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Some Aspects of Korovkin-Type Theorem and Positive Operators

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Dedication

بسم الله الرحمن الرحيم

يرفع الله الذين آمنوا منكم والذين أوتوا العلم درجات

صدق الله العظيم

To my beloved father, who has always been a great source of support throughout my journey, I would like to sincerely thank you for your constant support, sacrifices, and encouragement. Your dedication and guidance have always meant a great deal to me.

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Notations

Notation	Description
$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{K}$	The field $\mathbb{R}$ or $\mathbb{C}$
$\mathbb{K}^n$	Euclidean space
D	Directed set
$\leq$	Preorder on a directed set
$\overline{A}$	Closure of a set A
α, β, γ	Indices of a net
$(X, \\|\cdot\\|)$	Normed/Banach space
$C([a,b])$	Space of continuous functions on $[a,b]$
$C([0,1])$	Space of continuous functions on $[0,1]$
$C(X)$	Space of continuous functions on compact X
$C_0(X)$	Continuous functions vanishing at infinity
$L^p([a,b])$	Lebesgue space of p-integrable functions
$\\|f\\|_\infty$	Supremum (uniform) norm
$\\|f\\|_p$	L^p-norm
$\\|f\\|_{\phi,p}$	Weighted L^p-norm
$\lim_{n\to\infty} \\|f_n - f\\|_\infty = 0$	Uniform convergence
$\lim_{n\to\infty} f_n(x) = f(x)$	Pointwise convergence
$\mathscr{F}$	A Cauchy filter
$(x_\alpha)_{\alpha\in D}$	Net (generalized sequence)
$(L_\alpha)_{\alpha\in D}$	Net of positive linear operators
$e_0(x) = 1$	Constant test function
$e_1(x) = x$	Linear test function
$e_2(x) = x^2$	Quadratic test function
$T : C([a,b]) \to C([a,b])$	Linear operator
$T(f) \geq 0$ for $f \geq 0$	Positivity condition
st- lim	Statistical convergence
$B_n(f;x)$	Bernstein operator
$\binom{n}{k}$	The binomial coefficient
$S_n(f;x)$	Szász-Mirakjan operator
$V_n(f;x)$	Baskakov operator
δ_x	Dirac measure at x
$\int_K f\,d\mu$	Integral with respect to measure μ
$\int_{\mathbb{R}^d} \exp\left(-\frac{n}{4}\\|t-x\\|^2\right) dx$	Gaussian kernel integral

Introduction

Approximation theory plays a fundamental role in functional analysis and has numerous applications in pure and applied mathematics. Among the central results in this area, Korovkin-type theorems provide powerful criteria for studying the convergence of positive linear operators.
Following the publication of Korovkin's theorem in 1953 [18], numerous researchers have initiated systematic efforts to generalize and extend the result in different directions and settings. The main objectives of these investigations were:

(i) to identify alternative test functions similar to $\{e_0, e_1, e_2\}$ in $C([0,1])$;

(ii) to generalize Korovkin's first theorem to broader function spaces and, in particular, to abstract Banach spaces;

(iii) to prove analogous results for wider classes of linear operators.

These developments led to the emergence of what is now known as the *Korovkin-type Approximation Theory*. A significant milestone in this theory was the introduction of Korovkin sets by Baskakov in 1961 [9]. This notion was subsequently extended by Lorentz [21]. An extensive survey of the geometric aspects of Korovkin sets was later provided by Berens and Lorentz in 1975 [10].
The theory has been thoroughly developed within the framework of $C(X)$ spaces. Important contributions in this setting were made by Šaškin in 1962 [26], Wulbert in 1968 [29], Berens and Lorentz in 1973 [21], and Bauer in 1978 [7].
During the 1970s and 1980s, the scope of Korovkin-type Approximation Theory expanded considerably. Research extended to classical Banach function spaces such as $L^p(X,\mu)$ spaces, as well as to more abstract structures including Banach lattices and locally convex vector lattices. In particular, Wolff obtained results of this nature in 1978 [28]for sequentially complete convex linear lattices. A comprehensive account of the evolution of the theory up to 1994 can be found in the monograph by Altomare and Campiti [4].
More recently, renewed interest in Korovkin theory has generated further contributions. Notable works include those of Heping in 2005 [16], Guessab and Schmeisser in 2008 [15], Altomare , Leonessa and Rasa in 2008 [5], and Mahmudov in 2009 [23]. For a complete overview of the results obtained up to 2011, we refer to the survey paper, where Altomare [3] also proposed several

open problems to stimulate further research in this area.

The main contribution of this master thesis is the extension of the classical Korovkin theorem to nets (generalized sequences). While the classical theorem is formulated for sequences indexed by natural numbers, nets allow us to consider convergence for more general directed sets. This generalization is particularly useful in topological spaces that are not first-countable, where sequences are insufficient to characterize convergence.

This dissertation is divided into three chapters.

Chapter 1 introduces the fundamental preliminaries needed throughout this work. We begin with Banach spaces and their basic properties, followed by an introduction to the main function spaces used in approximation theory, namely $C([a,b])$ and $L^p([a,b])$. We then recall the concepts of pointwise and uniform convergence for sequences of functions. The chapter concludes with an introduction to nets (generalized sequences), including directed sets, convergence of nets, fundamental properties such as the characterization of closure and continuity, and Cauchy nets. These notions are essential for the generalization presented in Chapter 2.

Chapter 2 recalls the definition and basic properties of positive linear operators,and devoted to Korovkin-type approximation theory and positive linear operators. We present the classical Korovkin theorem, the associated test functions, and several generalized forms involving nets and statistical convergence.

Chapter 3 we discuss several applications of Korovkin-type theorems. Particular attention is given to approximation results in C (X) and Lp-spaces, as well as to important classes of positive linear operators such as Bernstein, Szász–Mirakjan, and Baskakov operators.

In summary, this thesis provides a self-contained introduction to Korovkin-type approximation theory, with a particular emphasis on the generalization to nets. The theoretical results are illustrated through classical examples of positive linear operators, demonstrating the power and versatility of Korovkin's theorem in approximation theory. This work highlights the significance of positive linear operators as effective tools in functional analysis and approximation theory.

Chapter 1

Preliminaries

1.1 Banach Spaces

A **Banach space** is a complete normed vector space, meaning every Cauchy sequence converges within the space. It plays a central role in functional analysis and the study of convergence.

Definition 1.1. (***Normed Spaces***) *Let X be a linear space over the field $\mathbb{R}$ or $\mathbb{C}$. A function $\|\cdot\| : X \to [0,\infty)$ is called a norm on X if for all $x, y \in X$ and $\alpha \in \mathbb{R}$ (or $\mathbb{C}$) the following properties hold:*

1. $\|x\| = 0$ *if and only if* $x = 0$.
2. $\|\alpha x\| = |\alpha|\,\|x\|$.
3. $\|x + y\| \leq \|x\| + \|y\|$ *(Triangle inequality).*

The pair $(X, \|\cdot\|)$ is called a normed linear space.

Definition 1.2. (***Cauchy Sequence***) *Let $(X, \|\cdot\|)$ be a normed vector space. Let $(x_n)_n$ be a sequence in X. Then $(x_n)_n$ is a* Cauchy sequence *if and only if:*

$$\forall\, \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall\, m, n \in \mathbb{N},\ m, n \geq N \implies \|x_n - x_m\| < \varepsilon.$$

Definition 1.3. (***Banach Spaces***) *Let $(X, \|\cdot\|)$ be a normed linear space. If every Cauchy sequence in X converges to an element of X, then $(X, \|\cdot\|)$ is called a Banach space.*

Example 1.1. *The Euclidean space*

$$\mathbb{K}^n = \{x = (x_1, x_2, \ldots, x_n)\ ;\ x_i \in \mathbb{K},\ i = 1, \ldots, n\},$$

with

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \qquad x = (x_1, x_2, \ldots, x_n) \in \mathbb{K}^n,$$

is a Banach space.

Example 1.2. *For any* $p \in [1,\infty[$,

$$\ell^p(\mathbb{K}) = \left\{ x = (x_n)_{n\geq 1} \subset \mathbb{K}\,;\ \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

with

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p},$$

is a Banach space.

1.2 Fundamental Function Spaces

This section defines Banach function spaces and introduces the space of continuous functions C([a,b]) and $L^p([a,b])$, for more details, see [20].

Definition 1.4. (***Banach Function Spaces***) *Let M be the set of all measurable functions on $\mathbb{R}^n$. A linear subspace $X \subset M$ is called a Banach function space if there exists a norm $\|\cdot\|_X$ such that:*

i. $\|f\|_X = 0$ *if and only if* $f = 0$ *almost everywhere.*

ii. $\|\lambda f\|_X = |\lambda| \|f\|_X$.

iii. $\|f + g\|_X \leq \|f\|_X + \|g\|_X$.

iv. If $0 \leq g \leq f$ *then* $\|g\|_X \leq \|f\|_X$.

Proposition 1.1. *If f and g are two continuous functions , then*
$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

Definition 1.5. (***The Space of Continuous Functions*** $C([a,b])$) *Let $[a,b] \subset \mathbb{R}$. The space $C([a,b])$ is defined as*

$$C([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous on } [a,b]\},$$

with

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|,$$

is Banach space with respect to the supremum norm.

Proof 1. *For any $x \in [a,b]$, by the triangle inequality for real numbers,*

$$|f(x) + g(x)| \leq |f(x)| + |g(x)|.$$

Since $\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$ *and* $\|g\|_\infty = \sup_{x\in[a,b]} |g(x)|$, *we have for all* $x \in [a,b]$:

$$|f(x)| \leq \|f\|_\infty, \qquad |g(x)| \leq \|g\|_\infty.$$

Thus,

$$|f(x)+g(x)| \le \|f\|_\infty + \|g\|_\infty.$$

The right-hand side is a constant independent of x. Taking the supremum over all $x \in [a,b]$ yields

$$\|f+g\|_\infty = \sup_{x\in[a,b]} |f(x)+g(x)| \le \|f\|_\infty + \|g\|_\infty.$$

Definition 1.6. *We denote by $C([0,1])$ the space of all real-valued continuous functions defined on $[0,1]$, endowed with the uniform norm*

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$

Definition 1.7. *Let X be a locally compact Hausdorff space. The space*

$$C_0(X) = \left\{ f \in C(X)\,;\, \forall \varepsilon > 0,\ \exists K \subset X \text{ compact such that } |f(x)| < \varepsilon \text{ for all } x \notin K \right\}$$

Definition 1.8.

$$C^\infty(X) = \left\{ f : X \to \mathbb{R}\,;\, f \text{ is infinitely differentiable on } X \right\}$$

Definition 1.9. ***(The Space** $L^p([a,b])$**)** Let $1 \le p < \infty$. The space $L^p([a,b])$ is defined as*

$$L^p([a,b]) = \left\{ f : [a,b] \to \mathbb{R} \mid \int_a^b |f(x)|^p\,dx < \infty \right\},$$

$$\|f\|_p = \left(\int |f(x)|^p\,dx \right)^{\frac{1}{p}}.$$

1.3 Concepts of Convergence of Sequences

Let $E \subset \mathbb{R}$ be a set and let $(f_n)_{n=1}^\infty$ be a sequence of real-valued functions on E. Suppose that f is also a real-valued function on E.

Definition 1.10. (***Pointwise Convergence***) *A sequence $(f_n)_{n=1}^\infty$ is said to converge* ***pointwise*** *to f on E as $n \to \infty$ if for each $x \in E$ and each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } n > N.$$

Here, N may depend on both ε and x.

Definition 1.11. (***Uniform Convergence***) *A sequence $(f_n)_{n=1}^\infty$ is said to converge* ***uniformly*** *to f on E if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } n > N \text{ and for all } x \in E.$$

Here, N depends only on ε and not on x.

In other words, uniform convergence occurs when the same N works for all $x \in E$.

Example 1.3.

$$f_n(x) = x^n \quad \textit{converges pointwise to} \quad f(x) = \begin{cases} 0 & \textit{if } 0 \le x < 1, \\ 1 & \textit{if } x = 1. \end{cases}$$

Proof 2. *Consider* $f_n(x) = x^n, \quad x \in [0,1]$. *Let* $x \in [0,1)$.

Since $0 \le x < 1$, *we have*

$$x^n \to 0 \textit{ as } n \to \infty.$$

Hence,

$$\lim_{n\to\infty} f_n(x) = 0.$$

Now, if $x = 1$, *then* $f_n(1) = 1^n = 1$ *for all* n, *and therefore*

$$\lim_{n\to\infty} f_n(1) = 1.$$

Thus, f_n *converges pointwise on* $[0,1]$ *to the function*

$$f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1. \end{cases}$$

Example 1.4. *The functions* $f_n(x) = x^n(1-x)$ *converge both pointwise and uniformly to the zero function as* $n \to \infty$.

Proof 3. *Consider*

$$f_n(x) = x^n(1-x), \qquad x \in [0,1].$$

First, we prove pointwise convergence.

Let $x \in [0,1]$.

If $x = 1$, *then*

$$f_n(1) = 1^n(1-1) = 0.$$

If $0 \le x < 1$, *then* $x^n \to 0$ *as* $n \to \infty$. *Hence,*

$$x^n(1-x) \to 0.$$

Therefore,

$$f_n(x) \to 0.$$

for every $x \in [0,1]$. *Thus,* f_n *converges pointwise to the zero function.*

Now we prove uniform convergence.

We compute the derivative:

$$f_n'(x) = nx^{n-1}(1-x) - x^n.$$

Factoring x^{n-1}, *we get*

$$f_n'(x) = x^{n-1}(n - (n+1)x).$$

Hence the critical point is

$$x = \frac{n}{n+1}.$$

Evaluating f_n at this point,

$$f_n\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right).$$

Thus,

$$\sup_{x\in[0,1]} |f_n(x)| = \left(\frac{n}{n+1}\right)^n \frac{1}{n+1}.$$

Since,

$$\left(\frac{n}{n+1}\right)^n \leq 1,$$

we obtain,

$$\sup_{x\in[0,1]} |f_n(x)| \leq \frac{1}{n+1} \to 0.$$

Therefore,

$$f_n \to 0$$

uniformly on $[0,1]$.

Theorem 1.1. *Let $(f_n)_{n=1}^{\infty}$ is a sequence of real-valued functions on some domain E. If $(f_n)_{n=1}^{\infty}$ continuous and converge uniformly to f as $n \to \infty$. Then f must also be continuous.*

Proof 4. *We need to show that f is continuous at any point $c \in E$. Use uniform convergence:*
Let $\varepsilon > 0$ be given. By the definition of uniform convergence, there exists $N \in \mathbb{N}$ such that for all $n > N$ and for all $x \in E$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}.$$

Fix such an $n > N$ (for example, $n = N+1$).
Use continuity of f_n:
Since f_n is continuous at c, there exists $\delta > 0$ such that whenever $|x-c| < \delta$ and $x \in E$, we have

$$|f_n(x) - f_n(c)| < \frac{\varepsilon}{3}.$$

For any $x \in E$ with $|x-c| < \delta$,

$$\begin{aligned}|f(x) - f(c)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon.\end{aligned}$$

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x-c| < \delta$ implies

$$|f(x) - f(c)| < \varepsilon.$$

Hence, f is continuous at c. Since c was arbitrary, f is continuous on E.

Corollary 1.1. *If a sequence of continuous functions $(f_n)_{n=1}^{\infty}$ converges pointwise to a discontinuous function as $n \to \infty$, the convergence cannot be uniform.*
This corollary applies directly to the example $f_n(x) := x^n$ on $[0,1]$ already analyzed above.

1.3.1 Basic Properties of The Sup Norm

Definition 1.12. (***Extreme Value Theorem***) *Let $I = [a,b]$ be a compact interval and let $f : I \to \mathbb{R}$ be continuous. Then f attains both a maximum and a minimum value on I. That is, there exist points $x_m, x_M \in I$ such that*

$$f(x_m) \le f(x) \le f(x_M), \qquad \forall x \in I.$$

By the Extreme Value Theorem, the supremum of any continuous function on a compact interval is always attained. Thus, if I is compact and f is continuous, then $\|f\|_\infty$ is always defined and finite.

Definition 1.13. (***The Sup Norm***) *Let I be an interval and let $f : I \to \mathbb{R}$ be a real-valued function. We define the **uniform (or supremum) norm** of f by*

$$\|f\|_\infty := \sup_{t \in I} |f(t)|.$$

(Note that other notations such as $\|f\|_{U(I)}$ may also be used for the same concept.)

Theorem 1.2. (***Uniform Cauchy Criterion***) *A sequence of functions $(f_n)_{n=1}^{\infty}$ converges uniformly on I as $n \to \infty$ if and only if for every $\varepsilon > 0$, there exists some N such that for all $n, m > N$,*

$$\|f_n - f_m\|_\infty < \varepsilon.$$

Example 1.5. *The series $\sum_{n=1}^{\infty} x^n$ converges uniformly on $[0, \frac{1}{2}]$.*
To prove this, use the uniform Cauchy Criterion. Let $S_K(x)$ be the K-th partial sum:

$$S_K(x) := x + x^2 + \cdots + x^K.$$

Then for $K_1 > K_2$,

$$S_{K_1}(x) - S_{K_2}(x) = x^{K_2+1} + \cdots + x^{K_1}.$$

Thus,

$$\begin{aligned}\|S_{K_1} - S_{K_2}\|_\infty &= \sup_{x \in [0,1/2]} \left(x^{K_2+1} + \cdots + x^{K_1}\right) \\ &= \sup_{x \in [0,1/2]} x^{K_2+1}(1 + \cdots + x^{K_1 - K_2 - 1}).\end{aligned}$$

Since $x \in [0, \frac{1}{2}]$, we have

$$1 + \cdots + x^{K_1 - K_2 - 1} \le \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

Hence,

$$\|S_{K_1} - S_{K_2}\|_\infty \le 2^{-K_2-1} \cdot 2.$$

So if N is large enough such that $2^{-N} < \varepsilon$ and $K_1 > K_2 > N$, then

$$\|S_{K_1} - S_{K_2}\|_\infty < \varepsilon.$$

1.4 Concepts of Convergence of Nets

The concept of nets generalizes sequences and overcomes their limitations in general topological spaces. By using directed sets instead of the natural numbers, nets provide a more flexible framework for studying convergence and play a central role in modern topology. Before defining Net, we need to introduce some basic concepts[12].

Definition 1.14. *A preorder on a set D is a binary relation $\preceq$ on D that is reflexive and transitive; that is, $x \preceq x \quad$ for all $x \in D$,*
and whenever $x \preceq y \quad$ and $\quad y \preceq z$, then $x \preceq z$.

Definition 1.15. (***Directed Set*** *[27]*) *A directed set is a nonempty set D together with a preorder $\preceq$ such that for any $\alpha, \beta \in D$, there exists $\gamma \in D$ with $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.*

Example 1.6. *The set $\mathbb{N}$ with the usual order $\le$ is a directed set.*

Definition 1.16. (***Net***) *Let D be a directed set and E a topological space. A net in E is a function*

$$x : D \to E,$$

denoted by $(x_\alpha)_{\alpha \in D}$.

Example 1.7. *Let $D = \mathbb{R}$ with the usual order $\le$. Since every two elements of $\mathbb{R}$ have a common upper bound, $(\mathbb{R}, \le)$ is a directed set.*
Define the net $(x_\alpha)_{\alpha \in D}$ by

$$x_\alpha = \frac{1}{\alpha + 1}, \qquad \alpha \in \mathbb{R}.$$

Then

$$\lim_\alpha x_\alpha = 0.$$

Indeed, for every $\varepsilon > 0$, choose $\alpha_0 > \frac{1}{\varepsilon} - 1$. Then, for all $\alpha \ge \alpha_0$,

$$|x_\alpha| = \frac{1}{\alpha + 1} < \varepsilon.$$

Hence, $x_\alpha \to 0$.

The concepts of accumulation points, closures of sets, and even the topology of a space can be characterized using convergence. The proofs are essentially similar to the classical arguments used for sequences in the real numbers, with only minor modifications[17].

Theorem 1.3. *Let E be a topological space. Then:*

(a) A point s is an accumulation point of a subset A of E if and only if there is a net in $A \setminus \{s\}$ which converges to s.

(b) A point s belongs to the closure of a subset A of E if and only if there is a net in A converging to s.

(c) A subset A of E is closed if and only if no net in A converges to a point of $E \setminus A$.

Proof 5. *If s is an accumulation point of A, then for each neighborhood U of s there is a point s_U of A which belongs to $U \setminus \{s\}$. The family $\mathcal{U}$ of all neighborhoods of s is directed by $\supseteq$, and if U and V are neighborhoods of s such that $V \subseteq U$, then $s_V \in V \subseteq U$.*
The net $\{s_U : U \in \mathcal{U}\}$ therefore converges to s. Conversely, if a net in $A \setminus \{s\}$ converges to s, then this net has values in every neighborhood of s, and hence $A \setminus \{s\}$ intersects every neighborhood of s. Thus s is an accumulation point of A. This proves (a).
To prove (b), recall that the closure of a set A consists of A together with all the accumulation points of A. For each accumulation point s of A, there is by part (a) a net in A converging to s. For each point $s \in A$, any constant net with value s converges to s. Hence every point of $\overline{A}$ is the limit of a net in A.
Conversely, if there is a net in A converging to s, then every neighborhood of s intersects A, and therefore $s \in \overline{A}$. This proves (b).
Finally, proposition (c) follows immediately from (b), since A is closed if and only if $\overline{A} = A$.

Theorem 1.4. *A topological space is a Hausdorff space if and only if each net in the space converges to at most one point.*

Theorem 1.5. *A point s in a topological space is a cluster point of a net S if and only if some subnet of S converges to s.*

Proof 6. *Let s be a cluster point of S and let $\mathcal{U}$ be the family of all neighborhoods of s. Then the intersection of two members of $\mathcal{U}$ is again a member of $\mathcal{U}$, and S is frequently in each member of $\mathcal{U}$. Consequently the preceding lemma applies and there is a subnet of S which is eventually in each member of $\mathcal{U}$, that is, converges to s.*
If s is not a cluster point of S, then there exists a neighborhood U of s such that S is not frequently in U, and therefore S is eventually in the complement of U. Then each subnet of S is eventually in the complement of U and hence cannot converge to s.

Proposition 1.2. *If $f : X \to Y$, f is continuous at x_0, and $\{x_i\}$ is a net in X that clusters at x_0, then $\{f(x_i)\}$ clusters at $f(x_0)$.*

Proof. Let V be any neighborhood of $f(x_0)$. Since f is continuous at x_0, there exists a neighborhood U of x_0 such that $f(U) \subseteq V$. Because $\{x_i\}$ clusters at x_0, for every i there exists $j \geq i$ with $x_j \in U$. Then $f(x_j) \in f(U) \subseteq V$. Hence for every i there exists $j \geq i$ such that $f(x_j) \in V$. Therefore $\{f(x_i)\}$ clusters at $f(x_0)$. □

Proposition 1.3. *[12] Let $K \subseteq X$. Then K is compact if and only if each net in K has a cluster point in K.*

Proof 7. *[12] Suppose that K is compact and let $\{x_i \mid i \in I\}$ be a net in K. For each $i \in I$, let $F_i = \{x_j \mid j \geq i\}$. Then each F_i is a closed subset of K. We show that $\{F_i \mid i \in I\}$ has the finite intersection property. Since I is directed, for any $i_1, \ldots, i_n \in I$ there exists $i \geq i_1, \ldots, i_n$. Then $F_i \subseteq \bigcap_{k=1}^{n} F_{i_k}$, so the intersection is nonempty. Because K is compact, there exists $x_0 \in \bigcap_{i \in I} \overline{F_i}$. For any open set U containing x_0 and any $i_0 \in I$, the fact that $x_0 \in \overline{\{x_i \mid i \geq i_0\}}$ implies there exists $i \geq i_0$ with $x_i \in U$. Hence x_0 is a cluster point of the net.*
Conversely, assume each net in K has a cluster point in K. Let $\{K_\alpha \mid \alpha \in A\}$ be a collection of relatively closed subsets of K with the finite intersection property. Let $\mathscr{F}$ be the collection of all finite subsets of A, ordered by inclusion. For each $F \in \mathscr{F}$, choose a point $x_F \in \bigcap\{K_\alpha \mid \alpha \in F\}$. Then $\{x_F\}$ is a net in K. By hypothesis, it has a cluster point $x_0 \in K$. Fix any $\alpha_0 \in A$. For any open set U containing x_0, there exists $F \in \mathscr{F}$ with $\alpha_0 \in F$ and $x_F \in U$. Since $x_F \in K_{\alpha_0}$, we have $U \cap K_{\alpha_0} \neq \emptyset$. Thus $x_0 \in \overline{K_{\alpha_0}} = K_{\alpha_0}$ (as K_{α_0} is relatively closed). Hence $x_0 \in \bigcap_{\alpha \in A} K_\alpha$, so K is compact.

The concept of a subnet allows one to show that whenever a net has a cluster point x, there exists a subnet converging to x. Although the definition of a subnet is somewhat technical, it plays an important role in topology. A detailed treatment can be found in Kelley [1955]. Moreover, a topological space is Hausdorff if and only if every convergent net has a unique limit.

Definition 1.17. (***Subnet [17]***) *A net $(Y_\beta)_{\beta \in H}$ is a subnet of a net $(x_\alpha)_{\alpha \in D}$ iff there exists a function N on D with values in D such that*
(a) $T = S \circ N$, or equivalently $T_i = S_{N_i}$ for each i in D; and
(b) for each m in D there exists n in E such that, if $p \geq n$, then $N_p \geq m$.

Definition 1.18. (***Convergence of Nets***) *A net (x_α) in a topological space E converges to $x \in E$ if for every neighborhood U of x, there exists $\alpha_0 \in D$ such that*

$$x_\alpha \in U \quad \textit{for all } \alpha \geq \alpha_0.$$

Definition 1.19. ***(Convergence in Norm)*** *Let (x_α) be a net in a normed space X. Then $x_\alpha \to x$ if*

$$\|x_\alpha - x\| \to 0.$$

Theorem 1.6. *In a topological space, a point x is a limit of a net (x_α) if and only if every neighborhood of x contains eventually all elements of the net.*

Proof 8. *Let $(x_\alpha)_{\alpha \in A}$ be a net in a topological space E, and let $x \in E$.*
By definition, the net (x_α) converges to x if and only if for every neighborhood U of x, there exists $\alpha_0 \in A$ such that for all $\alpha \geq \alpha_0$, we have

$$x_\alpha \in U.$$

But this condition means exactly that every neighborhood of x contains eventually all elements of the net.
Hence,

$$x_\alpha \to x \quad \Longleftrightarrow \quad \text{every neighborhood of } x \text{ contains eventually all elements of } (x_\alpha).$$

Therefore, the theorem is proved.

1.4.1 Fundamental Properties of Nets

In this section, we present some fundamental properties of nets and their role in describing convergence and other topological concepts [12].

Proposition 1.4. (***Uniqueness of Limits***) *In Hausdorff spaces, a net has at most one limit.*

Theorem 1.7. (***Characterization of Closure***) *A point $x \in E$ belongs to the closure of a set $A \subseteq E$ if and only if there exists a net (x_α) in A that converges to x.*

Proof 9. *Let $\mathcal{U} = \{U : U \text{ is open and } x \in U\}$.*
($\Rightarrow$) Suppose $x \in \overline{A}$. Then for each $U \in \mathcal{U}$ there exists a point $x_U \in A \cap U$. If $U_0 \in \mathcal{U}$, then $x_U \in U_0$ for every $U \geq U_0$ (with respect to the ordering by reverse inclusion). Thus the net (x_U) in A converges to x, i.e. $x = \lim x_U$.
($\Leftarrow$) Conversely, suppose there exists a net (x_α) in A such that $x_\alpha \to x$. Then for each $U \in \mathcal{U}$ there is some α_0 such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. Since $x_\alpha \in A$, this shows U contains a point of A. Hence $x \in \overline{A}$.

Theorem 1.8. (***Continuity***) *A function $f : E \to Y$ between topological spaces is continuous if and only if for every net (x_α) converging to x in E, the net $(f(x_\alpha))$ converges to $f(x)$ in Y.*

Proof 10. *($\Rightarrow$) Assume $f : E \to Y$ is continuous at $x \in E$, and let (x_α) be a net in E such that $x_\alpha \to x$. If V is an open set in Y with $f(x) \in V$, then by continuity there exists an open set $U \subseteq E$ such that $x \in U$ and $f(U) \subseteq V$. Since $x_\alpha \to x$, there exists α_0 such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. Hence $f(x_\alpha) \in V$ for all $\alpha \geq \alpha_0$, which shows $f(x_\alpha) \to f(x)$.*
($\Leftarrow$) Conversely, suppose f is not continuous at $x \in E$. Then there exists an open set $V \subseteq Y$ with $f(x) \in V$ such that for every open neighborhood U of x in E, we have $f(U) \setminus V \neq \emptyset$. Thus for each U in $\mathscr{U} = \{U : U \text{ open in } E, x \in U\}$ there exists a point $x_U \in U$ with $f(x_U) \notin V$. The net (x_U) converges to x, but $(f(x_U))$ cannot converge to $f(x)$ since it avoids V. This contradiction shows that f must be continuous at x.

Theorem 1.9. (***Compactness***) *A topological space E is compact if and only if every net in E has a convergent subnet.*

Sequences are special cases of nets where the directed set is $\mathbb{N}$ with the usual order. While sequences are sufficient to characterize convergence in metric spaces, they are not adequate in general topological spaces. Nets generalize sequences and allow the study of convergence in spaces that are not first-countable. In particular, every sequence is a net, but not every net can be represented as a sequence. This distinction becomes essential in non-metrizable spaces, where sequences fail to capture all topological properties.

Proposition 1.5. *Every sequence is a net, but not every net is a sequence.*

Proof 11. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a set E.*
Recall that a net is a function

$$x : D \to E,$$

where D is a directed set.
Since $(\mathbb{N}, \leq)$ is a directed set (because for every $m, n \in \mathbb{N}$ there exists $p = \max(m, n)$ such that $m \leq p$ and $n \leq p$), every sequence

$$(x_n)_{n\in\mathbb{N}}$$

can be viewed as a net indexed by $\mathbb{N}$. Hence e very sequence is a net.
Now consider the directed set $(\mathbb{R}, \leq)$ and define

$$x_\alpha = \alpha, \qquad \alpha \in \mathbb{R}.$$

Then $(x_\alpha)_{\alpha\in\mathbb{R}}$ is a net, since $\mathbb{R}$ is directed under the usual order.
However, it is not a sequence because its index set is $\mathbb{R}$ rather than $\mathbb{N}$. Therefore, not every net is a sequence.

1.4.2 Cauchy Nets

Cauchy nets play a fundamental role in the study of completeness in uniform and topological spaces, generalizing the notion of Cauchy sequences to more general directed sets [17].

Definition 1.20. *A net $\{S_n, n \in D\}$ in the uniform space (X, u) is a Cauchy net iff for each member U of u there is N in D such that $(S_m, S_n) \in U$ whenever both m and n follow N in the ordering of D.*

Chapter 2

Korovkin-Type Theorem and Positive Linear Operators

The Korovkin theorem is a fundamental result in approximation theory. It provides a simple criterion to verify the convergence of a sequence of positive linear operators. In this chapter, we study Korovkin-Type approximation theorems and their extensions using different modes of convergence [14].

2.1 Positive Linear Operators

We begin by introducing positive linear operator: Boundedness ensures that positive linear operators not only preserve order but also act continuously, making them stable tools in approximation theory[2], [6].

2.1.1 Defition and Basic Properties

Definition 2.1. (***Positive Linear Operator***) *A linear operator $T : C([a,b]) \to C([a,b])$ is called positive if*

$$\forall x \in [a,b],\ f(x) \geq 0 \implies T(f)(x) \geq 0.$$

Definition 2.2. (***Sequence of Positive Linear Operators***) *A sequence $(L_n)_n$ of linear operators is called a sequence of positive linear operators if each L_n is a positive linear operator. For $n \in \mathbb{N}$.*

Positive operators preserve order: if one function lies below another, the operator keeps this inequality intact.

Proposition 2.1. (***Monotonicity***) *Let $T : C([a,b]) \to C([a,b])$ be a positive linear operator. If $f, g \in C([a,b])$ such that $f(x) \leq g(x)$ for all $x \in [a,b]$, then*

$$T(f)(x) \leq T(g)(x), \quad \forall x \in [a,b].$$

Proof 12. *Suppose that $f(x) \leq g(x)$ for all $x \in [a,b]$. Then*

$$g(x) - f(x) \geq 0.$$

Since T is a positive linear operator, we have

$$T(g-f)(x) \geq 0.$$

By linearity,

$$T(g)(x) - T(f)(x) \geq 0,$$

Hence

$$T(f)(x) \leq T(g)(x).$$

Boundedness ensures that positive linear operators not only preserve order but also act continuously, making them stable tools in approximation theory [2].

Theorem 2.1. (***Boundedness***) *Every positive linear operator $T : C([a,b]) \to C([a,b])$ is bounded.*

Proof 13. *Let $f \in C([a,b])$. Then we have*

$$|f(x)| \leq \|f\|_\infty, \quad \forall x \in [a,b].$$

Thus,

$$-\|f\|_\infty \leq f(x) \leq \|f\|_\infty.$$

By monotonicity (Proposition 2.1) of T, we get

$$T(-\|f\|_\infty) \leq T(f)(x) \leq T(\|f\|_\infty).$$

$$T(\|f\|_\infty) = \|f\|_\infty T(1),$$

and

$$T(-\|f\|_\infty) = -\|f\|_\infty T(1).$$

Hence,

$$|T(f)(x)| \leq \|f\|_\infty \|T(1)\|_\infty.$$

Therefore,

$$\|T(f)\|_\infty \leq \|T(1)\|_\infty \|f\|_\infty.$$

which shows that T is bounded.

2.1.2 Examples

We now present some examples :

Example 2.1. (*The Identity Operator*)
$I : C([a,b]) \to C([a,b])$ is defined by

$$I(f) = f, \quad \forall f \in C([a,b]).$$

is a positive linear operator.
Indeed, I is linear since we have

$$I(\alpha f + \beta g) = \alpha f + \beta g = \alpha I(f) + \beta I(g).$$

And it is positive for If $f(x) \geq 0$, then $I(f)(x) = f(x) \geq 0$. Norm:

$$\|I(f)\|_\infty = \|f\|_\infty,$$

so $\|I\| = 1$.

Example 2.2. (*Integral Operator*)
Define the operator $T : C([0,1]) \to C([0,1])$ by

$$T(f)(x) = \int_0^x f(t)\,dt, \quad x \in [0,1].$$

Then T is a linear operator since for all $f, g \in C([0,1])$ and $a, b \in \mathbb{R}$, we have

$$T(af + bg)(x) = \int_0^x (af(t) + bg(t))dt = aT(f)(x) + bT(g)(x).$$

2.2 Classical Korovkin Theorem

2.2.1 Test Functions

We introduce the test functions used in Korovkin's theorem, which are sufficient to determine the convergence behavior of the considered operators [4].

Definition 2.3. (***Korovkin Test Functions***) *Let $C([a,b])$ be the space of all continuous real-valued functions on the compact interval $[a,b]$. The classical Korovkin test functions are defined by*

$$f_0(x) = 1, \qquad f_1(x) = x, \qquad f_2(x) = x^2.$$

These functions play a fundamental role in Korovkin-type approximation theory. If a sequence of positive linear operators (T_n) satisfies

$$\lim_{n\to\infty} \| T_n(f_i) - f_i \| = 0, \quad i = 0, 1, 2,$$

then for every $f \in C([a,b])$ *we have*

$$\lim_{n\to\infty} \| T_n(f) - f \| = 0.$$

Hence, the convergence of (T_n) *on the three test functions* $1, x, x^2$ *guarantees the convergence for all functions in* $C([a,b])$.

2.2.2 Geometric Interpretation

The classical Korovkin theorem admits a natural geometric interpretation. The set of test functions $\{1, x, x^2\}$ forms a three-dimensional subspace of $C([a,b])$ that is sufficient to control the uniform convergence of any sequence of positive linear operators.

- The constant function $f_0(x) = 1$ captures the **vertical shift** and ensures the preservation of mass or scaling.
- The identity function $f_1(x) = x$ captures the **linear behavior** (slope), allowing the operators to correctly approximate first-order moments.
- The quadratic function $f_2(x) = x^2$ captures the **curvature** (second-order moment). This function is crucial because it allows bounding the pointwise difference $|f(t) - f(x)|$ by a quadratic expression via uniform continuity.

More precisely, for any $f \in C([a,b])$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|t - x| < \delta$, then $|f(t) - f(x)| < \varepsilon$. For the case $|t - x| \geq \delta$, we use the boundedness of f to obtain the classical estimate:

$$|f(t) - f(x)| \leq \varepsilon + M(t-x)^2,$$

where $M = 2\|f\|_\infty \delta^{-2}$.

This quadratic estimate, combined with the positivity and linearity of the operators T_n, yields:

$$|T_n(f;x) - f(x)| \leq \varepsilon\, T_n(1;x) + M\, T_n((\cdot - x)^2; x) + |f(x)|\,|T_n(1;x) - 1|.$$

Since $T_n(1) \to 1$ and $T_n((\cdot - x)^2) \to 0$ uniformly (the latter follows from convergence on $1, x, x^2$), we obtain $T_n(f) \to f$ uniformly. Thus, Korovkin's theorem shows that approximating these three basic geometric shapes (constant, linear, and quadratic) is sufficient to approximate *every* continuous function uniformly on $[a,b]$.

Remark 1. *Geometrically, the three test functions represent the building blocks of approximation: translation (1), slope (x), and curvature (x²). Their role is analogous to the first three monomials in the Weierstrass approximation theorem, but here they are specifically designed to work with* positive linear operators.

2.2.3 Statement and Proof

We now present the classical Korovkin theorem, which provides a fundamental characterization of the convergence of positive linear operators in terms of simple test functions[19], [18].

Theorem 2.2. (***P. P. Korovkin 1953***)
Consider $x_0(t) = 1$, $x_1(t) = t$, $x_2(t) = t^2$ for $t \in [a,b]$. For $n = 1,2,\ldots$, let

$$T_n : C([a,b]) \longrightarrow C([a,b])$$

be a positive linear operator. If $T_n(x_j) \to x_j$ uniformly on $[a,b]$ for $j = 0,1,2$, then $T_n(x) \to x$ uniformly on $[a,b]$ for every $x \in C([a,b])$.

2.2.4 Proof of Korovkin Theorem

The proof of the theorem based on positivity arguments and uniform convergence techniques in spaces of continuous functions.
If $x \in C([a,b])$, then $x = \operatorname{Re} x + i \operatorname{Im} x$, where $\operatorname{Re} x, \operatorname{Im} x \in C([a,b])$ are real-valued. Since $T_n(x) = T_n(\operatorname{Re} x) + i\, T_n(\operatorname{Im} x)$ for each n, it is enough to prove that $T_n(x) \to x$ when x is real-valued. Let then $x \in C([a,b])$ be real-valued.
Since x is bounded, there exists some $\alpha \in \mathbb{R}$ such that $|x(t)| \le \alpha$ for all $t \in [a,b]$. For $t, s \in [a,b]$, we have

$$-2\alpha \le x(t) - x(s) \le 2\alpha.$$

Let $\varepsilon > 0$. Since x is uniformly continuous on $[a,b]$, there exists $\delta > 0$ such that for $t, s \in [a,b]$ with $|t-s| < \delta$, we have

$$-\varepsilon < x(t) - x(s) < \varepsilon.$$

Now fix $s \in [a,b]$ and consider the function

$$y_s(t) = (t-s)^2, \qquad t \in [a,b].$$

Then for $|t-s| \ge \delta$, we have

$$y_s(t) \ge \delta^2.$$

Combining these inequalities, we see that for all $t \in [a, b]$,

$$-\varepsilon - \frac{2\alpha}{\delta^2} y_s(t) \le x(t) - x(s) \le \varepsilon + \frac{2\alpha}{\delta^2} y_s(t).$$

Since each T_n is positive and linear, we have

$$-\varepsilon T_n(x_0) - \frac{2\alpha}{\delta^2} T_n(y_s) \le T_n(x) - x(s) T_n(x_0) \le \varepsilon T_n(x_0) + \frac{2\alpha}{\delta^2} T_n(y_s).$$

By assumption, $(T_n(x_0)(s))$ converges to 1 uniformly for $s \in [a, b]$.

Also, since

$$y_s = x_2 - 2s x_1 + s^2 x_0,$$

we have

$$T_n(y_s)(s) = T_n(x_2)(s) - 2s T_n(x_1)(s) + s^2 T_n(x_0)(s).$$

Hence by assumption, $(T_n(y_s)(s))$ converges to

$$s^2 - 2s \cdot s + s^2 \cdot 1 = 0$$

uniformly for $s \in [a, b]$.

Thus $(T_n(x)(s))$ converges to $x(s)$ uniformly for $s \in [a, b]$.

2.3 Korovkin-Type Theory for Nets (Generalized Sequences)

The classical Korovkin theorem is formulated for sequences of positive linear operators. However, in more general topological settings, sequences may not be sufficient to characterize convergence. For this reason, the notion of nets(that is introduced in Section 1.4) allows a natural extension of Korovkin-type approximation theory [4].

Theorem 2.3. *Let $(L_\alpha)_{\alpha \in D}$ be a net of positive linear operators on $C([a, b])$. If*

$$L_\alpha(f_i) \to f_i \quad (i = 0, 1, 2)$$

uniformly on $[a, b]$, where

$$f_0(x) = 1, \quad f_1(x) = x, \quad f_2(x) = x^2,$$

then

$$L_\alpha(f) \to f$$

uniformly for every $f \in C([a, b])$.

Proof 14. *Let $f \in C([a, b])$. Since L_α is positive and linear,*

$$|L_\alpha(f; x) - f(x)| \le |L_\alpha(1; x) - 1||f(x)| + L_\alpha(|f(t) - f(x)|; x).$$

By continuity of f, for every $\varepsilon > 0$, there exists $C > 0$ such that

$$|f(t) - f(x)| \leq \varepsilon + C(t-x)^2.$$

Hence,

$$|L_\alpha(f;x) - f(x)| \leq \varepsilon L_\alpha(1;x) + C\,L_\alpha((t-x)^2;x) + |f(x)||L_\alpha(1;x) - 1|.$$

Now,

$$L_\alpha((t-x)^2;x) = L_\alpha(t^2;x) - 2xL_\alpha(t;x) + x^2L_\alpha(1;x).$$

Using

$$L_\alpha(1) \to 1, \qquad L_\alpha(t) \to t, \qquad L_\alpha(t^2) \to t^2$$

uniformly on $[a,b]$, we get

$$L_\alpha((t-x)^2;x) \to 0.$$

Therefore,

$$\|L_\alpha(f) - f\|_\infty \to 0.$$

Thus,

$$L_\alpha(f) \to f$$

uniformly on $[a,b]$.

2.4 Statistical Korovkin-Type Theorem

Statistical convergence generalizes classical convergence by allowing a small subset of terms to deviate from the limit [24].

Definition 2.4. *A sequence (x_k) is said to be statistically convergent to L if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

2.4.1 Classical Statistical Korovkin Theorem

Theorem 2.4. *Let (L_n) be a sequence of positive linear operators on $C([0,1])$. If*

$$st - \lim_{n\to\infty} \|L_n(f_i) - f_i\| = 0, \qquad i = 0,1,2,$$

then

$$st - \lim_{n\to\infty} \|L_n(f) - f\| = 0$$

for all $f \in C([0,1])$.

The following result gives a Korovkin-type approximation theorem on an infinite interval by using exponential test functions.

Theorem 2.5. *Let (L_k) be a sequence of positive linear operators from $C(I)$ into $C(I)$. Then for all $f \in C(I)$,*

$$st - \lim_{k\to\infty} \|L_k(f,x) - f(x)\|_\infty = 0$$

if and only if

$$st - \lim_{k\to\infty} \|L_k(1,x) - 1\|_\infty = 0,$$

$$st - \lim_{k\to\infty} \|L_k(e^{-s},x) - e^{-x}\|_\infty = 0,$$

$$st - \lim_{k\to\infty} \|L_k(e^{-2s},x) - e^{-2x}\|_\infty = 0.$$

Remark 2. *Theorem 2.5 extends the classical statistical Korovkin theorem to functions defined on infinite intervals by using the test functions*

$$1,\ e^{-s},\ e^{-2s}.$$

Chapter 3

Applications

In this chapter, we present classical examples of positive linear operators to which Korovkin-type theorems apply. These operators are fundamental in approximation theory and serve as illustrations of the theoretical results established in Chapter 2.

3.1 Korovkin-Type Theorem for The Identity Operator on $C(X)$, X Compact

Korovkin-Type theorems give practical conditions to study the convergence of sequences of positive linear operators on $C(X)$, where X is compact. Instead of checking convergence for all functions, it is enough to verify it on a small set of test functions. In this context, for the identity operator, such results describe subsets of $C(X)$ that ensure uniform convergence of the operators [3].

Theorem 3.1. *Let M be a subset of $C(X)$ that separates the points of X. Then*

$$\{1\} \cup M \cup M^2$$

is a Korovkin subset of $C(X)$.
Moreover, if $M = \{f_1, \ldots, f_n\}$, $n \geq 1$, is finite, then

$$\left\{1, f_1, \ldots, f_n, \sum_{i=1}^{n} f_i^2\right\}$$

is a Korovkin subset of $C(X)$.
In particular, if $f \in C(X)$ is injective, then

$$\{1, f, f^2\}$$

is a Korovkin subset of $C(X)$.

As an immediate application of Theorem 3.1, we present the following examples of Korovkin subsets in $C([0,1])$.

Proposition 3.1. *1. If $0 < \lambda_1 < \lambda_2$, then $\{1, e_{\lambda_1}, e_{\lambda_2}\}$ is a Korovkin subset of $C([0,1])$, where*

$$e_{\lambda_k}(x) := x^{\lambda_k}, \quad x \in [0,1], \quad k = 1,2.$$

2. *If $u \in C([0,1])$ is strictly convex, then $\{1, e_1, u\}$ is a Korovkin subset of $C([0,1])$.*

Note that the space $A(K)$ includes all constant functions together with the restrictions to K of continuous linear functionals defined on E. By virtue of the Hahn–Banach theorem, these functionals separate the points of K. Consequently, Theorem 3.1 implies that

Corollary 3.1. *$A(K) \cup A(K)^2$ is a Korovkin subset of $C(K)$.*

Now, we consider a continuous selection $(\tilde{\mu}_x)_{x\in K}$ of probability Borel measures on K, i.e., for every $f \in C(K)$ the function

$$x \mapsto \int_K f \, d\tilde{\mu}_x$$

is continuous on K. This function will be denoted by $T(f)$, that is

$$T(f)(x) := \int_K f \, d\tilde{\mu}_x, \quad (x \in K).$$

Note that T is a positive linear operator from $C(K)$ into $C(K)$ and $T(1) = 1$.

Examples of Bernstein–Schnabl Operators

Example 3.1. *Consider the d-dimensional simplex K_d of $\mathbb{R}^d$. For every $x = (x_1, \ldots, x_d) \in K_d$, define*

$$\tilde{\mu}_x := \left(1 - \sum_{i=1}^{d} x_i\right)\delta_0 + \sum_{i=1}^{d} x_i \delta_{a_i},$$

where $a_i = (\delta_{ij})_{1\le j\le d}$.

Then the Bernstein–Schnabl operators associated with $(\tilde{\mu}_x)$ coincide with the classical Bernstein operators on K_d.

Example 3.2. *Consider the hypercube $Q_d = [0,1]^d$. For every $x = (x_1, \ldots, x_d)$ define*

$$\tilde{\mu}_x := \sum_{h_1,\ldots,h_d \in \{0,1\}} x_1^{h_1}(1-x_1)^{1-h_1} \cdots x_d^{h_d}(1-x_d)^{1-h_d}\, \delta_{b_{h_1,\ldots,h_d}},$$

where $b_{h_1,\ldots,h_d} = (\delta_{h_1 1}, \ldots, \delta_{h_d 1})$.

Then the Bernstein–Schnabl operators reduce to the classical Bernstein operators on Q_d.

Theorem 3.2. *For every $f \in C(K)$,*

$$\lim_{n\to\infty} B_n(f) = f \quad \text{uniformly on } K.$$

Further properties of the Bernstein–Schnabl operators, such as quantitative estimates of the rate of convergence, asymptotic formulas, and shape-preserving features, as well as their connections with the approximation of positive semigroups and the study of evolution equations.

3.2 Korovkin-Type Theorem in L^p Spaces

Korovkin-type theorems provide a powerful and efficient tool to study the convergence of sequences of positive linear operators. While the classical version is formulated in the space $C([0,1])$ of continuous functions, these results can be extended to the Lebesgue spaces $L^p([0,1])$, where $1 \le p < \infty$, allowing the approximation of integrable functions [4].

Theorem 3.3. *Let $(L_n)_{n\ge1}$ be a sequence of positive linear operators acting on $L^p([0,1])$, $1 \le p < \infty$. Assume that*

$$\lim_{n\to\infty} \|L_n(g) - g\|_p = 0$$

for each test function $g \in \{1, t, t^2\}$. Then, for every $f \in L^p([0,1])$, it follows that

$$\lim_{n\to\infty} \|L_n(f) - f\|_p = 0.$$

We present the following application of korovkin-type Theorems in weighted L^p Spaces [3].

An Application of Korovkin-Type Theorem

Theorem 3.4. *Let $\varphi : \mathbb{R}^d \to (0, +\infty)$ be a measurable function such that*

$$C_\varphi := \sup_{n\ge1,\, t\in\mathbb{R}^d} \left\{ \frac{1}{(4n\pi)^{d/2}} \int_{\mathbb{R}^d} \varphi(x) \exp\left(-\frac{n}{4}\|t-x\|^2\right) dx \right\} < +\infty.$$

For every $n \ge 1$, define the operators G_n by

$$G_n(f)(x) = \int_{\mathbb{R}^d} K_n(x,t)\, f(t)\, d\mu_{e^\varphi}(t),$$

where

$$K_n(x,t) = \frac{1}{\varphi(t)(4n\pi)^{d/2}} \exp\left(-\frac{n}{4}\|t-x\|^2\right).$$

Then, for every $p \in [1,+\infty)$ and $f \in L^p(\mathbb{R}^d, \mu_{e^\varphi})$, the operator $G_n(f)$ is well-defined, belongs to $L^p(\mathbb{R}^d, \mu_{e^\varphi})$, and satisfies

$$\|G_n(f)\|_{\varphi,p} \le \max\{1, C_\varphi\}\, \|f\|_{\varphi,p}.$$

Moreover, if the functions $1, x_1, \ldots, x_d, \sum_{i=1}^d x_i^2$ belong to $L^p(\mathbb{R}^d, \mu_{e^\varphi})$, then

$$\lim_{n\to\infty} \|G_n(f) - f\|_{\varphi,p} = 0,$$

for every $f \in L^p(\mathbb{R}^d, \mu_{e^\varphi})$.

Remark 3. *The above result provides an application of Korovkin-type theorems in weighted L^p spaces using a sequence of positive linear integral operators with Gaussian kernels. Typical examples of admissible weight functions include*

$$\varphi(x) = (1 + \|x\|^m)^{-1} \quad or \quad \varphi(x) = e^{-m\|x\|}, \quad m \ge 1.$$

Korovkin-type theorems in L^p spaces extend classical approximation results to integrable functions. By reducing convergence verification to a small set of test functions, they provide an effective criterion widely used in approximation theory.

3.3 Bernstein Operators

Bernstein operators are among the most important examples of positive linear operators used in approximation theory. They provide a constructive proof of the Weierstrass approximation theorem.
These operators also play a central role in Korovkin-type approximation theory, as they satisfy the Korovkin test functions conditions [25].

Definition 3.1. *Let $f \in C([0,1])$. The Bernstein operator $B_n(f)$ is defined by*

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1].$$

The Bernstein operators possess several fundamental properties which follow directly from their definition. First, they are linear operators.

Proposition 3.2. *Bernstein operators are positive linear operators.*

Proof 15. *Let $f, g \in C([0,1])$ and $a, b \in \mathbb{R}$. Then*

$$\begin{aligned} B_n(af+bg;x) &= \sum_{k=0}^{n}\left(af\left(\frac{k}{n}\right)+bg\left(\frac{k}{n}\right)\right)\binom{n}{k}x^k(1-x)^{n-k} \\ &= a\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} + b\sum_{k=0}^{n} g\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} \\ &= aB_n(f;x)+bB_n(g;x). \end{aligned}$$

B_n is a linear operator.
Now, suppose that $f(x) \geq 0$ for all $x \in [0,1]$. Since

$$f\left(\frac{k}{n}\right) \geq 0, \qquad \binom{n}{k}x^k(1-x)^{n-k} \geq 0,$$

for every $k = 0, \ldots, n$ and $x \in [0,1]$, it follows that

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} \geq 0.$$

Therefore, B_n is a positive operator. Consequently, the Bernstein operators are positive linear operators.

3.3.1 Application of Korovkin Theorem

To apply Korovkin's theorem, we check the test functions:

Lemma 3.1. *Let*

$$e_0(t) = 1, \qquad e_1(t) = t, \qquad e_2(t) = t^2.$$

Then the Bernstein operators B_n satisfy

$$B_n(e_0; x) = 1,$$

$$B_n(e_1; x) = x,$$

and

$$B_n(e_2; x) = x^2 + \frac{x(1-x)}{n}.$$

Proof 16. *Using the definition of the Bernstein operators*

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k},$$

we obtain directly

$$B_n(e_0; x) = 1, \qquad B_n(e_1; x) = x, \qquad B_n(e_2; x) = x^2 + \frac{x(1-x)}{n}.$$

3.3.2 Convergence of Bernstein Operators

For every function $f \in C([0,1])$, the sequence $B_n(f)$ converges uniformly to f on $[0,1]$. This result is known as Bernstein's proof of the Weierstrass approximation theorem [22].

Theorem 3.5. (***Bernstein***) *Let $f \in C([0,1])$. Define*

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Then

$$\lim_{n\to\infty} B_n(f; x) = f(x)$$

uniformly on $[0,1]$.

Proof 17. *Consider a random variable $X_n \sim$ Binomial(n, x). Then*

$$B_n(f; x) = \mathbb{E}\left[f\left(\frac{X_n}{n}\right)\right]. \quad \mathbb{E} : \text{is expected value.}$$

Since f is uniformly continuous, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon.$$

Hence

$$|B_n(f;x)-f(x)| \le \varepsilon + 2\|f\|_\infty P\left(\left|\frac{X_n}{n}-x\right| \ge \delta\right).$$

Using Chebyshev inequality,

$$P\left(\left|\frac{X_n}{n}-x\right| \ge \delta\right) \le \frac{x(1-x)}{n\delta^2}. \quad \textit{With P : is probability function.}$$

With P : is probability function Thus

$$|B_n(f;x)-f(x)| \le \varepsilon + \frac{C}{n}.$$

Letting $n \to \infty$ gives the result.

$$f_0(x)=1, \quad f_1(x)=x, \quad f_2(x)=x^2$$

We compute:

$$B_n(1;x)=1$$

$$B_n(x;x)=x$$

$$B_n(x^2;x)=x^2+\frac{x(1-x)}{n}$$

Thus,

$$\lim_{n\to\infty} B_n(f_i;x)=f_i(x), \quad i=0,1,2$$

By Korovkin's theorem, it follows that:

$$\lim_{n\to\infty} B_n(f;x)=f(x)$$

uniformly for all $f \in C([0,1])$.

3.4 Szász–Mirakjan Operators

The Szász–Mirakjan operators are an important class of positive linear operators used in approximation theory on the unbounded interval $[0,\infty)$. They were introduced by O. Szász (1950) as a generalization of Bernstein polynomials from a finite interval to an infinite interval [1].

Definition 3.2. *Let $f \in C([0,\infty))$ and $x \ge 0$. The Szász–Mirakjan operators are defined by:*

$$S_n(f;x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}.$$

Proposition 3.3. *The operators S_n are positive linear operators on $C([0,+\infty))$.*

Proof 18. *Let $f, g \in C([0,+\infty))$ and $a, b \in \mathbb{R}$. Then*

$$\begin{aligned} S_n(af+bg;x) &= e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\left(af\left(\frac{k}{n}\right)+bg\left(\frac{k}{n}\right)\right) \\ &= ae^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right)+be^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}g\left(\frac{k}{n}\right) \\ &= aS_n(f;x)+bS_n(g;x). \end{aligned}$$

S_n is a linear operator.

Now, let $f \geq 0$ on $[0,+\infty)$. Then

$$f\left(\frac{k}{n}\right) \geq 0, \qquad e^{-nx}\frac{(nx)^k}{k!} \geq 0$$

for every $k \geq 0$ and $x \geq 0$. Therefore,

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right) \geq 0.$$

Thus, S_n is a positive operator. Consequently, the Szász–Mirakjan operators are positive linear operators on $C([0,+\infty))$.

3.4.1 Application of Korovkin Theorem

To apply Korovkin's theorem, we check the test functions:

Lemma 3.2. *For the test functions*

$$e_0(t) = 1, \qquad e_1(t) = t, \qquad e_2(t) = t^2,$$

the Szász operators S_n satisfy

$$S_n(e_0;x) = 1, \qquad S_n(e_1;x) = x, \qquad S_n(e_2;x) = x^2 + \frac{x}{n}.$$

Proof 19. *Using the definition of the Szász operators,*

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),$$

a direct computation gives

$$S_n(e_0;x) = 1, \qquad S_n(e_1;x) = x, \qquad S_n(e_2;x) = x^2 + \frac{x}{n}.$$

3.4.2 Convergence of Szász–Mirakjan Operators

If $f \in C([0,\infty))$ and $\lim_{x\to\infty} f(x)$ exists, then

$$\lim_{n\to\infty} S_n(f;x) = f(x)$$

uniformly on compact subsets of $[0,\infty)$.

Theorem 3.6. (***Szász–Mirakjan***) *For every $f \in C([0,+\infty))$ such that f is bounded and continuous, we have*

$$\lim_{n\to\infty} S_n(f;x) = f(x)$$

uniformly on every compact subset of $[0,+\infty)$.

Proof 20. *By the classical Korovkin theorem, it suffices to verify the convergence for the test functions $e_0(t) = 1$, $e_1(t) = t$, and $e_2(t) = t^2$. From the lemma, we have:*

$$S_n(e_0;x) = 1 \to 1,$$

$$S_n(e_1;x) = x \to x,$$

$$S_n(e_2;x) = x^2 + \frac{x}{n} \to x^2.$$

Thus, all the conditions of Korovkin's theorem are satisfied, and hence

$$S_n(f;x) \to f(x)$$

uniformly on compact subsets of $[0,+\infty)$.

3.5 Baskakov Operators

Baskakov operators are an important class of positive linear operators used in approximation theory. They provide approximation of continuous functions on the interval $[0,\infty)$ and play a key role in Korovkin-type theorems [8] , [14].

Definition 3.3. *The Baskakov operators V_n are defined for a function $f \in C([0,\infty))$ and $x \geq 0$ by*

$$V_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) b_{n,k}(x),$$

where

$$b_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}, \quad k = 0,1,2,\ldots$$

Proposition 3.4. *Baskakov operators are positive linear operators and preserve the constant function.*

Proof 21. *Let $f, g \in C([0,\infty))$ and constants $a, b \in \mathbb{R}$, we have*

$$V_n(af+bg;x) = \sum_{k=0}^{\infty}(af+bg)\left(\frac{k}{n}\right)b_{n,k}(x)$$

$$= a\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)b_{n,k}(x) + b\sum_{k=0}^{\infty} g\left(\frac{k}{n}\right)b_{n,k}(x)$$

$$= aV_n(f;x) + bV_n(g;x).$$

Moreover, since

$$b_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} \geq 0,$$

for all $x \geq 0$ and $k \geq 0$, it follows that if $f(x) \geq 0$, then

$$V_n(f;x) \geq 0.$$

Hence, the operators V_n are positive linear operators.

Finally, for the constant function $f(t) = 1$, we obtain

$$V_n(1;x) = \sum_{k=0}^{\infty} b_{n,k}(x) = 1.$$

Therefore, the Baskakov operators preserve the constant function.

3.5.1 Application of Korovkin Theorem

To apply Korovkin's theorem, we check the test functions:

Lemma 3.3. *For the test functions*

$$e_0(t) = 1, \qquad e_1(t) = t, \qquad e_2(t) = t^2.$$

the Baskakov operators V_n satisfy

$$V_n(e_0;x) = 1,$$

$$V_n(e_1;x) = x,$$

and

$$V_n(e_2;x) = x^2 + \frac{x}{n}.$$

Proof 22. *Using the definition of the Baskakov operators*

$$V_n(f;x) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}f\left(\frac{k}{n}\right),$$

we compute the images of the test functions.

For $e_0(t) = 1$,

$$V_n(e_0;x) = 1.$$

For $e_1(t) = t$,

$$V_n(e_1; x) = x.$$

For $e_2(t) = t^2$,

$$V_n(e_2; x) = x^2 + \frac{x(1+x)}{n}.$$

Hence,

$$V_n(e_0; x) = 1, \qquad V_n(e_1; x) = x, \qquad V_n(e_2; x) = x^2 + \frac{x(1+x)}{n}.$$

3.5.2 Convergence of Baskakov Operators

Let $f \in C([0,\infty))$ be a continuous function. Then

$$\lim_{n\to\infty} V_n(f; x) = f(x)$$

uniformly on every compact subset of $[0,\infty)$.

Theorem 3.7. (***Baskakov***) *Let* V_n *be the Baskakov operators defined for* $f \in C([0,\infty))$ *by*

$$V_n(f; x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), \quad x \geq 0.$$

Then, for the test functions $e_0(x) = 1$, $e_1(x) = x$, *and* $e_2(x) = x^2$, *we have*

$$V_n(e_0; x) = 1, \quad V_n(e_1; x) = x, \quad V_n(e_2; x) = x^2 + \frac{x}{n}.$$

Proof 23. *Using the linearity of the operator* V_n, *we compute:*

First, for $e_0(x) = 1$, *we obtain*

$$V_n(e_0; x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} = 1,$$

since the series represents the binomial expansion of $(1+x)^{-n}(1+x)^n$.

Next, for $e_1(x) = x$, *we have*

$$V_n(e_1; x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \frac{k}{n}.$$

Using known identities for moments of Baskakov operators, it follows that

$$V_n(e_1; x) = x.$$

Finally, for $e_2(x) = x^2$, *we compute*

$$V_n(e_2; x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \left(\frac{k}{n}\right)^2,$$

which gives

$$V_n(e_2; x) = x^2 + \frac{x}{n}.$$

Remark 4. *Baskakov operators form an important example of positive linear operators and satisfy Korovkin-type approximation conditions. Hence, they are widely used in studying convergence and approximation behavior of functions.*

3.6 Weierstrass Approximation

The Weierstrass Approximation Theorem and Korovkin's Theorem are foundational pillars of mathematical analysis that deal with approximating complex continuous functions using simpler, well-behaved functions like polynomials [11].

Definition 3.4. *Let $C([a,b])$ be the space of continuous real-valued functions on a closed interval $[a,b]$. We say that a subspace $\mathscr{A} \subset C([a,b])$ has the Weierstrass approximation property if for every function $f \in C([a,b])$ and every $\varepsilon > 0$, there exists a function $g \in \mathscr{A}$ such that*

$$\|f - g\|_\infty < \varepsilon.$$

In particular, the set of algebraic polynomials satisfies this property in $C([a,b])$.

Theorem 3.8. (***Weierstrass Approximation Theorem***) *Let $f \in C([a,b])$. Then for every $\varepsilon > 0$, there exists a polynomial p such that*

$$\sup_{x\in[a,b]} |f(x) - p(x)| < \varepsilon.$$

Proof 24. *Let $f \in C([a,b])$. We reduce the problem to the interval $[0,1]$.*
Define the change of variable

$$t = \frac{x-a}{b-a}, \quad x = a + t(b-a),$$

and define a function g on $[0,1]$ by

$$g(t) = f(a + t(b-a)).$$

Then $g \in C([0,1])$.
Now define the Bernstein polynomials of g:

$$B_n(g;t) = \sum_{k=0}^{n} g\left(\frac{k}{n}\right)\binom{n}{k} t^k (1-t)^{n-k}.$$

It is known that $B_n(g;t) \to g(t)$ uniformly on $[0,1]$ as $n \to \infty$.
Hence, for every $\varepsilon > 0$, there exists n such that

$$\sup_{t\in[0,1]} |B_n(g;t) - g(t)| < \varepsilon.$$

Now define a polynomial on $[a,b]$ by

$$p_n(x) = B_n\left(g; \frac{x-a}{b-a}\right).$$

Then p_n is a polynomial in x, and we have

$$|p_n(x) - f(x)| = \left|B_n\left(g; \frac{x-a}{b-a}\right) - g\left(\frac{x-a}{b-a}\right)\right|.$$

Taking supremum over $x \in [a,b]$, we obtain

$$\sup_{x\in[a,b]} |p_n(x) - f(x)| = \sup_{t\in[0,1]} |B_n(g;t) - g(t)| < \varepsilon.$$

Thus, p_n converges uniformly to f on $[a,b]$, which proves the theorem.

Theorem 3.9. (*Stone–Weierstrass Theorem [3]*) *Let X be a locally compact Hausdorff space with a countable base, and let M be a closed subalgebra of $C_0(X)$.*
If M separates strongly the points of X, i.e.,

- ***i.*** *for every $x, y \in X$ with $x \neq y$, there exists $f \in M$ such that $f(x) \neq f(y)$,*

- ***ii.*** *for every $x \in X$, there exists $f \in M$ such that $f(x) \neq 0$,*

then

$$M = C_0(X).$$

Proof 25. *The proof is based on showing that A is dense in $C_0(X)$ and, since M is closed, it must coincide with $C_0(X)$.*
First, using the strong separation property and the separability of $C_0(X)$, one constructs a strictly positive function $f_0 \in C_0(X)$, i.e., $f_0(x) > 0$ for all $x \in X$.
Next, one proves that M is a Korovkin subspace by showing that the set $\{f_0\} \cup f_0 M \cup f_0 M^2$ is contained in M. This allows the use of a Korovkin-type theorem to approximate any function in $C_0(X)$.
Finally, given $f \in C_0(X)$ and $\varepsilon > 0$, one constructs functions in M that approximate f uniformly within ε. Hence f belongs to the closure of M, and since M is closed, we conclude that $M = C_0(X)$.

Conclusion

In this dissertation, we studied several fundamental aspects of Korovkin-type approximation theory and positive linear operators, which constitute an important area in functional analysis and approximation theory. The main objective of this work was to present the classical Korovkin theorem together with some generalized forms and applications in different functional spaces.

We began by introducing the preliminary notions required throughout this work. In particular, we recalled basic concepts related to Banach spaces, function spaces, convergence of sequences, and convergence of nets. We also discussed several fundamental properties of nets and Cauchy nets, emphasizing their importance in the study of convergence in general topological spaces.

Next, we focused on Korovkin-type approximation theory and positive linear operators. We presented the classical Korovkin theorem and explained the essential role of the test functions 1, x, and x^2 in determining the convergence of sequences of positive linear operators. We also studied generalized versions of Korovkin-type theorems involving nets, statistical convergence. These extensions illustrate the flexibility and richness of Korovkin-type approximation methods in different mathematical settings.

Finally, we examined several applications of these results in spaces such as $C(X)$ and L^p-spaces. Particular attention was devoted to important classes of positive linear operators, including Bernstein, Szász–Mirakjan, and Baskakov operators, which play a central role in approximation processes. We also discussed some applications related to summability methods and Weierstrass approximation.

This work highlights the significance of positive linear operators as effective tools in approximation theory and demonstrates the importance of Korovkin-type theorems in studying convergence phenomena in functional spaces. The theory remains an active area of research, especially with the development of new generalized forms of convergence and new applications in analysis and operator theory.

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Abstract

This thesis is concerned with Korovkin-type approximation theory and its applications using positive linear operators. In particular, if (T_n) is a sequence of positive linear operators on $C([0,1])$ such that $T_n(f)$ converges uniformly to f for $f \in \{1, x, x^2\}$, then this is sufficient to ensure the uniform convergence of $T_n(f)$ for all $f \in C([0,1])$. Moreover, the work investigates some convergence criteria for nets (or generalized sequences) of positive operators on function spaces. It also presents several new results and applications that allow readers to quickly grasp the main classical results of the theory as well as some recent developments.

Keywords: Korovkin-type theorem, positive linear operator, approximation by positive operators, nets, continuous function space, L^p-space.

Résumé

Cette thèse porte sur la théorie d'approximation de type Korovkin et ses applications à l'aide des opérateurs linéaires positifs. En particulier, si (T_n) est une suite d'opérateurs linéaires positifs sur $C([0,1])$ telle que $T_n(f)$ converge uniformément vers f pour $f \in \{1, x, x^2\}$, alors cela suffit pour assurer la convergence uniforme de $T_n(f)$ pour tout $f \in C([0,1])$. De plus, ce travail étudie certains critères de convergence pour les filets (ou suites généralisées) d'opérateurs positifs sur des espaces de fonctions. Il présente également plusieurs nouveaux résultats et applications permettant au lecteur de saisir rapidement les principaux résultats classiques de la théorie ainsi que certains développements récents.

Mots-clés : théorème de type Korovkin, opérateur linéaire positif, approximation par opérateurs positifs, filets, espace des fonctions continues, espace L^p.

الملخص

تتناول هذه المذكرة نظرية التقريب من نوع كوروفكين وتطبيقاتها باستعمال المؤثرات الخطية الموجبة. وتتمثل الفكرة الأساسية في أنه إذا كانت (T_n) متتالية من المؤثرات الخطية الموجبة على الفضاء $C([0,1])$ وتحقق أن $T_n(f)$ تتقارب بانتظام إلى f لكل دالة $f \in \{1, x, x^2\}$ ، فإن ذلك يكفي لضمان التقارب المنتظم لـ $T_n(f)$ لكل دالة $f \in C([0,1])$. وعلى وجه الخصوص، تبحث هذه الدراسة في بعض معايير التقارب للشبكات (أو المتتاليات المعممة) من المؤثرات الموجبة على فضاءات الدوال. كما تتضمن عرضًا لعدد من النتائج والتطبيقات الجديدة التي تمكّن القارئ من استيعاب أهم نتائج النظرية وبعض الإسهامات الحديثة فيها.

الكلمات المفتاحية: مبرهنة كوروفكين من النوع، المؤثرات الخطية الموجبة، التقريب بالمؤثرات الموجبة، الشبكات، فضاء الدوال المستمرة، فضاءات L^p .