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Theme

Genocchi Polynomials for Solving Linear Fredholm Integral Equation

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• ملخص

لعبت المعادلات التكاملية الخطية دورا حاسما في نمذجة مجموعة واسعة من الظواهر عبر مجالات مختلفة مثل الهندسة والبيولوجيا والفيزياء. ومن ثم أصبح تطوير الطرق العددية ضروريا لإيجاد حلول تقريبية سريعة ودقيقة لهذه المعادلات. في هذه الأطروحة نقدم نهجا عدديا يعتمد على كثيرات حدود جينوتشي. تتمثل عملية الطريقة في تحويل المعادلة التكاملية الخطية إلى معادلة مصفوفية. ولتوضيح فعالية ودقة هذه الطريقة نعرض عدة أمثلة عددية.

الكلمات المفتاحية : معادلات فريدهولم التكاملية، التقريب، طريقة التجميع، متعددات حدود جينوتشي

• Résumé

Les équations intégrales linéaires (EILs) jouent un rôle crucial dans la modélisation d'une vaste gamme de phénomènes dans des domaines aussi variés que l'ingénierie, la biologie et la physique. Pour cette raison, le développement de méthodes numériques est devenu nécessaire afin de trouver des solutions approchées rapides et précises à de telles équations. Dans cette thèse, nous présentons une approche numérique basée sur les polynômes de Genocchi. Le procédé de la méthode consiste à transformer l'équation intégrale linéaire en une équation matricielle. Pour illustrer l'efficacité et la précision de cette méthode, nous fournissons plusieurs exemples numériques.

Mots clés : Équations intégrales de Fredholm, approximatin, méthode de collocation,polynômes de Genocchi

• Abstract

Linear integral equations (LIEs) play a crucial role in modeling a wide range of phenomena across different fields such as engineering, biology, and physics. For this purpose, the development of numerical methods has become necessary to find fast and accurate approximate solutions to such equations. In this thesis, we present a numerical approach based on Genocchi polynomials. The process of the method is to transform the LIE into a matrix equation. To illustrate the effectiveness and accuracy of this method, we provide several numerical examples.

Keys Words :Fredholm integral equations, approximation, collocation method, Genocchi polynomials.

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khaoula

Dedication

With God’s help,
I was able to accomplish this humble work, which I lovingly dedicate to:
My role model and source of inspiration, who sacrificed everything for my happiness my dear father, **Ismaïl**.
The other half of my soul, the source of my strength and love, who always advises me and sacrifices for me my dear and perfect mother **Kamla**
My **husband** and our precious children, **Adam** and **Ayham**.
My brothers and sisters **Rima**, **Soumia**, **Ossama**, **Nada**, and **Ayman** for their constant support and love.
To all the teachers who have taught me throughout my journey.
And to all my friends, for their encouragement and companionship

Khaoula

List of Symbols

In what follows, we will use the following notations.

H	Hilbert space
A	Linear Operator.
I	Identity Operator
a_i	Coefficient of the Genocchi polynomial.
λ	Constant.
$\varphi(x)$	The Exact Solution.
$\varphi_i(x)$	The Approached Solution.
$[a; b]$	Actual interval.
$K(x; y)$	kernel of the integral.
$f(x)$	Given Function.
$G_n(x)$	Genocchi Polynomials.
$LFIE$	Linear Fredholm's Integral Equation.

Introduction

In mathematics, an equation in which the unknown function appears under an integral sign is called an *integral equation*. Integral equations are generally classified into two types: *Fredholm* and *Volterra*.

Many physical problems are modeled in the form of integral equations. These include potential theory, Dirichlet problems, electrostatics, contact problems, astrophysics problems and radiative heat transfer problems. (For more details see [1, 5]).

Solving such equations can be mathematically challenging and often requires specialized techniques to obtain accurate and efficient solutions. Researchers, scientists, and engineers have shown great interest in studying integral equations due to their broad scope of applications.

Some integral equations are too complex to be solved analytically, which has led researchers to develop numerical methods for handling cases that lack closed-form solutions. In the literature, various numerical approaches have been proposed for solving Linear Fredholm integral equations, including quadrature-based methods using hybrid functions [6], Haar wavelets [7], sinc-collocation method [8] and the sinc-Galerkin method [9]. These methods have proven to be effective and reliable for obtaining approximate solutions to Linear Fredholm integral equations of the second kind. The purpose of this thesis is to present a numerical method for solving Fredholm integral equations of the second kind using the collocation method with Ginocchi polynomials. The Linear Fredholm integral equation is transformed into a fundamental matrix equation, which is then solved, and the required values of the coefficient matrix of the Ginocchi polynomials are found. This method has an advantage over other methods because it transforms the problem into a system of linear equations. Larger matrices can be easily solved using computational software such as MATLAB, MAPLE SOFT, and MATHEMATICA.

This thesis is divided into three chapters as follows :

►**Chapter 01 :** This chapter introduces the fundamental concepts and theoretical foundations of functional analysis, including Hilbert and Banach spaces, as well as the theory of bounded, compact, and integral operators. These elements are essential for understanding Linear Fredholm integral equations and the methods used for their resolution.

►**Chapter 02 :** We investigate the existence and uniqueness of solutions to Linear Fredholm integral equations and present several analytical solution techniques, including the series expansion method and the Adomian decomposition method. Each method is accompanied by illustrative examples to clarify the approach.

► **Chapter 03 :** This chapter focuses on a numerical collocation method based on Ginocchi polynomials. Numerical examples are provided to highlight the efficiency and accuracy of this approach in solving Linear Fredholm integral equations, We solved them using MATLAB and presented the results in this chapter.

CHAPTER 1

PRELIMINARIES

We begin by reviewing some fundamental concepts of functional analysis that are essential for this work .

1.1 Recalls of functional spaces

1.1.1 Normed vector space

Definition 1.1.1. *Let* V *be a vector space over* $\mathbb{K}(\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C})$. *A mapping* $\|.\| : V \to \mathbb{R}$ *is called a norm on* V *if it satisfies the following condition for all vectors* $x, y \in V$ *and* $\alpha \in \mathbb{K}$ *.*

(i) $\|x\| = 0 \quad \Leftrightarrow \quad x = 0.$

(ii) $\|\alpha x\| = |\alpha|\, \|x\|$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

A vector space with a norm $(V, ||x||)$ *is called a normed space .*

• ***Examples of Normes :***

1. *The absolute value is a norm on* $\mathbb{R}$ *, the modulus is a norm on* $\mathbb{C}$ *.*
2. *The following applictions are norms on* $\mathbb{R}^n$ *.*

$$||x||_1 = \sum_{i=1}^{n} |x_i| \quad , \quad ||x||_2 = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}} \quad , \quad ||x||_\infty = \sup_{1 \leq i \leq n} |x_i|$$

3. *Let* $V = C([a, b], \mathbb{R})$ *denote the space of real-valued continuous functions defined on the interval* $([a, b])$. *For any function* $f \in V$*, the following functions define norms on* V *:*

$$||f||_1 = \int_a^b |f(x)|\mathrm{d}x \quad , \quad ||f||_2 = (\int_a^b |f(x)|^2 \mathrm{d}x)^{\frac{1}{2}} \quad , \quad ||f||_\infty = \max_{a \leq x \leq b} |f(x)|$$

1.1.2 Banach space

Definition 1.1.2. *(Cauchy sequence).Let* $(V, ||.||)$ *be a normed space.A sequence* $(x_n)_{n\in\mathbb{N}}$ *called is a Cauchy sequence if :*

$$\forall\varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N}; p > q \geq n_0 \Longrightarrow ||x_p - x_q|| \leqslant \varepsilon$$

Definition 1.1.3. *(complete space or Banach space). A normed space* $(V, ||.||)$ *is complet if every Cauchy sequence in* V *converges.*

- **Examples of Banach spaces :**

1. $C([a, b], \mathbb{R})$:The space of all continuous real-valued functions on the interval $[a; b]$,with the sup-norm is a Banach space.
2. $l^p (1 \leq p < \infty)$ The space of all sequences (x_n)such that $\sum_1^\infty |x_n|^p$ is finite, with the p-norm

   $$||x_n||_p = (\sum_1^\infty |x_n|^p)^{\frac{1}{p}}$$

   is a Banach space.
3. $L^2([a, b], \mathbb{R})$ The space of all real valued functions f such that $\int_a^b |f(x)|^2 \mathrm{d}x < \infty$, with the norm:

   $$||f(x)||_2 = (\int_a^b |f(x)|^2 \mathrm{d}x)^{\frac{1}{2}}$$

   is a Banach space.

Theorem 1.1.4. *Every finite-dimensional normed vector space is a Banach space.*

1.1.3 Hilbert space

Definition 1.1.5. *Let* V *be a vector space. A mapping* $\langle\cdot, \cdot\rangle : V \times V \to \mathbb{R}$ *is called an inner product on* V *if it satisfies the following conditions for all vectors* φ_1, φ_2 *and* $\varphi_3 \in V$ *and scalars* $\beta_1, \beta_2 \in \mathbb{R}$*:*

(i) $\langle\varphi_1, \varphi_2\rangle = \langle\varphi_2, \varphi_1\rangle$

(ii) $\langle\varphi_1, \varphi_1\rangle > 0$ *for all* $\varphi_1 \neq 0$

(iii) $\langle\beta_1\varphi_1 + \beta_2\varphi_2, \varphi_3\rangle = \beta_1\langle\varphi_1, \varphi_3\rangle + \beta_2\langle\varphi_2, \varphi_3\rangle$

Definition 1.1.6. *A norm on* V *is defined as* $\|\varphi\| = \sqrt{\langle \varphi, \varphi \rangle}$*. If the space* $(V, \|\cdot\|)$ *is complete, it is called a Hilbert space.*

Theorem 1.1.7. *the space* $L^2(\Omega)$ *is a Hilbert space equipped with the inner product*

$$\langle \varphi_1, \varphi_2 \rangle_{L^2(\Omega)} = \int_{\Omega} \varphi_1(x)\varphi_2(x)\, dx.$$

1.2 Concepts of operators

1.2.1 Linear operators

Definition 1.2.1. *Let* E *and* F *be two normed spaces, an operator* $A : E \longrightarrow F$ *is called to be linear if :*

$$\forall \varphi_1, \varphi_2 \in E \quad \textit{and} \quad \lambda_1, \lambda_2 \in \mathbb{K}, \textit{we have} \quad A(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1 A(\varphi_1) + \lambda_2 A(\varphi_2).$$

Definition 1.2.2. *(Continuity of linear operators) Let* E *and* F *be two normed spaces, a linear operator* A *defined on a subset* $G \subset E$ *to* F *is called continuous in* x *of* G *if:*

$\triangleright$ *for all sequence* (x_n) *in* G *converges to* x_0 *, the sequence* $A(x_n)$ *converges to* $A(x_0)$ *that mean :*

$$\lim_{n \longrightarrow \infty} A(x_n) = A(\lim_{n \longrightarrow \infty} x_n) = A(x_0).$$

Remark 1.2.3. *The linear operator* A *is called continuous in* G *if it is continuous in each point of the set* G *.*

1.2.2 Bounded operators

Definition 1.2.4. *A linear operator* $A : E \longrightarrow F$ *is called bounded if there exists a positive constant* $C > 0$ *such that:*

$$||A(x)||_F \leq C||x||_E, \forall x \in E.$$

Theorem 1.2.5. *A linear operator* A *is continuous if and only if it is bounded.*

1.2.3 Compact operators

Definition 1.2.6. *(compact linear operators) A linear operator* A *defined from a normed space* E *into a normed space* F *is called a compact operator if for every bounded subset of* Ω *in* E*, the image* $A(\Omega)$ *is relatively compact in* F *, in other words, the closure* $\overline{A(\Omega)}$ *is compact.*

Theorem 1.2.7. *The linear combination $A = A_1 + A_2$ of compact operators A_1 and A_2 is compact operator, for every scalars α and β .*

Proof. Let φ_n be a bounded sequence in E and let $A\varphi_n$ be a sequence in F then :

$$A\varphi_n = \alpha A_1\varphi_n(x) + \beta A_2\varphi_n(x), \quad \text{with}\varphi_n \in E, n \in \mathbb{N}.$$

the operators A_1 and A_2 are compact, one can extract from $A_1\varphi_n$ and $A_2\varphi_n$ two convergent sub sequences wich given by thier sum a convergent subsequence of $A\varphi_n$ .Hence A is compact . □

Theorem 1.2.8. **(*finite dimensional domain*)**. *Let A be bounded operator defined from E into F with the domain E has a finite dimension, then the operator A is compact.*

Proof. Indeed the space E has a finite dimension implies the finite dimensional range $A(E)$, say

$$dimA(E) < dimE$$

it follows, A is a compact operator . □

Theorem 1.2.9. *The identity operator I_d defined from a normed space E into E is compact if and only if the space E has a finite dimension.*

Corollary 1.2.10. *A bounded operator A in a normed space E is not generally a compact operator. Indeed, see the identity operator $A = I$ in the infinitely dimensional normed space is not compact.*

1.2.4 Integral operators

Integral operators are fundamental objects in functional analysis, where they are particularly useful in transforming function into simpler version for easier resolution. The integral operator is defined as any linear operator A defined on a normed space E with values in a normed space F , given in the form :

$$A\varphi(x) = \int_{G_2} k(x,y)\varphi(y)\mathrm{d}y, \quad x \in G_1$$

the function $k(x,y)$ is a measurable function defined on a measurable set $G_1 \times G_2$ and $\varphi(y)$ is a measurable function defined on G_2 .
$k(x,y)$ is called the kernel of the integral operator A .

Remark 1.2.11. *A kernel K is said to be separable or degenerate if it can be written in the form :*

$$K(x,y) = \sum_{i=0}^{n} a_i(x) b_i(y)$$

Definition 1.2.12. *(Norms of integral operators) The norm of the integral operators defined on $L^p(G_1)$, here p and q are conjugate exponents($\frac{1}{p} + \frac{1}{q} = 1$) with $(1 \leq p, q \leq \infty)$,the norm of the operator A is given by :*

$$||A||_p = \begin{cases} \left(\int_{G_1} \left(\int_{G_2} |k(x,y)|^q \, dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} & \text{for } 1 < p < \infty \\ \int_{G_1} \operatorname*{ess\,sup}_{y} |k(x,y)| \, dx & \text{for } p = 1 \\ \operatorname*{ess\,sup}_{x} \int_{G_2} |k(x,y)| \, dy & \text{for } p = \infty \end{cases}$$

Theorem 1.2.13. *Let A is an integral operator with a finite norm*

$$||A||_p < \infty$$

then the integral operator A is a continous linear operator from $L^p(G_1)$ to $L^p(G_2)$ Additionlly we have

$$||A\varphi||_p \leq ||A||_p ||\varphi||_p$$

CHAPTER 2

INTEGRALS EQUATIONS

2.1 Introduction to linear integral equations

An equation in which the unknown function of one or more variables appears under the integral sign is called an integral equation this general definition takes into account many naturally occurring forms arising from the modeling of various problems in mechanics and mathematical physics. Or as a reformulation of an important class of problems previously expressed using differential operators .

Definition 2.1.1. *A linear integral equation is a functional equation of the from :*

$$\varphi(x) = f(x) + \lambda \int_a^b k(x,t)\varphi(t)\mathrm{d}t$$

where $f(x)$,$k(x,t)$ *are the given functions, and* $\varphi(x)$ *is the unknown function that needs to be determined , and* λ *is a real or complex parameter different from zero. It can be written in operator form:*

$$(I - \lambda A)\varphi(x) = f(x)$$

where I *is the identity operator.*

2.2 Classification of integral equations

Integral equations appear in many types.The types depend mainly on the limits of integration and the kernel of the equation.

2.2.1 Fredholm integral equations

In Fredholm integral equations , the limits of integration are fixed. Moreover, the unknown function $\varphi(x)$ may appear only inside the integral in the form :

$$f(x) = \int_a^b K(x,t)\varphi(t)dt$$

This is called a Fredholm integral equation of the first kind.However, For Fredholm integral equations of the second kind, the unknown function $\varphi(x)$ appears both inside and outside the integral sign. The general form is :

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\varphi(t)dt \tag{2.1}$$

Example 2.2.1. *of Fredholm integral equations of the second kind are*

$$\varphi(x) = x + \frac{1}{2}\int_{-1}^{1}(x-t)\varphi(t)dt$$

2.2.2 Volterra integral equations

In Volterra integral equations, at least one of the limits of integration is a variable. For the first kind, the unknown function $\varphi(x)$ appears only under the integral sign :

$$f(x) = \int_0^x K(x,t)\varphi(t)dt$$

For the second kind, $\varphi(x)$ appears both inside and outside the integral :

$$\varphi(x) = f(x) + \lambda \int_0^x K(x,t)\varphi(t)dt \tag{2.2}$$

Example 2.2.2. *of Volterra integral equations of the second kind are :*

$$\varphi(x) = x + \int_0^x (x-t)\varphi(t)dt$$

2.2.3 Volterra-Fredholm integral equations

The Volterra-Fredholm integral equation, is equation include both a Volterra and Fredholm integral operators the general form of this equation is given by :

$$\varphi(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)\varphi(t)\mathrm{d}t + \lambda_2 \int_a^b k_2(x,t)\varphi(t)\mathrm{d}t \quad ; \quad x \in [a,b].$$

where the functions $k_1(x,t)$, $k_2(x,t)$ and $f(x)$ are known, $\varphi(x)$ is the unknown function and λ_1, λ_2 are non-zero parameters .

Example 2.2.3.

$$\varphi(x) = 6x + 3x^2 + 2 - \int_0^x x\varphi(t)dt - \int_0^1 t\varphi(t)\,dt$$

2.2.4 Singular integral equation

A singular integral equation is an equation in which one or both limits of integration are infinite or when the kernel becomes infinite at one or more points within the range of integration. For example, the integral equation,

$$\varphi(x) = f(x) + \lambda \int_{-\infty}^{\infty} \varphi(t)\, dt$$

is a singular integral equation of the second kind.

2.3 Existence and uniqueness of the solution of the Fredholm integral equations

Theorem 2.3.1. *[10] (Fredholm Alternative Theorem) If the homogeneous Fredholm integral equation*

$$\varphi(x) = \lambda \int_a^b k(x,t)\varphi(t)\mathrm{d}t \tag{2.3}$$

has only the trivial solution $\varphi(x) = 0$ *, then the corresponding non homogeneous Fredholm equation*

$$\varphi(x) = f(x) + \lambda \int_a^b k(x,t)\varphi(t)\mathrm{d}t \tag{2.4}$$

has always a unique solution. This theorem is known by the Fredholm alternative theorem

Theorem 2.3.2. *[10] (Unique Solution)*

If the kernel $K(x,t)$ *in Fredholm integral equation(2.1) is continuous, real valued function, bounded in the square* $a \leq x \leq b$ *and* $a \leq t \leq b$*, and if* $f(x)$ *is a continuous real valued function, then a necessary condition for the existence of a unique solution for Fredholm integral equation (2.1) is given by :*

$$|\lambda| M(b-a) < 1 \tag{2.5}$$

where

$$|k(x,t)| \leq M \in \mathbb{R} \tag{2.6}$$

On the contrary, if the necessary condition (2.5) does not hold, then a continuous solution may exist for Fredholm integral equation. To illustrate this, we consider the Fredholm integral equation :

$$\varphi(x) = -2 - 3x + \int_0^1 (3x+t)\varphi(t)\mathrm{d}t \tag{2.7}$$

It is clear that $\lambda = 1$,$|k(x,t)| \leq 4$ *and* $(b-a) = 1$ *.This gives*

$$|\lambda| M(b-a) = 4 \nless 1 \tag{2.8}$$

However, the Fredholm equation (2.7) has an exact solution given by

$$\varphi(x) = 6x$$

2.4 Analytical and numerical methods for solving linear Fredholm integral equations

2.4.1 Fredholm equations of the second kind

In this section, we will present some methods for solving Fredholm integral equations of the second kind

• Adomian decomposition method :

George Adomian introduced and developed a decomposition method, known as the Adomian method, which consists of writing the unknown function $\varphi(x)$ in the form of a series:

$$\varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x) \tag{2.9}$$

where $\varphi_n(x)$, $n \geq 0$, is defined recursively. By replacing equation (2.9) into (2.4), we obtain :

$$\sum_{n=0}^{\infty} \varphi_n(x) = f(x) + \lambda \int_a^b k(x,t) \left[\sum_{n=0}^{\infty} \varphi_n(t)\right] \mathrm{d}t \tag{2.10}$$

or equivalently

$$\varphi_0(x) + \varphi_1(x) + ... = f(x) + \lambda \int_a^b k(x,t)[\varphi_0(t) + \varphi_1(t) + ...]dt$$

Then, by identification, we have the following recurrence relation :

$$\begin{cases} \varphi_0(x) = f(x) \neq 0 \\ \varphi_{n+1}(x) = \lambda \int_a^b k(x,t) \varphi_n(t) \mathrm{d}t \end{cases} \tag{2.11}$$

Example 2.4.1. *Solve the Fredholm integral equation :*

$$\varphi(x) = e^x - x + x \int_0^1 t\varphi(t) \mathrm{d}t$$

By the (ADM) $\varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x)$ *such that :*

$$\begin{cases} \varphi_0(x) = e^x - x \\ \varphi_{n+1}(x) = x \int_0^1 t\varphi_n(t) \mathrm{d}t \end{cases}$$

Hence :

$$\begin{aligned}
\varphi_0(x) &= e^x - x \\
\varphi_1(x) &= x\int_0^1 t\varphi_0(t)\mathrm{d}t = x\int_0^1 t(e^t - t)\mathrm{d}t = \frac{2}{3}x \\
\varphi_2(x) &= x\int_0^1 t\varphi_1(t)\mathrm{d}t = x\int_0^1 t(\frac{2}{3}t)\,\mathrm{d}t = \frac{2}{9}x = \frac{2}{3^2}x \\
\varphi_3(x) &= x\int_0^1 t\varphi_2(t)\mathrm{d}t = x\int_0^1 t(\frac{2}{9}t)\mathrm{d}t = \frac{2}{27}x = \frac{2}{3^3}x
\end{aligned}$$

We deduce that :

$$\varphi_n(x) = \frac{2}{3^n}x$$

This result can be easily proven by induction. Thus, we have :

$$\begin{aligned}
\varphi(x) &= \sum_{n=0}^{\infty} \varphi_n(x) \\
&= e^x - x + \frac{2}{3}x\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ... + \frac{1}{3^n}\right)
\end{aligned} \tag{2.12}$$

We notice that :$\sum_{n=0}^{\infty} \frac{1}{3^n}$ a geometric series with ratio $r = \frac{1}{3}$ and with first term 1 such that the partial sum

$$S_n = \sum_{k=0}^{n} \varphi_k(x) = \frac{1-\left(\frac{1}{3}\right)^{n+1}}{(1-\frac{1}{3})}$$

Then :

$$\sum_{n=0}^{\infty} \varphi_n(x) = \lim_{n\longrightarrow\infty} = \frac{1}{(1-\frac{1}{3})} = \frac{3}{2}$$

Hence, we obtain :

$$\varphi(x) = e^x$$

• Modified Adomain decomposition method :

In the Adomain decomposition method, if the function $f(x)$ is a polynomial function of at most two terms, the components $\varphi_n(x), n \geq 0$ are easy to calculate. In the case where the function $f(x)$ is a polynomial function of more than two terms, or a trigonometric, hyperbolic function, etc., the calculation of $\varphi_n(x)$ will be more difficult. To simplify the calculations, Adomain developed the previous method defined as follows, where $\varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x)$:

$$\begin{aligned} f(x) &= f_1(x) + f_2(x) \\ \varphi_0(x) &= f_1(x) \\ \varphi_1(x) &= f_2(x) + \lambda \int_a^b k(x,t)\varphi_0(t)\mathrm{d}t \\ \varphi_{n+1}(x) &= \lambda \int_a^b k(x,t)\varphi_n(t)\mathrm{d}t, n \geq 1. \end{aligned} \tag{2.13}$$

Example 2.4.2. *Solve Fredholm is integral equation*

$$\varphi(x) = 3x + e^{4x} - \frac{1}{16}(17 + 3e^4) + \int_0^1 t\varphi(t)\mathrm{d}t$$

Using the modified ADM : $\varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x)$ *such that :*

$$f(x) = 3x + e^{4x} - \frac{1}{16}(17 + 3e^4) = f_1(x) + f_2(x)$$

Let :

$$\begin{aligned} f_1(x) &= 3x + e^{4x} \\ f_2(x) &= -\frac{1}{16}(17 + 3e^4) \end{aligned}$$

Where

$$\begin{aligned} \varphi_0(x) &= f_1(x) = 3x + e^{4x} \\ \varphi_1(x) &= f_2(x) + \lambda \int_a^b k(x,t)\varphi_0(t)\mathrm{d}t \\ &= -\frac{1}{16}(17 + 3e^4) + \int_0^1 t(3t + e^{4t})\mathrm{d}t \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \varphi_{n+1} &= \lambda \int_a^b k(x,t)\varphi_n(t)\mathrm{d}t, n \geq 1 \\ &= 0, \forall n \geq 1 \end{aligned}$$

then the solution : $\varphi(x) = 3x + e^{4x}$.

• The successive approximations method :

Let us consider the Fredholm integral equation of the second kind

$$\varphi(x) = f(x) + \lambda \int_a^b k(x,t)\, \varphi(t)\, dt \tag{2.14}$$

Its solution can be expressed as the limit of a sequence:

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x) \tag{2.15}$$

where the sequence $\{\varphi_n(x)\}$ is defined recursively by :

$$\begin{cases} \varphi_0(x) = \text{an arbitrary real-valued function,} \\ \varphi_{n+1}(x) = f(x) + \lambda \int_a^b k(x,t)\,\varphi_n(t)\,dt \end{cases} \tag{2.16}$$

In practice, $\varphi_0(x)$ is often chosen to be a simple function such as 0, 1 or x.

The solution is determined using the limit:

$$\varphi(x) = \lim_{n\to\infty} \varphi_{n+1}(x). \tag{2.17}$$

It is worth noting that the Adomian Decomposition Method allows the use of an iterative formula of the following form:

$$\begin{cases} \varphi_0(x) = \text{all terms not included under the integral sign,} \\ \\ \varphi_1(x) = \lambda \int_a^b k(x,t)\varphi_0(t)dt. \\ \\ \varphi_2(x) = \lambda \int_a^b k(x,t)\varphi_1(t)dt. \\ \\ \vdots \\ \\ \varphi_{n+1}(x) = \lambda \int_a^b k(x,t)\varphi_n(t)dt. \end{cases}$$

Example 2.4.3. *Solve the following equation by the Successive Approximations Method :*

$$\varphi(x) = x + e^x - \int_0^1 xt\varphi(t)dt \tag{2.18}$$

For the zeroth approximation $\varphi_0(x)$, we can select:

$$\varphi_0(x) = 0 \tag{2.19}$$

The method of successive approximations admits the use of the iteration formula:

$$\varphi_{n+1}(x) = x + e^x - \int_0^1 xt\varphi_n(t)\,dt, \quad n \geq 0 \tag{2.20}$$

Substituting (2.19) *into* (2.20) *we obtain: We consider the following sequence:*

$$
\begin{aligned}
\varphi_1(x) &= x + e^x - \int_0^1 xt\varphi_0(t)\,dt = e^x + x,\\
\varphi_2(x) &= x + e^x - \int_0^1 xt\varphi_1(t)\,dt = e^x - \frac{1}{3}x,\\
\varphi_3(x) &= x + e^x - \int_0^1 xt\varphi_2(t)\,dt = e^x + \frac{1}{9}x,\\
&\vdots\\
\varphi_{n+1}(x) &= x + e^x - \int_0^1 xt\varphi_n(t)\,dt = e^x + (-1)^n\frac{1}{3^n}x.
\end{aligned}
$$

Consequently, the solution $\varphi(x)$*of equation (2.18) is given by:*

$$\varphi(x) = \lim_{n\to\infty} \varphi_{n+1}(x) = e^x. \tag{2.21}$$

CHAPTER 3

NUMERICAL SOLUTION OF THE LINEAR FREDHOLM INTEGRAL EQUATION BY GENOCCHI POLYNOMIALS

3.1 Genocchi polynomials

During 1817-1889, Italian mathematician Angelo Genocchi first introduced Genocchi numbers and polynomials. Genocchi numbers have been widely studied across various fields of mathematics .

Definition 3.1.1. *[13] The Genocchi numbers G_n can be defined via the generating function*

$$\Omega(t) = \sum_{n=1}^{\infty} G_n \frac{t^n}{n!} = \frac{2}{e^t + 1} \qquad (|t| < \pi)\,.$$

These numbers are integers and are related to the Bernoulli numbers through the following formula .

$$G_n = 2(1 - 2^n)B_n$$

The initial Genocchi numbers are

G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}	G_{11}	G_{12}	G_{14}	G_{16}
1	*-1*	*0*	*1*	*0*	*-3*	*0*	*17*	*0*	*-155*	*0*	*2073*	*-38227*	*929569*

We must also note that $G_{2n+1} = 0$, for $n = 1, 2, 3, ...$ and the signs of G_n alternate for even values of n .

In general, it holds that $G_3 = G_5 = ... = 0$, while the even coefficients are given by :

$$G_{2n} = 2(1 - 2^{2n})B_{2n} = 2nE_{2n-1}$$

where B_n and E_n represent the well-known Bernoulli and Euler numbers, respectively.

Definition 3.1.2. *[13] The Genocchi polynomials $G_n(x)$ can be defined via the generating function :*

$$\Omega(t,x) = \frac{2te^{xt}}{e^t+1} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \qquad (|t| < \pi)\,.$$

where $G_n(x)$ denotes the Genocchi polynomials of order n .

Additionally, the Genocchi polynomials can be expressed as follows

$$G_n(x) = \sum_{m=1}^{n} \binom{n}{m-1} G_{n-m+1}x^{m-1}, \qquad n = 1, 2, ..., N$$

We present the first few Genocchi polynomials, which are as follows

$$G_1(x) = 1.$$

$$G_2(x) = 2x - 1.$$

$$G_3(x) = 3x^2 - 3x.$$

$$G_4(x) = 4x^3 - 6x^2 + 1$$

$$G_5(x) = 5x^4 - 10x^3 + 5x$$

$$G_6(x) = 6x^5 - 15x^4 + 15x^2 - 3$$

In the special case when $x = 0$, the Genocchi polynomials yield the Genocchi numbers : $G_n(0) = G_n$.

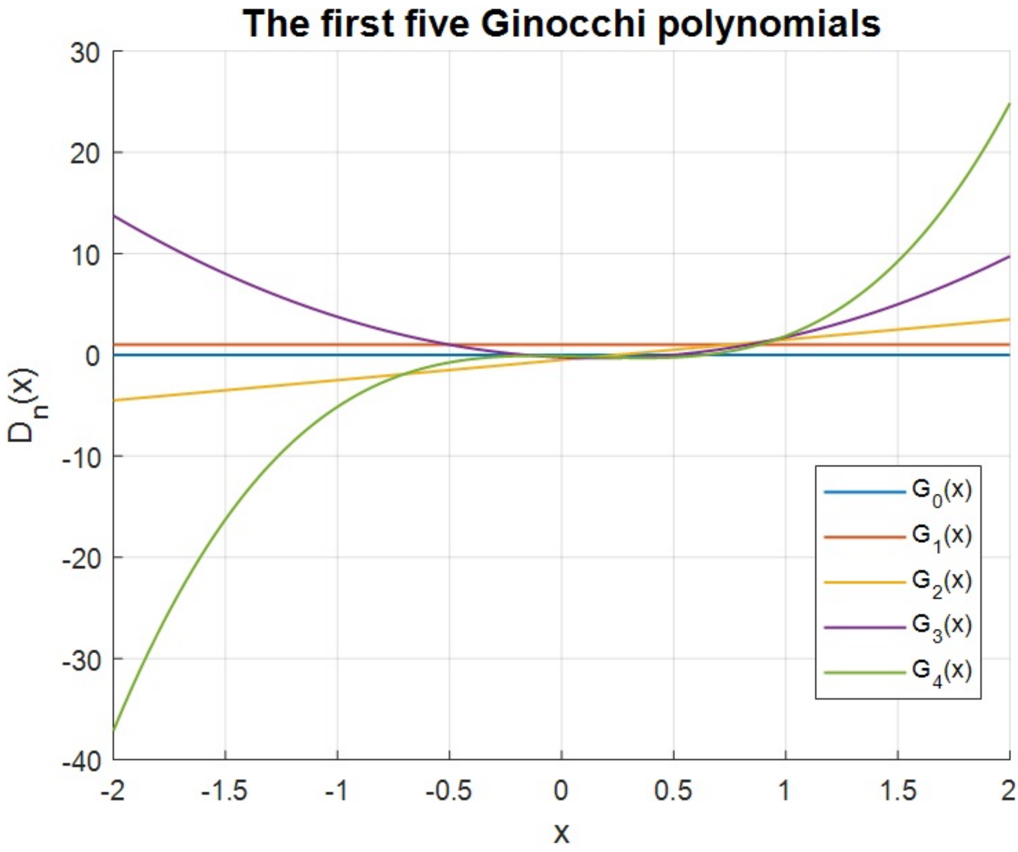

Figure 3.1: The first five Genocchi polynomials

Some of the essential basic properties of the Genocchi polynomials are as follows

1. $G_n(1) + G_n(0) = 0\,, \quad n > 1.$
2. $\frac{dG_n(x)}{dx} = nG_{n-1}(x)\,, \quad n \geq 1\,.$
3. $\displaystyle\int_0^1 G_j(x)G_k(x)dx = \frac{2(-1)^j j!k!}{(j+k)!}G_{j+k}, \quad j,k \geq 1\,.$
4. $\displaystyle\int_a^b G_n(x)d = \frac{1}{n+1}(G_{n+1}(b) - G_{n+1}(a))\,.$

This polynomial was chosen to find an approximate solution to the linear Fredholm integral equation due to its advantages, including :

- *The coefficients of each term in Genocchi polynomials are integers, which eliminates the possibility of computational error. In contrast, most polynomials, such as Euler and Bernoulli polynomials, have non-integer coefficients.*
- *Genocchi polynomials have fewer terms compared to other polynomials. For instance, $G_6(x)$ contains 4 terms, whereas $B_6(x)$ (Bernoulli polynomials) has 5 terms, and both shifted Chebyshev polynomials $T_6(x)$ and shifted Legendre polynomials $L_6(x)$ have 7 terms. As a result, when approximating arbitrary functions, Genocchi polynomials require less CPU time compared to Bernoulli, shifted Chebyshev, and shifted Legendre polynomials.*

3.2 Approximation of a function using Genocchi polynomials

In this part, we require the following linear independence, upon which the subsequent theoretical results are founded.

Suppose that $\{G_1(x), G_2(x), ..., G_N(x)\}$ be the set of Genocchi polynomials .

Lemma 3.2.1. *[13] The set $L = \{G_1(x), G_2(x), ..., G_N(x)\} \subset L^2\,[0,1]$ is linearly independent in $L^2\,[0,1]$*

Démonstration :[13]

To demonstrate that L consists of linearly independent elements in $L^2\,[0,1]$, it suffices to show that the Gram determinant is non zero. Specifically $Gram\,(G_1, G_2, ..., G_N)$.

The Gram determinant is given by :

$$Gram\left(G_1, G_2, ..., G_N\right) = \begin{vmatrix} \langle G_1, G_1\rangle & \langle G_1, G_2\rangle & ..... & \langle G_1, G_N\rangle \\ \langle G_2, G_1\rangle & \langle G_2, G_2\rangle & ..... & \langle G_2, G_N\rangle \\ . & . & ..... & . \\ . & . & ..... & . \\ . & . & ..... & . \\ \langle G_N, G_1\rangle & \langle G_N, G_2\rangle & ..... & \langle G_N, G_N\rangle \end{vmatrix}$$

where $\langle , \rangle$represents the inner product .

To prove that this determinant is nonzero, we first transform the Gram matrix into an upper triangular matrix using Gaussian elimination. It is straightforward to observe that the diagonal elements of the reduced matrix are given by :

$$l(m) = \frac{\left[(m-1)! \times (m!)\right]^2}{(2m-2)! \times (2m-1)!}, \quad m \in \mathbb{N}$$

It is evident that for any $m \in \mathbb{N}$, $l(m) \neq 0$. Therefore, the determinant expressed by $\prod_{m=1}^{N} l(m)$, is non zero. Thus, the set L comprises linearly independent elements .

Theorem 3.2.2. *[13] Let S represent the set Span $\{(G_1(x), G_2(x), ..., G_N(x)\} = Span(L)$, and let $\varphi(x)$ be an arbitrary element of $L^2[0,1]$. Since S is a finite-dimensional subspace (vector space) of $L^2[0,1]$,there exists a unique best approximation of $\varphi(x)$ in S, denoted by $\varphi_N(x)$, such that:*

$$\varphi(x) \simeq \varphi_N(x) = \sum_{n=1}^{N} a_n G_n(x) = G(x)A \tag{3.1}$$

$G(x) = \begin{bmatrix} G_1(x) & G_2(x) & ..... & G_N(x) \end{bmatrix}$, $A = \begin{bmatrix} a_1 & a_2 & ..... & a_N \end{bmatrix}^T$

3.2.1 Description of the method

The collocation method is one of the most widely used techniques for approximating the solutions of integral equations. This method approximates the solution by expressing it as a linear combination of a specific set of basis functions. The process involves substituting this approximate solution into the original equation at a finite set of points within the domain, resulting in an algebraic system of equations that can be solved.

We consider the following the Fredholm integral equation :

$$\varphi(x) = f(x) + \lambda \int_a^b k(x,t)\varphi(t)dt \tag{3.2}$$

In this work, we suppose the approximate solution to problem (3.2) has the form of a truncated series of Genocchi polynomials:

$$\varphi(x) \approx \varphi_N(x) = \sum_{n=1}^{N} a_n G_n(x), \quad a \leq x \leq b \tag{3.3}$$

Where G_n are Genocchi polynomials and a_n are their coefficients. Genocchi polynomials are defined by the explicit form for $n \geq 1$:

$$G_n(x) = \sum_{m=1}^{n} \binom{n}{m-1} G_{n-m+1} x^{m-1} \tag{3.4}$$

From the approximate solution $\varphi(x)$ of (3.2), given as a truncated series of Genocchi polynomials, we convert the finite series (3.3) into matrix form:

$$\varphi(x) \approx [G_1(x)\, G_2(x)\, \ldots\, G_i(x)]A = G(x)A \tag{3.5}$$

Where $A = [a_1\, a_2\, \ldots, a_N]^T$. Using the definition of Genocchi polynomials, we write:

$$G(x) = X(x)P \tag{3.6}$$

Substituting (3.6) into (3.5) :

$$\varphi(x) \approx X(x)PA \tag{3.7}$$

where

$$P = \begin{pmatrix} \binom{1}{0}G_1 & \binom{2}{0}G_2 & \binom{3}{0}G_3 & \cdots & \binom{N}{0}G_N \\ 0 & \binom{2}{1}G_1 & \binom{3}{1}G_2 & \cdots & \binom{N}{1}G_{N-1} \\ 0 & 0 & \binom{3}{2}G_1 & \cdots & \binom{N}{2}G_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N-1}G_1 \end{pmatrix}, X(x) = \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ \vdots \\ \vdots \\ x^{N-1} \end{bmatrix}^T$$

We introduce the matrix form of the kernel function:

$$k(x,t) \approx \sum_{i=1}^{N} \sum_{j=1}^{N} k_{ij} G_i(x) G_j(t) = G(x) K_G G^T(t) \tag{3.8}$$

where

$$K_G = [k_{ij}]_{N \times N}$$

$$k_{ij} = \frac{1}{4(i!j!)} \left(k^{(i-1,j-1)}(0,0) + k^{(i-1,j-1)}(0,1) + k^{(i-1,j-1)}(1,0) + k^{(i-1,j-1)}(1,1) \right)$$

for $\quad i, j = 0, 1, 2, \ldots, N.$

Using (3.5), the matrix relation for $\varphi(t)$ is:

$$\varphi(t) = G(t)A \tag{3.9}$$

Substituting (3.7) ,(3.8) , and (3.9) into equation (3.2):

$$\begin{aligned} X(x)PA &= f(x) + \lambda \int_0^1 G(x)K_G G^T(t)G(t)A dt \\ X(x)PA &= f(x) + \lambda G(x)K_G \underbrace{\left(\int_0^1 G^T(t)G(t)dt\right)}_{M} A \end{aligned}$$

$$XPA - \lambda XPK_G MA = f(x) \tag{3.10}$$

Define the matrix M as:

$$\begin{aligned} M &= \int_0^1 G^T(t)G(t)dt \\ &= \int_0^1 \begin{bmatrix} G_1(t) \\ \vdots \\ G_N(t) \end{bmatrix} \begin{bmatrix} G_1(t) & \cdots & G_N(t) \end{bmatrix} dt \\ &= [M_{ij}]_{N\times N} = \frac{2(-1)^j j!k!}{(j+k)!} G_{i+j} \end{aligned}$$

To obtain the Genocchi polynomials solution of equation (3.2) , we compute the Genocchi coefficients using collocation points defined by:

$$x_i = \frac{i-1}{N-1}, \quad i = 1, 2, \ldots, N$$

Substitute the matrix relations into the collocation form of (3.2):

$$\varphi(x_i) = f(x_i) + \lambda \int_0^1 k(x_i, t)\varphi(t)dt$$

Then:

$$X(x_i)PA = f(x_i) + \lambda X(x_i)PK_G MA$$

$$(X(x_i)P - \lambda X(x_i)PK_G M)A = f(x_i) \tag{3.11}$$

where

$$X = X(x_i) = \begin{bmatrix} X(x_1) \\ X(x_2) \\ \vdots \\ X(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^{N-1} \end{bmatrix}, \quad F = f(x_i) = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix}, \quad A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

Equation (3.11) becomes:

$$WA = F \Leftrightarrow [W;F], \quad \text{where } W = XP - \lambda XPK_GM \tag{3.12}$$

In augmented matrix form:

$$[W;F] = \begin{bmatrix} W_{11} & W_{12} & \cdots & W_{1N} & ; & f_1 \\ W_{21} & W_{22} & \cdots & W_{2N} & ; & f_2 \\ \vdots & \vdots & & & & \vdots \\ W_{N1} & W_{N2} & \cdots & W_{NN} & ; & f_N \end{bmatrix}$$

The system (3.12) has a unique solution if and only if $\det(W) \neq 0$.

Then :

$$A = W^{-1}F \tag{3.13}$$

3.3 Illustrative examples

Example 3.3.1. *Consider the following Fredholm integral equation:*

$$\varphi(x) = x + \int_0^1 (xt - x^2)\varphi(t)dt, \quad 0 \leq x, t \leq 1 \tag{3.14}$$

The exact solution of equation (3.14) *is :*

$$\varphi(x) = \frac{96}{73}x - \frac{36}{73}x^2$$

From equation $(3.14$ *), we obtain* $f(x) = x$ *and* $k(x,t) = xt - x^2$ *, which are continuous functions in the interval* $[0,1]$ *.*

To find the numerical solution of this problem, the Genocchi polynomial series method is used for $N = 3$ *. The approximate solution is given by :*

$$\varphi(x) = \varphi_3(x) = \sum_{n=1}^{3} \alpha_n G_n(x)$$

The fundamental matrix equation for this problem is written as:

$$(XP - \lambda XPK_GM)A = F, \quad \textit{if} \quad W = XP - \lambda XPK_GM, \ \textit{then} \quad WA = F$$

The collocation points are computed as:

$$\begin{cases} x_1 = 0, \\ x_2 = \frac{1}{2}, \\ x_3 = 1 \end{cases}$$

Where:

$$K_G = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{3} & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{3} & 0 \\ -\frac{1}{2} & 0 & \frac{3}{10} \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -3 \\ 0 & 0 & 3 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \lambda = 1$$

After performing the necessary operations on the fundamental matrix equation, the following augmented matrix is obtained:

$$[W, F] = \begin{bmatrix} 1 & -1 & 0 & ; & 0 \\ 1 & -\frac{1}{12} & -\frac{3}{4} & ; & \frac{1}{2} \\ \frac{3}{2} & \frac{5}{6} & -\frac{1}{4} & ; & 1 \end{bmatrix}$$

The solution of the system of three unknowns in the above matrix is:

$$A = \begin{bmatrix} \frac{30}{73} & \frac{30}{73} & -\frac{12}{73} \end{bmatrix}^T$$

Hence, the numerical solution of equation (3.14) is obtained as:

$$\varphi_2(x) = \frac{30}{73} + \frac{30}{73}(2x-1) + \left(\frac{-12}{73}\right)(3x^2 - 3x) = \frac{96}{73}x - \frac{36}{73}x^2$$

This is exactly the exact solution.

Example 3.3.2. *Let us consider the linear Fredholm integral equation of the second kind:*

$$\varphi(x) - \int_0^1 (xt - x^2)\varphi(t)dt = e^x - x + x^2(e-1) \tag{3.15}$$

The exact solution is given by:

$$\varphi(x) = e^x$$

We identify: $f(x) = e^x - x + x^2(e-1)$, *and* $K(x,t) = xt - x^2$. *The approximate solution* $\varphi_n(x)$ *is obtained using the collocation method based on Genocchi polynomials.*

***Table 1 :** Absolute errors for $N = 4$ and 8 using Genocchi polynomials.*

x	errors $N = 4$	errors $N = 8$
0.0	0.00	0.00
0.1	7.728e-04	2.053e-09
0.2	6.651e-04	0.860e-09
0.3	1.851e-04	0.185e-09
0.4	2.808e-04	0.221e-09
0.5	4.819e-04	0.312e-09
0.6	3.166e-04	0.164e-09
0.7	1.514e-04	0.069e-09
0.8	6.760e-04	1.107e-09
0.9	8.091e-04	1.995e-09
1	1.198e-04	0.308e-09

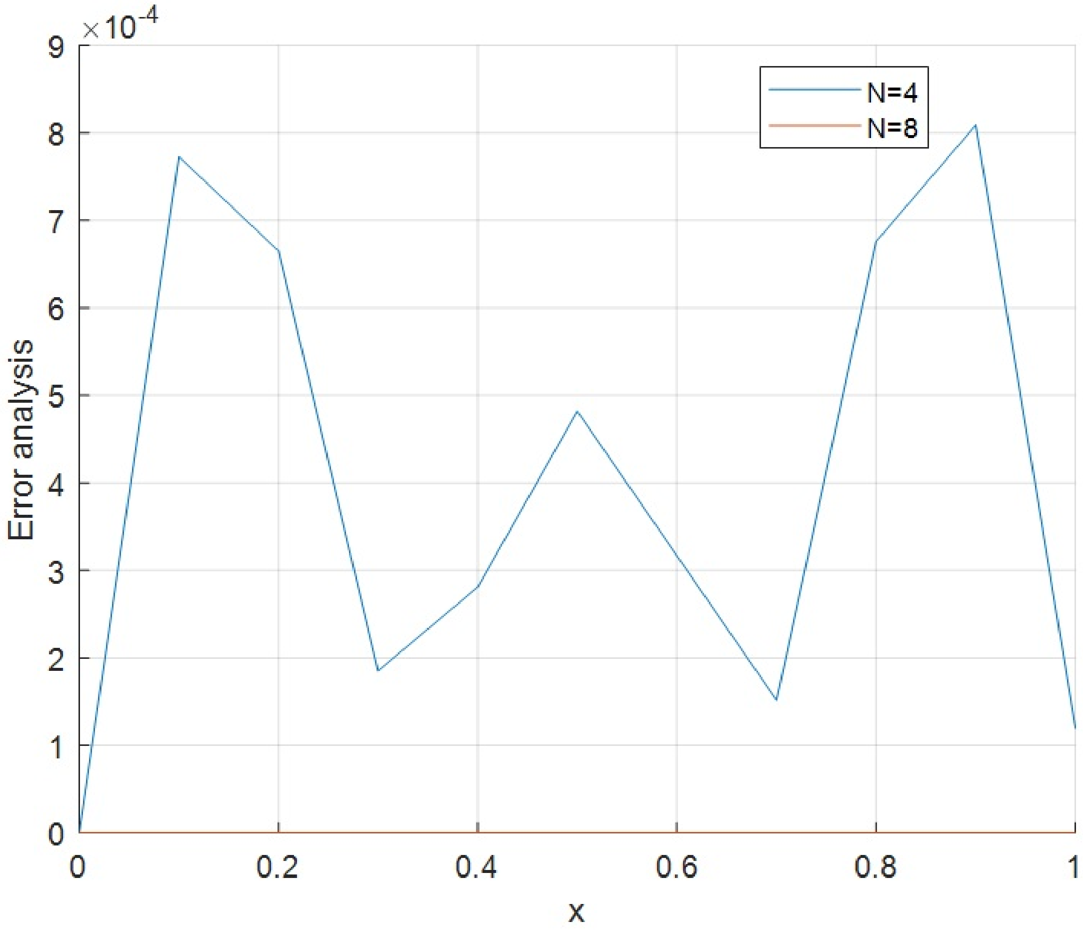

Figure 3.2: Absolute errors for $N = 4$ and 8 using Genocchi polynomials.

Example 3.3.3. *Let us consider the linear Fredholm integral equation of the second kind:*

$$\varphi(x) = 3x + 3x^2 + \frac{1}{2}\int_0^1 x^2 t\varphi(t)dt \tag{3.16}$$

The exact solution is given by:

$$\varphi(x) = 3x + 4x^2$$

We identify: $f(x) = 3x + 3x^2$, *and* $K(x,t) = x^2t$. *The approximate solution* $\varphi_n(x)$ *is obtained using the collocation method based on Genocchi polynomials.*

Table 2 : *Absolute errors for* $N = 2$ *and* 4 *using Genocchi polynomials.*

x	errors $N = 2$	errors $N = 4$
0.0	0.00	0.00
0.1	0.3800	0
0.2	0.6800	0
0.3	0.9000	0
0.4	1.0400	0
0.5	1.1000	0
0.6	1.0800	0
0.7	0.9800	0
0.8	0.8000	0
0.9	0.5400	0
1	0.2000	0

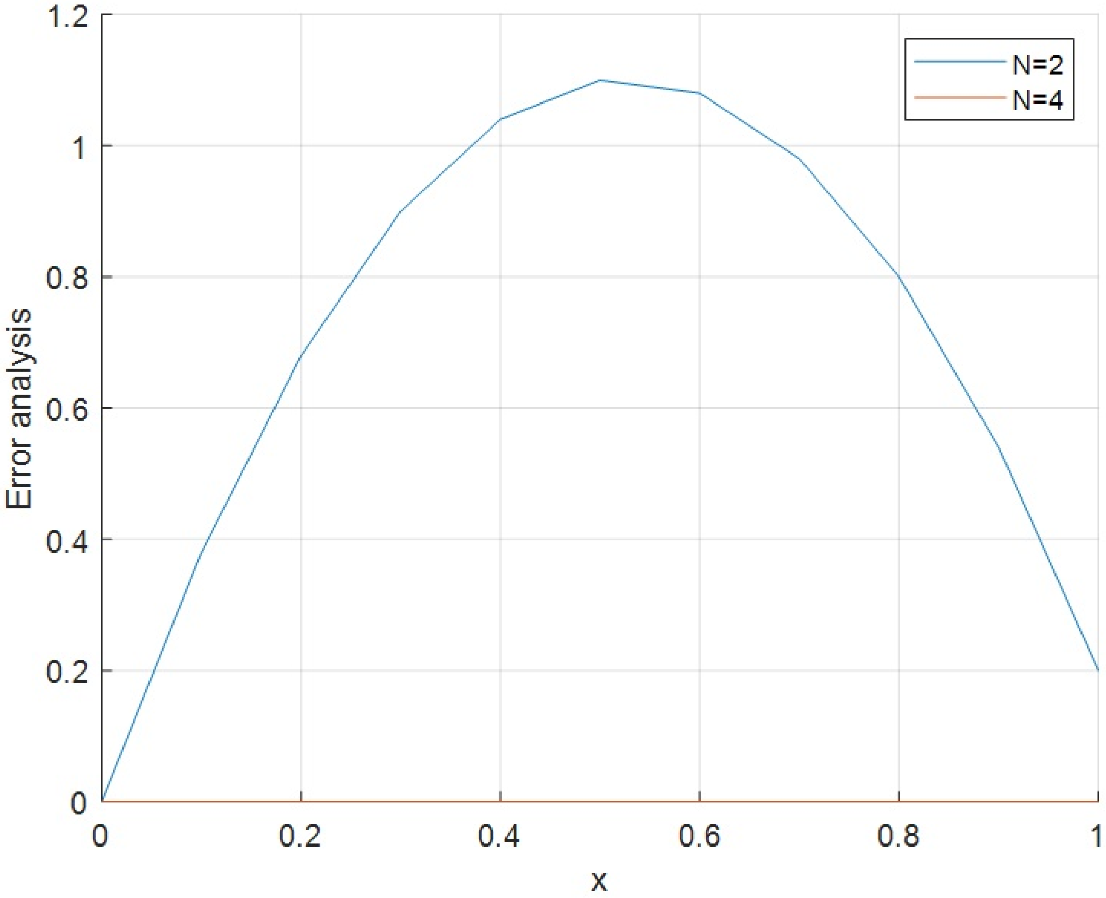

Figure 3.3: Absolute errors for $N = 2$ and 4 using Genocchi polynomials.

Conclusion

In this thesis, we presented a detailed study of linear Fredholm integral equations and proposed an effective numerical method for their solution using Genocchi polynomials. The work began by reviewing the foundational concepts of functional analysis, including normed spaces, operators, and the theory of integral equations.

We explored the classification of integral equations and highlighted the theoretical conditions required for the existence and uniqueness of their solutions. Special attention was given to Fredholm equations of both the first and second kind due to their wide applications in mathematical modeling and applied sciences.

The core contribution of this work lies in Chapter 3, where we employed Genocchi polynomials as a basis for constructing numerical approximations using the collocation method. Thanks to their integer coefficients and compact structure, Genocchi polynomials allowed for efficient computation and high accuracy. The formulation was supported by matrix representations and was validated through a worked example, where the numerical solution exactly matched the analytical one.

The results confirm that the Genocchi polynomial method is a powerful and practical approach for numerically solving Fredholm integral equations. It offers a promising alternative to traditional polynomial bases such as Bernoulli or Chebyshev, especially in cases where computational efficiency and stability are crucial.

Future work may include extending this method to systems of integral equations, nonlinear problems, or applying it to real-world models in physics and engineering.

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