.
\end{aligned}
$$

From the fact that $U_n(\alpha,\beta) = (\alpha+\beta)\,L_n(\alpha,\beta) = PL_n(\alpha,\beta)$ when n is even and the property $U_{2k} = U_k V_k$, we have

$$
\begin{aligned}
L_{620} &= \frac{U_{620}}{-1} = -U_{620} = -U_{2\times 310} \\
&= -U_{310}V_{310} \\
&= -\left(\frac{\left(\frac{1-\sqrt{13}}{2}\right)^{310} - \left(\frac{1+\sqrt{13}}{2}\right)^{310}}{\left(\frac{1-\sqrt{13}}{2}\right) - \left(\frac{1+\sqrt{13}}{2}\right)}\right)\left(\left(\frac{1-\sqrt{13}}{2}\right)^{310} + \left(\frac{1+\sqrt{13}}{2}\right)^{310}\right) \\
&= (-5508\,488\,401\,185\,359\,167\,292\,856\,797\,689\,523\,759\,586\,417\,140\,920\,068\,041\,684\,820\,042\,742 \\
&\quad 644\,512\,291\,966\,890\,036\,439\,133\,756\,273\,279\,658\,475\,179\,591) \times (19\,861\,137\,380\,772\,462\,896\,387 \\
&\quad 122\,256\,321\,005\,079\,534\,522\,650\,362\,901\,131\,094\,211\,102\,983\,558\,642\,112\,144\,023\,514\,380\,353\,371 \\
&\quad 949\,085\,501\,181\,662\,543)\,.
\end{aligned}
$$

In this calculation U_{310} negative and V_{310} positive, then $L_{620} = -U_{310}V_{310}$ positive.

- n is written as $n = 2^{\omega}p = 2^{7}11$

$$
\begin{aligned}
L_{2^{7}11}(\alpha,\beta) &= \left(\frac{\left(\frac{1-\sqrt{13}}{2}\right)^{2^{7}11} - \left(\frac{1+\sqrt{13}}{2}\right)^{2^{7}11}}{\left(\frac{1-\sqrt{13}}{2}\right)^{2} - \left(\frac{1+\sqrt{13}}{2}\right)^{2}}\right) \\
&= -U_{2^{7}11} \\
&= -U_{2^{6}11}V_{2^{6}11} \\
&= 0 + \omega_1.\omega_2
\end{aligned}
$$

where $\omega_1 = -U_{2^{6}11}$ and $\omega_2 = V_{2^{6}11}$ are two positive integers.

Theorem 4.1.2 *For any unlimited* n *,* F_n *differs by a limited integer from a product of two unlimited integers.*

Proof. Since $F_n = U_n(1,1)$, we obtain this result by the theorem 4.1.1. ∎

Illustration By the theorem 2.2.3, the following relations illustrate the factorization of terms of Fibonacci according to our required model, with $s = \pm 1$, since the terms of Lehmer and Lucas are unlimited for every n unlimited.

$$\begin{aligned}
F_{4n} &= -1+F_{2n-1}L_{2n+1} = s+\omega_1\omega_2. \\
F_{4n} &= 1+F_{2n+1}L_{2n-1} = s+\omega_1\omega_2. \\
F_{4n+1} &= -1+F_{2n+1}L_{2n} = s+\omega_1\omega_2. \\
F_{4n+1} &= 1+F_{2n}L_{2n+1} = s+\omega_1\omega_2. \\
F_{4n+2} &= -1+F_{2n+2}L_{2n} = s+\omega_1\omega_2. \\
F_{4n+2} &= 1+F_{2n}L_{2n+2} = s+\omega_1\omega_2.
\end{aligned}$$

Remark 4.1.1 *The theorem 2.2.3 serves, in the model $s+\omega_1\omega_2$, to specify the size of s which is ± 1.*

4.2 Representation of Pell and Pell-Lucas numbers as $s+\omega_1.\omega_2$

We see that Pell numbers are exactly Lucas numbers with $P = 2$, $Q = -1$ i.e, $P_n = U_n(2,1)$ [9]. Hence we have the following corollary.

Corollary 4.2.1 *All Pell and Pell Lucas numbers are decomposable according to the model: standard+unlimited× unlimited.*

Illustration Let s be standard integer, ω_1, ω_2 are unlimited integers. The following relations show the factorization in the requiring model.

$$P_{2n} = 2P_n q_n = s+\omega_1\omega_2.$$

We have also[9]

$$Q_{m+n} = 2Q_mQ_n - (-1)^n Q_{m-n}.$$

Hence

$$\begin{aligned}
Q_{3n} &= Q_{2n+n} = 2Q_{2n}Q_n - (-1)^n Q_n \\
&= 2Q_n\left[2Q_n^2 - (-1)^n\right] - (-1)^n Q_n \\
&= 4Q_n^3 - 2Q_n(-1)^n - (-1)^n Q_n \\
&= Q_n^3 + 3Q_n^3 - 3Q_n(-1)^n \\
&= Q_n^3 + 3Q_n\left[Q_n^2 - (1)^n\right] \\
&= Q_n\left(Q_n^2 + 3\left[Q_n^2 - (1)^n\right]\right) \\
&= s + \omega_1\omega_2.
\end{aligned}$$

4.3 Applications: Some special kind of integers written as $s+\omega_1.\omega_2$

in this section we will make representation of some types of unlimited integers according to the model mentioned above. as a reference of these considered integers in this section , we quote[23]

Proposition 4.3.1 *Let p and q be two different prime integers such that at least one of them is unlimited. Then the unlimited integer $p^{q-1}+q^{p-1}$ can be represented as $s+\omega_1.\omega_2$, where s is a standard integer and ω_1, ω_2 are unlimited integers.*

Proof. We distinguish the following cases

A) Let p and q be two unlimited prime numbers. According to the little theorem of Fermat we have

$$p^{q-1} \equiv 1 \pmod{q}.$$

So,

$$p^{q-1} = 1 + lq.$$

Then

$$\begin{aligned}
p^{q-1} + q^{p-1} &= 1 + lq + q^{p-1} \\
&\equiv 1 \pmod{q}.
\end{aligned}$$

In the same way, we have

$$p^{q-1}+q^{p-1}\equiv 1\,(\mathrm{mod}\, p)\,.$$

Since $(p,q)=1$, then we have

$$p^{q-1}+q^{p-1}\equiv 1\,(\mathrm{mod}\, pq)\,.$$

Hence

$$p^{q-1}+q^{p-1}=s+\omega_1.\omega_2.$$

Where s is standard and ω_1, ω_2 are unlimited.

B) $p\cong+\infty$ and $q\geq 2$ is standard

Considering in this case that

$$q^{p-1}=e^{(p-1)\ln(q)}.$$

But

$$e^{(p-1)\ln(q)}=\omega.p$$

where $\omega\cong\infty$ is generally real not necessarily integer. Since $p\cong\infty$, then

$$p^{q-1}+q^{p-1}=1+\alpha pq=s+\omega_1.\omega_2$$

where s is standard and $\omega_1=p\cong\infty$, $\omega_2=\alpha q\cong\infty$.

The case where $q\cong\infty$ and $P\geq 2$ standard, we treat by the same way. ■

Proposition 4.3.2 *Let* $N=\left(p^{q-1}\right)^m+\left(q^{p-1}\right)^n$, *where* m, n *are two positive integers and* p, q *are two different primes whose at least one of them is unlimited. Then* N *is written as* $s+\omega_1.\omega_2$, *where* s *is a standard integer and* ω_1, ω_2 *are unlimited integers.*

Proof. Showing that

$$\left(p^{q-1}\right)^m+\left(q^{p-1}\right)^n\equiv 1\,(\mathrm{mod}\, pq)\,.$$

By the little theorem of Fermat we have

$$p^{q-1}\equiv 1\,(\mathrm{mod}\, q)\,.$$

Then

$$\left(p^{q-1}\right)^m\equiv 1\,(\mathrm{mod}\, q)\,.$$

Hence

$$\left(p^{q-1}\right)^m+\left(q^{p-1}\right)^n\equiv 1\,(\mathrm{mod}\,q)\,.$$

In the same way we give

$$\left(p^{q-1}\right)^m+\left(q^{p-1}\right)^n\equiv 1\,(\mathrm{mod}\,p)\,.$$

Since $(p,q)=1$, then we have

$$\left(p^{q-1}\right)^m+\left(q^{p-1}\right)^n\equiv 1\,(\mathrm{mod}\,pq)\,.$$

Then

$$\left(p^{q-1}\right)^m+\left(q^{p-1}\right)^n=1+\alpha pq,\text{ with }\alpha\cong\infty$$

We end the proof by a cases study, exactly as we did for the proposition 4.3.1. ■

Proposition 4.3.3 *The positive integer* $n^{\phi(m)}+m^{\phi(n)}$ *with* $m\geq 2$ *and* $n\geq 2$ *are two coprime integers whose at least one of them is unlimited, is decomposable according the model: standard + unlimited×unlimited.*

Proof. We distinguish two cases

A) n and m are both unlimited

In this case, by Euler's theorem and the same ideas previously used, we have

$$n^{\phi(m)}+m^{\phi(n)}\equiv 1\,(\mathrm{mod}\,mn)\,.$$

Then

$$n^{\phi(m)}+m^{\phi(n)}\equiv 1\,(\mathrm{mod}\,mn)=1+\alpha mn=s+\omega_1.\omega_2$$

where s is a standard integer and ω_1, ω_2 are unlimited integers.

B) n is standard and $m\cong\infty$

Recall that for every $k\in\mathbb{N}$, we have[23]

$$\frac{1}{2}\sqrt{k}\leq\phi\left(k\right)\leq k.$$

From this last inequality, we have

$$n^{\phi(m)}\geq n^{\frac{1}{2}\sqrt{m}}=e^{\frac{1}{2}\sqrt{m}\ln n}$$

$e^{\frac{1}{2}\sqrt{m}\ln n} = \gamma m$, where $\gamma \cong \infty$ is generally real not necessarily integer. Hence

$$n^{\phi(m)} + m^{\phi(n)} \equiv 1 \,(\mathrm{mod}\, mn) = 1 + (\alpha n)\, m = s + \omega_1.\omega_2$$

With $\alpha \cong \infty$. Where s is standard, and ω_1, ω_2 are unlimited. ■

Proposition 4.3.4 *Let $m = m_1.m_2...m_r$, where m_i, $m_i \geq 2$ for $(1 \leq i \leq r)$, are relatively primes two by two, whose at least one of them is unlimited and r is standard. Then the positive integer $m_1^{\phi(m)/\phi(m_1)} + m_2^{\phi(m)/\phi(m_2)} + ... + m_r^{\phi(m)/\phi(m_r)} = s + \omega_1.\omega_2$ where s is standard, and ω_1, ω_2 are unlimited.*

Proof. We will show that

$$m_1^{\phi(m)/\phi(m_1)} + m_2^{\phi(m)/\phi(m_2)} + ... + m_r^{\phi(m)/\phi(m_r)} \equiv r - 1 \,(\mathrm{mod}\, m)\,.$$

We have

$$\begin{aligned} m_1^{\phi(m)/\phi(m_1)} &\equiv 0 \,(\mathrm{mod}\, m_1) \\ m_2^{\phi(m)/\phi(m_2)} &\equiv 1 \,(\mathrm{mod}\, m_1)\,,\ (m_1, m_2) = 1 \\ &\vdots \\ m_r^{\phi(m)/\phi(m_r)} &\equiv 1 \,(\mathrm{mod}\, m_1)\,,\ (m_1, m_r) = 1. \end{aligned}$$

Hence

$$m_1^{\phi(m)/\phi(m_1)} + m_2^{\phi(m)/\phi(m_2)} + ... + m_r^{\phi(m)/\phi(m_r)} \equiv r - 1 \,(\mathrm{mod}\, m_1)\,.$$

Similarly, we will have

$$m_1^{\phi(m)/\phi(m_1)} + m_2^{\phi(m)/\phi(m_2)} + ... + m_r^{\phi(m)/\phi(m_r)} \equiv r - 1 \,(\mathrm{mod}\, m_j)\,, \text{ when } j = 2, ...r.$$

Since $(m_i)_{1 \leq j \leq r}$ are relatively primes two by two, then

$$m_1^{\phi(m)/\phi(m_1)} + m_2^{\phi(m)/\phi(m_2)} + ... + m_r^{\phi(m)/\phi(m_r)} \equiv r - 1 \,(\mathrm{mod}\, m)\,.$$

Then we distinguish the following cases

a) At least two of the $(m_i)_{1 \leq j \leq r}$ are unlimited, (which are for example $m_l \cong \infty$, $m_k \cong \infty$), then

$$m_1^{\phi(m)/\phi(m_1)} + m_2^{\phi(m)/\phi(m_2)} + ... + m_r^{\phi(m)/\phi(m_r)} = r - 1 + \alpha m_l m_k = s + \omega_1.\omega_2.$$

b) There is a unique $i_0 \in \{1, 2, ...r\}$ such that $m_{i_0} \cong \infty$, without losing the generality we can assume that $m_{i_0} = m_1$, so $m = s.m_1$, when s is standard, then

$$\begin{aligned} m_1^{\phi(m)/\phi(m_1)} + m_2^{\phi(m)/\phi(m_2)} + .. + m_r^{\phi(m)/\phi(m_r)} &= m_1^{\phi(m_2)\phi(m_3)...\phi(m_r)} + m_2^{\phi(m_1)\phi(m_3)...\phi(m_r)} + R \\ &\geq m_2^{\phi(m_1)\phi(m_3)...\phi(m_r)} \\ &\geq m_2^{\phi(m_1)} \\ &\geq m_2^{\frac{1}{2}\sqrt{m_1}} = e^{\frac{1}{2}\sqrt{m_1}\ln m_2} = m_1\gamma, \end{aligned}$$

where $\gamma \cong \infty$ is generally real not necessarily integer.

Then

$$m_1^{\phi(m)/\phi(m_1)} + m_2^{\phi(m)/\phi(m_2)} + ... + m_r^{\phi(m)/\phi(m_r)} - (r-1) \geq m_1\gamma - (r-1).$$

On other hand, we have

$$m_1^{\phi(m)/\phi(m_1)} + m_2^{\phi(m)/\phi(m_2)} + ... + m_r^{\phi(m)/\phi(m_r)} - (r-1) = \lambda m_1 m_2 ... m_r = \lambda t m_1$$

where $t = m_2 m_3 ... m_r$, λ is a positive integer.

Since from the last inequality we have $\lambda t m_1 \geq m_1\gamma - (r-1)$ with $\gamma \cong \infty$ and $(r-1)$ is standard, then λt must be unlimited.

Hence

$$m_1^{\phi(m)/\phi(m_1)} + m_2^{\phi(m)/\phi(m_2)} + ... + m_r^{\phi(m)/\phi(m_r)} = (r-1) + \omega_1\omega_2,$$

where $r-1$ is standard, $\omega_1 = \lambda t \cong \infty$ and $\omega_2 = m_1$.

■

Proposition 4.3.5 *Let $N = a_1x_1 + a_2x_2 + ... + a_Nx_N$, where $(x_i)_{i\geq 1}$ is a standard sequence strictly decreasing of positive integers such that $x_i \mid x_{i+1}$ for $i \geq 1$ and $(a_i)_{i\geq 1}$ is standard sequence of positive integers. If there exists st (i_0) such that a_{i_0+1}, a_{i_0+2},...,a_{i_0+l} are null with $l \cong \infty$ and $\frac{l}{N} \cong 0$, then N is written as the model $s + \omega_1.\omega_2$ where s is standard, and ω_1, ω_2 are unlimited.*

Proof. We have

$$\begin{aligned}
N &= a_1x_1 + a_2x_2 + ... + a_Nx_N \\
N &= a_1x_1 + a_2x_2 + ... + a_{i_0}x_{i_0} + a_{i_0+1}x_{i_0+1} + ...a_{i_0+l}x_{i_0+l} + a_{i_0+l+1}x_{i_0+l+1}... + a_Nx_N \\
N &= a_1x_1 + a_2x_2 + ... + a_{i_0}x_{i_0} + 0 + a_{i_0+l+1}x_{i_0+l+1}... + a_Nx_N \\
&= (a_1x_1 + a_2x_2 + .. + a_{i_0}x_{i_0}) + x_{i_0+l+1}\left(a_{i_0+l+1} + a_{i_0+l+2}\frac{x_{i_0+l+2}}{x_{i_0+l+1}} + a_{i_0+l+3}\frac{x_{i_0+l+3}}{x_{i_0+l+1}} + .. + a_N\frac{x_N}{x_{i_0+l+1}}\right) \\
&= s + \omega_1.\omega_2.
\end{aligned}$$

■

Example 4.3.1 *Recall that the Cantor representation of a positive integer* $N \cong \infty$ *is given by*

$$\begin{aligned}
N &= a_m m! + a_{m-1}(m-1)! + ... + a_2 2! + a_1 1! \\
&= a_1 1! + a_2 2! + ... + a_m m!.
\end{aligned}$$

Let us consider $(x_i)_{i\geq1}$ *the sequence defined by* $x_i = i!$. *If the hypotheses described in the proposition 4.3.5 concerning sequences* $(x_i)_{i\geq1}$*and* $(a_i)_{i\geq1}$ *are valid, then*

$$N = s + \omega_1.\omega_2,$$

where s *is standard, and* ω_1, ω_2 *are unlimited.*

Theorem 4.3.1 *Let* m *and* n *be two positive amicable integers. Then*

1. m and n are both unlimited or are both standard.
2. If m and n be two unlimited with $\frac{m}{n} \cong \infty$, then one of them written as $s+\omega_1.\omega_2$, where s is standard, and ω_1, ω_2 are unlimited.

Proof. 1. Assume that $m \cong \infty$ and n standard, then $\sigma(m) \cong +\infty$, and $\sigma(n)$ standard. But $\sigma(n) = n + m = \sigma(m) \cong +\infty$, contradiction.

2. We have $\sigma(m) = m + n = \sigma(n)$. Assuming that $m = s_1p_1$ and $n = s_2p_2$, where s_1, s_2 are standard integers and p_1, p_2 are unlimited prime integers. Then $\frac{m}{n} = \frac{s_1p_1}{s_2p_2} \cong \infty$ which implies that $\frac{p_1}{p_2} \cong \infty$. On the other hand we have

$$\sigma(m)=\sigma(s_1p_1)=\sigma(s_1)(p_1+1)$$

and

$$\sigma(n)=\sigma(s_2p_2)=\sigma(s_2)(p_2+1).$$

Since $\sigma(m)=\sigma(n)$, then

$$\frac{p_1+1}{p_2+1}=\frac{\sigma(s_2)}{\sigma(s_1)}.$$

Which implies

$$\frac{p_1}{p_2}=\frac{\sigma(s_2)}{\sigma(s_1)}\frac{\left(1+\frac{1}{p_2}\right)}{\left(1+\frac{1}{p_1}\right)}.$$

That is, $\frac{p_1}{p_2}$ is appreciable, contradiction. Hence at least one of m and n can be written as $s+\omega_1.\omega_2$, where s is standard, and ω_1, ω_2 are unlimited. ■

Suggestions and perspectives

The aim of this work is, in one hand, the representation of a positive integer by using the terms of a linear recurrent sequences like sequences of Fibonacci, tribonacci, Lucas,... . On the other hand the representation of unlimited integers and unlimited terms of linear recurrent sequences according to the model $s+\omega_1\times\omega_2$.

As a research topic we can try to represent, according to our models, by using the sequence of Horadam, generalized Fibonacci sequence (GFS), generalized tribonacci sequence (GTS), k-Fibonacci sequence,... . Also, we have to represent the unlimited integers of every family of integers according to the model $s+\omega_1\times\omega_2$. This last representation is very vast, where we can easily find unlimited integers to represent.

4.4 Conclusion

Our work in this thesis concerns the representation of a positive integer N by two manner.

- Firstly, we represent every positive integer as a sum of elements of a given sequence. This work is well known in the mathematical literature; Indeed, this field of mathematics focuses on the study of words and formal languages combinatorics on words affects various areas of mathematics study, including algebra and computer science.

- Secondly, we represent N, when N is unlimited, according to the model

$$N = s + \omega_1 \times \omega_2. \tag{1}$$

  This representation have a deep relation with the cryptography. The researches in this field is far from complete because, on the one hand the problem of representing an unlimited integer according to (1) is still an open problem and on the other hand the representation of a positive integer N by a sum of elements of a given sequence is broad and promising.

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Abstract

The representation of a positive integer N by a sum of elements of a given sequence is an interesting problem which is well known in the mathematical literature, namely the unique representation as a sum of distinct Lucas numbers (Brown, Jr, 1969), Fibonacci and Lucas representation (Pihko, 1986), Cantor's development (Mercier, 2004),... . In addition, in the framework of the nonstandard analysis, A. Boudaoud proposed another method of representation of N; It is to write it according to the following model :

$$N = s + \omega_1.\omega_2$$

where s is a standard integer and ω_1, ω_2 are two unlimited integers.

Our work in this thesis concerns the representation of a positive integer N according to the model cited above. In this sense we have shown that if N satisfies certain constraints, then it can be represented by a finite sum of elements of the sequence of tribonacci. We observe that the techniques used can be employed in other problems for the same purposes; for example, the cases of higher orders (Pentanacci, hexanacci, ...k-Fibonacci sequence..).

In the last part of this thesis we have given several families of integers in which each unlimited integer is written according to the model

$$N = s + \omega_1.\omega_2.$$

As a motivation, this field of mathematics focuses on the study of words and formal languages combinatorics on words affects various areas of mathematics study, including algebra and computer science. Combinatorics of words is connected to many modern, as well as classical, fields of mathematics. Connections to combinatorics, actually being part of it, are obvious but also connections to algebra are deep.

Résumé

La représentation d'un entier positif illimité N par une somme d'éléments d'une suite donnée est un problème intéressant et très connu dans la littérature mathématique, à savoir la représentation unique comme une somme de nombres distincts de Lucas (Brown, Jr, 1969), la représentation de Fibonacci et de Lucas (Pihko, 1986), le développement de Cantor (Mercier, 2004),... . De plus, dans le cadre de l'analyse non standard, A. Boudaoud avait proposé un autre modèle de présentation d'un entier N; il consiste à l'écrire selon le modèle suivant

$$N = s + \omega_1.\omega_2 \tag{1}$$

où s est un entier standard et ω_1, ω_2 sont deux entiers illimités.

Notre travail dans cette thèse concerne la représentation de N selon les modèles évoqués ci-dessus. Dans ce sens nous avons montré que si N satisfait quelques contraintes, alors il peut être représenté par une somme finie d'éléments de la suite de tribonacci. Nous avons remarqué que les techniques utilisées peuvent être utilisées dans d'autres problèmes ayant le même but; par exemple les cas d'ordre supérieurs (Pentanacci, hexanacci, ...k-Fibonacci sequence...).

Dans la dernière partie de cette thèse nous avons donné plusieurs familles d'entiers dans lesquelles chaque entier illimité peut être écrit selon le modèle

$$N = s + \omega_1.\omega_2 \qquad (1)$$

où s est un entier standard et ω_1, ω_2 sont deux entiers illimités.

En tant que motivation, ce domaine de mathématiques qui se concentre sur l'étude des mots et la combinatoire des langages formels sur les mots affecte divers domaines de l'étude des mathématiques, y compris l'algèbre et l'informatique. La combinatoire de mots est liée à de nombreux domaines de mathématiques modernes, aussi bien que classiques. Les connexions à la combinatoire, actuellement devenues partie intégrante, sont évidentes, mais les connexions à l'algèbre sont profondes.

ملخص المذكرة

تمثيل الأعداد الصحيحة على شكل مجموع عناصر متتالية مسألة مشهورة في الرياضيات، يضاف اليها الوحدانية. كتمثيل عدد صحيح على شكل مجموع حدود مختلفة لمتتالية ليكاس (براون 1969).أيضا على شكل مجموع حدود لحدود متتالية فيبوناتشي (بيهكو1986) و كذالك تمثيل كانتور (ميرسيي 2004)....، وفي التحليل غير المعياري فنجد أن عبد المجيد بوداود اقترح نمودج تمثيل اخر لتمثيل عدد طبيعي وفق النمودج

$$N=s+\omega_1\times\omega_2$$

حيث s عدد صحيح معياري و ω_1 ، ω_2 عددان صحيحان متناهيان في الكبر يرتكز عملنا في هاته المذكرة على على تمثيل N وفق النمودج الأخير، وفي هذا الإطار توصلنا عند شروط معينة يمكن تمثيله بطريقة وحيدة على شكل مجموع عناصر لمتتالية تريبوناتشي، كما لاحظنا ان التقنيات المستعملة والقيود المرتبطة بها يمكن ان نصل بها الى تمثيلات جديدة باستعمال متتاليات خطية في شكلها العام . في الجزء الاخير قمنا بعرض عائلة من الاعداد الصحيحة بحيث يمكن للحدود المتناهية في الكبر منها أن تكتب على الشكل

$$N=s+\omega_1\times\omega_2$$

ونحن نثري في تمثيل الاعداد الصحيحة المتناهية في الكبر، إذ نقدم اضافات لنظرية اللغات وذالك لان كل تمثيل يوافقه لغة من اللغات تقوم عليه، كما نقوم باعطاء دفع للتحليل التوافقي وتطبيقات الجبر الحديث وما يتعلق به.

Abstract

The representation of a positive integer N by a sum of elements of a given sequence is an interesting problem which is well known in the mathematical literature, namely the unique representation as a sum of distinct Lucas numbers (Brown, Jr, 1969), Fibonacci and Lucas representation (Pihko, 1986), Cantor's development (Mercier, 2004),... . In addition, in the framework of the nonstandard analysis, A. Boudaoud proposed another method of representation of N ; It is to write it according to the following model :

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Our work in this thesis concerns the representation of a positive integer N according to the model cited above. In this sense we have shown that if N satisfies certain constraints, then it can be represented by a finite sum of elements of the sequence of tribonacci. We observe that the techniques used can be employed in other problems for the same purposes; for example, the cases of higher orders (Pentanacci, hexanacci, ... k -Fibonacci sequence..).

In the last part of this thesis we have given several families of integers in which each unlimited integer is written according to the model

$$N = s + \omega_1.\omega_2.$$

As a motivation, this field of mathematics focuses on the study of words and formal languages combinatorics on words affects various areas of mathematics study, including algebra and computer science. Combinatorics of words is connected to many modern, as well as classical, fields of mathematics. Connections to combinatorics, actually being part of it, are obvious but also connections to algebra are deep.

Résumé

La représentation d'un entier positif illimité N par une somme d'éléments d'une suite donnée est un problème intéressant et très connu dans la littérature mathématique, à savoir la représentation unique comme une somme de nombres distincts de Lucas (Brown, Jr, 1969), la représentation de Fibonacci et de Lucas (Pihko, 1986), le développement de Cantor (Mercier, 2004),... . De plus, dans le cadre de l'analyse non standard, A. Boudaoud avait proposé un autre modèle de présentation d'un entier N ; il consiste à l'écrire selon le modèle suivant

$$N = s + \omega_1.\omega_2$$

où s est un entier standard et ω_1 , ω_2 sont deux entiers illimités.

Notre travail dans cette thèse concerne la représentation de N selon les modèles évoqués ci-dessus. Dans ce sens nous avons montré que si N satisfait quelques contraintes, alors il peut être représenté par une somme finie d'éléments de la suite de tribonacci. Nous avons remarqué que les techniques utilisées peuvent être utilisées dans d'autres problèmes ayant le même but; par exemple les cas d'ordre supérieurs (Pentanacci, hexanacci, ... k -Fibonacci sequence...).

Dans la dernière partie de cette thèse nous avons donné plusieurs familles d'entiers dans lesquelles chaque entier illimité peut être écrit selon le modèle

$$N = s + \omega_1.\omega_2$$

où s est un entier standard et ω_1 , ω_2 sont deux entiers illimités.

En tant que motivation, ce domaine de mathématiques qui se concentre sur l'étude des mots et la combinatoire des langages formels sur les mots affecte divers domaines de l'étude des mathématiques, y compris l'algèbre et l'informatique. La combinatoire de mots est liée à de nombreux domaines de mathématiques modernes, aussi bien que classiques. Les connexions à la combinatoire, actuellement devenues partie intégrante, sont évidentes, mais les connexions à l'algèbre sont profondes.