People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

University of Mohammed Boudiaf - M'sila

Faculty of Mathematics and Informatics

Department of Mathematics

Speciality: Mathematical and Numerical Analysis

Master's Thesis on:

Numerical Treatment of Laplace Equation

Presented by: Ammari Hayat

Members of the jury:

Mr. Tahar Blizak	University of Mohammed Boudiaf - M'sila	Supervisor
Mme. Houria Serguine	University of Mohammed Boudiaf - M'sila	Examiner
Mme. Wahiba Delloum	University of Mohammed Boudiaf - M'sila	President

Academic Year: **2024/2025.**

Acknowledgment

All praise is due to Allah, by whose grace good deeds are perfected. May peace and blessings be upon the best of creation, our Prophet Muhammad, and upon his noble family and companions.

I would like to express my sincere gratitude to my supervisor, Mr. Belizak Taher, for his continuous support, valuable guidance, and insightful suggestions throughout the preparation of this work. I am also grateful to the esteemed members of the examination committee for their time, attention, and constructive academic feedback.

A special word of thanks is extended to my companion, Nour, who was a true model of patience and dedication. Her unwavering support and presence were a constant source of strength throughout every stage of this journey. May Allah reward her abundantly on my behalf.

Finally, my deepest gratitude goes to my beloved family. Your love, prayers, and steadfast encouragement have been the foundation upon which this achievement stands. You have always been my source of strength and inspiration. I ask Allah to reward you generously for all that you have given me.

Dedication

In the name of Allah, the Most Gracious, the Most Merciful.

All praise is due to Allah alone, without partner, and peace and blessings be upon His Prophet, the intercessor of this nation.

I dedicate this achievement to:

My family

My foundation, my hope, and the embodiment of my dreams.

To my mother, who sees her own success reflected in ours: *Ammari Fahima*

To my father, the steadfast support and shelter beneath which my dreams took shape: *Ammari Mabrouk*

To my elder brother, my strength, my guide, my challenger, and my role model: *Ammar*

To my second brother, the joy of my life and my companion in this world: *AbdNour*

To my sister and closest friend, the light of my eyes and my soul's companion: *Imane*

To my younger siblings: *Salaheddine and Ibtisseme*, the blooming buds of our family

To my sister not by blood but by bond, my dearest friend: *Imane*

To my cherished companions throughout this year's journey: *Marwa, Maryam, and all those who shared this path with me*

To my professors, those who left an enduring impact on me and contributed to the moment I stand in.

Every ending is simply the beginning of something greater.

I pray that this diploma will be a stepping stone toward even greater accomplishments.

To my dear friend and devoted companion throughout the journey of our graduation project, Nour.

ملخص

تعالج هذه المذكرة معادلة لابلاس، إحدى أهم المعادلات التفاضلية الجزئية $(PDEs)$ ، تحت شروط حدودية من نوع ديريشليه، وذلك باستخدام طريقتين:

- طريقة التحليل (فصل المتغيرات) للحصول على حل دقيق.
- وطريقة الفروق المتهية (FDM) للحصول على حل عددي.

في الفصل الثاني، تمت دراسة المعادلة داخل مجال مستطيلي، حيث أُنشئ الحل التحليلي بالاعتماد على فصل المتغيرات، والحل العددي باستخدام الفروق المتهية، وتمت برمجة كلا النموذجين باستخدام $MATLAB$ بهدف المقارنة بين الدقة والكفاءة.

أما الفصل الثالث، فتناول معادلة لابلاس في مجال دائري، حيث استُخدمت الإحداثيات القطبية

الكلمات المفتاحية: المعادلات التفاضلية الجزئية - معادلة لابلاس - شروط ديريشليه - طريقة الفروق المتهية (FDM) - فصل المتغيرات - الحل العددي - الحل التحليلي - $MATLAB$ - المجال المستطيلي

Résumé

Ce mémoire porte sur le traitement analytique et numérique de l'équation de Laplace, une équation différentielle partielle (EDP) fondamentale, sous conditions aux limites de Dirichlet. Deux méthodes sont explorées :

- la solution analytique par séparation des variables.

- et l'approximation numérique via la méthode des différences finies (FDM).

Au chapitre 2, l'équation est étudiée dans un domaine rectangulaire. Les deux solutions sont formulées et implémentées dans MATLAB, permettant une comparaison rigoureuse des résultats.

Le chapitre 3 traite du cas d'un domaine circulaire, en recourant aux coordonnées polaires, avec la même approche comparative.

Mots-clés : Équations différentielles partielles – Équation de Laplace – Conditions de Dirichlet – Méthode des différences finies (FDM) – Séparation des variables – Solution numérique – Solution analytique – MATLAB – Domaine rectangulaire – Domaine circulaire.

Abstract

This thesis addresses Laplace's equation, one of the most fundamental partial differential equations (PDEs), under Dirichlet boundary conditions. Two main approaches are employed:

- The analytical solution using the method of separation of variables.

- And the numerical approximation using the Finite Difference Method (FDM).

In Chapter 2, the equation is studied over a rectangular domain. The analytical solution is derived through variable separation, while the numerical solution is constructed via FDM. Both models are implemented in `MATLAB` to evaluate and compare their accuracy and efficiency.

Chapter 3 extends the analysis to a circular domain. By switching to polar coordinates, the same analytical and numerical techniques are applied, with a focus on analyzing the outcomes and comparing the numerical performance to the theoretical solution.

Keywords: Partial Differential Equations – Laplace's Equation – Dirichlet Boundary Conditions – Finite Difference Method (FDM) – Separation of Variables – Numerical Solution – Analytical Solution – `MATLAB` – Rectangular Domain – Circular Domain.

Contents

Introduction

According to [4], Partial Differential Equations (PDEs) are essential tools for describing a wide range of physical and engineering phenomena. From modeling heat transfer and electric potential to simulating fluid dynamics and structural deformation, PDEs form the backbone of many scientific applications. However, finding exact (analytical) solutions is often not feasible, especially when the problem involves complex geometries or boundary conditions. This limitation has led to an increasing reliance on numerical methods.

Numerical analysis, a key area in applied mathematics, focuses on developing reliable and efficient techniques for approximating the solutions of such equations. These methods aim to strike a balance between accuracy, stability, and computational efficiency. Among these techniques, the Finite Difference Method (FDM) stands out for its simplicity and effectiveness, particularly when the domain can be discretized into a regular grid—as is the case with rectangular or relatively uniform regions.

Laplace's equation is one of the most widely studied PDEs, with applications in fields such as electrostatics, fluid mechanics, and potential theory. In many practical situations—especially when the domain is irregular or curved—numerical solutions of Laplace's equation become necessary. In such cases, the Finite Difference Method provides an efficient framework for obtaining approximate solutions, particularly when an appropriately fine grid is used.

This chapter presents the theoretical background needed to apply the Finite Difference Method to the numerical solution of Laplace's equation in two dimensions. It begins with an overview of PDEs and their classification, followed by a focused discussion of the mathematical and physical characteristics of Laplace's equation. Then, the core ideas of the finite difference approach are introduced, including the discretization of the domain into a regular mesh, the derivation of finite difference approximations using central differences, the construction of the resulting linear system, and its solution to obtain a numerical approximation.

By the end of this chapter, the reader will have a solid theoretical foundation for understanding and implementing the Finite Difference Method to solve Laplace's equation. This foundation will support the practical applications and numerical results discussed in the following chapters.

CHAPTER 1

Partial Differential Equations and Laplace's Equation

1.1 Partial Differential Equation (PDE)

Definition

A partial differential equation (PDE) is an equation involving an unknown function of multiple variables and its partial derivatives. A general form of a k-th order PDE, where $k \geq 1$, can be written as:

$$F(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x) = 0, \quad x \in U \subset \mathbb{R}^n$$

where:

- $u : U \to \mathbb{R}$ is the unknown function.
- $U \subset \mathbb{R}^n$ is an open domain.
- $D^k u(x)$ represents the collection of all partial derivatives of u of order k.
- F is a given function depending on u, its derivatives up to order k, and the point $x \in U$. [5]

This general structure encompasses many physical and engineering models. Solving such an equation typically requires finding all functions u that satisfy the relation under appropriate boundary or initial conditions [4].

Classification of Partial Differential Equations

Second-order linear PDEs in two variables can be classified into three main types, based on the discriminant of their second-order terms:
Given the general form [5]:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{1.1}$$

where A, B, C, D, E, F,and G are given functions of x and y or constants. The classification depends on the sign of the discriminant $D = B^2 - AC$.

- **Elliptic PDEs:** If $D < 0$, e.g., Laplace's equation.
- **Parabolic PDEs:** If $D = 0$, e.g., the heat equation.
- **Hyperbolic PDEs:** If $D > 0$, e.g., the wave equation.

This classification helps determine the behavior of the solution and the appropriate numerical or analytical techniques used.

- Elliptic Equations
- Hyperbolic Equations
- Parabolic Equations

Elliptic Equations

In more than one space dimension, the steady-state equations discussed generalize naturally to *elliptic* partial differential equations. In two space dimensions a constant-coefficient elliptic equation has the form

$$a_1u_{xx} + a_2u_{xy} + a_3u_{yy} + a_4u_x + a_5u_y + a_6u = f \tag{1.2}$$

$$a_2^2 - 4a_1a_3 < 0 \tag{1.3}$$

This equation must be satisfied for all (x, y) in some region of the plane Ω, together with some boundary conditions on $\partial\Omega$, the boundary of Ω. For example, we may have Dirichlet boundary conditions in which case $u(x, y)$ is given at all points $(x, y) \in \partial\Omega$. If the ellipticity condition (1.3) is satisfied, then this gives a well-posed problem. If the coefficients vary with x and y, then the ellipticity condition must be satisfied at each point in Ω [7].

1.2 Laplace's Equation

1.2.1 General Form in Two Dimensions

In Cartesian coordinates:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

In two-dimensional polar coordinates (r, θ), Laplace's equation takes the form:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

This form arises from transforming the classical Cartesian form of Laplace's equation:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

into polar coordinates using the standard variable substitution:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Any function satisfying this is harmonic.

1.2.2 Physical Properties

Appears in equilibrium states like heat distribution, electrostatics, and fluid flow.

1.2.3 Maximum Principle

A harmonic function attains its extrema on the boundary unless it is constant [4].

1.3 Conditions in Laplace's Equation

To uniquely determine the solution of Laplace's equation, appropriate boundary conditions must be imposed on the domain's boundary $\partial\Omega$. The most common types of boundary conditions are the Dirichlet and Neumann conditions:

1.3.1 Dirichlet Condition

In the Dirichlet problem, the function u itself is specified on the boundary:

$$u = f \quad \text{on} \quad \partial\Omega$$

This condition is used when the exact value of the solution is known along the boundary. It defines a so-called *Dirichlet problem* for Laplace's equation [4].

1.3.2 uniqueness of solution of laplasce's equation

Theorem 1.1 *(Zero Value Property of Harmonic Functions)*
Let $f(x, y, z)$ be harmonic in some domain D, and suppose $f = 0$ at every point of a piecewise smooth, closed, orientable surface $S \subset D$, where the entire region T enclosed by S also lies in D. Then f is identically zero in T.

Theorem 1.2 *(Uniqueness Theorem for Laplace's Equatio)*
Let T be a region satisfying the assumptions of the divergence theorem, and let $f(x, y, z)$ be a harmonic function in a domain D that contains T and its boundary surface S. Then f is uniquely determined in T by its values on S.

Theorem 1.3 *(Uniqueness of the Dirichlet Problem Solution)*
If the assumptions in Theorem 1.2 are satisfied and the Dirichlet problem for the Laplace equation has

a solution in T, then this solution is unique [3].

1.4 Solving an Elliptic Partial Differential Equation (PDE)

1.4.1 Finite Difference Method (FDM)

We consider the following Laplace equation defined on a rectangular domain:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{for } (x, y) \in [a, b] \times [c, d]$$

To approximate the solution numerically, we discretize the domain into a uniform grid using finite difference methods.

Discretization of the Domain

On the x-axis:

- Divide the interval $[a, b]$ into n_x equal subintervals.
- Grid spacing:

$$h_x = \frac{b-a}{n_x}, \quad x_i = a + i \cdot h_x, \quad i = 0, 1, \ldots, n_x$$

On the y-axis:

- Divide the interval $[c, d]$ into n_y equal subintervals.
- Grid spacing:

$$h_y = \frac{d-c}{n_y}, \quad y_j = c + j \cdot h_y, \quad j = 0, 1, \ldots, n_y$$

Note: Later, we will use shorthand notations such as:

$$x_{i\pm1} = x_i \pm h_x \quad , \quad y_{j\pm1} = y_j \pm h_y$$

We approximate derivatives in the PDE using Taylor series expansions, assuming the function $f(x, y)$ is smooth (infinitely differentiable).

Approximation of First-Order Derivatives with Respect to x:

- **Forward Difference:**

$$\frac{\partial f}{\partial x}(x,y) \approx \frac{f(x+h_x,y)-f(x,y)}{h_x}$$

- **Backward Difference:**

$$\frac{\partial f}{\partial x}(x,y) \approx \frac{f(x,y)-f(x-h_x,y)}{h_x}$$

- **Central Difference (More Accurate):**

$$\frac{\partial f}{\partial x}(x,y) \approx \frac{f(x+h_x,y)-f(x-h_x,y)}{2h_x}$$

Approximation of Second-Order Derivatives: [2]

- With respect to x:

$$\frac{\partial^2 f}{\partial x^2}(x,y) \approx \frac{f(x+h_x,y)-2f(x,y)+f(x-h_x,y)}{h_x^2}$$

- With respect to y:

$$\frac{\partial^2 f}{\partial y^2}(x,y) \approx \frac{f(x,y+h_y)-2f(x,y)+f(x,y-h_y)}{h_y^2}$$

Applying to the Laplace Equation

Using the finite difference approximations, we rewrite the Laplace equation as [1]:

$$\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h_x^2}+\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h_y^2}=0$$

Where:

$$u_{i,j}=u(x_i,y_j)$$

Special Case: Uniform Grid $(h_x = h_y = h)$

In the case where grid spacing is equal in both directions:

$$h_x = h_y = h$$

The equation simplifies to:

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$$

Or, equivalently:

$$\frac{1}{h^2}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}\right) = 0$$

This form is known as the **five-point stencil**.

Five-Point Stencil Representation

This method uses a grid point and its four immediate neighbors to approximate the Laplacian:

$$\begin{array}{ccc} & u_{i,j+1} & \\ & | & \\ u_{i-1,j} - & u_{i,j} & - u_{i+1,j} \\ & | & \\ & u_{i,j-1} & \end{array}$$

Stencil Matrix Representation:

$$\frac{1}{h^2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot U_{i,j} = 0$$

Final Notes

- The resulting system of equations (one for each interior point) is typically solved using iterative numerical methods such as:

- Jacobi Iteration
- Gauss-Seidel Method
- Successive Over-Relaxation (SOR)

- Boundary conditions must be incorporated based on the specific PDE problem (Dirichlet, Neumann, or mixed).

Finite Difference in Polar Coordinates

The application of finite difference methods in p;lar coordinates is a significant topic in numerical analysis, particularly when dealing with problems that exhibit radial symmetry or involve circular domains.In such problems,periodic boundary conditions often arise, Making polar coordinates a natural choice for the solution process. As an illustrative example, consider Laplace's equation, which in polar coordinates (r, θ) following form:

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0$$

To discretize this equation using finite differences, we define a polar grid formed by the intersections of concentric circles $r = ih$ and radial lines $\theta = j\delta\theta$. The resulting mesh allows for the numerical approximation of the equation at each grid point.
The Laplace's equation at the point can be approximated using the finite difference method in polar coordinates as follows:

$$\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{1}{ih} \cdot \frac{U_{i+1,j} - U_{i-1,j}}{2h} + \frac{1}{(i\delta\theta)^2} \cdot \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{(i\delta\theta)^2} = 0$$

It follows from this equation that:

$$\left(1 - \frac{1}{2i}\right) U_{i-1,j} + \left(1 + \frac{1}{2i}\right) U_{i+1,j} + \frac{1}{(i\delta\theta)^2}(U_{i,j+1} + U_{i,j-1}) - 2\left[1 + \frac{1}{(i\delta\theta)^2}\right] U_{i,j} = 0$$

Assuming this equation applies for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, and given that the boundary conditions are known for $i = 0$, $i = n + 1$, $j = 0$ and $j = m + 1$.

We can write the system in matrix form $AU = B$, where U is the column vector of unknowns, and A is a matrix written in the following block-tridiagonal form:

$$A = \begin{bmatrix} B_1 & \left(1+\frac{1}{2}\right)I & 0 & 0 & \cdots & \cdots 0 \\ \left(1-\frac{1}{4}\right)I & B_2 & \left(1+\frac{1}{4}\right)I & 0 & \cdots & \cdots 0 \\ 0 & \left(1-\frac{1}{6}\right)I & B_3 & \left(1+\frac{1}{6}\right)I & 0 & \cdots 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \vdots \\ 0 & \cdots & \left(1-\frac{1}{2(n-1)}\right)I & B_{n-1} & \left(1+\frac{1}{2(n-1)}\right)I & 0 \\ 0 & \cdots & 0 & \left(1-\frac{1}{2n}\right)I & B_n & \end{bmatrix}$$

Each $B = \{B_i\}$ is an mm matrix defined as:

$$B_i = \begin{bmatrix} -2\left(1+\frac{1}{(i\,\delta\theta)^2}\right) & \frac{1}{(i\,\delta\theta)^2} & 0 & \cdots & 0 & 0 \\ \frac{1}{(i\,\delta\theta)^2} & -2\left(1+\frac{1}{(i\,\delta\theta)^2}\right) & \frac{1}{(i\,\delta\theta)^2} & \cdots & 0 & 0 \\ 0 & \frac{1}{(i\,\delta\theta)^2} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{1}{(i\,\delta\theta)^2} & 0 \\ 0 & \cdots & \cdots & \frac{1}{(i\,\delta\theta)^2} & -2\left(1+\frac{1}{(i\,\delta\theta)^2}\right) & \frac{1}{(i\,\delta\theta)^2} \\ 0 & 0 & \cdots & 0 & \frac{1}{(i\,\delta\theta)^2} & -2\left(1+\frac{1}{(i\,\delta\theta)^2}\right) \end{bmatrix}$$

1.4.2 Iteration Methods for Laplace's Equation

Gauss-Seidel Iteration

In general, we can accelerate the convergence of the numerical iteration process by immediately using the newly computed values during the application of the formula. That is, after providing an initial guess, we update the values according to a formula that partially depends on the values just computed within the same iteration cycle, not only on the previous ones.

$$u_{ij}^{(k+1)} = \frac{1}{4}(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k+1)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k+1)}) \tag{1.4}$$

This method is known as the Gauss-Seidel method, or Liebmann's method, and it is considered an efficient technique that typically leads to faster convergence toward the numerical solution using the finite difference method.

1.4.3 Separation of Variables Theorem for Solving Laplace's Equation

The separation of variables theorem is a classical and effective analytical method for solving Laplace's equation, particularly in regular domains with homogeneous boundary conditions. This method assumes that the solution can be expressed as a product of functions, each depending on a single variable:

$$u(x,y) = X(x)Y(y)$$

This factorization allows the partial differential equation (PDE) to be split into two independent ordinary differential equations (ODEs), one for each spatial coordinate. By substituting this form into Laplace's equation and rearranging, we obtain equations that can be solved separately, with their solutions combined to meet the boundary conditions. The method leverages the linearity of the equation and is particularly suited for problems with homogeneous or partially homogeneous boundaries.

1.5 Fourier series

Theorem 1.4 *Let f be a function satisfying the following conditions:*

1. $f \in D([k, k+2\ell])$, *where* $k \in \mathbb{R}$, *i.e.,* f *is piecewise continuously differentiable on the open interval* $(k, k+2\ell)$.

2. $f \in C([k, k+2\ell])$, *i.e.,* f *is continuous on the closed interval* $[k, k+2\ell]$.

3. f *is periodic with period* $T = 2\ell$, *meaning that* $f(x+2\ell) = f(x)$ *for all* $x \in \mathbb{R}$.

Then, the function f can be represented by the following trigonometric Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

*which is called the **Fourier series** of f, where the Fourier coefficients are given by:*

$$\forall n \in \mathbb{N} : \begin{cases} a_n = \frac{1}{\ell} \int_k^{k+2\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) dx \\ b_n = \frac{1}{\ell} \int_k^{k+2\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx \end{cases}$$

Remark 1.1 *It is sometimes possible to represent the function f as a **Fourier sine series** or a **Fourier cosine series**, depending on the symmetry of f over the interval $[-\ell, \ell]$.*

- *If f is even, i.e., $f(-x) = f(x)$, then:*

$$b_n = 0, \quad a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) dx$$

- *If f is odd, i.e., $f(-x) = -f(x)$, then:*

$$a_n = 0, \quad b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

Therefore, to study f on the interval $[-\ell, \ell]$, it suffices to define it on the half-interval $[0, \ell]$, then extend it either evenly or oddly to the full interval $[-\ell, \ell]$, depending on the desired type of Fourier expansion [9].

1.6 Euler - Cauchy Differential Equation

The Euler-C equation is a second-order linear differential equation with variable coefficients. Its general form is:

$$x^2 y'' + axy' + by = 0$$

where a and b are constants, and $y(x)$ is the unknown function.

In this section we need a specific form of the Euler-C Equation called Radial Equation where [6]:

$$\begin{cases} x = r \\ a = 1 \\ b = -\lambda^2 \end{cases}$$

SOLUTION OF EULER'S EQUATION

$$(r^2R'' + rR' - \lambda^2 R = 0)$$

CASE 1 $(\lambda = 0)$ Here Euler's equation reduces to:

$$r^2R'' + rR' = 0$$

and it is easy to see that the general solution is:

$$R(r) = a + b \ln r$$

This can be found by letting $V(r) = R'(r)$ and solving the new equation for $V(r)$. After finding $V(r) = \frac{c_1}{r}$ (use the ODE technique of separation of variables), we substitute back to get:

$$R(r) = c_1 \ln r + c_2$$

CASE 2 $(\lambda > 0)$ Here Euler's equation is:

$$r^2R'' + rR' - \lambda^2 R = 0$$

and to solve this, we look for solutions of the form $R(r) = r^\alpha$. The goal is to find two values of α (say α_1 and α_2) so that the general solution will be:

$$R(r) = ar^{\alpha_1} + br^{\alpha_2}$$

Plugging $R(r) = r^\alpha$ into Euler's equation yields $\alpha = \lambda, -\lambda$ and, hence,

$$R(r) = ar^\lambda + br^{-\lambda}$$

CHAPTER 2

Laplace's Equation on a Rectangle

In two dimensions the heat equation [1] is:

$$u_t = \alpha(u_{xx} + u_{yy}) = \alpha \Delta u$$

where $\Delta u = u_{xx} + u_{yy}$ is the **Laplacian** of u (the operator Δ is the 'Laplacian'). If the solution reaches an equilibrium, the resulting steady state will satisfy

$$u_{xx} + u_{yy} = 0$$

This equation is **Laplace's equation** in two dimensions, one of the essential equations in applied mathematics (and the most important for time-independent problems). Note that in general, the Laplacian for a function $u(x_1, \cdots, x_n)$ in $\mathbb{R}^n \to \mathbb{R}$ is defined to be the sum of the second partial derivatives:

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$$

Laplace's equation is then compactly written as:

$$\Delta u = 0$$

[1] The derivation follows the same argument as what we did in one dimension.

The inhomogeneous case, i.e.

$$\Delta u = f$$

the equation is called **Poisson's equation**. Innumerable physical systems are described by Laplace's equation or Poisson's equation, beyond steady states for the heat equation: inviscid fluid flow (e.g. flow past an airfoil), stress in a solid, electric fields, wavefunctions (time independence) in quantum mechanics, and more.
The two differences with the wave equation

$$u_{tt} = c^2 u_{xx}$$

are:

- We specify **boundary conditions** in both directions, not initial conditions in t.
- There is an opposite sign; we have $u_{xx} = -u_{yy}$ rather than $u_{tt} = c^2 u_{xx}$.

The first point changes the way the problem is solved slightly; the second point changes the answer. Note that there is also no coefficient, but this is not really important (we can just as easily solve $u_{xx} + k^2 u_{yy} = 0$).

2.1 Dirichlet Problem for a Rectangle

We first consider the boundary-value problem

$$\nabla^2 u = u_{xx} + u_{yy} = 0, \quad 0 < x < a,\ 0 < y < b \tag{2.1}$$

$$u(x, 0) = f(x), \quad u(x, b) = 0, \quad 0 \leq x \leq a \tag{2.2}$$

$$u(0, y) = 0, \quad u(a, y) = 0, \quad 0 \leq y \leq b \tag{2.3}$$

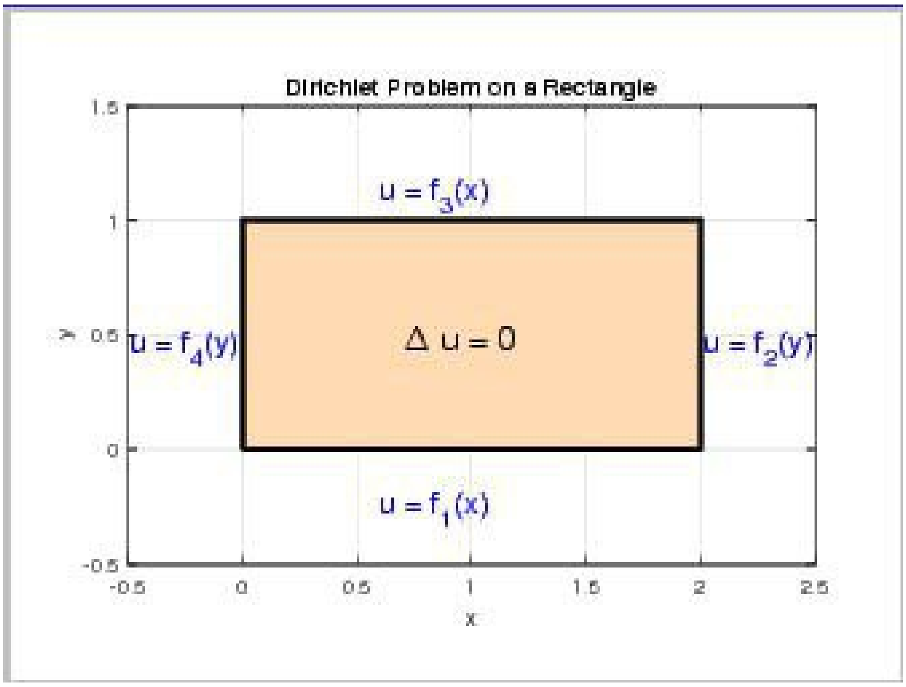


Figure 2.1: Dirichlet problem in a rectangle.

Example 2.1

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \textit{in } (x, y) \in [0, 1] \times [0, 1]$$

$$U(x, 0) = 0 \quad \textit{(bottom Boundary)}$$

$$U(x, 1) = \sin(\pi x) \quad \textit{(top Boundary)}$$

$$U(0, y) = 0 \quad \textit{(left Boundary)}$$

$$U(1, y) = 0 \quad \textit{(right Boundary)}$$

Let us solve the Laplace equation:

1. *Using the method of separation of variables.*

2. *Using the Finite Difference Method (FDM) with a grid step size of $h_x = h_y = h = 0.25$, [8].*

Analytical solution

by applying the method of separation of variables to a rectangle:

Detailed Derivation

Consider Laplace's equation in a rectangular domain $0 \leq x \leq a,\ 0 \leq y \leq b$, with the following boundary conditions:

- $u(0, y) = 0, u(a, y) = 0$
- $u(x, 0) = 0, u(x, b) = f(x)$

This setup represents a typical Dirichlet problem.

- **Insert the Product Solution:**

  Assume $u(x, y) = X(x)Y(y)$ and substitute into:

  $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

  Compute the second partial derivatives:

  $$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 X}{dx^2} Y(y), \quad \frac{\partial^2 u}{\partial y^2} = X(x) \frac{d^2 Y}{dy^2}$$

  Substituting yields:

  $$\frac{d^2 X}{dx^2} Y(y) + X(x) \frac{d^2 Y}{dy^2} = 0$$

- **Isolate Variables:**

  Divide both sides by $X(x)Y(y)$, assuming $X \neq 0$ and $Y \neq 0$:

  $$\frac{1}{X(x)} \frac{d^2 X}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y}{dy^2} = 0$$

  This implies:

  $$\frac{1}{X(x)} \frac{d^2 X}{dx^2} = -\frac{1}{Y(y)} \frac{d^2 Y}{dy^2}$$

  Since the left side depends only on (x) and the right side only on (y), both must equal a constant, say λ:

  $$\frac{1}{X(x)} \frac{d^2 X}{dx^2} = \lambda, \quad -\frac{1}{Y(y)} \frac{d^2 Y}{dy^2} = \lambda$$

This results in two ODEs:

- $X''(x) - \lambda X(x) = 0$
- $Y''(y) + \lambda Y(y) = 0$

The choice of λ affects the solution form. To align with the homogeneous boundary conditions, we assume $\lambda > 0$.

- **Solve the (X)-Equation:**

For the $X(x)$-equation:

$$X''(x) - \lambda X(x) = 0$$

Let $\lambda = \mu^2$ (where $\mu > 0$) to obtain oscillatory solutions:

$$X''(x) + \mu^2 X(x) = 0$$

The general solution is:

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

Apply the boundary conditions:

- $u(0, y) = X(0)Y(y) = 0 \Rightarrow X(0) = 0$

$$X(0) = A\cos(0) + B\sin(0) = A = 0 \Rightarrow A = 0$$

So, $X(x) = B\sin(\mu x)$

- $u(a, y) = X(a)Y(y) = 0 \Rightarrow X(a) = 0$

$$X(a) = B\sin(\mu a) = 0$$

For a non-trivial solution ($B \neq 0$):

$$\sin(\mu a) = 0 \Rightarrow \mu a = n\pi,\ n = 1, 2, 3, \ldots$$

So:

$$\mu_n = \frac{n\pi}{a}, \quad \lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{a}\right)^2$$

The eigenfunctions are:

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{a}\right)$$

- Solve the (Y)-Equation:

$$Y''(y) + \lambda_n Y(y) = 0$$

- Substitute $\lambda_n = -\left(\frac{n\pi}{a}\right)^2$:

$$Y''(y) - \left(\frac{n\pi}{a}\right)^2 Y(y) = 0$$

- The general solution is:

$$Y_n(y) = C_n e^{\frac{n\pi}{a}y} + D_n e^{-\frac{n\pi}{a}y}$$

- Alternatively, use hyperbolic functions:

$$Y_n(y) = E_n \sinh\left(\frac{n\pi y}{a}\right) + F_n \cosh\left(\frac{n\pi y}{a}\right)$$

- Apply the bottom boundary condition:

$$u_n(x, y) = X_n(x)Y_n(y) = \sin\left(\frac{n\pi x}{a}\right)\left[E_n \sinh\left(\frac{n\pi y}{a}\right) + F_n \cosh\left(\frac{n\pi y}{a}\right)\right]$$

- Apply $u(x, 0) = 0 \Rightarrow u_n(x, 0) = 0$:

$$u_n(x, 0) = \sin\left(\frac{n\pi x}{a}\right)[E_n \sinh(0) + F_n \cosh(0)] = \sin\left(\frac{n\pi x}{a}\right)$$

- Thus, $F_n = 0$, and:

$$u_n(x, y) = E_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

- From the general solution:

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

  where $A_n = E_n$.

- Apply the top boundary condition:

  At $y = b$, $u(x,b) = f(x)$:

$$u(x,b) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right) = f(x)$$

- This is a Fourier sine series. The coefficients are found as:

$$A_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

- Thus:

$$A_n = \frac{2}{\sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

Solution of Example 2.1:

This scenario might the electric potential in a capacitor with a sinusoidal potential on the top edge.

The solution is:

$$U(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \sinh(n\pi y)$$

At $y = 1$:

$$\sum_{n=1}^{\infty} A_n \sin(n\pi x) \sinh(n\pi) = \sin(\pi x)$$

The Fourier series has only one term $n = 1$:

$$A_1 \sinh(\pi) \sin(\pi x) = \sin(\pi x) \Rightarrow A_1 = \frac{1}{\sinh(\pi)}$$

$$A_n = 0 \quad \text{for } n \neq 1$$

Thus

$$u(x, y) = \frac{\sin(\pi x)\sinh(\pi y)}{\sinh(\pi)}$$

This solution describes a potential that varies sinusoidally in x and grows hyperbolically in y, consistent with the boundary conditions.

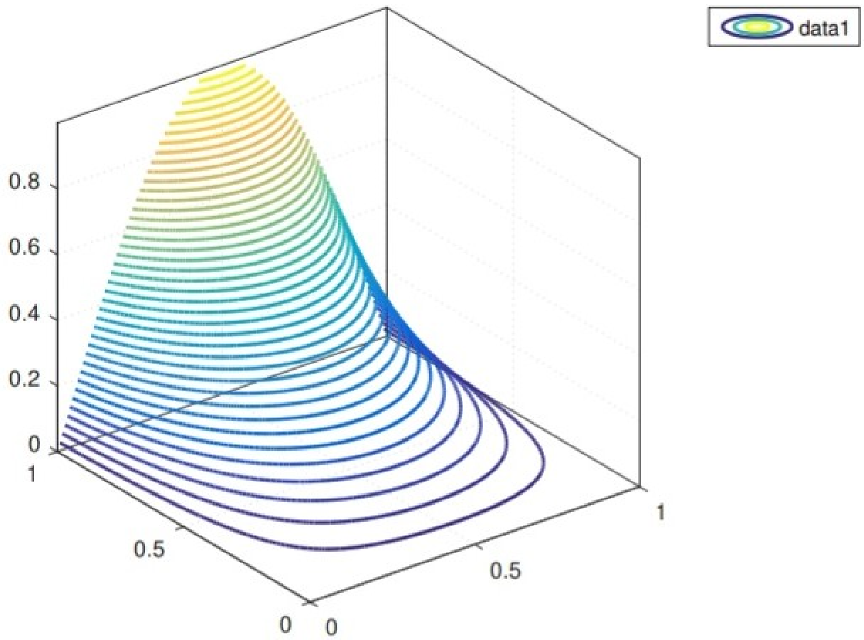


Figure 2.2: Analytical solution of the Laplace's equation in unit square

Numerical Solution

by applying finite difference method:

1. Grid Discretization

Along the X-axis We have

$$h_x = h_y = \frac{b-a}{n} \Rightarrow n = \frac{1-0}{0.25} = 4 \quad ; (n = nx = ny),$$

Number of nodes:

$$(n_x+1)(n_y+1) = (n+1)^2 = 25$$

$$\begin{cases} x_i = 0,\ 0.25,\ 0.5,\ 0.75,\ 1 \\ y_i = 0,\ 0.25,\ 0.5,\ 0.75,\ 1 \end{cases}$$

2. Interior Nodes

$$(n_x-1)(n_y-1) = (n-1)^2 = 3\times 3 = 9$$

j \ i	$x_1 = 0.25$	$x_2 = 0.5$	$x_3 = 0.75$
$y_1 = 0.25$	$U_{1,1}$	$U_{2,1}$	$U_{3,1}$
$y_2 = 0.5$	$U_{1,2}$	$U_{2,2}$	$U_{3,2}$
$y_3 = 0.75$	$U_{1,3}$	$U_{2,3}$	$U_{3,3}$

3. Boundary Conditions

Only the top boundary $y = 1$ has non-zero values:

$$U(x,1) = \sin(\pi x)$$

At the interval x values:

$$U_{1\,4} = U(x_1, 1) \Rightarrow U_{1,4} = \sin(0.25\pi) = \frac{\sqrt{2}}{2} \simeq 0.7071$$

$$U_{2,4} = U(x_2, 1) \Rightarrow U_{2,4} = \sin(0.5\pi) = 1$$

$$U_{3,4} = U(x_3, 1) \Rightarrow U_{3,4} = \sin(0.75\pi) = \frac{\sqrt{2}}{2} \simeq 0.7071$$

All other boundaries: $U = 0$

4. Finite Difference Scheme

We apply the standard 5-point stencil:

$$U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j} = 0$$

This leads to a 9×9 linear system:

$$AU = B$$

Where:

- A: Coefficients matrix 9×9
- B: Right-hand side vector including top boundary
- U: Column vector with the 9 unknowns

At the point $U_{1,1}$

$$U_{2,1} + U_{1,2} + U_{1,0} + U_{0,1} - 4U_{1,1} = 0$$

At the point $U_{1,2}$

$$U_{2,2} + U_{1,3} + U_{1,1} + U_{0,2} - 4U_{1,2} = 0$$

At the point $U_{1,3}$

$$U_{2,3} + U_{1,4} + U_{1,2} + U_{0,3} - 4U_{1,3} = 0$$

At the point $U_{2,1}$

$$U_{3,1} + U_{2,2} + U_{2,0} + U_{1,1} - 4U_{2,1} = 0$$

At the point $U_{2,2}$

$$U_{3,2} + U_{2,3} + U_{2,1} + U_{1,2} - 4U_{2,2} = 0$$

At the point $U_{2,3}$

$$U_{3,3} + U_{2,4} + U_{2,2} + U_{1,3} - 4U_{2,3} = 0$$

At the point $U_{3,1}$

$$U_{3,2} + U_{3,0} + U_{2,1} + U_{4,1} - 4U_{3,1} = 0$$

At the point $U_{3,2}$

$$U_{4,2} + U_{3,3} + U_{3,1} + U_{2,2} - 4U_{3,2} = 0$$

At the point $U_{3,3}$

$$U_{4,3} + U_{3,4} + U_{3,2} + U_{2,3} - 4U_{3,3} = 0$$

$$\begin{cases} U_{2,1} + U_{1,2} + U_{1,0} + U_{0,1} - 4U_{1,1} = 0 \\ U_{2,2} + U_{1,3} + U_{1,1} + U_{0,2} - 4U_{1,2} = 0 \\ U_{2,3} + \sin\left(\frac{\pi}{4}\right) + U_{1,2} + U_{0,3} - 4U_{1,3} = 0 \\ U_{3,1} + U_{2,2} + U_{2,0} + U_{1,1} - 4U_{2,1} = 0 \\ U_{3,2} + U_{2,3} + U_{2,1} + U_{1,2} - 4U_{2,2} = 0 \\ U_{3,3} + \sin\left(\frac{\pi}{2}\right) + U_{2,2} + U_{1,3} - 4U_{2,3} = 0 \\ U_{3,2} + U_{3,0} + U_{2,1} + U_{4,1} - 4U_{3,1} = 0 \\ U_{4,2} + U_{3,3} + U_{3,1} + U_{2,2} - 4U_{3,2} = 0 \\ U_{4,3} + \sin\left(\frac{3\pi}{4}\right) + U_{3,2} - 4U_{3,3} = 0 \end{cases}$$

$$
\begin{aligned}
&U_{1,0} = U_{0,1} = 0 && \\
&U_{0,2} = 0 && \\
&U_{0,3} = 0 && U_{1,4} = \sin\left(\tfrac{\pi}{4}\right) \\
&U_{2,0} = 0 && \\
&U_{2,4} = \sin\left(\tfrac{\pi}{2}\right) && \\
&U_{3,0} = 0 && U_{4,1} = 0 \\
& && U_{4,2} = 0 \\
&U_{4,3} = 0 && U_{2,4} = \sin\left(\tfrac{3\pi}{4}\right)
\end{aligned}
$$

$$
\begin{bmatrix}
-4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4
\end{bmatrix}
\begin{bmatrix}
U_{1,1} \\ U_{1,2} \\ U_{1,3} \\ U_{2,1} \\ U_{2,2} \\ U_{2,3} \\ U_{3,1} \\ U_{3,2} \\ U_{3,3}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ -\sin\left(\tfrac{\pi}{4}\right) \\ 0 \\ 0 \\ -\sin\left(\tfrac{\pi}{2}\right) \\ 0 \\ 0 \\ -\sin\left(\tfrac{3\pi}{4}\right)
\end{bmatrix}
$$

Using one of the methods studied in Numerical Analysis (for solving linear systems), we obtain the following solution:

$$U_{i,j} = \begin{bmatrix} 0.0584 \\ 0.1509 \\ 0.3318 \\ 0.0825 \\ 0.2134 \\ 0.4693 \\ 0.0584 \\ 0.1509 \\ 0.3318 \end{bmatrix} \qquad i,j = 1,2,3$$

The boundary conditions have brought us to the following grid in which we will seek the unknowns of the equation:

Approximate solution $U(i,j)$ using FDM with $h = 0.25$

y x	0.00	0.25	0.5	0.75	1.0
1.0	0.000	0.707	1.000	0.707	0.000
0.75	0.000	0.332	0.469	0.332	0.000
0.5	0.000	0.151	0.213	0.151	0.000
0.25	0.000	0.058	0.083	0.058	0.000
0.0	0.000	0.000	0.000	0.000	0.000

Grid values: $x, y = 0, 0.25, 0.5, 0.75, 1$ Using a small matlab program, we convert the matrix u into a vector to accurately solve the system AU=B

```
for j=1 : ny-1
 for i=1 : nx-1
v(k)=u(i.j);
k=k+1
 end
end
```

We opted for direct resolution as indicated by the title. below is the detailed matlab program that constructs the matrix based, on the computation of the vector's elements

```
clc;
clear;

% Grid setup
h = 0.25;
a = 0; b = 1;
c = 0; d = 1;

nx = (b - a)/h + 1;
ny = (d - c)/h + 1;
n = (nx - 2) * (ny - 2);
% number of interior points

% Initialize matrix A and vector B
A = zeros(n);
B = zeros(n,1);

% Construct matrix A and vector B
for j = 1:ny-2
    for i = 1:nx-2
        k = (j - 1)*(nx - 2) + i;
        A(k,k) = -4;

        if i > 1
            A(k, k - 1) = 1;
        end
        if i < (nx - 2)
            A(k, k + 1) = 1;
        end
        if j > 1
            A(k, k - (nx - 2)) = 1;
        end
        if j < (ny - 2)
            A(k, k + (nx - 2)) = 1;
```

```
        end

        % Apply top boundary condition: u(x,1) = sin(pi*x)
        if j == (ny - 2)
            x_i = i * h;
            B(k) = B(k) - sin(pi * x_i);
        end
    end
end

% Solve the linear system
V = A \ B;

% Fill the full solution matrix u including boundaries
u = zeros(ny, nx);
k = 1;
for j = 2:ny-1
    for i = 2:nx-1
        u(j,i) = V(k);
        k = k + 1;
    end
end

% Apply boundary conditions
x = linspace(a, b, nx);
u(end,:) = sin(pi * x);
% Top boundary: u(x,1)
% Other boundaries (u(x,0), u(0,y), u(1,y)) are already zero

% Plot the result
y = linspace(c, d, ny);
[X, Y] = meshgrid(x, y);
mesh(X, Y, u);
xlabel('x');
ylabel('y');
```

```
zlabel('u(x,y)');
title('Solution of Laplace''s Equation in the Unit Square Using FDM');
```

NOTE: to valudate the system at different values of simply replace the value h=0.25 with other values and rerun the program. this will generate the corresponding curve for each h.

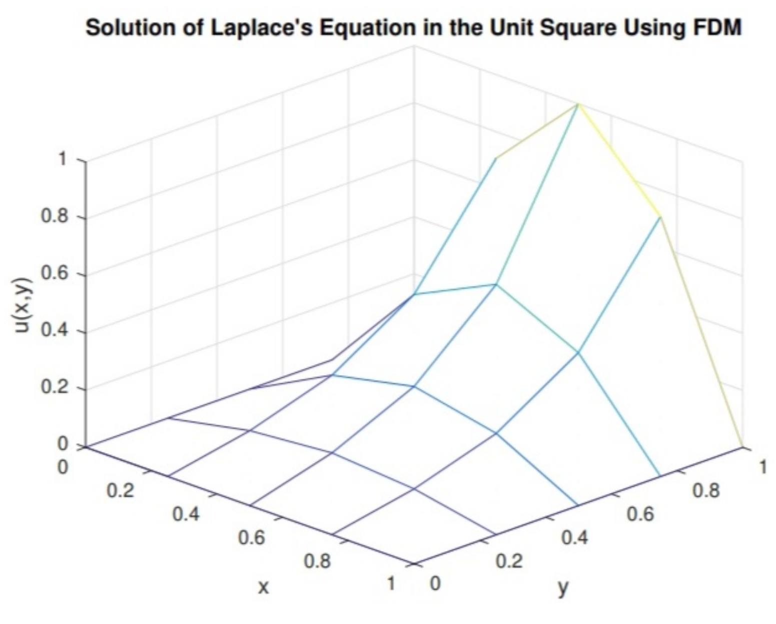


Figure 2.3: with h=0.25

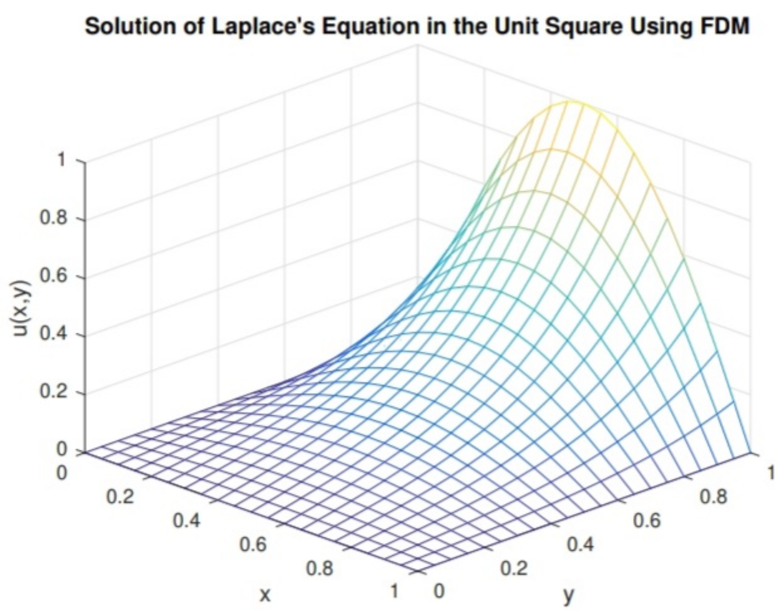


Figure 2.4: with h=0.05

After obtaining a numerical approximation of the partial differential equation using the finit difference method, it is important to evaluate the accuracy of this solution by comparing it with the analytical solution, if available. this comparaison to assess the effectiveness of the numerical method by calculating the error resulting from the numerical approximation and analyzing how this error is affected by the grid step size.

Maatlab program for the analytical solution

```
% Create the domain
x = linspace(0, 1, 100);
y = linspace(0, 1, 100);
[X, Y] = meshgrid(x, y);

% Analytical solution of Laplace's equation
U = (1 / sinh(pi)) * sin(pi * X) .* sinh(pi * Y);

% Plot 3D contour lines with enhanced appearance
figure('Color', 'w');
[~, h] = contour3(X, Y, U, 40);     % 40 contour levels for more detail
h.LineWidth = 1.5;                  % Increase line thickness
colormap turbo;                     % Use vivid colormap
colorbar;                           % Add colorbar

% Axis labels and title
xlabel('x', 'FontSize', 12);
ylabel('y', 'FontSize', 12);
zlabel('u(x,y)', 'FontSize', 12);
title('3D Contour Lines of the Analytical Solution of Laplace''s Equation', '
    FontSize', 14);
grid on;
view(45, 30);                       % Set view angle
```

x	y	Numerical solution	Exact solution	Absolute Error
0.25	0.25	0.058352	0.053187	0.005165
0.25	0.5	0.150887	0.140904	0.009983
0.25	0.75	0.331811	0.320099	0.011712
0.5	0.25	0.082522	0.075218	0.014118
0.5	0.5	0.213386	0.199268	0.016564
0.5	0.75	0.469251	0.452688	0.016564
0.75	0.25	0.058352	0.053187	0.005165
0.75	0.5	0.150887	0.140904	0.009983
0.75	0.75	0.331811	0.320099	0.011712

Table 2.1: comparative analysis of the analytical and numerical solution with h=0.25.

x	y	Numerical solution	Exact solution	Absolute Error
0.25	0.25	0.053361	0.053187	0.000174
0.25	0.5	0.141264	0.140904	0.000360
0.25	0.75	0.320554	0.320099	0.000455
0.5	0.25	0.075463	0.075218	0.000246
0.5	0.5	0.199778	0.199268	0.000509
0.5	0.75	0.453332	0.452688	0.000644
0.75	0.25	0.053361	0.053187	0.000174
0.75	0.5	0.141264	0.140904	0.000360
0.75	0.75	0.320554	0.320099	0.000455

Table 2.2: comparative analysis of the analytical and numerical solution with h=0.05.

CHAPTER 3

Laplace's Equation Inside a Circle

3.1 Laplace's equation in the polar coordinate system

It is a general principle in boundary value problems for PDEs to choose coordinates that make the formulas for the boundary as simple as possible. Here polar coordinates are used for this purpose as follows: since we want to discuss circular membranes, we first transform the Laplace's equation into polar coordinates.

Recall that **Laplace's equation** in $\mathbb{R}^2$ in terms of the usual (i.e., Cartesian) (x, y) coordinate system is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0 \tag{3.1.1}$$

The Cartesian coordinates can be represented by the polar coordinates as follows:

$$x = r\cos\theta, \quad y = r\sin\theta \tag{3.1.2}$$

Let us first compute the partial derivatives of x, y w.r.t. r, θ.

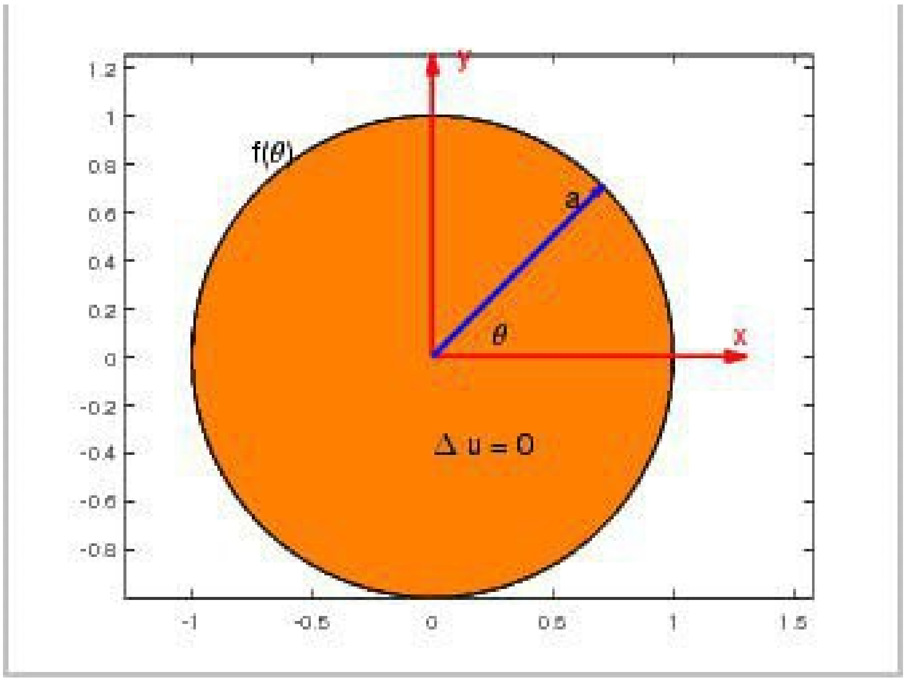
$$\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\ theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \tag{3.1.3}$$

In two-dimensional polar coordinates (r, θ), Laplace's equation takes the form:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

3.2 Dirichlet Problem inside a circle

We shall now establish the existence of the solution of the Dirichlet Problem for a circle.



The Dirichlet problem is

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 \le r < a, \quad 0 < \theta \le 2\pi \tag{3.2.1}$$

$$u(a, \theta) = f(\theta) \text{ for all } \theta \text{ in } [0, 2\pi] \tag{3.2.2}$$

Example 3.1 *Consider the interior Dirichlet problem*

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\begin{cases} 0 \le r < 3 \\ u(3, \theta) = 2\sin 4\theta - 3\cos 7\theta \end{cases}$$

Analytical solution:

By the method of separation of variables, We seek partial nontrivial solutions of the Laplace equation represented as a product [2]:

$$u(r, \theta) = R(r)\Theta(\theta).$$

Substituting this product into the Laplace equation, we get upon multiplication by $r^2 R(r)^{-1}(r)\Theta^{-1}(\theta)$ *that*

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0 \iff \frac{r^2 R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

Separation of variables yields:

$$\frac{r^2 R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

where, as usual, we denote a constant by λ*. Then we get two ordinary differential equations for each unknown function:*

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0, \quad \Theta''(\theta) + \lambda\Theta(\theta) = 0$$

We want to find solutions that are periodic: $\Theta(\theta) = \Theta(\theta + 2\pi)$*, which leads to the Sturm–Liouville problem for function* Θ*:*

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad \Theta(\theta) = \Theta(\theta + 2\pi)$$

Since the differential equation has a periodic solution only when constant λ *is not negative,*

$$\Theta(\theta) = A\cos(\sqrt{\lambda}\theta) + B\sin(\sqrt{\lambda}\theta), \quad \textit{if } \lambda \textit{ is positive}.$$

$$\Theta(\theta) = A + B\theta, \quad \textit{if } \lambda \textit{ is zero}.$$

We conclude λ *must be an integer; its square root branch is convenient (but not necessarily) to choose nonnegative. Therefore, we get a sequence of eigenvalues* $\lambda = n^2, \quad n = 0, 1, 2, \ldots$*, and corresponding* *eigenfunctions ("circular harmonics"):*

$$\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta), \quad \text{for } n = 1, 2, \ldots$$

$$\Theta_0(\theta) = A_0, \quad \text{for } n = 0.$$

with some arbitrary constants A's and B's. The above two formulae could be united into one:

$$\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta), \quad n = 0, 1, 2, \ldots$$

because $\sin(0) = 0$ *and* $\cos(0) = 1$. *Substituting instead of λ its eigenvalue into the radial differential equation for R, we get:*

$$r^2 R''(r) + rR'(r) - n^2 R(r) = 0$$

The above equation is an example of Euler's equation; so we seek its solution in the form $R(r) = r^\alpha$. Upon its substitution into the radial differential equation, we obtain:

$$\alpha(\alpha - 1) + \alpha - n^2 = 0 \quad \Longleftrightarrow \quad \alpha^2 - n^2 = 0$$

Thus, we have two distinct solutions $\alpha = \pm n$ when n is positive, and one double root $\alpha = 0$ when n is zero. Correspondingly, we get the general solution for the radial Euler's equation:

$$R_n(r) = C_n r^n + D_n r^{-n}, \quad \text{for } n = 1, 2, 3, \ldots.$$

$$R_0(r) = C_0 + D_0 \ln r, \quad \text{for } n = 0.$$

Now we conclude that the most general solution of the Laplace partial differential equation would be a sum of the solutions found for all separation constants:

$$u(r, \theta) = C_0 + D_0 \ln r + \sum_{n \geq 1}^{\infty} [A_n \cos(n\theta) + B_n \sin(n\theta)] \left(C_n r^n + D_n r^{-n}\right)$$

or:

$$u(r,\theta) = \frac{a_0}{2} + d_0 \ln r + \sum_{n\geq 1}^{\infty} (\frac{r}{a})^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] + \sum_{n\geq 1}^{\infty} (\frac{a}{r})^n \left[c_n \cos(n\theta) + d_n \sin(n\theta)\right]$$

Solution of Example 3.1:

its solution is known to be:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n\leq 1} \left(\frac{r}{3}\right)^n \left[a_n \cos n\theta + b_n \sin n\theta\right]$$

When

$$0 \leq r \leq 3; \quad 0 \leq \theta \leq 2\pi$$

where f is a combination of eigenfunctions so we know that all coefficients in the above expansion are zeroes except $n = 4$ and $n = 7$.
Hence $b_4 = 2; a_7 = -3$ this yields

$$u(r,\theta) = 2\left(\frac{r}{3}\right)^4 \sin 4\theta - 3\left(\frac{r}{3}\right)^7 \cos 7\theta$$

Numerical Solution

by applying finite difference method:

1. Grid Discretization

Consider the interior Dirichlet Problem:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

when

$$\begin{cases} 0 \leq r < 3 \\ u(3,\theta) = 2\sin(4\theta) - 3\cos(7\theta) \end{cases}$$

We discretize the circle into a grid

- N_r: Points in the radial direction $r \in [0,3]$.

- N_θ : Points in the angular direction $\theta \in [0,2\pi]$.

Let we take:

$$N_r = 4; r = 0,1,2,3$$

$$N_\theta = 4; \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

We compute the step sizes:

$$h_r = \frac{3}{N_r - 1} = \frac{3}{(4-1)} = 1$$

$$h_\theta = \frac{2\pi}{N_\theta} = \frac{2\pi}{4} = \frac{\pi}{2}$$

2.Apply Boundary Conditions

We have a Dirichlet condition at $r = 3$, which we apply to the last column in the solution matrix:

$$u(3,\theta_j) = 2\sin(4\theta_j) - 3\cos(7\theta_j)$$

θ_j	$u(3,\theta_j) = 2\sin(4\theta_j) - 3\cos(7\theta_j)$
0	$2\sin(0) - 3\cos(0) = -3$
$\frac{\pi}{2}$	$2\sin(2\pi) - 3\cos\left(7\frac{\pi}{2}\right) = 0$
π	$2\sin(4\pi) - 3\cos(7\pi) = +3$
$\frac{3\pi}{2}$	$2\sin\left(12\frac{\pi}{2}\right) - 3\cos\left(21\frac{\pi}{2}\right) = 0$

Then :

$$\text{Boundary Conditions :} \begin{cases} u_{3,0} = -3 \\ u_{3,1} = u_{3,3} = 0 \\ u_{3,2} = 3 \end{cases}$$

3. Finite Difference Approximation

$$\frac{1}{r_i}\frac{(r_i+\frac{1}{2})u_{i+1,j}-2r_iu_{i,j}+(r_i-\frac{1}{2})u_{i-1,j}}{h_r^2}+\frac{1}{r_i^2}\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h_\theta^2}=0$$

At the point $U_{2,0}$:

$$r_i=2;\theta_j=0;h_\theta^2=\left(\frac{\pi}{2}\right)^2;h_r=1$$

$$\frac{1}{2}\left[\frac{(2.5)(u_{3,0}-4u_{2,0})+(1.5)u_{1,0}}{1}\right]+\frac{1}{4}\left[\frac{u_{2,1}-2u_{2,0}+u_{2,3}}{h_\theta^2}\right]=0$$

Apply the boundary conditions:

$$\frac{1}{2}\left[\frac{(2.5)(-3)-4u_{2,0}+(1.5)(0)}{1}\right]+\frac{1}{4}\left[\frac{0-2u_{2,0}+0}{\frac{\pi^2}{4}}\right]=0$$

$$\iff\frac{1}{2}\left[-\frac{15}{2}-4u_{2,0}\right]+\frac{1}{4}\left[\frac{-2u_{2,0}}{\frac{\pi^2}{4}}\right]=0$$

$$\iff\frac{1}{2}\left[-7.5-4u_{2,0}\right]+\left[\frac{-2u_{2,0}}{\pi^2}\right]=0$$

$$\iff -3.75-2u_{2,0}-\frac{2u_{2,0}}{\pi^2}=0$$

$$\iff -3.75=u_{2,0}\left(2+\frac{2}{\pi^2}\right)$$

$$\implies u_{2,0}=\frac{-3.75}{2+\frac{2}{\pi^2}}$$

$$\implies u_{2,0}\simeq -1.703$$

In the same way we calculate the next point.

Where we find full Table of Approximate Values:

r_i \ θ_j	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
3 (outer boundary)	-3	0.000	3	0.000
2	-1.703	0.000	1.703	0.000
1	-0.460	0.000	0.460	0.000
0 (center)	0.000	0.000	0.000	0.000

Conclusion

In this thesis, we addressed the treatment of Laplace's equation, a fundamental equation in the mathematical modeling of steady-state physical systems, using both analytical and numerical approaches.

On the analytical side, we employed the separation of variables method, a systematic and powerful technique that transforms partial differential equations into ordinary differential equations. This enabled us to obtain exact solutions expressed as Fourier series, both in rectangular and circular domains, under Dirichlet boundary conditions.

On the numerical side, we used the Finite Difference Method (FDM), which allowed us to discretize the domain and approximate partial derivatives with sufficient accuracy. Numerical implementations were carried out using MATLAB, which facilitated the visualization and direct comparison with analytical solutions.

The study included two main cases:

In Chapter 2, Laplace's equation was solved in a rectangular domain using both approaches, followed by a quantitative comparison.

In Chapter 3, the same comparison was repeated for a circular domain, after converting the equation into polar coordinates.

The comparison demonstrated that FDM yields results that closely match the analytical solutions, especially when fine grids are used—highlighting its effectiveness in regular geometries.

However, the separation of variables method presents certain limitations:

It becomes more complex when dealing with non-homogeneous boundary conditions, often requiring transformations or additional series expansions.

It is less effective in irregular or complex geometries, where conformal mapping or numerical

methods are preferred.
The method increases in complexity in polar or spherical coordinates, involving special functions such as Bessel or Legendre functions.
In contrast, numerical methods like FDM offer a practical and powerful alternative, particularly when analytical solutions are not feasible.

In conclusion, this study provides an integrated framework combining both theoretical and numerical approaches for solving Laplace's equation. It enhances the researcher's ability to tackle a wide range of steady-state problems in physics and engineering and lays a solid foundation for future work involving more complex geometries and nonlinear boundary conditions.

Bibliography

[1] Ahmad M. D. Al-Eybani, *Analytical Solution of Laplace's Equation via Separation of Variables*, *International Journal of Mathematics and Physical Sciences Research*, Vol. 13, No. 1, April–September 2025, pp. 50–54. ISSN 2348-5736.

[2] Curtis F. Gerald and Patrick O. Wheatley, *Applied Numerical Analysis*, 7th ed., Addison-Wesley, Boston, 2004. ISBN: 978-0-321-18766-3.

[3] Erwin Kreyszig, *Advanced Engineering Mathematics*, 10th ed., John Wiley & Sons, Hoboken, NJ, 2006.

[4] Evans, L. C. *Partial Differential Equations*. 2nd ed., American Mathematical Society, 2010.

[5] Logan, J. David, *Applied Partial Differential Equations*, Springer, 3rd edition, 2015.

[6] Mostafa Chekak, *Numerical Series*, École Normale Supérieure de l'Enseignement Technique, Oran. (A title of book in Arabic translated to english))

[7] Randall J. LeVeque, *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems*, SIAM, Philadelphia, 2007.

[8] Stanley J. Farlow, *Partial Differential Equations for Scientists and Engineers*, Dover Publications, New York, 1982.

[9] Tyn Myint-U and Lokenath Debnath, *Linear Partial Differential Equations for Scientists and Engineers*, 4th ed., Birkhäuser, Boston, 2007.