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General Introduction

In recent years , the fractional calculus has eaught the attention of many researchers over the last few decades as it a solid and growing work both in theory and in its application.The history of the theory goes back to the $17th$ century,when in 1965 the derivative of order $\alpha = \frac{1}{2}$ was described by *LEIBNITZ* in his letter to L'HOPITAL.

Since then , the new thoery turend out to be very attractive to mathematicians as well as physicsts,biologists,engineers and economists

The calasical fractional calculus is based on several definitions for the operators of integration and derivatives such as : *Riemann-Liouville, Caputo, Hadamarad* and *Katugampola*

This thesis is organized as follows , in the first chapter which is concerned with some basic concepts of fractional calculus, and some definitions and results that are important for the study ,also we give definitions and some properties of fractional integrale and derivative and its properties.

We will discuss in the second chapter , the existence and uniqueness of solutions for the :

$$\begin{cases} D_{0^+}^{\alpha,\rho}\left(u(x,t) - u(x,0)\right) = u_{xx} + a(t)f(x,t) \\ u(x,0) = \varphi(x) \ ; \quad 0 < x < 1 \\ u(0,t) = u(1,t) \ , \ u_x(1,t) = 0 \ ; \quad 0 < t \leq T \end{cases}$$

and the over deternmination condition

$$\int_0^1 u(x,t)dx = g(t)$$

finaly chapter , the existence and uniqueness of solutions for the :

$$\begin{cases} D_{0^+}^{\alpha}\left(u(x,t) - u(x,0)\right) = u_{xx} + a(t)u(x,t) + c(t)F(x,t) \\ u(x,0) = \varphi(x) \ ; \quad x \in [0.1] \\ u(0,t) = u(1,t) \ , \ u_x(1,t) = 0 \ ; \quad t \in [0.T] \\ \int_0^1 u(x,t) = E(t) \ \ t \in [0,T] \end{cases}$$

CHAPTER 1

PRELIMINAIRES

1.1 Basic fractional calculus

In this section , we present the most commonly used fractional integration operators and definitions of fractional derivatives and give the most important properties of these concepts .we begin by giving some definitions on special functions and functional spaces.

1.1.1 special functions of fractional calculus

In the following , we present the Euler Gamma function, the Beta function ,these functions play a very important role in the theory of fractional calculus .

Euler Gamma function

One of the basic function of the fractional calculus is Euler gamma function $\Gamma(z)$, the simplest interpretation of the gamma function is the genrealization of the notion of factorial , for real numbers. which generalizes the factorial $n!$ and allows n to take also non-integer and even complex values.

Definition 1.1. *(Euler gamma function[14]).the Euler gamma is defined on $\mathbb{C}$, by the improper integral*

$$\Gamma(\alpha) = \int_0^\infty \tau^{\alpha-1} e^{-\tau} d\tau, \quad (R(\alpha) > 0) \tag{1.1}$$

which converges on the complex half-planes $R(\alpha) > 0$

Properties of Euler gamma function : we give some properties of the Euler gamma function :

1. $\Gamma(1) = 1, \Gamma(n+1) = n!, \forall n \in \mathbb{N}$
2. The Euler gamma function $\Gamma(\alpha)$ is a monotonous and strictly decreasing for $0 < \alpha \leq 1$.

3. The Euler gamma function is a monotonous and strictly increasing function for $\alpha \geq 2$, so it is convex for $\alpha \in]0, +\infty[$, with point of minimum equal to $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$
4. By integrating by part , we obtain

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha), \;\; R(\alpha) > 0. \tag{1.2}$$

Beta function

Also ,known as integral Euler of the first type , it shares the form that typically resembles the fractional integral derivative of many functions.

Definition 1.2. *(Beta function).for a positive values of the two parameters $p, q \in \mathbb{C}$, the Beta function is defined on $\mathbb{C}\mathbb{C}$, by the following integral*

$$B(p, q) = \int_0^1 \tau^{p-1}(1-\tau)^{q-1}d\tau, \;\; (R(p) > 0 \; and \; R(q) > 0) \tag{1.3}$$

Properties of Beta function: For all $p, q \in \mathbb{C}$ with $(R(p) > 0 \; and \; R(q) > 0)$

1. $B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, (Connection between the Euler gamma function and Beta function).
2. The Beta function is symmetric , i.e,

$$B(p, q) = B(q, p)$$

The Mittag-leffter function

Definition 1.3. *([14]) The Mittag-leffter function of two parameters is defined as*

$$E_{\xi,\eta}(x) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\xi k + \eta)}, z \in \mathbb{C}, \;\; R(\xi) > 0, R(\eta) > 0$$

For $\eta = 1$, the Mittag-leffter function is reduced to classical one parameter Mittag-leffter function that is

$$E_{\xi,1} = E_{\xi} = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\xi k + 1)}$$

Let $e_{\xi}(t, \mu) = E_{\xi}(-\mu t^{\xi})$ and $e_{\xi,\eta}(t, \mu) = t^{\eta-1}E_{\xi,\eta}(-\mu t^{\xi})$, where μ is a positive real number The Mittag-leffter function $e_{\xi}(t, \mu), e_{\xi,\eta}(t, \mu)$ for $0 < \xi \leq 1, 0 < \xi \leq \eta \leq 1$, respectively , are completely monotone function , i,e

$$(-1)^n \frac{\partial^n}{\partial t^n}[e_{\xi}(t, \mu)] \geq 0 \; and \; (-1)^n \frac{\partial^n}{\partial t^n}[e_{\xi,\eta}(t, \mu)] \geq 0, n \in \mathbb{N}$$

Using Theorem 1.6 *in* [6] *we can have the following estimate*

$$|\mu e_{\xi,\xi}(t,\mu)| \leq \frac{N\mu t^{\xi}}{t(1+\mu^{\xi})} \leq \frac{N}{t} \leq C,\ t \in]\epsilon, T] \tag{1.4}$$

Where $\epsilon > 0$ *,*N *and* C *are some constants*

Definition 1.4. ([4, 1.15]) *Let* $[a,b]$ *be a finite intervale .Then* $AC[a,b]$ *is the space of absolute continuous function on* $[a,b]$ *, definde by*

$$AC[a,b] = \left\{ f : [a,b] \longrightarrow \mathbb{R} \text{ such that } f(x) = c + \int_a^x \varphi(t)dt,\ \varphi \in L^1(a,b) \right\}$$

1.1.2 Riemann-Liouville fractional integrals

Definition 1.5. *([14]).Let* $u \in L^1([a,b]),\mathbb{R})$ *and* $\alpha > 0$ *.Then ,for any* $t \in [a,b]$ *, the integrals*

$$I^{\alpha}_{a^+}u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1}u(\tau)d\tau,\ t \in [a,b] \tag{1.5}$$

is called the left-sided Riemann-Liouville fractional integral of order $\alpha > 0$*.*

The extension to the real axis $\mathbb{R}$ and $\mathbb{R}^+$ known by fractional integrals on the real line are noted I^{α}_{+} and $I^{\alpha}_{0^+}$ (respectively) and given by the following

$$I^{\alpha}_{+}u(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t-\tau)^{\alpha-1}u(\tau)d\tau,\ t \in \mathbb{R} \tag{1.6}$$

and

$$I^{\alpha}_{0^+}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}u(\tau)d\tau,\ t \in [a,b] \tag{1.7}$$

Definition 1.6. *([14]).The right-sided Riemann-Liouville fractional integral of order* $\alpha > 0$ *of function* $u \in L^1([a,b],\mathbb{R})$ *is given by :*

$$I^{\alpha}_{b^-}u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1}u(\tau)d\tau,\ t \in [a,b] \tag{1.8}$$

Then extension on $[a,+\infty[$ and $\mathbb{R}$ is noted I^{α}_{-}

$$I^{\alpha}_{-}u(t) = \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} (\tau-t)^{\alpha-1}u(\tau)d\tau,\ t \in \mathbb{R} \tag{1.9}$$

Properties of fractional integrals

The fractional integral I^{α} of arbitrary real order $\alpha > 0$ defined by (1.5) and (1.8) , has the following important properties(see[14])

1. For any $t \in (a,b), \alpha > 0$ and $\beta > 0$, we have :
$$\left(I^{\alpha}_{a^+}(x-a)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(t-a)^{\beta+\alpha-1}$$
$$\left(I^{\alpha}_{b^-}(b-x)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(b-t)^{\beta+\alpha-1}$$
2. Semigroup property : for $u \in L^1([a,b])$
$$I^{\alpha}_{a+}I^{\beta}_{a+}u = I^{\alpha+\beta}_{a+}u,\ \ I^{\alpha}_{b-}I^{\beta}_{b-}u = I^{\alpha+\beta}_{b-}u,\ \ \alpha > 0,\ \ \beta > 0.$$
3. Fractional integration by parts formula : Let $u \in L^p([a,b])$ and $v \in L^q([a,b])$ either with $\alpha \geq 1, p = q = 1$ or with $0 < \alpha < 1, \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha,\ p,q > 1.$ Then
$$\int_a^b u(\tau)I^{\alpha}_{a+}v(\tau)d\tau = \int_a^b I^{\alpha}_{a+}u(\tau)v(\tau)d\tau$$

1.1.3 Riemann-Liouville fractional derivatives

The most frequently used definitions of the fractional calculs involves the Riemann-Liouville fractional derivative , Then to define a fractional derivative there is no formula for the n th derivative analogous to (??) so , we need to generalize the derivative through a fractional integral.

For $\alpha > 0$, we denote $[\alpha]$ the integer part of $\alpha, [\alpha]$ is the unique integer satisfying
$$[\alpha] \leq \alpha < [\alpha] + 1$$
Let $u : [a,b] \longrightarrow \mathbb{R}$. From the clasic relationship $\frac{d}{dt} = \frac{d^2}{dt^2} \circ I^1_{a^+}$ we can define a fractional derivative of order $0 \leq \alpha < 1$ by
$$\frac{d^{\alpha}}{dt^{\alpha}} = \frac{d}{dt} \circ I^{1-\alpha}_{a^+}$$
More generally , if $\alpha > 0$ and if $n = [\alpha] + 1$, we can put
$$\frac{d^{\alpha}}{dt^{\alpha}} = \left(\frac{d}{dt}\right)^n \circ I^{n-\alpha}_{a^+} \tag{1.10}$$
we get exactly the left-sided Riemann-Liouville derivative given by the following definition

Definition 1.7. *([14]) Let $\alpha > 0$ and $n = [\alpha]+1$ The left-sided Rieman-Liouville fractional derivative of order α of a continuous function $u : [a,b] \longrightarrow \mathbb{R}$ is given by*
$$D^{\alpha}_{a^+}u(t) = \left(\frac{d}{dt}\right)^n \circ I^{n-\alpha}_{a^+}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1}u(\tau)d\tau \tag{1.11}$$

Moreover the right integral was associated with $\left(-\frac{d}{dt}\right)$. The previous reasoning therefore leads to the following definition :

Definition 1.8. *([14]).Let $\alpha > 0$ and $n = [\alpha] + 1$ The right-sided Riemann-Liouville fractional derivative of order α of a continuous function $u : [a, b] \longrightarrow \mathbb{R}$ is given by*

$$D^{\alpha}_{b^-}u(t) = \left(-\frac{d}{dt}\right)^n \circ I^{n-\alpha}_{a^+}u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_t^b (\tau - t)^{n-\alpha-1}u(\tau)d\tau \qquad (1.12)$$

If $u : \mathbb{R} \longrightarrow \mathbb{R}$ the previous definitions are directly generalized and are called Liouville derivative

Definition 1.9. *Let $\alpha > 0$ and $n = [\alpha] + 1$ .The left-sided Liouville fractional derivative on the real line is given by*

$$D^{\alpha}_{+}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_{-\infty}^t (t - \tau)^{n-\alpha-1}u(\tau)d\tau$$

Moreover , the right-integral was associated with $\left(-\frac{d}{dt}\right)$ .The preceding reasoning therefore leads to the following definition :

Definition 1.10. *.Let $\alpha > 0$ and $n = [\alpha] + 1$.The right-sided Liouville fractional derivative on the real line is given by*

$$D^{\alpha}_{-}u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_t^{+\infty} (\tau - t)^{n-\alpha-1}u(\tau)d\tau$$

Properties of fractional derivative

The fractional derivative $D^{\alpha}_{a^+}$ has the following properties

1. In general the fractional derivative of Riemann-Liouville of a constant function is neither zero nor constant , we have

$$D^{\alpha}(C) = \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha}$$

2. Let $0 < \alpha < 1$, we have

   - for any $u \in L^1([a, b])$, we that $D^{\alpha}_{a^+}I^{\alpha}_{a^+}u = u$.
   - The latter can be generalized .In fact , if the function $I^{1-\alpha}_{a^+}u$ is absolutely continuous on $[a, b]$, then

$$I^{\alpha}_{a^+}D^{\alpha}_{a^+}u(t) = u(t) - \frac{I^{1-\alpha}_{a^+}}{\Gamma(\alpha)}(t-a)^{\alpha-1}, \ t \in (a, b)$$

   where $I^{1-\alpha}_{a^+}u(a) = lim_{\tau \longrightarrow a^+}(I^{1-\alpha}_{a^+}u)(s)$, which is in general non zero

3. For any $0 \leq n - 1 < \alpha < n$ and $\beta > -1$, the Riemann-Liouville fractional derivative of the function $u(t)$ such that : $u(t) = (t - a)^{\beta}$ is given by :

$$D^{\alpha}_{a^+}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1}(t-a)^{\beta-\alpha} \qquad (1.13)$$

1.1.4 Caputo-type fractional derivative

The definition of caputo fractional derivative is based on the inversion of the compositions in the right sided of (1.10) it also seems reasonable to define the fractional derivative called the caputo derivative , wich is given by :

Definition 1.11. *([?]) .Let $\alpha > 0$ and $n = [\alpha] + 1$.The left-sided Caputo fractional derivative of order α of a function $u \in C^n([a,b],\mathbb{R})$ is given by*

$$^cD^{\alpha}_{a^+}u(t) = I^{n-\alpha}_{a^+} \circ \left(\frac{d}{dt}\right)^n u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n u(\tau)d\tau \tag{1.14}$$

The right-sided Caputo fractional derivative of order α of a function $u \in C^n([a,b],\mathbb{R})$ is given by

$$^cD^{\alpha}_{b^-}u(t) = I^{n-\alpha}_{b^-} \circ \left(-\frac{d}{dt}\right)^n u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n u(\tau)d\tau \tag{1.15}$$

Definition 1.12. *([15]) Let $\alpha > 0$ and $n = [\alpha] + 1$.The left-sided Caputo fractional derivative of order α of a functionu $\in C^n(\mathbb{R},\mathbb{R}$ is given by*

$$^cD^{\alpha}_{+}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^t (t-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n u(\tau)d\tau$$

The right-sided Caputo fractional derivative of order α of a function $u \in C^n(\mathbb{R},\mathbb{R}$ is given by

$$^cD^{\alpha}_{-}u(t) = \frac{(-1)}{\Gamma(n-\alpha) \int_t^{+\infty} (\tau-t)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n u(\tau)d\tau}$$

Properties of Caputo fractional deratives

1. The Caputo fractional derivative of a constant function is zero.
2. Let $\alpha \in \mathbb{R}^+, n \in \mathbb{N}$, such that $n = [\alpha] + 1$ If $u \in AC^n([a,b])$, then

$$lim_{\alpha \longrightarrow n^-}\ {^cD^{\alpha}_{a^+}}u(t) = u^{(n)}(t)$$

$$lim_{\alpha \longrightarrow n^-}\ {^cD^{\alpha}_{b^-}}u(t) = (-1)^n u^{(n)}(t)$$

almost everywhere

3. If $\alpha \notin \mathbb{N}$ and $u(t)$ is a function , for which the Caputo fractional derivatives ${}^cD^{\alpha}_{a^+}u(t)$ and ${}^cD^{\alpha}_{b^-}u(t)$ of order $\alpha > 0$ exist together with the Riemann-Liouville fractional derivatives ${}^cD^{\alpha}_{a^+}u(t)$ and ${}^cD^{\alpha}_{b^-}u(t)$,then we have

$$ {}^cD^{\alpha}_{a^+}u(t) = D^{\alpha}_{a^+}u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha},\ n = [\alpha]+1 \tag{1.16} $$

and

$$ {}^cD^{\alpha}_{b^-}u(t) = D^{\alpha}_{b^-}u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-t)^{k-\alpha},\ n = [\alpha]+1 \tag{1.17} $$

The Caputo fractional derivatives ${}^cD^{\alpha}_{a^+}u(t)$ and ${}^cD^{\alpha}_{b^-}u(t)$,coincide with the Riemann-Liouville fractional derivatives $D^{\alpha}_{a^+}u(t)$ and $D^{\alpha}_{b^-}u(t)$,in the following cases :

$$ {}^cD^{\alpha}_{a^+}u(t) = D^{\alpha}_{a^+}u(t), \quad if\ u(a) = u'(a) = \cdots u^{(n-1)}(a) = 0 $$

and

$$ {}^cD^{\alpha}_{b^-}u(t) = D^{\alpha}_{b^-}u(t), \quad if\ u(b) = u'(b) = \cdots u^{(n-1)}(b) = 0 $$

4. For α and β , such that : $0 \le n-1 < \alpha < n$ and $\beta > n-1$,the Caputo fractional derivative of the function $u(t)$ and $v(t)$ where , $u(t) = (t-a)^{\beta}$ and $v(t) = (b-t)^{\beta}$, are given by :

$$ {}^cD^{\alpha}_{a^+}u(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha} $$
$$ {}^cD^{\alpha}_{a^+}v(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(b-t)^{\beta-\alpha} $$

1.1.5 Katugampola fractionl integrals and fractional derivative

Introduced by UDITA katugampola (2011), which generalizes the Riemann-Liouville and Hadamard fractional integral into a single form (see kilbas).The generalized fractional integral is based on the observation that , for $n \in \mathbb{N}$,

$$ \int_a^t \tau_1^{\rho-1}d\tau_1 \int_a^{\tau_1} \tau_2^{\rho}d\tau_2 \cdots \int_a^{\tau_{n-1}} \tau_n^{\rho-1}u(\tau_n)d\tau_n = \frac{\rho^{1-n}}{(n-1)!}\int_a^t \frac{\tau^{\rho-1}y(\tau)}{(t^{\rho}-\tau^{\rho}})^{1-n}d\tau \tag{1.18} $$

by replacing the integer n by positive real number α , we obtain the following definition

Definition 1.13. *(katugampola fractional integral) . (see [12]) Let* $\Omega = [a,b](-\infty < a < b < \infty)$ *be afinite interval on the real axis* $\mathbb{R}$ *for* $(1 \le p < \infty)$. *The generalizedfractional integral* ${}^{\rho}I^{\alpha}_{a^+}u(t)$ *of order* $\alpha \in \mathbb{C}(R(\alpha) > 0)$ *of* $u(t) \in X^p_c(a,b)$ *is defined by*

$$ ({}^{\rho}I^{\alpha}_{a^+}u)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{\tau^{\rho-1}u(\tau)}{(t^{\rho}-\tau^{\rho})^{1-\alpha}}d\tau \tag{1.19} $$

for $t > a$ and $\rho > 0$. This integral is called the left-sided fractional integral. Similarly we can define the right-sided fractional integral ${}^{\rho}I_{b^-}^{\alpha}u$ by

$$({}^{\rho}I_{b^-}^{\alpha}u)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_t^b \frac{\tau^{\rho-1}u(\tau)}{(\tau^{\rho}-t\rho)^{1-\alpha}}d\tau \tag{1.20}$$

for $t < b$ and $R(\alpha) > 0$.

Definition 1.14. *(Katugampola fractional derivative). (see[12]) Let* $\alpha \in \mathbb{C}, R(\alpha) \geq 0, n = [R(\alpha)] + 1$ *and* $\rho > 0$ *The generalized fractional derivative for* $0 \leq a < t < b \leq \infty$ *is defined by*

$$\begin{aligned} ({}^{\rho}D_{a^+}^{\alpha}u)(t) &= \left(t^{1-\rho}\frac{d}{dt}\right)^n \left({}^{\rho}I_{a^+}^{n-\alpha}u\right)(t) \\ &= \frac{\rho^{n-\alpha+1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t \frac{\tau^{\rho-1}u(\tau)}{(t^{\rho}-\tau^{\rho})^{\alpha-n+1}}d\tau \end{aligned} \tag{1.21}$$

respectively

$$\begin{aligned} ({}^{\rho}D_{b^-}^{\alpha}u)(t) &= \left(-t^{1-\rho}\frac{d}{dt}\right)^n \left({}^{\rho}I_{b^-}^{n-\alpha}u\right)(t) \\ &= \frac{\rho^{n-\alpha+1}}{\Gamma(n-\alpha)}\left(-t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t \frac{\tau^{\rho-1}u(\tau)}{(\tau^{\rho}-t^{\rho})^{\alpha-n+1}}d\tau \end{aligned} \tag{1.22}$$

Properties of Katugampola Fractional operators

Here , we give some lemmas and remarks that present the main properties of the generalized fractional operators , we mainly restrict our attention to the left-sided operators (1.19), (1.20) (of course, the right-sided katugampola operators (1.20), (1.22) possesses similar properties).

Theoreme 1.1. *Let* $\alpha, \beta \in \mathbb{C}$ *be such that* $0 < R(\alpha) < 1$ *and* $0 < R(\beta) < 1$ *: if* $0 < a < b < \infty$ *and* $1 \leq p \leq \infty$ *then for any* $u, v \in X_c^p(a,b), \rho > 0$ *we have*

- Index property :

$${}^{\rho}I_{a^+}^{\alpha\rho}I_{a^+}^{\beta}u = {}^{\rho}I_{a^+}^{\alpha+\beta}u \text{ and } {}^{\rho}D_{a^+}^{\alpha\rho}D_{a^+}^{\beta}u = {}^{\rho}D_{a^+}^{\alpha+\beta}u$$

- Composition property : for $0 < R(\alpha) < R(\beta) < 1$ and $u \in L^p(a,b)$

$${}^{\rho}D_{a^+}^{\alpha\rho}I_{a^+}^{\beta}u = {}^{\rho}I_{a^+}^{\beta-\alpha}u \text{ and } {}^{\rho}D_{b^-}^{\alpha\rho}I_{b^-}^{\beta}u = {}^{\rho}I_{b^-}^{\beta-\alpha}u$$

- Linearity property :

$${}^{\rho}I_{a^+}^{\alpha}(u+v) = {}^{\rho}I_{a^+}^{\alpha} + {}^{\rho}I_{a^+}^{\alpha}v$$

and

$${}^{\rho}D_{a^+}^{\alpha}(u+v) = {}^{\rho}D_{a^+}^{\alpha} + {}^{\rho}D_{a^+}^{\alpha}v$$

Theoreme 1.2. *Let $\alpha \in \mathbb{C}, R(\alpha) \geq 0, n = R[(\alpha)] + 1$ and $\rho > 0$ then*

$$lim_{\rho \longrightarrow 1}(\,^{\rho}I^{\alpha}_{a^+}u)(t) \;=\; ^{RL}I^{\alpha}_{a^+}u(t) \;=\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}u(\tau)d\tau$$

$$lim_{\rho \longrightarrow 1}(\,^{\rho}D^{\alpha}_{a^+}u)(t) \;=\; ^{RL}D^{\alpha}_{a^+}u(t) \;=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t (t-\tau)^{n-\alpha-1}u(\tau)d\tau$$

Proposition 1.1. *For $\alpha, \rho > 0$ and $\upsilon > -\rho$ we quote*

1. $^{\rho}D^{\alpha}_{0^+}t^{\upsilon} = \frac{\Gamma\left(1+\frac{\upsilon}{\rho}\right)\rho^{\alpha-1}}{\Gamma\left(1+\frac{\upsilon}{\rho}-\alpha\right)}t^{\upsilon-\alpha\rho}$ *Let us give in particular*

$$^{\rho}D^{\alpha}_{0^+}t^{\rho(\alpha-m)} = 0 \; foreachm = 1, 2..., n$$

2. $$^{\rho}I^{\alpha}_{0^+}t^{\upsilon} = \frac{\rho^{-\alpha}\Gamma\left(1+\frac{\upsilon}{\rho}\right)}{\Gamma\left(1+\frac{\upsilon}{\rho}+\alpha\right)}t^{\upsilon+\alpha\rho}? \; for \; all \; \upsilon > -\rho$$

Caputo type of katugampola fractional derivatives

The Caputo-Katugampola fractional derivatives eare given by the following deffinition

Definition 1.15. *(Caputo-Katugampola fractional derivative).([13]). Let n be the smallest integer greather than α .Then, the left and right Caputo-Katugampola fractional derivative of order $\alpha > 0$ are defined by*

$$^{c}D^{\alpha,\rho}_{a^+}u(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_a^t \frac{\tau^{(\rho-1)(1-n)}}{(t^{\rho}-\tau^{\rho})^{\alpha-n+1}}u^{(n)}(\tau)d\tau \tag{1.23}$$

and

$$^{c}D^{\alpha,\rho}_{b^-}u(t) = \frac{(-1)^n\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_a^t \frac{\tau^{(\rho-1)(1-n)}}{(\tau^{\rho}-t^{\rho})^{\alpha-n+1}}u^{(n)}(\tau)d\tau \tag{1.24}$$

In this section we define and give some basic properties of the Caputo type modification of the Katugampola fractional derivative.

Theoreme 1.3. *The relationship between the Caputo-Katugampola derivative and the Katugampola fractional integral of order α is given by*

- Let $u \in C([a, b])$. Then

$$^{c}D^{\alpha,\rho\rho}_{a^+}I^{\alpha}_{a^+}u(t) = u(t)$$

- Let $u \in C^1([a, b])$. Then

$$I^{\alpha}_{a^+}\; ^{c}D^{\alpha,\rho}_{a^+}u(t) = u(t) - u(a)$$

Lemma 1.1. *For $\upsilon > 0$ and $\alpha \in (0.1)$, define*

$$u(t) = \left(\frac{t^\rho - a^\rho}{\rho}\right)^\upsilon \text{ and } y(t) = \left(\frac{b^\rho - t^\rho}{\rho}\right)^\upsilon$$

Then

$$^cD^{\alpha;\rho}_{a^+}u(t) = \frac{\rho^{\alpha-\upsilon}\Gamma(1+\upsilon)}{\Gamma(1-\alpha+\upsilon)}(t^\rho - a^\rho)^{\upsilon-\alpha} \tag{1.25}$$

Performing the change of variables $\prod = \left(\frac{\tau^\rho - a^\rho}{t^\rho - a^\rho}\right)$ and by the definition of the beta function we get

$$\begin{aligned} ^cD^{\alpha;\rho}_{a^+}u(t) &= \tfrac{\rho^{\alpha-\upsilon}}{\Gamma(1-\alpha)}(t^\rho - a^\rho)^{\upsilon-\alpha}\int_0^1(1-\prod)^{-\alpha}\prod^{\upsilon-1}d\prod \\ &= \tfrac{\rho^{\alpha-\upsilon}}{\Gamma(1-\alpha)}(t^\rho - a^\rho)^{\upsilon-\alpha}\beta(1-\alpha,\upsilon) \\ &\tfrac{\rho^{\alpha-\upsilon}\Gamma(1+\upsilon)}{\Gamma(1-\alpha+\upsilon)}(t^\rho - a^\rho)^{\upsilon-\alpha} \end{aligned}$$

Similary, for all $\alpha, \rho > 0$ we have

$$^cD^{\alpha,\rho}_{b^-}y(t) = \frac{\rho^{\alpha-\upsilon}\Gamma(1+\upsilon)}{\Gamma(1-\alpha+\upsilon)}(b^\rho - t^\rho)^{\upsilon-\alpha}$$

Remark 1.1. *From definition* (1.1) *and* (1.23)

1. *If we put $\rho = 1$, then we abtain the left and right Caputo fractional derivative* (1.14) *and* (1.15)
2. *Attending that $lim_{\rho\longrightarrow 0^+}\left(\frac{t^\rho-\tau^\rho}{\rho}\right) = ln\left(\frac{t}{\tau}\right)$ we obtain the left and right Caputo-Hadamard fractional derivatives as defined in* (??) *and* (??)

Definition 1.16. *([1]) Let $f : [0,+\infty[\longrightarrow \mathbb{R}$ be a real valued function.The $\rho-$ Laplace transform of is defined by*

$$L_\rho\left\{f(t)\right\}_s = \int_0^\infty e^{-s\frac{t^\rho}{\rho}}f(t)t^{\rho-1}dt \qquad \rho > 0$$

for all values of s ,the integrale is valid

Theoreme 1.4. *([1]) If the ρ-Laplace transform of $f : [0,+\infty[\longrightarrow \mathbb{R}$ exists for $s > c_1$ and the ρ-Laplace transform of $g : [0,+\infty[\longrightarrow \mathbb{R}$ for $s > c_2$.Then, for any constant a and b , the ρ-Laplace transform of $af + bg$ exists and*

$$L_\rho\{af(t) + bg(t)\}(s) = aL_\rho\{f(t)\}(s) + bL_\rho\{g(t)\}(s), \text{ } for \text{ } s > max\{c_1, c_2\}.$$

Definition 1.17. *([5]) Let f and g be two functions which are piecewise continuous at each intrval $[0,T]$ We define the ρconvolution of f and g by*

$$(f * g)(t) = \int_0^t f\left[(t^\rho - s^\rho)^{1/\rho}\right] g(s)s^{\rho-1}ds$$

Theoreme 1.5. *([5]) Let f and g be two functions which are piecewise continuous at each intrval $[0,T]$ We define the ρconvolution of f and g by*

$$L_\rho\{(f * g)(t)\} = L_\rho\{f(t)\}L_\rho\{g(t)\}$$

Theoreme 1.6. *([5]) Let $\alpha > 0$ and $f \in AC[0,T]$, Then*

$$L_\rho\{({}_0^cD^{\alpha,\rho}f)(t)\}(s) = s^\alpha L_\rho\{f(t)\} - s^{\alpha-1}f(0)$$

Chapter 2

A new inverse source problem fractional diffusion equation with a nonlocal boundary conditions

let $T > 0$ *be a Fixed number and* $D_T = \left\{(x,t) : \begin{array}{c} 0 < x < 1 \\ 0 < t < T \end{array}\right\}$ *we are concerned with the following fractional differential equation in* $\bar{D}_T$:

$$D_{0^+}^{\alpha,\rho}\left(u(x,t) - u(x,0)\right) = u_{xx} + a(t)f(x,t) \tag{2.1}$$

with the initial condition

$$u(x,0) = \varphi(x) \ ; \quad 0 < x < 1 \tag{2.2}$$

and the boundary condition

$$u(0,t) = u(1,t) \ , \ u_x(1,t) = 0 \ ; \quad 0 < t \leq T \tag{2.3}$$

and the over deternmination condition

$$\int_0^1 u(x,t)dx = g(t) \tag{2.4}$$

wehere $D_{0^+}^{\alpha,\rho}$ *refers to the Katugambola of order* $0 < \alpha < 1$ *in the time defini by*

$$D_{0^+}^{\alpha,\rho}u(t) = \frac{\gamma}{\Gamma(1-\alpha)} \int_0^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{-\alpha} u(\tau)\tau^{\rho-1}d\tau \ ; \gamma = t^{1-\rho}\frac{d}{dt}$$

where $g \in AC[0,T]$ *(the spaceof absolutely continuous functions). The solvability of invers source problems with such condition has been considered earllier* $[2,4,7,11]$.

$f(x,t), g(t), a(t)$ *are given function*

When we want to solve the inverse source problem $(2.1) - (2.4)$ *using Fourier's method*

, we have to consider the spectral problem

$$\begin{cases} X'' = -\lambda X \quad x \in (0.1) \\ X(0) = X(1), \quad X'(1) = 0 \end{cases} \tag{2.5}$$

According to $[10, p.73]$ *the boundary conditions in* (2.5) *are regular but not strongly regular The boundary-value problem* (2.5) *is non self-adjoint , it admits the following adjoint problem*

$$\begin{cases} Y'' = -\lambda Y \quad Y \in (0.1) \\ Y'(0) = Y'(1), \quad Y(0) = 0 \end{cases} \tag{2.6}$$

Theoreme 2.1. *([5]) The Cauchy problem*

$$\begin{cases} D^{\alpha,\rho}(u(t) - u(0)) + \lambda u(t) = f(t), \quad t > 0, \ 0 < \alpha < 1, \ \rho > 0, \ \lambda \in \mathbb{R} \\ u(0) = u_0, \ u_0 \in \mathbb{R} \end{cases}$$

has the solution

$$u(t) = u_0 e_\alpha \left(\frac{t^\rho}{\rho}, \lambda \right) + \int_0^t e_{\alpha,\alpha} \left(\frac{t^\rho - \tau^\rho}{\rho} \right) f(\tau) \tau^{\rho - 1} d\tau$$

Proposition 2.1. *for* $0 < \alpha \leq \beta \leq 1$*the following hold :*

1. *for* $\lambda > 0; t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)$*is completely monotomic function*
2. $E_{\alpha,\alpha}(-\lambda t^\alpha) < +\infty$ *and* $\int_0^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^\alpha\right)\tau^{\rho-1}d\tau = \frac{1}{\lambda}\left[1 - E_\alpha\left(-\lambda\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^\alpha\right)\right] < +\infty$
3. *for* $\lambda \in \mathbb{R}^+, t \in]0, \tau] : E_\alpha\left(-\lambda\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^\alpha\right) \leq \frac{\rho(t^\rho-\tau^\rho)^{\alpha-1}}{\rho^\alpha+(t^\rho-\tau^\rho)^\alpha} < +\infty$

Main results

This section is devoted to give an existence,uniqueness and satability results of solution for the invers problem $(1) - (4)$

2.1 A bi-orthogonal system of functions

the sets of function

$$\{2, \{4cos(2\pi nx)\}_{n=1}^\infty, \{4(1-x)sin(2\pi nx)\}_{n=1}^\infty\} \tag{2.7}$$

and

$$\{x, \{xcos(2\pi nx)\}_{n=1}^\infty, \{sin(2\pi nx)\}_{n=1}^\infty\} \tag{2.8}$$

are obtained from the non-self-adjoint spectral problem (2.5) *and its adjoint problem*

$$Y'' = -\lambda Y \qquad x \in (0,1) \tag{2.9}$$

$$Y'(0) = Y'(1), \qquad Y(0) = 0 \tag{2.10}$$

The set of functions (2.7) *and* (2.8) *is complete in* $L^2(0,1)$*and forms a Riesz basis in* $L^2(0,1)$. *Furthermore,set of function* (2.7) − (2.8) *constitutes a bi-orthogonal system with the one to one correspodence*

$$\begin{array}{ccc} \{\underbrace{2}, & \underbrace{\{4cos(2\pi nx)\}_{n=1}^{\infty}}, & \underbrace{\{4(1-x)sin(2\pi nx)\}_{n=1}^{\infty}}\} \\ \downarrow & \downarrow & \downarrow \\ \{\quad x, & \{xcos(2\pi nx)\}_{n=1}^{\infty}, & \{\quad sin(2\pi nx)\}_{n=1}^{\infty} \quad \} \end{array}$$

The set of bi-orthogonal functions formed from (2.7) *and* (2.8) *plays an important role in proving*

2.2 Existence and uniqueness of the solution of the invers problem

for the proof of the main result , i.e., Theorem 2.2 *we will use properties of the bi-ortogonal system of functions and application of the banach fixed point theorem .*
we have the following theorem

Theoreme 2.2. *suppose the following condition hold :*

$(A1)\varphi \in C^4([0.1]), \varphi(1) = \varphi(0), \varphi'(1) = 0, \varphi''(0) = \varphi''(1), \varphi'''(1) = 0;$

$(A2)f \in C^4([\bar{Q}_T, \mathbb{R}]), f(0,t) = f(1,t), f_x(1,t) = 0, f_{xx}(0,t) = f_{xx}(1,t),$

$f_{xxx}(1,t) = 0, \int_0^1 f(x,t)dx \neq 0$ *and*

$$0 < \frac{1}{M_1} \leq \left| \int_0^1 f(x,t)dx \right|$$

$(A3)g \in AC([0,T])$, *and* $g(t)$ *satisfies the consistency condition* $\int_0^1 \varphi(x)dx = g(0)$
If the following condition

$$T < \left(\frac{\alpha \rho^{\alpha}}{MC'} \right)^{1/\rho\alpha} \tag{2.11}$$

where C' *is defined in* (2.26) *, then the inverse problem* (2.1) − (2.4) *has a unique solution*

Proof.According to assumptions $(A1)-(A3)$, *there are positive constants,* $L_1, L_2, M_i, i = 0, ..., 2$
such that

$$\begin{aligned} L_1 &= max_{0\le t\le T}\ e_\alpha\left(\frac{t^\rho}{\rho}, \lambda_n\right) \\ L_2 &= max_{0\le \tau\le t\le T}\ E_{\alpha,\alpha}\left[-\lambda_n\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^\alpha\right] \\ M_0 &= ||r||_{C(0.T)} \\ M_1 &= max\left(||f_0||_{C(0.T)}, ||f_{1n}^{(4)}||_{C(0.T)}, ||f_{2n}^{(4)}||_{C(0.T)}\right) \\ M_2 &= max\left(||\varphi_0||, ||\varphi_{1n}^{(4)}||, ||\varphi_{2n}^{(4)}||\right) \end{aligned}$$

The proof of this theorem takes place in three steps:

Step 1 :Construction of solution

By applying the Fourier's method , the solution $u(x,t)$ *of the direct problem* $(2.1)-(2.3)$ *ca be developed in uniformly convergent series from using the eigen functions* (2.7) *in* $L^2(0.1)$ *as follows*

$$u(x,t) = 2u_0(t) + \sum_{n=1}^{\infty} u_{1n}(t)4cos(2\pi nx) + \sum_{n=1}^{\infty} u_{2n}(t)4(1-x)sin(2\pi nx) \tag{2.12}$$

where $u_0(t), u_{1n}(t), u_{2n}(t)$ *for* $n \in \mathbb{N}$ *are to be determined.*
Let $\{f_0(t), f_{1n}(t), f_{2n}(t)\}$ *be the coefficients of the series expansion of* $f(x,t)$ *in the basis* (3.1) *which are given by*

$$\begin{gathered} f_0(t) = \textstyle\int_0^1 f(x,t)xdx, f_{1n}(t) = \int_0^1 f(x,t)xcos(2\pi nx)dx, \\ f_{2n}(t) = \textstyle\int_0^1 f(x,t)sin(2\pi nx)dx \end{gathered} \tag{2.13}$$

Using properties of the bi-orthogonal system we have

$$u_0(t) = (u(x,t), x) \tag{2.14}$$

where $(f,g) := \int_0^1 f(x)g(x)dx$ *is the scalar product in* $L^2(0.1)$. *by virtue of* (2.14), *we have*

$$D_{0^+}^{\alpha,\rho}(u_0(t) - u_0(0)) = (D_{0^+}^{\alpha,\rho}(u(x,t) - u(x,0)), x).$$

Using (2.1) *we can write*

$$D_{0^+}^{\alpha,\rho}(u_0(t) - u_0(0)) = ((u_{xx} + a(t)f(x,t)), x).$$

On computing we obtain the following linear fractional differential equation

$$D_{0^+}^{\alpha,\rho}(u_0(t) - u_0(0)) = a(t)f_0(t) \tag{2.15}$$

Alike ,we obtain the linear fractional equation

$$D_{0^+}^{\alpha,\rho}(u_{2n}(t)-u_{2n}(0))+(2\pi n)^2u_{2n}(t)=a(t)f_{2n}(t) \tag{2.16}$$

$$D_{0^+}^{\alpha,\rho}(u_{1n}(t)-u_{1n}(0))+4\pi^2n^2u_{1n}(t)+4\pi n u_{2n}(t)=a(t)f_{1n}(t) \tag{2.17}$$

The solution of the linear fractional differential equation (2.15) *is*

$$u_0(t)=\varphi_0+\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}\tau^{\rho-1}a(\tau)f_0(\tau)d\tau \tag{2.18}$$

the solution of the linear fractional differential equation $(2.15)-(2.16)$ *are*

$$u_{2n}(t)=\varphi_{2n}e_\alpha\left(\frac{t^\rho}{\rho},\lambda_n\right)+\int_0^t e_{\alpha,\alpha}\left(\frac{t^\rho-\tau^\rho}{\rho},\lambda_n\right)\tau^{\rho-1}a(\tau)f_{2n}(\tau)d\tau \tag{2.19}$$

$$u_{1n}(t)=\varphi_{1n}e_\alpha\left(\frac{t^\rho}{\rho},\lambda_n\right)+\int_0^t e_{\alpha,\alpha}\left(\frac{t^\rho-\tau^\rho}{\rho},\lambda_n\right)\tau^{\rho-1}a(\tau)f_{1n}(\tau)d\tau \tag{2.20}$$

$$-4\pi n\int_0^t e_{\alpha,\alpha}\left(\frac{t^\rho-\tau^\rho}{\rho},\lambda_n\right)\tau^{\rho-1}u_{2n}(\tau)d\tau$$

*where we have used Theorem*2.1, $\lambda_n=(2\pi n)^2$

$$\varphi_0=\int_0^1\varphi(x)dx,\quad \varphi_{1n}=\int_0^1\varphi(x)cos(2\pi nx)dx,\quad \varphi_{2n}=\int_0^1\varphi(x)sin(2\pi nx)dx$$

After substituting expressions $u_0(t),u_{2n}(t)$ *and* $u_{1n}(t)$ *respectivley described by* $(2.18)-(2.19)-(2.20)$, *into* (2.12) *we have*

$$\begin{aligned}
&u(x,t)=2\varphi_0+\tfrac{2}{\Gamma(\alpha)}\textstyle\int_0^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}\tau^{\rho-1}a(\tau)f_0(\tau)d\tau\\
&+\textstyle\sum_{n=1}^{+\infty}\left\{\varphi_{2n}e_\alpha\left(\frac{t^\rho}{\rho},\lambda_n\right)+\int_0^t e_{\alpha,\alpha}\left(\frac{t^\rho-\tau^\rho}{\rho},\lambda_n\right)\tau^{\rho-1}a(\tau)f_{2n}(\tau)d\tau\right\}4(1-x)sin(2\pi nx)\\
&+\textstyle\sum_{n=1}^{+\infty}\left\{\varphi_{1n}e_\alpha\left(\frac{t^\rho}{\rho},\lambda_n\right)+\int_0^t e_{\alpha,\alpha}\left(\frac{t^\rho-\tau^\rho}{\rho},\lambda_n\right)\tau^{\rho-1}a(\tau)f_{1n}(\tau)d\tau\right.\\
&\left.-4\pi n\textstyle\int_0^t e_{\alpha,\alpha}\left(\frac{t^\rho-\tau^\rho}{\rho},\lambda_n\right)\tau^{\rho-1}u_{2n}(\tau)d\tau\right\}4cos(2\pi nx)
\end{aligned} \tag{2.21}$$

Taking fractional derivative $D_{0^+}^{\alpha,\rho}$ *under the integral sign of the over-determination condition* (2.4) *and in view of the consistency relation we have*

$$\int_0^t D_{0^+}^{\alpha,\rho}(u(x,t)-u(x,0))dx=D_{0^+}^{\alpha,\rho}(g(t)-g(0))$$

which by using (2.1) *and integration by parts leads to*

$$a(t)=\left(\int_0^t f(x,t)dx\right)^{-1}(D_{0^+}^{\alpha,\rho}(g(t)-g(0))+u_x(0,t)) \tag{2.22}$$

Recall that $\int_0^t f(x,t)dx \neq 0$ *and we have*

$$f(x,t) = 2f_0(t) + \sum_{n=1}^{\infty} f_{1n}(t)4cos(2\pi nx) + \sum_{n=1}^{\infty} f_{2n}(t)4(1-x)sin(2\pi nx)$$

where $f_0(t), f_{1n}(t)$ *and* $f_{2n}(t)$ *are given by* (2.13) *then*

$$\int_0^t f(x,t)dx = 2f_0(t) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{f_{2n}}{n} \tag{2.23}$$

and

$$u_x(0,t) = \sum_{n=1}^{\infty} 8\pi n\varphi_{2n}e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_n\right) + \sum_{n=1}^{\infty} 8\pi n\int_0^t e_{\alpha,\alpha}\left(\frac{t^{\rho}-\tau^{\rho}}{\rho},\lambda_n\right)\tau^{\rho-1}a(\tau)f_{2n}(\tau) \tag{2.24}$$

$$a(t) = \left(\int_0^t f(x,t)dx\right)^{-1}\left(D_{0^+}^{\alpha,\rho}(g(t)-g(0)) + u_x(0,t)\right)$$

by (2.19) *and* (2.20) *we have the volterra integral equation*

$$a(t) = F(t) + \left(2f_0(t) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{f_{2n}(t)}{n}\right)^{-1}\int_0^t K(t,\tau)a(\tau)\tau^{\rho-1}d\tau \tag{2.25}$$

where

$$F(t) = \left(\int_0^t f(x,t)dx\right)^{-1}\left(D_{0^+}^{\alpha,\rho}(g(t)-g(0)) + \sum_{n=1}^{\infty} 8\pi n\varphi_{2n}e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_n\right)\right) \tag{2.26}$$

and

$$K(t,\tau) = \left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1}\sum_{n=1}^{\infty} 8\pi nf_{2n}(\tau)e_{\alpha,\alpha}\left(\frac{t^{\rho}-\tau^{\rho}}{\rho},\lambda_n\right) \tag{2.27}$$

Step 2 :Existence and uniqueness of the solution of the inverse problem

Befor presenting the existence result for $u(x,t)$ *, we will use banach Fixed Point Theorem to prove the unique existence of* $a(t)$ *, for which consider the ollowing map :*

$$B(a(t)) = a(t)$$

on the space$C[0,T]$*, where* $a(t)$ *is given by equation* (2.25)*, with norm*

$$||\phi|| = max_{0\leq t\leq T}|\phi(t)|$$

where the operator B *is*

$$B(a(t)) = F(t) + \left(2f_0(t) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{f_{2n}(t)}{n}\right)^{-1}\int_0^t K(t,\tau)a(\tau)\tau^{\rho-1}d\tau$$

To show B is well defined ,first ,we will prove that for $a \in C[0,T], B(a(t))$*represents a continuous function.*

Since ,under the assumptions $(A1)$ *and integration by parts four times ,we obtaian*

$$\sum_{n=1}^{\infty} 8\pi n \varphi_{2n} e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_n\right) \leq \sum_{n=1}^{+\infty} \frac{L_1|\varphi_{2n}^{(4)}|}{2\pi^3 n^3}, t \in [0,T] \tag{2.28}$$

where $\varphi_{2n}^{(4)} = \int_0^1 \varphi^{(4)}(x) sin(2\pi n x)dx,$

Under the assumptions $(A2)$ *and integration by parts four times ,we obtain*

$$\sum_{n=1}^{\infty} 8\pi n f_{2n}(\tau) e_{\alpha,\alpha}\left(\frac{t^{\rho}-\tau^{\rho}}{\rho},\lambda_n\right) \leq \sum_{n=1}^{+\infty} \frac{L_2|f_{2n}^{(4)}(\tau)|}{2\pi^3 n^3}, t,\tau \in [0,T] \tag{2.29}$$

where $f_{2n}^{(4)}(t) = \int_0^1 \frac{\partial^4 f(x,t)}{\partial x^4} sin(2\pi n x)dx,$

Using the Cauchy-schwarz and Bessel inequalities, we obtain

$$\begin{aligned} \sum_{n=1}^{+\infty} \frac{L_2|f_{2n}^{(4)}(\tau)|}{2\pi^3 n^3} &\leq \left[\sum_{n=1}^{+\infty} \frac{L_2^2}{4\pi^6 n^6}\right]^{1/2} \left[\sum_{n=1}^{+\infty} \left(f_{2n}^{(4)}(\tau)\right)^2\right]^{1/2} \\ &\leq c \left\|\frac{\partial^4 f(x,t)}{\partial x^4}\right\|_{L^2(0,1)} \end{aligned}$$

where c is a constant independent of t and n . Thus , we have

$$\sum_{n=1}^{\infty} 8\pi n f_{2n}(\tau) e_{\alpha,\alpha}\left(\frac{t^{\rho}-\tau^{\rho}}{\rho},\lambda_n\right) \leq C' \tag{2.30}$$

where $C' = c \max_{0\leq t\leq T} \left\|\frac{\partial^4 f(x,t)}{\partial x^4}\right\|_{L^2(0,1)}$

By (2.29) *and* (2.30) *the series function*

$$\sum_{n=1}^{+\infty} 8\pi n f_{2n}(\tau) e_{\alpha,\alpha}\left(\frac{t^{\rho}-\tau^{\rho}}{\rho},\lambda_n\right) and \sum_{n=1}^{\infty} 8\pi n \varphi_{2n} e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_n\right)$$

are uniformly convergent. Then ,$F(t)$ *and* $K(t,\tau)$ *are continuous functions on* $[0,T]$ *and* $[0,T][0,T]$*, respectively.Hense , the operator B is well defined .*

Next to schow B is contraction .Let $a_1, a_2 \in C(0,T)$ *we have*

$$\begin{aligned} |B(a_1(t) - B(a_2(t))| &\leq \left|\left[\int_0^1 f(x,t)dx\right]^{-1}\right| \int_0^t |k(t,\tau)||a_1(\tau)-a_2(\tau)|\tau^{\rho-1}d\tau \\ &\leq MC' \int_0^T \left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1} |a_1(\tau)-a_2(\tau)|\tau^{\rho-1}d\tau \\ &\leq MC' \max_{0\leq t\leq T} |a_1(\tau)-a_2(\tau)| \int_0^t \left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1} \tau^{\rho-1}d\tau \end{aligned}$$

$$||B(a_1) - B(a_2)|| \leq \frac{MC'T^{\rho\alpha}}{\alpha\rho^{\alpha}}||a_1 - a_2|| \tag{2.31}$$

With the condition (2.11) , $\frac{MC'T^{\rho\alpha}}{\alpha\rho^{\alpha}} < 1$, *the the mapping* B *is a contraction .Consequently by Banach Fixed point Theorem , the mapping* B *has unique fixed point* $a \in C[0,T]$.
To establish the regularity of the obtained solution , it remains to show

$$u(x,t), u_x(x,t), u_{xx}(x,t), D_t^{\alpha,\rho}u(x,t) \in C(\Omega_T)$$

Under assumptions $(A1)-(A2)$ *and integration by parts four times, we have*

$$\begin{aligned} f_{2n}(t) &= \frac{f_{2n}^{(4)}(t)}{16\pi^4 n^4}, f_{2n-1}(t) = \frac{1}{16\pi^4 n^4}\left(f_{2n-1}^{(4)}(t) - \frac{2}{\pi n} f_{2n}^{(4)}(t)\right) \\ \varphi_{2n}(t) &= \frac{\varphi_{2n}^{(4)}(t)}{16\pi^4 n^4}, \varphi_{2n-1}(t) = \frac{1}{16\pi^4 n^4}\left(\varphi_{2n-1}^{(4)}(t) - \frac{2}{\pi n}\varphi_{2n}^{(4)}(t)\right) \end{aligned} \tag{2.32}$$

From $(2.18)-(2.20)-(2.32)$ *and* (1.4) *we get*

$$\begin{aligned} |u_0(t)| &\leq M_2 + \frac{M_0 M_1 T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \; t \in [0,T] \\ |u_{2n}(t)| &\leq \frac{L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha}}{16\pi^4 n^4}, \; t \in [0,T] \\ |u_{2n-1}(t)| &\leq \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha})(3CT^{\rho}/\rho)}{16\pi^4 n^4}, \; t \in [\epsilon, T], \epsilon > 0 \end{aligned} \tag{2.33}$$

By using (2.12) *and* (2.33) *, following relations hold for* $x \in [0,1]$ *and* $t \in [\epsilon, T]$ *with* $\epsilon > 0$ *such that*

$$\begin{aligned} |u(x,t)| &\leq 2M_1 + \sum_{n=1}^{+\infty} \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha})(2+CT^{\rho}/\rho)}{4\pi^4 n^4} \\ &+ \sum_{n=1}^{+\infty} \frac{2(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha})}{4\pi^4 n^4} \\ |u_x(x,t)| &\leq \sum_{n=1}^{+\infty} \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha})(2+CT^{\rho}/\rho)}{2\pi^3 n^3} \\ &+ \sum_{n=1}^{+\infty} \frac{(1+4\pi n)(2)(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha})}{4\pi^4 n^4} \\ |u_{xx}(x,t)| &\leq \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha})(2+CT^{\rho}/\rho)}{\pi n^2} \\ &+ \sum_{n=1}^{+\infty} \frac{(1+2\pi n)(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha})}{\pi^3 n^3} \end{aligned} \tag{2.34}$$

From $(2.15)-(2.17), (2.33)$ *and for* $t \in [\epsilon, T]$ *, we have*

$$\begin{aligned} |D_{0^+}^{\alpha,\rho}(u_0(t) - u_0(0))| &\leq M_0 M_2 \\ |D_{0^+}^{\alpha,\rho}(u_{2n}(t) - u_{2n}(0)| &\leq \frac{M_0 M_2}{16\pi^4 n^4} + \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha})}{4\pi^2 n^2} \\ |u_{2n-1}(t) - u_{2n-1}(0)| &\leq \frac{3M_0 M_2}{16\pi^4 n^4} + \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha}) + (2+CT^{\rho}/\rho)}{4\pi^2 n^2} \\ &+ \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha})}{4\pi^3 n^3} \end{aligned}$$

Consequently,

$$\begin{aligned} |D_{0^+}^{\alpha;\rho}(u(x,t)-u(x,0)| &\leq 2M_0M_2 + \sum_{n=1}^{+\infty}\frac{4M_0M_2}{4\pi^4n^4} \\ &+ \sum_{n=1}^{+\infty}\frac{(L_1M_2+L_2M_0M_1T^{\rho\alpha}/\alpha\rho^{\alpha})}{\pi^3n^3} \\ &+ \sum_{n=1}^{+\infty}\frac{(L_1M_2+L_2M_0M_1T^{\rho\alpha}/\alpha\rho^{\alpha})(4+CT^{\rho}/\rho)}{\pi^2n^2} \end{aligned} \tag{2.35}$$

From $(2.34)-(2.35)$ *and by Weiestrass M-test , the series corresponding to* $u(x,t), u_x(x,t)u_{xx}(x,t), D_{0^+}^{\alpha;\rho}u$ *are uniformly convergent on* $[0,1][\epsilon,T]$ *for* $\epsilon > 0$. *Hence ,* $u(x,t), u_x(x,t)u_{xx}(x,t), D_{0^+}^{\alpha,\rho}(u(x,t)-u(x,0)$ *are continuous functionon* Ω_T.

Step 3 : Uniqueness of the solution

Let $\{u(x,t), a_1(t)\}$ *and* $\{v(x,t), a_2(t)\}$ *be two solution sets of the inverse problem* $(2.1)-(2.4)$ *by using* (2.12) *we obtain*

$$\begin{aligned} u(x,t)-v(x,t) &= 2(u_0(t)-v_0(t)) + \sum_{n=1}^{+\infty}(u_{1n}(t)-v_{1n}(t))X_{1n}(x) \\ &+ \sum_{n=1}^{+\infty}(u_{2n}(t)-v_{2n}(t))X_{2n}(x) \end{aligned} \tag{2.36}$$

Due to the estimat (2.31) *and condition* (2.11) *we have* $a_1 = a_2$ *and by substituting* $a_1 = a_2$ *in* (2.36) *and* $(2.18)-(2.20)$ *we obtain* $u = v$

2.2.1 Continuous dependence of the solution on the data

Let H *be the of triples* φ, f, g *where the functions* φ, f *and* g *satisfy the assumptions of Theorem* (2.2) *and*

$$||\varphi||_{C^4(0.1)} \leq M_4, ||f||_{C^4(\Omega_T)} \leq M_5, ||g||_{C^1(0.1)} \leq M_6.$$

For $\phi \in H$ *we define the norm*

$$||\phi||_H = ||\varphi||_{C^4(0.1)} + ||f||_{C^4(\Omega_T)} + ||g||_{C^1(0.1)}$$

By using the Cauchy-schwarz and bessel inequalities , the series function

$$\sum_{n=1}^{+\infty}\frac{|f_{2n}^{(4)}(\tau)|}{2\pi^3n^3} \leq M_7$$

is uniformly convergent ,where $\varphi_{2n}^{(4)}$ *are the coefficients of the sine Fourier expansion of the function* $\varphi_{2n}^{(4)}$

Now , we are present the result on the stability of the solution of the inverse problem $(2.1)-(2.4)$

Theoreme 2.3. *The solution* $\{u(x,t), a(t)\}$ *of the inverse problem* $(2.1) - (2.4)$ *under the assumptions on theorem* (2.2) *depends continously upon the data for*
$T < \left(\frac{\alpha\rho^{\alpha}}{MC'}\right)^{1/\rho\alpha}$

proof .Let $\{u(x,t), a(t)\}$ *and* $\{\tilde{u}(x,t), \tilde{a}(t)\}$ *be two solution sets of the inverse problem* $(2.1) - (2.4)$ *corresponding to the data* $\phi = \{\varphi, f, g\}$ *and* $\phi = \{\tilde{\varphi}, \tilde{f}, \tilde{g}\}$ *respectively*
For $g, \tilde{g} \in C^1(0,T)$ *we have*

$$||D_{0^+}^{\alpha,\rho}(g(t) - g(0)) - D_{0^+}^{\alpha,\rho}(\tilde{g}(t) - \tilde{g}(0)||_{C(0.T)} \leq M_8||g - \tilde{g}||_{C^1(0.T)} \tag{2.37}$$

where $M_8 = \frac{T^{1-\rho\alpha}}{\rho^{1-\alpha}\Gamma(2-\alpha)}$ *from* (2.26) *we have*

$$\begin{aligned}
F(t) - \tilde{F}(t) &= \left(\int_0^t f(x,t)dx\right)^{-1}\left(D_{0^+}^{\alpha,\rho}(g(t) - g(0)) + \textstyle\sum_{n=1}^{\infty} 8\pi n\varphi_{2n}e_\alpha\left(\frac{t^\rho - \tau^\rho}{\rho}, \lambda_n\right)\right)\\
&- \left(\int_0^t \tilde{f}(x,t)dx\right)^{-1}\left(D_{0^+}^{\alpha,\rho}(\tilde{g}(t) - \tilde{g}(0)) + \textstyle\sum_{n=1}^{\infty} 8\pi n\tilde{\varphi}_{2n}e_\alpha\left(\frac{t^\rho - \tau^\rho}{\rho}, \lambda_n\right)\right)\\
&= \left(\int_0^1 f(x,t)dx \int_0^1 \tilde{f}(x,t)dx\right)^{-1}\left(\int_0^1 \tilde{f}(x,t)dx\,(D_{0^+}^{\alpha,\rho}(g(t) - g(0))\right.\\
&\left. + \textstyle\sum_{n=1}^{\infty} 8\pi n\varphi_{2n}e_\alpha\left(\frac{t^\rho - \tau\rho}{\rho}, \lambda_n\right)\right)\Big)\\
&- \int_0^1 f(x,t)dx\left(D_{0^+}^{\alpha,\rho}(\tilde{g}(t) - \tilde{g}(0)) + \textstyle\sum_{n=1}^{\infty} 8\pi n\tilde{\varphi}_{2n}e_\alpha\left(\frac{t^\rho - \tau^\rho}{\rho}, \lambda_n\right)\right)\\
&= \left(\int_0^1 f(x,t)dx \int_0^1 \tilde{f}(x,t)dx\right)^{-1}\left(\int_0^1 \tilde{f}(x,t)dx\,(D_{0^+}^{\alpha,\rho}(g(t) - g(0)) - D_{0^+}^{\alpha,\rho}(\tilde{g}(t) - \tilde{g}(0)) -\right.\\
&\left. + \textstyle\sum_{n=1}^{\infty} 8\pi n(\varphi_{2n} - \tilde{\varphi}_{2n})e_\alpha\left(\frac{t^\rho - \tau\rho}{\rho}, \lambda_n\right)\right)\Big)\\
&+ D_{0^+}^{\alpha,\rho}(\tilde{g}(t) - \tilde{g}(0))\left(\int_0^1 \tilde{f}(x,t)dx - \int_0^1 f(x,t)dx\right)\\
&+ \textstyle\sum_{n=1}^{\infty} 8\pi n\tilde{\varphi}_{2n}e_\alpha\left(\frac{t^\rho - \tau\rho}{\rho}, \lambda_n\right)\Big)\left(\int_0^1 \tilde{f}(x,t)dx - \int_0^1 f(x,t)dx\right)
\end{aligned}$$

From (2.32) *we have*

$$\varphi_{2n} - \tilde{\varphi}_{2n} = \int_0^1 (\varphi(x) - \tilde{\varphi}(x))X_{2n}(x)dx = \frac{\varphi_{2n}^{(4)} - \tilde{\varphi}_{2n}^{(4)}}{16\pi^4 n^4}$$

we have the estimate

$$||F - \tilde{F}|| \leq N_1||\varphi - \tilde{\varphi}||_{C^4(0.1)} + N_2||f - \tilde{f}||_{C(\Omega_T)} + N_3||g - \tilde{g}||_{C^1(0.1)}$$

Where $N_1 = M^2L_1C^*, N_2 = M^2(M_7L_1 + M_6M_8), N_3 = M^2M_5M_8$

From (2.25) *,we obtain*

$$
\begin{aligned}
a(t)-\tilde{a}(t) \quad &= F(t)-\tilde{F}(t)+\left(\int_0^1 f(x,t)dx\right)^{-1}\int_0^t K(t,\tau)a(\tau)\tau^{\rho-1}d\tau \\
&-\left(\int_0^1 \tilde{f}(x,t)dx\right)^{-1}\int_0^t \tilde{K}(t,\tau)\tilde{a}(\tau)\tau^{\rho-1}d\tau \\
&= F(t)-\tilde{F}(t) \\
&+\left(\int_0^1 f(x,t)dx\right)^{-1}\left(\int_0^1 \tilde{f}(x,t)dx\right)^{-1}\left(\int_0^1 \tilde{f}(x,t)dx\int_0^t K(t,\tau)a(\tau)\tau^{\rho-1}d\tau\right. \\
&\left.-\int_0^1 f(x,t)dx\int_0^t \tilde{K}(t,\tau)\tilde{a}(\tau)\tau^{\rho-1}d\tau\right) \\
&= F(t)-\tilde{F}(t) \\
&+\left(\int_0^1 f(x,t)dx\right)^{-1}\left(\int_0^1 \tilde{f}(x,t)dx\right)^{-1}\left(\int_0^1 \tilde{f}(x,t)dx\int_0^t(K(t,\tau)-\tilde{K}(t,\tau))a(\tau)\tau^{\rho-1}d\tau\right. \\
&+\int_0^1 \tilde{f}(x,t)dx\int_0^t \tilde{K}(t,\tau)(a(\tau)-\tilde{a}(\tau))\tau^{\rho-1}d\tau \\
&\left.\left(\int_0^t \tilde{K}(t,\tau)\tilde{a}(\tau)\tau^{\rho-1}d\tau\right)\left(\int_0^1 \tilde{f}(x,t)dx-\int_0^1 f(x,t)dx\right)\right)
\end{aligned}
$$

We have the estimate

$$
\begin{aligned}
||a-\tilde{a}|| \quad &\leq ||F-\tilde{F}||+M||a||\int_0^t|K(t,\tau)-\tilde{K}(t,\tau)|\tau^{\rho-1}d\tau \\
&+M||a-\tilde{a}||\int_0^t|\tilde{K}(t,\tau)|\tau^{\rho-1}d\tau+M^2||\tilde{a}||\ ||f-\tilde{f}||_{C(\Omega_T)}\int_0^t|\tilde{K}(t,\tau)|\tau^{\rho-1}d\tau \\
&\leq ||F-\tilde{F}||+\frac{MM_0T^{\rho\alpha}}{\alpha\rho^{\alpha}}||f^{(4)}-\tilde{f}^{(4)}||_{C(\Omega_T)}+\frac{MC'T^{\rho\alpha}}{\alpha\rho^{\alpha}}||a-\tilde{a}|| \\
&+\frac{M^2M_0C'T^{\rho\alpha}}{\alpha\rho^{\alpha}}||f-\tilde{f}||_{C(\Omega_T)}
\end{aligned}
$$

Due to the estimate of $||F-\tilde{F}||$ *, we have*

$$
\begin{aligned}
\left(1-\frac{MC'T^{\rho\alpha}}{\alpha\rho^{\alpha}}\right)||a-\tilde{a}|| \quad &\leq N_1||\varphi-\tilde{\varphi}||_{C^4(0.1)} \\
&+\left(N_2+\frac{MM_0T^{\rho\alpha}}{\alpha\rho^{\alpha}}+\frac{M^2M_0C'T^{\rho\alpha}}{\alpha\rho^{\alpha}}\right)||f-\tilde{f}||_{c^4(\Omega_T)} \\
&+N_3\ ||g-\tilde{g}||_{C^1(0.1)}
\end{aligned}
$$

Hence

$$\left(1-\frac{MC'T^{\rho\alpha}}{\alpha\rho^{\alpha}}\right)||a-\tilde{a}||\leq N_4||\phi-\tilde{\phi}||_H$$

Where $N_4=max\left\{N_1,N_2+\frac{MM_0T^{\rho\alpha}}{\alpha\rho^{\alpha}}+\frac{M_0^MC'T^{\rho\alpha}}{\alpha\rho^{\alpha}},N_3\right\}$

For $T<\left(\frac{\alpha\rho^{\alpha}}{MC'}\right)^{1/\rho\alpha}$ *we have*

$$||a-\tilde{a}||\leq\frac{N_4}{1-\frac{MC'T^{\rho\alpha}}{\alpha\rho^{\alpha}}}||\phi-\tilde{\phi}||_H$$

From (2.12) *a similar estimate can be also obtained for the difference* $u - \tilde{u}$

$$||u - \tilde{u}||_{C(\bar{\Omega}_T)} \leq N_5 ||\phi - \tilde{\phi}||_H$$

Chapter 3

An invers coefficient-source problem for a time-fractional diffusion equation

The fractional diffusion equation are believed to be more realistic in describing anomalous diffusion in heterogeneous porus media than the classical diffusion ones, Thus,they have drawn attention of researchers from various disciplines of science and engeneering.The readers may refer to Cheng et al.(2009);J NaKagawa et al. (2010) and references therein.
Invers source problems are the problems that consist of finding the unknown source term via an additional data .Some works on fractional invers diffusion problems have been published. We refer to Aleroev et al .(2013);Chenget al .(2009);Ftama and Mansur (2012); J.Nakagawa et al .(2010) ; Ozbilge et al .(2016); Tuan (2011)
Here, we consider the so-called fractional diffusion problem involving the linear nonhomogeneous equation

$$D_{0^+}^{\alpha}\left(u(x,t)-u(x,0)\right)=u_{xx}+a(t)u(x,t)+c(t)F(x,t) \tag{3.1}$$

with initial and nonlocal condition

$$u(x,0)=\varphi(x)\ ;\quad x\in[0.1] \tag{3.2}$$

$$u(0,t)=u(1,t)\ ,\ u_x(1,t)=0\ ;\quad t\in[0.T] \tag{3.3}$$

D_{0+}^{α} *is the Riemann-Liouville fractional derivative of order* $0<\alpha<1$ *in the the time variable* $a(t),t>0$ *is the source control term ,*$F(x,t)$ *is the known source term ,* $\varphi(x)$ *is the initial temperature.*
our invers problem consists of determining the time dependent unknown coifficient of the source term $c(t)$ *and the tempurature distribution* $u(x,t)$ *,from the initial tempurature* (3.2) *and the boundary conditions* (3.3) *needs to be determined uniquely by the over-determination condition*

$$\int_0^1 u(x,t)=E(t)\quad t\in[0,T] \tag{3.4}$$

which is the additional of the thermal energy $E(t)$ *.We note that in the papers Aleroev et al .(2013); Fatma and Mansur (2012); lonkin (1977) the time-dependent source coefficient is determined from such condition.*
The used approach is based on the expansion of the solution by using a orthogonal system of function obtained frome the nonlocal boundary conditions.we are motived by the works Aleroev et al .(2013) where the authors considered an in invers time-fractional diffusion problem with $a(t) = 0$ *and Fatma and Mansur (2012) where the authors considered an in invers problem with* $a(t) = 0$ *and* $\alpha = 0$
The outline of the paper is as follows .In section 2 , some necessary preliminaries are given . In section 2 , the exsitence and uniqueness of the solution of the inverse problem $(3.1) - (3.4)$ *is etablished by using the Fourier method and the Banch Fixed point theorem , In section 3, the continuous dependence of the solution of the invers problem upon the data of* $\{a(t), F(x,t), \varphi(x), E(t)\}$ *is shown*
Theoreme 3.1. *The solution* $u \in AC[0,T]$ *of the lineare nonhomogenous fractional problem*

$$\begin{cases} D^{\alpha}_{0+}(u(t) - u(0)) + \lambda u(t) = f(t) \quad t \in (0.T], \ \lambda > 0 \\ u(0+) = c \end{cases} \tag{3.5}$$

where $f \in L^1[0,T]$ *is given by the integral expression*

$$u(t) = cE_{\alpha,1}(-\lambda t^{\alpha}) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha}) f(\tau) d\tau \tag{3.6}$$

3.1 Existence and uniqueness result

First note that for the non-selfadjoint operator $AX = -X'' = -\lambda_n X$ *with* $X(0) = X(1); X'(1) = 0$ *the eigenvalues are* $\lambda_0 = 0, \ \lambda_n = (2n\pi)^2, n \geq 1$*. Thus,we construct the biorthogonal sysetem of function*

$$\{X_0(x) = 2, X_{1n}(x) = 4cos(2\pi nx), X_{2n} = 4(1-x)sin(2\pi nx); n \geq 1\} \tag{3.7}$$

and

$$\{Y_0(x) = x, Y_{1n}(x) = xcos(2\pi nx), Y_{2n}(x) = sin(2\pi nx); n \geq 1\} \tag{3.8}$$

Which are Risez bases in $L^2[0,1]$ *.For more details ,the reader can consult Aleroev et al .(2013) ll'in (1997) ; lonkin (1977)*
Let use define the following space :

$$\begin{aligned} C^{2,\alpha}([0,1](0,T] &= \{u(t,x) \in C^2[0.1]; \ t \in [0.T] \\ &\quad and \ D^{\alpha}_{0+}(u(x,t) - u(x,0)) \in C(0,T]; x \in [0.1]\} \end{aligned}$$

The function a, φ, F *and* E *satisfy the following assumptions*
$(A1) a(t) \in C[0,T]$
$(A2) F(x,.) \in C[0,T]$ *and for* $t \in [0,T], F(.,t) \in C^4[0,1], F(0,t) = F(1,t), F_x(1,t) = 0, F_{xx}(0,t) = F_{xx}(1,t), F_{xxx}(1,t) = 0$ *and* $\int_0^1 F(x,t)dx \neq 0$

$(A3) \varphi \in C^4([0.1]), \varphi(1) = \varphi(0), \varphi'(1) = 0, \varphi''(0) = \varphi''(1), \varphi'''(1) = 0;$

$(A4) E \in AC([0,T])$ *and* $\int_0^1 \varphi(x)dx = E(0)$
From the above , there exsit some positive constants $L_j j = 1,2;\ M_i i = 0,...,5; k$ *such that*

$$
\begin{aligned}
L_1 &= max_{0\leq t\leq T}\ E_{\alpha,1}(-\lambda_n t^\alpha) \\
L_2 &= max_{0\leq \tau\leq t\leq T}\ E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha) \\
M_0 &= ||a||_{C[0,T]} \\
M_1 &= ||f^{-1}||_{C[0,T]} \\
M_2 &= max\left(D^\alpha_{0^+}(E(t)-E(0))||_{C[0,T]}, ||E||_{C[0,T]}\right);\ M_3 = ||c||_{c[0,T]} \\
M_4 &= max\left(||F_0||_{C[0,T]}; ||F_{i,n}||_{C[0,T]}\right);\ i = 1,2;\ n = 1,... \\
M_5 &= max\left(|\varphi_0|; |\varphi_{i,n}|\right)\ i = 1,2\ n = 1,..
\end{aligned}
$$

$$
max_{0<\tau<t\leq T}\lambda_n (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha) \leq k \tag{3.9}
$$

Theoreme 3.2. *Let* $A(1) - A(4)$ *be satisfied . Then the inverse problem* $(3.1) - (3.4)$ *has a unique solution* $(u(x,t), c(t))$ *for some small* T *.Furthermore thise solution is in* $[C^{2,\alpha}([0,1](0,T]) \cap C([0,1][0,T])]$

proof.Let us start with the existence of the solution of direct problem
Step 1: *Determination of* $u(x,t)$ *for arbitary* $c(t)$ *in* $C[0,T]$
By applying the Fourier's method , the solution $u(x,t)$ *of the direct problem* $(3.1) - (3.3)$ (3.1) *can be expanded in a uniformly convergent series in term of eigenfunctions of* (3.7) *in* $L^2(0.1)$ *on the form*

$$
u(x,t) = 2u_0(t) + \sum_{n=1}^{\infty} u_{1n}(t) 4cos(2\pi nx) + \sum_{n=1}^{\infty} u_{2n}(t) 4(1-x) sin(2\pi nx) \tag{3.10}
$$

The coefficients $u_0(t), u_{1n}(t), u_{2n}(t)$ *for* $n \geq 1$ *are to be found by making use of the orthogonality of the eigenfunction .Namely, we multiply* (3.1) *by the eigenfunctions of* (3.8)

and integral over $(0,1)$. *Recall that the scalar product in* $L^2(0.1)$ *is defined by* $(f,g) = \int_0^1 f(x)g(x)dx$.

Let us note the expansion coefficients of $F(x,t)$ *and* $\varphi(x)$ *in the eigenfunctions of* (3.8) *for* $n \geq 1$ *respectivley by*

$$\begin{cases} F_0(t) &= (F(x,t), Y_0(x)) \\ F_{1n}(t) &= (F(x,t), Y_{1n}(x)) \\ F_{2n}(t) &= (F(x,t), Y_{2n}(x)) \end{cases} \quad and \quad \begin{cases} \varphi_0 &= (\varphi(x), Y_0(x)) \\ \varphi_{1n} &= (\varphi(x), Y_{1n}(x)) \\ \varphi_{2n} &= (\varphi(x), Y_{2n}(x)) \end{cases}$$

we obtain view of (3.1) *and with* $(u(x,t), Y_0(x)) = u_0(t)$

$$\begin{cases} D^{\alpha}_{0^+}(u_0(t) - u_0(0)) = a(t)u_0(t) + c(t)F_0(t) \\ u_0(0) = \varphi_0 \end{cases} \tag{3.11}$$

for $u_{1n}(t) = (u(x,t), Y_{1n}(x)); n \geq 1$ *in view of* (3.1) *we have*

$$\begin{aligned} &(D^{\alpha}_{0^+}(u_0(t) - u_0(0)), Y_{1n}(x)) \\ &= (u_{xx}(x,t) + a(t)u(x,t) + c(t)F(x,t), Y_{1n}(x)) \\ &= (u_{1n}(t)X''_{1n}(x) + u_{2n}(t)X''_{2n}(x), Y_{1n}) + a(t)u_{1n} + c(t)F_{1n} \\ &= -\lambda_n u_{1n}(t) - 4\pi n u_{2n} + a(t)u_{1n} + c(t)F_{1n} \end{aligned}$$

The linear fractional differential equation satisfied by $u_{1n}(t)$; $n \geq 1$ *are*

$$\begin{cases} (D^{\alpha}_{0^+}(u_{1n}(t) - u_{1n}(0)) = -\lambda_n u_{1n}(t) - 4\pi n u_{2n}(t) \\ +a(t)u_{1n}(t) + c(t)F_{1n}(t) \\ u_{1n}(0) = \varphi_{1n} \end{cases} \tag{3.12}$$

Also ,the linear fractional differential equation satisfid by $u_{2n}(t); n \geq 1$ *are*

$$\begin{cases} (D^{\alpha}_{0^+}(u_{2n}(t) - u_{2n}(0)) = -\lambda_n u_{2n}(t) + a(t)u_{2n}(t) + c(t)F_{2n}(t) \\ u_{2n}(0) = \varphi_{2n} \end{cases} \tag{3.13}$$

Applying $I^{\alpha}_{0^+}$ *to* (3.11) *we get the following Volterra integral equation satisfide by the solution*

$$u_0(t) = \varphi_0 + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} c(\tau)F_0(\tau)d\tau + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} a(\tau)u_0(\tau)d\tau \tag{3.14}$$

This solution is bounded in $C[0,T]$ *in view of* $(A1)-(A3)$. *we have*

$$\begin{aligned} ||u_0||_{C[0,T]} \leq \quad & |\varphi_0| + \tfrac{T^{\alpha}}{\alpha\Gamma(\alpha)}||F_0||_{C[0,T]}||c||_{C[0,T]} \\ & + \tfrac{T^{\alpha}}{\alpha\Gamma(\alpha)}||a||_{C[0,T]}||u_0||_{C[0,T]} \end{aligned}$$

Hence

$$||u_0||_{C[0,T]} \leq \left[M_5 + \frac{T^{\alpha}}{\alpha\Gamma(\alpha)}M_4M_3\right][1-\Psi_0]^{-1} = \psi_0 \tag{3.15}$$

for

$$\Psi_0 = \frac{T^{\alpha}}{\alpha\Gamma(\alpha)}M_0 < \frac{1}{2} \tag{3.16}$$

In view of theorem 3.1 *the problem* (3.13) *admits a solution in* $C[0,T]$ *satisfying*

$$u_{2n}(t) = \varphi_{2n}E_{\alpha,1}(-\lambda_n t^{\alpha} + \textstyle\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha}) \quad (a(\tau)u_{2n}(\tau) + c(\tau)F_{2n}(\tau))d\tau \tag{3.17}$$

Under $(A1)-(A3)$, $u_{2n}(t)$ *is bounded in* $C[0,T]$ *as follows*

$$||u_{2n}||_{C[0,T]} \leq |\varphi_{2n}|L_1 + L_2\tfrac{T^{\alpha}}{\alpha}||F_{2n}||_{C[0,T]}||c||_{C[0,T]} + L_2\tfrac{T^{\alpha}}{\alpha}||a||_{C[0,T]}||u_{2n}||_{C[0,T]} \tag{3.18}$$

Then we have for $n \geq 1$

$$||u_{2n}||_{C[0,T]} \leq \left[M_5L_1 + L_2\frac{T^{\alpha}}{\alpha}M_4M_3\right][1-\Psi_1]^{-1} = \psi_2 \tag{3.19}$$

for

$$\Psi_1 = L_2\frac{T^{\alpha}}{\alpha}M_0 < \frac{1}{2} \tag{3.20}$$

The problem (3.12) *admits a solution which is the solution in* $C[0,T]$ *of the integral equation*

$$u_{1n}(t) = \varphi_{1n}E_{\alpha,1}(-\lambda_n t^{\alpha}) - 4\pi n\textstyle\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha})u_{2n}(\tau)d\tau + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha})(a(\tau)u_{1n}(\tau) + c(\tau)F_{1n}(\tau))d\tau \tag{3.21}$$

Under conditions $(A1)-(A3), u_{1n}$ *is bounded in* $C[0,T]$ *as follows*

$$||u_{1n}||_{C[0,T]} \leq |\varphi_{1n}|L_1 + L_2\tfrac{T^{\alpha}}{\alpha}||F_{1n}||_{C[0,T]}||c||_{C[0,T]} + T\tfrac{k}{\pi n}||u_{2n}||_{C[0,T]} + L_2\tfrac{T^{\alpha}}{\alpha}||a||_{C[0,T]}||u_{1n}||_{C[0,T]} \tag{3.22}$$

As previous we get for $n \geq 1$

$$||u_{1n}||_{C[0,T]} \leq \left[M_5L_1 + L_2\frac{T^{\alpha}}{\alpha}M_4M_3 + T\frac{k}{\pi}\psi_2\right][1-\Psi_1]^{-1} \tag{3.23}$$

Let us use the product Banach space $[C[0,T]]^3$ *endowed with its norme to prove the existence and uniquess of the solution under this from* $(u_0, u_{1n}, u_{2n}) \in [C[0,T]]^3$*.Define the operator* Γ *on* $[C[0,T]]^3$ *by* $\Gamma(u_0, u_{1n}, u_{2n}) = (P_0u_0(t), P_1u_{1n}(t), P_2u_{2n}(t))$ *where the operators* P_0, P_1, P_2 *are defined on* $C[0,T]$ *by the right side of* $(3.14)-(3.21)$ *and* (3.17) *respectively.*
In view of $(3.15)-(3.23)$ *and* $(3.19)\Gamma : [C[0,T]]^3 \longrightarrow [C[0,T]]^3$
Prove that Γ *is contraction on* $[C[0,T]]^3$ *.So ,for each* $(u_0, u_{1n}, u_{2n}); (v_0, v_{1n}, v_{2n}) \in [C[0,T]]^3$ *we have*

$$||\Gamma(u_0, u_{1n}, u_{2n}) - \Gamma(v_0, v_{1n}, v_{2n})||_{[C[0,T]]^3}$$

$$\leq max\left(||P_0u_0 - P_0v_0||_{C[0,T]}; ||P_1u_{1n} - P_1v_{1n}||_{C[0,T]}; ||P_2u_{2n} - P_2v_{2n}||_{C[0,T]}\right)$$

First, we get easly

$$||P_0u_0 - P_0v_0||_{C[0,T]} \leq \frac{T^\alpha}{\alpha\Gamma(\alpha)}||a||_{C[0,T]}||u_0 - v_0||_{C[0,T]} \leq \Psi_0||u_0 - v_0||_{C[0,T]} \tag{3.24}$$

For P_1 *by* (3.9) *we have for each* $t \in [0,T]$

$$|P_1u_{1n} - P_1v_{1n}|$$

$$\leq \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)|a(\tau)||u_{1n}(\tau) - v_{1n}(\tau)|d\tau$$

$$+4\pi n \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)|u_{2n}(\tau) - v_{2n}(\tau)|d\tau$$

$$\leq L_2\frac{t^\alpha}{\alpha}||a||_{C[0,T]}||u_{1n} - v_{1n}||_{C[0,T]} + t\frac{k}{\pi n}||u_{2n} - v_{2n}||_{C[0,T]}|$$

Hence for $n \geq 1$

$$||P_1u_{1n} - P_1v_{1n}||_{C[0,T]} \leq \Psi_1||u_{1n} - v_{1n}||_{C[0,T]} + T\frac{k}{\pi n}||u_{2n} - v_{2n}||_{C[0,T]}| \tag{3.25}$$

Similary for each $t \in [0,T]$

$$|P_2u_{2n} - P_2v_{2n}| \leq \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)|a(\tau)||u_{2n}(\tau) - v_{2n}(\tau)|d\tau$$

which gives for $n \geq 1$

$$||P_2u_{2n} - P_2v_{2n}||_{C[0,T]} \leq \Psi_1||u_{2n} - v_{2n}||_{C[0,T]} \tag{3.26}$$

Consequently

$$||\Gamma(u_0, u_{1n}, u_{2n}) - \Gamma(v_0, v_{1n}, v_{2n})||_{[C[0,T]]^3}$$

$$\leq max\left[\Big(\Psi_0||u_0 - v_0||_{C[0,T]}; \Psi_1||u_{1n} - v_{1n}||_{C[0,T]}; \Psi_1||u_{2n} - v_{2n}||_{C[0,T]}\Big)\right.$$

$$\left.+T\frac{k}{\pi n}\Big(0, ||u_{2n} - v_{2n}||_{C[0,T]}; 0\Big)\right]$$

$$\leq \left[max(\Psi_0, \Psi_1) + T\frac{k}{\pi}\right]||(u_0, u_{1n}, u_{2n}) - (v_0, v_{1n}, v_{2n})||_{[C[0,T]]^3}$$

According to (3.16) − (3.20)

$$max(\Psi_0, \Psi_1) + \frac{T}{\alpha}\frac{k}{\pi} < 1 \ for \ T\frac{k}{\alpha\pi} < \frac{1}{2} \tag{3.27}$$

Then , Γ *is a contraction on* $[C[0,T]]^3$ *and has a unique fixed point which is the coeffients* (u_0, u_{1n}, u_{2n}) *of the solution* (3.10)*. Then ,there exists a unique solution of* (3.1) − (3.3) *for arbitray* $c(t)$ *bounded in* $C[0,T]$

Step2: *Determination of the coefficient* $c(t)$ *in* $C[0,T]$

Applying D^{α}_{0+} *to the over-determination condition* (3.4) *we obtain the following equation*

$$\begin{aligned} D^{\alpha}_{0+}(E(t)-E(0)) &= \int_0^1 D^{\alpha}_{0+}(u(x,t)-u(x,0))dx \\ &= c(t)\int_0^1 F(x,t)dx + u_x(0,t) + a(t)E(t) \end{aligned}$$

Which yields

$$c(t) = [f(t)]^{-1}[D^{\alpha}_{0+}(E(t)-E(0)) - a(t)E(t) - u_x(0,t)]$$

Next ,we calculate $f(t) = \int_0^1 F(x,t)dx$ *and find*

$$\begin{aligned} f(t) &= F_0(t)\int_0^1 X_0(x)dx + \sum_{n=1}^{\infty} F_{1n}(t)\int_0^1 X_{1n}(x)dx \\ &+ \sum_{n=1}^{\infty} F_{2n}(t)\int_0^1 X_{2n}(x)dx \qquad (3.28) \\ &= 2F_0(t) + \sum_{n=1}^{\infty} \frac{2}{n\pi}F_{2n}(t) \end{aligned}$$

Then , we derive (3.10) *with respect to* x *and get* $u_x(0,t) = \sum_{n=1}^{\infty} 8\pi n u_{2n}(t)$*; where* $u_{2n}(t); n \geq 1$ *are given by* (3.17)*.The obtained equation is an integral equation with respect to* $c(t)$

$$\begin{aligned} c(t) = \quad & H_0(t) + H_1(t) \\ & - \sum_{n=1}^{\infty} \frac{8\pi n}{f(t)}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha})a(\tau)u_{2n}(\tau)d\tau \qquad (3.29) \\ & - \sum_{n=1}^{\infty} \frac{8\pi n}{f(t)}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha})c(\tau)F_{2n}(\tau)d\tau \end{aligned}$$

Where

$$H_0(t) = \frac{1}{f(t)}(D^{\alpha}_{0+}(E(t)-E(0)) - a(t)E(t))$$

$$H_1(t) = -\sum_{n=1}^{\infty} \frac{8\pi n}{f(t)}\varphi_{2n}E_{\alpha,1}(-\lambda_n t^{\alpha})$$

we have under $(A1)-(A4)$ *and* (3.9)

$$||H_0||_{C[0,T]} \leq (1+M_0)M_1M_2$$

$$||H_1||_{C[0,T]} \leq 4M_1L_1\sum_{n=1}^{\infty} 2\pi n|\varphi_{2n}|$$

Under $(A1)-(A3)\sum_{n=1}^{\infty} 2\pi n|\varphi_{2n}|$ *is convergent. In fact , we have* $\varphi \in C^4[0,1]$ *and* $\varphi_{2n} = \frac{\varphi^{(4)}_{2n}}{(2n\pi)^4}$ *where* $\varphi^{(4)}_{2n}$ *are the coefficient of Fourier series of the function* $\varphi^{(4)}$ *with respect to the basis* (3.8)*.The Cauchy-Schwartz inequality yields*

$$\begin{aligned} K_1 = \quad & \sum_{n=1}^{\infty} 2\pi n|\varphi_{2n}| = \sum_{n=1}^{\infty} \frac{|\varphi^{(4)}_{2n}|}{(2n\pi)^3} \\ & \leq \left(\sum_{n=1}^{\infty} \frac{1}{(2n\pi)^6}\right)^{1/2}\left(\sum_{n=1}^{\infty} |\varphi^{(4)}_{2n}|^2\right)^{1/2} < \infty \end{aligned}$$

Recall that by the Bessel's inequality , we have $\sum_{n=1}^{\infty}|\varphi_{2n}^{(4)}|^2 \leq C||\varphi^{(4)}||^2_{L^2(0,1)}$. *Thus*

$$||c||_{C[0,T]} \leq ||H_0||_{C[0,T]} + ||H_1||_{C[0,T]} + 2M_1kM_0T\sum_{n=1}^{\infty}\frac{||u_{2n}||_{C[0,T]}}{n\pi}$$
$$+2M_1kT||c||_{C[0,T]}\sum_{n=1}^{\infty}\frac{||F_{2n}||_{C[0,T]}}{n\pi}$$

where

$$\sum_{n=1}^{\infty}\frac{||u_{2n}||_{C[0,T]}}{n\pi} \leq \frac{L_1}{[1-\Psi_1]}\sum_{n=1}^{\infty}\frac{|\varphi_{2n}|}{n\pi} + \frac{L_2T^{\alpha}||c||_{C[0,T]}}{\alpha[1-\Psi_1]}\sum_{n=1}^{\infty}\frac{||F_{2n}||_{C[0,T]}}{n\pi} \tag{3.30}$$

The convergence of the series

$$\sum_{n=1}^{\infty}\frac{|\varphi_{2n}|}{n\pi} = K_3 \text{ and } \sum_{n=1}^{\infty}\frac{||F_{2n}||_{C[0,T]}}{n\pi} = K_2$$

is clear by the Cauchy-Schwartz inquality.Hence, for

$$\Psi_c = 2M_1kTK_2\left[1 + \frac{L_2M_0T^{\alpha}}{\alpha[1-\Psi_1]}\right] < 1 \tag{3.31}$$

we have

$$||c||_{C[0,T]} \leq [(1+M_0)M_1M_2 + 4M_1L_1K_1][1-\Psi_c]^{-1} + \frac{2M_1M_0TkL_1K_3}{[1-\Psi_1][1-\Psi_c]}$$

Hence $c(t)$ *is bounded in* $C[0,T]$.*Now, we define the operator* P *on* $C[0,T]$ *by the right side of* (3.29).

Let us show that P *is a contraction mapping in* $C[0,T]$.*For each* $c, b \in C[0,T]$ *with* $u_{2n}v_{2n}$ *defined by* (3.17) *relativley to* c, b *respectivley, we have*

$$||u_{2n} - v_{2n}||_{C[0,T]} \leq L_2\frac{T^{\alpha}}{\alpha[1-\Psi_1]}||c-b||_{C[0,T]}||F_{2n}||_{C[0,T]}$$

and

$$||P_c - P_b||_{C[0,T]} \leq 2M_1kT\sum_{n=1}^{\infty}\frac{||F_{2n}||_{C[0,T]}}{n\pi}||c-b||_{C[0,T]}$$
$$+2M_1kM_0T\sum_{n=1}^{\infty}\frac{||u_{2n}-v_{2n}||_{C[0,T]}}{n\pi}$$
$$2M_1kTK_2\left[1 + \frac{L_2M_0T^{\alpha}}{\alpha[1-\Psi_1]}\right]||c-b||_{C[0,T]}$$

According to (3.31) *the operator* P *has a unique fixed point* $c(t)$ *in* $C[0,T]$ *, by the banach fixed point theorem*

Step 3: *Estimation of the time of the local existence*
According to $(3.16)-(3.20)-(3.31)$ *and* (3.27) T^* *must satisfy this approximation*

$$T^* < inf\left[\left(\frac{\alpha\Gamma(\alpha)}{2M_0}\right)^{1/\alpha}; \left(\frac{\alpha}{2M_0L_2}\right)^{1/\alpha}; \frac{\alpha\pi}{2k}; \frac{1}{4M_1kK_2}; \left(\frac{\alpha}{2M_0L_2K_2}\right)^{1/\alpha+1}\right]$$

to ensure the existence of the solution on $[0,T]$ *for each* $T < T^*$
Step 4: *Convergence of the solution series* (3.10)
As it was proved ,in view of $(A1)-(A4)$*, the coefficients* $u_0(t), u_{1n}(t)$ *and* $u_{2n}(t); n \geq 1$ *are bounded in* $C[0,T]$ *.Thus , the series expression* (3.10) *of* $u(x,t)$ *gives*

$$sup_{x\in[0,1]}|u(x,t)| = |u(t)| \leq 2|u_0(t)| + 4\sum_{n=1}^{\infty}|u_{1n}(t)| + 4\sum_{n=1}^{\infty}|u_{2n}(t)| \tag{3.32}$$

$(A2)-(A3)$ *implies that* $\sum_{n=1}^{\infty}|\varphi_{in}|, \sum_{n=1}^{\infty}2\pi n|\varphi_{in}|$ *and* $\sum_{n=1}^{\infty}|F_{in}(t)|; \sum_{n=1}^{\infty}2\pi n|F_{in}(t)| i = 1,2$ *converge uniformly.*
In consequent by $(3.22), (3.18)$ *and* (3.30) *the series* $u(x,t)$ *and its partial derivative* $u_x(x,t)$ *are uniformly convergent in* $[0,1][\epsilon,T]$ *for any* $\epsilon > 0$*. Therefore ,their sums are in* $C[0,T]$ *for* $x \in [0,1]$ *.*
Also , its scend partial derivative $u_{xx}(x,t)$ *is uniformly convergent in* $[0,1][\epsilon,T]$ *for any* $\epsilon > 0$ *by the Cauchy-Schwartz inequality and the Bessel's inequality in view of the fact that* $\varphi_{2n} = \frac{\varphi_{2n}^{(4)}}{(2\pi n)^4}$ *and* $F_{in}(t) = \frac{F_{in}^{(4)}(t)}{(2\pi n)^4}; i = 1,2$ *Then , the uniformly convergence of* $\sum_{n=1}^{\infty}u_{in}(t), \sum_{n=1}^{\infty}\lambda_n u_{in}(t); i = 1,2$ *and the inequalities*

$$\begin{aligned}
||D_{0^+}^{\alpha}(u(t)-u_0(0))||_{C[0,T]} &\leq M_0||u_0||_{C[0,T]} + M_4M_3\\
||D_{0^+}^{\alpha}(u_{1n}(t)-u_{1n}(0))||_{C[0,T]} &\leq (\lambda_n + M_0)||u_{1n}||_{C[0,T]}\\
&+4n\pi||u_{2n}||_{C[0,T]} + M_3||F_{1n}||_{C[0,T]}\\
||D_{0^+}^{\alpha}(u_{2n}(t)-u_{2n}(0))||_{C[0,T]} &\leq (\lambda_n + M_0)||u_{2n}||_{C[0,T]} + M_3||F_{2n}||_{C[0,T]}
\end{aligned}$$

obtained from $(3.11), (3.12), (3.13)$*, imply that* $\sum_{n=1}^{\infty}(u_{in}(t)-u_{in}(0))$ *and* $\sum_{n=1}^{\infty}D_{0^+}^{\alpha}(u_{in}(t)-u_{in}(0)), i = 1,2$ *are uniformly convergent on* $[\epsilon,T]$ *For any* $\epsilon > 0$ *.*
In view of Theorem theα*-partial derivative* $D_{0^+}^{\alpha}$ *of the series* (3.10) *is uniformly convergent for* $t \in [\epsilon,T]$ *for any* $\epsilon > 0$ *and* $x \in [0,1]$*.*
Thus $u(x,t) \in C^{2,\alpha}[0,1](0,T] \cap C([0,1][0,T])$ *and satisfies the condition* $(3.1)-(3.3)$ *for arbitrayc*$(t) \in C[0,T]$*.*
Step 5 : *Uniqueness of the solution* $(u(x,t), c(t))$*.*
Assume that the pairs of functions $(u(x,t), c(t))$ *and* $(v(x,t), b(t))$ *are solutionsof the inverse*

problem $(3.1)-(3.4)$ *.Let us use the product Banach space* $[C[0,T]]^4$ *endowed with its norme to prove the uniquness of the solutions under this from* $(u_0,u_{1n},u_{2n},c)\in[C[0,T]]^4$*.We have*

$$||(u_0,u_{1n},u_{2n},c)-(v_0,v_{1n},v_{2n},b)||_{[C[0,T]]^4}$$
$$\leq max(\Psi_0;\Psi_1;\Psi_c)||(u_0,u_{1n},u_{2n},c)-(v_0,v_{1n},v_{2n},b)||_{[C[0,T]]^4}$$

In view of $(3.16),(3.20)$ *and* (3.31)

$$||(u_0,u_{1n},u_{2n},c)-(v_0,v_{1n},v_{2n},b)||_{[C[0,T]]^4}$$

This implies that $u(x,t=v((x,t)$ *and* $c(t)=b(t)t\in[0,T]$*. This completess the proof.*

3.2 Continuous Dependence on the Data

Theoreme 3.3. *Under assumption* $(A1)-(A4)$ *the solution* $(u(x,t),c(t))$ *of the problem* $(3.1)-(3.4)$ *depends continuously upon the data of* $\Phi(t)=\{a(t),F(x,t);\varphi(x);E(t)$

proof.Let $(u(x,t),c(t));(\bar{u}(x,t),\bar{c}(t))$ *be the solutions of the inverse problem* $(3.1)-(3.4)$ *corresponding to the data* $\Phi(t)$ *and* $\bar{\Phi}(t)$ *respectively.*
Let us denote $||\Phi||=||a||_{C[0,T]}+||E||_{AC[0,T]}+||\varphi||_{C[0,1]}+||F||_{C([0,1][0,T])}$
First , we estimate each coefficient of $u(x,t)-\bar{u}(x,t)$ *in* $C[0,T]$.
We will use this from $|AG-\bar{AG}|=|AG-A\bar{G}+A\bar{G}-\bar{AG}|\leq|A||G-A\bar{G}|+|\bar{G}||A-\bar{A}|$*.Then ,for some constants* $\theta_i, i=1,...,4$

$$\begin{aligned}||u_0-\bar{u}_0||_{C[0,T]} &\leq \tfrac{\theta_1}{[1-\Psi_0]}|\varphi_0-\bar{\varphi}_0|+\tfrac{\theta_2}{[1-\Psi_0]}||c-\bar{c}||_{C[0,T]}\\ &+\tfrac{\theta_3}{[1-\Psi_0]}||F_0-\bar{F}_0||_{C[0,T]}\\ &+\tfrac{\theta_4}{[1-\Psi_0]}||a-\bar{a}||_{C[0,T]}\end{aligned} \tag{3.33}$$

By similar calculus we can obtain for some constants $\beta_i, i=,...,5$ *and* $\delta_i, i=1,...,4$

$$\begin{aligned}||u_{1n}-\bar{u}_{1n}||_{C[0,T]} &\leq \tfrac{\beta_1}{[1-\Psi_1]}|\varphi_{1n}-\bar{\varphi}_{1n}|+\tfrac{\beta_2||F_{1n}||_{C[0,T]}}{[1-\Psi_1]}||c-\bar{c}||_{C[0,T]}\\ &+\tfrac{\beta_3}{[1-\Psi_1]}||F_{1n}-\bar{F}_{1n}||_{C[0,T]}+\tfrac{\beta_4||u_{1n}||_{C[0,T]}}{[1-\Psi_1]}||a-\bar{a}||_{C[0,T]}\\ &+\tfrac{\beta_5}{[1-\Psi_1]}||u_{2n}-\bar{u}_{2n}||_{C[0,T]}\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}||u_{2n}-\bar{u}_{2n}||_{C[0,T]} &\leq \tfrac{\delta_1}{[1-\Psi_1]}|\varphi_{2n}-\bar{\varphi}_{2n}|+\tfrac{\delta_2||F_{2n}||_{C[0,T]}}{[1-\Psi_1]}||c-\bar{c}||_{C[0,T]}\\ &+\tfrac{\delta_3}{[1-\Psi_1]}||F_{2n}-\bar{F}_{2n}||_{C[0,T]}+\tfrac{\delta_4||u_{2n}||_{C[0,T]}}{[1-\Psi_1]}||a-\bar{a}||_{C[0,T]}\end{aligned} \tag{3.35}$$

We use (3.33), (3.34), (3.36) *in* (3.32) *to get for some positive constants* $\eta_i i = 1, ..., 4$ *the estimate of* $u(x,t) - \bar{u}(x,t)$ *in* $C[0,1][0,T]$

$$\begin{aligned} ||u - \bar{u}||_{(C[0,1][0,T])} &\leq \eta_1 ||\varphi - \bar{\varphi}||_{C^2[0,1]} + \eta_2 ||c - \bar{c}||_{C[0,T]} \\ &+ \eta_3 ||F - \bar{F}||_{C^2([0,1][0,T]} + \eta_4 ||a - \bar{a}||_{C[0,T]} \end{aligned}$$

In addition , we have

$$\begin{aligned} &|D^\alpha_{0^+,t}(u(x,t) - u(x,0)) - D^\alpha_{0^+,t}(\bar{u}(x,t) - \bar{u}(x,0))| \\ &\leq ||u_{xx} - \bar{u}_{xx}||_{C([0,1][]0,T])} \\ &+ ||a - \bar{a}||_{C[0,T]} ||u||_{C([0,1][0,T])} + ||\bar{a}||_{C[0,T]} ||u - \bar{u}||_{C([0,1][]0,T])} \\ &+ ||c - \bar{c}||_{C[0,T]} ||F||_{C([0,1][0,T])} + ||\bar{c}||_{C[0,T]} ||F - \bar{F}||_{C([0,1][]0,T])} \end{aligned}$$

we get for some positive constants $B_1; B_2$

$$||u - \bar{u}||_{C^{2,\alpha}([0,1][0,T])} \leq B_1 ||\Phi - \bar{\Phi}|| + B_2 ||c - \bar{c}||_{C[0,T]} \tag{3.36}$$

Next , we have to estimate each term of the difference $|c(t) - \bar{c}(t).|$ *Let us not* $E(t) - E(0) = A(t)$*, then we have by* (3.29)

$$\begin{aligned} ||H_0 - \bar{H}_0||_{C[0,T]} \leq \quad & M_1 ||D^\alpha_{0^+} A(t) - D^\alpha_{0^+} \bar{A}(t)||_{C[0,T]} + M_1^2 M_2 ||\bar{f} - f||_{C[0,T]} \\ & + M_1 M_2 ||\bar{a} - a||_{C[0,T]} + M_0 M_1 ||E - \bar{E}||_{C[0,T]} \\ & + M_0 M_1^2 M_2 ||\bar{f} - f||_{C[0,T]} \end{aligned}$$

Also we get by (3.29)

$$||H_0 - \bar{H}_0||_{C[0,T]} \leq 8 M_1 L_1 \sum_{n=1}^{\infty} n\pi |\varphi_{2n} - \bar{\varphi}_{2n}| + 8 M_1^2 L_1 \sum_{n=1}^{\infty} n\pi |\bar{\varphi}_{2n}| \ ||\bar{f} - f||_{C[0,T]}$$

The estimate of the first integral part is

$$\left| \sum_{n=1}^{\infty} 8\pi n \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha) \left[\frac{c(\tau) F_{2n}(\tau)}{f(t)} - \frac{\bar{c}(\tau) - \bar{F}_{2n}(\tau)}{\bar{f}(t)} \right] d\tau \right|$$

$$\leq 2TKM_1 M_3 \sum_{n=1}^{\infty} \frac{||F_{2n} - \bar{F}_{2n}||_{C[0,T]}}{n\pi} + 2TKM_1 ||c - \bar{c}||_{C[0,T]} \sum_{n=1}^{\infty} \frac{||\bar{F}_{2n}||_{C[0,T]}}{n\pi}$$

$$+ 2TKM_1^2 M_3 \sum_{n=1}^{\infty} \frac{||\bar{F}_{2n}||_{C[0,T]}}{n\pi} ||\bar{f} - f||_{C[0,T]}$$

For some positive constants $\prod_i, i = 1, ..., 6$ *we have*

$$\begin{aligned} ||c - \bar{c}||_{C[0,T]} \leq \quad & \tfrac{\prod_1}{[1-\Psi_c]} ||E - \bar{E}||_{AC[0,T]} + \tfrac{\prod_2}{[1-\Psi_c]} \sum_{n=1}^{\infty} n\pi |\varphi_{2n} - \bar{\varphi}_{2n}| \\ & + \tfrac{\prod_3}{[1-\Psi_c]} ||\bar{f} - f||_{C[0,T]} + \tfrac{\prod_4}{[1-\Psi_c]} \sum_{n=1}^{\infty} \tfrac{||F_{2n} - \bar{F}_{2n}||_{C[0,T]}}{n\pi} \\ & + \tfrac{\prod_5}{[1-\Psi_c]} ||\bar{a} - a||_{C[0,T]} + \tfrac{\prod_6}{[1-\Psi_c]} \sum_{n=1}^{\infty} \tfrac{||u_{2n} - \bar{u}_{2n}||_{C[0,T]}}{n\pi} \end{aligned}$$

we can obtain for some positive constants B_3 *and* B_4 *and with* (3.36)

$$\begin{aligned} ||c-\bar{c}||_{C[0,T]} \leq & \quad B_3||\phi-\bar{\phi}||+B_4||u-\bar{u}||_{C^{2,\alpha}([0,1][0,T])} \\ & \quad [B_3+B_4B_1]||\Phi-\bar{\Phi}||+B_2B_4||c-\bar{c}||_{C[0,T]} \end{aligned}$$

Then

$$\begin{aligned} ||c-\bar{c}||_{C[0,T]} \leq & \quad \tfrac{[B_3+B_4B_1]}{1-B_4B_2}||\Phi-\bar{\Phi}|| \\ ||u-\bar{u}||_{C^{2,\alpha}([0,1][0,T])} \leq & \quad \left[B_1+\tfrac{[B_3+B_4B_1]}{1-B_4B_2}\right]||\Phi-\bar{\Phi}|| \end{aligned}$$

The Theorem has been proved.

Conclusion

The inverse problem regarding the simultaneous identification of the time-dependent source coefficient with the tempurature distrubution in a one-demensional sub-diffusion equation with nonlocal boundary and integral overdetermination conditions has been considered.The nonlocal boundary conditions,the Riemann-Liouville and Katugampola fractional derivative and the control coefficient made our problem more difficult.The conditions for the existence,uniqueness and continuous dependence upon the data of the problem have been established by using the Fourier method with some bi-orthogonal system, an associated Riemann-Liouville and Katugampola fractional derivative which contains an initial data and the Banach fixed point theorem for a product of Banach spaces

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[16] *Loubna Settara on 02 April 2018* An invers coefficient-source problem for a time-fractional diffusion equation

Astract

In the present work , we study two classes of inverse problems for diffusion equation with source term, where the partial derivative is fractional in the time.
The from of EDP problems are called sub-diffusion problems.
The first investigation is devoted to the determination of the source term coefficient dependent on time of an inverse source problem with non-local boundary conditions and integral condition.
We establish results of existence, uniquenss and continuous dependence data.Tools used for demonstration are based on one hand , the Fourier method for bi-orthogonal systems , the operator being not self-adjoint , in author hand the fixed point theory.
The scend investigation is devoted to determination of source term coefficient dependent on the space for sub-diffusion problem with homogeneous boundary conditions and an initial weighted condition.
For direct problem , the key point in our analysis is the use of Duhamel principle in addition Fourier method , to show existence, uniquenss of weak solution, then the question of regularity is treated.
to determine a unique coefficient, we add an integral condition to introduce input output mapping.
The inverse problem is reduces to the problem of ineversibility of the input output mapping,which shoud be monotone and it's invers is bijectif