الجمهوريــة الجزائريــة الديمقراطيــة الشعبيــة
وزارة التعليـم العالــي والبحــث العلمــي

جامعة محمد بوضياف بالمسيلـــة
كليـــة العلـــــوم

رقم 73/ك ع/2025

جامعة محمد بوضياف - المسيلة
Université Mohamed Boudiaf - M'sila

المسيــلة في: 2025/02/11

شهادة موافقة علمية على مطبوعة بيداغوجية

للأستاذة صابر ساعد السعود - أستاذ محاضر أ -

يشهد رئيس المجلس العلمي لكلية العلوم بجامعة محمد بوضياف بالمسيلة، أنه بعد الإطــلاع على تقاريــر الخبـــرة الواردة من طرف الخبـــراء من صف الأستاذيـــة:

- السيد مزاهي فاطمة الزهراء ، أستاذ التعليم العالي بجامعة محمد بوضياف- المسيلة.
- السيد بوحمادو عبد المجيد ، أستاذ التعليم العالي بجامعة سطيف 1

والمعينين من طرف المجلس العلمي لكلية العلوم في الاجتماع المنعقد في دورته العادية يوم2024/11/25 لإجراء الخبرة للمطبوعة البيداغوجية الخاصة بالأستاذ: صابر ساعد السعود - أستاذ محاضر- أ- بقسم بالجذع المشترك علوم المادة والمتعلقة بخبرة للمطبوعة البيداغوجية للمـــادة المعنونة بـــ:

PHYSICS 1: Mechanics Of Material Point والمقررة في برنامج التكوين ليسانس، تخصص: « **L1 (SM et MI)** » و المفتوح بالجذع المشترك علوم المادة

تـمت الموافقـة عليها شكـلا ومضمونـا.

رئيس المجلس العلمي لكلية العلوم

بعزيز حكيم

People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

Mohamed Boudiaf University of M'sila

PEDAGOGICAL HAND-OUT IN:

PHYSICS 1: Mechanics of material point

For 1st year LMD students

Prepared by : Dr. Saber Saad Essaoud

Academic Year 2023-2024

Preface: *This manual is intended for first-year university students and offers a comprehensive series of detailed lessons and solved exercises. It is designed to align with the ministry's corresponding LMD "Academic/Licence" training model. The primary goal of this manual is to equip students with the necessary foundational knowledge to understand and master the basic concepts of particle mechanics, enabling them to apply these principles in analyzing various other physical phenomena. In addition, students will develop essential intellectual skills that allow them to bridge what they have learned in this module to broader academic fields and subjects as they continue in their academic journey.*

The manual is structured to gradually build up the students' understanding, beginning with essential theoretical knowledge and progressively integrating practical applications. Each chapter is accompanied by exercises, reinforcing the material covered and helping students apply theoretical concepts to real-world situations. The following chapters are included in the manuscript:

- ***Chapter One*** *provides an overview of the main concepts of dimensional analysis. It includes a mathematical review and offers explanations for the fundamental concepts of vector calculus, ensuring students have a solid foundation in the necessary mathematical tools.*
- ***Chapter Two*** *introduces the kinematics of a material point, laying out the principles that govern the motion of objects and how these can be described in mathematical terms.*
- ***Chapter Three*** *delves into the dynamics of a material point, illustrating the key principles of Newtonian mechanics through its three fundamental laws of motion, and explaining how forces affect the motion of objects.*
- ***Chapter Four*** *focuses on the work and energy associated with a material point, exploring how energy is transferred and transformed in mechanical systems.*

Each chapter is supported by a series of corrected exercises that are designed to enhance the student's understanding of the material, allowing them to apply the theoretical knowledge gained in practical scenarios. These exercises build on the concepts introduced in the chapters, promoting a deeper comprehension of mechanical applications. Through the study of this manual, students will not only grasp the core principles of mechanics but also acquire the analytical skills necessary for future academic challenges in the field of physics and beyond.

Detailed Contents

Chapter 01: Fundamentals (Units and Vectors)

Chapter 02: Kinematics

Chapter 03: Dynamics

Chapter 04: Work and Energy

Chapter One: Fundamentals (Units and Vectors)

FIRST PART: UNITS

Understanding physics easily, especially in classical mechanics, requires giving significant importance to building accurate concepts and providing a physical perspective on mathematical relationships. Therefore, our goal through this compilation of lessons is to reanalyse and review many of the physical principles and connect some mathematical concepts with their physical meanings.

1) physical quantity

In physics, a ***physical quantity*** *is any* ***measurable property*** *that describes a specific aspect of an object or a system.*

Examples of physical quantities: *mass, time, distance, area, volume, energy, density, pressure...velocity, acceleration, force and temperature …. Etc*

The physical quantities can be categorized into two types:

1-1) Scalar quantities

Scalar quantities are properties described just by their ***magnitude*** *(numerical value) and the appropriate* ***unit of measurement****. Examples of scalar quantities include:*

Mass (e.g., 15 kg), Temperature (e.g., 30°C), Energy (e.g., 100 J), Time (e.g., 10 seconds).

1.2) Vector quantities

Vector quantities are properties described by specifying their ***direction****, in addition to their* ***magnitude*** *(numerical value) and their appropriate* ***unit of measurement.***

other definition : are properties that cannot be fully described without specifying the direction (in addition to their unit and numerical value (magnitude)).

Examples of vector quantities include:

Velocity, Force, Displacement, Acceleration.

2) Units of measurement

Units of measurement are symbols added to values to give them specific physical meanings and also for distinguishing between quantities. Units are divided into two main groups:

2-1) Fundamental Units

*These are the basic building blocks of measurement and are inherently independent, meaning they can't be expressed in terms of other units. They include crucial physical quantities like **units** of mass, time, electric charge, length, and distance.*

2-2) Derived Units

Derived units are units that can be expressed as a combination of fundamental units. Examples include:

The newton (N), derived from kilogram, meter, and second, measures force.

The joule (J), a combination of Newton and meter, quantifies energy.

The watt (W), obtained from joule and second, is used for power measurement.

The coulomb (C), derived from ampere and second, quantifies electric charge.

The hertz (Hz), based on the reciprocal of the second (1/s), represents frequency.

3) Systems of units in physics

Among the systems of units, we mention:

3-1) Centimeter-Gram-Second (CGS) System:

The CGS system includes three basic quantities: length, mass, and time, as well as units such as centimetres, grammes, and seconds.

3-2) meter-kilogram-second-ampere system

MKSA system consists of four quantities and units: length, mass, time, and electrical current. Units are meters, kilogrammes, seconds, and amperes.

3-3) The International System of Units (SI)

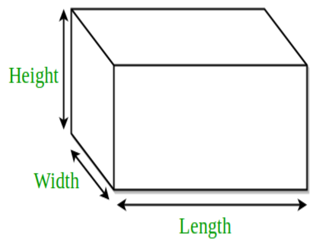
Measurement units for physical quantities can vary from one country to another. For example, in the United States, units like inches (1 inch = 2.54 cm) and pounds (1 pound = 0.45 kg) are used to measure distance and mass. These units may not be widely recognized in many other countries, leading to challenges in scientific communication. Therefore, to establish a uniform unit system, a consensus has been reached, resulting in the International System of Units (SI), which consists of seven fundamental units as outlined in the table below.

Table01: The seven independent SI base units

Base quantity	***Unit name***	***Unit Symbol***
Length	*Meter*	*m*
Time	*Second*	*s*
Mass	*Kilogram*	*Kg*
Temperature	*Kelvin*	*K*
Electric current	*Ampere*	*A*
Amount of substance	*Mole*	*mol*
Luminous intensity	*Candela*	*cd*

Example 01:

Find the derived unit for the following quantities



- *Volume = Lenght × Width × Hight*

Volume unit ≡ Length unit × Width unit × Hight unit≡ $m \times m \times m \equiv m^3$

- *Velocity = distance / time* ⇒ *Velocity unit ≡ distance unit / time unit = m/s*
- *Acceleration = velocity/ time* ⇒ *Acceleration unit ≡ velocity unit / time unit ≡* m/s^2
- *Force = mass× acceleration* ⇒ *Force unit ≡ mass unit × acceleration unit≡* $Kg.m/s^2$ ≡ **N**

4) Dimensional analysis

4-1) Dimension definition

In physics, the term "dimension" signifies the inherent property of a physical quantity. we use dimension to classify the physical quantities according to their types regardless of their values or the unit used for them. For instance, the measurement of distance between two points can be done using units like feet, meters, or kilometres, with each unit representing a unique aspect of the dimension known as length. Similarly, when measuring the mass of an object, it can be quantified using units like kilogram, gram, pound, or milligram. These units all share a common nature, as they represent mass, and therefore they have a mass dimension. To denote the dimension, we typically write the symbol of the quantity within square brackets ***[quantity].***

Table02: Base Quantities and Their Dimensions

Base quantity	Symbol for dimension
Length	L
Time	T
Mass	M
Temperature	Θ
Electric current	I
Amount of substance	N
Luminous intensity	J

The dimension of the derived physical quantity (Q) can be expressed as:

$$[Q] = L^a M^b L^c T^d I^e \Theta^f N^g J^h$$

In this expression, the variables a, b, c, d, e, f, and g represent the powers associated with the fundamental physical dimensions.

4-2) The dimensional equation

The dimensional equation is an equation that expresses the relationship between different physical quantities through their dimensions side. We can obtain the dimensional equation by rewriting the original equation in a way that represents each physical quantity in *terms of its dimension. In other words, the symbols used for physical quantities are replaced with symbols representing their dimensions.*

Through this process, we can verify that the relationship between the physical quantities in the equation is correctly aligned in terms of dimensions and units, ensuring the accuracy and compliance of the equation with the laws of physics.

Example 02:

let as assume we have a physics equation that relates force (F) to mass (m) and acceleration (a). This dimensional equation can be represented as follows: $\vec{F} = m \cdot \vec{a}$

In this equation:

$\vec{F}$ *represents force and can be expressed in units of Newton (N).*

m represents mass and can be expressed in units of kilograms (kg).

$\vec{a}$ *represents acceleration and can be expressed in units of meters per second squared* (m/s^2).

If we analyze the dimensions in this dimensional equation, we find that they align correctly:

$[F] = [m]\,[a] = M.L.T^{-2} \equiv N$

This indicates that the unit of Newton for force correctly corresponds to the units of mass and acceleration. Thus, the equation is dimensionally and physically correct.

4-3) Properties of dimensional equation

"Let F, A,B, and Care different physical quantities, and 'm' and 'n' are real numbers.

- *The equation* $F + C = B + A$ *is dimensionally correct only if all the quantities F, C, B, and A, have the same dimensions*($[F] = [C] = [B] = [A]$).
- *The dimension of any constant real number is dimensionless (If* $F = n \Rightarrow [F] = 1$).
- $F = n \times A$ *we can analyse its dimension as* $[F] = [n] \times [A] \Rightarrow [F] = 1 \times [A]$. *This demonstrates that F possesses the same dimension as A."*
- $F = B \times A \times C$, *we can determine its dimension as '*$[F] = [B] \times [A] \times [C]$*.' This equation displays how the dimension of F is a product of the dimensions of A, B, and C."*
- $F = A^m \Rightarrow [F] = [A]^m$. *This illustrates that F's dimension is related to A raised to the power of 'm'*
- $F = dA/dx \Rightarrow [F] = [A]/[x]$

- $F = \int (A\, dy) \Rightarrow [F] = [A] \times [y]$
- $F = \sqrt{A} \Rightarrow [F] = [\sqrt{A}] = \sqrt{[A]}$
- $A = \frac{df}{dt} \Rightarrow [A] = \left[\frac{df}{dt}\right] \Rightarrow \left[\frac{df}{dt}\right] = \frac{[f]}{[t]}$
- $A = \int g\, dx \Rightarrow [A] = [\int g\, dx\,] = [g]\,[dx\,]$
- $A = number \Rightarrow [A] = 1$
- $A = cos\,(B) \Rightarrow [A] = 1$
- $A = sin\,(B) \Rightarrow [A] = 1$
- $A = tan\,(B) \Rightarrow [A] = 1$

Note: *When physical quantities have very large or very small values, it is preferable to express those using powers of 10 or by adding a prefix to the unit that represents the power of 10, as shown in the following table:*

Table03: Prefixes for Powers of Ten

Factor	Prefix	Symbol
10^{-24}	yocto	y
10^{-21}	zepto	z
10^{-18}	atto	a
10^{-15}	femto	f
10^{-12}	pico	p
10^{-9}	nano	n
10^{-6}	micro	μ
10^{-3}	milli	m
1^{--2}	centi	c
10^{-1}	deci	d

Factor	Prefix	Symbol
10^{1}	deka	da
10^{2}	hecto	h
10^{3}	kilo	k
10^{6}	mega	M
10^{9}	giga	G
10^{12}	tera	T
10^{15}	peta	P
10^{18}	exa	E
10^{21}	zetta	z

Example 03:

Find the dimension of the following quantities: velocity, acceleration, force, Charge quantity, , Kinetic Energy , and Gravitational potential energy.

- ***Velocity:*** $V=x/t \Rightarrow [V]= [x]/[t]= L/T$
- ***Acceleration:*** $a=v/t \Rightarrow [a]= [v]/[t]=(L/T)/T = L.T^{-2}$

- **Force:** $\vec{F} = m\,\vec{a} \Rightarrow [F] = [m].[a] = M.L.T^{-2}$
- **Charge quantity:** $Q = i \times t \Rightarrow [Q] = [i] \times [t] \Rightarrow [Q] = I.T$
- **Kinetic Energy:** $E = \frac{1}{2} mV^2 \Rightarrow [E] = \left[\frac{1}{2}\right][m][V^2] = M\,L^2T^2$
- **Gravitational potential energy:** $E = m.g.h \Rightarrow [E] = [m][g][h] \Rightarrow [E] = M.L.T^{-2}.L = M\,L^2T^2$

12) Serie of Solved Exercises

Exercise01:

The following equation expresses the change in displacement with respect to time and the acceleration of a moving object.

$$x = \frac{1}{2} at^2$$

1 - Write the homogeneity condition of the equation.

2 - Prove its dimensional consistency using the dimensional analysis method.

Solution:

1. The homogeneity condition of the equation :

$$[dimenion\ of\ the\ left\ side] = [dimension\ of\ the\ right\ side]$$

2- Prove its dimensional consistency using the dimensional analysis method.

The left side of the equation $[x] = L$

The right side of the equation $\left[\frac{1}{2}\right][a][t^2] = [a][t]^2 = L\,T^{-2}T^2 = L$

The dimension of the left side equals the dimension of the right side, indicating that the equation is dimensionally correct.

Exercise02:

What are the dimensions of both k and v_0 *constants in the following equation?*

$v = kt + v_0$ *where v represents velocity and t represents time*

Solution:

The equation is dimensionally correct if both sides have the same dimension.

$$[v] = [k][t] = [v_0]$$

$$LT^{-1} = [k]T = [v_0]$$

$$LT^{-1} = [v_0]\ and\ LT^{-2} = [k]$$

Exercise03:

Determine the physical dimension of the spring constant and then verify the homogeneity of the equation that relates the spring constant to the elastic potential energy

Solution:

1) *The physical dimension of the spring constant (elasticity constant) can be found using the formula related the force exerted by a spring to its displacement*

$$F = -Kx$$

Where

F *is the force applied to the spring.*

K *is the spring constant.*

x *is the displacement from the equilibrium position.*

To find the dimension of K*, we can rearrange the equation as follows:*

$$K = -\frac{F}{x}$$

So, we can find the dimension of the spring constant K *as follows:*

$$[K] = \frac{[F]}{[x]} = \frac{M.L.T^{-2}}{L} = M.T^{-2}$$

So, the physical dimension of the spring constant (K) is $M.T^{-2}$

2) To verify the homogeneity of the equation connecting the spring constant to elastic potential energy, you can follow these steps:

- Express the equation for elastic potential energy (U) in terms of the spring constant (K) and the displacement (x).

$$U = \frac{1}{2}Kx^2$$

- *Determine the dimensions of each variable involved:*

The dimension of elastic potential energy $[U]$ *is energy, which is represented as* $[U] = M\ L^2T^2$

We already found the dimension of the spring constant $[K]$ *in a previous response:* $[K] = M.T^{-2}$

The dimension of displacement (x) is length, $[x] = L$

Now, substitute K and x dimensions into the equation for elastic potential energy:

$$[K][x]^2 = M.T^{-2}.L^2 = [U]$$

The equation is homogeneous because the dimensions on both sides match. Therefore, the equation connecting the spring constant to elastic potential energy is consistent in terms of dimensions.

Exercise 04:

Determine the dimension of the two parameters A and B which appear in this equation:

$f = A.m.v + B\,v^2$ *where f represents the force, m represents the mass, and v represents the velocity*

Solution:

$$f = A.m.v + B\,v^2 \Rightarrow [f] = [A.m.v] = [B\,v^2]$$

$$[f] = [A.m.v] = [A].[m].[v]$$

$$[A] = \frac{[f]}{[m][v]}$$

$$[A] = \frac{M\,L\,T^{-2}}{M\,L\,T^{-1}} = T^{-1}$$

$$[f] = [B\,v^2] = [B\,][\,v^2] = [B\,]\;[\,v]^2$$

$$[B] = \frac{[f\,]}{[\,v]^2} = \frac{M\,L\,T^{-2}}{L^2\,T^{-2}} = ML^{-1}$$

Exercise 05:

The equation of state for an ideal gas is given by: $PV = nRT$

- Write the dimensions of each parameter in this equation.

Solution:

1) ***Pressure*** *:* $P = F/S$ *(F: force and S : area)*

$[P] = \frac{[F]}{[S]} \Rightarrow \begin{cases} F = m\,a \Rightarrow [F] = [m]\,[a] \\ S = L \times W \Rightarrow [S] = [L] \times [W] \end{cases}$ *(m signifies mass, a represents acceleration, L denotes length, and w represents width.)*

$$\begin{cases} [F] = [m]\,[a] = M\,L\,T^{-2} \\ \Rightarrow [S] = [L] \times [W] = L^2 \end{cases} \Rightarrow [P] = \frac{[F]}{[S]} = \frac{M\,L\,T^{-2}}{L^2} = M\,L^{-1}\,T^{-2}$$

2) ***volume*** *:* $V = L \times W \times H \Rightarrow [V] = [L] \times [W] \times [H] = L^3$

3) ***Amount of substance*** *:* $[n] = N$

4) ***Temperature*** *:* $[T] = \theta$

5) The ***ideal gas constant****, also known as the* ***universal gas constant***

$$R = \frac{PV}{nT} \Rightarrow [R] = \frac{[P][V]}{[n][T]} = \frac{M\,L^{-1}\,T^{-2}\,L^{3}}{N\,\theta} = M\,L^{2}\,T^{-2}\,N^{-1}\theta^{-1}$$

Exercise 06:

Using dimensional analysis, find the dimensions of both parameters E and G that ensure the dimensional consistency of the following equation:

$$K = G\,v\,f + E\,p^{-1}\,m^{-2}g^{-1}$$

where, f represents the force, v represents the velocity, m represents the mass, p represents the pressure, g represents the acceleration due to gravity. The dimensions of the quantity K are given as: $[K] = M^{2}\,L^{2}\,T^{-3}$

Solution:

1) Dimensions of the given parameters :

- Force f: The dimension of force f is MLT^{-2} *(since force = mass m × acceleration a).*

- Velocity v: The dimension of velocity is LT^{-1} *(since velocity = distance / time).*

- Mass m: The dimensions of mass are M.

- Pressure p: The dimension of pressure is $M\,L^{-1}\,T^{-2}$ *(since pressure = force / area).*

- Acceleration due to gravity g: The dimension of acceleration is LT^{-2}.

2) Finding the dimensions of G and E:

o ensure the dimensional consistency of the equation, the dimensions of the following terms must match the dimension of K

therefore,

$$[K] = [G\,v\,f] = [E\,p^{-1}\,m^{-2}g^{-1}] \Rightarrow \begin{cases} [G] = \dfrac{[K]}{[\,f][\,v]} \\ [E] = \dfrac{[K]}{[\,p]^{-1}\,[\,m]^{-2}[\,g]^{-1}} \end{cases}$$

$$\begin{cases} [G] = \dfrac{M^{2}\,L^{2}\,T^{-3}}{MLT^{-2}\,LT^{-1}} \\ [E] = \dfrac{M^{2}\,L^{2}\,T^{-3}}{M^{-1}\,L\,T^{2}\,M^{-2}L^{-1}T^{2}} \end{cases} \Rightarrow \begin{matrix} [G] = M \\ [E] = M^{5}\,L^{2}\,T^{-7} \end{matrix}$$

SECOND PART : VECTORS

1) Definition of vector:

A vector is an oriented segment that has both magnitude (length) and direction. Graphically, vectors are often represented by arrows. The length of the arrow represents the magnitude of the vector, and the direction of the arrow indicates the direction of the vector.In handwritten equations or mathematical expressions, vectors are typically represented by placing a letter with an arrow above itsuch as $\vec{V}$*.In physics, vectors offer a comprehensive representation of certain quantities that cannot be adequately described solely by stating their values and units. Vectors enable us to model and represent these quantities, allowing us to understand how they change in various spatial directions, essentially giving them a descriptive form associated with spatial orientations.*

Tail (origin) ⟶ Tip (Head)

2) characteristics of a vector

*1- **Magnitude (Length)**: Magnitude represents the length of the vector and expresses the scalar quantity of the vector. Its value is always positive. The symbol used to represent the magnitude or absolute value of a vector is indeed typically written as double vertical bars (‖ ‖)surrounding the vector* $\vec{\mathbf{V}}$*, like this:* $\|\vec{\mathbf{V}}\|$

*2- **Direction**: Direction indicates the orientation of the scalar quantity of the vector, geometrically defined by the angle between the vector and a specific axis or the arrow's direction.*

*3- **Support**: Support refers to the line connecting the starting point (tailor origin) of the vector to its endpoint (head or tip). In other words, the line of action of the vector is the straight line along which the vector lies.*

3) Specifying a vector

The vector is typically defined using one of the following methods:

- ✓ *By specifying its length (magnitude) and the angle it makes with a specific axis (see figure 01).*

- ✓ *By determining its starting point (tail) and endpoint (head) using their coordinates in space (see figure 02).*
- ✓ *By describing its components in space, such as its three-dimensional coordinates (We will address this point later).*

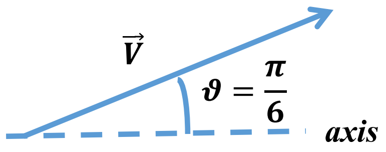


figure 01

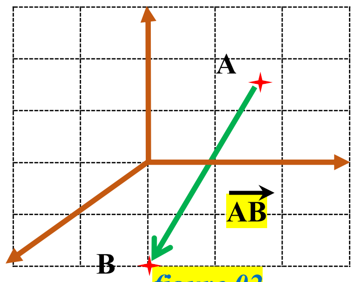
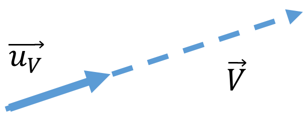


figure 02

4) The Unit Vector

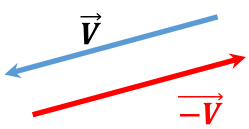


The unit vector notation for vector $\vec{V}$, *which is denoted as* $\overrightarrow{u_V}$, *exhibits the following characteristics:*

Magnitude: *It has a magnitude of exactly one, denoted as* $\|\overrightarrow{u_V}\| = 1$. *This signifies that its length is normalized to unity, meaning it doesn't carry any specific units.*

Direction: $\overrightarrow{u_V}$ *aligns precisely in the same direction as the original vector* $\vec{V}$. *In mathematical terms, this can be expressed as* $\overrightarrow{u_V} = \vec{V}/\|\vec{V}\|$, *where* $\|\vec{V}\|$*represents the magnitude (length) of vector* $\vec{V}$.

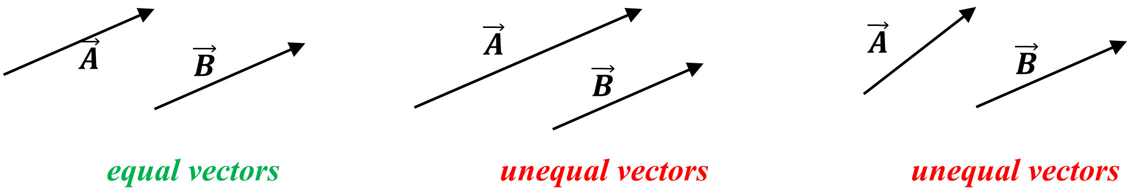
5) The Negative of a Vector



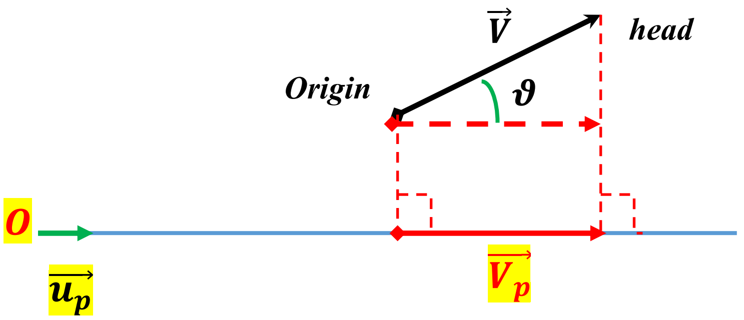
The negative vector is a vector with an equal magnitude to the original vector but directed in the opposite way.

5) Equality of Two Vectors

we said $\vec{A}$ *and* $\vec{B}$ *are two equal vectors* ($\vec{A} = \vec{B}$) *only if they have the same magnitude* ($\|\vec{A}\| = \|\vec{B}\|$) *and direction, whether or not their initial points are the same (where, the lines of action of any two equal vectors are parallel.), as shown the following Figure*



6) The vertical projection of a vector onto an axis



The vertical projection of a vector $\vec{V}$ *onto an axis*(OP) *is a vector* $\vec{V_p}$ *aligned with that axis. Its starting point (origin or tail) is the projection of the vector's originpoint onto the axis, and its endpoint is the projection of the vector's head(tip) onto the axis. Its length is the result of multiplying the vector's length by the cosine of the angle* (ϑ)*enclosed between them.*

$$\|\vec{V_p}\| = \|\vec{V}\| \times cos(\vartheta)$$

If the axis onto which a vector is projected is oriented with a specific direction, such as being represented by a unit vector, we can express the projection by taking the scalar product (dot product) between the original vector and the unit vector of the axis. We'll delve into the concept of the dot product between two vectors in more detail later on.,and therefore, we can write the following: $\vec{V_p} = \|\vec{V_p}\|\vec{u_p} = \|\vec{V}\| \times cos(\vartheta)\vec{u_p}$

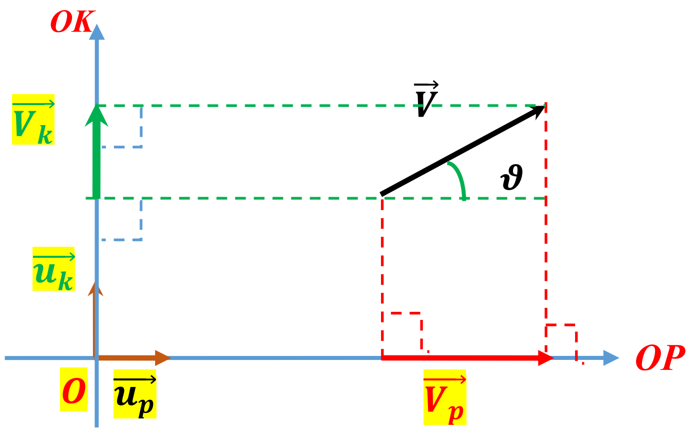
The length of the vector obtained through the projection can be called the ***'component*** *of the original vector on the axis.This component represents a part of the original vector and allows you to analyze the original vector into its* ***subcomponents*** *along multiple axes.*

7) The analytical representation of a vector

The analytical representation of a vector means representing the vector using its individual components relative to specific axes. These components are typically the outcomes of projecting the vector orthogonally onto these axes. In other words, the analytical representation of a vector is the process of rewriting it using its components and unit vectors extending in specific directions along particular axes.

Example:

The vector $\vec{V}$ *in the corresponding figure has a magnitude of 2 and makes an angle of 30 degrees with the (OP) axis.We can find its components along the two axes (OP) and (OK), which carry the unit vectors* $\overrightarrow{u_p}$ *and* $\overrightarrow{u_k}$*, respectively. We can write its analytical expression along these two axes using the following method:*



To find the component along the (op) axis. Firstly, we need to perform orthogonal projections of the original vector onto both axes, as detailed in the figure.

On (OP) axis $\Rightarrow \quad \overrightarrow{V_p} = \|\overrightarrow{V_p}\|\overrightarrow{u_p} = \|\vec{V}\| \times cos(\vartheta)\overrightarrow{u_p} = \frac{2\sqrt{3}}{2}\overrightarrow{u_p} = \sqrt{3}\overrightarrow{u_p}$

On (OK) axis $\Rightarrow \quad \overrightarrow{V_k} = \|\overrightarrow{V_k}\|\overrightarrow{u_k} = \|\vec{V}\| \times sin(\vartheta)\overrightarrow{u_k} = 2 \times \frac{1}{2}\overrightarrow{u_k} = \overrightarrow{u_k}$

$$\vec{V} = \overrightarrow{V_p} + \overrightarrow{V_k} = \sqrt{3}\overrightarrow{u_p} + \overrightarrow{u_k}$$

So $\sqrt{3}$ and 1 *values resulting from projecting the vector* $\vec{V}$ *on the two principal axes (OP, and OK) are called the components of the vector* $\vec{V}$.

- *When vector $\vec{A}$ analyzed into its components in a Cartesian coordinate system* $(O,\vec{\imath},\vec{\jmath},\vec{k})$*, the analytical expression for this vector is written as follows:*

$$\vec{A} = X_A\vec{\imath} + Y_A\vec{\jmath} + Z_A\vec{k} \text{ or } \vec{A} = \begin{pmatrix} X_A \\ Y_A \\ Z_A \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix}$$

Where the relationship between the components of the vector, its magnitude, and the angle it forms with a given axis is expressed as follows:

$$\begin{cases} X_A = |\vec{A}| \cos\alpha \\ Y_A = |\vec{A}| \cos\beta \\ Z_A = |\vec{A}| \cos\gamma \end{cases}$$

Where X_A is the projection of $\vec{A}$ on (Ox), Y_A is the projection of $\vec{A}$ on (Oy), and Z_A is the projection of $\vec{A}$ on (Oz). α, β and γ are the angles between the vector $\vec{A}$ and the axis (Ox), (Oy) and (Oz) respectively.

7-1) Belonging of vector to a specific plane determined by two vectors

We classify the vector $\vec{V}$ as part of the plane (P) defined by vectors $\vec{V_1}$ and $\vec{V_2}$ when we can represent it using these two vectors. In other words, if we can find values "a" and "b" such that $\vec{V} = a\,\vec{V_1} + b\,\vec{V_2}$, we can conclude that vector $\vec{V}$ can be expressed in relation to vectors $\vec{V_1}$ and $\vec{V_2}$ and is situated within the specified plane.

Example

We consider the following vectors:

$$\vec{V_1} = 2\vec{\imath} + \vec{\jmath}\vec{V_2} = \vec{\imath} + 3\vec{\jmath} + \vec{k}\vec{V_3} = 4\vec{\imath} + 7\vec{\jmath} + 2\vec{k}$$

To prove that the vector $\vec{V_3}$ belongs to the plane (P) formed by vectors $\vec{V_1}$ and $\vec{V_2}$

$$\vec{V_3} = a\,\vec{V_1} + b\,\vec{V_2} = 2a\,\vec{\imath} + a\,\vec{\jmath} + b\,\vec{\imath} + 3b\,\vec{\jmath} + b\,\vec{k} = (2a+b)\,\vec{\imath} + (a+3b)\,\vec{\jmath} + b\,\vec{k}$$

$$(2a+b)\vec{\imath} + (a+3b)\vec{\jmath} + b\,\vec{k} = 4\vec{\imath} + 7\vec{\jmath} + 2\vec{k} \Rightarrow \begin{cases} 2a+b = 4 \\ a+3b = 7 \\ b = 2 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 2 \end{cases}$$

$\vec{V_3} = \vec{V_1} + 2\,\vec{V_2}$ *we can write $\vec{V_3}$ as function of both $\vec{V_1}$ and $\vec{V_2}$, thus, the vector $\vec{V_4}$ belongs to the plane (P)*

7-2) The linear dependence of two vectors

We say that two vectors are linearly dependent if one of them can be expressed as the scalar product of the other by a real number. In other term, the vectors $\vec{A}$ *and* $\vec{B}$ *are linearly dependent if there exists a real number* $\boldsymbol{\alpha}$ *that satisfies the following relationship:*$\vec{A} = \alpha.\ \vec{B}$

Example:

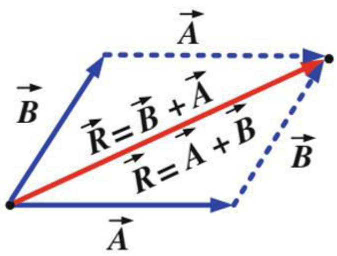
We consider the following vectors:$\vec{A} = \vec{\imath} + 2\vec{\jmath} + \vec{k}\vec{B} = 4\vec{\imath} + 8\vec{\jmath} + 2\vec{k}$

- *Verify whether the two vectors are linearly dependent*

$\vec{A} = \alpha\,\vec{B} \Rightarrow \alpha\left(\vec{\imath} + 2\vec{\jmath} + \vec{k}\right) = 4\vec{\imath} + 8\vec{\jmath} + 2\vec{k} \Rightarrow \begin{cases} \alpha = 4 \\ 2\,\alpha = 8 \\ \alpha = 2 \end{cases} \Rightarrow = \begin{matrix} \alpha = 4 \\ \alpha = 4 \\ \alpha = 2 \end{matrix}$ *Therefore, It can be said that these two vectors are not linearly dependent, as there is no single real number that satisfies the relation* $\vec{A} = \alpha\,\vec{B}$

8) Operations on vectors and their properties

8-1) Vector Addition:



The sum of two vectors $\vec{A}$ and $\vec{B}$ *is a vector with its tail at the beginning of the first vector* $\vec{A}$*and its head at the end of the second vector*$\vec{B}$*, while its length is obtained as follows:*

$$\vec{R} = \vec{A} + \vec{B} \Rightarrow (\vec{R})^2 = (\vec{A} + \vec{B})^2 = (\vec{A})^2 + (\vec{B})^2 + 2\,(\vec{A}.\vec{B})$$

$\|\vec{R}\|^2 = \|\vec{A}\|^2 + \|\vec{B}\|^2 + 2\,\|\vec{A}\|.\|\vec{B}\|\,\cos(\vartheta)$ *where* $\boldsymbol{\vartheta}$ *denotes the angle enclosed between* $\vec{A}$ and $\vec{B}$ *vectors*

$$\|\vec{R}\| = \sqrt{\|\vec{A}\|^2 + \|\vec{B}\|^2 + 2\,\|\vec{A}\|.\|\vec{B}\|\,\cos(\vartheta)}$$

Geometrically, *vectors can be added by parallel displacement of the second vector to the first vector, such that the end of the first vector (the head) matches the starting point of the second vector (the tail). The resulting vector from the addition has its starting point at the origin of the first vector and its endpoint at the head of the second vector, as shown in the following figure where* $\vec{R} = \vec{A} + \vec{B}$

Analytically:

Let us consider $\vec{A}$ *and* $\vec{B}$ *are two vectors have their components in the Cartesian coordinates system (O, XYZ), where "O" is the origin, and "XYZ" represents the three axes with three basis orthogonal unit vectors* $(\vec{\imath}, \vec{\jmath}, \vec{k})$ *that correspond to the three main coordinate axes. We can sum* $\vec{A}$ *and* $\vec{B}$ *analytically by adding all the components multiplied by the same unit vector as follows:*

$$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k} \qquad \vec{B} = B_x\vec{\imath} + B_y\vec{\jmath} + B_z\vec{k}$$

$$\vec{A} + \vec{B} = (A_x + B_x)\vec{\imath} + (A_y + B_y)\vec{\jmath} + (A_z + B_z)\vec{k}$$

We can use another writing shape for vectors sum

$$\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix} \quad \text{and} \quad \vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix} \Rightarrow \vec{A} + \vec{B}\begin{pmatrix} A_x + B_x \\ A_y + B_y \\ A_z + B_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix}$$

The magnitude of $(\vec{A} + \vec{B})$ *vector is given as follow:*

$$\|\vec{A} + \vec{B}\| = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2 + (A_z + B_z)^2}$$

To find the coordinates of the starting point of the vector sum of vectors $\vec{A}$ *and* $\vec{B}$*, where vector* $\vec{A}$ *has its tail at point* (x_1, y_1) *and its head at point* (x_2, y_2)*, and vector* $\vec{B}$ *has its tail at point* (x_3, y_3) *and its head at point* (x_4, y_4)*, the resultant of their addition, vector* $\vec{C} = \vec{A} + \vec{B}$*, will have its starting coordinates at* (x_1+x_3, y_1+y_3) *and its head at* $(x_2 + x_4, y_2 + y_4)$*.*

Important Note:

1- *Subtracting two vectors is a special case of vector addition. Geometrically, we add the first vector to the negative of the second vector. Analytically, it is expressed as follows:*

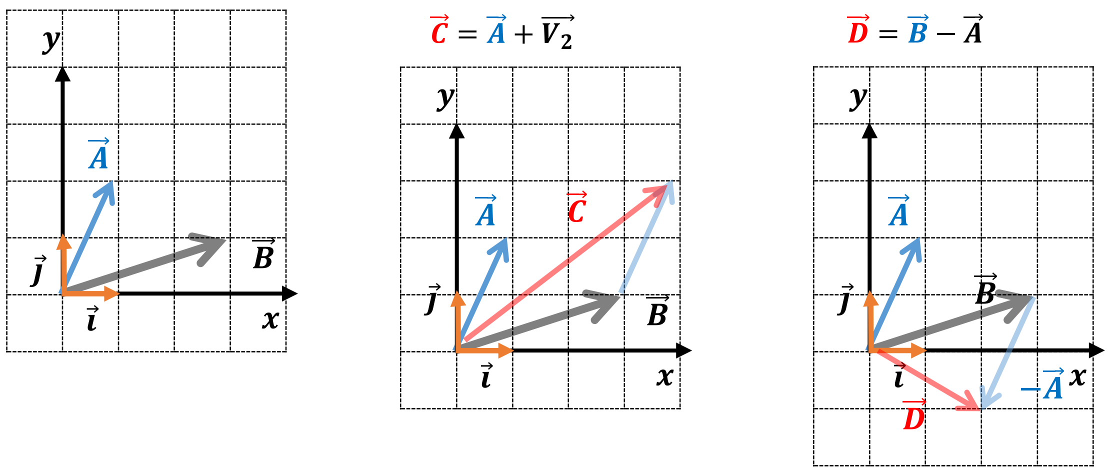
$$\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix} \quad and \quad \vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix} \Rightarrow \vec{A} - \vec{B} \begin{pmatrix} A_x - B_x \\ A_y - B_y \\ A_z - B_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix}$$

- ✓ *Vector addition has the commutative property* $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
- ✓ *Vector addition has the associative property* $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$
- ✓ *Vector addition has the distributive property* $\lambda \times (\vec{A} + \vec{B}) = \lambda \times \vec{A} + \lambda \times \vec{B}$

Example:

We consider the following vectors: $\vec{A} = 3\vec{\imath} + 1\,\vec{\jmath}\,\vec{B} = 1\vec{\imath} + 2\,\vec{\jmath}$

Using the graphical and analytical methods, find the sum and subtraction of these vectors



Addition **Subtraction**

Analytically:

$$\vec{C} = \vec{A} + \vec{B} = 3\vec{\imath} + 1\,\vec{\jmath} + 1\vec{\imath} + 2\,\vec{\jmath} = 4\vec{\imath} + 3\,\vec{\jmath}$$

$$\vec{D} = \vec{B} - \vec{A} = 1\vec{\imath} + 2\,\vec{\jmath} - (3\vec{\imath} + 1\,\vec{\jmath}) = -2\vec{\imath} + 1\,\vec{\jmath}$$

8-2) Scalar (Dot) Product, Vector (Cross) Product, And Mixed Product

8-2.1) Scalar (Dot) Product

- ✓ *The scalar (dot) product of two vectors* $\vec{A}$ and $\vec{B}$ *is denoted* $\vec{A} \cdot \vec{B}$
- ✓ *The scalar (dot) product of two vectors* $\vec{A}$ and $\vec{B}$*produces a* **scalar**.
- ✓ *The scalar product of vectors* $\vec{A}$ and $\vec{B}$ *is given in terms of their magnitudes and the angle (*ϑ*) enclosed between them as follows:*

$$\vec{A} . \vec{B} = \|\vec{A}\| . \|\vec{B}\| \cos(\vartheta)$$

We can distinguish three special cases based on the value of the angle

$$\vec{A}.\vec{B} = \|\vec{A}\|.\|\vec{B}\| \cos(\vartheta) = \begin{cases} \|\vec{A}\|.\|\vec{B}\| \ if\ \vartheta = 0\ \vec{A} \text{ and } \vec{B} \ \ are\ parallel\ vectors \\ -\|\vec{A}\|.\|\vec{B}\| \ if\ \vartheta = \pi\ \vec{A} \text{ and } \vec{B} \ \ are\ anti\ parallel\ vectors \\ 0\ \ if\ \vartheta = \frac{\pi}{2} \ \ so\ \vec{A} \perp \vec{B} \end{cases}$$

- ✓ *The scalar product of vectors* $\vec{A}$ and $\vec{B}$ *is given in terms of the components of* $\vec{A}$ and $\vec{B}$ *as follows:*

$$\vec{A}.\vec{B} = (A_x \times B_x) + (A_y \times B_y) + (A_z \times B_z)$$

- ✓ *The scalar product has the commutative property* $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- ✓ *The scalar product has the distributive property* $\vec{C} \cdot (\vec{A} + \vec{B}) = (\vec{C} \cdot \vec{A}) + (\vec{C} \cdot \vec{B})$
- ✓ *The scalar product of the unit vector is given as follow:*

$$\vec{\imath}.\vec{\jmath} = \vec{\imath}.\vec{\jmath} = \vec{\imath}.\vec{k} = \vec{k}.\vec{\imath} = \vec{k}.\vec{\jmath} = \vec{\jmath}.\vec{k} = 0$$

$$\vec{\imath}.\vec{\imath} = \vec{\jmath}.\vec{\jmath} = \vec{k}.\vec{k} = 1$$

- ✓ *If the components of vectors* $\vec{A}$ and $\vec{B}$ *are time-dependent, the derivative of their scalar product is expressed as follows:*

$$\frac{d(\vec{A} \cdot \vec{B})}{dt} = \frac{d(\vec{A})}{dt} \cdot \vec{B} + \vec{A} \cdot \frac{d(\vec{B})}{dt}$$

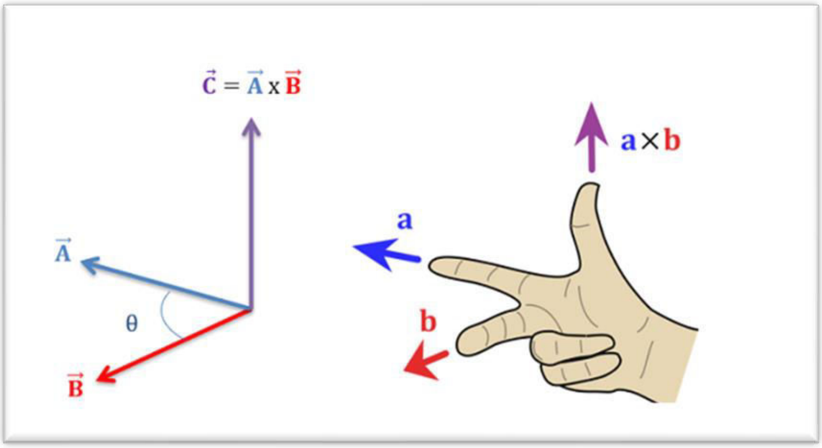
8-2.2) Cross Product

- ✓ *The Vector (Cross) product of two vectors* $\vec{A}$ and $\vec{B}$ *is denoted* $\vec{A} \wedge \vec{B}$
- ✓ *The Vector (Cross) product of two vectors* $\vec{A}$ and $\vec{B}$ *produces a* ***vector*** *that is perpendicular on the plane formed by* $\vec{A}$ and $\vec{B}$*, and its direction is determined using* ***the right-hand rule****. Its magnitude is given in terms of is given in term of* $\vec{A}$ and $\vec{B}$ *magnitudes and the angle (**ϑ**) enclosed between them as follows:*

$$\|\vec{C}\| = \|\vec{A} \wedge \vec{B}\| = \|\vec{A}\|.\|\vec{B}\| \sin(\vartheta)$$

We can distinguish two special cases based on the value of the angle

$$\|\vec{A} \wedge \vec{B}\| = \|\vec{A}\|.\|\vec{B}\| \sin(\vartheta) = \begin{cases} 0 \quad if\ \ \vec{A} \text{ and } \vec{B}\ are\ parallel\ vectors\ \ (\vec{A} \,|\, |\vec{B}) \\ \pm\|\vec{A}\|.\|\vec{B}\| \quad if\ \ \vec{A} \text{ and } \vec{B}\ are\ orthogonal\ vectors\ (\vec{A} \perp \vec{B}) \end{cases}$$



- ✓ *The vector(Cross) product has not the commutative property* $\vec{A} \wedge \vec{B} = -\vec{B} \wedge \vec{A}$
- ✓ *The vector(Cross) product has the distributive property* $\vec{C} \wedge (\vec{A} + \vec{B}) = (\vec{C} \wedge \vec{A}) + (\vec{C} \wedge \vec{B})$
- ✓ *The vector(Cross) product of vectors* $\vec{A}$ and $\vec{B}$ *is given in terms of the determinant as follows:*

$$\vec{\boldsymbol{A}} \wedge \vec{\boldsymbol{B}} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \vec{\imath} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \vec{\jmath} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \vec{k}$$

$$\vec{\boldsymbol{A}} \wedge \vec{\boldsymbol{B}} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - B_y A_z)\vec{\imath} - (A_x B_z - B_x A_z)\vec{\jmath} + (A_x B_y - B_x A_y)\vec{k}$$

- ✓ *The vector (Cross) product of the unit vector is given as follow:*

$$\vec{\imath} \wedge \vec{\jmath} = -\vec{\jmath} \wedge \vec{\imath} = \vec{k}$$
$$\vec{\jmath} \wedge \vec{k} = -\vec{k} \wedge \vec{\jmath} = \vec{\imath}$$
$$\vec{k} \wedge \vec{\imath} = -\vec{\imath} \wedge \vec{k} = \vec{\jmath}$$
$$\vec{\imath} \wedge \vec{\imath} = \vec{\jmath} \wedge \vec{\jmath} = \vec{k} \wedge \vec{k} = \vec{0}$$

- ✓ *If the components of vectors* $\vec{A}$ and $\vec{B}$ *are time-dependent, the derivative of the vector (cross) product is expressed as follows:*

$$\frac{d(\vec{A} \wedge \vec{B})}{dt} = \frac{d(\vec{A})}{dt} \wedge \vec{B} + \vec{A} \wedge \frac{d(\vec{B})}{dt}$$

8-2.3) Mixed Product

The result of the mixed product of vectors is a scalar value, calculated by the determinant as follows:

$$\vec{C}\cdot(\vec{A}\wedge\vec{B}) = \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} C_x - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} C_y + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} C_z$$

$$\vec{C}\cdot(\vec{A}\wedge\vec{B}) = (A_yB_z - B_yA_z)C_x - (A_xB_z - B_xA_z)C_y + (A_xB_y - B_xA_y)C_z$$

9) Perpendicular and Parallel Vectors

Perpendicular Vectors ($\vec{A}\perp\vec{B}$): *Two nonzero vectors* $\vec{A}$ ***and*** $\vec{B}$ *are said to be perpendicular (orthogonal) if their scalar (dot) product is equal to zero. Mathematically, this condition is expressed as* $\vec{A}.\vec{B} = \|\vec{A}\|.\|\vec{B}\| \cos(\theta) = 0$, *which can also be written in term of components as* $A_x B_x+A_y B_y+A_z B_z=0$, *where* A_x, A_y, A_z *and* B_x, B_y, B_z *are the respective components of vectors* $\vec{A}$ ***and*** $\vec{B}$ *along the x, y, and z axes. This indicates that the angle between the two vectors is 90 degrees* ($\theta = \frac{\pi}{2} \Rightarrow \cos(\theta) = 0$) , *meaning they are orthogonal in space.*

Parallel Vectors ($\vec{A} \parallel \vec{B}$):

two nonzero vectors $\vec{A}$ ***and*** $\vec{B}$ *are considered to be parallel if their cross product equals zero. In mathematical terms, this condition is written as* $\vec{A}\wedge\vec{B} = \vec{0}$ *which can be also written in term of their components as* $(A_yB_z - B_yA_z)\vec{\imath} - (A_xB_z - B_xA_z)\vec{\jmath} + (A_xB_y - B_xA_y)\vec{k} = \vec{0}$. *When the cross product is zero, it implies that the vectors are either pointing in the same direction or in exactly opposite directions, making them linearly dependent. Parallel vectors lie along the same or directly opposite lines of action, meaning the angle between them is either 0 or 180 degrees.*

Important note: *this equation* $(A_yB_z - B_yA_z)\vec{\imath} - (A_xB_z - B_xA_z)\vec{\jmath} + (A_xB_y - B_xA_y)\vec{k} = \vec{0}$

is only satisfied when these conditions are met together:

$$\begin{cases} A_yB_z - B_yA_z = 0 \Rightarrow A_yB_z = B_yA_z \Rightarrow \dfrac{A_y}{B_y} = \dfrac{A_z}{B_z} \\ A_xB_z - B_xA_z = 0 \Rightarrow A_xB_z = B_xA_z \Rightarrow \dfrac{A_x}{B_x} = \dfrac{A_z}{B_z} \\ A_xB_y - B_xA_y = 0 \Rightarrow A_xB_z = B_xA_z \Rightarrow \dfrac{A_x}{B_x} = \dfrac{A_z}{B_z} \end{cases} \Rightarrow \frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z} = \lambda$$

$$\vec{A} = \lambda.\vec{B}$$

Accordingly, the two vectors $\vec{A}$ and $\vec{B}$ are parallel if there is a number λ that satisfies the condition of linear dependence $\vec{A} = \lambda.\ \vec{B}$ or $\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z} = \lambda$

9) Other operations on vectors

In addition to the well-known operations such as vector addition, subtraction, and multiplication (either by a scalar or vector, whether dot product or cross product), other operations can be performed on vectors such as integration, derivation, rotation, divergence, and the application of certain operators.

9-1) Derivative of Vector

Let $\vec{\boldsymbol{A}}$ be a vector whose three components are expressed in terms of the unit vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$ as a function of the scalar variable t as follows:

$$\vec{A}(t) = A_x(t)\vec{\imath} + A_y(t)\,\vec{\jmath} + A_z(t)\vec{k}$$

The derivative of the vector $\vec{\boldsymbol{A}}$ with respect to the scalar variable $\boldsymbol{t}$ is given as follows:

$$\frac{d\vec{A}(t)}{dt} = \frac{dA_x(t)}{dt}\,\vec{\imath} + \frac{dA_y(t)}{dt}\,\vec{\jmath} + \frac{dA_z(t)}{dt}\vec{k}$$

9-2) vector integral

Let $\vec{\boldsymbol{B}}$ be a vector whose three components are expressed in terms of the unit vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$ as a function of the scalar variable t as follows:

$$\vec{B}(t) = B_x(t)\vec{\imath} + B_y(t)\,\vec{\jmath} + B_z(t)\vec{k}$$

Vector integration is the process of integrating each component of the vector separately.

Then the integral with respect to the scalar variable t is calculated as:

$$\int \vec{B}(t)\,dt = \int B_x(t)dt\ \vec{\imath} + \int B_y(t)dt\ \vec{\jmath} + \int B_z(t)dt\,\vec{k}$$

9-3) Gradient, Divergence, and Curl

9-3-1) Gradient of scalar field

- *The gradient operator ($\overrightarrow{grad}$ or $\vec{\nabla}$) is a **differential vector operator** and is defined as follows:*

$$\overrightarrow{grad} = \vec{\nabla} = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$$

$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$ *are partial derivatives*

- *The gradient operator is used to analyze how a scalar field f(x, y, z) varies in various directions in space.*
- *Applying the gradient operator to a scalar quantity gives a vector field with the following relation:*

$$\overrightarrow{grad}f(x,y,z) = \vec{\nabla}f(x,y,z) = \frac{\partial f(x,y,z)}{\partial x}\vec{\imath} + \frac{\partial f(x,y,z)}{\partial y}\vec{\jmath} + \frac{\partial f(x,y,z)}{\partial z}\vec{k}$$

in this equation ,

$\frac{\partial f(x,y,z)}{\partial x} \Rightarrow$ *is the partial derivative of the scalar field function with respect to the variable **x**, with the variables **y** and **z** considered as constants.*

$\frac{\partial f(x,y,z)}{\partial y} \Rightarrow$ *is the partial derivative of the scalar field function with respect to the variable **y**, with the variables **x** and **z** considered as constants.*

$\frac{\partial f(x,y,z)}{\partial z} \Rightarrow$ *is the partial derivative of the scalar field function with respect to the variable **z**, with the variables **x** and **y** considered as constants.*

Example:

The expression for the field function is given as follows:

$P(x,y,z) = P_0 + \rho g z$ *(P_0, ρ, and g are constants)*

- Calculate the change of this field $P(x,y,z)$ in the different directions of space (calculate the gradient of this field)

$$\vec{\nabla}P(x,y,z) = \frac{\partial P(x,y,z)}{\partial x}\vec{\imath} + \frac{\partial P(x,y,z)}{\partial y}\vec{\jmath} + \frac{\partial P(x,y,z)}{\partial z}\vec{k}$$

$$\begin{cases} \dfrac{\partial P(x,y,z)}{\partial x} = 0 \\ \dfrac{\partial P(x,y,z)}{\partial y} = 0 \\ \dfrac{\partial P(x,y,z)}{\partial z} = \rho g \end{cases}$$

$$\vec{\nabla} P(x,y,z) = \rho g \vec{k}$$

Exercise (home work):

1- *calculate the gradient of these scalar fields:*

- ✓ $U(x,y,z) = \frac{k}{\sqrt{x^2+y^2+z^2}}$ *K = constant*
- ✓ $f(x,y,z) = 2x^2 + y^3 + z$
- ✓ $T(x,y,z) = y^2x^2 + yz$

9-3-2) Divergence of Vector field

Divergence is denoted by $\overrightarrow{div}$, *and is applied to a vector field function* $\vec{V} = V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}$ *and its result is a scalar value and its general meaning is a measure of the flow or spread of the vector field to or from a certain region of space. The expression for the divergence of a vector field is given by the scalar product between the Nabla operator* $\vec{\nabla}$ *and the vector field as follows:*

$$\overrightarrow{div}.\vec{V} = \vec{\nabla}.\vec{V} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right).\left(V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}\right) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Example:

The expression for the vector field $\vec{V}$ *is given as follows:*

$$\vec{V} = x^2\,\vec{\imath} + y^2\,\vec{\jmath} + z^2\,\vec{k}$$

**- Calculate the divergence of this vector field* $\vec{V}$ *at the point (0 , 1 , 1)*

$$\overrightarrow{div}.\vec{V} = \vec{\nabla}.\vec{V} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right).\left(V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}\right) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\begin{cases} \frac{\partial V_x}{\partial x} = 2x \\ \frac{\partial V_y}{\partial y} = 2y \\ \frac{\partial V_z}{\partial z} = 2z \end{cases}$$

at the point (0 , 1 , 1) $\Rightarrow \overrightarrow{div}.\vec{V} = 0 + 2 + 2 = 4$

9-3-3) The curl of Vector field

Rotation of a vector field is denoted by $\overrightarrow{Rot}$*, and is applied to a vector field function* $\vec{V} = V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}$. *It expresses the rotation of this vector field about a given point. The expression for the rotation of the vector field about a given point is given by the vector (cross) product between the operator Naples and the vector field, and is written as follows:*

$$\overrightarrow{Rot}\,\vec{V} = \vec{\nabla} \wedge \vec{V} = curl\,\vec{V} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

Example

The expression for the vector field $\vec{V}$ *is given as follows:*

$$\vec{V} = x^2\,\vec{\imath} - 2xy\,\vec{\jmath} + z\,\vec{k}$$

**- Calculate the curl (rotation) of this vector field* $\vec{V}$ *at the point (1 , 1 , 1)*

$$curl\,\vec{V} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\vec{\imath} + \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right)\vec{\jmath} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\vec{k}$$

$V_x = x^2$, $V_y = -2xy$ *and* $V_z = z$

$$\begin{cases} \frac{\partial V_x}{\partial y} = 0 \\ \frac{\partial V_x}{\partial z} = 0 \end{cases} \quad \begin{cases} \frac{\partial V_y}{\partial x} = -2y \\ \frac{\partial V_y}{\partial z} = 0 \end{cases} \quad \begin{cases} \frac{\partial V_z}{\partial x} = 0 \\ \frac{\partial V_z}{\partial y} = 0 \end{cases}$$

$curl\,\vec{V} = -2y\vec{k}$

at the point (1 , 1 , 1) $\Rightarrow curl\,\vec{V} = -2\vec{k}$

10) Series of Solved Exercises

Exercise01:

Consider the following vectors:

$$\vec{A} = 4\vec{\imath} + 2\,\vec{\jmath}\vec{B} = 2\vec{\imath} + \vec{\jmath} - \vec{k}$$

2- *Calculate the magnitude of* $\vec{A}$ *and* $\vec{B}$, *the dot product* $\vec{A}.\vec{B}$*and the cross product* $\vec{A}\Lambda\vec{B}$

3- *Calculate the angle* θ *formed by the two vectors*

Solution:

1- *Calculate the magnitude of* $\vec{A}$ *and* $\vec{B}$, *the dot product* $\vec{A}.\vec{B}$*and the cross product* $\vec{A}\Lambda\vec{B}$

$$\|\vec{A}\| = \sqrt{(4)^2 + (2)^2} = \sqrt{20}$$

$$\|\vec{B}\| = \sqrt{(2)^2 + (1)^2 + (-1)^2} = \sqrt{6}$$

$$\vec{A}\,.\,\vec{B} = \ 4 * 2 + 2 * 1 + 0 * (-1) = 10$$

$$\vec{\boldsymbol{A}} \wedge \vec{\boldsymbol{B}} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 4 & 2 & 0 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix}\vec{\imath} - \begin{vmatrix} 4 & 0 \\ 2 & -1 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix}\vec{k} = -2\vec{\imath} - 4\vec{\jmath} + 0\,\vec{k}$$

2- *Calculate the angle* θ *formed by the two vectors*

$$\vec{A}\,.\,\vec{B} = \|\vec{A}\| \times \|\vec{B}\|\,cos(\vartheta) \ = 10$$

$$\|\vec{A}\| \times \|\vec{B}\|\,cos(\vartheta) \ = \frac{\vec{A}\,.\,\vec{B}}{\|\vec{A}\| \times \|\vec{B}\|} = \frac{10}{\sqrt{20 \times 6}} = 0.912$$

$$\vartheta = 24.21^0$$

Exercise02:

Given that $\vec{\boldsymbol{A}} = 2\boldsymbol{\imath} - 3\boldsymbol{\jmath} - \boldsymbol{k}, \vec{\boldsymbol{B}} = 3\boldsymbol{\imath} - \boldsymbol{\jmath}$ *and* $\vec{\boldsymbol{C}} = \boldsymbol{\jmath} - 4\boldsymbol{k}$, *find the expression of :*

(1) $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}$

(2)$(\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}) \times \vec{\boldsymbol{C}}$

(3) $\vec{\boldsymbol{A}} \cdot (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}})$

Solution:

1) $\vec{A} \times \vec{B} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & -3 & -1 \\ 3 & -1 & 0 \end{vmatrix} = -\vec{\imath} - 3\vec{\jmath} + 7\vec{k}$

2) $(\vec{A} \times \vec{B}) \times \vec{C} == (-\vec{\imath} - 3\vec{\jmath} + 7\vec{k}) \times (\vec{\jmath} - 4\vec{k}) = 5\vec{\imath} - 4\vec{\jmath} - \vec{k}$

3) $(\vec{B} \times \vec{C}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3 & -1 & 0 \\ 0 & 1 & -4 \end{vmatrix} = 4\vec{\imath} + 12\vec{\jmath} + 3\vec{k}$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (2\vec{\imath} - 3\vec{\jmath} - \vec{k}).(4\vec{\imath} + 12\vec{\jmath} + 3\vec{k}) = 8 - 36 - 3 = -31$$

Exercise03:

Prove that these three vectors are perpendicular?

$\vec{A} = -2\vec{\imath} + \vec{\jmath}$, $\vec{B} = -3\vec{k}$ *and* $\vec{C} = -\vec{\imath} - 2\vec{\jmath}$

Solution:

$\vec{A}.\vec{B} = (-2 \times 0) + (1 \times 0) + (0 \times (-3)) = 0 \quad \Rightarrow \vec{A} \perp \vec{B}$

$\vec{A}.\vec{C} = (-2 \times (-1)) + (1 \times (-2)) = 0 \quad \Rightarrow \vec{A} \perp \vec{C}$

$\vec{B}.\vec{C} = (0 \times -1) + (0 \times (-2)) + (-3 \times 0) = 0 \quad \Rightarrow \vec{B} \perp \vec{C}$

Exercise 05:

Which of these Vectors are perpendicular and which are parallel?

$\vec{A} = -2\vec{\imath} + 2\vec{k} \quad \vec{B} = -2\vec{\jmath} \quad \vec{C} = -3\vec{\imath} - 3\vec{\jmath} \quad \vec{D} = \vec{\imath} - \vec{k} \quad \vec{E} = \frac{2}{3}\vec{\imath} - \frac{2}{3}\vec{k}, \quad \vec{f} = \vec{\imath} + 5\vec{k}$

Solution:

$\vec{A}, \vec{D}, \vec{f}$ *and* $\vec{E}$ *are vectors written in terms of the same unit vectors* $\vec{\imath}$ *and* $\vec{k}$ *, so we can check their linear dependence (parallelism condition using this rule* $\frac{A_x}{D_x} = \frac{A_y}{D_y} = \frac{A_z}{D_z} = \lambda$

$\frac{A_x}{D_x} = \frac{-2}{1} = -2, \frac{A_z}{D_z} = \frac{2}{-1} = -2 \Rightarrow \frac{A_x}{D_x} = \frac{A_z}{D_z} = \lambda \Rightarrow \vec{A} = -2\vec{D} \Rightarrow \vec{A} \parallel \vec{D}$ *(* $\vec{A}$ *and* $\vec{D}$ *parallel vectors*

$\frac{A_x}{f_x} = \frac{-2}{1} = -2, \frac{A_z}{f_z} = \frac{2}{5} = \frac{2}{5} \Rightarrow \frac{A_x}{D_x} \neq \frac{A_z}{D_z} \Rightarrow \vec{A}$ *and* $\vec{f}$, $\vec{D}$ *and* $\vec{f}$, $\vec{E}$ *and* $\vec{f}$ *are unparallel vectors*

$\frac{A_x}{E_x} = \frac{-2}{2/3} = -3, \frac{A_z}{E_z} = \frac{2}{-2/3} = -3 \Rightarrow \frac{A_x}{D_x} = \frac{A_z}{D_z} = -3 \Rightarrow \vec{A} = -3\vec{D} \Rightarrow \vec{A} \parallel \vec{D}$

($\vec{A}$ *and* $\vec{E}$ *parallel vectors*

$\vec{A}.\vec{f} = -2\times 1 + 2\times 5 = 8 \neq 0 \Rightarrow \vec{A}$ and $\vec{f}$, $\vec{D}$ and $\vec{f}$, $\vec{E}$ and $\vec{f}$ are Non-perpendicular vectors

$\vec{A}.\vec{B} = -2\times 0 + 2\times 0 + 2\times 0 = 0 \Rightarrow \vec{A}$ and $\vec{B}$, $\vec{D}$ and $\vec{B}$, $\vec{E}$ and $\vec{B}$ are perpendicular vectors

$\vec{A}.\vec{C} = -2\times(-3) + (-3)\times 0 + 2\times 0 = +6 \neq 0$ $\vec{A}$ and $\vec{C}$, $\vec{D}$ and $\vec{C}$, $\vec{E}$ and $\vec{C}$ are Non-perpendicular vectors

10) Additional exercises

FIRST PART: DEMENSIONAL ANALYSIS

EXERCISE 01:

Check the homogeneity of the following equations:

- ✓ $C = P + \rho.g.z$ *In which P represents pressure, p stands for density, z denotes height, and C remains a constant.*
- ✓ $2(x_0 - vt) = gt^2 \sin(\theta)$
- ✓ $v = -\frac{f}{R}gt + \sqrt{2Lg\sin(\theta)}$

Where x_0 *is the initial position, v is velocity, L is distance, f and R are reaction forces,* θ *is an angle, and t and T are times.*

Solution:

Check the homogeneity of the following equations:

1) $C = P + \rho.g.z$

This equation exhibits dimensional non-consistency

$$[C] = 1 \neq [P] = \frac{[F]}{[S]} = \frac{[m][a]}{[S]} = \frac{M\,L\,T^{-2}}{L^2} = M\,L^{-1}T^{-2}$$

2) $2(x_0 - vt) = gt^2 \sin(\theta)$

This equation exhibits dimensional consistency

$$[2] = 1\,;\ [x_0] = L\,;\ [vt] = L\,T^{-1}\,T = L$$

$$[gt^2 \sin(\theta)] = [g][t^2][\sin(\theta)] = L\,T^{-2}T^2 = L$$

3) $v = -\frac{f}{R}gt + \sqrt{2Lg\sin(\theta)}$

This equation exhibits dimensional consistency

$$[v] = L\,T^{-1}$$

$$\left[\frac{f}{R}\,gt\right] = \frac{[f]}{[R]}[g][t] = L\,T^{-2}T = L\,T^{-1}$$

$$\left[\sqrt{2Lgsin(\theta)}\right] = \sqrt{[2Lgsin(\theta)]} = \sqrt{[g][L][sin(\theta)]} = \sqrt{L\,T^{-2}L} = L\,T^{-1}$$

EXERCISE 02 :

Consider the physical quantities s, v, a and t with dimensions [s]= L , [v]= LT^{-1}, [a]= L T^{-2} , and [t]= T. Check whether each of the following equations is dimensionally consistent:

- *s= v t +0.5 a t^2*
- *s = vt^2+0.5 a.t*
- *v = sin (at^2/s)*

Solution:

1) *s= v t +0.5 a t^2*

This equations is dimensionally consistent

$$[s] = L \;\; ; [vt] = LT^{-1}\,T = L\,; [0.5at^2] = LT^{-2}T^2 = L$$

2) *s = vt^2+0.5 a.t*

This equations is not dimensionally consistent

$$[s] = L; [vt^2] = LT^{-1}T^2 = LT\;; [0.5at] = LT^{-2}T = LT^{-1}$$

3) *v = sin (at^2/s)*

This equations is not dimensionally consistent

$$[v] = LT^{-1}; \left[sin\left(\frac{at^2}{s}\right)\right] = 1$$

EXERCISE 03:

Determine the dimension of the variable 'X' that achieve dimensional consistency for the equation, given that 'h' represents height, "v" is the velocity and 'm' represents mass.

$$\frac{1}{2}m\,v^2 = m\,X\,h$$

Solution:

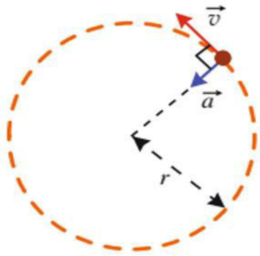
$$\frac{1}{2}m\,v^2 = m\,X\,h$$

$$X = \frac{v^2}{h}$$

$$[X] = \frac{[v^2]}{[h]} = \frac{(LT^{-1})^2}{L} = LT^{-2}$$

EXERCISE 04:

A particle moves with a constant velocity v in a circular orbit of a radius r as shown in the facing figure. The magnitude of its acceleration is proportional to some power of $r(r^n)$and some power of v (v^m). Determine both powers n and m of r and v respectively.



Solution:

$$a = k\, r^n v^m$$

$$[a] = [k][r^n][v^m]$$

$$LT^{-2} = L^n\,(LT^{-1})^m = L^{n+m}T^{-m}\begin{cases}-m = -2\\ n+m = 1\end{cases}\begin{cases}m = 2\\ n = -1\end{cases}$$

$$a = k\frac{v^2}{r}$$

EXERCISE 05:

The volumetric density ρ of a cylinder with mass (m), radius (R), and length (l) is given by the following equation:

$$\rho = \pi^{-1}\, m^x\, l^y\, R^{-2}$$

- Using the dimensions method, calculate the two constant values of x and y , and rewrite the precise expression of the density:

Solution:

$$[\rho] = [\pi^{-1}][l^y][m^x][R^{-2}]$$

$$[\rho] = ML^{-3}, [\pi^{-1}] = 1, [l^y] = L^y, [m^x] = M^x, [R^{-2}] = L^{-2}$$

$ML^{-3} = L^y M^x L^{-2} \Rightarrow x=1$, $y-2=-3 \Rightarrow x=1$ ad $y = -1$

$$\rho = \pi\, m\, l^{-1} R^{-2}$$

$$\rho = \frac{m}{\pi\ l\, R^2}$$

SECOND PART: VECTORS

EXERCISE 01:

Consider the following points: A (1, 1, 1), B (2, -1, 0), and C (0, 2, 2).

1- *Represent these points in a Cartesian coordinates system (O, xyz)*

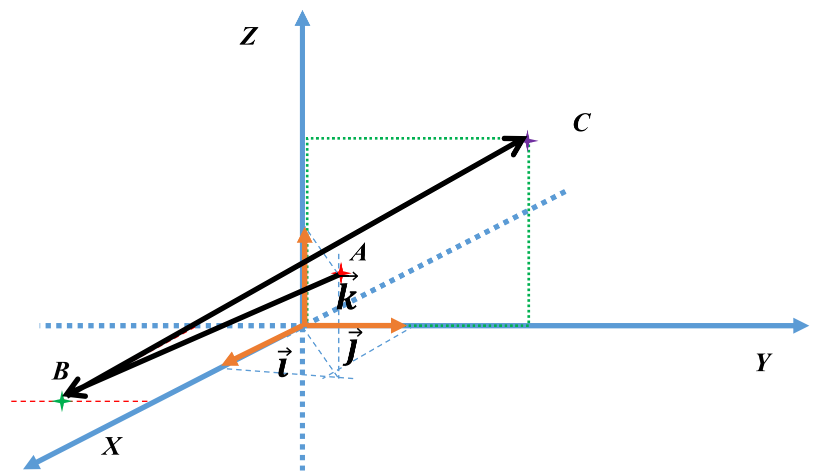
2- *Determine the components of the vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{BC}$

3- *Calculate the angle M between the two vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{BC}$.

Solution:

Consider the following points: A (1, 1, 1), B (2, -1, 0), and C (0, 2, 2).

1) Represent these points in a Cartesian coordinates system (O, xyz)



2) Determine the components of the vectors $\overrightarrow{AB}$ *and* $\overrightarrow{BC}$

$$\overrightarrow{AB} = \begin{pmatrix} B_x - A_x \\ B_y - A_y \\ B_z - A_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix}$$

Or

$$\overrightarrow{AB} = (B_x - A_x)\vec{\imath} + (B_y - A_y)\vec{\jmath} + (B_z - A_z)\,\vec{k} = \vec{\imath} - 2\vec{\jmath} - \vec{k}$$

$$\overrightarrow{BC} = \begin{pmatrix} C_x - B_x \\ C_y - B_y \\ C_z - B_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix}$$

Or

$$\overrightarrow{BC} = (C_x - B_x)\vec{\imath} + (C_y - B_y)\vec{\jmath} + (C_z - B_z)\,\vec{k} = -2\vec{\imath} + 3\vec{\jmath} + 2\,\vec{k}$$

4- *Calculate the angle M between the two vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{BC}$.

$$\overrightarrow{AB}\,.\,\overrightarrow{BC} = \|\overrightarrow{AB}\|.\|\overrightarrow{BC}\|\cos(\vartheta)$$

$$= (B_x - A_x)(C_x - B_x) + (B_y - A_y)(C_y - B_y) + (B_z - A_z)(C_z - B_z)$$

$$\cos(\vartheta) = \frac{(B_x - A_x)(C_x - B_x) + (B_y - A_y)(C_y - B_y) + (B_z - A_z)(C_z - B_z)}{\|\overrightarrow{AB}\|.\|\overrightarrow{BC}\|}$$

$$\|\overrightarrow{AB}\| = \sqrt{(B_x - A_x)^2 + (B_y - A_y)^2 + (B_z - A_z)^2} = \sqrt{6}$$

$$\|\overrightarrow{BC}\| = \sqrt{(C_x - B_x)^2 + (C_y - B_y)^2 + (C_z - B_z)^2} = \sqrt{17}$$

$$\cos(\vartheta) = \frac{-2 - 6 - 2}{\sqrt{6}.\sqrt{17}} = -0.9901 \Rightarrow \vartheta = 171.93^0$$

EXERCISE 02:

1) Using the graphical and analytical methods, find the sum and subtraction of the following vectors

- ✓ $\vec{V_1} = 3\vec{\imath} + 3\vec{\jmath}$ $\vec{V_2} = 2\vec{\imath} + 2\vec{\jmath}$
- ✓ $\vec{V_1} = 3\vec{\imath} + 1\,\vec{\jmath}$ $\vec{V_2} = 1\vec{\imath} + 2\,\vec{\jmath}$

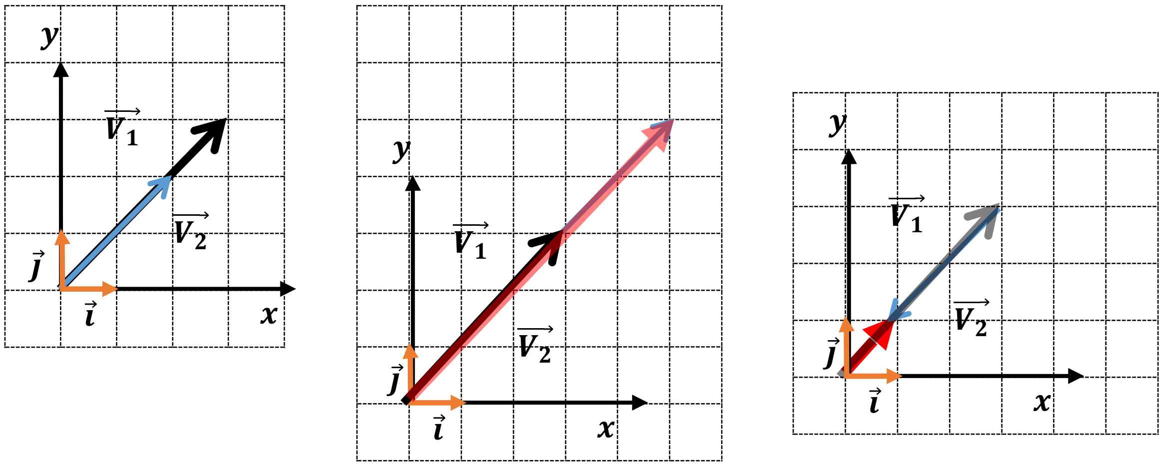
2) Find the angle formed by $\vec{V_1}$ *and* $\vec{V_2}$

3) Calculate the dot (scalar) product and the cross (vector) product of $\vec{V_1}$ *and* $\vec{V_2}$

Solution:

1) Using the graphical and analytical methods, find the sum and subtraction of the following vectors

- ✓ $\vec{V_1} = 3\vec{\imath} + 3\,\vec{\jmath}\vec{V_2} = 2\vec{\imath} + 2\,\vec{\jmath}$

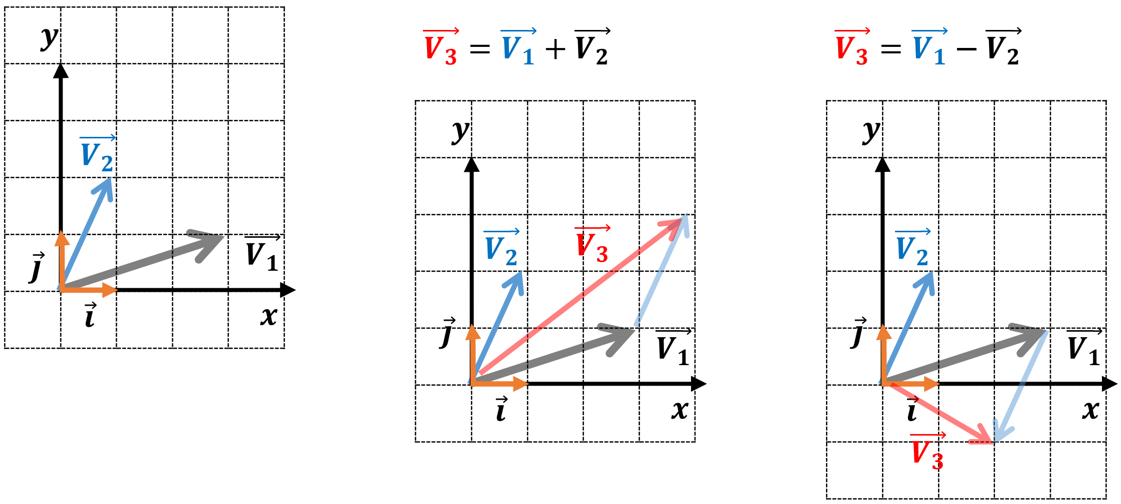


addition *subtraction*

$\vec{V_3} = \vec{V_1} + \vec{V_2} = 3\vec{\imath} + 3\vec{\jmath} + 2\vec{\imath} + 2\vec{\jmath} = 5\vec{\imath} + 5\vec{\jmath}$

$\vec{V_3} = \vec{V_1} + \vec{V_2} = 3\vec{\imath} + 3\vec{\jmath} - (2\vec{\imath} + 2\vec{\jmath}) = \vec{\imath} + \vec{\jmath}$

✓ $\vec{V_1} = 3\vec{\imath} + 1\vec{\jmath}$ $\quad \vec{V_2} = 1\vec{\imath} + 2\vec{\jmath}$



addition *subtraction*

$\vec{V_3} = \vec{V_1} + \vec{V_2} = 3\vec{\imath} + 1\vec{\jmath} + 1\vec{\imath} + 2\vec{\jmath} = 4\vec{\imath} + 3\vec{\jmath}$

$\vec{V_3} = \vec{V_1} + \vec{V_2} = 3\vec{\imath} + 1\vec{\jmath} - (1\vec{\imath} + 2\vec{\jmath}) = 2\vec{\imath} - 1\vec{\jmath}$

2) Find the angle formed by $\vec{V_1}$ *and* $\vec{V_2}$

Geometrically, we can see that the two vector carriers are identical, so the angle between them is zero

Remember this information: *We can assert that two vectors are linearly related when one vector can be represented as a scalar multiple of the other. Mathematically, if vector* $\vec{A}$ *equals vector* $\vec{B}$ *multiplied by a scalar factor 'k,' we express this as*

$\vec{A} = k\,\vec{B}.$

$$\overrightarrow{V_1} = 3\vec{\imath} + 3\,\vec{\jmath} \qquad \overrightarrow{V_2} = 2\vec{\imath} + 2\,\vec{\jmath}$$

$$\text{We can see that } \overrightarrow{V_1} = 3\vec{\imath} + 3\,\vec{\jmath} = \tfrac{3}{2}(2\vec{\imath} + 2\,\vec{\jmath}) = \tfrac{3}{2}\overrightarrow{V_2}$$

So these two vectors are linearly related, these two vectors are parallel, and the angle between them is equal to zero

3) Calculate the cross (vector) product of $\overrightarrow{V_1}$ *and* $\overrightarrow{V_2}$

$$\overrightarrow{V_1}\,.\overrightarrow{V_2} = \left\|\overrightarrow{V_1}\right\|.\left\|\overrightarrow{V_2}\right\|\cos(\vartheta) = (3\vec{\imath} + 3\,\vec{\jmath}).(2\vec{\imath} + 2\,\vec{\jmath}) = V_{1x}V_{2x} + V_{1y}V_{2y}$$

$$\begin{cases} \vec{\imath}.\vec{\jmath} = \vec{\imath}.\vec{\jmath} = 0 \\ \vec{\imath}.\vec{\imath} = \vec{\jmath}.\vec{\jmath} = 1 \end{cases}$$

$$\overrightarrow{V_1}\,.\overrightarrow{V_2} = 6 + 6 = 12$$

We can confirm the angle value

$$\cos(\vartheta) = \frac{\overrightarrow{V_1}\,.\overrightarrow{V_2}}{\left\|\overrightarrow{V_1}\right\|.\left\|\overrightarrow{V_2}\right\|}$$

$$\left\|\overrightarrow{V_1}\right\| = \sqrt{18}$$

$$\left\|\overrightarrow{V_2}\right\| = \sqrt{8}$$

$$\cos(\vartheta) = \frac{12}{\sqrt{8}.\sqrt{18}} 1 \Rightarrow \vartheta = 0^0$$

$$\overrightarrow{V_1} \wedge \overrightarrow{V_2} = (3\vec{\imath} + 3\,\vec{\jmath}) \wedge (2\vec{\imath} + 2\,\vec{\jmath})$$

method 01:

$$\overrightarrow{V_1} \wedge \overrightarrow{V_2} = (3\vec{\imath} + 3\,\vec{\jmath}) \wedge (2\vec{\imath} + 2\,\vec{\jmath}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3 & 3 & 0 \\ 2 & 2 & 0 \end{vmatrix} = \vec{0} \quad \text{We can conduct that}$$

$\overrightarrow{V_1}$and $\overrightarrow{V_2}$*are parallel*

Method 02: cross product is ***distributive***

$$\overrightarrow{V_1} \wedge \overrightarrow{V_2} = (3\vec{\imath} + 3\,\vec{\jmath}) \wedge (2\vec{\imath} + 2\,\vec{\jmath}) = 3\vec{\imath} \wedge (2\vec{\imath} + 2\,\vec{\jmath}) + 3\,\vec{\jmath} \wedge (2\vec{\imath} + 2\,\vec{\jmath})$$

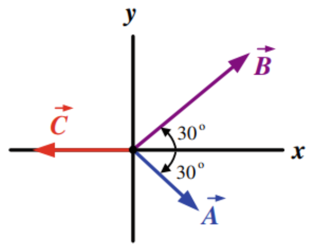
$$\text{depending on the following relations} \begin{cases} \vec{\imath} \wedge \vec{\jmath} = -\vec{\jmath} \wedge \vec{\imath} = \vec{k} \\ \vec{\jmath} \wedge \vec{k} = -\vec{k} \wedge \vec{\jmath} = \vec{\imath} \\ \vec{k} \wedge \vec{\imath} = -\vec{\imath} \wedge \vec{k} = \vec{\jmath} \\ \vec{\imath} \wedge \vec{\imath} = \vec{\jmath}.\wedge \vec{\jmath} = \vec{k} \wedge \vec{k} = \vec{0} \end{cases}$$

$$\overrightarrow{V_1} \wedge \overrightarrow{V_2} = (3\vec{\imath} + 3\,\vec{\jmath}) \wedge (2\vec{\imath} + 2\,\vec{\jmath}) = \vec{0}$$

EXERCISE 03:

1- *Vector* $\vec{A}$ *has x and y components of 4 cm and -5 cm, respectively. Vector* $\vec{B}$ *has x and y components of -2 cm and 1 cm, respectively. If* $\vec{A} - \vec{B} + 3\,\vec{C} = \vec{0}$, *then what are the components of the vector* $\vec{C}$.

2- *Three vectors are oriented as shown in Figure below, where A = 10, B = 20, and C = 15 units. Find: (a) the x and y components of the resultant vector* $\vec{D} = \vec{A} + \vec{B} + \vec{C}$, *(b) the magnitude and direction of the resultant vector* $\vec{D}$.



Solution:.

1) $\vec{A} = 4\,\vec{\imath} - 5\,\vec{\jmath}$

$\vec{B} = -2\,\vec{\imath} + \vec{\jmath}$

$$3\,\vec{C} = -\vec{A} + \vec{B} = \frac{1}{3}\left(-\vec{A} + \vec{B}\right) = \frac{1}{3}(-4\,\vec{\imath} + 5\,\vec{\jmath} - 2\,\vec{\imath} + \vec{\jmath}) = -2\,\vec{\imath} + 2\,\vec{\jmath}$$

2) $\vec{A} = A\cos\left(\frac{\pi}{6}\right)\vec{\imath} - A\sin\left(\frac{\pi}{6}\right)\vec{\jmath}$

$\vec{B} = B\cos\left(\frac{\pi}{6}\right)\vec{\imath} + B\sin\left(\frac{\pi}{6}\right)\vec{\jmath}$

$3\,\vec{C} = -C\,\vec{\imath}$

$$\vec{D} = \vec{A} + \vec{B} + \vec{C} = A\cos\left(\frac{\pi}{6}\right)\vec{\imath} - A\sin\left(\frac{\pi}{6}\right)\vec{\jmath} + B\cos\left(\frac{\pi}{6}\right)\vec{\imath} + B\sin\left(\frac{\pi}{6}\right)\vec{\jmath} - C\,\vec{\imath}$$

$$\vec{D} = \left(A\cos\left(\frac{\pi}{6}\right) + B\cos\left(\frac{\pi}{6}\right) - C\right)\vec{\imath} + \left(B\sin\left(\frac{\pi}{6}\right) - A\sin\left(\frac{\pi}{6}\right)\right)\vec{\jmath}$$

$$\vec{D} = \left(10\sqrt{3}/2 + 20\sqrt{3}/2 - 15\right)\vec{\imath} + \left(20\times\frac{1}{2} - 10\times\frac{1}{2}\right)\vec{\jmath} = 15\left(\sqrt{3}-1\right)\vec{\imath} + 5\vec{\jmath}$$

$$\|\vec{D}\| = \sqrt{(D_x)^2 + \left(D_y\right)^2} = \sqrt{\left(15\left(\sqrt{3}-1\right)\right)^2 + (5)^2}$$

$$\tan(\vartheta) = \frac{5}{15\left(\sqrt{3}-1\right)} = \frac{1}{3\left(\sqrt{3}-1\right)}$$

EXERCISE 04:

In a direct orthonormal coordinate system $\Re(\vec{\imath}, \vec{\jmath}, \vec{k})$ we consider the following vectors:

$$\vec{V_1} = 3\vec{\imath} + 3\vec{\jmath} \qquad \vec{V_2} = \vec{\imath} + 3\vec{\jmath} + \vec{k} \qquad \vec{V_3} = \vec{\imath} - \vec{\jmath} + 2\vec{k} \qquad \vec{V_4} = 2\vec{\imath} - \vec{k}$$

- ✓ *Represent the vectors $\vec{V_1}$and $\vec{V_2}$.*
- ✓ *Calculate the magnitude of $\vec{V_1}$and $\vec{V_2}$, the dot product $\vec{V_1}.\vec{V_2}$and the cross product $\vec{V_1}\Lambda\vec{V_2}$.*
- ✓ *Calculate the angle θ formed by the vectors $\vec{V_1}$and $\vec{V_2}$.*
- ✓ *Prove that the vector $\vec{V_3}$ is perpendicular to the plane (P) formed by vectors $\vec{V_1}$and $\vec{V_2}$.*
- ✓ *Provethat the vector $\vec{V_4}$ belongs to the plane (P).*

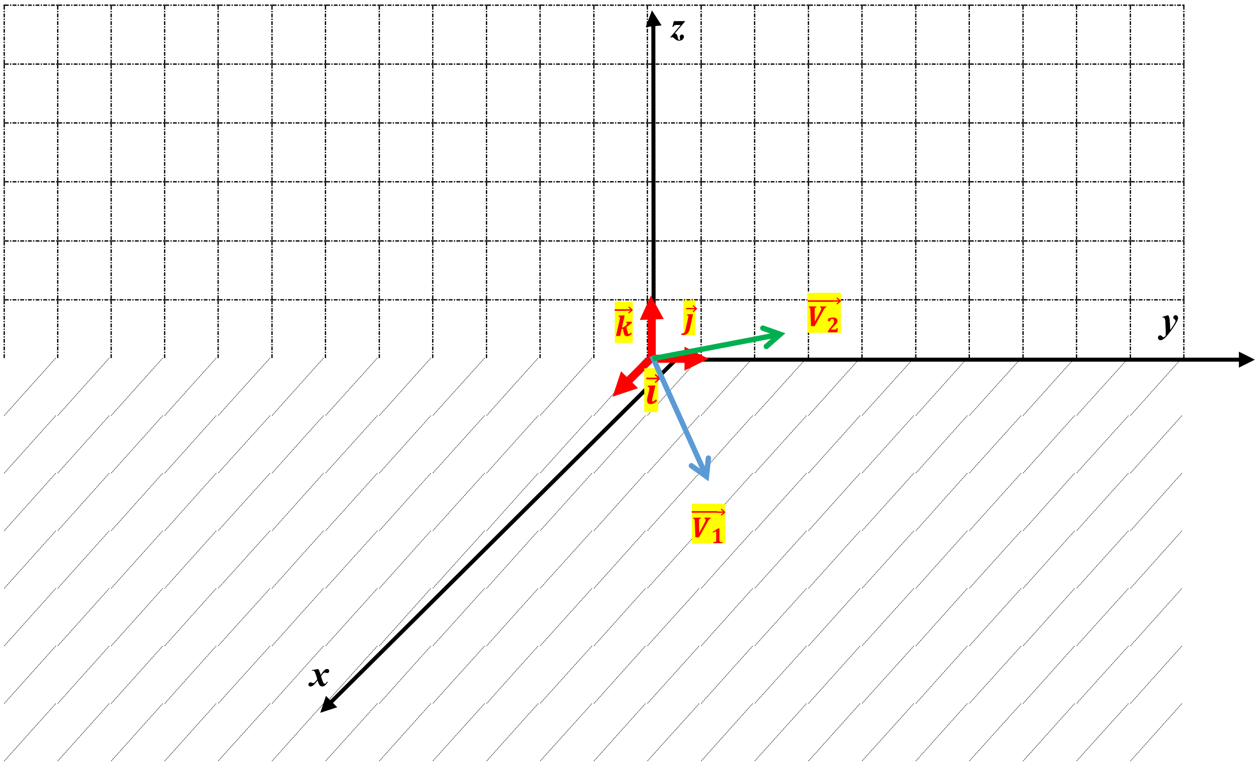
Determine the unit vector $\vec{U}$ carried by the vector $\vec{V_1}$and $\vec{V_2}$.

Solution:

In a direct orthonormal coordinate system $\Re(\vec{\imath}, \vec{\jmath}, \vec{k})$ we consider the following vectors:

$$\vec{V_1} = 3\vec{\imath} + 3\,\vec{\jmath}\vec{V_2} = \vec{\imath} + 3\,\vec{\jmath} + \vec{k}\vec{V_3} = \vec{\imath} - \vec{\jmath} + 2\vec{k}\vec{V_4} = 2\vec{\imath} - \vec{k}$$

- ✓ *Represent the vectors $\vec{V_1}$and $\vec{V_2}$.*



- ✓ *Calculate the magnitude of* $\vec{V_1}$ *and* $\vec{V_2}$, *the dot product* $\vec{V_1}.\vec{V_2}$ *and the cross product* $\vec{V_1}\Lambda\vec{V_2}$.

$$\|\vec{V_1}\| = \sqrt{(3)^2 + (3)^2} = \sqrt{18}$$

$$\|\vec{V_2}\| = \sqrt{(1)^2 + (3)^2 + (1)^2} = \sqrt{11}$$

$$\vec{V_1}\,.\vec{V_1} = 3*1 + 3*3 + 0*1 = 12$$

$$\vec{V_1} \wedge \vec{V_1} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3 & 3 & 0 \\ 1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 3 & 1 \end{vmatrix}\vec{\imath} - \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} 3 & 3 \\ 1 & 3 \end{vmatrix}\vec{k} = 3\vec{\imath} - 3\vec{\jmath} + 6\vec{k}$$

Calculate the angle θ formed by the vectors $\vec{V_1}$ *and* $\vec{V_2}$.

$$\vec{V_1}\,.\vec{V_1} = \|\vec{V_1}\| \times \|\vec{V_2}\|\, cos(\vartheta) = 12$$

$$\|\vec{V_1}\| \times \|\vec{V_2}\|\, cos(\vartheta) = \frac{\vec{V_1}\,.\vec{V_1}}{\|\vec{V_1}\| \times \|\vec{V_2}\|} = \frac{12}{\sqrt{18 \times 11}} = 0.852$$

$$\vartheta = 31.57^0$$

- ✓ *Prove that the vector* $\vec{V_3}$ *is perpendicular to the plane (P) formed by vectors* $\vec{V_1}$ *and* $\vec{V_2}$.

$$\vec{V_1} = 3\vec{\imath} + 3\,\vec{\jmath}\,\vec{V_2} = \vec{\imath} + 3\,\vec{\jmath} + \vec{k}\vec{V_3} = \vec{\imath} - \vec{\jmath} + 2\vec{k}$$

$$\begin{cases} \vec{V_3}\,.\vec{V_1} = 3*1 - 3*1 + 2*0 = 0 \\ \vec{V_3}\,.\vec{V_2} = 1*1 - 3*1 + 2*1 = 0 \end{cases} \quad \vec{V_3} \perp \vec{V_1} \; and \; \vec{V_3} \perp \vec{V_2}$$

✓ *Prove that the vector $\vec{V_4}$ belongs to the plane (P).*

$$\vec{V_4} = 2\vec{\imath} - \vec{k}$$

$$\vec{V_4} = a\,\vec{V_1} + b\,\vec{V_2} = 3a\,\vec{\imath} + 3a\,\vec{\jmath} + b\,\vec{\imath} + 3b\vec{\jmath} + b\,\vec{k} = (3a+b)\vec{\imath} + (3a+3b)\vec{\jmath} + b\,\vec{k}$$

$$(3a+b)\vec{\imath} + (3a+3b)\vec{\jmath} + b\,\vec{k} = 2\vec{\imath} - \vec{k} \;\Rightarrow \begin{cases} 3a + b = 2 \\ 3a + 3b = 0 \\ b = -1 \end{cases} \Rightarrow \begin{cases} a = -b = 1 \\ b = -1 \end{cases}$$

$\vec{V_4} = \vec{V_1} - \vec{V_2}$ *we can write* $\vec{V_4}$*as function of both* $\vec{V_1}$ *and* $\vec{V_2}$ *, thus , the vector* $\vec{V_4}$ *belongs to the plane (P)*

✓ *Determine the unit vector $\vec{U}$ carried by the vector $\vec{V_1}$ and $\vec{V_2}$.*

$$\vec{U_1} = \frac{\vec{V_1}}{\|\vec{V_1}\|} = \frac{1}{\sqrt{18}}(3\vec{\imath} + 3\,\vec{\jmath})$$

$$\vec{U_2} = \frac{\vec{V_2}}{\|\vec{V_2}\|} = \frac{1}{\sqrt{11}}\vec{\imath} + 3\,\vec{\jmath} + \vec{k}$$

EXERCISE 05:

Calculate the volume of the parallelepiped formed by the following vectors:

$\vec{A} = -2\vec{\imath} + \vec{\jmath} + 2\,\vec{k}$ $\quad \vec{B} = -\vec{\imath} + 2\vec{\jmath} + 2\,\vec{k}$ $\quad \vec{C} = -3\,\vec{\imath} - 3\,\vec{\jmath} + \vec{k}$

Solution:

$$\boldsymbol{V} = \vec{\boldsymbol{A}} \cdot (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}})$$

$$(\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -1 & 2 & 2 \\ -3 & -3 & 1 \end{vmatrix} = 8\vec{\boldsymbol{\imath}} - 5\vec{\boldsymbol{\jmath}} + 9\,\vec{\boldsymbol{k}}$$

$$\vec{\boldsymbol{A}} \cdot (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}) = (-2\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}} + 2\,\vec{\boldsymbol{k}}).(8\vec{\boldsymbol{\imath}} - 5\vec{\boldsymbol{\jmath}} + 9\,\vec{\boldsymbol{k}}) = -3 \Rightarrow V = |-3| = 3$$

EXERCISE 06:

Calculate the area of the parallelogram formed by the two following vectors:

$\vec{B} = 3\vec{\imath} + 2\vec{\jmath} + \vec{k}$ $\quad \vec{C} = \vec{\imath} - \vec{\jmath} + 2\vec{k}$

Solution:

The area of the parallelogram formed by $\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}$ *vectors is given as follow:*

$$\vec{\boldsymbol{S}} = \vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}$$

$$(\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 5\vec{\boldsymbol{\imath}} - 5\vec{\boldsymbol{\jmath}} - 5\,\vec{\boldsymbol{k}}$$

The magnitude of the cross product represents the area of the parallelogram

$$\vec{\boldsymbol{S}} = \|\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}\| = \sqrt{\boldsymbol{5^2 + (-5)^2 + (-5)^2}} = \boldsymbol{5\sqrt{3}}$$

EXERCISE 07:

1- Calculate the derivative and the integral of the $\vec{V}$ *vector with respect to the variable t:*

$$\vec{V} = t^2\vec{\imath} + 2\vec{\jmath} + t\,\vec{k}$$

2- The expression for the field function is given as follows:

$H(x, y, z) = x^2y + z$

- Calculate the change of this field $H(x, y, z)$*in the different directions of space (calculate the gradient of this field)*

3- The expression for the vector field $\vec{V}$ *is given as follows:*

$$\vec{V} = y\,x^2\,\vec{\imath} + z^2\,\vec{k}$$

- Calculate the divergence and the curl (rotation) of this vector field $\vec{V}$ *at the point (0 , 0 , 1)*

Solution:

1- The derivative

$$\frac{d\vec{V}}{dt} = \frac{d}{dt}\left(t^2\vec{\imath} + 2\vec{\jmath} + t\,\vec{k}\right) = 2t\,\vec{\imath} + \vec{k}$$

The integral

$$\int \vec{V}\,dt = \int \left(t^2\vec{\imath} + 2\vec{\jmath} + t\,\vec{k}\right)dt = \frac{1}{3}t^3\vec{\imath} + 2t\,\vec{\jmath} + \frac{1}{2}t^2\,\vec{k} + \vec{c}$$

2- The gradient of field $H(x,y,z) = x^2y + z$

$$\vec{\nabla}H(x,y,z) = \frac{\partial H(x,y,z)}{\partial x}\,\vec{\imath} + \frac{\partial H(x,y,z)}{\partial y}\,\vec{\jmath} + \frac{\partial H(x,y,z)}{\partial z}\,\vec{k}$$

$$\begin{cases} \dfrac{\partial H(x,y,z)}{\partial x} = 2xy \\ \dfrac{\partial H(x,y,z)}{\partial y} = x^2 \\ \dfrac{\partial H(x,y,z)}{\partial z} = 1 \end{cases}$$

$$\vec{\nabla}H(x,y,z) = exy\,\vec{\imath} + x^2\vec{\jmath} + \vec{k}$$

3- Calculate the divergence of this vector field $\vec{V} = y\,x^2\,\vec{\imath} + z^2\,\vec{k}$ *at the point (0 , 0 , 1)*

$$\overrightarrow{div}.\vec{V} = \vec{\nabla}.\vec{V} = \left(\frac{\partial}{\partial x}\,\vec{\imath} + \frac{\partial}{\partial y}\,\vec{\jmath} + \frac{\partial}{\partial z}\,\vec{k}\right).\left(V_x\vec{\imath} + V_y\,\vec{\jmath} + V_z\vec{k}\right) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\begin{cases} \dfrac{\partial V_x}{\partial x} = 2xy \\ \dfrac{\partial V_y}{\partial y} = 0 \\ \dfrac{\partial V_z}{\partial z} = 2z \end{cases} \Rightarrow \overrightarrow{div}.\vec{V} = 2xy + 2z$$

at the point (0 , 0, 1) $\Rightarrow \overrightarrow{div}.\vec{V} = 0 + 0 + 2 = 2$

**- Calculate the curl (rotation) of this vector field* $\vec{V} = y\,x^2\,\vec{\imath} + z^2\,\vec{k}$ *at the point (0, 0 , 1)*

$$curl\,\vec{V} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\vec{\imath} + \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right)\vec{\jmath} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\vec{k}$$

$V_x = yx^2$, $V_y = 0$ *and* $V_z = z^2$

$$\begin{cases} \dfrac{\partial V_x}{\partial y} = x^2 \\ \dfrac{\partial V_x}{\partial z} = 0 \end{cases} \quad \begin{cases} \dfrac{\partial V_y}{\partial x} = 0 \\ \dfrac{\partial V_y}{\partial z} = 0 \end{cases} \quad \begin{cases} \dfrac{\partial V_z}{\partial x} = 0 \\ \dfrac{\partial V_z}{\partial y} = 0 \end{cases}$$

$$curl\,\vec{V} = (0 - 0)\vec{\imath} + (0 - 0)\vec{\jmath} + \left(0 - x^2\right)\vec{k} = -x^2\,\vec{k}$$

at the point (0 , 0 , 1) $\Rightarrow curl\,\vec{V} = 0\,\vec{k}$

Chapter 02: Kinematics

FIRST PART: Coordinate Systems

1) Introduction

The first step in studying the motion of a particle begins with a detailed examination of the terms and concepts of the particle's kinematic properties without delving into the causes of this motion. During this study, the particle is considered dimensionless (meaning its dimensions are negligible compared to the surroundings), and parameters of the particle's motion, such as its positions, displacements, direction of motion, velocity, acceleration, and path relative to a spatial-temporal reference frame, are determined.

*The **spatiotemporal reference** is used to study the precise motion and analysis of a particle. This reference allows for the temporal and spatial determination, observation, and examination of the body's motion, significantly facilitating the understanding and accurate analysis of motion and its monitoring.*

*An example of this concept in practical application is in athletics, such as a footrace, where specific points in time and space (the starting line and the moment of the race's starting) are chosen as **spatial and temporal references** to track the athletes' performance accurately, where the athletes cannot commence the race freely from any location and at any moment they desire.*

2) Material point Concept

In the study of body kinematics, "material point" refers to a body with mass but whose dimensions are considered negligible compared to the distance it travels. This concept is used to simplify the study of the motion of a body by focusing only on its mass (which is the sum of the masses of all parts of the body) and the characteristics of its motion (such as speed, distance traveled, acceleration, time taken, etc.)

3) Reference Frames:

Examples of inertial reference frames include the solar-centered frame in the Kepler model, the terrestrial-centered frame, and the laboratory-centered frame.

3-1) The concept of the inertial reference frame (or Galilean frame) *refers to a coordinate system in which the motion of each isolated material point is considered straight and uniform. Although obtaining a completely isolated body is not possible experimentally, inertial frames are considered an ideal reference for studying motion.*

3-2) The Heliocentric reference frame (Kepler frame)

In the Kepler model, the sun is considered a fixed central point and its three main axes point towards three distant stars that can be considered fixed. This frame of reference is used in studying the motion of planets.

3-3) The terrestrial-centered reference frame: *Its center is the Earth and its axes point towards three distant stars that are considered fixed. This reference is used in studying the motion of objects close to the Earth, such as moons.*

3-4) The terrestrial laboratory reference frame: *Its center is the Earth's surface and its three perpendicular axes. It is used in laboratory studies of experiments.*

For laboratory studies, we know the frame of reference as follows:

The reference Frame *is the point of observation and monitoring used to study the motion of particles. In other words, it is a specific point in space (referred to as the origin and denoted as "O") chosen for the observation and monitoring of motion variables for other bodies.*

One, two, or three directed axes (one axis if the motion occurs in a single direction, two axes if the motion is in a plane, i.e., two directions, and three directed axes if the motion occurs in three-dimensional space) can be assigned to the origin point to facilitate the monitoring and analysis of these kinetic variables of the moving particle during its motion.

The combination of directed axes and the origin point is referred to as the "coordinate system.

4) Coordinate systems

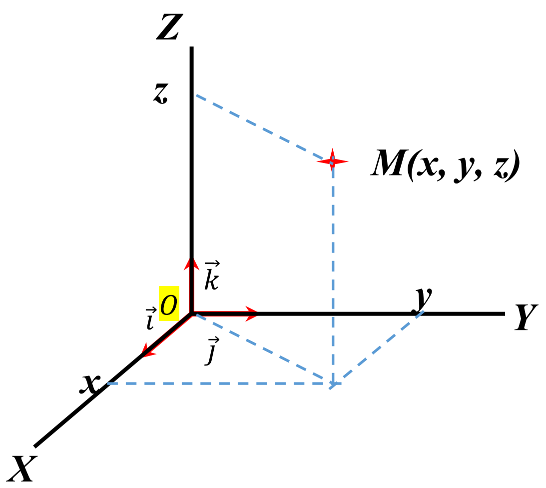
The Coordinate systems are reference frameworks used to represent numerical data geometrically. Every coordinate system includes a reference point known as the origin, from which one or more directed axes extend. In physics, coordinate systems are employed to represent the kinematic properties of particles, such as their position, displacement, velocity, and acceleration, in a way that makes them easily analyzable and simplified.

There are several coordinate systems used in the study of mechanical phenomena and particle motion, such as ***Cartesian****,* ***polar****,* ***cylindrical****, and* ***spherical*** *coordinate systems. Choosing the appropriate system for studying particle motion depends on the nature of the motion. Therefore, you should select the system that allows you to simplify the analysis of particle motion and avoid unnecessary complexity in calculations.*

4-1) *Cartesian coordinate system*

This system comprises a fixed reference point, known as the origin (usually denoted as the point '0'), from which emanate three basic orthogonal axes oriented (commonly labeled as the x-axis, the y-axis, and the z-axis) at 90-degree angles to each other, and carrying three unit vectors(commonly labeled as $\vec{\imath}, \vec{\jmath}$, *and* $\vec{k}$*).*

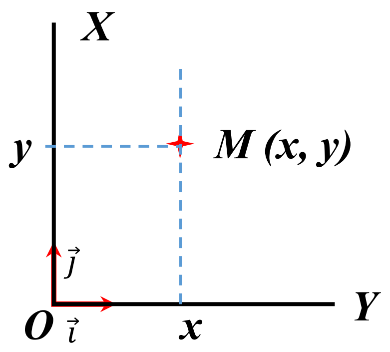
The coordinates of a point M in the Cartesian coordinate system are given as (x_M, y_M, z_M)*, where each of x, y, and z represents its projection along the three principal axes, namely, OX, OY, and OZ, respectively. This is illustrated in the following diagram:*



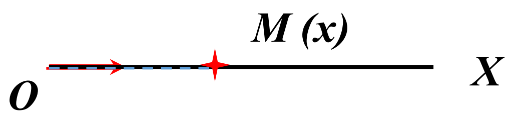
The position vector for any point can be written as follows:

$$\overrightarrow{OM} = \overrightarrow{r_M} = x_M\vec{\imath} + y_M\vec{\jmath} + z_M\vec{k}$$

Based on the nature of motion, we can use only the necessary coordinates to determine the position of a point during its motion. For example, if the motion occurs in the plane (OXY), the third coordinate (z) can be omitted as it typically does not have a significant role in this context. Similarly, if the motion is straight linear, it can be studied by considering only a single axis (OX). This approach simplifies the analysis by using the coordinates that are most relevant to the specific motion, making it more efficient and straightforward to describe and understand.



Motion in plane (XOY)



4-2) *Polar coordinate system*

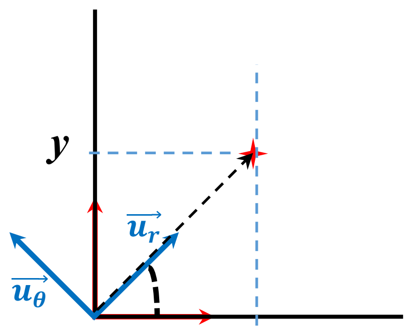
If the particle's motion occurs within the (OXY) plane, an alternative coordinate system can be employed the polar coordinate system with unit radial vectors ($\vec{u_r}$) and angular vectors ($\vec{u_\theta}$).

Radial unit vector ($\vec{u_r}$): *It possesses a length of 1 unit and aligns with the position vector OM.*

Angular unit vector ($\vec{u_\theta}$): *Also with a unit length of 1, it stands perpendicular to the unit radial vector ($\vec{u_r}$), indicating a counterclockwise direction to signify angular changes (θ).*

The angle θ is the angle enclosed between the position vector ($\overrightarrow{OM}$) and the (OX) axis.

*The polar coordinates of point M represented by both Radial **(r)** and Angular **(θ)** coordinates, and given as **M (r, θ).***

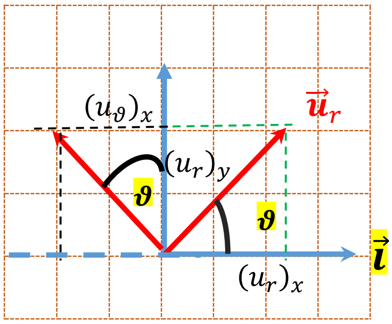


The position vector for the point M can be written in the polar coordinate system as follows:

$$\overrightarrow{OM} = \vec{r_M} = r\,\vec{u_r} \ \ where \ \ r = \|\overrightarrow{OM}\| \quad and\ \vec{u_r} = \frac{\vec{r_M}}{r} = \frac{\overrightarrow{OM}}{\|\overrightarrow{OM}\|}$$

$$r = \|\overrightarrow{OM}\| = \sqrt{x^2 + y^2} \ and \ \tan\theta = \frac{y}{x}$$

4-2-1) Transformation of unit vectors from the polar system ($\vec{u}_r$ and $\vec{u}_\theta$) to the Cartesian system ($\vec{\imath}$ and $\vec{\jmath}$)



We can obtain the expressions for the unit vectors in polar coordinates in terms of the unit vectors in Cartesian coordinates through the process of orthogonal projection (meaning we find the components of the polar unit vectors in Cartesian coordinates) as follows:

$\vec{u}_r = (u_r)_x \times \vec{\imath} + (u_r)_y \times \vec{\jmath}$

$(u_r)_x$*and* $(u_r)_y$*are the components of the polar unit vectors* $\vec{u}_r$ *in Cartesian coordinates*

$(u_r)_x = \|\vec{u}_r\| \times \cos\vartheta;\ (u_r)_y = \|\vec{u}_r\| \times \sin\vartheta\ \|\vec{u}_r\| = 1$

$$\vec{u}_r = \cos\vartheta \times \vec{\imath} + \sin\vartheta \times \vec{\jmath}$$

$\vec{u}_\vartheta = -(u_\vartheta)_x \times \vec{\imath} + (u_\vartheta)_y \times \vec{\jmath}$

$(u_\vartheta)_x$*and* $(u_\vartheta)_y$ *are the components of the polar unit vectors* $\vec{u}_\vartheta$ *in Cartesian coordinates*

$(u_\vartheta)_x = \|\vec{u}_\vartheta\| \times \sin\vartheta;\ (u_\vartheta)_y = \|\vec{u}_\vartheta\| \times \cos\vartheta\ \|\vec{u}_\vartheta\| = 1$

$$\vec{u}_\vartheta = -\sin\vartheta \times \vec{\imath} + \cos\vartheta \times \vec{\jmath}$$

4-2-2) Transformation of the Cartesian coordinates (x, y) to the polar coordinates (r , θ)

*The coordinates of point M can be expressed in either Cartesian or polar coordinates. The relationship between the Cartesian coordinates (**x, y**) and the polar coordinates (**r** , **θ**) for point M can be expressed as follows:*

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Example 01:

*Convert the coordinates of the position of M point from polar **M (4, π/6)** to the Cartesian coordinates*

Solution:

$$x = \boldsymbol{r}\cos\boldsymbol{\theta} \qquad \text{where } \boldsymbol{r = 4 \text{ and } \theta = \frac{\pi}{6}}$$
$$y = \boldsymbol{r}\sin\boldsymbol{\theta}$$

$$\overrightarrow{OM} = 4\,\overrightarrow{\boldsymbol{u_r}}$$

$$M\,(4, \pi/6) \;\Rightarrow \begin{cases} x = 4\,cos\left(\frac{\pi}{6}\right) = \frac{4\sqrt{3}}{2} = \sqrt{3} \\ y = 4\,sin\left(\frac{\pi}{6}\right) = 4 \times \frac{1}{2} = 2 \end{cases}$$

$$\overrightarrow{OM} = \sqrt{\mathbf{3}}\vec{\imath} + 2\,\vec{\jmath}$$

Example 02:

The Cartesian coordinates of a particle at two separate times are (3.4) and (-2.2). Find this moving particle's polar coordinates.

Solution:

$$r = \left\|\overrightarrow{OM}\right\| = \sqrt{x^2 + y^2} \;\; and \;\; tan\,\theta = \frac{y}{x} \;\;\Rightarrow\;\; \theta = arctan(\frac{y}{x})$$

$$(3.4) \Rightarrow \begin{cases} r = \sqrt{3^2 + 4^2} = 5 \\ \theta = arctan\left(\frac{4}{3}\right) = 0.92\ Rad \end{cases}$$

$$(2, -2) \Rightarrow \begin{cases} r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2} \\ \theta = arctan\left(\frac{-2}{2}\right) = 2.35\ Rad \end{cases}$$

4-3) *cylindrical coordinate system*

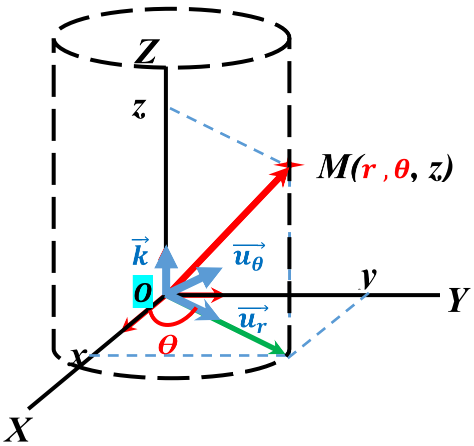
The cylindrical coordinate system is a three-dimensional reference system that isn't fixed in place (correlated to the position of the moving object)and uses three cylindrical coordinates to describe the positions of objects in space:

- ***Radial Coordinate (r)***: *It represents the length of the vector resulting from projecting the position vector onto the plane formed by the unit radial vector ($\overrightarrow{\boldsymbol{u_r}}$) and the unit azimuthal vector ($\overrightarrow{\boldsymbol{u_\theta}}$).*
- ***Angular Coordinate (θ)***: *This represents the angle between the radial line and the positive axis of the Cartesian coordinates (usually measured counterclockwise) and is depicted as an angle that starts from the positive Cartesian axis.*
- ***Coordinate (z)***: *This indicates the vertical height of the point above the origin plane, which corresponds to the Cartesian coordinate plane.*

The cylindrical coordinate system relies on three fundamental unit vectors, which are:

- ***Radial unit vector*** ($\overrightarrow{u_r}$)***:*** *It possesses a length of 1 unit and aligns with the position vector OM.*
- ***Angular unit vector*** ($\overrightarrow{u_\theta}$)***:*** *Also with a unit length of 1, it stands perpendicular to the unit radial vector ($\overrightarrow{u_r}$), indicating a counterclockwise direction to signify angular changes (θ).*
- ***Unit Vertical Vector*** ($\vec{k}$)*: Also with a unit length of 1, it points vertically, and perpendicular to both ($\overrightarrow{u_r}$) and ($\overrightarrow{u_\theta}$) unit vectors.*

The cylindrical coordinates of point M represented by both Radial ***(r)*** *and Angular* ***(θ)*** *and* ***(z)*** *coordinates, and given as* ***M (r, θ, z)*** *as shown in the following figure*

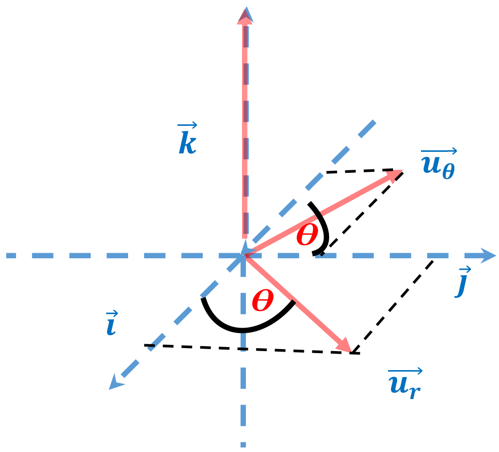


The position vector for the point M can be written in the polar coordinate system as follows:

$$\overrightarrow{OM} = \overrightarrow{r_M} = r\overrightarrow{u_r} + z\,\vec{k} \text{ where } r = \sqrt{x^2 + y^2}\ ,\ \|\overrightarrow{OM}\| = \sqrt{r^2 + z^2} \text{ and } \tan\theta = \frac{y}{x}$$

4-2-1) Transformation of the cylindrical unit vectors ($\vec{u}_r$, $\vec{u}_\theta$, and $\vec{k}$) to the Cartesian unit vectors ($\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$)

Using the same projection method employed to express the unit vectors in the polar coordinate system, we can obtain expressions for the unit vectors in the cylindrical coordinate system.



$$\begin{cases} \vec{u}_r = \cos\vartheta \times \vec{i} + \sin\vartheta \times \vec{j} \\ \vec{u}_\vartheta = -\sin\vartheta \times \vec{i} + \cos\vartheta \times \vec{j} \\ \vec{k} = \vec{k} \end{cases}$$

4-3-2) Transformation of the Cartesian coordinates (x, y, z) to the cylindrical coordinates (r , θ , z)

*The relationship between the Cartesian coordinates (**x, y, z**) and the cylindrical coordinates (**r , θ, z**) for point M can be expressed as follows:*

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Example 03:

*Convert the coordinates of the position of M point from cartesian **M (4, 3, 1)** to the cylindrical coordinates*

Solution:

$$\overrightarrow{OM} = r\,\overrightarrow{u_r} + z\,\vec{k}$$

$$\tan\theta = \frac{y}{x} = \frac{3}{4} = 0.75 = 36.86^0 = 0.643\ Rad$$

$$z = 1$$

$$\overrightarrow{OM} = 5\,\overrightarrow{u_r} + \vec{k}$$

Example 04:

Convert the coordinates of the position of M point from cylindrical coordinates $\boldsymbol{M}\ (\mathbf{2}, \frac{\pi}{2}\ \boldsymbol{rad}, \mathbf{1})$ *to the Cartesian coordinates*

Solution:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Rightarrow \begin{matrix} x = 2\cos\frac{\pi}{2} = 0 \\ y = 2\sin\frac{\pi}{2} = 2 \\ z = 1 \end{matrix} \Rightarrow M\ (0, 2, 1)$$

4-4) *Spherical coordinate system*

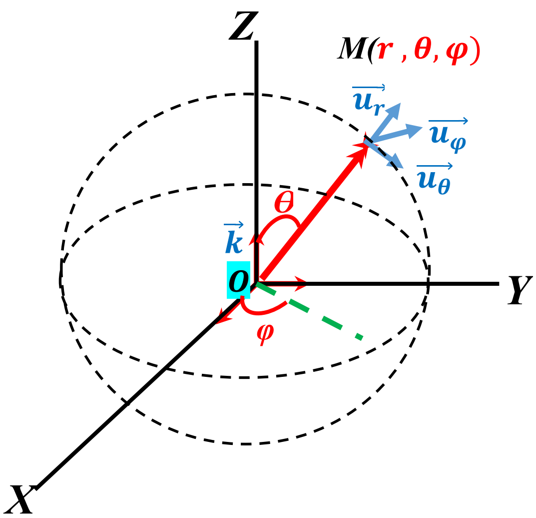
The spherical coordinate system is a three-dimensional reference system used to express the positions of points in three-dimensional space. This system relies on three main coordinates:

- ***Radial Coordinate (r)**: Represents the distance between the point and the origin (the pole) and is always positive.*
- ***Polar Angle Coordinate (θ)**: Indicates the angle enclosed between the radial line and the vertical axis (usually ranging from 0 to 180 degrees).*
- ***Azimuthal Angle Coordinate (φ)**: Represents the angle enclosed between the radial line and the horizontal plane (usually ranging from 0 to 360 degrees).*

The position vector for the point M can be written in the spherical coordinate system as follows:

$$\overrightarrow{OM} = \overrightarrow{r_M} = r\,\overrightarrow{u_r} \text{ where } r = \|\overrightarrow{OM}\| \quad \text{and } \overrightarrow{u_r} = \frac{\overrightarrow{r_M}}{r} = \frac{\overrightarrow{OM}}{\|\overrightarrow{OM}\|}$$

$$r = \|\overrightarrow{OM}\| = \sqrt{x^2 + y^2 + z^2} \text{ and } \tan\varphi = \frac{y}{x} \text{ and } \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$$



4-4-1) Transformation of the spherical unit vectors ($\vec{u}_r$, $\vec{u}_\theta$, and $\vec{u}_\varphi$) to the Cartesian unit vectors ($\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$)

using the projection method on the spherical unit vectors, we can obtain expressions for the unit vectors in the cylindrical coordinate system.

$$\begin{cases} \vec{u}_r = \sin\theta\cos\varphi\,\vec{\imath} + \sin\theta\,\sin\varphi\,\vec{\jmath} + \cos\theta\,\vec{k} \\ \vec{u}_\vartheta = \cos\theta\cos\varphi\,\vec{\imath} + \cos\theta\,\sin\varphi\,\vec{\jmath} - \sin\theta\,\vec{k} \\ \vec{u}_\varphi = -\sin\varphi\,\vec{\imath} + \cos\varphi\,\vec{\jmath} \end{cases}$$

4-4-2) Transformation of the spherical coordinates (r, θ, φ) to the Cartesian coordinates (x, y, z)

To convert from spherical coordinates to Cartesian coordinates, we substitute the values of r , θ , *and* φ *into the following set of equations:*

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\,\sin\varphi \\ z = r\cos\theta \end{cases}$$

Example 05:

Convert the coordinates of the position of M point from cartesian ***M (2, 1, 2)*** *to the spherical coordinates(* $\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\varphi}$*)*

Solution:

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \tan\varphi = \dfrac{y}{x} \\ \cos\theta = \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{cases} \Rightarrow \begin{cases} r = \sqrt{2^2+1^2+2^2} \\ \tan\varphi = \dfrac{1}{2} \\ \cos\theta = \dfrac{2}{3} \end{cases} \Rightarrow \begin{cases} r = 3 \\ \varphi = 0.46\ Rad \\ \theta = 0.84\ Rad \end{cases}$$

Example 06:

Convert the coordinates of the position of M point from cylindrical coordinates $\boldsymbol{M}\ (\boldsymbol{3}, \frac{\pi}{3}\ \boldsymbol{rad}, \frac{\pi}{6}\ \boldsymbol{rad})$ *to the Cartesian coordinates*

Solution:

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\ \sin\varphi \\ z = r\cos\theta \end{cases} \Rightarrow \begin{cases} x = 3\sin\dfrac{\pi}{3}\cos\dfrac{\pi}{6} \\ y = 3\sin\dfrac{\pi}{3}\ \sin\dfrac{\pi}{6} \\ z = 3\cos\dfrac{\pi}{3} \end{cases} \Rightarrow \begin{cases} x = 2.25 \\ y = 1.3 \\ z = 1.5 \end{cases}$$

4-5) Intrinsic coordinates system

The intrinsic coordinate system at for any point from the path is denoted by the base $(\vec{u}_T, \vec{u}_N)$ *where :*

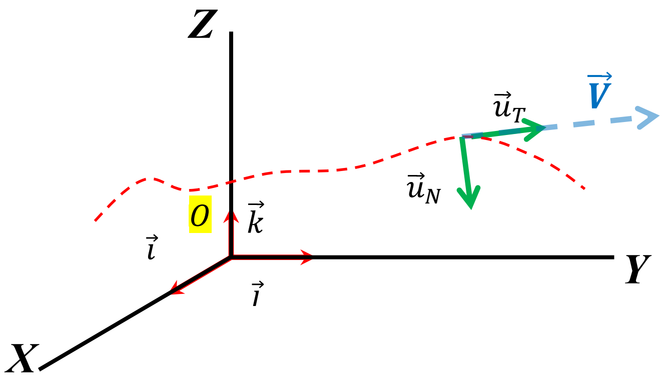
$(\vec{u}_T)$ *represent the normal unit vector and oriented towards the center of curvature.*

$(\vec{u}_N$ *represent the tangent unit vector and oriented in the direction of motion.*

The velocity vector is tangent to the trajectory and in the same direction as the movement, it carries the unit vector $\vec{u}_T$ *and is written:*

$$\vec{v} = \|\vec{v}\|\ \vec{u}_T = \frac{dS}{dt}\ \vec{u}_T$$

where dS is the curvilinear abscissa



SECOND PART: Kinematics

The kinetic properties of a moving particle can be studied by focusing on its kinetic features during its movement. This includes tracking its change in position relative to a frame of reference, examining its path (trajectory) of movement, measuring its velocity, and estimating its acceleration.

1) Position vector

The position of a moving particle at a specific moment (t) is determined with respect to a specified reference frame (origin) through its coordinates. If the motion occurs in one dimension, the position of the moving entity is expressed with a single coordinate. In the case of motion in a plane, it is represented by two coordinates. Meanwhile, three coordinates are employed to express motion in three-dimensional space.

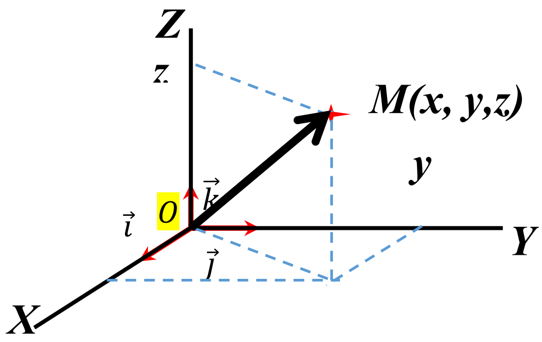
The position vector*for the moving particle 'M' at the moment 't' is a vector extending from the origin to the location of the point. It is symbolized by '*$\overrightarrow{OM}$*' and is analytically formulated as follows:*$\overrightarrow{OM} = \overrightarrow{r_M} = x\vec{i} + y\vec{j} + z\vec{k}$

The position vector *is characterized by:*

- ***Starting point (the tail) of the vector:*** *The origin point 'O'*
- ***Ending point (Head or Tip) of the vector:*** *The location of the moving particle 'M'.*
- ***Direction of the vector:*** *From point 'O' to position 'M'.*
- ***Support of the vector:*** *The straight line connecting the origin 'O' to the position 'M'.*

- ***The magnitude of the vector:*** $\|\overrightarrow{OM}\| = \sqrt{x^2 + y^2 + z^2}$

The position vector $\overrightarrow{OM}$ is represented in the Cartesian coordinate system as shown in the figure



2) The Displacement Vector

The displacement vector for a moving particle between two moments, t_i and t_f, is a vector that represents the particle's transition from position M_i to position M_f. It is symbolized by $\overrightarrow{M_iM_f}$ and extends from the initial position to the final position. It can be expressed as the difference between the position vectors, as follows:

$$\overrightarrow{M_iM_f} = \overrightarrow{OM_f} - \overrightarrow{OM_i}$$

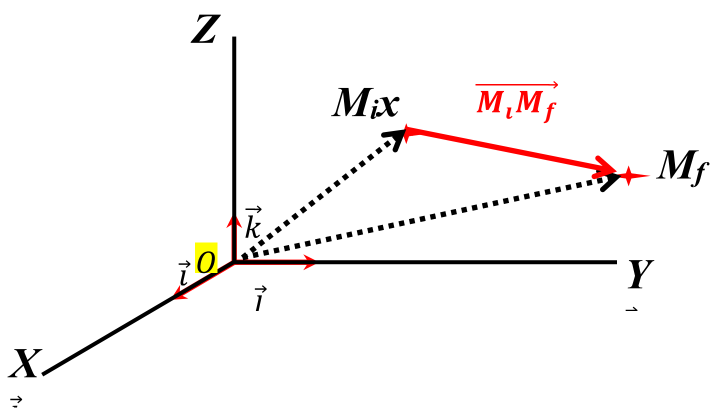
$$\overrightarrow{M_iM_f} = (x_f - x_i)\vec{i} + (y_f - y_i)\vec{j} + (z_f - z_i)\vec{k}$$

The displacement vector is characterized by:

- ***Starting point (Tail) of the vector:*** *The position of the particle 'M_i' at moment 't_i.'*
- ***Head (Tip) of the vector:*** *The position of the moving particle 'M_f' at moment 't_f.'*
- ***Direction of the vector:*** *From position M_i to position M_f*
- ***Support of the vector:*** *The straight line connecting position M_i to position M_f*

- ***The magnitude of the vector:*** $\|\overrightarrow{M_iM_f}\| = \sqrt{(x_f - x_i)^2 + (y_f - y_i)^2 + (z_f - z_i)^2}$

The position vector $\overrightarrow{M_iM_f}$ is represented in the Cartesian coordinate system as shown in the figure



Example 01:

(0, 1, 1) and (2, 4, 4) are the Cartesian coordinates of the positions of a moving particle (m) at the moments $t_1 = 5$ s and $t_2 = 10$ s respectively.

1) Write the expression for the position vector at each moment.

2) Write the expression for the displacement vector of the moving particle between the two positions

Solution:

1) The expression for the position vector at $t_1 = 5$ s.

$$\overrightarrow{OM}(t_1) = \overrightarrow{OM_1} = 0\vec{\imath} + 1\,\vec{\jmath} + 1\,\vec{k} = \vec{\jmath} + \ \vec{k}$$

- The expression for the position vector at $t_1 = 10$ s.

$$\overrightarrow{OM}(t_2) = \overrightarrow{OM_2} = 2\vec{\imath} + 4\,\vec{\jmath} + 4\,\vec{k}$$

2) The expression for the displacement vector of the moving particle between the two positions:

$$\overrightarrow{M_1M_2} = (x_2 - x_1)\vec{\imath} + (y_2 - y_1)\vec{\jmath} + (z_2 - z_1)\vec{k} = (2 - 0)\vec{\imath} + (4 - 1)\vec{\jmath} + (4 - 1)\vec{k}$$

$$\overrightarrow{M_1M_2} = 2\,\vec{\imath} + 3\,\vec{\jmath} + 3\vec{k}$$

3) Velocity vector

Generally, velocity represents the change in distance travelled by a moving object over a specific time duration. By means of calculating velocity, we can classify it into two forms: average velocity and instantaneous velocity.

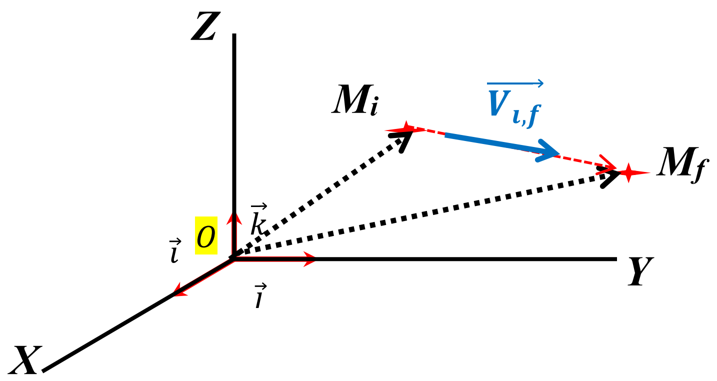
3-1) Average velocity

Average velocity, denoted as $\overrightarrow{\boldsymbol{V_a}}$ *or* $\overrightarrow{\boldsymbol{V_{\iota,f}}}$*, serves as a vector quantity to describe the velocity over an extended time interval between two distinct moments* $\boldsymbol{t_i}$ *and* $\boldsymbol{t_f}$*. Mathematically, it is articulatedas the ratio of the displacement vector* $\overrightarrow{\boldsymbol{M_\iota M_f}}$ *to the time interval* $\Delta \boldsymbol{t} = \boldsymbol{t_f} - \boldsymbol{t_i}$*during this displacement.*

The vector of the average velocity ($\overrightarrow{\boldsymbol{V_{\iota,f}}}$*) shares the same direction and support as the displacement vector (*$\overrightarrow{\boldsymbol{M_\iota M_f}}$*), while its magnitude is determined by the ratio of the displacement vector's length to the time interval.*

$$\overrightarrow{V_{\iota,f}} = \frac{\overrightarrow{M_\iota M_f}}{\Delta t} = \frac{\overrightarrow{OM_f} - \overrightarrow{OM_\iota}}{t_f - t_i} = \frac{(x_f - x_i)}{t_f - t_i}\vec{\imath} + \frac{(y_f - y_i)}{t_f - t_i}\vec{\jmath} + \frac{(z_f - z_i)}{t_f - t_i}\vec{k}$$

$$\left\|\overrightarrow{V_{\iota,f}}\right\| = \sqrt{\left(\frac{x_f - x_i}{t_f - t_i}\right)^2 + \left(\frac{y_f - y_i}{t_f - t_i}\right)^2 + \left(\frac{z_f - z_i}{t_f - t_i}\right)^2}$$



3-2) Instantaneous velocity

The instantaneous velocity, represented by $\vec{\boldsymbol{V}}$*, is a vector describing the velocity of an particle at a specific moment "* $\boldsymbol{t}$ *" or within an Infinitesimally small time interval (*$\boldsymbol{\Delta t}$*) approaching (tending to) zero. Mathematically, the vector of instantaneous velocity is defined as the limiting valueof the ratio of the displacement vector to the time interval as* $\boldsymbol{\Delta t}$ *approaches (tends to) zero.*

$$\vec{V}(t) = \lim_{\Delta t \to 0} \frac{\overrightarrow{M_\iota M_f}}{\Delta t}$$

In mathematical calculations, the limit process is expressed by the derivativeof the position vector with respect to time.

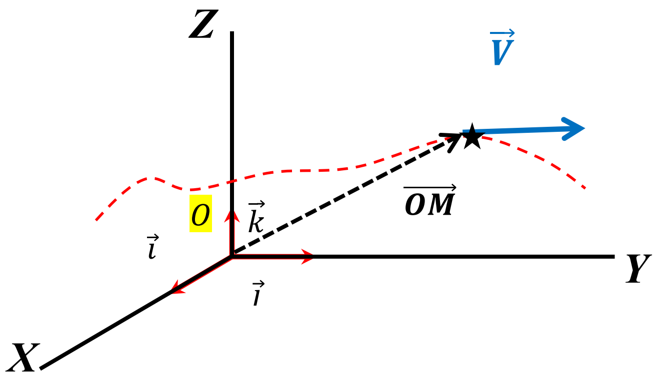
$$\vec{V}(t) = \frac{d(\overrightarrow{OM})}{dt} = \frac{d(x\vec{i} + y\,\vec{j} + z\,\vec{k})}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

$$\|\vec{V}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

The vector of instantaneous velocity represents a tangential vector to the path at the location where the instantaneous velocity is calculated.

Note: The vector of average velocity between two moments t_i and t_f equals the vector of instantaneous velocity at the midpoint of this interval $\frac{t_i+t_f}{2}$

$$\vec{V}\left(\frac{t_i + t_f}{2}\right) = \overrightarrow{V_{\iota,f}}$$



4) Acceleration vector

Acceleration is defined as the rate of change in velocity with respect to time. Based on how acceleration is calculated, we can distinguish between two types: average acceleration and instantaneous acceleration.

4-1) Average acceleration

Average acceleration, denoted as $\overrightarrow{a_a}$or $\overrightarrow{a_{\iota,f}}$, describes the change in velocity over a specific time interval, typically between two distinct momentst_i and t_f.Mathematically, the Average acceleration is expressed as the ratio of the change in velocity ($\overrightarrow{\Delta V_{\iota,f}} = \overrightarrow{V_f} - \overrightarrow{V_\iota}$) to the change *in time ($\Delta t = t_f - t_i$). The vector of average acceleration ($\overrightarrow{a_{i,f}}$) shares the same direction and support as the vector of velocity change ($\overrightarrow{\Delta V_{i,f}}$), While its magnitude is the ratio of the magnitude of the velocity change vector ($\|\overrightarrow{\Delta V_{i,f}}\|$) to the duration of the time interval $\Delta t = t_f - t_i$) .*

$$\overrightarrow{a_{i,f}} = \frac{\overrightarrow{\Delta V_{i,f}}}{\Delta t} = \frac{\overrightarrow{V_f} - \overrightarrow{V_i}}{t_f - t_i} = \frac{(V_{xf} - V_{xi})}{t_f - t_i}\vec{i} + \frac{(V_{yf} - V_{yi})}{t_f - t_i}\vec{j} + \frac{(V_{zf} - V_{zi})}{t_f - t_i}\vec{k}$$

$$\|\overrightarrow{a_{i,f}}\| = \frac{\|\overrightarrow{\Delta V_{i,f}}\|}{t_f - t_i}$$

4-2) Instantaneous acceleration

The instantaneous acceleration vector, denoted by $\vec{a}$, signifies the acceleration at a precise moment t, specifically within an infinitesimally small time interval (approaching or tending to zero).

The acceleration vector is mathematically defined as the limit of the rate of change in velocity over an infinitesimally small period of time, approaching or tending to zero.

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\overrightarrow{V_{i,f}}}{\Delta t}$$

In mathematical terms, when the time interval becomes extremely small, the limit of a function represents its derivative. Consequently, we can express the formula for instantaneous acceleration as follow

$$\vec{a}(t) = \frac{d(\vec{V})}{dt} = \frac{d(V_x\vec{i} + V_y\vec{j} + V_z\vec{k})}{dt} = \frac{dV_x}{dt}\vec{i} + \frac{dV_y}{dt}\vec{j} + \frac{dV_z}{dt}\vec{k}$$

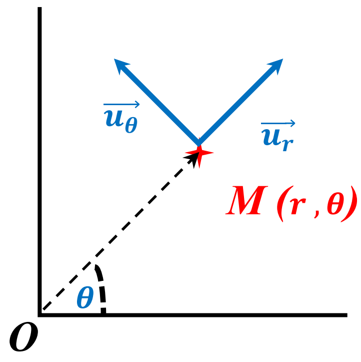
$$\|\vec{a}(t)\| = \sqrt{\left(\frac{dV_x}{dt}\right)^2 + \left(\frac{dV_y}{dt}\right)^2 + \left(\frac{dV_z}{dt}\right)^2}$$

Note: The vector of average acceleration between two moments t_i and t_f equals the vector of instantaneous acceleration at the midpoint of this interval $\frac{t_i+t_f}{2}$

$$\vec{a}\left(\frac{t_i + t_f}{2}\right) = \overrightarrow{a_{i,f}}$$

5) Velocity and acceleration Vectors in polar coordinate system

*The polar coordinates of point M represented by both Radial **(r)** and Angular **(θ)** coordinates, and given as **M (r, θ),** andthe position vector for the point M can be written in the polar coordinate system as follows:* $\overrightarrow{OM} = r\,\overrightarrow{u_r}$ ***where*** $r = \|\overrightarrow{OM}\|$ *and* $\overrightarrow{u_r} = \frac{\overrightarrow{r_M}}{\rho} = \frac{\overrightarrow{OM}}{\|\overrightarrow{OM}\|}$



*The expression for the instantaneous **velocity** vector in **polar** coordinates is given as follows:*

$$\vec{V}(t) = \frac{d(\overrightarrow{OM})}{dt} = \frac{d(r\overrightarrow{u_r})}{dt} = \frac{dr}{dt}\overrightarrow{u_r} + r\frac{d\overrightarrow{u_r}}{dt}$$

$$\vec{u}_r = \cos\theta \times \vec{i} + \sin\theta \times \vec{j}$$

$$\vec{u}_\vartheta = -\sin\theta \times \vec{i} + \cos\theta \times \vec{j}$$

$$\frac{d\,(\vec{u}_r)}{dt} = \frac{d(\cos\theta \times \vec{i} + \sin\theta \times \vec{j})}{dt} = -\frac{d\theta}{dt}\sin\theta \times \vec{i} + \frac{d\theta}{dt}\cos\theta \times \vec{j}$$

where $\frac{d\,\vec{i}}{dt} = \vec{0}\,\frac{d\,\vec{j}}{dt} = \vec{0}\,\frac{d\vartheta}{dt} = \dot{\theta}$

$$\frac{d\,(\vec{u}_r)}{dt} = \frac{d\theta}{dt}(-\sin\theta \times \vec{i} + \cos\theta \times \vec{j}) = \dot{\theta}\vec{u}_\theta$$

$$\frac{d\,(\vec{u}_\theta)}{dt} = \frac{d(-\sin\theta \times \vec{i} + \cos\theta \times \vec{j})}{dt} = -\frac{d\theta}{dt}\cos\theta \times \vec{i} - \frac{d\theta}{dt}\sin\theta \times \vec{j}$$

where

$$\frac{d\,\vec{i}}{dt} = \vec{0} \qquad \frac{d\,\vec{j}}{dt} = \vec{0} \qquad \frac{d\theta}{dt} = \dot{\theta}$$

$$\frac{d\,(\vec{u}_\theta)}{dt} = -\frac{d\theta}{dt}(\cos\theta \times \vec{i} + \sin\theta \times \vec{j}) = -\dot{\theta}\vec{u}_r$$

$$\vec{V}(t) = \frac{dr}{dt}\overrightarrow{u_r} + r\frac{d\theta}{dt}\vec{u}_\theta = \dot{r}\overrightarrow{u_r} + r\dot{\theta}\vec{u}_\theta$$

$$\|\vec{V}(t)\| = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2}$$

The expression for the instantaneous ***acceleration*** *vector in* ***polar*** *coordinates is given as follows:*

$$\vec{a}(t) = \frac{d(\vec{V})}{dt} = \frac{d(\dot{r}\vec{u_r} + r\dot{\theta}\vec{u}_\theta)}{dt}$$

$$= \frac{d\dot{r}}{dt}\vec{u_r} + \dot{r}\frac{d\vec{u_r}}{dt} + \frac{dr}{dt}\dot{\theta}\vec{u}_\theta + r\frac{d\dot{\theta}}{dt}\vec{u}_\theta + r\dot{\theta}\frac{d\vec{u}_\theta}{dt}$$

$$= \ddot{r}\vec{u_r} + \dot{r}\dot{\theta}\vec{u}_\theta + \dot{r}\dot{\theta}\vec{u}_\theta + r\ddot{\theta}\vec{u}_\theta - r\dot{\theta}\dot{\theta}\vec{u}_r$$

$$\vec{a}(t) = (\ddot{r} - r\dot{\theta}^2)\,\vec{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{u}_\theta$$

$$\|\vec{a}(t)\| = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (2\dot{r}\dot{\theta} + r\ddot{\theta})^2}$$

6) Velocity and acceleration in cylindrical coordinate system

$$\overrightarrow{OM} = r\vec{u_r} + z\,\vec{k}$$

$$\vec{V}(t) = \frac{d(r\vec{u_r} + z\,\vec{k})}{dt} = \frac{dr}{dt}\vec{u_r} + r\frac{d\vec{u_r}}{dt} + \frac{dz}{dt}\vec{k}$$

noting that $\frac{d\vec{u_r}}{dt} = \dot{\theta}\vec{u}_\theta$

$$\vec{V}(t) = \frac{dr}{dt}\vec{u_r} + r\frac{d\theta}{dt}\vec{u}_\theta + \frac{dz}{dt}\vec{k} = \dot{r}\vec{u_r} + r\dot{\theta}\vec{u}_\theta + \dot{z}\vec{k}$$

$$\|\vec{V}(t)\| = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2 + \dot{z}^2}$$

$$\vec{a}(t) = \frac{d(\vec{V})}{dt} = \frac{d(\dot{r}\vec{u_r} + r\dot{\theta}\vec{u}_\theta + \dot{z}\vec{k})}{dt}$$

$$\vec{a}(t) = \frac{d\dot{r}}{dt}\vec{u_r} + \dot{r}\frac{d\vec{u_r}}{dt} + \frac{dr}{dt}\dot{\theta}\vec{u}_\theta + \frac{d\dot{\theta}}{dt}r\vec{u}_\theta + \frac{d\vec{u}_\theta}{dt}r\dot{\theta} + \frac{d\dot{z}}{dt}\vec{k}$$

using these equqations $\begin{cases} \frac{d\,\vec{u}_r}{dt} = \dot{\theta}\vec{u}_\theta \\ \frac{d\,\vec{u}_\theta}{dt} = -\dot{\theta}\vec{u}_r \end{cases}$

$$\vec{a}(t) = \ddot{r}\,\vec{u_r} + \dot{r}\dot{\theta}\vec{u}_\theta + \dot{r}\dot{\theta}\vec{u}_\theta + \ddot{\theta}r\vec{u}_\theta - \dot{\theta}\vec{u}_r r\dot{\theta} + \ddot{z}\vec{k}$$

$$\vec{a}(t) = (\ddot{r} - r\dot{\theta}^2)\vec{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{u}_\theta + \ddot{z}\,\vec{k}$$

$$\|\vec{a}(t)\| = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (2\dot{r}\dot{\theta} + r\ddot{\theta})^2 + \ddot{z}^2}$$

7) Velocity and acceleration in spherical coordinate system

$$\overrightarrow{OM} = r\,\overrightarrow{u_r}$$

$$\vec{V}(t) = \frac{d(r\overrightarrow{u_r})}{dt} = \frac{dr}{dt}\overrightarrow{u_r} + r\frac{d\overrightarrow{u_r}}{dt}$$

$$\frac{d\overrightarrow{u_r}}{dt} = \frac{d}{dt}(\sin\theta\cos\varphi\;\vec{\imath} + \sin\theta\;\sin\varphi\;\vec{\jmath} + \cos\theta\,\vec{k})$$

$$\frac{d\overrightarrow{u_r}}{dt} = \cos\theta\,\frac{d\theta}{dt}\cos\varphi\;\vec{\imath} - \sin\theta\sin\varphi\frac{d\varphi}{dt}\;\vec{\imath} + \cos\theta\,\frac{d\theta}{dt}\sin\varphi\;\vec{\jmath} + \sin\theta\;\cos\varphi\frac{d\varphi}{dt}\;\vec{\jmath} - \frac{d\theta}{dt}\sin\theta\,\vec{k}$$

$$\frac{d\overrightarrow{u_r}}{dt} = \frac{d\theta}{dt}\left(\cos\theta\;\cos\varphi\;\vec{\imath} + \cos\theta\;\sin\varphi\;\vec{\jmath} - \sin\theta\,\vec{k}\right) + \sin\theta\frac{d\varphi}{dt}(-\sin\varphi\;\vec{\imath} + \cos\varphi\;\vec{\jmath})$$

$$\frac{d\overrightarrow{u_r}}{dt} = \frac{d\theta}{dt}(\vec{u}_\vartheta) + \sin\theta\frac{d\varphi}{dt}(\vec{u}_\varphi) = \dot{\theta}\,\vec{u}_\vartheta + \dot{\varphi}\sin\theta\;\vec{u}_\varphi$$

$$\vec{V}(t) = \dot{r}\overrightarrow{u_r} + r\,\dot{\theta}\,\vec{u}_\vartheta + r\,\dot{\varphi}\sin\theta\;\vec{u}_\varphi$$

$$\vec{a}(t) = \frac{d(\vec{V})}{dt} = \frac{d(\dot{r}\overrightarrow{u_r} + r\,\dot{\theta}\,\vec{u}_\vartheta + r\,\dot{\varphi}\sin\theta\;\vec{u}_\varphi)}{dt}$$

$$\vec{a}(t) = \frac{d\dot{r}}{dt}\overrightarrow{u_r} + \frac{d\overrightarrow{u_r}}{dt}\dot{r} + \frac{dr}{dt}\;\dot{\theta}\,\vec{u}_\vartheta + \frac{d\dot{\theta}}{dt}r\;\vec{u}_\vartheta + \frac{d\vec{u}_\vartheta}{dt}\;r\,\dot{\theta} + \frac{dr}{dt}\dot{\varphi}\sin\theta\;\vec{u}_\varphi + \frac{d\dot{\varphi}}{dt}\;r\;\sin\theta\;\vec{u}_\varphi + \frac{d\;\sin\theta}{dt}r\;\;\dot{\varphi}\;\vec{u}_\varphi + \frac{d\vec{u}_\varphi}{dt}r\,\dot{\varphi}\sin\theta$$

To simplify this equation we need to find the time derivatives of the following unit vectors:

- $\frac{d\overrightarrow{u_r}}{dt} = \dot{\theta}\,\vec{u}_\vartheta + \dot{\varphi}\sin\theta\;\vec{u}_\varphi$ *(It was previously calculated)*
- $\frac{d\vec{u}_\vartheta}{dt} = \frac{d}{dt}(\cos\theta\cos\varphi\,\vec{\imath} + \cos\theta\;\sin\varphi\;\vec{\jmath} - \sin\theta\,\vec{k})$

$$\frac{d\vec{u}_\vartheta}{dt} = \frac{d\cos\theta}{dt}\cos\varphi\,\vec{\imath} + \frac{d\cos\varphi}{dt}\cos\theta\,\vec{\imath} + \frac{d\cos\theta}{dt}\;\sin\varphi\,\vec{\jmath} + \frac{d\sin\varphi}{dt}\cos\theta\;\vec{\jmath} - \frac{d\sin\theta}{dt}\vec{k}$$

$$\frac{d\vec{u}_\vartheta}{dt} = -\frac{d\theta}{dt}\sin\theta\cos\varphi\,\vec{\imath} - \sin\varphi\;\frac{d\varphi}{dt}\cos\theta\,\vec{\imath} - \frac{d\theta}{dt}\sin\theta\;\sin\varphi\,\vec{\jmath} + \cos\varphi\frac{d\varphi}{dt}\;\cos\theta\;\vec{\jmath} - \cos\theta\frac{d\theta}{dt}\;\vec{k}$$

$$\frac{d\vec{u}_\vartheta}{dt} = \left(-\sin\theta\cos\varphi\,\vec{i} - \sin\theta\,\sin\varphi\,\vec{j} - \cos\theta\,\vec{k}\right)\frac{d\theta}{dt} + (-\sin\varphi\,\cos\theta\,\vec{i} + \cos\varphi\,\cos\theta\,\vec{j})\,\frac{d\varphi}{dt}$$

$$\frac{d\vec{u}_\vartheta}{dt} = -\frac{d\theta}{dt}\vec{u_r} + \frac{d\varphi}{dt}\cos\theta\,\vec{u}_\varphi = -\dot{\theta}\vec{u_r} + \dot{\varphi}\cos\theta\,\vec{u}_\varphi$$

- $\frac{d\vec{u}_\varphi}{dt} = \frac{d(-\sin\varphi\,\vec{i}+\cos\varphi\,\vec{j})}{dt}$

$$\frac{d\vec{u}_\varphi}{dt} = \frac{d(-\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j})}{dt}$$

$$\frac{d\vec{u}_\varphi}{dt} = -\frac{d\varphi}{dt}\cos\varphi\,\vec{i} - \frac{d\varphi}{dt}\sin\varphi\,\vec{j} == -\dot{\varphi}\cos\varphi\,\vec{i} - \dot{\varphi}\sin\varphi\,\vec{j}$$

We multiply both sides of the equation $\vec{u}_r = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\,\sin\varphi\,\vec{j} + \cos\theta\,\vec{k}$ *by* $\sin\theta$ *, and we get:*

$$\sin\theta \times \vec{u}_r = \sin\theta \times (\sin\theta\cos\varphi\,\vec{i} + \sin\theta\,\sin\varphi\,\vec{j} + \cos\theta\,\vec{k}) \qquad \textbf{Eq(a)}$$

Also, We multiply both sides of the equation $\vec{u}_\vartheta = \cos\theta\cos\varphi\,\vec{i} + \cos\theta\,\sin\varphi\,\vec{j} - \sin\theta\,\vec{k}$ *by* $\cos\theta$ *, and we get:*

$$\cos\theta \times \vec{u}_\vartheta = \cos\theta \times (\cos\theta\cos\varphi\,\vec{i} + \cos\theta\,\sin\varphi\,\vec{j} - \sin\theta\,\vec{k}) \qquad \textbf{Eq(b)}$$

The sum of the equations Eq(a) and Eq(b) gives:

$$\sin\theta \times \vec{u}_r + \cos\theta \times \vec{u}_\vartheta = \sin\theta^2\cos\varphi\,\vec{i} + \sin\theta^2\sin\varphi\,\vec{j} + \cos\theta^2\cos\varphi\,\vec{i} + \cos\theta^2\sin\varphi\,\vec{j}$$

$$\sin\theta \times \vec{u}_r + \cos\theta \times \vec{u}_\vartheta = (\sin\theta^2 + \cos\theta^2)\cos\varphi\,\vec{i} + (\sin\theta^2 + \cos\theta^2)\sin\varphi\,\vec{j}$$

$$\sin\theta \times \vec{u}_r + \cos\theta \times \vec{u}_\vartheta = \cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}$$

and therefore we can write ,

$$\frac{d\vec{u}_\varphi}{dt} = -\dot{\varphi}\cos\varphi\,\vec{i} - \dot{\varphi}\sin\varphi\,\vec{j} = -\dot{\varphi}\left(\cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}\right)$$

$$\frac{d\vec{u}_\varphi}{dt} == -\dot{\varphi}\,(\sin\theta \times \vec{u}_r + \cos\theta \times \vec{u}_\vartheta)$$

$$\begin{cases} \dfrac{d\overrightarrow{u_r}}{dt} = \dot{\theta}\,\vec{u}_\vartheta + \dot{\varphi}\sin\theta\,\vec{u}_\varphi \\ \dfrac{d\vec{u}_\vartheta}{dt} = -\dot{\theta}\overrightarrow{u_r} + \dot{\varphi}\cos\theta\,\vec{u}_\varphi \\ \dfrac{d\vec{u}_\varphi}{dt} = -\dot{\varphi}\,(\sin\theta \times \vec{u}_r + \cos\theta \times \vec{u}_\vartheta) \end{cases}$$

Using these time derivatives of these unit vectors in the previously calculated expression for acceleration in spherical coordinates, we find:

$$\vec{a}(t) = \ddot{r}\,\overrightarrow{u_r} + (\dot{\theta}\,\vec{u}_\vartheta + \dot{\varphi}\sin\theta\,\vec{u}_\varphi)\dot{r} + \dot{r}\,\dot{\theta}\,\vec{u}_\vartheta + \ddot{\theta}r\,\vec{u}_\vartheta + (-\dot{\theta}\overrightarrow{u_r} + \dot{\varphi}\cos\theta\,\vec{u}_\varphi)\,r\,\dot{\theta} + \dot{r}\,\dot{\varphi}\sin\theta\,\vec{u}_\varphi + \ddot{\varphi}\,r\sin\theta\,\vec{u}_\varphi + \dot{\theta}\cos\theta\,r\,\dot{\varphi}\,\vec{u}_\varphi - (\sin\theta \times \vec{u}_r + \cos\theta \times \vec{u}_\vartheta)r\,\dot{\varphi}^2\sin\theta$$

$$\vec{a}(t) = \ddot{r}\,\overrightarrow{u_r} + \dot{r}\dot{\theta}\,\vec{u}_\vartheta + \dot{r}\dot{\varphi}\sin\theta\,\vec{u}_\varphi + \dot{r}\,\dot{\theta}\,\vec{u}_\vartheta + \ddot{\theta}r\,\vec{u}_\vartheta + -r\,\dot{\theta}^2\overrightarrow{u_r} + r\,\dot{\theta}\dot{\varphi}\cos\theta\,\vec{u}_\varphi + \dot{r}\,\dot{\varphi}\sin\theta\,\vec{u}_\varphi + \ddot{\varphi}\,r\sin\theta\,\vec{u}_\varphi + \dot{\theta}\cos\theta\,r\,\dot{\varphi}\,\vec{u}_\varphi - r\,\dot{\varphi}^2\sin^2\theta \times \vec{u}_r - r\,\dot{\varphi}^2\sin\theta\cos\theta \times \vec{u}_\vartheta$$

$$\vec{a}(t) = (\ddot{r} - r\,\dot{\theta}^2 - r\,\dot{\varphi}^2\sin^2\theta)\overrightarrow{u_r} + (\ddot{\theta}r + 2\dot{r}\dot{\theta} - r\,\dot{\varphi}^2\sin\theta\cos\theta)\vec{u}_\vartheta + (r\sin\theta\,\ddot{\varphi} + 2\dot{r}\,\dot{\varphi}\sin\theta + 2r\,\dot{\theta}\dot{\varphi}\cos\theta\,)\vec{u}_\varphi$$

$$\vec{a}(t) = a_r\overrightarrow{u_r} + a_\theta\vec{u}_\vartheta + a_\varphi\,\vec{u}_\varphi \quad \text{where} \quad \begin{cases} a_r = \ddot{r} - r\,\dot{\theta}^2 - r\,\dot{\varphi}^2\sin^2\theta \\ a_\theta = \ddot{\theta}r + 2\dot{r}\dot{\theta} - r\,\dot{\varphi}^2\sin\theta\cos\theta \\ a_\varphi = r\sin\theta\,\ddot{\varphi} + 2\dot{r}\,\dot{\varphi}\sin\theta + 2r\,\dot{\theta}\dot{\varphi}\cos\theta \end{cases}$$

8) - Velocity and acceleration in Intrinsic coordinates system

The velocity vector is tangent to the path in the same direction as the movement, it carries the unit vector $\vec{u}_T$ and is written:

$$\vec{v} = \|\vec{v}\|\,\vec{u}_T = \frac{dS}{dt}\,\vec{u}_T$$

where dS is the curvilinear abscissa

The acceleration of the point M in is written:

$$\vec{a} = a_T\,\vec{u}_T + a_N\,\vec{u}_N$$

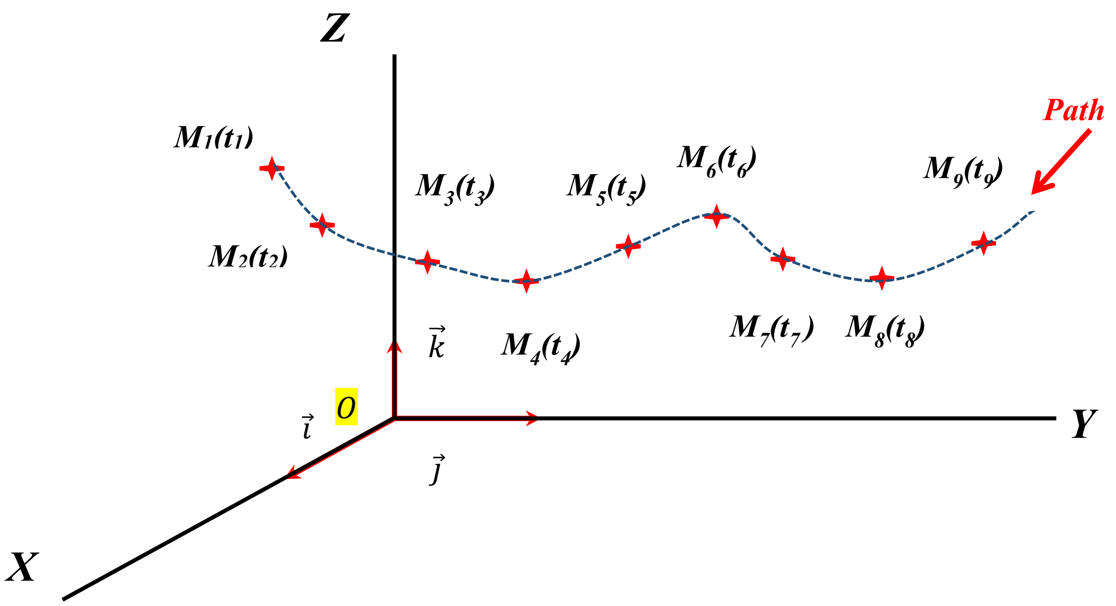
where:

$\boldsymbol{a_T}$ *is the tangential component of the acceleration :* $a_T = \frac{dv}{dt}$

$\boldsymbol{a_N}$ *is the normal component of the acceleration :* $a_N = \frac{v^2}{R}$ *here, R represent the curvature radius.*

8) Motion Path (trajectory)

The trajectory is a set of successive positions occupied by a moving particle at consecutive (successive) moments. In other words, the trajectory is the geometric locus of consecutive positions of the moving particle. It represents the connected line passing through all the positions of the moving particle during its motion in chronological order. By examining the shape of the trajectory followed by the moving particle, we can determine the type of its motion, whether it is straight, curved, or circular.



8-1) Path (trajectory) Equation

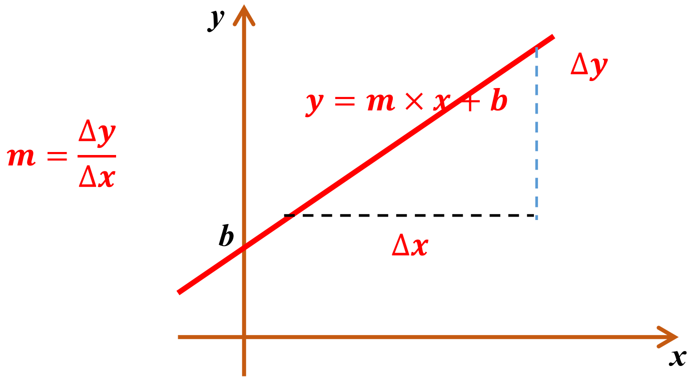
The path equation for a moving particle is a function that defines the relationship between the coordinates of this particle during its movement. Examples of some standard forms of the path equations:

- ***Straight Path Equation:***

In two-dimensional Cartesian coordinates, a straight path can be represented by the equation:

$$y = m \times x + b$$

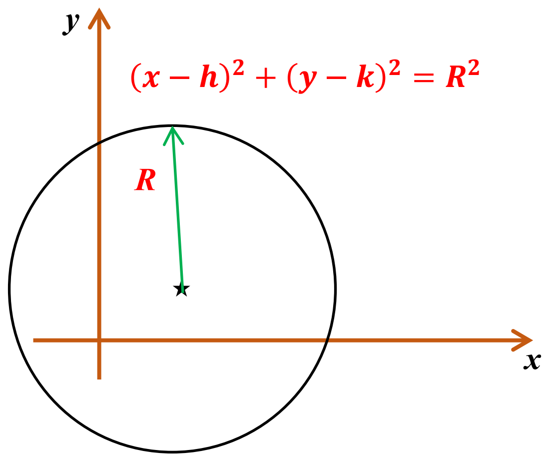
*where **m** is the slope of the line, and **b** is the y-intercept.*



- ***Circular Path Equation:***

*The equation for a circle with radius **R** centered at the pointC(h,k)*

$$(x-h)^2 + (y-k)^2 = R^2$$



- ***Parabolic Path Equation:***

The general equation for a parabola in standard form is:

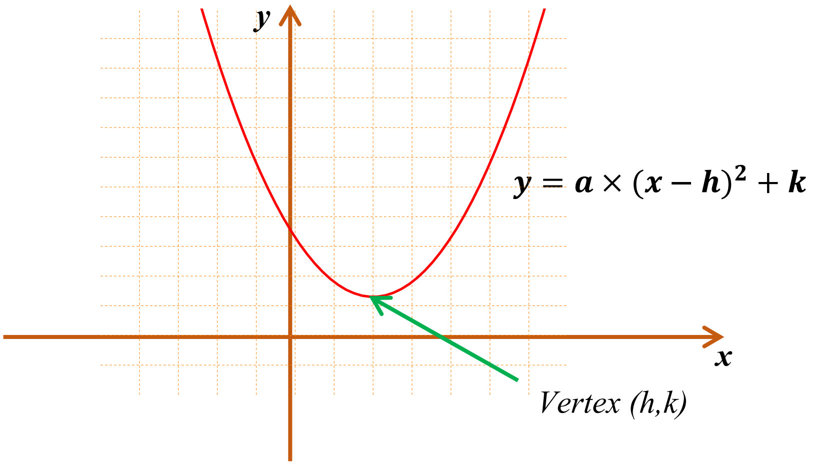
$$y = a \times x^2 + b \times x + c$$

*This is a quadratic equation, where **a**, **b**, and **c** are constants.*

The vertex form of a parabola is:

$$y = a \times (x-h)^2 + k$$

Where (h,k) is the vertex of the parabola.

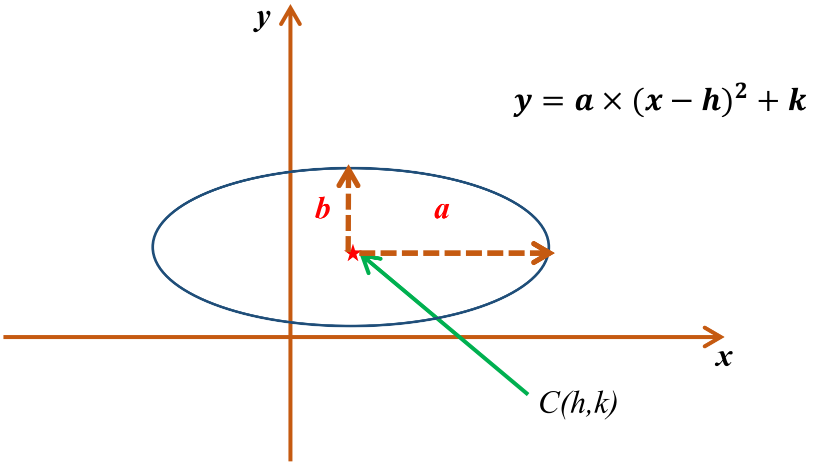


- ***Elliptical Path Equation:***

The general equation for a hyperbola is given by:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

*where, **(h, k)** is the center of the hyperbola, and **a** and **b** are constants determining the shape of the hyperbola.*



Example 01:

The position of a moving particle in the Cartesian coordinate system is described over time by the following equations: $x = 3(1 + cos(2t)), y = 3(2 + sin(2t))$

1- Determine the Path (trajectory) Equation and plot it.

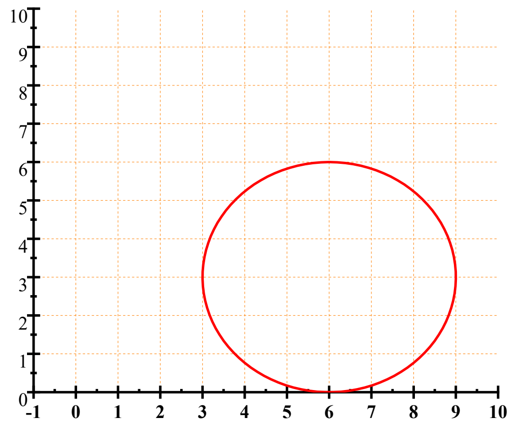
Solution:

$$\begin{cases} x = 3(1 + cos(2t)) \\ y = 2(2 + sin(2t)) \end{cases} \Rightarrow \begin{cases} x = 3 + 3\,cos(2t) \\ y = 6 + 3\,sin(2t) \end{cases} \Rightarrow \begin{cases} \frac{x-3}{3} = cos(2t) \\ \frac{y-6}{3} = sin(2t) \end{cases}$$

$$\begin{cases} \left(\frac{x-3}{3}\right)^2 = \left(cos(2t)\right)^2 \\ \left(\frac{y-6}{3}\right)^2 = \left(sin(2t)\right)^2 \end{cases} \Rightarrow \left(\frac{x-3}{3}\right)^2 + \left(\frac{y-6}{3}\right)^2 = 1$$

$$(x-3)^2 + (y-6)^2 = 3^2$$

The trajectory takes on a circular form as the derived equation represents a circle with a radius of 3, centered at the coordinates (3, 6)



9) Motion types

The nature of particle motion is determined by the shape of its path (straight, circular, or curved) and the variation in its velocity (constant, non-uniform, or uniformly varying).

9-1) Rectilinear motion (RM)

Rectilinear motion is the type of movement that occurs along a straight path, described by a single coordinate and studied in a one-dimensional coordinate system (dimension). The expressions for the kinematics characteristics (the position, velocity and acceleration) of a moving particle are given as follows:

$$\begin{cases} \overrightarrow{OM} = x(t)\,\vec{\imath} \\ \vec{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx(t)}{dt}\vec{\imath} = \dot{x}(t)\,\vec{\imath} \\ \vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2x(t)}{d^2t}\vec{\imath} = \ddot{x}(t)\vec{\imath} \end{cases}$$

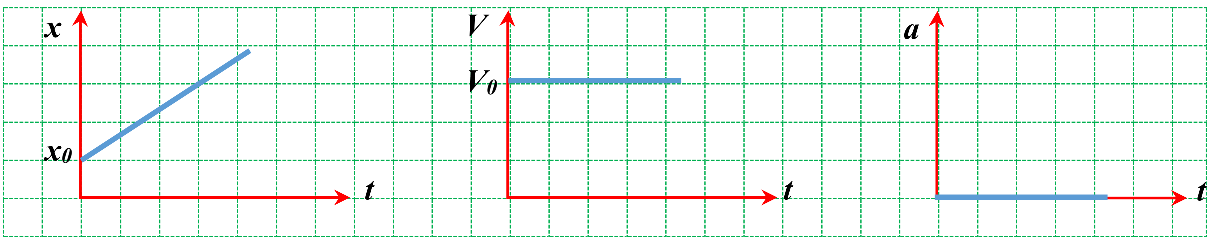
9-1-1) Uniform rectilinear motion (URM)

Uniform rectilinear motion occurs along a straight path with constant magnitude and direction of velocity (uniform velocity). This means that during this motion, the particle travels equal distances in equal time intervals. The expressions for the position, velocity, and acceleration of a moving particle are given as follows:

$$\begin{cases} \vec{a} = \dfrac{d\vec{V}}{dt} = \vec{0} \\ \vec{V} = \displaystyle\int \vec{a}\,dt = constant\ \vec{\imath} = V_0\vec{\imath} \\ \overrightarrow{OM} = \displaystyle\int \vec{V}\,dt = (V_0 \times t + x_0)\ \vec{\imath} \end{cases} \Rightarrow \begin{cases} \overrightarrow{OM} = x(t)\vec{\imath} = (V_0 \times t + x_0)\ \vec{\imath} \\ \vec{V} = \dfrac{d\overrightarrow{OM}}{dt} = constant\ \vec{\imath} = V_0\vec{\imath} \\ \vec{a} = \dfrac{d\vec{V}}{dt} = \vec{0} \end{cases}$$

The time equations of motion are given:

$$x(t) = V_0 \times t + x_0$$



9-1-2) Uniformly varied rectilinear motion (UVRM)

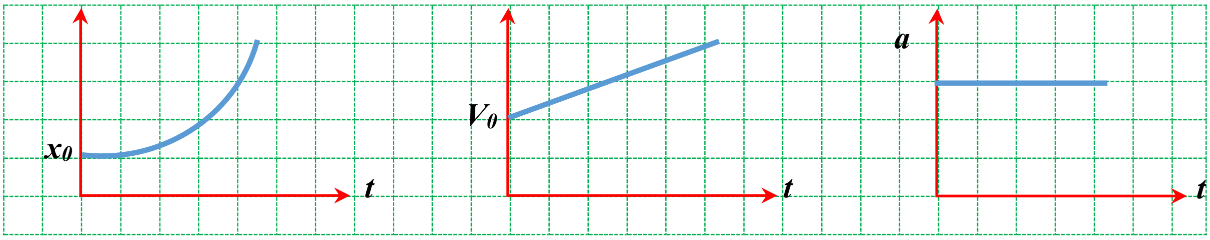
Uniformly Varied Rectilinear Motion occurs when an object moves along a straight path with a uniformly changing velocity, implying a constant acceleration. In simpler terms, during this motion, the particle's velocity undergoes a consistent change at a steady rate over equal time intervals. The expressions for the position, velocity, and acceleration of a moving particle are as follows:

$$\begin{cases} \vec{a} = \dfrac{d\vec{V}}{dt} = onstant\ \vec{\imath} = a\,\vec{\imath} \\ \vec{V} = \displaystyle\int \vec{a}\,dt = (a \times t + V_0)\,\vec{\imath} \\ \overrightarrow{OM} = \displaystyle\int \vec{V}\,dt = (\tfrac{1}{2}a \times t^2 + V_0 \times t + x_0)\ \vec{\imath} \end{cases} \Rightarrow \begin{cases} \overrightarrow{OM} = (\tfrac{1}{2}a \times t^2 + V_0 \times t + x_0)\,\vec{\imath} \\ \vec{V} = \dfrac{d\overrightarrow{OM}}{dt} = (a \times t + V_0)\,\vec{\imath} \\ \vec{a} = \dfrac{d\vec{V}}{dt} = a\,\vec{\imath} \end{cases}$$

The time equations of motion are given:

$$x(t) = (\tfrac{1}{2}a \times t^2 + V_0 \times t + x_0)$$

x ***V***

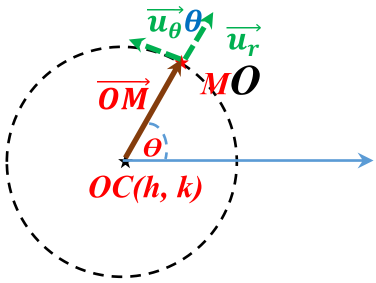


According to the direction of both the velocity and acceleration vectors of the particle's movement, we can distinguish two cases:

- ***Accelerated motion:*** *If acceleration and velocity are in the same direction* ⇒ $\vec{v}.\vec{a} > 0$
- ***Retarded motion:*** *If acceleration and velocity are in opposite directions*⇒ $\vec{v}.\vec{a} < 0$

9-2) Circular motion

*Circular motion is motion that takes place along a circular path that has a fixed radius **R**. This movement can be studied using the polar coordinate system.*



The expressions for the position, velocity and acceleration vectors of a moving particle are given as follows:

$$\overrightarrow{OM} = r\vec{u_r} = R\,\vec{u_r}$$

$$\vec{V}(t) = \frac{d(\overrightarrow{OM})}{dt} = \frac{d(R\vec{u_r})}{dt} = R\,\frac{d\vec{u_r}}{dt}$$

$$\frac{d\,(\vec{u}_r)}{dt} = \dot{\theta}\vec{u}_\theta$$

$$\frac{d\,(\vec{u}_\theta)}{dt} = -\dot{\theta}\vec{u}_r$$

$$\vec{V}(t) = R\,\dot{\theta}\vec{u}_\theta \quad where\ \ \dot{\theta} = \frac{d\theta}{dt}$$

$$\|\vec{V}(t)\| = \sqrt{(R\dot{\theta})^2} = R\dot{\theta}$$

$\dot{\theta}$ *is he angular velocity*

*for the instantaneous **acceleration** vector in **polar** coordinates is given as follows:*

$$\vec{a}(t) = \frac{d(\vec{V})}{dt} = \frac{d(R\dot{\theta}\vec{u}_\theta)}{dt}$$

$$= R\frac{d\dot{\theta}}{dt}\vec{u}_\theta + R\dot{\theta}\frac{d\vec{u}_\theta}{dt}$$

$$= R\ddot{\theta}\vec{u}_\theta - R\dot{\theta}\dot{\theta}\vec{u}_r$$

$$\vec{a}(t) = (-R\dot{\theta}^2)\,\vec{u}_r + (R\ddot{\theta})\vec{u}_\theta = (a_r)\,\vec{u}_r + (a_\theta)\vec{u}_\theta$$

$$\|\vec{a}(t)\| = \sqrt{\left(R\dot{\theta}^2\right)^2 + (R\ddot{\theta})^2}$$

$\ddot{\theta}$ *is the angular acceleration.*

Note: *For the intrinsic coordinates, we perform the same steps, only we replace the unit vectors as follows:* $\vec{u_r} = -\vec{u_N}$ ***and*** $\vec{u_\theta} = \vec{u_T}$

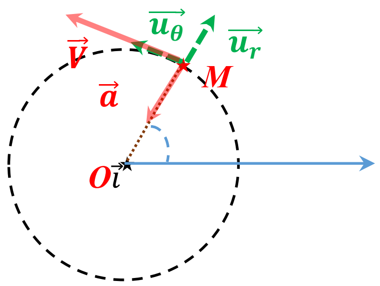
$$\vec{a}(t) = a_N\vec{u}_N + a_T\vec{u}_T$$

$$\begin{cases} a_N = R\dot{\theta}^2 \\ a_T = R\ddot{\theta} \end{cases}$$

9-2-1) Uniform circular motion UCM

The Uniform circular motion occurs along a circular path with a constant angular velocity ($\dot{\theta} = \frac{d\theta}{dt} = constant = \omega_0$). *This means that the particle displaces at constant angles during equal time intervals.*

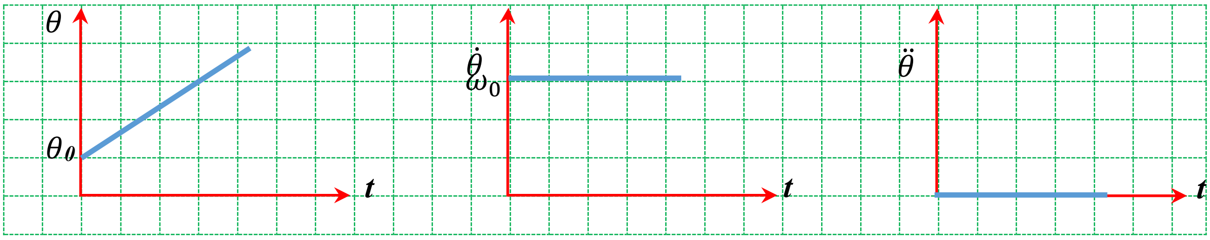
$$\begin{cases} \ddot{\theta} = \frac{d\dot{\theta}}{dt} = 0 \\ \dot{\theta} = \int \ddot{\theta}dt = constant = \omega_0 \end{cases} \Rightarrow \begin{cases} \overrightarrow{OM} = r\,\vec{u_r} = R\,\vec{u_r} \\ \vec{V}(t) = R\,\omega_0\vec{u}_\theta \\ \vec{a}(t) = (-R\dot{\theta}^2)\,\vec{u}_r \end{cases}$$



In uniform circular motion, the velocity vector stays consistently magnitude and tangential to the object's path at its position. Simultaneously, the acceleration is radial, pointing towards the center of the circle, with a zero tangential component.

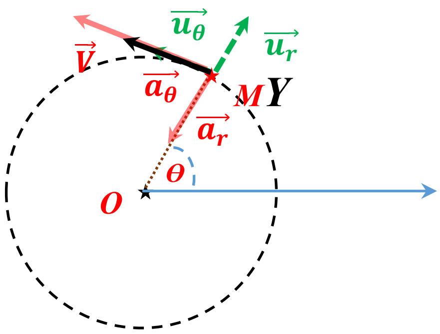
The time equations of motion are given:

$$\begin{cases} \ddot{\theta} = 0 \\ \dot{\theta} = \dfrac{d\theta}{dt} = \omega_0 \\ \theta(t) = \omega_0 \times t + \theta_0 \end{cases}$$



2-2-2) Uniformly varied circular motion (UVCM)

Uniformly varied circular motion (UVCM) is movement that takes place on a circular path with angular velocities that vary regularly (constant angular acceleration). In other words, the angular velocity changes at a constant rate over equal time intervals.

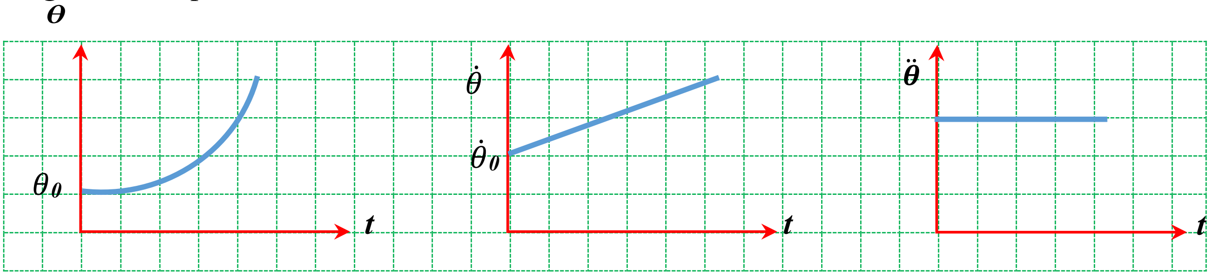


$$\begin{cases} \ddot{\theta} = \dfrac{d\dot{\theta}}{dt} = constant \\ \dot{\theta} = \displaystyle\int \ddot{\theta} dt = \ddot{\theta} \times t + \dot{\theta}_0 \end{cases} \Rightarrow \begin{cases} \overrightarrow{OM} = r\vec{u_r} = R\,\vec{u_r} \\ \vec{V}(t) = R\,\dot{\theta}\vec{u}_\theta \\ \vec{a}(t) = (-R\dot{\theta}^2)\,\vec{u}_r + (R\ddot{\theta})\vec{u}_\theta \end{cases}$$

The time equations of motion are given:

$$\begin{cases} \ddot{\theta} = constant \\ \dot{\theta} = \ddot{\theta} \times t + \dot{\theta}_0 \\ \theta(t) = \frac{1}{2}\ddot{\theta} \times t^2 + \dot{\theta}_0 \times t + \theta_0 \end{cases}$$

During uniformly varied circular motion, the acceleration vector has both radial and tangential components



10) Series of Solved Exercises

EXERCISE 01:

The coordinates of a material point (M) moving in the (xOy) plane are given in terms of time (t) as follows:

$$\begin{cases} x(t) = 2t^2 + 2t - 2 \\ y(t) = 3t + 2 \end{cases}$$

1) *Write the expression for the position vector*
2) *write the path equation of this material point motion.*
3) *Write the expression for its instantaneous velocity vector and its instantaneous acceleration vector*
4) *Write their expressions at the instant 2 s.*

Solution:

1) The expression for the position vector

$$\overrightarrow{OM}(t) = x(t)\,\vec{\imath} + y(t)\,\vec{\jmath}$$

$$\overrightarrow{OM}(t) = (2t^2 + 2t - 2)\vec{\imath} + (3t + 2)\,\vec{\jmath}$$

2) the path equation (trajectory)

$$\begin{cases} x(t) = 2t^2 + 2t - 2 \\ y(t) = 3t + 2 \end{cases} \Rightarrow t = \frac{y-2}{3} \Rightarrow x = 2(\frac{y-2}{3})^2 + 2\frac{y-2}{3} - 2$$

$$x = \left(\frac{2y^2}{9}\right) - \frac{2y}{9} - \frac{22}{9}$$

3) The expression for its instantaneous velocity vector and its instantaneous acceleration vector

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(2t^2 + 2t - 2)}{dt}\vec{\imath} + \frac{d(3t+2)}{dt}\vec{\jmath}$$

$$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt} = (4t + 2)\vec{\imath} + 3\vec{\jmath}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d(4t+2)}{dt}\vec{\imath} + \frac{d(3)}{dt}\vec{\jmath}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = 4\,\vec{\imath}$$

4) Their expressions at the instant 2 s.

$$\vec{v}(2) = (4 \times 2 + 2)\vec{\imath} + 3\vec{\jmath} = 10\vec{\imath} + 3\vec{\jmath}$$

$$\vec{a}(2) = 4\,\vec{\imath}$$

EXERCISE 02:

The Cartesian coordinates of a material point M in terms of time (t) are given as follows:

$$\begin{cases} x(t) = 2t \\ y(t) = \sqrt{4(1-t^2)} \end{cases}$$

1) *Write the equation of its path, the vector of its position, the vector of its instantaneous velocity and acceleration.*

Solution:

1. *the equation of its path*

$$\begin{cases} x(t) = 2t \\ y(t) = \sqrt{4(1-t^2)} \end{cases} \Rightarrow \begin{array}{l} x^2 = 4t^2 \\ y^2 = 4(1-t^2) \end{array}$$

By adding the two equations, we find:

$$x^2 + y^2 = 4t^2 + 4 - 4t^2$$

$$x^2 + y^2 = 4$$

this path is a circle with center O (0; 0) and radius R = 2

- *the vector of its position*

$$\overrightarrow{OM}(t) = x(t)\,\vec{\imath} + y(t)\,\vec{\jmath}$$

$$\overrightarrow{OM}(t) = 2t\,\vec{\imath} + \sqrt{4(1-t^2)}\,\vec{\jmath}$$

the vector of its instantaneous velocity and acceleration.

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(2t)}{dt}\vec{\imath} + \frac{d\sqrt{4(1-t^2)}}{dt}\vec{\jmath}$$

$$\vec{v}(t) = 2\,\vec{\imath} - \frac{8t}{2\sqrt{4(1-t^2)}}\vec{\jmath} = 2\,\vec{\imath} - \frac{2t}{\sqrt{(1-t^2)}}\vec{\jmath}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d(2\,)}{dt}\vec{\imath} + \frac{d(-\frac{2t}{\sqrt{(1-t^2)}})}{dt}\vec{\jmath}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{-2}{(1-t^2)^{\frac{3}{2}}}\vec{\jmath}$$

EXERCISE 03:

A particle moves along the x-axis following the equation $x(t) = 3t^2$. *Find :*

- ✓ *The displacement and average velocity of the particle during the time interval between t = 3 s and t = 5 s*
- ✓ *The instantaneous velocity of the particle as a function of time , and at t = 3 s , t = 4 s, and t = 5 s*
- ✓ *The average acceleration of the particle during the time interval between t = 3 s and t = 5 s*
- ✓ *The instantaneous acceleration of the particle as a function of time and at t = 4 s*

Solution:

- ✓ *The displacement and average velocity of the particle during the time interval between t = 3 s and t = 5 s*

$$\overrightarrow{\boldsymbol{M_3M_5}} = \overrightarrow{\boldsymbol{OM_5}} - \overrightarrow{\boldsymbol{OM_3}}$$

$$\overrightarrow{OM_i}(t) = x_i(t)\vec{\imath} = 3t^2\,\vec{\imath}$$

$$\begin{cases} \overrightarrow{OM_3} = \overrightarrow{OM}(t=3) = 3\times 9\,\vec{\imath} = 27\,\vec{\imath} \\ \overrightarrow{OM_5} = \overrightarrow{OM}(t=5) = 3\times 25\,\vec{\imath} = 75\,\vec{\imath} \end{cases} \Rightarrow \overrightarrow{\boldsymbol{M_3M_5}} = \overrightarrow{\boldsymbol{OM_5}} - \overrightarrow{\boldsymbol{OM_3}} = 75\,\vec{\imath} - 25\,\vec{\imath} = 48\,\vec{\imath}$$

$$\overrightarrow{V_{\iota,f}} = \frac{\overrightarrow{M_\iota M_f}}{\Delta t} = \frac{\overrightarrow{OM_f} - \overrightarrow{OM_\iota}}{t_f - t_i} = \frac{(x_f - x_i)}{t_f - t_i}\vec{\imath} = \frac{(48)}{5-3}\vec{\imath} = 24\,\vec{\imath}$$

- ✓ *The instantaneous velocity of the particle as a function of time , and at t = 3 s , t = 4 s, and t = 5 s*

$$\vec{V}(t) = \frac{d(\overrightarrow{OM})}{dt} = \frac{dx}{dt}\vec{\imath} = \frac{d(3t^2)}{dt}\,\vec{\imath} = 6t\,\vec{\imath}$$

at t = 3 s : $\vec{V}(t=3) = 6\times 3\ \vec{\imath} = 18\,\vec{\imath}$

at t = 4 s: $\vec{V}(t=4) = 6\times 4\ \vec{\imath} = 24\,\vec{\imath}$

at t = 5 s: $\vec{V}(t=5) = 6\times 5\ \vec{\imath} = 30\,\vec{\imath}$

- ✓ *The average acceleration of the particle during the time interval between t = 3 s and t = 5 s*

$$\overrightarrow{a_{\iota,f}} = \frac{\overrightarrow{\Delta V_{\iota,f}}}{\Delta t} = \frac{\vec{V_f} - \vec{V_\iota}}{t_f - t_i} = \frac{(V_{xf} - V_{xi})}{t_f - t_i}\vec{\imath} = \frac{(30 - 18)}{5-3}\vec{\imath} = 6\,\vec{\imath}$$

- ✓ *The instantaneous acceleration of the particle as a function of time and at t = 4 s*

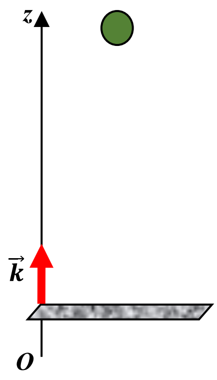
$$\vec{a}(t) = \frac{d(\vec{V})}{dt} = \frac{dV_x}{dt}\vec{\imath} = \frac{d(6t)}{dt}\,\vec{\imath} = 6\,\vec{\imath}$$

at t = 4 s

$$\vec{a}(t=4) = 6\,\vec{\imath}$$

EXERCISE 04:

A tennis ball is dropped from a building that is 60 m high. Find:



(a) Its position and velocity 3 s later;

(b) The total time it takes the ball to fall to the ground

(c) Its velocity just before it hits the ground.

Solution:

We have at $t = 0s, v_0 = 0\, m/s\,;\; z_0 = 60\, m$

$$\overrightarrow{OM} = z\,\vec{k} = (-\frac{1}{2}g\,t^2 + 60)\vec{k}$$

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(-\frac{1}{2}g\,t^2 + z_0)}{dt}\vec{k} = -g\,t\,\vec{k}$$

at t = 3s

$$\overrightarrow{OM}(t = 3s) = (-\frac{1}{2}g\,t^2 + z_0)\vec{k} = 15.9\,\vec{k}$$

$$\vec{v}(t = 3s) = -g\,t\,\vec{k} = -29.4\vec{k}$$

(b) The total time it takes the ball to fall to the ground

$$0\,\vec{k} = (-\frac{1}{2}g\,t^2 + 60)\vec{k}$$

$$-\frac{1}{2}g\,t^2 + 60 = 0 \;\Rightarrow\; t^2 = \frac{60}{4.9} \Rightarrow t = 3.49s$$

(c)its velocity just before it hits the ground

$$\vec{v}(t = 3.49s) = -9.8 \times 3.49\,\vec{k} = 34.2\, m/s$$

EXERCISE 05:

The polar coordinates of a material point moving in the plane are given as follows:

$$\begin{cases} r = \frac{1}{2}t^2 \\ \theta = t^2 \end{cases}$$

Find the expression for each of the position vector, velocity vector and acceleration vector.

Solution:

$$\overrightarrow{OM} = r\,\overrightarrow{u_r} = \frac{1}{2}t^2\,\overrightarrow{u_r}$$

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dr}{dt}\ \overrightarrow{u_r} + r\frac{d\ \overrightarrow{u_r}}{dt}$$

$$\frac{d\ \overrightarrow{u_r}}{dt} = \frac{d\theta}{dt}\ \overrightarrow{u_\theta} = \frac{d(t^2)}{dt}\ \overrightarrow{u_\theta} = 2t\ \overrightarrow{u_\theta}$$

$$\vec{v} = \frac{d(\frac{1}{2}t^2)}{dt}\ \overrightarrow{u_r} + \frac{1}{2}t^2 2t\ \overrightarrow{u_\theta} = \boldsymbol{t}\,\overrightarrow{u_r} + t^3\ \overrightarrow{u_\theta}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(\boldsymbol{t}\,\overrightarrow{u_r} + t^3\ \overrightarrow{u_\theta})}{dt} = \frac{d(t)}{dt}\overrightarrow{u_r}\ + t\ \frac{d(\overrightarrow{u_r})}{dt} + \frac{d(t^3)}{dt}\overrightarrow{u_\theta} + t^3\frac{d(\overrightarrow{u_\theta})}{dt}$$

$$\frac{d(\overrightarrow{u_\theta})}{dt} = -\frac{d\theta}{dt}\ \overrightarrow{u_r} = -2t\,\overrightarrow{u_r}$$

$$\vec{a} = \overrightarrow{u_r}\ + t \times\ 2t\ \overrightarrow{u_\theta}\ + 3t^2\,\overrightarrow{u_\theta} + t^3(-2t\,\overrightarrow{u_r})$$

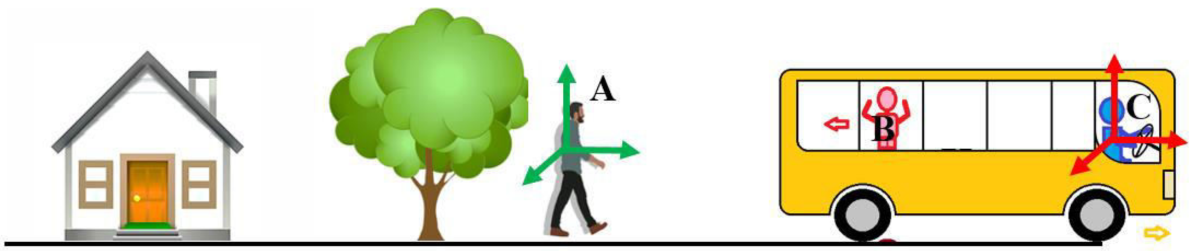
$$\vec{a} = (1 - 2t^4)\overrightarrow{u_r}\ + 5t^2\,\overrightarrow{u_\theta}$$

THIRD PART: Relative motion

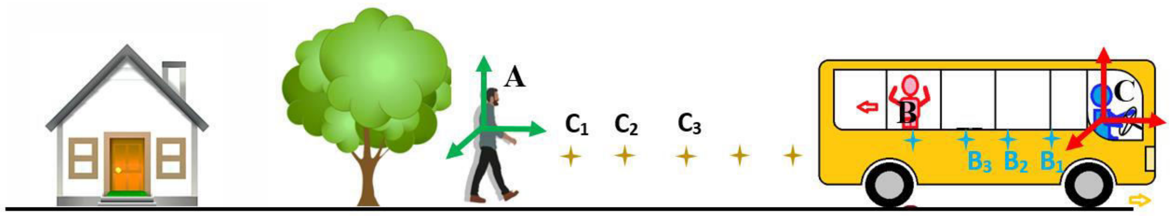
1) Relative motion

In this section, we will focus on studying the motion of a material particle relative to two reference frames, one of which is stationary (R'), and the other is moving (R).

As an example of relative motion, a person (A) sits under a tree to bid farewell to his friend (B), who is moving inside a bus that passes in front of him. In this case, the observer (A) sees that the movement of his friend (B) is composed of two movements: his accompanying movement and the one resulting from the movement of the bus. And its own movement inside the bus.



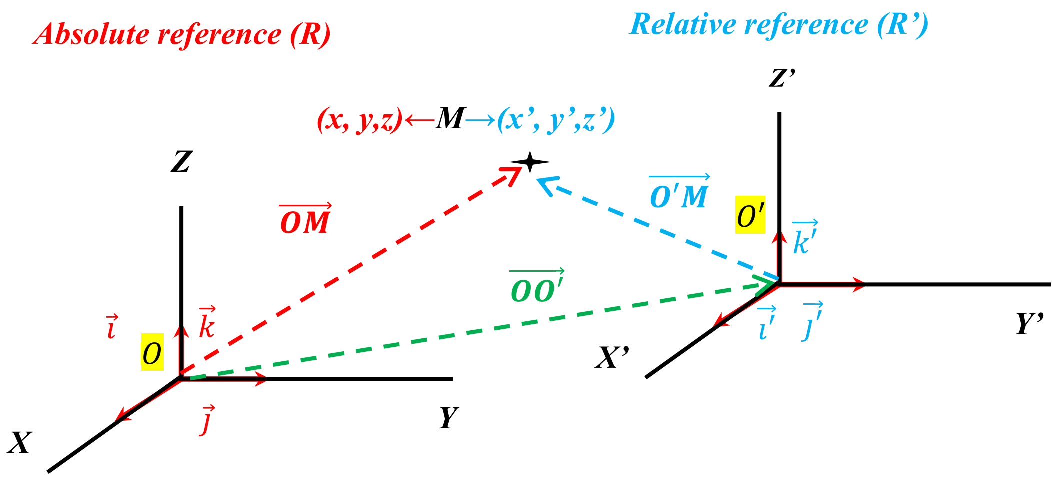
*As indicated in the figure below, the motion of person B (the passenger on the bus) with respect to person A (the observer) is a composite) superposition**)** motion. Therefore, their movement is influenced by the **positions**, **velocities**, **accelerations**, and the **path** of the moving reference, which is the bus. From the diagram, it is evident that the bus occupies different positions (C_1, C_2, C_3, etc.) concerning the stationary reference (observer). Simultaneously, person B moves to different positions (B_1, B_2, B_3,..., etc.) concerning the moving reference (observer C inside the bus).*



1-1) The absolute reference (stationary) and the relative reference (moving)

- *The stationary reference frame is called the **absolute** frame (R(O, x, y, z)),its origin **O**, andits unit vectors $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ remain constant relative to the origin O over time.*
- *The coordinates of the particle **M**with respect to the absolute reference are denoted as x, y, z*

$$\left.\frac{d\,\vec{\imath}}{dt}\right|_R = \vec{0}, \left.\frac{d\,\vec{\jmath}}{dt}\right|_R = \vec{0}, \left.\frac{d\,\vec{k}}{dt}\right|_R = \vec{0}$$



- *Therelative reference frame, is a moving reference with respect to the absolute frame, its origin O', and its unit vectors* $\vec{i'}$, $\vec{j'}$, and $\vec{k'}$ *remain constant relative to the origin O' over time, and are not constant relative to the absolute frame.*

$$\left.\frac{d\,\vec{i'}}{dt}\right|_{R'} = \vec{0}, \left.\frac{d\,\vec{j'}}{dt}\right|_{R'} = \vec{0}, \left.\frac{d\,\vec{k'}}{dt}\right|_{R'} = \vec{0}$$

$$\left.\frac{d\,\vec{i'}}{dt}\right|_{R} \neq \vec{0}, \left.\frac{d\,\vec{j'}}{dt}\right|_{R} \neq \vec{0}, \left.\frac{d\,\vec{k'}}{dt}\right|_{R} \neq \vec{0}$$

- *The coordinates of the particle **M** with respect to the relative reference R' are denoted as x', y', z'.*

✓ *The motion of M concerning R is labelled as absolute motion.*

✓ *The motion of M concerning* R' *is identified as relative motion.*

✓ *The motion of* R' *in relation to R is termed entrained motion.*

1-2) Analysing Particle Motion within an Absolute Frame of ReferenceR(O, x, y, z)

The expression provides the position vector, velocity, and acceleration for the moving particle M in the fixed or absolute frame (R) as follows:

1-2-1) Position Vector

Position vector of the particle M with respect to the absolute (fixed) frame (R) is expressed as:

$$\overrightarrow{OM} = \vec{r_a} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

1-2-2) Velocity vector (Absolute Velocity $\vec{V_a}(t)$)

Velocity vector of the particle M with respect to the absolute frame (R) is called absolute velocity $\vec{V_a}(t)$ *and is expressed as:*

$$\vec{V_a}(t) = \left.\frac{d(\vec{r_a})}{dt}\right|_{R} = \left.\frac{d(\overrightarrow{OM})}{dt}\right|_{R} = \frac{d(x\vec{i} + y\vec{j} + z\vec{k})}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

Where

$$\left.\frac{d\vec{i}}{dt}\right|_R = \vec{0}, \qquad \left.\frac{d\vec{j}}{dt}\right|_R = \vec{0}, \qquad \left.\frac{d\vec{k}}{dt}\right|_R = \vec{0}$$

$$\vec{V_a}(t) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

$$\vec{V_a}(t) = V_{a,x}\vec{i} + V_{a,y}\vec{j} + V_{a,z}\vec{k} \ \ where \begin{cases} V_{a,x} = \dfrac{dx}{dt} \\ V_{a,y} = \dfrac{dy}{dt} \\ V_{a,z} = \dfrac{dz}{dt} \end{cases}$$

1-2-3) Acceleration vector (Absolute Acceleration $\overrightarrow{a_a}(t)$)

Acceleration vector of the particle M with respect to the absolute frame (R) is called absolute acceleration $\overrightarrow{a_a}(t)$ and is defined as:

$$\overrightarrow{a_a}(t) = \left.\frac{d(\overrightarrow{V_a})}{dt}\right|_R = \left.\frac{d^2(\overrightarrow{OM})}{d^2t}\right|_R = \frac{d(\dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k})}{dt} = \frac{d(V_{a,x})}{dt}\vec{i} + \frac{d(V_{a,y})}{dt}\vec{j} + \frac{d(V_{a,z})}{dt}\vec{k}$$

$$\overrightarrow{a_a}(t) = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$$

$$\overrightarrow{a_a}(t) == a_{a,x}\vec{i} + a_{a,y}\vec{j} + a_{a,z}\vec{k} \ \ where \begin{cases} a_{a,x} = \dfrac{d(V_{a,x})}{dt} \\ a_{a,y} = \dfrac{d(V_{a,y})}{dt} \\ a_{a,z} = \dfrac{d(V_{a,z})}{dt} \end{cases}$$

2) Analysing Particle Motion within an Relative Frame of Reference R'(O', x', y', z')

The expression of the position vector, velocity, and acceleration for the moving particle M in the relative (moving) frame (R')are given as follows:

2-1) Position Vector

Position vector of the particleM with respect to the relative (moving) frame (R') is given as:

$$\overrightarrow{O'M} = \vec{r_r} = x'\vec{i'} + y'\vec{j'} + z'\vec{k'}$$

2-2) Velocity vector (Relative Velocity $\overrightarrow{V_r}(t)$)

Velocity vector of the particle M with respect to the relative frame (R') is called relative velocity $\overrightarrow{V_r}(t)$ *and is expressed as:*

$$\overrightarrow{V_r}(t) = \left.\frac{d(\overrightarrow{r_r})}{dt}\right|_{R'} = \left.\frac{d(\overrightarrow{O'M})}{dt}\right|_{R'} = \frac{d(x'\overrightarrow{i'} + y'\overrightarrow{j'} + z'\overrightarrow{k'})}{dt} = \frac{dx'}{dt}\overrightarrow{i'} + \frac{dy'}{dt}\overrightarrow{j'} + \frac{dz'}{dt}\overrightarrow{k'}$$

Where

$$\left.\frac{d\,\overrightarrow{i'}}{dt}\right|_{R'} = \vec{0}, \left.\frac{d\,\overrightarrow{j'}}{dt}\right|_{R'} = \vec{0}, \left.\frac{d\,\overrightarrow{k'}}{dt}\right|_{R'} = \vec{0}$$

$$\overrightarrow{V_r}(t) = \frac{dx'}{dt}\overrightarrow{i'} + \frac{dy'}{dt}\overrightarrow{j'} + \frac{dz'}{dt}\overrightarrow{k'} = \dot{x}'\overrightarrow{i'} + \dot{y}'\overrightarrow{j'} + \dot{z}'\overrightarrow{k'}$$

$$\overrightarrow{V_r}(t) == V_{r,x'}\overrightarrow{i'} + V_{r,y'}\overrightarrow{j'} + V_{r,z'}\overrightarrow{k'} \text{ where } \begin{cases} V_{r,x'} = \dfrac{dx'}{dt} \\ V_{r,y'} = \dfrac{dy'}{dt} \\ V_{r,z'} = \dfrac{dz'}{dt} \end{cases}$$

2-3) Acceleration vector (Relative Acceleration $\overrightarrow{a_r}(t)$)

The relative Acceleration vector $\overrightarrow{a_r}(t)$ *of the particle M with respect to the relative frame (R') is written as:*

$$\overrightarrow{a_r}(t) = \left.\frac{d(\overrightarrow{V_r})}{dt}\right|_{R'} = \left.\frac{d^2(\overrightarrow{O'M})}{d^2t}\right|_{R'} = \frac{d(\dot{x}'\overrightarrow{i'} + \dot{y}'\overrightarrow{j'} + \dot{z}'\,\overrightarrow{k'})}{dt}$$

$$= \frac{d(V_{r,x'})}{dt}\overrightarrow{i'} + \frac{d(V_{r,y'})}{dt}\overrightarrow{j'} + \frac{d(V_{r,z'})}{dt}\overrightarrow{k'}$$

$$\overrightarrow{a_r}(t) = \ddot{x}'\overrightarrow{i'} + \ddot{y}'\overrightarrow{j'} + \ddot{z}'\overrightarrow{k'}$$

$$\overrightarrow{a_r}(t) = a_{r,x'}\overrightarrow{i'} + a_{r,y'}\overrightarrow{j'} + a_{r,z'}\overrightarrow{k'} \text{ where } \begin{cases} a_{r,x'} = \dfrac{d(V_{r,x'})}{dt} \\ a_{r,y'} = \dfrac{d(V_{r,y'})}{dt} \\ a_{r,z'} = \dfrac{d(V_{r,z'})}{dt} \end{cases}$$

3) Relation between relative and absolute motions

Based on the principles of composition, it is possible to derive the mathematical formula that establishes the connections between the components of the position vector (velocity and acceleration) for the moving particle M in the absolute frame (R) and the components of the position vector (velocity and acceleration) for the moving particle M in a relative frame (R').

3-1) Relation between position Vectors ($\overrightarrow{OM}$ and $\overrightarrow{O'M}$)

The position vector ($\overrightarrow{OM}$) for the particle M in the absolute frame (R) is equal to the sum of the position vector ($\overrightarrow{O'M}$) for particle M in the relative frame (R') and the origin position vector ($\overrightarrow{OO'}$) of the relative frame with respect to the absolute frame. The relation between both vectors can be written according to the following expression:

$$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}$$

$$\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k} = \overrightarrow{OO'} + x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'}$$

$$\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k} = x_{O'}\vec{\imath} + y_{O'}\vec{\jmath} + z_{O'}\vec{k} + x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'}$$

Where ($x_{O'}, y_{O'}, z_{O'}$) are the coordinates of the origin O' with respect to the absolute reference frame (R).

In the case of irregular motion of the moving frame (R'), the unit vectors are unequal ($\vec{\imath} \neq \vec{\imath}', \vec{\jmath} \neq \vec{\jmath}'$, and $\vec{k} \neq \vec{k}'$)

3-2) Relation between velocity Vectors ($\vec{V_a}$ and $\vec{V_r}$)

When we derive the position vector for the moving particle M with respect to the absolute reference(R), we obtain the expression for the absolute velocity vector as follows:

$$\vec{V_a}(t) = \left.\frac{d(\overrightarrow{OM})}{dt}\right|_R = \left.\frac{d(\overrightarrow{OO'} + \overrightarrow{O'M})}{dt}\right|_R = \left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R + \left.\frac{d(\overrightarrow{O'M})}{dt}\right|_R$$

$$\vec{V_a}(t) = \left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R + \left.\frac{d(x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'})}{dt}\right|_R$$

The unit vectors ($\vec{\imath'}$, $\vec{\jmath'}$, and $\vec{k'}$) of the relative frame are constant with respect to the origin O' of the relative frame (R')($\left.\frac{d\,\vec{\imath'}}{dt}\right|_{R'} = \vec{0}, \left.\frac{d\,\vec{\jmath'}}{dt}\right|_{R'} = \vec{0}, \left.\frac{d\,\vec{k'}}{dt}\right|_{R'} = \vec{0}$), however, they are not constant with respect to the absolute frame (R). Therefore, the derivative of these unit vectors ($\vec{\imath'}$, $\vec{\jmath'}$, and $\vec{k'}$) with respect to the absolute frame (R) does not equal the zero vector ($\left.\frac{d\,\vec{\imath'}}{dt}\right|_{R} \neq \vec{0}, \left.\frac{d\,\vec{\jmath'}}{dt}\right|_{R} \neq \vec{0}, \left.\frac{d\,\vec{k'}}{dt}\right|_{R} \neq \vec{0}$).

$$\vec{V_a}(t) = \left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R + \frac{d(x')}{dt}\vec{\imath'} + x'\frac{d(\vec{\imath'})}{dt} + \frac{d(y')}{dt}\vec{\jmath'} + y'\frac{d(\vec{\jmath'})}{dt} + \frac{d(z')}{dt}\vec{k'} + z'\frac{d(\vec{k'})}{dt}$$

We rearrange the equation, and we get the following:

$$\vec{V_a}(t) = \underbrace{\left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt}}_{\vec{V_e}} + z'\frac{d\vec{k'}}{dt} + \underbrace{\frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'}}_{\vec{V_r}}$$

$$\vec{V_a}(t) = \underbrace{\left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt}}_{\vec{V_e}} + \underbrace{\dot{x}'\vec{\imath'} + \dot{y}'\vec{\jmath'} + \dot{z}'\vec{k'}}_{\vec{V_r}}$$

Hence, the absolute velocity vector ($\vec{V_a}(t)$) of the moving particle M is the sum of the relative ($\vec{V_r}(t)$) velocity vector and the entrained velocity vector $\vec{V_e}(t)$.

$$\vec{V_a}(t) = \vec{V_r}(t) + \vec{V_e}(t) \quad where \begin{cases} \vec{V_e}(t) = \left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt} \\ \vec{V_r}(t) = \frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'} \end{cases}$$

We can furtherdevelop the expression for the entrained velocity

$$\left.\frac{d(\vec{\imath'})}{dt}\right|_R = \frac{d\theta}{dt}\vec{\jmath'} \; where \; \frac{d\theta}{dt} = \omega \; (angular\ velocity) \; and \; \vec{\jmath'} = \vec{k'} \wedge \vec{\imath'}$$

$$\left.\frac{d(\vec{\imath'})}{dt}\right|_R = \frac{d\theta}{dt}\vec{\jmath'} = \omega\,\vec{k'} \wedge \vec{\imath'} = \vec{\omega} \wedge \vec{\imath'} \quad where \; = \omega\,\vec{k'} = \vec{\omega}$$

Using the same method we obtain

$$\left.\frac{d(\vec{j'})}{dt}\right|_R = \vec{\omega}\wedge\vec{j'} \text{and} \left.\frac{d(\vec{k'})}{dt}\right|_R = \vec{\omega}\wedge\vec{k'}$$

And therefore the entrained velocity can be written:

$$\vec{V_e}(t) = \left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R + x'\vec{\omega}\wedge\vec{i'} + y'\vec{\omega}\wedge\vec{j'} + z'\vec{\omega}\wedge\vec{k'}$$

$$\vec{V_e}(t) = \left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R + \vec{\omega}\wedge(x'\vec{i'}) + \vec{\omega}\wedge(y'\vec{j'}) + \vec{\omega}\wedge(z'\vec{k'})$$

$$\vec{V_e}(t) = \left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R + \vec{\omega}\wedge\left(x'\vec{i'} + y'\vec{j'} + z'\vec{k'}\right)$$

$$\vec{V_e}(t) = \left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R + \vec{\omega}\wedge\overrightarrow{O'M}$$

The entrained velocity vector ($\vec{V_e}(t)$) of the relative reference frame is related, on one hand, to the velocity vector of the origin O′ with respect to the absolute reference frame, and on the other hand, to the rotation of the unit vectors of the relative reference frame with respect to the absolute reference frame. Therefore, we must take into account both the translational and rotational motions of the relative reference frame.

3-3) Relation between acceleration Vectors ($\vec{a_a}$ and $\vec{a_r}$)

We derive the expression for absolute velocity with respect to time then, we simplify the obtained expressions as follows:

$$\vec{a_a}(t) = \frac{d}{dt}\left\{\left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R + x'\frac{d\vec{i'}}{dt} + y'\frac{d\vec{j'}}{dt} + z'\frac{d\vec{k'}}{dt} + \dot{x}'\vec{i'} + \dot{y}'\vec{j'} + \dot{z}'\vec{k'}\right\}$$

$$\vec{a_a}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + \frac{dx'}{dt}\frac{d\vec{i'}}{dt} + x'\frac{d^2\vec{i'}}{d^2t} + \frac{dy'}{dt}\frac{d\vec{j'}}{dt} + y'\frac{d^2\vec{j'}}{d^2t} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt} + z'\frac{d^2\vec{k'}}{d^2t}$$

$$+ d\frac{\dot{x}'}{dt}\vec{i'} + \dot{x}'\frac{d\vec{i'}}{dt} + d\frac{\dot{y}'}{dt}\vec{j'} + \dot{y}'\frac{d\vec{j'}}{dt} + d\frac{\dot{z}'}{dt}\vec{k'} + \dot{z}'\frac{d\vec{k'}}{dt}$$

We further simplify this equation and rearrange its terms, we obtain the following expression:

$$\vec{a_a}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + \dot{x}'\frac{d\vec{i'}}{dt} + \dot{y}'\frac{d\vec{j'}}{dt} + \dot{z}'\frac{d\vec{k'}}{dt} + x'\frac{d^2\vec{i'}}{d^2t} + y'\frac{d^2\vec{j'}}{d^2t} + z'\frac{d^2\vec{k'}}{d^2t}$$

$$+ \ddot{x}'\vec{i'} + \ddot{y}'\vec{j'} + \ddot{z}'\vec{k'} + \dot{x}'\frac{d\vec{i'}}{dt} + \dot{y}'\frac{d\vec{j'}}{dt} + \dot{z}'\frac{d\vec{k'}}{dt}$$

Finally, we obtain the following formula:

$$\vec{a_a}(t) = \underbrace{\left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + x'\frac{d^2\vec{i'}}{d^2t} + y'\frac{d^2\vec{j'}}{d^2t} + z'\frac{d^2\vec{k'}}{d^2t}}_{\vec{a_e}} + \underbrace{2\left(\dot{x}'\frac{d\vec{i'}}{dt} + \dot{y}'\frac{d\vec{j'}}{dt} + \dot{z}'\frac{d\vec{k'}}{dt}\right)}_{\vec{a_c}} + \underbrace{\ddot{x}'\vec{i'} + \ddot{y}'\vec{j'} + \ddot{z}'\vec{k'}}_{\vec{a_r}}$$

Consequently, the absolute acceleration vector ($\vec{a_a}$) for the moving particle M is the summation of the relative acceleration vector ($\vec{a_r}$), the entrained acceleration vector ($\vec{a_e}$), and the Coriolis acceleration vector ($\vec{a_c}$).

$$\vec{a_a}(t) = \vec{a_r}(t) + \vec{a_e}(t) + \vec{a_c}(t) \quad where \begin{cases} \vec{a_r}(t) = d\frac{\dot{x}'}{dt}\vec{i'} + d\frac{\dot{y}'}{dt}\vec{j'} + d\frac{\dot{z}'}{dt}\vec{k'} \\ \vec{a_e}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + x'\frac{d^2\vec{i'}}{d^2t} + y'\frac{d^2\vec{j'}}{d^2t} + z'\frac{d^2\vec{k'}}{d^2t} \\ \vec{a_c}(t) = 2\left(\dot{x}'\frac{d\vec{i'}}{dt} + \dot{y}'\frac{d\vec{j'}}{dt} + \dot{z}'\frac{d\vec{k'}}{dt}\right) \end{cases}$$

We can further simplify the Coriolis acceleration vector $\vec{a_c}(t)$and the entrained acceleration vector $\vec{a_e}(t)$as follows:

- *The Coriolis acceleration vector $\vec{a_c}(t)$*

  Using these relations

$$\left.\frac{d(\vec{i'})}{dt}\right|_R = \vec{\omega} \wedge \vec{i'} \;, \quad \left.\frac{d(\vec{j'})}{dt}\right|_R = \vec{\omega} \wedge \vec{j'} and \left.\frac{d(\vec{k'})}{dt}\right|_R = \vec{\omega} \wedge \vec{k'}$$

$$\vec{a_c}(t) = 2\left(\dot{x}'\frac{d\vec{i'}}{dt} + \dot{y}'\frac{d\vec{j'}}{dt} + \dot{z}'\frac{d\vec{k'}}{dt}\right) = 2\left(\dot{x}'\vec{\omega} \wedge \vec{i'} + \dot{y}'\vec{\omega} \wedge \vec{j'} + \dot{z}'\vec{\omega} \wedge \vec{k'}\right)$$

$$\vec{a_c}(t) = 2\left(\vec{\omega} \wedge (\dot{x}'\vec{i'} + \dot{y}'\vec{j'} + \dot{z}'\vec{k'})\right) = 2\vec{\omega} \wedge \vec{V_r}$$

- *The entrained acceleration vector $\vec{a_e}(t)$*

$$\vec{a_e}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + x'\frac{d^2\vec{i'}}{d^2t} + y'\frac{d^2\vec{j'}}{d^2t} + z'\frac{d^2\vec{k'}}{d^2t}$$

Using these relations

$$\begin{cases} \frac{d^2\vec{i'}}{d^2t} = \frac{d}{dt}\left(\vec{\omega} \wedge \vec{i'}\right) = \frac{d\vec{\omega}}{dt} \wedge \vec{i'} + \vec{\omega} \wedge \frac{d\vec{i'}}{dt} = \frac{d\vec{\omega}}{dt} \wedge \vec{i'} + \vec{\omega} \wedge \vec{\omega} \wedge \vec{i'} \\ \frac{d^2\vec{j'}}{d^2t} = \frac{d}{dt}\left(\vec{\omega} \wedge \vec{j'}\right) = \frac{d\vec{\omega}}{dt} \wedge \vec{j'} + \vec{\omega} \wedge \frac{d\vec{j'}}{dt} = \frac{d\vec{\omega}}{dt} \wedge \vec{j'} + \vec{\omega} \wedge \vec{\omega} \wedge \vec{j'} \\ \frac{d^2\vec{k'}}{d^2t} = \frac{d}{dt}\left(\vec{\omega} \wedge \vec{k'}\right) = \frac{d\vec{\omega}}{dt} \wedge \vec{k'} + \vec{\omega} \wedge \frac{d\vec{k'}}{dt} = \frac{d\vec{\omega}}{dt} \wedge \vec{k'} + \vec{\omega} \wedge \vec{\omega} \wedge \vec{k'} \end{cases}$$

$$\vec{a_e}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + x'(\frac{d\vec{\omega}}{dt} \wedge \vec{i'} + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{i'})) + y'(\frac{d\vec{\omega}}{dt} \wedge \vec{j'} + \vec{\omega} \wedge \ (\vec{\omega} \wedge \vec{j'})) + z'(\frac{d\vec{\omega}}{dt} \wedge \vec{k'} + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{k'}))$$

$$\vec{a_e}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + (\frac{d\vec{\omega}}{dt} \wedge x'\vec{i'} + \vec{\omega} \wedge (\vec{\omega} \wedge x'\vec{i'})) + \ (\frac{d\vec{\omega}}{dt} \wedge y'\vec{j'} + \vec{\omega} \wedge (\vec{\omega} \wedge y'\vec{j'})) + (\frac{d\vec{\omega}}{dt} \wedge z'\vec{k'} + \vec{\omega} \wedge (\vec{\omega} \wedge z'\vec{k'}))$$

$$\vec{a_e}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + (\frac{d\vec{\omega}}{dt} \wedge x'\vec{i'}) + \left(\frac{d\vec{\omega}}{dt} \wedge y'\vec{j'}\right) + (\frac{d\vec{\omega}}{dt} \wedge z'\vec{k'}) + \vec{\omega} \wedge (\vec{\omega} \wedge x'\vec{i'}) + \vec{\omega} \wedge (\vec{\omega} \wedge y'\vec{j'}) + (\vec{\omega} \wedge (\vec{\omega} \wedge z'\vec{k'}))$$

$$\vec{a_e}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + (\frac{d\vec{\omega}}{dt} \wedge (x'\vec{i'} + \ y'\vec{j'} + z'\vec{k'}) + \vec{\omega} \wedge (\vec{\omega} \wedge (x'\vec{i'} + \ y'\vec{j'} + \ z'\vec{k'})$$

$$\vec{a_e}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + (\frac{d\vec{\omega}}{dt} \wedge (\overrightarrow{O'M}) + \vec{\omega} \wedge \left(\vec{\omega} \wedge (\overrightarrow{O'M}\right)$$

Note: If the relative reference moves in a translation motion with respect to the absolutereference ($\vec{\omega} = \vec{0}$), then:

- $\vec{V_e}(t) = \left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R$
- $\vec{a_c}(t) = 2\left(\vec{\omega} \wedge (\dot{x'}\vec{i'} + \dot{y'}\vec{j'} + \dot{z'}\vec{k'}\right) = \vec{0}$
- $\vec{a_e}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R$

4) Solved exercises

EXERCISE 01:

The position vector expressions for a particle moving with respect to a fixed reference R and a reference moving horizontally with a constant velocity V_0 are given as follows:

$\overrightarrow{OM} = t\,\vec{\imath} + t^2\vec{\jmath} + (2t+3)\,\vec{k}$; $\overrightarrow{O'M} = t\,\vec{\imath'} + t^2\vec{\jmath'} + (4t+3)\,\vec{k'}$

- *Write the expression the position vector* $\overrightarrow{OO'}$
- *Write the expression of the absolute velocity and relative velocity of M.*
- *Deduce the entrained velocity and the nature of the movement of R' in relation to R.*
- *Determine the absolute acceleration, entrained acceleration, and relative acceleration of M*

Solution

-*Writing the expression the position vector* $\overrightarrow{OO'}$

$$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M} \Rightarrow \overrightarrow{OO'} = \overrightarrow{OM} - \overrightarrow{O'M}$$

$$\overrightarrow{OO'} = t\,\vec{\imath} + t^2\vec{\jmath} + (2t+3)\vec{k} - t\,\vec{\imath'} - t^2\vec{\jmath'} - (4t+3)\,\vec{k'}$$

$$\vec{\imath} = \vec{\imath}' \;\; ; \vec{\jmath} = \vec{\jmath}' \;\; ; \vec{k} = \vec{k}'$$

$$\overrightarrow{OO'} = -2t\,\vec{k}$$

- *The absolute velocity and relative velocity of M.*

$$\vec{v_a} = \frac{d(\overrightarrow{OM})}{dt} = \vec{\imath} + 2t\,\vec{\jmath} + 2\,\vec{k}$$

$$\vec{v_r} = \frac{d(\overrightarrow{O'M})}{dt} = \vec{\imath'} + 2t\,\vec{\jmath'} + 4\,\vec{k'}$$

$$\vec{\imath} = \vec{\imath}' \;\; ; \vec{\jmath} = \vec{\jmath}' \;\; ; \vec{k} = \vec{k}'$$

$$\vec{v_r} = \frac{d(\overrightarrow{O'M})}{dt} = \vec{\imath} + 2t\,\vec{\jmath} + 4\,\vec{k}$$

- *Deduce the entrained velocity and the nature of the movement of R' in relation to R.*

$$\vec{v_a} = \vec{v_r} + \vec{v_e}$$

$$\vec{v_e} = \vec{v_a} - \vec{v_r} = -2\,\vec{k}$$

or

$$\vec{v_e} = \frac{d(\overrightarrow{OO'})}{dt} = \frac{d(-2t\,\vec{k})}{dt} = -2\,\vec{k}$$

$$\|\vec{v_e}\| = 2$$

The motion of O' relative to O is uniform rectilinear motion

- The absolute acceleration, entrained acceleration, and relative acceleration of M

** The absolute acceleration*

$$\vec{a_a} = \frac{d(\vec{v_a})}{dt} = \frac{d(\vec{\imath} + 2t\,\vec{\jmath} + 2\,\vec{k})}{dt} = 2\,\vec{\jmath}$$

** The relative acceleration*

$$\vec{a_r} = \frac{d(\vec{v_r})}{dt} = \frac{d(\vec{\imath'} + 2t\,\vec{\jmath'} + 4\,\vec{k'})}{dt} = 2\,\vec{\jmath}' = 2\,\vec{\jmath}$$

** The entrained acceleration*

$\vec{a_a} = \vec{a_r} + \vec{a_e} + \vec{a_c}$ *in this case* $\vec{a_c} = \vec{0}$

$$\vec{a_e} = \vec{a_a} + \vec{a_r} = \vec{0}$$

Exercise 02:

The motion of a mobile is defined by the position vector in a mobile frame R':

$$\overrightarrow{O'M} = 5t\,\vec{\imath'} + (t^2 - t)\vec{\jmath'} - 2t\,\vec{k'}$$

This frame undergoes rectilinear translational motion with respect to a fixed reference R, characterized by a velocity vector $\vec{V_e} = 2t\,\vec{\imath} + \vec{\jmath} + \vec{k}$

- *Determine the expression for the absolute velocity of M concerning the reference R.*
- *Calculate both the relative and absolute acceleration of the mobile M. Deduce the coriolis acceleration*

Solution

- *Determining the expression for the absolute velocity of M concerning the reference R*

due to the rectilinear translational motion of R' with respect to R $\Rightarrow \vec{\imath} = \vec{\imath}'$; $\vec{\jmath} = \vec{\jmath}'$; $\vec{k} = \vec{k}'$

$\overrightarrow{O'M} = 5t\,\vec{\imath'} + (t^2 - t)\vec{\jmath'} - 2t\,\vec{k'}$; $\vec{V_e} = 2t\,\vec{\imath} + \vec{\jmath} + \vec{k}$

$$\vec{v_a} = \vec{v_r} + \vec{v_e}$$

$$\vec{v_r} = \frac{d(\overrightarrow{O'M})}{dt} = \frac{d(5t\,\vec{\imath'} + (t^2 - t)\vec{\jmath'} - 2t\,\vec{k'})}{dt} = 5\,\vec{\imath'} + (2t - 1)\vec{\jmath'} - 2\,\vec{k'}$$

$$\vec{v_r} = 5\,\vec{\imath} + (2t - 1)\vec{\jmath} - 2\vec{k}$$

$$\vec{v_a} = (5 + 2t)\vec{\imath} + 2t\,\vec{\jmath} - \vec{k}$$

- *Calculating the relative and absolute acceleration of the mobile M.*

**The absolute acceleration*

$$\vec{a_a} = \frac{d(\vec{v_a})}{dt} = \frac{d((5 + 2t)\vec{\imath} + 2t\,\vec{\jmath} - \vec{k})}{dt} = 2\,\vec{\imath} + 2\,\vec{\jmath}$$

* *The relative acceleration*

$$\overrightarrow{a_r} = \frac{d(\overrightarrow{v_r})}{dt} = \frac{d(5\,\vec{\imath} + (2t-1)\vec{\jmath} - 2\vec{k})}{dt} = 2\,\vec{\jmath}' = 2\,\vec{\jmath}$$

* *The entrained acceleration*

$$\overrightarrow{a_e} = 2\,\vec{\imath}$$

- *Deduce the coriolis acceleration*

$\overrightarrow{a_a} = \overrightarrow{a_r} + \overrightarrow{a_e} + \overrightarrow{a_c}$ *in this case* $\overrightarrow{a_c} = \overrightarrow{a_a} - \overrightarrow{a_r} - \overrightarrow{a_e}$

$$\overrightarrow{a_e} = 2\,\vec{\imath} + 2\,\vec{\jmath} - 2\,\vec{\imath} - 2\,\vec{\jmath} = \vec{0}$$

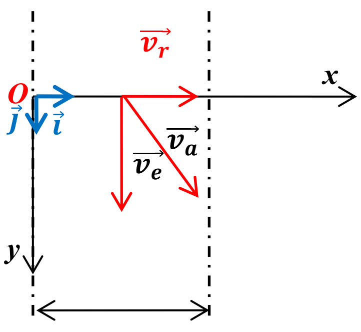
Exercise 03:

A swimmer is crossing a river with a width of L = 1 km, moving from one bank to the other perpendicular to the current at a constant velocity v = 0.5 km/h. The current velocity is = 2 km/h.

- *What does each of these two velocities signify?, Write their analytical expressions in the Cartesian coordinate system (xoy).*
- *Calculate the angle at which the swimmer deviates.*
- *Determine the swimmer's trajectory, velocity, and the time required to reach the opposite bank.*

Solution

- *The two velocities signified*



2 km/h⇒ the entrained velocity $\overrightarrow{v_e}$

0.5 km/h⇒ the relative velocity $\overrightarrow{v_r}$

- *The analytical expressions of the two velocities in the Cartesian coordinate system (xoy)*

$$\overrightarrow{v_e} = 2\,\vec{\jmath} \quad ; \overrightarrow{v_r} = 0.5\,\vec{\imath}$$

- *Calculate the angle at which the swimmer deviates.*

$$\overrightarrow{v_a} = \overrightarrow{v_e} + \overrightarrow{v_r}$$

$$\vec{v_a} = v_a \cos\alpha\, \vec{\imath} + v_a \sin\alpha\, \vec{\jmath} = 0.5\, \vec{\imath} + 2\, \vec{\jmath}$$

$$\begin{cases} v_a \cos\alpha = 0.5 \\ v_a \sin\alpha = 2 \end{cases} \Rightarrow \tan\alpha = \frac{2}{0.5} = 4 \Rightarrow \alpha = 75.9^0$$

$$v_a = \frac{2}{\sin\alpha} = 2.06\ km/h$$

- *Determining the swimmer's trajectory, velocity, and the time required to reach the opposite bank.*

$$\vec{v_a} = 0.5\, \vec{\imath} + 2\, \vec{\jmath} = \frac{d(\overrightarrow{OM})}{dt}$$

$$d(\overrightarrow{OM}) = (0.5\, \vec{\imath} + 2\, \vec{\jmath})\ dt$$

$$\overrightarrow{OM} = \int (0.5\, \vec{\imath} + 2\, \vec{\jmath})\, dt = 0.5t\, \vec{\imath} + 2t\, \vec{\jmath}$$

The trajectory of the swimmer

$$\begin{cases} x = 0.5t \\ y = 2t \end{cases} \Rightarrow \frac{y}{x} = 4 \Rightarrow y = 4x$$

$y = 4x \Rightarrow$ straight line (rectilinear motion

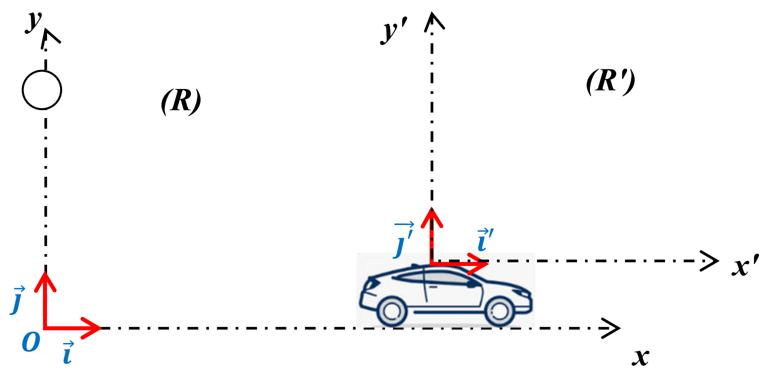
Exercise 04:

A ball falls without initial velocity from a building of height H, and its descent follows uniformly accelerated rectilinear motion with acceleration g.

- *Determine the trajectory of the ball in a reference frame linked to a car moving at a constant velocity, passing through the vertical drop point at the moment of release.*
- *Identify the trajectory of the ball in a reference frame tied to a car moving along a uniformly accelerated straight path with acceleration g, passing vertically under the point of descent at the moment of departure.*

Solution

- *Determining the trajectory of the ball in a reference frame linked to the moving car.*



in ht reference (R) :

$$\overrightarrow{OM} = (-\frac{1}{2}\,g\,t^2 + H)\,\vec{j}$$

$$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}$$

$$\overrightarrow{O'M} = \overrightarrow{OM} - \overrightarrow{OO'}$$

$$\vec{v_e} = v_0\,\vec{\imath} = \frac{d(\overrightarrow{OO'})}{dt}$$

$$d\overrightarrow{OO'} = v_0\,\vec{\imath}\;dt$$

$$\overrightarrow{OO'} = \int (v_0\,\vec{\imath})\,dt = v_0\,t\,\vec{\imath} + \overrightarrow{x_0}$$

at t =0 s , $\overrightarrow{OO'} = \vec{0} \Rightarrow \overrightarrow{OO'} = v_0\,t\,\vec{\imath}$

$$\overrightarrow{O'M} = \overrightarrow{OM} - \overrightarrow{OO'} = (-\frac{1}{2}\,g\,t^2 + H)\,\vec{j} - v_0\,t\,\vec{\imath}$$

$$\begin{cases} x' = -v_0\,t \\ y' = -\frac{1}{2}\,g\,t^2 + H \end{cases} \Rightarrow t = -\frac{x'}{v_0} \quad \Rightarrow y' = -\frac{1}{2}g\,\frac{x'^2}{{v_0}^2} + H$$

- *Identifying the trajectory of the ball*

$$\overrightarrow{OO'} = (\frac{1}{2}a\;t^2 + v_0 t + x_0)\;\vec{\imath}$$

$$\overrightarrow{OO'} = \frac{1}{2}a\;t^2\;\vec{\imath}$$

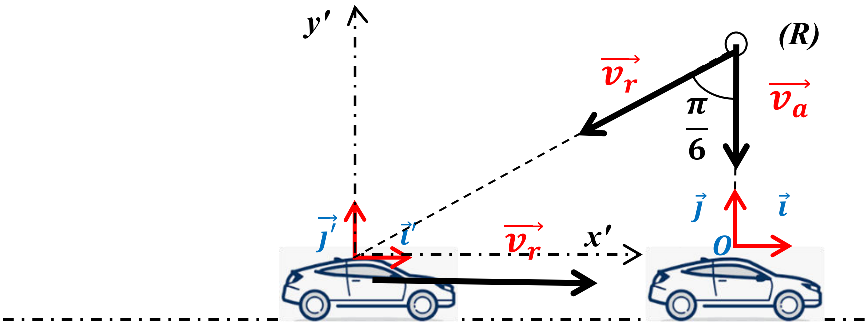
$$\overrightarrow{O'M} = \overrightarrow{OM} - \overrightarrow{OO'} = (\frac{1}{2}\,g\,t^2 + H)\,\vec{j} - \frac{1}{2}a\;t^2\,\vec{\imath}$$

$$\begin{cases} x' = -\frac{1}{2}a\;t^2 \\ y' = -\frac{1}{2}\,g\,t^2 + H \end{cases} \Rightarrow t^2 = -\frac{2x'}{a} \quad \Rightarrow y' = g\,\frac{x'}{a} + H$$

Exercise 05:

While driving in the rain at 100 km/h on a flat road, a driver notices that raindrops, seen through the side windows of the car, follow trajectories making an angle of $(\frac{\pi}{6} rad)$ with the vertical. Upon stopping the car, the driver observes that the rain is actually falling vertically. Calculate the velocity of the rain relative to the stationary car and relative to the car moving at 100 km/h.

Solution



- *The absolute velocity of the rain drops.*

$$\vec{v_a} = \vec{v_r} + \vec{v_e}$$

$$\vec{v_e} = v_e\,\vec{i} \quad ; \vec{v_a} = -v_a\,\vec{j}$$

$$\vec{v_r} = -v_r \sin\frac{\pi}{6}\,\vec{i} - v_r \cos\frac{\pi}{6}\,\vec{j}$$

$$-v_a\,\vec{j} = v_e\,\vec{i} + -v_r \sin\frac{\pi}{6}\,\vec{i} - v_r \cos\frac{\pi}{6}\,\vec{j}$$

$$\begin{cases} 0 = v_e + -v_r \sin\frac{\pi}{6} \\ -v_a = -v_r \cos\frac{\pi}{6} \end{cases} \Rightarrow \quad \begin{matrix} v_e = v_r \sin\frac{\pi}{6} \\ v_a = v_r \cos\frac{\pi}{6} \end{matrix}$$

$$v_r = \frac{v_e}{\sin\frac{\pi}{6}} = \frac{100}{0.5} = 200$$

$$v_a = v_r \cos\frac{\pi}{6} = 200\frac{\sqrt{3}}{2} = 100\sqrt{3}$$

Exercise 06:

In the coordinate system $R'(O', \vec{i'}, \vec{j'}, \vec{k'})$, the Cartesian coordinates of a material object M are expressed as functions of time: $x' = t^2 + 3t$, $y' = t$, $z' = -t^3$. The coordinate system R' moves in a uniform rectilinear translation with a velocity vector $\vec{V_e} = -3\vec{i} + 5\vec{k}$ relative to an absolute coordinate system R.

1. *Determine the expression for the velocity vector of M relative to coordinate system R.*

2. *Infer the coordinates of M in coordinate system R, given that att=0, in coordinate system R, M is located at the point (0, 1, 0).*
3. *Calculate the relative and absolute accelerations of M.*

Solution

- *The expression for the velocity vector of M relative to coordinate system R*

$$\vec{v_a} = \vec{v_r} + \vec{v_e}$$

$$\vec{v_r} = \frac{d(\overrightarrow{O'M})}{dt} = \frac{d(t^2 + 3t)\vec{\imath'} + (t)\vec{\jmath'} - t^3\vec{k'})}{dt} = (2t+3)\vec{\imath'} + \vec{\jmath'} - 3t^2\vec{k'}$$

$$\vec{v_r} = (2t+3)\vec{\imath} + \vec{\jmath} - 3t^2\vec{k}$$

$$\vec{v_a} = (2t+3)\vec{\imath} + \vec{\jmath} - 3t^2\vec{k} - 3\vec{\imath} + 5\vec{k} = (2t+3)\vec{\imath} + \vec{\jmath} + (-3t^2 + 5)\vec{k}$$

- *The coordinates of M in coordinate system R, given that att=0, in coordinate system R, M is located at the point (0, 1, 0).*

$$\vec{v_a} = (2t+3)\vec{\imath} + \vec{\jmath} + (-3t^2 + 5)\vec{k} = \frac{d(\overrightarrow{OM})}{dt}$$

$$d(\overrightarrow{OM}) = ((2t+3)\vec{\imath} + \vec{\jmath} + (-3t^2 + 5)\vec{k})\ dt$$

$$\overrightarrow{OM} = \int \left((2t+3)\vec{\imath} + \vec{\jmath} + (-3t^2 + 5)\vec{k}\right) dt$$

$$\overrightarrow{OM} = (t^2 + 3t + a)\vec{\imath} + (t + b)\vec{\jmath} + (-t^3 + 5t + c)\vec{k}$$

$$at\ t = 0s \Rightarrow \overrightarrow{OM} = (0)\vec{\imath} + (1)\vec{\jmath} + (0)\vec{k} \Rightarrow \begin{cases} a = 0 \\ b = 1 \\ c = 0 \end{cases}$$

$$\overrightarrow{OM} = (t^2 + 3t)\vec{\imath} + (t+1)\vec{\jmath} + (-t^3 + 5t)\vec{k}$$

- *The absolute and relative velocities*

**The absolute acceleration*

$$\vec{a_a} = \frac{d(\vec{v_a})}{dt} = \frac{d((2t+3)\vec{\imath} + \vec{\jmath} + (-3t^2 + 5)\vec{k})}{dt} = 2\vec{\imath} - 6t\vec{k}$$

** The relative acceleration*

$$\vec{a_r} = \frac{d(\vec{v_r})}{dt} = \frac{d((2t+3)\vec{\imath} + \vec{\jmath} - 3t^2\vec{k})}{dt} = 2\vec{\imath} - 6t\vec{k}$$

5) Additional exercises

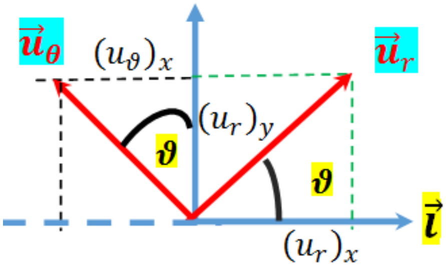
EXERCISE 01:

Using $\vec{\imath}$ *and* $\vec{\jmath}$ *as the unit vectors for the Cartesian coordinate system, and* $\vec{u}_r$ *and* $\vec{u}_\theta$ *as the unit vectors for the polar coordinate system (where* θ *is time-dependent),*

1) *Write the expressions for the unit vectors* $\vec{u}_r$ *and* $\vec{u}_\theta$ *in terms of* $\vec{\imath}$ *and* $\vec{\jmath}$.
2) *Calculate the derivatives of these unit vectors with respect to both time and* θ.
3) *Express the unit vectors* $\vec{\imath}$ *and* $\vec{\jmath}$ *in terms of* $\vec{u}_r$ *and* $\vec{u}_\theta$.
4) *Compute the derivatives of* $\vec{\imath}$ *and* $\vec{\jmath}$ *unit vectors with respect to both time and* θ

Solution:

1- the expressions for the unit vectors $\vec{\boldsymbol{u}}_r$ *and* $\vec{\boldsymbol{u}}_\theta$ *in terms of* $\vec{\boldsymbol{\imath}}$ *and* $\vec{\boldsymbol{\jmath}}$



We can obtain the expressions for the unit vectors in polar coordinates in terms of the unit vectors in Cartesian coordinates through the process of orthogonal projection (meaning we find the components of the polar unit vectors in Cartesian coordinates) as follows:

$$\vec{\boldsymbol{u}}_r = (u_r)_x \times \vec{\boldsymbol{\imath}} + (u_r)_y \times \vec{\boldsymbol{\jmath}}$$

$(u_r)_x$ *and* $(u_r)_y$ *are the components of the polar unit vectors* $\vec{\boldsymbol{u}}_r$ *in Cartesian coordinates*

$(u_r)_x = \|\vec{\boldsymbol{u}}_r\| \times \cos\vartheta$; $(u_r)_y = \|\vec{\boldsymbol{u}}_r\| \times \sin\vartheta$ $\|\vec{\boldsymbol{u}}_r\| = 1$

$$\vec{\boldsymbol{u}}_r = \cos\vartheta \times \vec{\boldsymbol{\imath}} + \sin\vartheta \times \vec{\boldsymbol{\jmath}}$$

$$\vec{\boldsymbol{u}}_\vartheta = -(u_\vartheta)_x \times \vec{\boldsymbol{\imath}} + (u_\vartheta)_y \times \vec{\boldsymbol{\jmath}}$$

$(u_\vartheta)_x$ *and* $(u_\vartheta)_y$ *are the components of the polar unit vectors* $\vec{\boldsymbol{u}}_\vartheta$ *in Cartesian coordinates*

$(u_\vartheta)_x = \|\vec{\boldsymbol{u}}_\vartheta\| \times \sin\vartheta$; $(u_\vartheta)_y = \|\vec{\boldsymbol{u}}_\vartheta\| \times \cos\vartheta$ $\|\vec{\boldsymbol{u}}_\vartheta\| = 1$

$$\vec{\boldsymbol{u}}_\vartheta = -\sin\vartheta \times \vec{\boldsymbol{\imath}} + \cos\vartheta \times \vec{\boldsymbol{\jmath}}$$

$$\begin{cases} \vec{\boldsymbol{u}}_r = \cos\vartheta \times \vec{\boldsymbol{\imath}} + \sin\vartheta \times \vec{\boldsymbol{\jmath}} \\ \vec{\boldsymbol{u}}_\vartheta = -\sin\vartheta \times \vec{\boldsymbol{\imath}} + \cos\vartheta \times \vec{\boldsymbol{\jmath}} \end{cases}$$

2- Calculate the derivatives of these unit vectors with respect to both time and $\boldsymbol{\theta}$*.*

$$\frac{d\,(\vec{u}_r)}{d\vartheta} = \frac{d(\cos\vartheta \times \vec{\imath} + \sin\vartheta \times \vec{\jmath})}{d\vartheta} = -\sin\vartheta \times \vec{\imath} + \cos\vartheta \times \vec{\jmath} = \vec{\boldsymbol{u}}_{\vartheta} \qquad where \quad \frac{d\,\vec{\imath}}{d\vartheta} = \vec{0}\,\frac{d\,\vec{\jmath}}{d\vartheta} = \vec{0}$$

$$\frac{d\,(\vec{u}_\vartheta)}{d\vartheta} = \frac{d(-\sin\vartheta \times \vec{\imath} + \cos\vartheta \times \vec{\jmath})}{d\vartheta} = -\cos\vartheta \times \vec{\imath} - \sin\vartheta \times \vec{\jmath} = -\vec{\boldsymbol{u}}_r$$

$$\frac{d\,(\vec{u}_r)}{dt} = \frac{d(\cos\vartheta \times \vec{\imath} + \sin\vartheta \times \vec{\jmath})}{dt} = -\frac{d\vartheta}{dt}\sin\vartheta \times \vec{\imath} + \frac{d\vartheta}{dt}\cos\vartheta \times \vec{\jmath}$$

where

$$\frac{d\,\vec{\imath}}{dt} = \vec{0}\,\frac{d\,\vec{\jmath}}{dt} = \vec{0}\,\frac{d\vartheta}{dt} = \dot{\boldsymbol{\vartheta}}$$

$$\frac{d\,(\vec{u}_r)}{dt} = = \frac{d\vartheta}{dt}(-\sin\vartheta \times \vec{\imath} + \cos\vartheta \times \vec{\jmath}) = \dot{\boldsymbol{\vartheta}}\vec{\boldsymbol{u}}_{\vartheta}$$

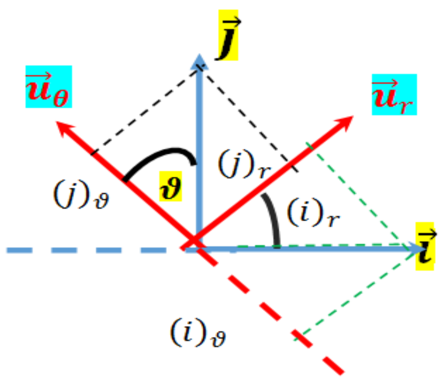
$$\frac{d\,(\vec{u}_r)}{dt} = \frac{d(-\sin\vartheta \times \vec{\imath} + \cos\vartheta \times \vec{\jmath})}{dt} = -\frac{d\vartheta}{dt}\cos\vartheta \times \vec{\imath} - \frac{d\vartheta}{dt}\sin\vartheta \times \vec{\jmath}$$

where

$$\frac{d\,\vec{\imath}}{dt} = \vec{0}\,\frac{d\,\vec{\jmath}}{dt} = \vec{0}\,\frac{d\vartheta}{dt} = \dot{\boldsymbol{\vartheta}}$$

$$\frac{d\,(\vec{u}_r)}{dt} = = -\frac{d\vartheta}{dt}(\cos\vartheta \times \vec{\imath} + \sin\vartheta \times \vec{\jmath}) = -\dot{\boldsymbol{\vartheta}}\vec{\boldsymbol{u}}_r$$

3- Express the unit vectors $\vec{\boldsymbol{\imath}}$ *and* $\vec{\boldsymbol{\jmath}}$ *in terms of* $\vec{\boldsymbol{u}}_r$ *and* $\vec{\boldsymbol{u}}_{\boldsymbol{\theta}}$*.*



Method 01: *We can obtain the expressions for the unit vectors in Cartesian coordinates in terms of the unit vectors in polar coordinates through the process of orthogonal projection (meaning we find the components of the Cartesian unit vectors in polar coordinates) as follows:*

$\vec{\boldsymbol{\imath}} = (i)_r \times \vec{\boldsymbol{u}}_r + (i)_\vartheta \times \vec{\boldsymbol{u}}_\vartheta$

$(i)_r$*and* $(i)_\vartheta$ *are the components of the cartesian unit vectors* $\vec{\boldsymbol{\imath}}$ *in polar coordinates*

$(i)_r = \|\vec{\boldsymbol{\imath}}\| \times \cos\vartheta;\ (i)_\vartheta = -\|\vec{\boldsymbol{\imath}}\| \times \sin\vartheta\ \|\vec{\boldsymbol{\imath}}\| = 1$

$$\vec{\boldsymbol{\imath}} = \cos\vartheta \times \vec{\boldsymbol{u}}_r - \sin\vartheta \times \vec{\boldsymbol{u}}_\vartheta$$

$\vec{\boldsymbol{\jmath}} = (j)_r \times \vec{\boldsymbol{u}}_r + (j)_\vartheta \times \vec{\boldsymbol{u}}_\vartheta$

$(j)_r$*and* $(j)_\vartheta$ *are the components of the cartesian unit vectors* $\vec{\boldsymbol{\jmath}}$ *in polar coordinates*

$(j)_r = \|\vec{\boldsymbol{\jmath}}\| \times \sin\vartheta;\ (j)_\vartheta = \|\vec{\boldsymbol{\jmath}}\| \times \cos\vartheta\ \|\vec{\boldsymbol{\jmath}}\| = 1$

$$\vec{\boldsymbol{\jmath}} = \sin\vartheta \times \vec{\boldsymbol{u}}_r + \cos\vartheta \times \vec{\boldsymbol{u}}_\vartheta$$

Method 02:

$$\begin{cases} \vec{\boldsymbol{u}}_r = \cos\vartheta \times \vec{\boldsymbol{\imath}} + \sin\vartheta \times \vec{\boldsymbol{\jmath}} \\ \vec{\boldsymbol{u}}_\vartheta = -\sin\vartheta \times \vec{\boldsymbol{\imath}} + \cos\vartheta \times \vec{\boldsymbol{\jmath}} \end{cases}$$

A system of two equations with variables, where the variables are $\vec{\imath}$ *and* $\vec{\jmath}$*, we can solve it as follow:*

$\frac{\vec{\boldsymbol{u}}_r - \cos\vartheta \times \vec{\boldsymbol{\imath}}}{\sin\vartheta} = \vec{\jmath}$ *we replace the expression of* $\vec{\jmath}$ *in the second equation*

$$\vec{\boldsymbol{u}}_\vartheta = -\sin\vartheta \times \vec{\boldsymbol{\imath}} + \cos\vartheta \times \left(\frac{\vec{\boldsymbol{u}}_r - \cos\vartheta \times \vec{\boldsymbol{\imath}}}{\sin\vartheta}\right)$$

$$\vec{\boldsymbol{u}}_\vartheta = -\frac{\sin\vartheta^2}{\sin\vartheta} \times \vec{\boldsymbol{\imath}} + \cos\vartheta \times \frac{\vec{\boldsymbol{u}}_r}{\sin\vartheta} - \left(\frac{\cos\vartheta^2 \times \vec{\boldsymbol{\imath}}}{\sin\vartheta}\right)$$

$$-\cos\vartheta \times \vec{\boldsymbol{u}}_r + \sin\vartheta \times \vec{\boldsymbol{u}}_\vartheta = (-\sin\vartheta^2 - \cos\vartheta^2) \times \vec{\boldsymbol{\imath}}$$

$$-\cos\vartheta \times \vec{\boldsymbol{u}}_r + \sin\vartheta \times \vec{\boldsymbol{u}}_\vartheta = (-1) \times \vec{\boldsymbol{\imath}}$$

$$\vec{\boldsymbol{\imath}} = +\cos\vartheta \times \vec{\boldsymbol{u}}_r - \sin\vartheta \times \vec{\boldsymbol{u}}_\vartheta$$

$$\vec{\jmath} = \frac{\vec{\boldsymbol{u}}_r - \cos\vartheta \times \vec{\boldsymbol{\imath}}}{\sin\vartheta} = \frac{\vec{\boldsymbol{u}}_r - \cos\vartheta \times (+\cos\vartheta \times \vec{\boldsymbol{u}}_r - \sin\vartheta \times \vec{\boldsymbol{u}}_\vartheta)}{\sin\vartheta}$$

$$\vec{\jmath} = \frac{\vec{\boldsymbol{u}}_r - \cos\vartheta \times (+\cos\vartheta \times \vec{\boldsymbol{u}}_r - \sin\vartheta \times \vec{\boldsymbol{u}}_\vartheta)}{\sin\vartheta}$$

$$\vec{\jmath} = \sin\vartheta\, \vec{\boldsymbol{u}}_r + \cos\vartheta \times \vec{\boldsymbol{u}}_\vartheta$$

4- Compute the derivatives of $\vec{\boldsymbol{\imath}}$ *and* $\vec{\boldsymbol{\jmath}}$ *unit vectors with respect to both time and* $\boldsymbol{\theta}$

$$\frac{d\,(\vec{\imath})}{d\vartheta}=\frac{d(\cos\vartheta\times\vec{u}_r-\sin\vartheta\times\vec{u}_\vartheta)}{d\vartheta}$$

$$=-\sin\vartheta\times\vec{u}_r+\cos\vartheta\frac{d(\vec{u}_r)}{d\vartheta}-\cos\vartheta\,\vec{u}_\vartheta-\sin\vartheta\times\frac{d(\vec{u}_\vartheta)}{d\vartheta}$$

$$\frac{d\,(\vec{\imath})}{d\vartheta}=-\sin\vartheta\times\vec{u}_r+\cos\vartheta\,\vec{u}_\vartheta-\cos\vartheta\,\vec{u}_\vartheta-\sin\vartheta\times(-\vec{u}_r)=\vec{0}$$

$$\frac{d\,(\vec{\jmath})}{d\vartheta}=\frac{d(\sin\vartheta\,\vec{u}_r+\cos\vartheta\times\vec{u}_\vartheta)}{d\vartheta}=\cos\vartheta\times\vec{u}_r+\sin\vartheta\frac{d(\vec{u}_r)}{d\vartheta}-\sin\vartheta\,\vec{u}_\vartheta+\cos\vartheta\times\frac{d(\vec{u}_\vartheta)}{d\vartheta}$$

$$\frac{d\,(\vec{\imath})}{d\vartheta}=\cos\vartheta\times\vec{u}_r+\sin\vartheta\,\vec{u}_\vartheta-\sin\vartheta\,\vec{u}_\vartheta+\cos\vartheta\times(-\vec{u}_r)=\vec{0}$$

$$\frac{d\,(\vec{\imath})}{dt}=\frac{d(\cos\vartheta\times\vec{u}_r-\sin\vartheta\times\vec{u}_\vartheta)}{dt}$$

$$=-\frac{d\vartheta}{dt}\sin\vartheta\times\vec{u}_r+\cos\vartheta\frac{d(\vec{u}_r)}{dt}-\frac{d\vartheta}{dt}\cos\vartheta\,\vec{u}_\vartheta-\sin\vartheta\times\frac{d(\vec{u}_\vartheta)}{d\vartheta}$$

$$\frac{d\,(\vec{\imath})}{dt}=-\frac{d\vartheta}{dt}\sin\vartheta\times\vec{u}_r+\cos\vartheta\frac{d\vartheta}{dt}\vec{u}_\vartheta-\frac{d\vartheta}{dt}\cos\vartheta\,\vec{u}_\vartheta-\sin\vartheta\times\left(-\frac{d\vartheta}{dt}\vec{u}_r\right)=\vec{0}$$

$$\frac{d\,(\vec{\jmath})}{dt}=\frac{d(\sin\vartheta\,\vec{u}_r+\cos\vartheta\times\vec{u}_\vartheta)}{dt}$$

$$=\frac{d\vartheta}{dt}\cos\vartheta\times\vec{u}_r+\sin\vartheta\frac{d(\vec{u}_r)}{dt}-\frac{d\vartheta}{dt}\sin\vartheta\,\vec{u}_\vartheta+\cos\vartheta\times\frac{d(\vec{u}_\vartheta)}{d\vartheta}$$

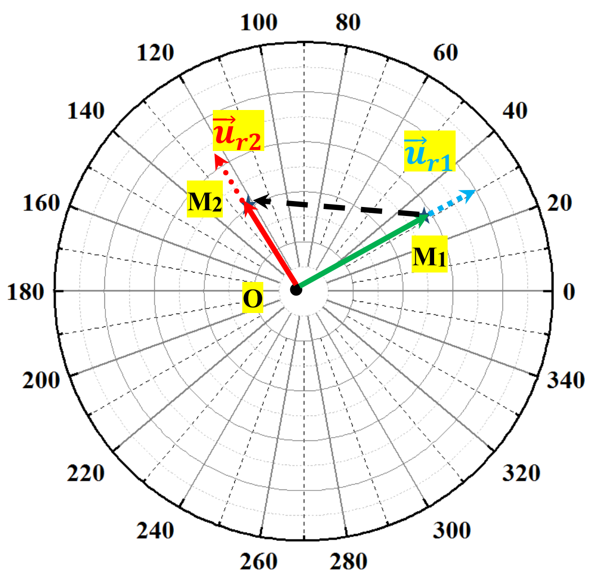
$$\frac{d\,(\vec{\jmath})}{dt}=\frac{d\vartheta}{dt}\cos\vartheta\times\vec{u}_r+\sin\vartheta\frac{d\vartheta}{dt}\vec{u}_\vartheta-\frac{d\vartheta}{dt}\sin\vartheta\,\vec{u}_\vartheta+\cos\vartheta\times\left(-\frac{d\vartheta}{dt}\vec{u}_r\right)=\vec{0}$$

EXERCISE 02:

*In the polar coordinate system with unit vectors $\vec{u}_r$ and $\vec{u}_\theta$, the positions of the moving object **M** at two different moments **t_1** and **t_2** are given as follows :**M_1 (3, π/6) M_2 (2, 2π/3)***

1) *Represent the positions of the moving object **M** in the polar coordinate system.*
2) *Provide the expressions for the position vector at **t_1** and **t_2** moments.*
3) *Determine the expression for the displacement vector from **M_1** to **M_2**.*
4) *Convert the coordinates of the two positions from polar to Cartesian coordinates, and rewrite the previous expressions in Cartesian coordinates.*

Solution:



2- the expressions for the position vector at t_1 and t_2 moments.

$$\overrightarrow{OM_1} = 3\,\vec{u}_{r1} \qquad \overrightarrow{OM_2} = 2\,\vec{u}_{r2}$$

3- the expression for the displacement vector from M_1 to M_2.

$$\overrightarrow{M_1M_2} = \overrightarrow{OM_2} - \overrightarrow{OM_1} = 2\,\vec{u}_{r2} - 3\,\vec{u}_{r1}$$

4- Convert the coordinates of the two positions from polar to Cartesian coordinates, and rewrite the previous expressions in Cartesian coordinates.

$M_1\ (3,\ \pi/6)\ \Rightarrow \begin{cases} x = 3\cos\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2} \\ y = 3\sin\left(\frac{\pi}{6}\right) = 3 \times \frac{1}{2} \end{cases} \Rightarrow \quad \overrightarrow{OM_1} = \frac{3\sqrt{3}}{2}\vec{\imath} + \frac{3}{2}\vec{\jmath}$

$M_2\ (2,\ 3\pi/2)\ \Rightarrow \begin{cases} x = 2\cos(3\pi/2) = -2\frac{1}{2} \\ y = 2\sin(3\pi/2) = 2 \times \frac{\sqrt{3}}{2} \end{cases} \Rightarrow \quad \overrightarrow{OM_2} = -\vec{\imath} + \sqrt{3}\vec{\jmath}$

$$\overrightarrow{M_1M_2} = \overrightarrow{OM_2} - \overrightarrow{OM_1} == -\vec{\imath} + \sqrt{3}\vec{\jmath} - \left(\frac{\mathbf{3\sqrt{3}}}{\mathbf{2}}\vec{\imath} + \frac{3}{2}\vec{\jmath}\right) = (-1 - \frac{\mathbf{3\sqrt{3}}}{\mathbf{2}})\vec{\imath} + (\sqrt{3} - \frac{3}{2})\vec{\jmath}$$

EXERCISE 03:

Consider a point material point M, its motion described by the following Cartesian coordinates:

$$x = R\big(1 + \cos(2\theta)\big), y = R\sin(2\theta), where\ \theta = \omega t$$

1- Find in Cartesian coordinates:

a- The equation of the trajectory and plot it.

b- The position, velocity, and acceleration vectors. Calculate the magnitudes of the velocity and acceleration.

2- Find in polar coordinates:

a- The equation of the trajectory $r = f(\theta)$

b- The position, velocity, and acceleration vectors. Calculate the magnitudes of the velocity and acceleration.

Solution:

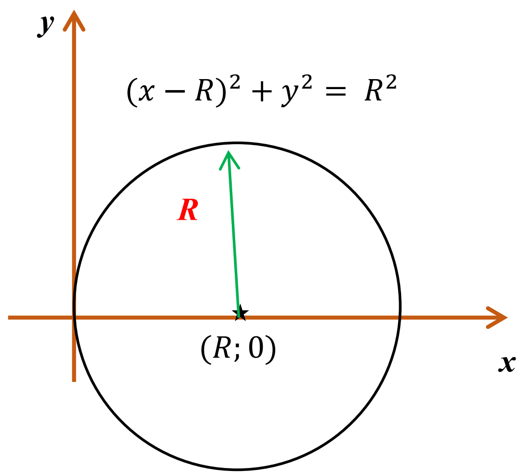
1- In Cartesian coordinates:

$$\begin{cases} x = R\big(1 + cos(2\theta)\big) \\ y = Rsin(2\theta) \end{cases} \qquad \theta = \omega t \Rightarrow \begin{cases} x - R = Rcos(2\theta) \\ y = Rsin(2\theta) \end{cases}$$

$$\begin{cases} (x-R)^2 = R^2cos^2(2\theta) \\ y^2 = R^2sin^2(2\theta) \end{cases} \Rightarrow (x-R)^2 + y^2 = R^2(cos^2(2\theta) + R^2sin^2(2\theta))$$

$$(x-R)^2 + y^2 = R^2$$

It is the equation of a circle with a center at point (R.0) and radius R.



b) The position, velocity, and acceleration vectors. Calculate the magnitudes of the velocity and acceleration

- *The position vector*

$$\overrightarrow{OM}(t) = x(t)\,\vec{\imath} + y(t)\,\vec{\jmath}$$

$$\overrightarrow{OM}(t) = R\big(1 + cos(2\theta)\big)\,\vec{\imath} + Rsin(2\theta)\,\vec{\jmath}$$

- *The velocity vector*

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\left(R\left(1 + cos(2\,\omega t)\right)\right)}{dt}\vec{\imath} + \frac{d(Rsin(2\,\omega t))}{dt}\vec{\jmath}$$

$$\vec{v} = -2\,R\omega\sin(2\omega t)\,\vec{\imath} + 2\omega R\,cos(2\omega t)\,\vec{\jmath} = = 2\,R\omega(-\sin(2\omega t)\,\vec{\imath} + cos(2\omega t)\,\vec{\jmath})$$

- *The acceleration vector*

$$\vec{a} = \frac{d\vec{v}}{dt} = 2\,R\omega\left[\frac{d(-\sin(2\omega t))}{dt}\vec{\imath} + \frac{d(cos(2\omega t))}{dt}\vec{\jmath}\right]$$

$$\vec{a} = 2\,R\omega\,(2\omega cos(2\omega t)\,\vec{\imath} - 2\omega\,\sin(2\omega t)\,\vec{\jmath})$$

$$\vec{a} = 4\,R\omega^2\,(cos(2\omega t)\,\vec{\imath} - \sin(2\omega t)\,\vec{\jmath})$$

2- In polar coordinates:

a- The equation of the trajectory ρ = f(θ)

$$r = \sqrt{x^2 + y^2} = \sqrt{(R(1 + cos(2\theta)))^2 + (Rsin(2\theta))^2}$$

$$r = R\sqrt{1 + 2\,cos(2\theta) + cos^2(2\theta) + sin^2(2\theta)} = R\sqrt{2\,(cos(2\theta) + 1)}$$

$$cos(2\theta) = cos^2(\theta) - sin^2(\theta) = cos^2(\theta) - \left(1 - cos^2(\theta)\right) = 2cos^2(\theta) - 1$$

$$r = R\sqrt{2(2cos^2(\theta) - 1 + 1)} = R\sqrt{4cos^2(\theta)}$$

$$r = 2R|\cos\theta|$$

b- The position, velocity, and acceleration vectors. Calculate the magnitudes of the velocity and acceleration:

- *The velocity vector*

$$\overrightarrow{OM} = r\,\overrightarrow{u_r} = 2R\cos\omega t\,\overrightarrow{u_r}$$

- *The velocity vector*

$$\vec{v} = \frac{d(r\,\overrightarrow{u_r})}{dt} = \frac{d(r\,)}{dt}\overrightarrow{u_r} + r\frac{d(\,\overrightarrow{u_r})}{dt} = \frac{d(r\,)}{dt}\overrightarrow{u_r} + r\frac{d\theta}{dt}\overrightarrow{u_\theta}$$

$$\frac{d\theta}{dt} = \omega$$

$$\vec{v} = -2R\omega\sin\omega t\,\overrightarrow{u_r} + 2R\omega\cos\omega t\,\overrightarrow{u_\theta}$$

$$\vec{v} = 2R\omega(-\sin\omega t\,\overrightarrow{u_r} + \cos\omega t\,\overrightarrow{u_\theta})$$

- *The acceleration vector*

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(2R\omega(-\sin\omega t\ \overrightarrow{u_r} + \cos\omega t\,\overrightarrow{u_\theta}))}{dt}$$

$$\vec{a} = 2R\omega(\frac{d(-\sin\omega t)}{dt}\overrightarrow{u_r} + -\sin\omega t\ \frac{d(\overrightarrow{u_r})}{dt} + \frac{d(\cos\omega t)}{dt}\ \overrightarrow{u_\theta}\ + \cos\omega t\frac{d(\overrightarrow{u_\theta})}{dt})$$

$$\vec{a} = 2R\omega(-\omega\cos\omega t\,\overrightarrow{u_r} + -\omega\sin\omega t\ \ \overrightarrow{u_\theta} - \omega\sin\omega t\ \ \overrightarrow{u_\theta} - \omega\cos\omega t\,\overrightarrow{u_r})$$

$$\vec{a} = -4R\omega^2(\cos\omega t\,\overrightarrow{u_r} + \sin\omega t\ \ \overrightarrow{u_\theta})$$

EXERCISE 04:

The position of a particle in the Cartesian coordinate system is described over time by the following equations: x = 4(1 + cos(9×t)), y = 4sin(9×t)

1- Find the expression of the equation of the trajectory and plot it.

2- Find the expression of the position, velocity, and acceleration vectors.

3- Calculate the magnitudes of the velocity and acceleration.

Solution:

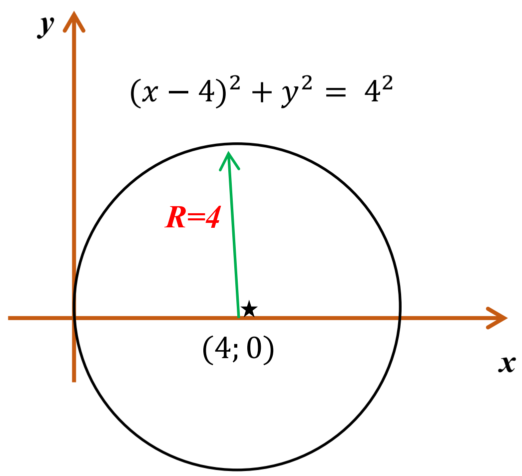
1- The path equation

$$\begin{cases} x = 4(1 + cos(9t)) \\ y = 4sin(9t) \end{cases} \Rightarrow \begin{cases} x - 4 = 4cos(9t) \\ y = 4sin(9t) \end{cases}$$

$$\begin{cases} (x-4)^2 = 4^2cos^2(9t) \\ y^2 = 4^2sin^2(9t) \end{cases} \Rightarrow (x-4)^2 + y^2 = 4^2(cos^2(9t) + 4^2sin^2(9t))$$

$$(x-4)^2 + y^2 = 4^2$$

It is the equation of a circle with a center at point (4.0) and radius R.



2- the expression of the position, velocity, and acceleration vectors.

- The position vector

$$\overrightarrow{OM}(t) = x(t)\,\vec{\imath} + y(t)\,\vec{\jmath}$$

$$\overrightarrow{OM}(t) = 4\big(1 + cos(9t)\big)\,\vec{\imath} + 4sin(9t)\,\vec{\jmath}$$

- The velocity vector

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\Big(4\big(1 + cos(9t)\big)\Big)}{dt}\vec{\imath} + \frac{d(4sin(9t))}{dt}\vec{\jmath}$$

$$\vec{v}(t) = -36\sin 9t\ \vec{\imath} + 36\cos 9t\ \vec{\jmath}$$

- The acceleration vector

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d(-36\sin 9t\ )}{dt}\vec{\imath} + \frac{d(36\cos 9t)}{dt}\vec{\jmath}$$

$$\vec{a}(t) = -324\,cos\,9t\ \vec{\imath} - 324\,sin\,9t\ \vec{\jmath}$$

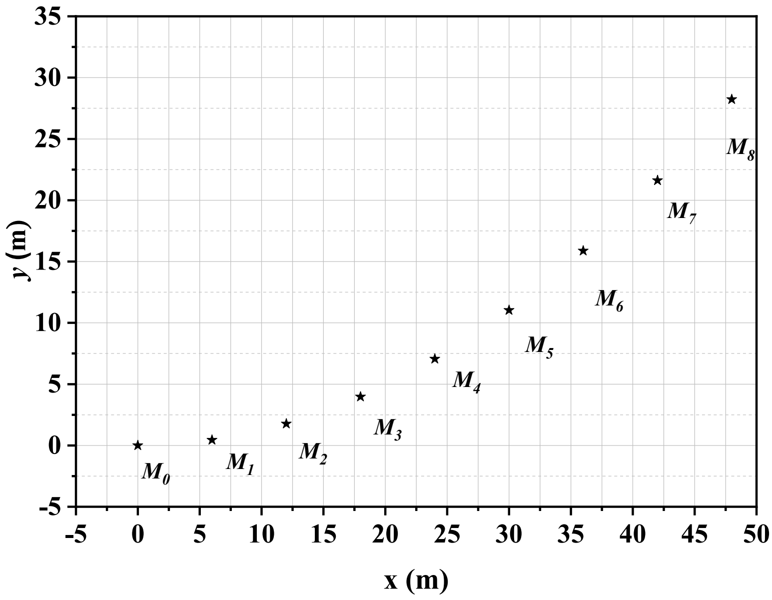
3- the magnitudes of the velocity and acceleration.

$$\|\vec{v}(t)\| = \sqrt{(-36\sin 9t\ )^2 + (36\cos 9t\ )^2} = 36$$

$$\|\vec{a}(t)\| = \sqrt{(-324\cos 9t\ )^2 + (-324\sin 9t\ )^2} = 324$$

EXERCISE 05:

Given the positions of a particle (m) every 0.3 s in the plane (OXY) as shown in the figure:



1- *Using this diagram, complete the table by specifying the time and the Cartesian coordinates* (x_i, y_i) *of the moving object at each moment* t_i

position	M_0	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
$t_i(s)$									
$t^2{}_i(s^2)$									
$x_i(m)$									
$y_i(m)$									

2- *Plot the curves x=f(t) and y= k(t^2)*

3- *Derive the parametric equations governing the particle's motion x(t)and y(t), then, providing details about the constants' values and the nature of each constant.*

4- *write the expression of the vector $\overrightarrow{OM}$ (position vector).*

5- *5- Represent the displacement vector of the moving object from M_1 to M_3 on the diagram, then write the expression for the displacement vector. Derive the expression for the average velocity between moments t_1 and t_3.*

6- *Write the expressions for instantaneous velocity and acceleration.*

7- *Calculate the instantaneous velocity at position M2.*

8- *Express the average acceleration between positions M4 and M6.*

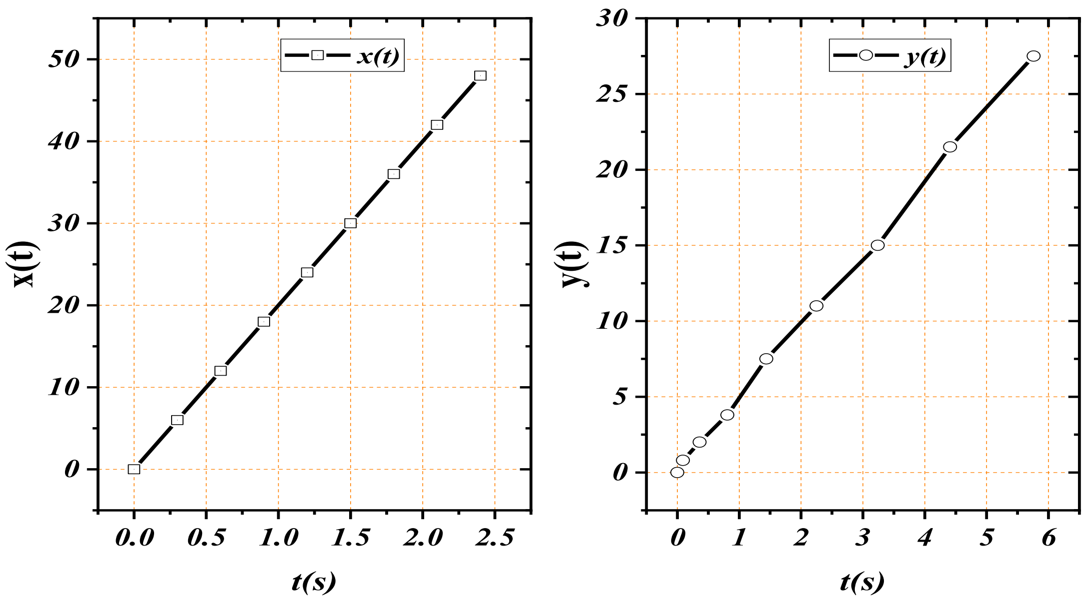
9- *Calculate the instantaneous acceleration at position M5.*

Solution:

1- *Filling the table by specifying the time and the Cartesian coordinates (x_i, y_i) of the moving object at each moment t_i*

position	M_0	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
$t_i(s)$	0	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4
$t^2{}_i(s^2)$	0	0.09	0.36	0.81	1.44	2.25	3.24	4.41	5.76
$x_i(m)$	0	6	12	18	24	30	36	42	48
$y_i(m)$	0	0.4	1.8	3.9	7.0	11	15.9	21.5	28.3

2- *Plot the curves x=f(t) and y= k(t^2)*



3- Derivation of Parametric Equations for Particle Motion and Analysis of Constants

According to the graph x=f(t), we notice that the curve is a straight line passing through the origin with an equation of the form $x = a \times t$. *Where a represents the slope of the curve and is calculated by the tangent of the angle α*

$$\tan\alpha = \frac{36-0}{1.8-0} = 20$$

the dimension of the constant a

$a = \frac{x}{t} \Rightarrow [a] = \frac{[x]}{[t]} = L.T^{-1}$ $\Rightarrow$ *The constant a has the same dimension as velocity.*

According to the graph y=f(t²), we notice that the curve is a straight line passing through the origin with an equation of the form $y = b \times t^2$. *Where b represents the slope of the curve and is calculated by the tangent of the angle α*

$$\tan\alpha = \frac{11-0}{2.25-0} = 4.88$$

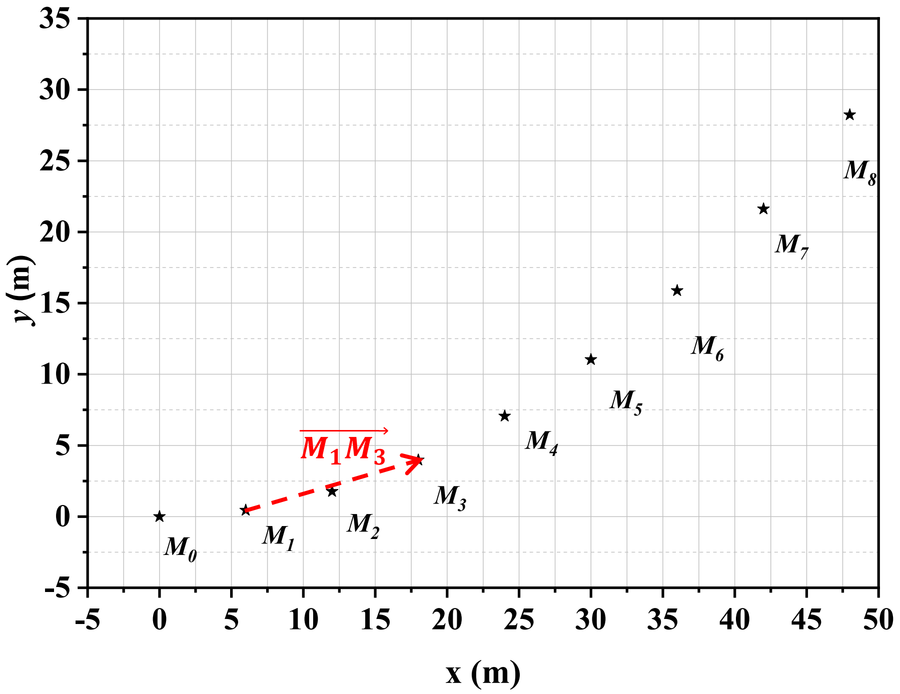
$b = \frac{y}{t^2} \Rightarrow [b] = \frac{[y]}{[t]^2} = L.T^{-2}$ $\Rightarrow$ *The constant b has the same dimension as acceleration.*

4- the expression of the vector $\overrightarrow{OM}$ *(position vector).*

$$\overrightarrow{OM}(t) = x(t)\,\vec{\imath} + y(t)\,\vec{\jmath}$$

$$\overrightarrow{OM}(t) = (20t)\vec{\imath} + (4.88t^2)\,\vec{\jmath}$$

5- Representing the displacement vector of the moving object from M_1 *to* M_3 *on the diagram*



Writing the expression for the displacement vector

$$\begin{cases} \overrightarrow{OM_1} = 6\,\vec{\imath} + 0.4\,\vec{\jmath} \\ \overrightarrow{OM_3} = 18\,\vec{\imath} + 3.9\,\vec{\jmath} \end{cases}$$

$$\overrightarrow{M_1M_3} = \overrightarrow{OM_3} - \overrightarrow{OM_1} = (18 - 6)\vec{\imath} + (3.9 - 0.5)\vec{\jmath} = 12\,\vec{\imath} + 3.5\,\vec{\jmath}$$

Deriving the expression for the average velocity between moments t_1 and t_3.

$$\overrightarrow{V_{1,3}} = \frac{\overrightarrow{M_1M_3}}{\Delta t} = \frac{\overrightarrow{OM_3} - \overrightarrow{OM_1}}{t_3 - t_1} = \frac{(x_3 - x_1)}{t_3 - t_1}\vec{\imath} + \frac{(y_3 - y_1)}{t_3 - t_1}\vec{\jmath}$$

$$\overrightarrow{V_{1,3}} = \frac{\overrightarrow{M_1M_3}}{\Delta t} = \frac{12\,\vec{\imath} + 3.4\,\vec{\jmath}}{0.9 - 0.3} = 20\,\vec{\imath} + 5.83\,\vec{\jmath}$$

6- Writing the expressions for instantaneous velocity and acceleration.

$$\vec{V}(t) = \frac{d(\overrightarrow{OM})}{dt} = \frac{d(20t\,\vec{\imath} + 4.88t^2\,\vec{\jmath})}{dt} = 20\,\vec{\imath} + 9.76t\,\vec{\jmath}$$

$$\vec{a}(t) = \frac{d(\vec{v})}{dt} = \frac{d(20\,\vec{\imath} + 9.76t\,\vec{\jmath})}{dt} = 9.76\,\vec{\jmath}$$

7- calculating the instantaneous velocity at position M2 (at t= t_2=0.6s)

$$\vec{V}(t_2) = 20\,\vec{\imath} + 9.76 \times 0.6\,\vec{\jmath} = 20\,\vec{\imath} + 5.85\,\vec{\jmath}$$

$$\vec{V}(t_2) = \overrightarrow{V_{1,3}}$$

8-Determine the average acceleration between positions M4 and M6.

$$\overrightarrow{a_{4,6}} = \frac{\overrightarrow{\Delta V_{4,6}}}{\Delta t} = \frac{\overrightarrow{V_6} - \overrightarrow{V_4}}{t_6 - t_4} = \frac{(V_{6x} - V_{4x})}{t_6 - t_4}\vec{\imath} + \frac{(V_{6y} - V_{4y})}{t_6 - t_4}\vec{\jmath}$$

$$\overrightarrow{V_6} = \vec{V}(t_6) = 20\,\vec{\imath} + 9.76 \times 1.8\ \vec{\jmath} = 20\,\vec{\imath} + 17.56\ \vec{\jmath}$$

$$\overrightarrow{V_4} = \vec{V}(t_4) = 20\,\vec{\imath} + 9.76 \times 1.2\ \vec{\jmath} = 20\,\vec{\imath} + 11.71\ \vec{\jmath}$$

$$\overrightarrow{a_{4,6}} = \frac{(20 - 20)}{1.8 - 1.2}\vec{\imath} + \frac{(17.56 - 11.71)}{1.8 - 1.2}\vec{\jmath} = 9.75\,\vec{\jmath}$$

9- calculating the instantaneous velocity at position M5 (at t= t_5=1.5s)

$$\vec{a}(t_5) = 9.76\ \vec{\jmath}$$

$$\vec{a}(t_5) = \overrightarrow{a_{4,6}}$$

EXERCISE 06:

The motion of a material point M in the plane is described using the following polar coordinates:

$$\begin{cases} r = a\,e^{\left(-\frac{t}{\tau}\right)} \\ \theta(t) = \omega t \end{cases} \quad \text{where } a, \tau \text{ and } \omega \text{ are positive constants}$$

a) Calculate the velocity and acceleration of point M in the polar coordinates

b) Calculate the velocity and acceleration of point M in the Cartesian coordinates

Solution

a) Calculating the velocity and acceleration of point M in polar coordinates

the position vector in the polar coordinates is given as follow:

$$\overrightarrow{OM} = r\,\overrightarrow{u_r} = a^{\left(-\frac{t}{\tau}\right)}\,\overrightarrow{u_r}$$

the velocity vector:

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dr}{dt}\,\overrightarrow{u_r} + r\frac{d\,\overrightarrow{u_r}}{dt}$$

$$\frac{d\,\overrightarrow{u_r}}{dt} = \frac{d\theta}{dt}\,\overrightarrow{u_\theta} = \frac{d(\omega t)}{dt}\,\overrightarrow{u_\theta} = \omega\ \overrightarrow{u_\theta}$$

$$\vec{v} = \frac{d(a\, e^{\left(-\frac{t}{\tau}\right)})}{dt}\ \overrightarrow{u_r} + \omega\, a^{\left(-\frac{t}{\tau}\right)}\ \overrightarrow{u_\theta} = -\frac{a}{\tau} e^{\left(-\frac{t}{\tau}\right)}\ \overrightarrow{u_r} + \omega\, a\, e^{\left(-\frac{t}{\tau}\right)}\ \overrightarrow{u_\theta}$$

$$\vec{v} = a\, e^{\left(-\frac{t}{\tau}\right)}(\omega\ \overrightarrow{u_\theta} - \frac{1}{\tau}\ \overrightarrow{u_r})$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(a\, e^{\left(-\frac{t}{\tau}\right)}(\omega\ \overrightarrow{u_\theta} - \frac{1}{\tau}\ \overrightarrow{u_r}))}{dt}$$

$$\vec{a} = \frac{d(a\, e^{\left(-\frac{t}{\tau}\right)})}{dt}(\omega\ \overrightarrow{u_\theta} - \frac{1}{\tau}\ \overrightarrow{u_r})\ + a\, e^{\left(-\frac{t}{\tau}\right)}\ \frac{d(\omega\ \overrightarrow{u_\theta} - \frac{1}{\tau}\ \overrightarrow{u_r})}{dt}$$

$$\vec{a} = -\frac{a}{\tau}\ e^{\left(-\frac{t}{\tau}\right)}(\omega\ \overrightarrow{u_\theta} - \frac{1}{\tau}\ \overrightarrow{u_r})\ + a\, e^{\left(-\frac{t}{\tau}\right)}(\omega\ \frac{d(\ \overrightarrow{u_\theta})}{dt} - \frac{1}{\tau}\frac{d(\ \overrightarrow{u_r})}{dt})$$

$$\begin{cases} \dfrac{d\,\overrightarrow{u_r}}{dt} = \dfrac{d\theta}{dt}\ \overrightarrow{u_\theta} = \dfrac{d(\omega t)}{dt}\ \overrightarrow{u_\theta} = \omega\ \overrightarrow{u_\theta} \\ \dfrac{d\,\overrightarrow{u_\theta}}{dt} = -\dfrac{d\theta}{dt}\ \overrightarrow{u_r} = -\dfrac{d(\omega t)}{dt}\ \overrightarrow{u_r} = -\omega\ \overrightarrow{u_r} \end{cases}$$

$$\vec{a} = -\frac{a}{\tau}\ e^{\left(-\frac{t}{\tau}\right)}(\omega\ \overrightarrow{u_\theta} - \frac{1}{\tau}\ \overrightarrow{u_r})\ + a\, e^{\left(-\frac{t}{\tau}\right)}(-\omega^2\ \overrightarrow{u_r} - \frac{1}{\tau}\omega\ \overrightarrow{u_\theta})$$

$$\vec{a} = -\frac{a}{\tau}\ e^{\left(-\frac{t}{\tau}\right)}\left\{(\omega\ \overrightarrow{u_\theta} - \frac{1}{\tau}\ \overrightarrow{u_r})\ + (\omega^2\,\tau\ \overrightarrow{u_r} + \omega\ \overrightarrow{u_\theta})\right\}$$

$$\vec{a} = -\frac{a}{\tau}\ e^{\left(-\frac{t}{\tau}\right)}\left\{2\omega\ \overrightarrow{u_\theta} + (\omega^2\,\tau - \frac{1}{\tau}\ )\ \overrightarrow{u_r}\right\}$$

b) Calculating the velocity and acceleration of point M in the Cartesian coordinates

According to exercise 1 we have:
$$\begin{cases} \vec{\boldsymbol{u}}_r = \cos\vartheta \times \vec{\boldsymbol{\imath}} + \sin\vartheta \times \vec{\boldsymbol{\jmath}} \\ \vec{\boldsymbol{u}}_\vartheta = -\sin\vartheta \times \vec{\boldsymbol{\imath}} + \cos\vartheta \times \vec{\boldsymbol{\jmath}} \end{cases}$$

By substituting the expressions for the unit vectors in the polar coordinates written in terms of the unit vectors in the Cartesian coordinates in the expressions for velocity and acceleration, we get:

$$\vec{v} = a\, e^{\left(-\frac{t}{\tau}\right)}\left(\omega\ \overrightarrow{u_\theta} - \frac{1}{\tau}\ \overrightarrow{u_r}\right)$$

$$\vec{v} = a\, e^{\left(-\frac{t}{\tau}\right)}\left(\omega\ \ (-\sin\vartheta \times \vec{\boldsymbol{\imath}} + \cos\vartheta \times \vec{\boldsymbol{\jmath}}) - \frac{1}{\tau}\ \ (\cos\vartheta \times \vec{\boldsymbol{\imath}} + \sin\vartheta \times \vec{\boldsymbol{\jmath}})\right)$$

$$\vec{v} = -\omega\, a\, e^{\left(-\frac{t}{\tau}\right)} \sin\vartheta \times \vec{\imath} + \omega\, a\, e^{\left(-\frac{t}{\tau}\right)} \cos\vartheta \times \vec{\jmath} - \frac{a\, e^{\left(-\frac{t}{\tau}\right)}}{\tau} \cos\vartheta \times \vec{\imath} - \frac{a\, e^{\left(-\frac{t}{\tau}\right)}}{\tau} \sin\vartheta \times \vec{\jmath}$$

$$\vec{v} = \left(-\frac{a\, e^{\left(-\frac{t}{\tau}\right)}}{\tau} \cos\vartheta - \omega\, a\, e^{\left(-\frac{t}{\tau}\right)} \sin\vartheta\right)\vec{\imath} + \left(\omega\, a\, e^{\left(-\frac{t}{\tau}\right)} \cos\vartheta - \frac{a\, e^{\left(-\frac{t}{\tau}\right)}}{\tau} \sin\vartheta\right)\vec{\jmath}$$

for the acceleration

$$\vec{a} = -\frac{a}{\tau}\, e^{\left(-\frac{t}{\tau}\right)} \left\{2\omega\ \ \overrightarrow{u_\theta} + \left(\omega^2\, \tau - \frac{1}{\tau}\right)\ \overrightarrow{u_r}\right\}$$

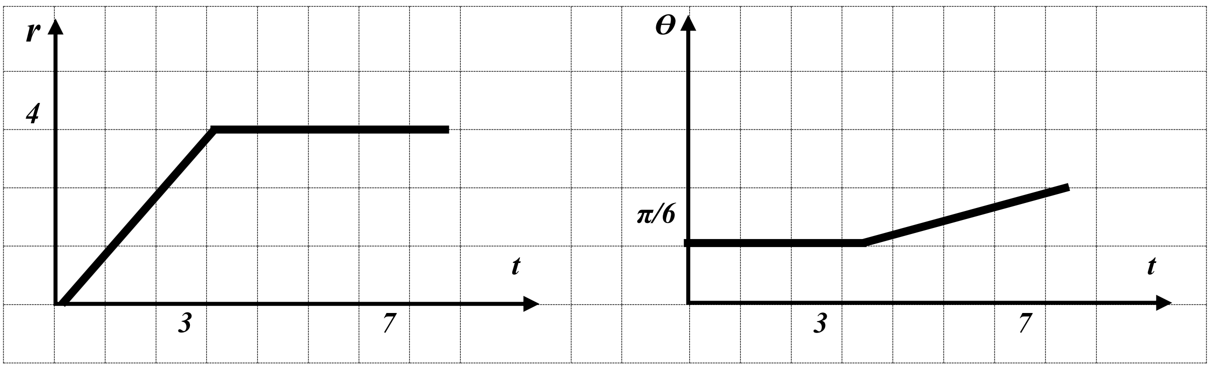
$$\vec{a} = -\frac{a}{\tau}\, e^{\left(-\frac{t}{\tau}\right)} \left\{\left(2\omega\ \left(-\sin\vartheta \times \vec{\imath} + \cos\vartheta \times \vec{\jmath}\right) + \left(\omega^2\, \tau - \frac{1}{\tau}\right)\left(\cos\vartheta \times \vec{\imath} + \sin\vartheta \times \vec{\jmath}\right)\right)\right\}$$

$$\vec{a} = -\frac{a}{\tau}\, e^{\left(-\frac{t}{\tau}\right)} \left\{\left(\ \left(\omega^2\, \tau - \frac{1}{\tau}\right)\cos\vartheta - 2\omega\sin\vartheta\right)\vec{\imath} + \left(\left(\omega^2\, \tau - \frac{1}{\tau}\right)\sin\vartheta + 2\omega\cos\vartheta\right)\vec{\jmath}\right)\right\}$$

EXERCISE 07:

The motion of a particle (M) is defined by its polar coordinates r(t) and θ(t) given by the graphs

1. *write the equations for each phase of the motion*
2. *Express the position vector for each phase in the polar coordinate system and compute the velocity and acceleration components within this coordinate system, thereby deriving their magnitudes*



Solution

1- writing the equations of motion for each time phase

- *1st Phase : t* $\epsilon\ [0; 3s]$

The curve r=f(t) is a straight line passing through the origin, with its equation in the form r=at, where a is a constant determined by calculating the slope. On the other hand, the curve for the angle θ=g(t) is a horizontal straight line with its equation in the form θ=b, where b is a constant

$$\begin{cases} r = a\,t \quad a = \tan\alpha = \frac{4}{3} \\ \theta = b = \frac{\pi}{6} \end{cases} \Rightarrow \begin{cases} r = \frac{4}{3}\,t \\ \theta = \frac{\pi}{6} \end{cases}$$

- *2nd Phase :* $t \in [3; 7s]$

The curve θ =f(t) is a straight line passing through the origin, with its equation in the form θ =at+b, where a is a constant determined by calculating the slope. On the other hand, the curve for the angle r=g(t) is a horizontal straight line with its equation in the form r=c, where c is a constant

$$\begin{cases} r = c \\ \theta = at + b \quad a = \tan\alpha = \frac{\frac{\pi}{3} - \frac{\pi}{6}}{7-3} = \frac{\pi}{24} \end{cases}$$

$$at\ t = 3s \Rightarrow \theta(3) = \frac{\pi}{6} \Rightarrow \frac{\pi}{6} = \frac{\pi}{24} \times 3 + b \Rightarrow b = \frac{\pi}{24}$$

$$\begin{cases} r = 4 \\ \theta = \frac{\pi}{24}\,t + \frac{\pi}{24} \end{cases}$$

2- calculating the position vector , the velocity, and acceleration and their magnitudes

- *1st Phase :* $t \in [0; 3s]$

- the position vector:

$$\overrightarrow{OM} = r\,\overrightarrow{u_r} = \frac{4}{3}\boldsymbol{t}\,\overrightarrow{u_r}$$

- the velocity vector:

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dr}{dt}\,\overrightarrow{u_r} + r\frac{d\,\overrightarrow{u_r}}{dt}$$

$$\frac{d\,\overrightarrow{u_r}}{dt} = \frac{d\theta}{dt}\,\overrightarrow{u_\theta} = \frac{d(\frac{\pi}{6})}{dt}\,\overrightarrow{u_\theta} = 0\,\overrightarrow{u_\theta}$$

$$\vec{v} = \frac{4}{3}\,\overrightarrow{u_r}\,;\quad \|\vec{v}\| = \frac{4}{3}$$

- The acceleration vector:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(\frac{4}{3}\ \overrightarrow{u_r})}{dt}$$

$$\vec{a} = \frac{4}{3}\frac{d(\ \overrightarrow{u_r})}{dt} = \vec{0}\ \ ,\ \|\vec{a}\,\| = 0$$

- *2nd Phase : t* ϵ $[3; 7s]$

- The position vector:

$$\overrightarrow{OM}\ = r\,\overrightarrow{u_r} = 4\,\overrightarrow{u_r}$$

- The velocity vector:

$$\vec{v}\ = \frac{d\overrightarrow{OM}}{dt} = 4\frac{d\,\overrightarrow{u_r}}{dt}$$

$$\frac{d\,\overrightarrow{u_r}}{dt} = \frac{d\theta}{dt}\ \overrightarrow{u_\theta} = \frac{d(\frac{\pi}{24}\ t + \frac{\pi}{24})}{dt}\ \overrightarrow{u_\theta} = \frac{\pi}{24}\ \overrightarrow{u_\theta}$$

$$\vec{v}\ = \frac{\pi}{6}\ \overrightarrow{u_\theta}\ \ ;\ \ \|\vec{v}\,\| = \frac{\pi}{6}$$

- The acceleration vector:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(\frac{\pi}{6}\ \overrightarrow{u_\theta})}{dt}$$

$$\frac{d\,\overrightarrow{u_\theta}}{dt} = -\frac{d\theta}{dt}\ \overrightarrow{u_r} = -\ \frac{d(\frac{\pi}{24}\ t + \frac{\pi}{24})}{dt}\ \overrightarrow{u_r} = -\frac{\pi}{24}\ \overrightarrow{u_r}$$

$$\vec{a} = -\frac{\pi}{6}\ \frac{\pi}{24}\overrightarrow{u_r} = -\frac{\pi^2}{144}\overrightarrow{u_r}\ \ ,\ \|\vec{a}\,\| = \frac{\pi^2}{144}$$

EXERCISE 08:

The time equations for the motion of a particle in polar coordinates are given by:

$$r = \ a\,,\theta(t) = \ \pi t$$

where "a" is a positive constant.

1. *Find the expression for the Cartesian coordinates of the moving particle, deduce the shape of the path.*

2. *Use the polar coordinates to calculate the time it takes for the particle to complete two and a half cycles ($t_{2+0.5}$).*
3. *Write expressions of the velocity and the acceleration, then calculate their magnitudes.*

Solution

1- the Cartesian coordinates of the moving particle:

$$M\ (a,\ \pi t)\ \Rightarrow \begin{cases} x = a\cos\pi t \\ y = a\sin\pi t \end{cases}$$

- The path equation

$$\begin{cases} x^2 = a^2\cos^2\pi t \\ y^2 = a^2\sin^2\pi t \end{cases} \Rightarrow x^2 + y^2 = a^2\,(\cos^2\pi t + \sin^2\pi t) = a^2$$

$x^2 + y^2 = a^2$ *(path of circle with a center at point (0.0) and radius R=a)*

2- the time it takes for the particle to complete two and a half cycles ($t_{2+0.5}$)

1 turn = 2 π ⇒ 2.5 turns = 5π

$5\pi = \pi t \quad \Rightarrow t = 5\frac{\pi}{\pi} = 5s$

3- the velocity and the acceleration, and their magnitudes

$$\overrightarrow{OM} = r\,\overrightarrow{u_r} = a\,\overrightarrow{u_r}$$

- The velocity vector:

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = a\frac{d\,\overrightarrow{u_r}}{dt}$$

$$\frac{d\,\overrightarrow{u_r}}{dt} = \frac{d\theta}{dt}\,\overrightarrow{u_\theta} = \frac{d(\pi t)}{dt}\,\overrightarrow{u_\theta} = \pi\,\overrightarrow{u_\theta}$$

$$\vec{v} = a\pi\,\overrightarrow{u_\theta}\ ;\ \ \|\vec{v}\| = a\pi$$

- The acceleration vector:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(a\pi\,\overrightarrow{u_\theta})}{dt}$$

$$\frac{d\,\overrightarrow{u_\theta}}{dt} = -\frac{d\theta}{dt}\,\overrightarrow{u_r} = -\frac{d(\pi t)}{dt}\,\overrightarrow{u_r} = -\pi\,\overrightarrow{u_r}$$

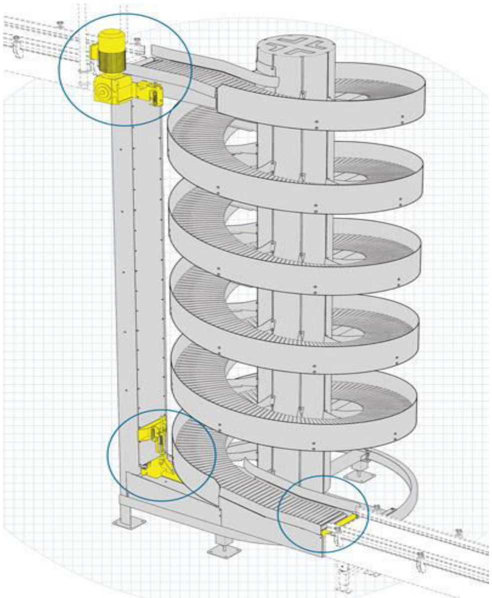
$$\vec{a} = -a\pi\,\pi\,\overrightarrow{u_r} = -a\pi^2\,\overrightarrow{u_r}\ ,\ \|\vec{a}\| = a\pi^2$$

EXERCISE 09:

A person slides on a helical path as shown in the figure, and its coordinates in the Cartesian system vary with time as follows:

x(t) = 4 cos (2×t) , y = 4 sin(2×t), z(t) = 8×t

1- *Write expressions for the position vector, velocity, and acceleration. Calculate the time required to complete half a cycle. Determine the height of the object when completing half cycle.*
2- *Re-answer the same questions (1 and 2) using cylindrical coordinates.*



solution

1- *The expressions of the position vector, velocity, and acceleration.*

$$\overrightarrow{OM}(t) = 4\cos 2t\ \vec{\imath} + \ 4\sin 2t\ \vec{\jmath} + 8t\,\vec{k}$$

$$\vec{V}(t) = \frac{d(\overrightarrow{OM})}{dt} = \frac{d(4\cos 2t)}{dt}\vec{\imath} + \frac{d(\ 4\sin 2t\ )}{dt}\vec{\jmath} + \frac{d(8t)}{dt}\vec{k}$$

$$\vec{V}(t) = -8\sin 2t\,\vec{\imath} + 8\cos 2t\,\vec{\jmath} + 8\vec{k}$$

$$\vec{a}(t) = \frac{d(\vec{v})}{dt} = \frac{d(-8\sin 2t)}{dt}\vec{\imath} + \frac{d(\ 8\cos 2t)}{dt}\vec{\jmath} + \frac{d(8)}{dt}\vec{k}$$

$$\vec{a}(t) = -16\ \cos 2t\,\vec{\imath} - 16\sin 2t\ \vec{\jmath}$$

- *Calculating the time required to complete half a cycle.*

at t = 0s $\Rightarrow x(0) = 4$, $y(0) = 0$, $z(0) = 0$, *to achieve half cycle x= -4*

$4\cos 2t = -4 \Rightarrow \cos 2t = -1 \Rightarrow 2t = (2n+1)\pi$ *where n is the number of cycles = 0,1,2....n*

for half cycle $n = 0 \Rightarrow t = \frac{\pi}{2}$

- Calculating the height of the object when completing half a cycle and a full cycle.

$$z(t) = 8\,t$$

$$t = \frac{\pi}{2} \quad \Rightarrow z\left(\frac{\pi}{2}\right) = 8\,\frac{\pi}{2} = 4\pi = 12.5$$

2- *The same questions using cylindrical coordinates.*

in the cylindrical coordinates

$$M\,(x,y\,,z\,) \Rightarrow M\,(r,\theta\,,z\,)\ \ where \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \dfrac{y}{x} \\ z = z \end{cases}$$

$$\begin{cases} r = \sqrt{16\cos^2(2t) + 16\ \sin^2(2t)} = 4 \\ \tan\theta = \dfrac{4\sin 2t}{4\cos 2t} = \tan 2t \ \Rightarrow\ \theta = 2t \\ z = 8t \end{cases}$$

$$M\,(r,\theta\,,z\,) = M\,(4\,,2t\,,8t)$$

$$\overrightarrow{OM}(t) = 4\,\overrightarrow{u_r} + 8t\,\vec{k}$$

$$\vec{V}(t) = \frac{d(\overrightarrow{OM})}{dt} = \frac{d(4\,\overrightarrow{u_r})}{dt} + \frac{d(8t)}{dt}\vec{k}$$

$$\vec{V}(t) = 4\frac{d\overrightarrow{u_r}}{dt} + 8\vec{k}$$

$$\frac{d\overrightarrow{u_r}}{dt} = 2\,\overrightarrow{u_\theta}$$

$$\vec{V}(t) = 8\,\overrightarrow{u_\theta} + 8\vec{k}$$

$$\vec{a}(t) = \frac{d(\vec{v})}{dt} = \frac{d(8\,\overrightarrow{u_\theta})}{dt} + \frac{d(8)}{dt}\vec{k}$$

$$\frac{d\overrightarrow{u_\theta}}{dt} = -2\,\overrightarrow{u_r}$$

$$\vec{a}(t) = \frac{d(\vec{v})}{dt} = -16\,\overrightarrow{u_r}$$

Calculating the time required to complete half a cycle.

$\theta = 2t$, *then to achieve a half cycle* $\theta = \pi$

$$2t = \pi \quad \Rightarrow t = \frac{\pi}{2}$$

Home work

Answer the same questions of exercies 08 in the following cases :

- ✓ $r = v_0 \times t \,, \theta(t) = \pi$
- ✓ $r = v_0 \times t \,, \theta(t) = \pi \times t$

Chapter 03: Dynamics

1) Introduction

In the previous chapter, we discussed the motion of a particle by considering only the changes in its position, velocity, acceleration, and trajectory, without delving into the reasons for these changes. In this chapter, we will address the laws of motion that connect the characteristics of motion (position, velocity, acceleration) on one hand, and the factors affecting motion and what influences them (forces and the mass of the particle) on the other hand.

2) NEWTON'S LAWS OF MOTION

Newton's Laws of Motion are a set of three fundamental principles in classical physics that describe the relationship between the motion of objects and the forces acting on them. These laws provide a framework for understanding the motion of objects and the interactions between them. These laws were formulated by the English physicist Sir Isaac Newton in the 17th century and remain a cornerstone of classical mechanics. Here are the explanations of Newton's Laws of Motion.

Before delving into the explanation of Newton's Three Laws, we will elucidate some terms associated with these laws:

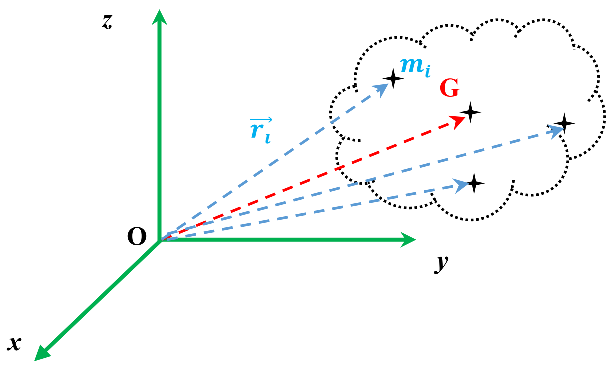
2-1)Mass

***Mass** is a scalar quantity that determines the amount of matter present in a body. From a dynamic standpoint, it represents the inertia of the body, meaning it is a measure of the resistance to any change in the motion of the body.*

2-2) Center of inertia (G) :

Considering a body as a collection of material points (each with mass m_i and position vectors $\vec{r}_i$ relative to a reference point O), the center of mass of the body coincides with the center of mass (the centroid) where it satisfies the following relationship:

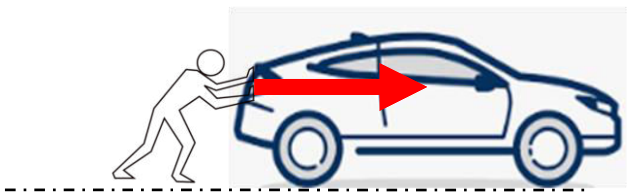
$$\overrightarrow{OG} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$



2-3) Forces:

- ***Force*** *is any effect that leads to a change in the motion state of an object (change in the velocity, direction, path, or stopping of the object) or a change in the shape of the object.*
- ***Force*** *is a vector quantity used to describe the interaction between two bodies, represented by a vector* $\vec{F}$, *where this vector represents the characteristics of force, including direction, support, and magnitude.*
- ***Forces****in nature are one of two:*

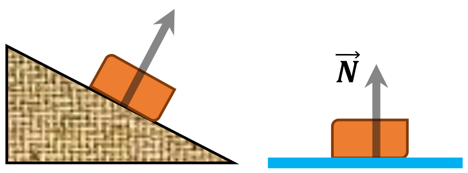
2-3-1)Contact forces *resulting from direct contact between two objects (for example, pulling an object with a rope, pushing a cart)*



We list below the properties of some forces that act at a distance:

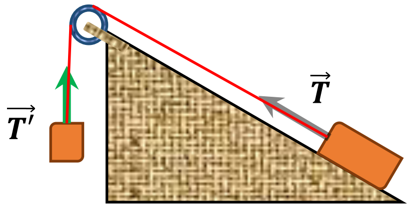
a) Normal Force ($\vec{N}$)

this force represents the effect exerted by the supporting surface on the body and its direction is perpendicular to the surfaceof contact.



b) Tension force($\vec{T}$):

The tension force $\vec{T}$ is the force that a cord, rope, or cable exerts on an object attached to it. Thisforce is directed along the rope away from the object at thepoint where the rope is attached.



c) Friction Force:

The frictional force is due to the interaction between the surface atoms of any two bodies in contact. The direction of this force is always parallel to the surface of contact, opposing the motion or the planned motion of one object relative to the other. Hence, the normal and frictional forces are both contact forces and they are always perpendicular to each other.

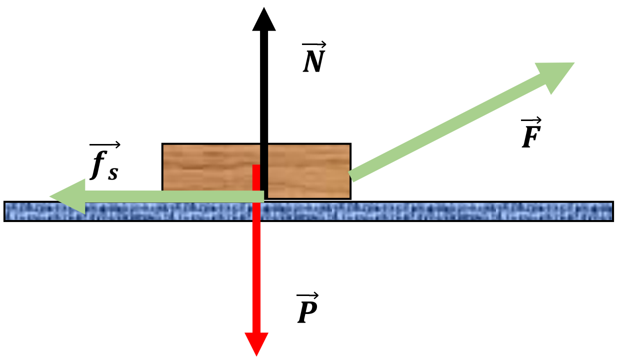
There are two types of frictional forces:

- ✓ ***The statistical frictional force*** $\vec{f}_s$

  The name statistical comes from the fact that the body remains stationary. Statical frictional force is directly proportional to the magnitude of the normal force, $\vec{N}$, according to the following relationship

$$\vec{f_s} = \mu_s \vec{N}$$

where μ_s *coefficient of static friction.*



- ✓ ***The kinetic frictional force*** $\vec{f}_k$

When the body moves, the retarding frictional force is then called the kinetic frictional force $\vec{f}_k$ *, The expression for this force is given by the following relation:*

$$\vec{f}_k = \mu_k \vec{N}$$

where μ_k *is the coefficient of kinetic friction.*

The dimensionless coefficients μ_s *and* μ_k *depend on the nature of the surfaces in contact.*

d) The elastic force of a spring

The elastic force of a spring, also known as the spring force or restoring force, is the force generated by a spring when it is displaced from its equilibrium position (compressed or extended). The expression for the elastic force is given by the following relation:

$$\|\vec{F}\| = -k\,\Delta x$$

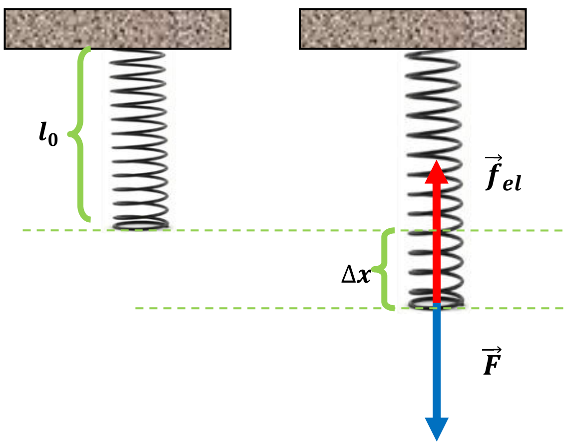
where k *is the spring constant, and* Δx *is the displacement from the equilibrium position.*

The negative sign indicates that the force opposes the direction of displacement.

Direction and Orientation of the elastic force:

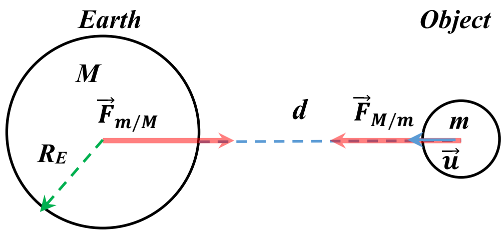
Direction: The elastic force $\vec{f}_{el}$ *is always in the opposite direction of the displacement, meaning it is a restoring force that works to bring the spring back to its equilibrium position.*

Orientation: If the spring is stretched (positive displacement), the elastic force is directed inward (toward the equilibrium position). If the spring is compressed (negative displacement), the force is directed outward (toward the equilibrium position)



2-3-2) *Field forces (Forces acting at a distance):* *these forces do not require contact between the two bodies, such as the force of gravity between two objects,*

Attraction and repulsion between two magnets, and the electric force between two electric charges).



We list below the properties of some forces that act at a distance:

a) Weight ($\vec{w}$)

- *The weight of and object (The gravitational force) is a non-contact force exerted by the Earth on an object. Its magnitude is proportional to the mass of the object and the value of the gravitational constant* $\vec{g}$.
- *The weight force is a consequence of the universal law of gravitation between two bodies (the Earth with a mass **M** and the object with a mass **m**), separated by a distance **d**.*

$$\vec{w} = \vec{F}_{M/m} = -\vec{F}_{\frac{m}{M}} = G\,\frac{M \times m}{(R_E+d)^2}\vec{u}$$

Where $G=6.67\times10^{-11}N\,m^2.\,Kg^{-2}$, $M = 5{,}98\times10^{24}$ *Kg, and* $R_E = 6.37\times10^6 m$

For $d=0 \rightarrow \vec{w} = G\,\frac{M\times m}{(R_E)^2}\vec{u} \rightarrow \vec{w} = m\,\vec{g}$ *where* $\vec{g} = G\,\frac{M}{(R_E)^2}\vec{u} = 9.8\,\vec{u}$

- *The weight force is denoted by a vector* $\vec{w}$ *and itexpression is given by:*

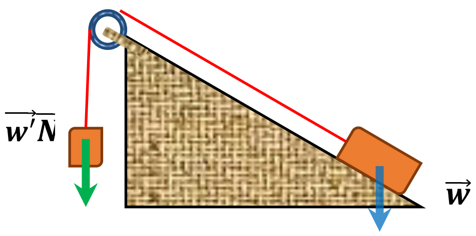
$$\vec{w} = m\,\vec{g}$$

- *The characteristics of the gravitational force vector are:*

***Direction**: Vertically towards the center of the Earth.*

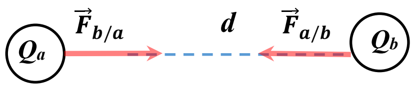
***Support**: The line connecting the center of mass of the object and the center of the Earth.*

***Magnitude**: Proportional to the mass of the object and the gravitational acceleration of magnitude 9.8m/s².* $\|\vec{w}\| = m\,\|\vec{g}\| = m\ \times 9.8$



b) Electrostatic forces (Coulomb force):

These forces arise between two charged bodies $\boldsymbol{Q_a}$ *and* $\boldsymbol{Q_b}$*separated by a distance* **d.**



$$\vec{F}_{a/b} = -\vec{F}_{\frac{b}{a}} = K\frac{Q_aQ_b}{\|\overrightarrow{AB}\|^3}\overrightarrow{AB}$$

c)The electromagnetic force:

This force arises due to the presence of a charged body **Q** *moving with a velocity* $\vec{v}$ *within an electric field* $\vec{E}$ *and a magnetic field* $\vec{B}$

$$\vec{F} = Q(\vec{E} + \vec{v}\wedge\vec{B})$$

3) Linear momentum

Linear momentum $\vec{P}$ *for a moving object is a vector quantity equal to the product of the mass of the object and its velocity.* $\vec{P} = m\,\vec{V}$

If the system consists of a collection of bodies, the total linear momentum of the system is the sum of the linear momentum of its individual bodies

$$\vec{P} = \sum \vec{P}_i = \sum m_i.\vec{V}_i$$

3-1) Conservation of Linear Momentum

The linear momentum of a system is conserved if the derivative of the linear momentum with respect to time is zero. The linear momentum of an isolated system is conserved if its value and direction remain constant throughout the entire time. the law of conservation of momentum, which can be written as:

$$\vec{P}(t_i) = \vec{P}\ (t_f) \Rightarrow \frac{d\vec{P}}{dt} = \vec{0}$$

where the subscripts refer to the total momentum of the system at initial time i and final time f

4) Newton's Laws

4-1) Newton's First Law (The Law of Inertia):

Newton's first law states that if the net force on an object is zero, it must stay at rest or move with constant velocity.

$$1^{st}\ law \quad \Rightarrow \quad \sum \overrightarrow{F_{ext}} = \vec{0} \begin{cases} \vec{v} = \vec{0} \\ \vec{v} = constant \end{cases}$$

4-2) Newton's Second Law

Newton's Second Law states that the change in the linear momentum of an object is directly proportional to the net external acting forces. The second law of Newton can be formulated mathematically as follows:

$$\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt}$$

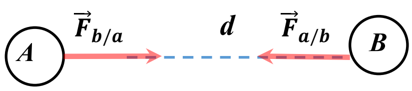
$$\sum \vec{F}_{ext} = \frac{d(m\,\vec{v})}{dt} = m\frac{d(\,\vec{v})}{dt} + \vec{v}\frac{d(m\,)}{dt}$$

$$\frac{d(m\,)}{dt} = 0\ ;\ \frac{d(\,\vec{v})}{dt} = \vec{a}$$

The acceleration of an object, $\vec{a}$, is related to its mass, m, and the net external acting forces on it $\sum \vec{F}_{ext}$. this second law can be expressed by the relation:

$$\sum \vec{F}_{ext} = m\,\vec{a}$$

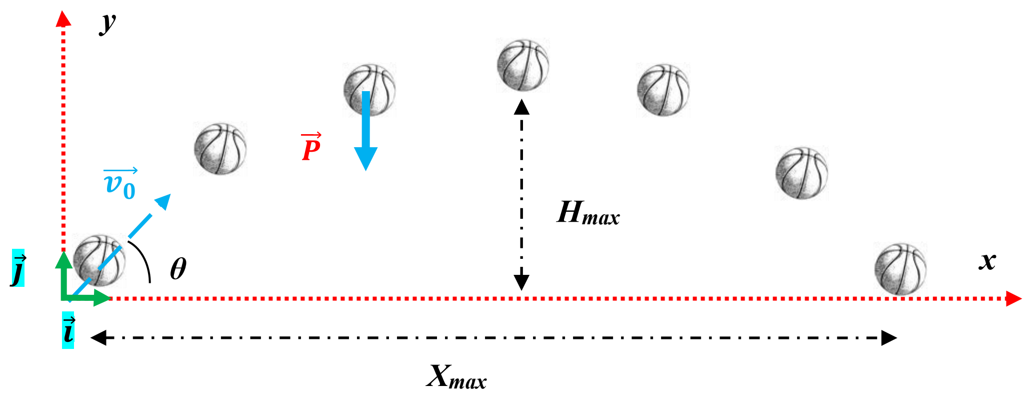
4-3) Newton's Third Law (Action and Reaction)



For every action (force) on an object, there is an equal and opposite reaction (force).The third law of Newton can be formulated mathematically as follows:

$$\vec{F}_{A/B} = -\vec{F}_{B/A}$$

4-4) Application : Projectile Motion



Study of the motion of a projectile subject only to the force of gravity as it is thrown upwards with an initial velocity and at an angle of α . The motion of this projectile is in the plane and is studied in a Cartesian axis (oxy)

Based on the basic principle of motion, we can write:

$$\sum \vec{F}_{ext} = m\,\vec{a} \Rightarrow \vec{P} = m\,\vec{a} \Rightarrow -m\ g\,\vec{j} = m\,\vec{a} \Rightarrow -g\,\vec{j} = \vec{a} = \overrightarrow{cnstant}$$

To find the equations of motion, we follow the steps:

$$\vec{a} = \frac{d\vec{v}}{dt} = -\ g\,\vec{j} \Rightarrow d\vec{v} = -\ g\ dt\ \vec{j} \Rightarrow \int_{\vec{v}_0}^{\vec{v}} d\vec{v} = \int_0^t -\ g\ dt\ \vec{j}$$

$$\vec{v} - \vec{v}_0 = -\ g\ t\ \vec{j} \Rightarrow \vec{v} = -\ g\ t\ \vec{j} + \vec{v}_0$$

$$\vec{v} = \frac{d\vec{r}}{dt} = -gt\,\vec{j} + \vec{v}_0 \Rightarrow d\vec{r} = (-gt\,\vec{j} + \vec{v}_0)dt \Rightarrow \int_{\vec{r}_0}^{\vec{r}} d\vec{r} = \int_0^t (-gt\,\vec{j} + \vec{v}_0)dt$$

$$\vec{r} - \vec{r}_0 = -\frac{1}{2}gt^2\,\vec{j} + \vec{v}_0\,t \Rightarrow \vec{r} = -\frac{1}{2}gt^2\,\vec{j} + \vec{v}_0\,t + \vec{r}_0$$

According to initial data, at the moment $\vec{r}_0 = \vec{0};\ \vec{v}_0 = v_0 \cos\alpha\,\vec{i} + v_0 \sin\alpha\,\vec{j}$

So the equations of motion for the basketball are as follows:

$$\begin{cases} \vec{r} = x\,\vec{i} + y\,\vec{j} = -\frac{1}{2}gt^2\,\vec{j} + (v_0\cos\alpha\,\vec{i} + v_0\sin\alpha\,\vec{j})\,t \\ \vec{v} = v_x\,\vec{i} + v_y\,\vec{j} = -\ g\ t\ \vec{j} + v_0\cos\alpha\,\vec{i} + v_0\sin\alpha\,\vec{j} \\ \vec{a} = a_x\,\vec{i} + a_y\,\vec{j} = -\ g\,\vec{j} \end{cases}$$

$$\begin{cases} \vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} = (-\frac{1}{2}gt^2 + tv_0 \sin\alpha\,)\,\vec{\jmath} + \ t\ v_0 \cos\alpha\,\vec{\imath} \\ \vec{v} = v_x\,\vec{\imath} + v_y\,\vec{\jmath} = (-\ g\,t + v_0 \sin\alpha\,\vec{\jmath})\ \ \vec{\jmath} + \ v_0 \cos\alpha\,\vec{\imath} \\ \vec{a} = \ a_x\,\vec{\imath} + \ a_y\,\vec{\jmath} = -\ g\,\vec{\jmath} \end{cases}$$

Therefore, the projectile's motion is straight and uniform with respect to the (Ox) axis; and straight and uniformly variable with respect to the (Oy) axis. Its motion equations with respect to each axis are given as follows:

$$\Rightarrow along\ (Ox) \begin{cases} a_x = 0 \\ v_x = v_0 \cos\alpha \\ x = \ t\ \ v_0 \cos\alpha \end{cases} \quad and\ along\ (Oy) \begin{cases} a_y = -g \\ v_y = -gt + v_0 \cos\alpha \\ y = \ -\frac{1}{2}gt^2 + tv_0 \sin\alpha \end{cases}$$

At the maximum height the projectile reaches, the velocity v_y *becomes null*

$$-gt + v_0 \cos\alpha \ = 0 \ \Rightarrow t = \frac{v_0 \cos\alpha}{g}$$

The projectile reaches its maximum height at the moment $t = \frac{v_0 \cos\alpha}{g}$

$$H_{max} = \ -\frac{{v^2}_0}{2g}\cos^2\alpha \ + \frac{\cos\alpha}{g}{v^2}_0 \sin\alpha$$

The distance at which a projectile falls:

$$-\frac{1}{2}gt^2 + tv_0 \sin\alpha = 0 \ \Rightarrow \frac{1}{2}gt^2 \ = tv_0 \sin\alpha \ \Rightarrow \frac{1}{2}gt \ = v_0 \sin\alpha \ \Rightarrow t \ = \frac{2v_0}{g}\sin\alpha$$

The moment at which the projectile touches the ground's surface is $t = \frac{2v_0}{g}\sin\alpha$

And it touches the ground surface after covering a distance of

$$x = \ t\ \ v_0 \cos\alpha \ = \frac{2{v^2}_0}{g}\sin\alpha\cos\alpha$$

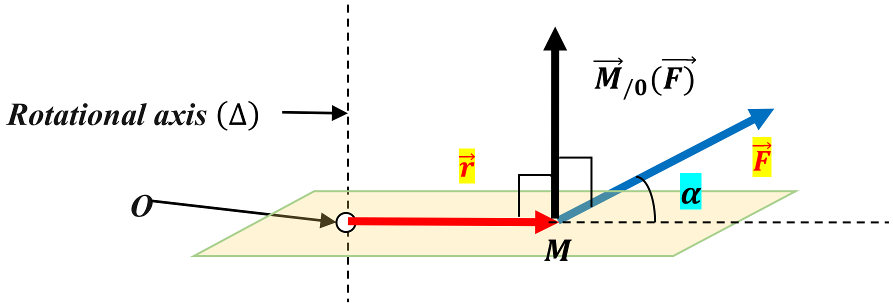
5) The torque

Torque is a vector quantity that represents the ability (tendency) of a force to induce rotational motion of an object around an axis (Δ)*passing through point O .*

Torque is denoted by $\vec{\mathcal{M}}_{/0}(\vec{F})$, *where* $\vec{F}$ *is the applied force* O *is the point through which the axis of rotation passes, and* $\vec{r} = \overrightarrow{OM}$ *is the vector extending from point* O *(the point on the axis of rotation) to the* M *point where the force is applied. The expression for torque is given by the cross product between* $\vec{r}$ *and* $\vec{F}$ *as follow :*

$$\vec{\mathcal{M}}_{/0}(\vec{F}) = \vec{r} \wedge \vec{F}$$

The direction of $\vec{\mathcal{M}}_{/0}(\vec{F})$ *is perpendicular to the plane formed by* $\vec{r}$ *and* $\vec{F}$ *and its sense is given by the right-hand rule or of advance of a right-handed screw rotating from* $\vec{r}$ *to* $\vec{F}$.
the magnitude of the torque : $\|\vec{\mathcal{M}}_{/0}(\vec{F})\| = \|\vec{r}\| \, \|\vec{F}\| \sin\alpha$ *, where* α *is the smaller angle between* $\vec{r}$ *and* $\vec{F}$.

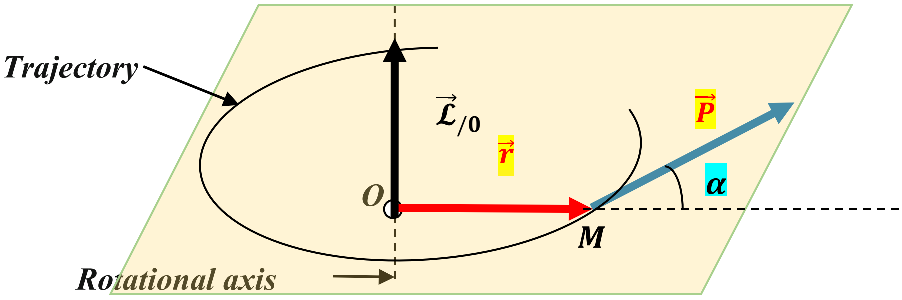


6) Angular momentum

The angular momentum of the point M in rotation around a fixed point O is vector quantity denoted $\vec{\mathcal{L}}_{/0}$ *, its expression is given by is given by the cross product between the vector* $\vec{r}$ *and the linear momentum* $\vec{P}$ *as follow :*

$$\vec{\mathcal{L}}_{/0} = \vec{r} \wedge \vec{P}$$

- ✓ *The angular momentum is a vector perpendicular to the plane containing the vectors* $\vec{r}$ *and* $\vec{P}$.
- ✓ *The magnitude of the angular momentum:* $\|\vec{\mathcal{L}}_{/0}\| = \|\vec{r}\| \, \|\vec{P}\| \sin\alpha$ *, where* α *is the smaller angle between* $\vec{r}$ *and* $\vec{P}$.



The angular momentum expression can be written in Cartesian coordinates as follows:

$$\begin{cases} \vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k} \\ \vec{P} = P_x\,\vec{\imath} + P_y\,\vec{\jmath} + P_z\,\vec{k} \end{cases}$$

$$\vec{\mathcal{L}}_{/0} = \vec{r} \wedge \vec{P} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix} = \begin{vmatrix} y & z \\ P_y & P_z \end{vmatrix}\vec{\imath} - \begin{vmatrix} x & z \\ P_x & P_z \end{vmatrix}\vec{\jmath} + \begin{vmatrix} x & y \\ P_x & P_y \end{vmatrix}\vec{k}$$

$$\vec{\mathcal{L}}_{/0} = \underbrace{(yP_z - P_y z)}_{\mathcal{L}_x}\vec{\imath} - \underbrace{(xP_z - P_x z)}_{\mathcal{L}_y}\vec{\jmath} + \underbrace{(xP_y - P_x y)}_{\mathcal{L}_z}\vec{k}$$

If the system consists of a number of particles, the angular momentum of the system is the sum of the angular momentum of these particles $\vec{\mathcal{L}}_{/0} = \sum_i \vec{\mathcal{L}_{\imath}}_{/0}$

6-1) The angular momentum theorem

The angular momentum theorem states that the derivative of the total angular momentum of a system with respect to time is equal to the sum of the external torques applied to the system. The law of the angular momentum theorem can be derived as follows:

$$\vec{\mathcal{L}}_{/0} = \vec{r} \wedge \vec{P}$$

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \frac{d}{dt}(\vec{r} \wedge \vec{P}) = \frac{d\vec{r}}{dt} \wedge \vec{P} + \vec{r} \wedge \frac{d\vec{P}}{dt} = \vec{v} \wedge \boldsymbol{m}\vec{v} + \vec{r} \wedge \frac{d\vec{P}}{dt}$$

$$\vec{v} \wedge \boldsymbol{m}\vec{v} = \vec{0}$$

$$\frac{d\vec{P}}{dt} = \vec{F}$$

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \vec{r} \wedge \vec{F} = \overrightarrow{\boldsymbol{\mathcal{M}}}_{/0}(\vec{\boldsymbol{F}})$$

If the system consists of n particles, then the derivative of the angular momentum of the system is equal to the sum of the torques of all external forces acting on these particles.

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \sum_{i}^{N} \overrightarrow{\mathcal{M}_i}_{/0}(\vec{F})$$

6-2) Conservation of Angular Momentum

The angular momentum of a system is conserved if the derivative of the angular momentum with respect to time is zero. Therefore, we can write:

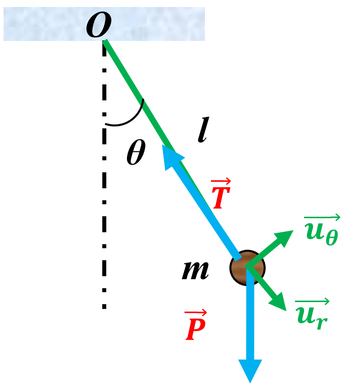
$$\vec{\mathcal{L}}_{/0} = \vec{r} \wedge \vec{P}$$

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \vec{0} \quad \text{if} \quad \overrightarrow{\mathcal{M}}_{/0}(\vec{F}) = \vec{0}$$

If the net external torque acting on a system is zero (i.e. an isolated system), the total angular momentum of the system remains constant in both magnitude and direction.

6-3) Application : simple pendulum

A simple pendulum consists of a small mass suspended by a thread of length L of massless and non-stretchable fixed at the other end. If the mass is pulled to the right or left from its equilibrium position and then released, the pendulum will swing in a vertical plane about an axis passing through O.



the forces applied to the mass m are the weight $\vec{P}$ *and the tension* $\vec{T}$*, their expressions in the polar coordinates are given as :*

$$\begin{cases} \vec{P} = P\cos\theta\,\vec{u_r} - P\sin\theta\,\vec{u_\theta} \\ \vec{T} = T\,\vec{u_r} \end{cases}$$

By Applying the angular momentum theorem:

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \sum_{i}^{N} \overrightarrow{\mathcal{M}_{\mathrm{i}}}_{/0}(\vec{F}) = \overrightarrow{\mathcal{M}_{\mathrm{i}}}_{/0}(\vec{P}) + \overrightarrow{\mathcal{M}_{\mathrm{i}}}_{/0}(\vec{T})$$

the vector $\overrightarrow{OM} = l\,\overrightarrow{u_r}$

$$\begin{cases} \overrightarrow{\mathcal{M}_{\mathrm{i}}}_{/0}(\vec{P}) = \overrightarrow{OM} \wedge \vec{P} \\ \overrightarrow{\mathcal{M}_{\mathrm{i}}}_{/0}(\vec{T}) = \overrightarrow{OM} \wedge \vec{T} \end{cases}$$

$$\overrightarrow{\mathcal{M}_{i}}_{/0}(\vec{P}) = \overrightarrow{OM} \wedge \vec{P} = \begin{vmatrix} \overrightarrow{u_r} & \overrightarrow{u_\theta} & \vec{k} \\ l & 0 & 0 \\ P\cos\theta & -P\sin\theta & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 \\ -P\sin\theta & 0 \end{vmatrix}\overrightarrow{u_r} - \begin{vmatrix} l & 0 \\ P\cos\theta & 0 \end{vmatrix}\overrightarrow{u_\theta} + \begin{vmatrix} l & 0 \\ P\cos\theta & -P\sin\theta \end{vmatrix}\vec{k} = -l\,P\sin\theta\,\vec{k}$$

$$\overrightarrow{\mathcal{M}_{i}}_{/0}(\vec{T}) = \overrightarrow{OM} \wedge \vec{T} = \begin{vmatrix} \overrightarrow{u_r} & \overrightarrow{u_\theta} & \vec{k} \\ l & 0 & 0 \\ T & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}\overrightarrow{u_r} - \begin{vmatrix} l & 0 \\ T & 0 \end{vmatrix}\overrightarrow{u_\theta} + \begin{vmatrix} l & 0 \\ T & 0 \end{vmatrix}\vec{k} = 0\,\vec{k}$$

$$\vec{\mathcal{L}}_{/0} = \overrightarrow{OM} \wedge \vec{P} = \overrightarrow{OM} \wedge \boldsymbol{m}\,\vec{v}$$

$$\vec{v} = \frac{d(\overrightarrow{OM})}{dt} = l\frac{d\overrightarrow{u_r}}{dt} = l\frac{d\theta}{dt}\,\overrightarrow{u_\theta} \Rightarrow \vec{P} = \boldsymbol{m}\,\vec{v} = m\,l\frac{d\theta}{dt}\,\overrightarrow{u_\theta}$$

$$\vec{\mathcal{L}}_{/0} = \overrightarrow{OM} \wedge \vec{P} = \begin{vmatrix} \overrightarrow{u_r} & \overrightarrow{u_\theta} & \vec{k} \\ l & 0 & 0 \\ 0 & m\,l\frac{d\theta}{dt} & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ m\,l\frac{d\theta}{dt} & 0 \end{vmatrix}\overrightarrow{u_r} - \begin{vmatrix} l & 0 \\ 0 & 0 \end{vmatrix}\overrightarrow{u_\theta} + \begin{vmatrix} l & 0 \\ 0 & m\,l\frac{d\theta}{dt} \end{vmatrix}\vec{k}$$

$$\vec{\mathcal{L}}_{/0} = m\,l^2\frac{d\theta}{dt}\,\vec{k}$$

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \frac{d(m\,l^2\frac{d\theta}{dt}\,\vec{k})}{dt} = m\,l^2\frac{d^2\theta}{d^2t}\,\vec{k} = m\,l^2\ddot{\theta}\,\vec{k}$$

$$m\,l^2\ddot{\theta}\,\vec{k} = -l\,P\sin\theta\,\vec{k} \Rightarrow ml^2\ddot{\theta} = -l\,P\sin\theta$$

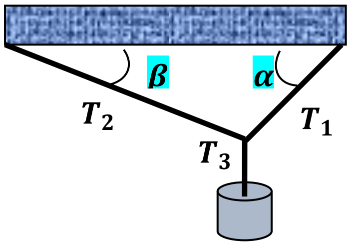
For small values of angle $\theta \Rightarrow \sin\theta \approx \theta$

$$\ddot{\theta} + l\frac{mg}{ml^2}\theta = 0 \quad \Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$$

7) solved exercises

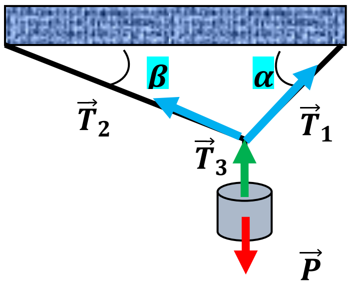
EXERCISE 01

A body of mass m= 3 Kg suspends by three cords, as illustrated in the figure:



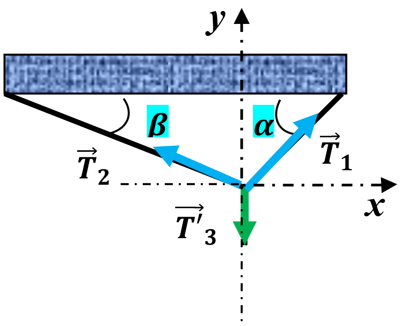
If the angles **α** *and* **β** *formed by cords 1 and 2 with the horizontal plane are* $\frac{\pi}{6}$ *and* $\frac{\pi}{3}$*, respectively, find the tension force of the three cords.*

Solution



$$\sum \vec{F}_{ext} = \vec{0} \Rightarrow \vec{P} + \vec{T}_3 = \vec{0} \Rightarrow T_3 = m\,g = 3 \times 9.8 = 29.4\ N$$

at the knot



$$\sum \vec{F}_{ext} = \vec{0} \Rightarrow \vec{T}_1 + \vec{T}_2 + \vec{T'}_3 = \vec{0} \quad \text{where } \vec{T'}_3 = -\vec{T}_3$$

$$\begin{cases} \vec{T}_1 = T_1 \cos\alpha\ \vec{\imath} + T_1 \sin\alpha\ \vec{\jmath} \\ \vec{T}_2 = -T_2 \cos\beta\ \vec{\imath} + T_2 \sin\beta\ \vec{\jmath} \\ \vec{T'}_3 = -T_3\ \vec{\jmath} \end{cases}$$

$$\vec{T}_1 + \vec{T}_2 + \vec{T'}_3 = \vec{0} \Rightarrow T_1 \cos\alpha\ \vec{\imath} + T_1 \sin\alpha\ \vec{\jmath} - T_2 \cos\beta\ \vec{\imath} + T_2 \sin\beta\ \vec{\jmath} - T_3\ \vec{\jmath} = \vec{0}$$

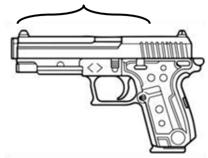
$$(T_1 \cos\alpha - T_2 \cos\beta)\,\vec{\imath} + (T_1 \sin\alpha + T_2 \sin\beta - T_3)\,\vec{\jmath} = \vec{0}$$

$$\begin{cases} T_1 \cos\alpha - T_2 \cos\beta = 0 \\ T_1 \sin\alpha + T_2 \sin\beta - T_3 = 0 \end{cases} \Rightarrow \begin{cases} \frac{T_1\sqrt{3}}{2} - \frac{T_2}{2} = 0 \\ \frac{T_1}{2} + \frac{T_2\sqrt{3}}{2} - T_3 = 0 \end{cases}$$

$$\begin{cases} T_1\sqrt{3} = T_2 \\ \frac{T_1}{2} + \frac{T_1\sqrt{3}\sqrt{3}}{2} = T_3 \end{cases} \Rightarrow \begin{cases} T_1\sqrt{3} = T_2 \\ \frac{T_1}{2} + \frac{3T_1}{2} = T_3 \end{cases} \Rightarrow \begin{cases} T_2 = T_1\sqrt{3} = \frac{T_3\sqrt{3}}{2} = 25.46N \\ T_1 = \frac{T_3}{2} = 14.7\ N \end{cases}$$

EXERCISE 02

A billet with a mass of 5 g is launched from rest and reached a velocity of 200 m/s at the exit of handgun barrel of 8 cm length .



1- Assuming that the acceleration of the bullet is constant inside the barrel, calculate the acceleration of the bullet.

2- Calculate the force applied to the bullet that causes this acceleration.

Solution

1- Since the bullet's motion is straight and uniformly varied with constant acceleration, we can write the motion equations of this bullet:

$$\begin{cases} x = \frac{1}{2}a\,t^2 + v_0\,t + x_0 \\ v = at + v_0 \end{cases} \text{ where } v_0 = 0\,\frac{m}{s} \text{ and } x_0 = 0$$

$$v = at \Rightarrow t = \frac{v}{a} \text{ and } x = \frac{1}{2}a\,t^2 = x = \frac{1}{2}a\,(\frac{v}{a})^2$$

$$x = \frac{v^2}{2a} \Rightarrow a = \frac{v^2}{2x}$$

x is the length of the gun = 8cm =0.08m

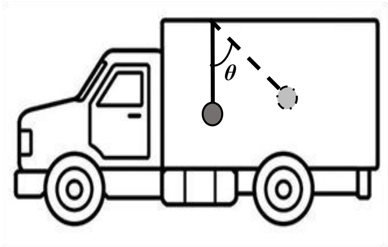
$$a = \frac{(200)^2}{2 \times 0.08} = 2500000 \text{ m/s}^2$$

2- The applied force

$$F = m \times a = 0.005 \times 2500000 = 12500 \text{ N}$$

EXERCISE 03

A small ball of mass 10 g is suspended by a massless thread from the roof of a truck at rest.



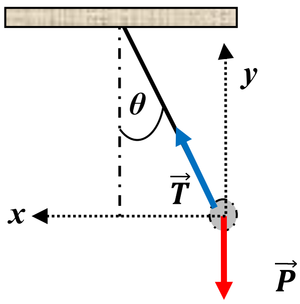
1- Calculate the angle θ made by the thread when the truck is moving with an acceleration of 3 m/s²

, Determine the value of the thread tension.

2- Calculate the angle θ made by the thread when the truck is moving with a constant speed of 50 m/s , Determine the value of the thread tension.

Solution

1- Calculating the angle θ made by the thread when the truck is moving with an acceleration of 3 m/s², and determining the thread tension value



$$\sum \vec{F}_{ext} = m\,\vec{a} \Rightarrow \vec{P} + \vec{T} = m\,\vec{a} \text{ where } \begin{cases} \vec{P} = -mg\,\vec{j} \\ \vec{T} = T\sin\theta\,\vec{i} + T\cos\theta\,\vec{j} \\ \vec{a} = a\,\vec{i} \end{cases}$$

$$-mg\,\vec{j} + T\sin\theta\,\vec{i} + T\cos\theta\,\vec{j} = m\,a\,\vec{i} \Rightarrow T\sin\theta\,\vec{i} + (T\cos\theta - mg)\,\vec{j} = m\,a\,\vec{i}$$

$$\Rightarrow \begin{cases} T\ sin\,\theta = ma \\ T\ cos\,\theta\ - mg = 0 \end{cases} \Rightarrow \begin{cases} T\ sin\,\theta = ma \\ T\ cos\,\theta = mg \end{cases} \Rightarrow \frac{T\ sin\,\theta}{T\ cos\,\theta} = \frac{ma}{m\,g} \Rightarrow \tan\theta = \frac{a}{g} = \frac{3}{9.8}$$

$$\theta = 17.0^0$$

$$T\ cos\,\theta = mg\ \Rightarrow T\ = \frac{mg}{cos\,\theta} = \frac{0.01 \times 9.8}{0.95} = 0.103\ N$$

2- Calculating the angle θ made by the thread when the truck is moving with a constant speed of 50 m/s , and determining the thread tension value

in this case the acceleration is zero ($\vec{a} = \vec{0}$)since the truck is moving at a constant velocity

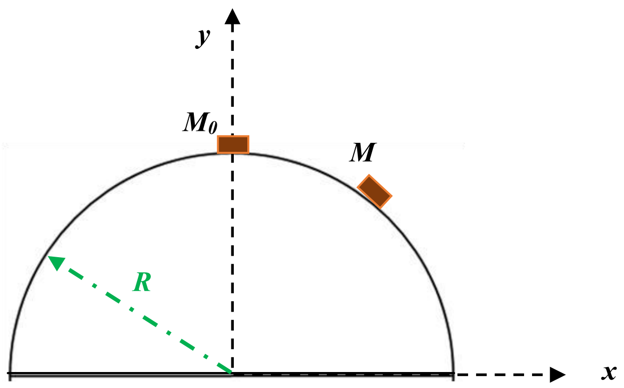
$$\sum \vec{F}_{ext} = \vec{0}\ \Rightarrow\ \vec{P} + \vec{T} = \vec{0}\ \ where \begin{cases} \vec{P} = -mg\,\vec{j} \\ \vec{T} = T\ \sin\theta\,\vec{i} + T\ \cos\theta\,\vec{j} \\ \vec{a} = 0\,\vec{i} \end{cases}$$

$$-mg\,\vec{j} + T\ sin\,\theta\,\vec{i} + T\ cos\,\theta\ \vec{j} = 0\,\vec{i}\ \Rightarrow\ T\ sin\,\theta\,\vec{i} + (T\ cos\,\theta\ - mg\,)\vec{j} = 0\,\vec{i}$$

$$\Rightarrow \begin{cases} T\ sin\,\theta = 0 \\ T\ cos\,\theta\ - mg = 0 \end{cases} \Rightarrow \begin{cases} sin\,\theta = 0 \\ T\ cos\,\theta = mg \end{cases} \Rightarrow \begin{cases} \theta = 0 \\ T = mg = 0.098\ N \end{cases}$$

EXERCISE 04

*A block of 20 g slides without friction from the highest point of a Surface of a hemisphere of radius **R** (as shown in the figure), starting at position M_0 at time t = 0 s.*



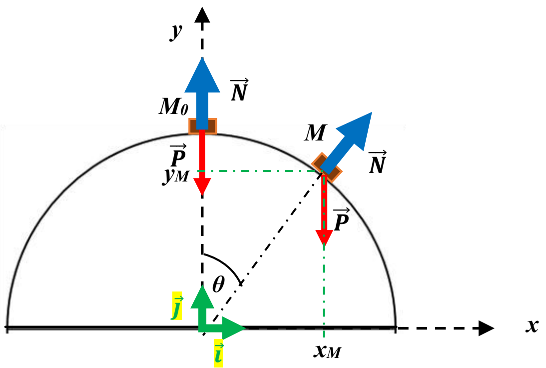
1. *represent the forces acting on the body at positions M_0 and M.*
2. *Using Cartesian coordinates (xy), write the position vector, instantaneous velocity, and acceleration of the body, then calculate their magnitudes.*

3. *Using Cartesian coordinates (xy), write the analytical expression for each force at the point M.*
4. *Write the velocity as a function of g R and cos θ, then deduce the formula for the normal force.*
5. *Calculate the angle at which the body stops touching the surface of the sphere.*
6. *Re-answer all the previous questions using the polar coordinates (r, θ).*

Solution

1- Representing the forces acting on the body at positions M_0 and M.

(the forces acting on the block are: the gravitational force (the weigh force) $\vec{P}$ *(vertical towards the centre of the earth), and the normal force* $\vec{N}$ *(perpendicular to the surface)*



2- Writing the position, instantaneous velocity, and instantaneous acceleration vectors of the body using Cartesian coordinates (xy).

- Position vector

$$\overrightarrow{OM} = \vec{r} = x_M\,\vec{\imath} + y_M\vec{\jmath} \Rightarrow \begin{cases} x_M = R\sin\theta \\ y_M = R\cos\theta \end{cases} \Rightarrow \overrightarrow{OM} = \vec{r} = R\sin\theta\;\vec{\imath} + R\cos\theta\,\vec{\jmath}$$

$$\left\|\overrightarrow{OM}\right\| = r = \sqrt{(R\sin\theta)^2 + (R\cos\theta)^2} = \sqrt{(R)^2\,(\cos^2\theta + \sin^2\theta)} = R$$

- Velocity vector

$$\vec{v} = v_x\,\vec{\imath} + v_y\vec{\jmath} = \frac{d\vec{r}}{dt} = \frac{dx_M}{dt}\,\vec{\imath} + \frac{dy_M}{dt}\vec{\jmath}$$

$$\Rightarrow \begin{cases} v_x = \dfrac{dx_M}{dt} = \dot{x}_M = \dfrac{d\theta}{dt} R\cos\theta = \dot{\theta} R\cos\theta \\ v_y = \dfrac{dx_M}{dt} = \dot{y}_M = -\dfrac{d\theta}{dt} R\sin\theta = -\dot{\theta} R\sin\theta \end{cases}$$

$$\vec{v} = \frac{d\theta}{dt} R\cos\theta\,\vec{\imath} - \frac{d\theta}{dt} R\sin\theta\,\vec{\jmath} = R\dot{\theta}\cos\theta\,\vec{\imath} - R\dot{\theta}\sin\theta\,\vec{\jmath}$$

$$\|\vec{v}\| = v = \sqrt{(R\dot{\theta}\cos\theta)^2 + (-R\dot{\theta}\sin\theta)^2} = \sqrt{(R\dot{\theta})^2\,(cos^2\theta + sin^2\theta)} = R\dot{\theta}$$

- Acceleration vector

$$\vec{a} = a_x\,\vec{\imath} + a_y\vec{\jmath} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\,\vec{\imath} + \frac{dv_y}{dt}\vec{\jmath} \Rightarrow \begin{cases} a_x = \dfrac{dv_x}{dt} = \dot{v}_x = \dfrac{d(\dot{\theta}R\cos\theta)}{dt} \\ a_y = \dfrac{dv_y}{dt} = \dot{v}_y = \dfrac{d(-\dot{\theta}R\sin\theta)}{dt} \end{cases}$$

$$\begin{cases} a_x = \dfrac{d(\dot{\theta}R\cos\theta)}{dt} = R\cos\theta\dfrac{d(\dot{\theta})}{dt} + \dot{\theta}\dfrac{d(R\cos\theta)}{dt} = R\cos\theta\,\ddot{\theta} - R\dot{\theta}^2\sin\theta \\ a_y = \dfrac{d(-\dot{\theta}R\sin\theta)}{dt} = -R\sin\theta\dfrac{d(\dot{\theta})}{dt} - \dot{\theta}\dfrac{d(R\sin\theta)}{dt} = -R\sin\theta\,\ddot{\theta} - R\dot{\theta}^2\cos\theta \end{cases}$$

$$\vec{a} = (R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta)\vec{\imath} + (-R\ddot{\theta}\sin\theta - R\dot{\theta}^2\cos\theta)\vec{\jmath}$$

$$\|\vec{a}\| = a = \sqrt{(R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta)^2 + (-R\ddot{\theta}\sin\theta - R\dot{\theta}^2\cos\theta)^2}$$

$$a = \sqrt{(R\ddot{\theta})^2 + (R\dot{\theta}^2)^2}$$

3- Writing the analytical expression for each force at the point M using Cartesian coordinates (xy).

$$\begin{cases} \vec{P} = -mg\,\vec{\jmath} \\ \vec{N} = N\sin\theta\,\vec{\imath} + N\cos\theta\,\vec{\jmath} \end{cases}$$

4- Writing the velocity as a function of g R and cosθ

$$\sum \vec{F}_{ext} = m\,\vec{a} \Rightarrow \vec{P} + \vec{N} = m\,\vec{a}$$

$$N\sin\theta\ \vec{\imath} + N\cos\theta\,\vec{\jmath} - mg\,\vec{\jmath} = m\left(R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta\right)\vec{\imath} + m(-R\ddot{\theta}\sin\theta - R\dot{\theta}^2\cos\theta)\vec{\jmath}$$

$$\begin{cases} N\sin\theta = m\left(R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta\right) \\ N\cos\theta = m(-R\ddot{\theta}\sin\theta - R\dot{\theta}^2\cos\theta) - mg \end{cases}$$

By multiplying both sides of the 1st equations by $\cos\theta$ *, and both sides of the 2nd equation by* $\sin\theta$ *, we get:*

$$\begin{cases} \cos\theta\,N\sin\theta = \cos\theta\,mR\ddot{\theta}\cos\theta - \cos\theta\ mR\dot{\theta}^2\sin\theta \\ \sin\theta\ N\cos\theta = \sin\theta\ mg - \sin\theta\ mR\ddot{\theta}\sin\theta - \sin\theta\ mR\dot{\theta}^2\cos\theta \end{cases}$$

$$\cos\theta\,mR\ddot{\theta}\cos\theta - \cos\theta\ mR\dot{\theta}^2\sin\theta = \sin\theta\ mg - \sin\theta\ mR\ddot{\theta}\sin\theta - \sin\theta\ mR\dot{\theta}^2\cos\theta$$

$$\sin\theta\ mR\ddot{\theta}\sin\theta + \cos\theta\,mR\ddot{\theta}\cos\theta = \sin\theta\ mg - \sin\theta\ mR\dot{\theta}^2\cos\theta + \cos\theta\ mR\dot{\theta}^2\sin\theta$$

$$mR\ddot{\theta}(\sin^2\theta + \cos^2\theta) = \sin\theta\ mg + mR\dot{\theta}^2(-\sin\theta\cos\theta + \cos\theta\ \sin\theta)$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$R\ddot{\theta} = \sin\theta\ g \quad \Rightarrow R\frac{d\dot{\theta}}{dt} = \sin\theta\ g$$

$$Note\ that\ \frac{d\theta}{dt} = \dot{\theta} = \frac{v}{R} \Rightarrow \frac{R}{R}\frac{dv}{dt} = \sin\theta\ g \Rightarrow \frac{dv}{dt} = \sin\theta\ g$$

By multiplying both sides of last equation by $d\theta$

$$d\theta\frac{dv}{dt} = g\sin\theta\ d\theta \Rightarrow dv\frac{d\theta}{dt} = g\sin\theta\ d\theta \Rightarrow v\,dv = Rg\sin\theta\,d\theta$$

$$\int_0^v v\,dv = \int_0^\theta Rg\sin\theta\,d\theta \Rightarrow \frac{1}{2}v^2 = [-Rg\cos\theta]_0^\theta = Rg\,(1-\cos\theta)$$

$$\boldsymbol{v = \sqrt{2Rg\,(1-\cos\theta)}}$$

** Deducing the formula for the normal force.*

$$N\sin\theta = mR\ddot{\theta}\cos\theta - mR\dot{\theta}^2\sin\theta$$

$$\text{note that} \quad R\ddot{\theta} = \sin\theta \; g \;\Rightarrow N\sin\theta = mg\sin\theta \; \cos\theta - mR\dot{\theta}^2 \sin\theta$$

$$N = mg\;\cos\theta - mR\dot{\theta}^2 \quad \text{Note that } \frac{d\theta}{dt} = \;\dot{\theta} = \frac{v}{R} \;\Rightarrow\; N = mg\;\cos\theta - mR\left(\frac{v}{R}\right)^2$$

$$N = mg\;\cos\theta - \frac{m}{R}v^2 \quad \text{note that } \boldsymbol{v} = \sqrt{\mathbf{2}\boldsymbol{Rg}\;(\mathbf{1} - \mathbf{cos}\,\boldsymbol{\theta})}$$

$$N = mg\;\cos\theta - \frac{m}{R}\mathbf{2}\boldsymbol{Rg}\;(\mathbf{1} - \mathbf{cos}\,\boldsymbol{\theta}) \;\Rightarrow N = mg\;\cos\theta - 2m\;\boldsymbol{g}\;(\mathbf{1} - \mathbf{cos}\,\boldsymbol{\theta})$$

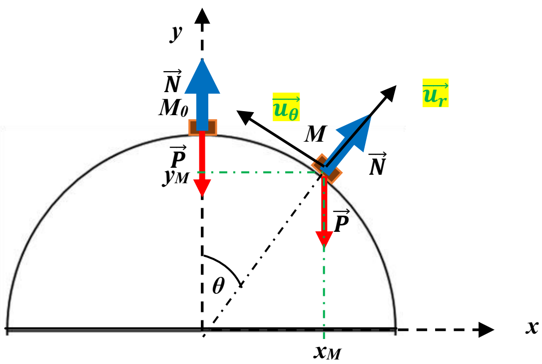
$$\boldsymbol{N = mg\;(cos\,\theta - 2 + 2\cos\theta) = m\;g\;(-2 + 3\;cos\;\theta\;)}$$

5- Calculate the angle at which the body stops touching the surface of the sphere.

$$N = 0 = m\;g\;(-2 + 3\;cos\;\theta\;) \Rightarrow -2 + 3\;cos\;\theta = 0 \Rightarrow 3\;cos\;\theta = 2 \Rightarrow cos\;\theta = \frac{2}{3} \Rightarrow \theta = 48^0$$

6- Re-answer all the previous questions using ***the polar coordinates (r, θ).***

a- Writing the position, instantaneous velocity, and instantaneous acceleration vectors of the body, using Cartesian coordinates (r, θ).



- Position vector

$$\overrightarrow{OM} = R\,\vec{u}_r$$

- Velocity vector

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dR}{dt}\vec{u}_r + \frac{d\vec{u}_r}{dt}R \Rightarrow \begin{cases} \dfrac{dR}{dt} = 0 \\ \dfrac{d\vec{u}_r}{dt} = \dfrac{d\theta}{dt}\vec{u}_\theta \end{cases} \Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = R\frac{d\theta}{dt}\vec{u}_\theta = R\dot{\theta}\vec{u}_\theta$$

- Acceleration vector

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(R\frac{d\theta}{dt}\vec{u}_\theta)}{dt} = \frac{d\theta}{dt}\vec{u}_\theta\frac{d(R)}{dt} + R\frac{d\theta}{dt}\frac{d(\vec{u}_\theta)}{dt} + R\vec{u}_\theta\frac{d(\frac{d\theta}{dt})}{dt}$$

$$\Rightarrow \begin{cases} \dfrac{dR}{dt} = 0 \\ \dfrac{d\vec{u}_\theta}{dt} = -\dfrac{d\theta}{dt}\vec{u}_r = -\dot{\theta}\,\vec{u}_r \\ d\left(\dfrac{d\theta}{dt}\right) = \dfrac{d^2\theta}{d^2t} = \ddot{\theta} \end{cases}$$

$$\vec{a} = -R\dot{\theta}\,\dot{\theta}\,\vec{u}_r + R\vec{u}_\theta\ddot{\theta} = -R\dot{\theta}^2\,\vec{u}_r + R\ddot{\theta}\,\vec{u}_\theta$$

b- Writing the analytical expression for each force at the point M using Cartesian coordinates (r, θ).

$$\begin{cases} \vec{N} = N\,\vec{u}_r \\ \vec{p} = -P\cos\theta\ \vec{u}_r - P\sin\theta\,\vec{u}_\theta \end{cases}$$

c- Writing the velocity as a function of g R and cosθ

$$\sum \vec{F}_{ext} = m\,\vec{a} \Rightarrow \vec{P} + \vec{N} = m\,\vec{a}$$

$$N\,\vec{u}_r - P\cos\theta\ \vec{u}_r - P\sin\theta\,\vec{u}_\theta = -m\,R\dot{\theta}^2\,\vec{u}_r + mR\ddot{\theta}\,\vec{u}_\theta$$

$$(N - P\cos\theta)\,\vec{u}_r - P\sin\theta\,\vec{u}_\theta = -m\,R\dot{\theta}^2\,\vec{u}_r + mR\ddot{\theta}\,\vec{u}_\theta$$

$$\Rightarrow \begin{cases} N - P\cos\theta = -m\,R\dot{\theta}^2 \\ -P\sin\theta = mR\ddot{\theta} \end{cases}$$

$$N = P\cos\theta - m\,R\dot{\theta}^2$$

$$-P\sin\theta = mR\ddot{\theta} \Rightarrow R\ddot{\theta} = \sin\theta\ g \quad \Rightarrow R\frac{d\dot{\theta}}{dt} = \sin\theta\ g$$

$$Note\ that\ \frac{d\theta}{dt} = \dot{\theta} = \frac{v}{R} \Rightarrow \frac{R}{R}\frac{dv}{dt} = \sin\theta\ g \Rightarrow \frac{dv}{dt} = \sin\theta\ g$$

By multiplying both sides of last equation by $d\theta$

$$d\theta\ \frac{dv}{dt} = g\sin\theta\ d\theta \Rightarrow dv\ \frac{d\theta}{dt} = g\sin\theta\ d\theta \Rightarrow v\ dv = Rg\sin\theta\ d\theta$$

$$\int_0^v v\ dv = \int_0^\theta Rg\sin\theta\ d\theta \Rightarrow \frac{1}{2}v^2 = [-Rg\cos\theta]_0^\theta = Rg\ (1-\cos\theta)$$

$$v = \sqrt{2Rg\ (1-\cos\theta)}$$

** Deducing the formula for the normal force*

$$N - P\cos\theta = -m\,R\dot{\theta}^2 \Rightarrow N = P\cos\theta - m\,R\dot{\theta}^2$$

$$Note\ that\ \frac{d\theta}{dt} = \dot{\theta} = \frac{v}{R} \Rightarrow N = mg\ \cos\theta - mR(\frac{v}{R})^2$$

$$N = mg\ \cos\theta - \frac{m}{R}v^2\ \ note\ that\ \boldsymbol{v} = \sqrt{\boldsymbol{2Rg\ (1-\cos\theta)}}$$

$$N = mg\ \cos\theta - \frac{m}{R}\boldsymbol{2Rg\ (1-\cos\theta)} \Rightarrow N = mg\ \cos\theta - 2m\ \boldsymbol{g\ (1-\cos\theta)}$$

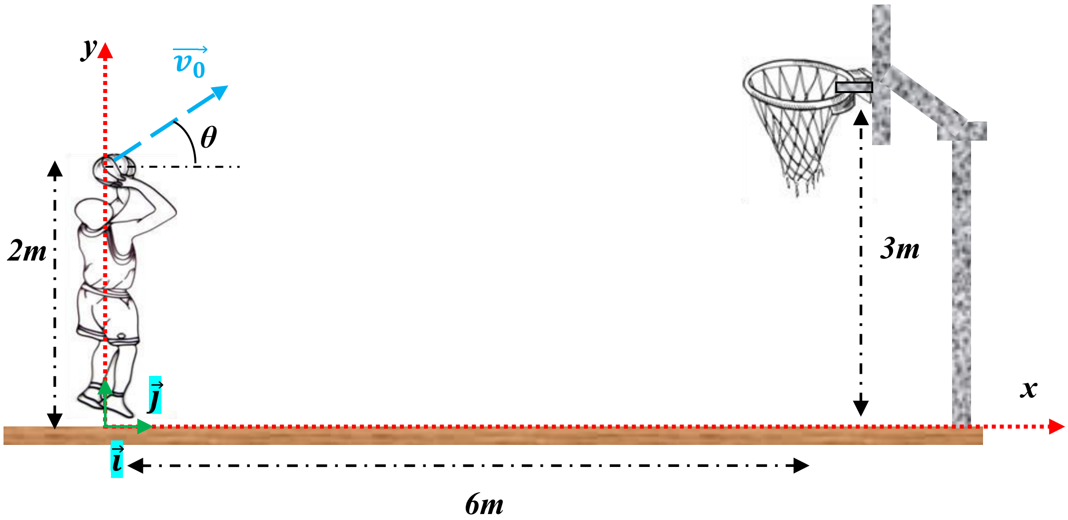
$$N = mg\ (\cos\theta - 2 + 2\cos\theta) = m\ g\ (-2 + 3\cos\theta)$$

5- Calculate the angle at which the body stops touching the surface of the sphere.

$$N = 0 = m\ g\ (-2 + 3\cos\theta) \Rightarrow -2 + 3\cos\theta = 0 \Rightarrow 3\cos\theta = 2 \Rightarrow \cos\theta = \frac{2}{3} \Rightarrow \theta = 48^0$$

EXERCISE 05

A basketball player, standing 2 meters tall, throws a ball at an angle of 30^0, as shown in the figure. The basket is 3 meters above the ground and 6 meters away from the player.



1. Study the motion of this ball and write the time-dependent equations that describe its motion.

2. Find the velocity required for the ball to reach the basket? And how long does it take to reach the basket?

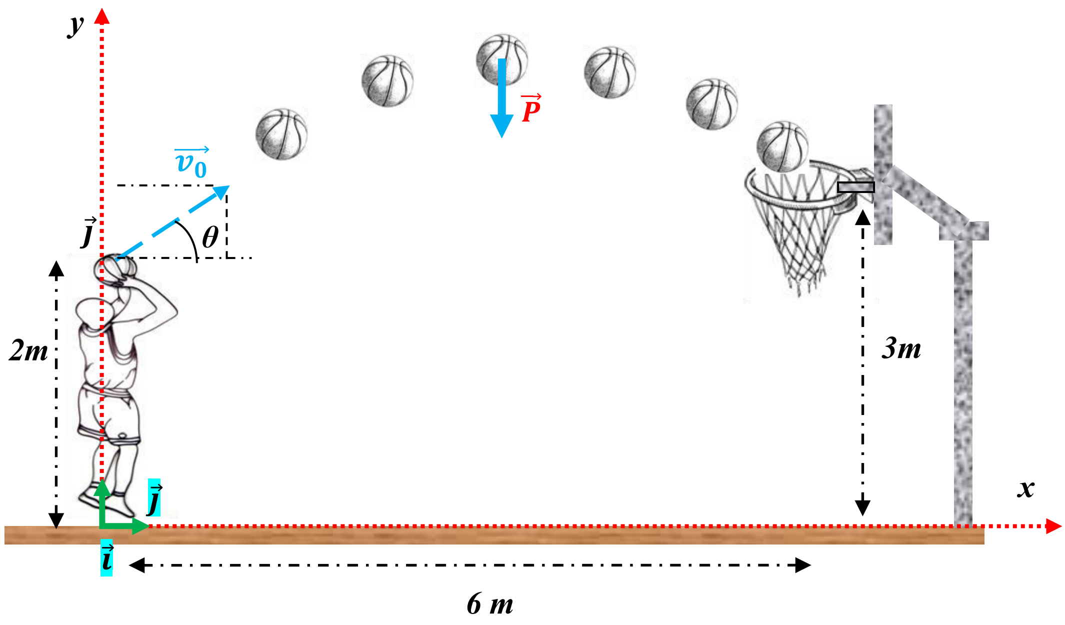
3. Write the equation of a basketball's trajectory, then find the maximum height the ball can reach.

4. If the player throws the ball at initial velocity = 10 m/s at an angle of 45^0, find the distance at which he must be positioned for the ball to go into the basket.

Solution

1- *Writing the time-dependent equations that describe its motion.*

$$\sum \vec{F}_{ext} = m\,\vec{a} \Rightarrow \vec{P} = m\,\vec{a} \Rightarrow -m\ g\,\vec{j} = m\,\vec{a} = m\,a_x\,\vec{i} + m\,a_y\,\vec{j}$$



To find the equations of motion, we follow the steps:

$$\vec{a} = -g\,\vec{j}$$

$$\vec{a} = a_x\,\vec{i} + a_y\,\vec{j} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\,\vec{i} + \frac{dv_y}{dt}\vec{j} = -g\,\vec{j} \Rightarrow \begin{cases} \frac{dv_x}{dt} = 0 \\ \frac{dv_y}{dt} = -g \end{cases}$$

$$\Rightarrow \begin{cases} dv_x = 0\,dt \\ dv_y = -g\,dt \end{cases} \Rightarrow \begin{cases} \int_{v_{x_0}}^{v_x} dv_x = \int_0^t 0\,dt \\ \int_{v_{y_0}}^{v_y} dv_y = \int_0^t -g\,dt \end{cases} \Rightarrow \begin{cases} v_x - v_{x_0} = 0 \\ v_y - v_{y_0} = -gt \end{cases}$$

$$\begin{cases} v_x = v_{x_0} = v_0\cos\theta = \frac{v_0\sqrt{3}}{2} \\ v_y = -gt + v_{y_0} = -g\,t + v_0\sin\theta = -gt + \frac{v_0}{2} \end{cases}$$

$$\vec{v} = \frac{v_0\sqrt{3}}{2}\,\vec{i} + \left(-gt + \frac{v_0}{2}\right)\vec{j}$$

$$\vec{v} = v_x\,\vec{i} + v_y\,\vec{j} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\vec{i} + \frac{dy}{dt}\vec{j} = \frac{v_0\sqrt{3}}{2}\,\vec{i} + \left(-gt + \frac{v_0}{2}\right)\vec{j}$$

$$\Rightarrow \begin{cases} \frac{dx}{dt} = \frac{v_0\sqrt{3}}{2} \\ \frac{dy}{dt} = -gt + \frac{v_0}{2} \end{cases} \Rightarrow \begin{cases} dx = \frac{v_0\sqrt{3}}{2}\,dt \\ dy = \left(-gt + \frac{v_0}{2}\right)dt \end{cases} \Rightarrow \begin{cases} \int_{x_0}^{x} dx = \int_0^t \frac{v_0\sqrt{3}}{2}\,dt \\ \int_{y_0}^{y} dy = \int_0^t \left(-gt + \frac{v_0}{2}\right)dt \end{cases}$$

$$\Rightarrow \begin{cases} x - x_0 = \dfrac{v_0\sqrt{3}}{2}t \\ y - y_0 = -\dfrac{1}{2}gt^2 + \dfrac{v_0}{2}t \end{cases} \Rightarrow \begin{cases} x = \dfrac{v_0\sqrt{3}}{2}t \ + x_0 \\ y = -\dfrac{1}{2}gt^2 + \dfrac{v_0}{2}t + y_0 \end{cases}$$

According to the given data, at the moment $t = 0s$ $x_0 = 0\ m$ and $y_0 = 2\ m$

$$\begin{cases} x = \dfrac{v_0\sqrt{3}}{2}t \\ y = -\dfrac{1}{2}gt^2 + \dfrac{v_0}{2}t + 2 \end{cases}$$

So the equations of motion for the basketball are as follows:

$$\begin{cases} \vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} = \dfrac{v_0\sqrt{3}}{2}t\,\vec{\imath} + (-\dfrac{1}{2}gt^2 + \dfrac{v_0}{2}t + 2)\,\vec{\jmath} \\ \vec{v} = v_x\,\vec{\imath} + v_y\,\vec{\jmath} = \dfrac{v_0\sqrt{3}}{2}\,\vec{\imath} + \left(-gt + \dfrac{v_0}{2}\right)\vec{\jmath} \\ \vec{a} = a_x\,\vec{\imath} + a_y\,\vec{\jmath} = -\ g\,\vec{\jmath} \end{cases}$$

2- Finding the velocity required for the ball to reach the basket

To reach the basket which has a height of 3 meters, the ball must cover a distance of 6 meters.

$$6 = \frac{v_0\sqrt{3}}{2}t \ \Rightarrow t = \frac{12}{v_0\sqrt{3}}$$

$$3 = -\frac{1}{2}g(\frac{12}{v_0\sqrt{3}})^2 + \frac{v_0}{2}\frac{12}{v_0\sqrt{3}} + 2 \ \Rightarrow g\frac{72}{v^2{}_0} = 6\sqrt{3} - 3$$

$$v^2{}_0 = 9.8\frac{72}{6\sqrt{3} - 3} = 9.7\ \text{m/s}$$

* *Finding the time taken by the basketball to reach the basket*

To reach the basket which has a height of 3 meters, the ball must cover a distance of 6 meters.

$$t = \frac{12}{v_0\sqrt{3}} = \frac{12}{9.7\sqrt{3}} = 0.7s$$

3- Writing the equation of a basketball's trajectory,

$$t = \frac{2x}{v_0\sqrt{3}}$$

$$y = -\frac{1}{2}g(\frac{2x}{v_0\sqrt{3}})^2 + \frac{v_0}{2}\frac{2x}{v_0\sqrt{3}} + 2 \quad \Rightarrow y = -\frac{2g}{3\,v^2{}_0}\,x^2 + \frac{x}{\sqrt{3}} + 2$$

$$y = -\frac{2g}{3\,v^2{}_0}\,x^2 + \frac{x}{\sqrt{3}} + 2$$

**Finding the maximum height the ball can reach.*

$$v_y = 0 = \; -gt + \frac{v_0}{2} \Rightarrow \; gt = \frac{v_0}{2} \Rightarrow t = \frac{v_0}{2g}$$

$$y = -\frac{1}{2}g(\frac{v_0}{2g})^2 + \frac{v_0}{2}\left(\frac{v_0}{2g}\right) + 2 \Rightarrow y = -\frac{v^2{}_0}{8g} + \frac{v^2{}_0}{4g} + 2 \Rightarrow 3.2\,m$$

4- Finding the distance at which he must be positioned for the ball to go into the basket, in case when the player If the player throws the ball at initial velocity $v_0 = \; 6\,m/s$

$$\vec{v} = v_x\,\vec{\imath} + \; v_y\,\vec{\jmath} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\vec{\imath} + \frac{dy}{dt}\vec{\jmath} \Rightarrow \begin{cases} v_x = \dfrac{dx}{dt} = \; v_0\cos\theta \\ v_y = \dfrac{dy}{dt} = \; -gt \; + v_{y_0} = \; -\,g\,t + v_0\sin\theta \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = v_0\cos\theta \\ \dfrac{dy}{dt} = -\,g\,t + v_0\sin\theta \end{cases} \Rightarrow \begin{cases} dx = v_0\cos\theta \; dt \\ dy = (-\,g\,t + v_0\sin\theta\,)dt \end{cases}$$

$$\Rightarrow \begin{cases} \displaystyle\int_{x_0}^{x} dx = \int_0^t v_0\cos\theta \; dt \\ \displaystyle\int_{y_0}^{y} dy = \int_0^t (-\,g\,t + v_0\sin\theta\;\,)dt \end{cases}$$

$$\Rightarrow \begin{cases} x - x_0 = v_0\cos\theta\,t \\ y - y_0 = -\dfrac{1}{2}gt^2 + v_0\sin\theta\,t \end{cases} \Rightarrow \begin{cases} x = v_0\cos\theta\,t \;\; + x_0 \\ y = -\dfrac{1}{2}gt^2 + v_0\sin\theta\,t + y_0 \end{cases}$$

According to the given data, at the moment $t = 0s$ $x_0 = 0\,m$ *and* $y_0 = 2\,m$

$$\begin{cases} x = v_0\cos\theta\,t \\ y = -\dfrac{1}{2}gt^2 + v_0\sin\theta\,t + 2 \end{cases}$$

To reach the basket, the ball must arrive at the coordinates of the basket (x, 3). Therefore we can write:

$$\begin{cases} x = 10\,t\,\cos(\dfrac{\pi}{4}) \\ 3 = -\dfrac{1}{2}gt^2 + 8\,t\;\sin(\dfrac{\pi}{4}) + 2 \end{cases} \Rightarrow \begin{cases} x = 10\,t\dfrac{\sqrt{2}}{2} = 5\sqrt{2}\;t \\ 1 = -\dfrac{1}{2}gt^2 + 5\sqrt{2}\,t \end{cases}$$

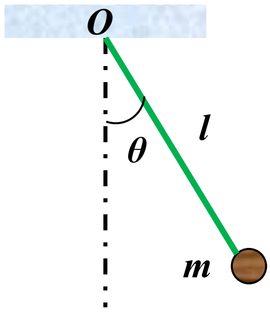
$$-\frac{1}{2}gt^2 + 5\sqrt{2}\,t - 1 = 0$$

$$\Delta = 30.36 \Rightarrow \sqrt{\Delta} = 5.51\,,\, t = 1.28s$$

$$x = 5\sqrt{2}\;t = 9.07\;m$$

EXERCISE 06

A small ball of mass 10 g is suspended by a massless thread from the roof as shown in the following figure:



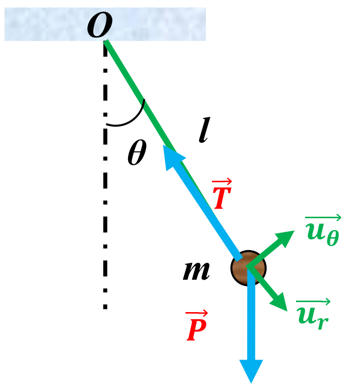
1- Using both methods, demonstrate that the differential equation for the motion of a simple pendulum is as follows:

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

a- The first method involves using polar coordinates and Newton's second law.
b- The second method employs Cartesian coordinates and the angular momentum theorem .

Solution

a- The first method involves using polar coordinates and Newton's second law.



$$\overrightarrow{OM} = l\,\overrightarrow{u_r} \quad \Rightarrow \vec{v} = \frac{d(\overrightarrow{OM})}{dt} = l\frac{d\overrightarrow{u_r}}{dt} = l\frac{d\theta}{dt}\,\overrightarrow{u_\theta} \quad \Rightarrow \vec{a} = \frac{d(\vec{v})}{dt} = l\frac{d^2\theta}{dt^2}\,\overrightarrow{u_\theta} + l\frac{d\theta}{dt}\frac{d\overrightarrow{u_\theta}}{dt}$$

$$\frac{d\overrightarrow{u_\theta}}{dt} = -\frac{d\theta}{dt}\,\overrightarrow{u_r}$$

$$\vec{a} = l\ddot{\theta}\,\overrightarrow{u_\theta} - l\dot{\theta}^2\overrightarrow{u_r}$$

$$\sum \vec{F}_{ext} = m\,\vec{a} \;\Rightarrow\; \vec{P} + \vec{T} = m\,\vec{a} \;\; where \begin{cases} \vec{P} = P\cos\theta\,\overrightarrow{u_r} - P\sin\theta\,\overrightarrow{u_\theta} \\ \vec{T} = T\,\overrightarrow{u_r} \end{cases}$$

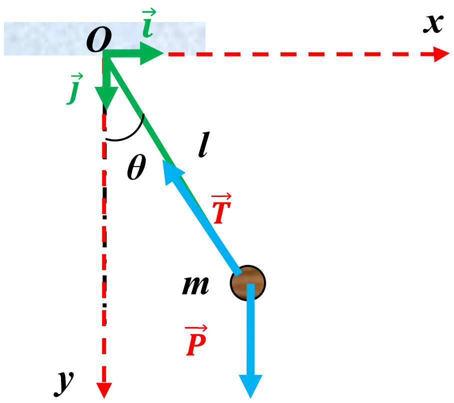
$$P\cos\theta\,\overrightarrow{u_r} - P\sin\theta\,\overrightarrow{u_\theta} + T\,\overrightarrow{u_r} = m\,l\ddot{\theta}\,\overrightarrow{u_\theta} - ml\dot{\theta}^2\overrightarrow{u_r}$$

$$\begin{cases} m\,l\ddot{\theta} = -P\sin\theta \\ -ml\dot{\theta}^2 = T + P\cos\theta \end{cases}$$

or small values of angle $\theta \Rightarrow \sin\theta \approx \theta$

$$m\,l\ddot{\theta} + P\sin\theta = 0 \;\Rightarrow\; \ddot{\theta} + \frac{g}{l}\theta = 0$$

b- The second method employs Cartesian coordinates and the angular momentum theorem



By Applying the angular momentum theorem:

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \sum_i^N \overrightarrow{\mathcal{M}_i}_{/0}(\vec{F}) = \overrightarrow{\mathcal{M}_i}_{/0}(\vec{P}) + \overrightarrow{\mathcal{M}_i}_{/0}(\vec{T})$$

the vector $\overrightarrow{OM} = x\,\vec{\imath} + y\,\vec{\jmath} = l\sin\theta\,\vec{\imath} + l\cos\theta\,\vec{\jmath}$

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\theta}{dt}\,l\cos\theta\,\vec{\imath} - \frac{d\theta}{dt}\,l\sin\theta\,\vec{\jmath}$$

the forces applied to the mass m are the weight $\vec{P}$ *and the tension* $\vec{T}$*, their expressions in the polar coordinates are given as :*

$$\begin{cases} \vec{P} = mg\,\vec{j} \\ \vec{T} = -T\sin\theta\,\vec{\imath} - T\cos\theta\,\vec{j} \end{cases}$$

By Applying the angular momentum theorem:

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \sum_i^N \overrightarrow{\mathcal{M}_i}_{/0}(\vec{F}) = \overrightarrow{\mathcal{M}_i}_{/0}(\vec{P}) + \overrightarrow{\mathcal{M}_i}_{/0}(\vec{T})$$

$$\begin{cases} \overrightarrow{\mathcal{M}_i}_{/0}(\vec{P}) = \overrightarrow{OM} \wedge \vec{P} \\ \overrightarrow{\mathcal{M}_i}_{/0}(\vec{T}) = \overrightarrow{OM} \wedge \vec{T} \end{cases}$$

$$\overrightarrow{\mathcal{M}_i}_{/0}(\vec{P}) = \overrightarrow{OM} \wedge \vec{P} = \begin{vmatrix} \vec{\imath} & \vec{j} & \vec{k} \\ l\sin\theta & l\cos\theta & 0 \\ 0 & mg & 0 \end{vmatrix}$$

$$= \begin{vmatrix} l\cos\theta & 0 \\ mg & 0 \end{vmatrix}\vec{\imath} - \begin{vmatrix} l\sin\theta & 0 \\ 0 & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} l\sin\theta & l\cos\theta \\ 0 & mg \end{vmatrix}\vec{k} = l\,mg\sin\theta\,\vec{k}$$

$$\overrightarrow{\mathcal{M}_i}_{/0}(\vec{T}) = \overrightarrow{OM} \wedge \vec{T} = \begin{vmatrix} \vec{\imath} & \vec{j} & \vec{k} \\ l\sin\theta & l\cos\theta & 0 \\ -T\sin\theta & -T\cos\theta & 0 \end{vmatrix}$$

$$= \begin{vmatrix} l\cos\theta & 0 \\ -T\cos\theta & 0 \end{vmatrix}\vec{\imath} - \begin{vmatrix} l\sin\theta & 0 \\ -T\sin\theta & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} l\sin\theta & l\cos\theta \\ -T\sin\theta & -T\cos\theta \end{vmatrix}\vec{k} = 0\,\vec{k}$$

$$\vec{\mathcal{L}}_{/0} = \overrightarrow{OM} \wedge \vec{P} = \overrightarrow{OM} \wedge \boldsymbol{m}\,\vec{v}$$

$$\vec{\mathcal{L}}_{/0} = \overrightarrow{OM} \wedge \vec{P} = \begin{vmatrix} \vec{\imath} & \vec{j} & \vec{k} \\ l\sin\theta & l\cos\theta & 0 \\ \frac{d\theta}{dt}\,ml\cos\theta & -\frac{d\theta}{dt}\,m\,l\,\sin\theta & 0 \end{vmatrix}$$

$$= \begin{vmatrix} l\cos\theta & 0 \\ -\frac{d\theta}{dt}\,m\,l\,\sin\theta & 0 \end{vmatrix}\vec{\imath} - \begin{vmatrix} l\sin\theta & 0 \\ \frac{d\theta}{dt}\,ml\cos\theta & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} l\sin\theta & l\cos\theta \\ \frac{d\theta}{dt}\,ml\cos\theta & -\frac{d\theta}{dt}\,m\,l\,\sin\theta \end{vmatrix}\vec{k}$$

$$\vec{\mathcal{L}}_{/0} = \left(-\frac{d\theta}{dt}\,m\,l^2\,\sin^2\theta + \frac{d\theta}{dt}\,ml^2\cos^2\theta\right)\vec{k} = -\frac{d\theta}{dt}\,m\,l^2\,\vec{k}$$

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \frac{d(-\frac{d\theta}{dt}\,m\,l^2\,\vec{k})}{dt} = -\,m\,l^2\frac{d^2\theta}{d^2t}\,\vec{k} = -m\,l^2\ddot{\theta}\,\vec{k}$$

$$-m\,l^2\ddot{\theta}\,\vec{k} = l\,mg\sin\theta\,\vec{k} \Rightarrow -ml^2\ddot{\theta} = l\,mg\sin\theta$$

For small values of angle $\theta \Rightarrow \sin\theta \approx \theta$

$$\ddot{\theta} + l\frac{mg}{ml^2}\theta = 0 \quad \Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$$

EXERCISE 07

The position vector of a body weighing 2 kg is given as follows.

$$\overrightarrow{OM} = (t^2 + 4t + 3)\vec{\imath} + (2t^2)\vec{\jmath} + (t + 3)\vec{k}$$

1- Find the mathematical expressions for:

- *The velocity*
- *The acceleration*
- *The linear momentum*
- *The applied force*
- *The torque of this force*
- *The angular momentum*

2- Calculate the derivative of the linear momentum with respect to time, and state your conclusions.

3- Calculate the derivative of the angular momentum with respect to time, and state your conclusions from these calculations.

Solution

♣ *The velocity*

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d((t^2 + 4t + 3)\vec{\imath} + (2t^2)\vec{\jmath} + (t + 3)\overrightarrow{k})}{dt} = (2t + 4)\vec{\imath} + 4t\,\vec{\jmath} + \vec{k}$$

♣ *The acceleration*

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d((2t + 4)\vec{\imath} + 4t\,\vec{\jmath} + \vec{k})}{dt} = 2\,\vec{\imath} + 4\,\vec{\jmath}$$

♣ *The linear momentum*

$$\vec{P} = m\ \vec{v} = 2\left((2t + 4)\vec{\imath} + 4t\,\vec{\jmath} + \vec{k}\right) = (4t + 8)\vec{\imath} + 8t\,\vec{\jmath} + 2\,\vec{k}$$

♣ *The applied force*

$$\vec{F} = m\ \vec{a} = 2(2\,\vec{\imath} + 4\,\vec{\jmath}) = 4\,\vec{\imath} + 8\,\vec{\jmath}$$

♣ *The torque of this force*

$$\overrightarrow{\mathcal{M}_i}_{/0}(\vec{F}) = \overrightarrow{OM} \wedge \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ t^2+4t+3 & 2t^2 & t+3 \\ 4 & 8 & 0 \end{vmatrix}$$

$$\overrightarrow{\mathcal{M}_i}_{/0}(\vec{F}) = \begin{vmatrix} 2t^2 & t+3 \\ 8 & 0 \end{vmatrix}\vec{\imath} - \begin{vmatrix} t^2+4t+3 & t+3 \\ 4 & 0 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} t^2+4t+3 & 2t^2 \\ 4 & 8 \end{vmatrix}\vec{k}$$

$$= (-8t - 24)\,\vec{\imath} + (-4t - 12)\vec{\jmath} + (32t + 24)\,\vec{k}$$

♣ *The angular momentum*

2. $$\vec{\mathcal{L}}_{/0} = \overrightarrow{OM} \wedge \vec{P} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ t^2+4t+3 & 2t^2 & t+3 \\ 4t+8 & 8t & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2t^2 & t+3 \\ 8t & 2 \end{vmatrix}\vec{\imath} - \begin{vmatrix} t^2+4t+3 & t+3 \\ 4t+8 & 2 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} t^2+4t+3 & 2t^2 \\ 4t+8 & 8t \end{vmatrix}\vec{k}$$

$$= (-4t^2 - 24\,t)\,\vec{\imath} + (-2t^2 - 12t - 18)\vec{\jmath} + (16t^2 + 24t)\,\vec{k}$$

2- Calculate the derivative of the linear momentum with respect to time, and state your conclusions.

$$\frac{d\vec{P}}{dt} = \frac{d((4t+8)\vec{\imath} + 8t\,\vec{\jmath} + 2\,\vec{k})}{dt} = 4\,\vec{\imath} + 8\,\vec{\jmath}$$

we conclude that:

$$\frac{d\vec{P}}{dt} = m\,\vec{a} = \vec{F}$$

3- Calculate the derivative of the angular momentum with respect to time, and state your conclusions from these calculations.

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \frac{d((-4t^2 - 24\,t)\,\vec{\imath} + (-2t^2 - 12t - 18)\vec{\jmath} + (16t^2 + 24t)\,\vec{k}\,)}{dt}$$

$$= \;(-8t - 24\,)\,\vec{\imath} + (-4t - 12)\vec{\jmath} + (32t + 24)\,\vec{k}$$

we conclude that:

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \overrightarrow{\boldsymbol{\mathcal{M}}_{i}}_{/0}(\vec{F})$$

Chapter 04: Work and Energy

1) Introduction

Through this chapter, we aim to advance from studying the motion a material point (kinematics) and the forces causing this motion (dynamics), to exploring energy changes and the work done by applying force. when a force affects an object, it performs work that results in the transfer or conversion of energy. thus, any applied force on a system has a visible effect 'the work done by the force) and hidden effect , which involves the transfer or conversion of energy.

2) Work of a force

Any action of a force that results in the displacement of an object is known as work of force. Work of force is a means of transferring or transforming energy from one form to another.

Work is a scalar quantity, and is symbolized by the symbol W.

The physical dimension of work is: $[W] = M\,L^2\,T^{-2}$, *where:*

M *represents mass.*

L *represents distance or displacement.*

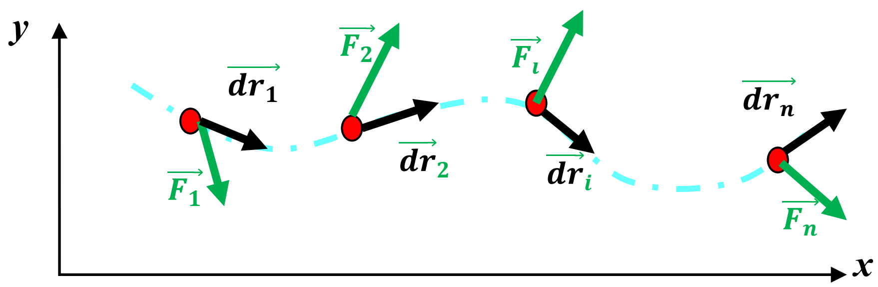
T *represents time.*

The unit of work in the International System of Units (SI) is the joule (J), and the joule can be expressed as follows:

$$1\,joule = 1\,kg \cdot m^2/s^2$$

2-1) Work Done by a Varying Force

consider object moving along a curved path and affected by a force may vary in magnitude or in direction or in both as shown the figure:



We divide the path of the body's motion into elementary displacements $\overrightarrow{\boldsymbol{dr}}$ *(* $\overrightarrow{\boldsymbol{dr}}$ *tangent to the path) where for each displacement, the force can be approximated to be constant in both magnitude and direction, and the elementary work* dW *done by the force* $\overrightarrow{\boldsymbol{F_i}}$ *is written as follows:*

$$dW_i(\overrightarrow{\boldsymbol{F_i}}) = \overrightarrow{\boldsymbol{F_i}}\,.\,\overrightarrow{\boldsymbol{dr_i}}$$

The total work done as the object moves from A to B is the sum of all the elementary works done along each elementary displacement:

$$W_{A\to B}\left(\vec{F}\right) = \sum_{i=1}^{n} W_i = \int_A^B dW_1 + \cdots + \int_A^B dW_i + \cdots + \int_A^B dW_n$$

$$= \int_{\boldsymbol{A}}^{\boldsymbol{B}} \overrightarrow{\boldsymbol{F_1}}\,.\,\overrightarrow{\boldsymbol{dr_1}} + \ldots + \int_{\boldsymbol{A}}^{\boldsymbol{B}} \overrightarrow{\boldsymbol{F_i}}\,.\,\overrightarrow{\boldsymbol{dr_i}} + \cdots + \int_{\boldsymbol{A}}^{\boldsymbol{B}} \overrightarrow{\boldsymbol{F_n}}\,.\,\overrightarrow{\boldsymbol{dr_n}}$$

By dividing the path into a large number "n" of elemental displacements, therefore to calculate the total work done by the force along the path, we calculate the integral dW_i

$$W_{A\to B}\left(\vec{F}\right) = \sum_{i=1}^{n} W_i = \int_A^B \sum_{i=1}^{n} dW_i = \int_{\boldsymbol{A}}^{\boldsymbol{B}} \sum_{i=1}^{n} \overrightarrow{\boldsymbol{F_i}}\,.\,\overrightarrow{\boldsymbol{dr}} \qquad (\boldsymbol{i = 1, \ldots \ldots, n})$$

To accomplish this integration, it is sufficient to write the components of both the force and displacement vectors, and this is according to the coordinate system adopted in the study. For example:

- ***Cartesian coordinates***

In the ***Cartesian coordinates*** *elementary displacement vector* $\overrightarrow{\boldsymbol{dr}}$ *and the force vector* $\vec{\boldsymbol{F}}$ *are written as :*

$$\begin{cases} \overrightarrow{dr} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k} \\ \vec{F} = F_x\,\vec{\imath} + F_y\,\vec{\jmath} + F_z\,\vec{k} \end{cases}$$

$$\Rightarrow W_{A\to B}\left(\vec{F}\right) = \int_{x_A}^{x_B} F_x\,dx + \int_{y_A}^{y_B} F_y\,dy + \int_{z_A}^{z_B} F_z\,dz$$

- ***Cylindrical coordinate***

In the ***Cylindrical coordinates*** *elementary displacement vector* $\overrightarrow{\boldsymbol{dr}}$ *and the force vector* $\vec{\boldsymbol{F}}$ *are written as :*

$$\begin{cases} \overrightarrow{dr} = dr\,\overrightarrow{u_r} + r\,d\theta\,\overrightarrow{u_\theta} + dz\,\vec{k} \\ \vec{F} = F_r\,\overrightarrow{u_r} + F_\theta\,\overrightarrow{u_\theta} + F_z\,\vec{k} \end{cases}$$

$$\Rightarrow W_{A\to B}(\vec{F}) = \int_{r_A}^{r_B} F_r\,dr + \int_{\theta_A}^{\theta_B} rF_\theta\,d\theta + \int_{z_A}^{z_B} F_z\,dz$$

❖ ***Spherical coordinates***

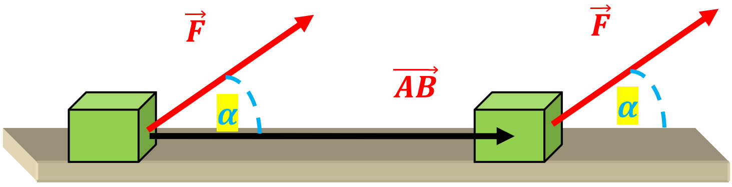
*In the **Spherical coordinates** elementary displacement vector $\overrightarrow{\mathbf{dr}}$ and the force vector $\vec{\mathbf{F}}$ are written as :*

$$\begin{cases} \overrightarrow{dr} = dr\,\overrightarrow{u_r} + r\,d\theta\,\overrightarrow{u_\theta} + r\sin\theta\,d\varphi\,\overrightarrow{u_\varphi} \\ \vec{F} = F_r\,\overrightarrow{u_r} + F_\theta\,\overrightarrow{u_\theta} + F_\varphi\,\overrightarrow{u_\varphi} \end{cases}$$

$$\Rightarrow W_{A\to B}(\vec{F}) = \int_{r_A}^{r_B} F_r\,dr + \int_{\theta_A}^{\theta_B} rF_\theta\,d\theta + \int_{\varphi_A}^{\varphi_B} r\sin\theta\,F_\varphi\,d\varphi$$

2-2) Work Done by a Constant Force

Consider an abject moved along a straight line and affected by a constant force $\vec{F}$ (in both magnitude and direction that causes it to move from position A to position B, where $\overrightarrow{AB}$ represents the vector of its displacement, as shown in the following figure:



Work done by a constant force can be defined as the scalar product between the force $\vec{F}$ and its displacement $\overrightarrow{AB}$. the mathematical expression of the force work is given as:

$$W_{A\to B}(\vec{F}) = \vec{F}\cdot\overrightarrow{AB} = \|\vec{F}\|\cdot\|\overrightarrow{AB}\|\cos\alpha = \begin{cases} \|\vec{F}\|\cdot\|\overrightarrow{AB}\| & if\ \alpha = 0 \\ 0 & if\ \alpha = \frac{\pi}{2} \\ -\|\vec{F}\|\cdot\|\overrightarrow{AB}\| & if\ \alpha = \pi \end{cases}$$

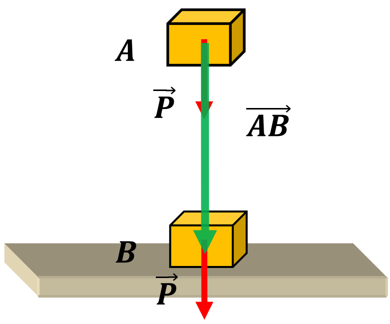
where α is the angle between $\vec{F}$ and $\overrightarrow{AB}$

***A key point:* If the value of the work done by a force is positive, this means that the work is *driving* and the force is in the same direction as the motion (. If the work is negative, this indicates that the work is *resistive*, and the force is in the opposite direction to the motion.**

3) Examples of some force works

3-1) Work Done by a Weight

the work of the weight force *$\vec{P}$ of a block falls from point A to point B as shown in the figure.*

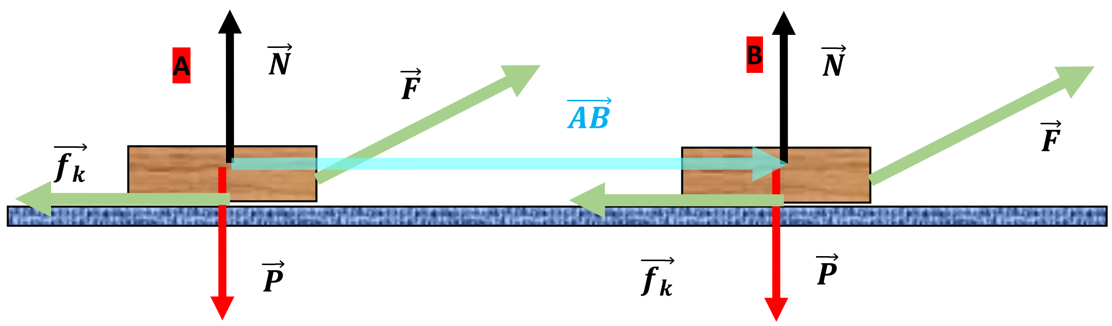


$$W_{A\to B}\left(\vec{P}\right) = \vec{P}\,.\overrightarrow{AB} = m\,g\,.\left\|\overrightarrow{AB}\right\|\,cos\,0$$

$$W_{A\to B}\left(\vec{P}\right) = \vec{P}\,.\overrightarrow{AB} = m\,g\,.\left\|\overrightarrow{AB}\right\|$$

3-2) Work Done by Friction

A block moves on a rough surface where it is subjected during its movement to the weight force $\vec{\boldsymbol{P}}$, the normal force $\vec{\boldsymbol{N}}$, the force of tension $\vec{\boldsymbol{T}}$, and the force of kinetic friction $\vec{\boldsymbol{f}}_{\boldsymbol{k}}$ as shown in the figure.

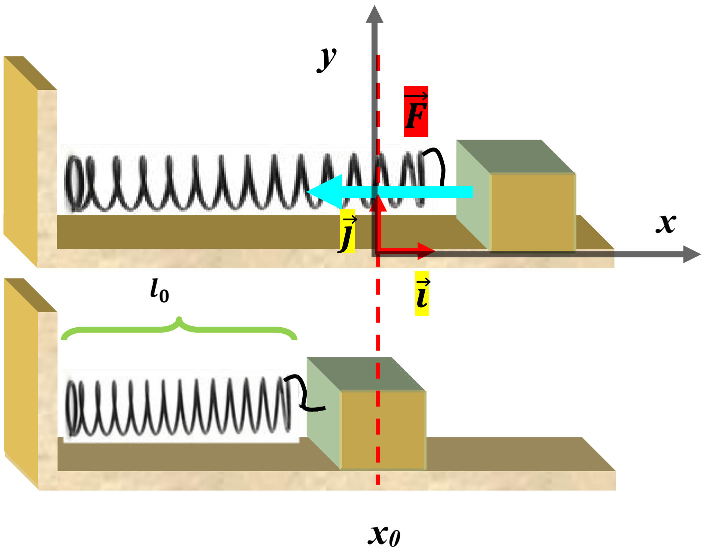


The work of the kinetic friction force during the movement of the mass from position A to position B is given by the relation:

$$W_{A\to B}\left(\vec{\boldsymbol{f}}_{\boldsymbol{k}}\right) = \vec{\boldsymbol{f}}_{\boldsymbol{k}}\,.\overrightarrow{AB} = \left\|\vec{\boldsymbol{f}}_{\boldsymbol{k}}\right\|.\left\|\overrightarrow{AB}\right\|\,cos\,\alpha = -\left\|\vec{\boldsymbol{f}}_{\boldsymbol{k}}\right\|.\left\|\overrightarrow{AB}\right\|$$

2-3) Work Done by a Spring Force

The following figure represents a block connected to a light spring of constant K fixed at the other end to a horizontal, frictionless surface.



The expression for the spring force, which is a non-constant force in both magnitude and direction, is written as follows:

$$\vec{F} = -\,k\,x\,\vec{i}$$

*where **x** represents the displacement from the equilibrium position, Thus, the expression for the work of the spring force when a displacement occurs from position **x** to the equilibrium position $x_0 = 0$ is as follows is as follows:*

$$\begin{cases} \overrightarrow{dr} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k} \\ \vec{F} = F_x\,\vec{i} = -\,k\,x\,\vec{i} \end{cases}$$

$$W_{x\to x_0}\left(\vec{F}\right) = \int_x^{x_0} F_x\,dx = \int_x^{x_0} -\,k\,x\,dx = -\frac{1}{2}\,k\,x^2\Big|_x^{x_0} = -\frac{1}{2}\,k\,x^2$$

3) Conservative and Nonconservative Forces

*We say that a force is **conservative** if the work it does when a body moves from one position to another is independent of the path followed but only on the initial and final positions.*

*We say that a force is **non-conservative** if the work it does from one place to another changes according to the path followed.*

*Conservative forces are also distinguished by the fact that they are derived from potential **U**, unlike non-conservative forces. We write:*

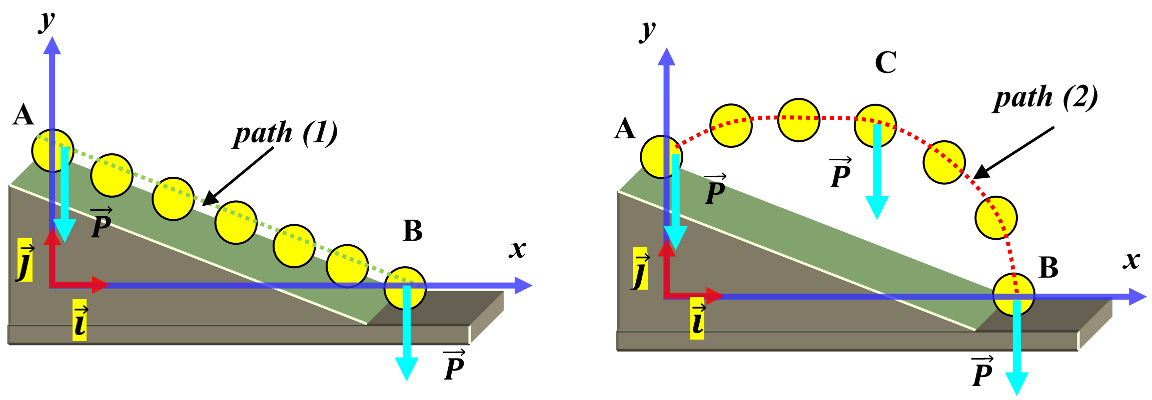
$$\text{conservative force} \Rightarrow \vec{F} = -\overrightarrow{grad}\,U$$

the rotational of conservative force equal to zero $\overrightarrow{rot}\,\vec{F} = \vec{\nabla} \wedge \vec{F} = curl\,\vec{F} = \vec{0}$

where the formula of $\vec{\nabla}$ is given in the Cartesian, Cylindrical, and spherical coordinates systems as follow:

$$\begin{cases} Cartesian: \quad \vec{\nabla} = \dfrac{\partial}{\partial x}\vec{i} + \dfrac{\partial}{\partial y}\vec{j} + \dfrac{\partial}{\partial z}\vec{k} \\ Cylindrical: \quad \vec{\nabla} = \dfrac{\partial}{\partial r}\overrightarrow{u_r} + \dfrac{1}{r}\dfrac{\partial}{\partial \theta}\overrightarrow{u_\theta} + \dfrac{\partial}{\partial z}\vec{k} \\ Spherical: \quad \vec{\nabla} = \dfrac{\partial}{\partial r}\overrightarrow{u_r} + \dfrac{1}{r}\dfrac{\partial}{\partial \theta}\overrightarrow{u_\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \varphi}\overrightarrow{u_\varphi} \end{cases}$$

As an example of a conservative force, consider the following example, which involves the motion of a block from position A to position B via two different paths.



a) Calculating the work done by the weight force according to path (1)

$$\text{the postion } A \begin{cases} x_A = 0 \\ y_A \end{cases} \quad \text{the postion } B \begin{cases} x_B \\ y_B = 0 \end{cases} \quad \text{the weight force vector } \vec{P} = -mg\,\vec{j}$$

the displacement vector $\overrightarrow{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} = (x_B)\vec{i} + (-y_A)\vec{j}$

$$W_{A\to B}\left(\vec{P}\right) = \vec{P}\,.\overrightarrow{AB} = mgy_A$$

b) Calculating the work done by the weight force according to path (2)

$$\text{the postion } A \begin{cases} x_A = 0 \\ y_A \end{cases} \quad \text{the postion } C \begin{cases} x_C \\ y_C \end{cases} \quad \text{the postion } B \begin{cases} x_B \\ y_B = 0 \end{cases}$$

$$\text{the displacement vector } \overrightarrow{AC} = (x_C - x_A)\vec{\imath} + (y_C - y_A)\vec{\jmath} = (x_C)\vec{\imath} + (y_C - y_A)\vec{\jmath}$$

$$\text{the displacement vector } \overrightarrow{CB} = (x_B - x_C)\vec{\imath} + (y_B - y_C)\vec{\jmath} = (x_B - x_C)\vec{\imath} + (-y_C)\vec{\jmath}$$

$$\text{the weight force vector } \vec{P} = -mg\,\vec{\jmath}$$

$$W_{A\to C}\left(\vec{P}\right) = \vec{P}\,.\overrightarrow{AC} = (-mg\,\vec{\jmath})\,.\left((x_C)\vec{\imath} + (y_C - y_A)\vec{\jmath}\right) = -mgy_C + mg\,y_A$$

$$W_{C\to B}\left(\vec{P}\right) = \vec{P}\,.\overrightarrow{CB} = (-mg\,\vec{\jmath})\,.\left((x_B - x_C)\vec{\imath} + (-y_C)\vec{\jmath}\right) = +mgy_C$$

$$W_{A\to C}\left(\vec{P}\right) + W_{C\to B}\left(\vec{P}\right) = -mgy_C + mg\,y_A + mgy_C = mg\,y_A = W_{A\to B}\left(\vec{P}\right)$$

4) Power of a force

The Power is defined the time rate of doing work, it expresses by the derivative of work with respect to time, and is written as follows:

$$P = \frac{dW}{dt} = \frac{\vec{F}\,.\overrightarrow{dr}}{dt} = \vec{F}\,.\vec{v}$$

The SI unit of power is joules per second (J/s) and is called the watt (W).

1 W = 1 J/s = 1 kg.m^2/s^3

5) Work-Energy Theorem

The work done by this net force on a moving block from an initial position $\vec{r_i}$ to a final position $\vec{r_f}$ can be calculated using the following steps :

- Using the fundamental principle of motion:

$$\vec{F} = \frac{d\vec{P}}{dt} \Rightarrow d\vec{P} = \vec{F}\,dt$$

The effect of the force $\vec{F}$ on the block results in a change in the linear momentum $\vec{P}$ by $d\vec{P}$.

$$\vec{P} = m\,\vec{v}$$

$$d\vec{P} = m\,d\vec{v} + \vec{v}\,dm$$

If the mass is constant, the velocity of the moving block changes:

$$d\vec{P} = m\, d\vec{v}$$

$$\begin{cases} d\vec{P} = \vec{F}\, dt \\ d\vec{P} = m\, d\vec{v} \end{cases} \Rightarrow \vec{F}\, dt = m\, d\vec{v}$$

We multiply both sides of this equation by $\vec{v}$, we find:

$$\vec{F}.\vec{v}\, dt = m\vec{v}\,.d\vec{v}$$

given that $\vec{v} = \frac{d\vec{r}}{dt}$

$$\vec{F}.\frac{d\vec{r}}{dt}\, dt = m\vec{v}\,.d\vec{v}$$

$$\vec{v}\,.d\vec{v} = \|\vec{v}\|\,.\|d\vec{v}\|\cos\theta$$

where θ confined between $\vec{v}$ *and* $d\vec{v}$ and, $\|\vec{v}\| = v$ *and* $\|d\vec{v}\| = dv$

$$\vec{F}\,.d\vec{r} = mv\,.dv \quad \Rightarrow \int_{\vec{r_i}}^{\vec{r_f}} \vec{F}\,.d\vec{r} = \int_{v_i}^{v_f} mv\,.dv$$

$$W_{\vec{r_i}\to\vec{r_f}}\left(\vec{F}\right) = \frac{1}{2}m\,v^2\Big|_{v_i}^{v_f}$$

The quantity $\frac{1}{2}m\,v^2$ *expresses the kinetic energy of an object of mass* **m** *moving with a velocity of* v*. According to the expression obtained* ($W_{\vec{r_i}\to\vec{r_f}}\left(\vec{F}_{external}\right) = \Delta E_K$) *the change in kinetic energy between two moments is equal to the net work done by the* ***external*** *forces acting on the body.*

This statement relates to the ***work-energy theorem****, which states that the net work done on an object is equal to the change in its kinetic energy.*

5-1) Translational Kinetic energy

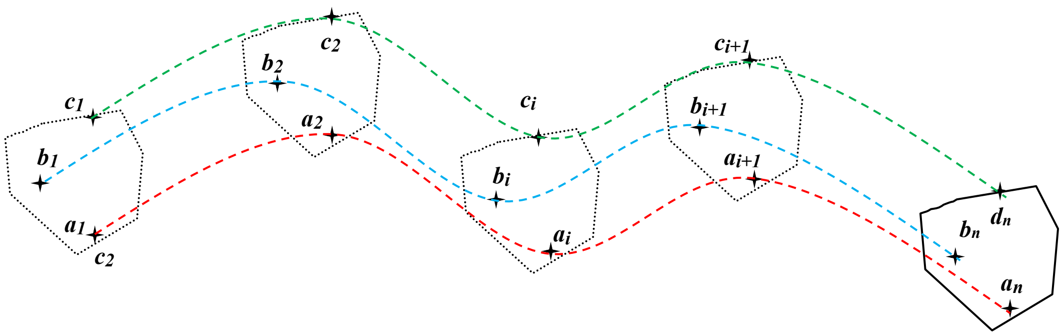
A block is said to be in translational motion if all points of the body, during their motion:

1. *Move along identical paths*

2. *Travel the same distance in the same period of time.*

The following figure *represents the paths of three points on a block moving in translational motion, where these points follow identical paths. Also, all points travel the same distance in the same amount of time.*

$$\text{within a period of time } \Delta t = t_{i+1} - t_1 \Rightarrow \|\overrightarrow{a_i a_{i+1}}\| = \|\overrightarrow{b_i b_{i+1}}\| = \|\overrightarrow{c_i c_{i+1}}\|$$



The expression for the kinetic energy of a body moving in a translational motion is written as follows:

$$E_K = \sum_{i=1}^{n} \frac{1}{2} m_i \, {v_i}^2$$

$$\text{where } v_1 = v_2 = \cdots . v_i \ldots = v \quad \text{, and} \quad m = \sum_{i=1}^{N} m_i$$

$$E_K = \frac{1}{2} m \, v^2$$

5-2) Rotational Kinetic Energy

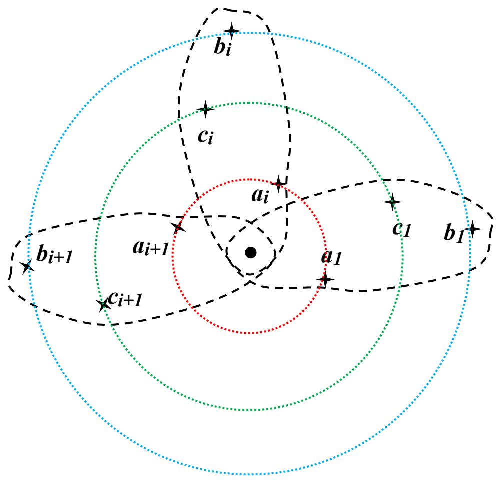
A block is said to be in rotational motion if all points of the block during their motion:

1. Move along circular paths

2. All circular paths have the same center of rotation.

3. The distance traveled increases over the same time interval as one moves away from the axis of rotation.

The following figure represents the paths of three points on a block moving in rotational motion, where all points trace circular paths with the same center, and all points cover different distances that increase as we move farther from the center of rotation.



$$\text{within a period of time } \Delta t = t_{i+1} - t_1 \Rightarrow \|\overrightarrow{a_i a_{i+1}}\| < \|\overrightarrow{b_i b_{i+1}}\| < \|\overrightarrow{c_i c_{i+1}}\|$$

The expression for the kinetic energy of a body moving in a translational motion is written as follows:

$$E_K = \sum_{i=1}^{n} \frac{1}{2} m_i \, {v_i}^2$$

$\text{where } v_i = r_i \, \dot{\theta} \quad (\dot{\theta} \text{ is the angulat velocity})$

$$E_K = \sum_{i=1}^{n} \frac{1}{2} m_i \, {r_i}^2 \, \dot{\theta}^2 = \frac{1}{2} \, \dot{\theta}^2 \sum_{i=1}^{n} m_i \, {r_i}^2 = \frac{1}{2} \, \dot{\theta}^2 I \text{ where } I = \sum_{i=1}^{n} m_i \, {r_i}^2 \text{ is moment of inertia}$$

6) Potential energy

A body is said to have potential energy if its state or position enables it to perform work (or movement) under the influence of conservative forces, such as gravity or elastic force, without the need for additional forces to generate this work. For example: the state of an elevated body allows it to move downward under the influence of its conservative gravitational force. Similarly, the state of a compressed spring allows it to move and return to its natural state under the influence of elastic force.

The work of the conservative forces can be expressed from a state function called potential energy. Thus, the change in potential energy is written as follows:

$$\Delta E_P = E_{P_f} - E_{P_i} = -W_{i\to f}\left(\vec{F}_{conservative}\right)$$

the negative sign arises because the work done is opposite to the change in potential energy. That is, if the object moves from a state of low potential energy to a state of high potential energy, the work must be negative.

For an elementary change in potential energy, we write:

$$dE_P = -\,dW(\vec{F}) = -\vec{F}\,.\overrightarrow{dr} \;\Rightarrow E_P = -\int \vec{F}\,.\overrightarrow{dr}$$

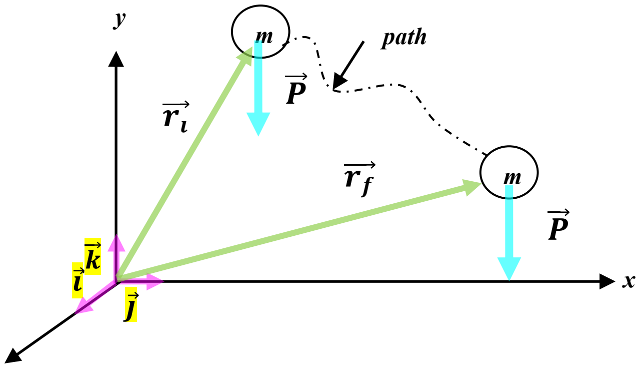
$$\vec{F} = -\overrightarrow{grad}E_P = -\vec{\nabla}E_P$$

$\vec{\nabla}$ *is given in the Cartesian, Cylindrical, and spherical coordinates systems as follow:*

$$\begin{cases} Cartesian: \quad \vec{\nabla} = \dfrac{\partial}{\partial x}\vec{\imath} + \dfrac{\partial}{\partial y}\vec{\jmath} + \dfrac{\partial}{\partial z}\vec{k} \\ Cylindrical: \quad \vec{\nabla} = \dfrac{\partial}{\partial r}\overrightarrow{u_r} + \dfrac{1}{r}\dfrac{\partial}{\partial \theta}\overrightarrow{u_\theta} + \dfrac{\partial}{\partial z}\vec{k} \\ Spherical: \quad \vec{\nabla} = \dfrac{\partial}{\partial r}\overrightarrow{u_r} + \dfrac{1}{r}\dfrac{\partial}{\partial \theta}\overrightarrow{u_\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \varphi}\overrightarrow{u_\varphi} \end{cases}$$

6-1) Gravitational Potential Energy

To calculate the gravitational potential energy of an object, we write:



$$\vec{F} = -\vec{\nabla}E_P = -\left(\frac{dE_P}{dx}\vec{\imath} + \frac{dE_P}{dy}\vec{\jmath} + \frac{dE_P}{dz}\vec{k}\right) \text{ and } \vec{P} = -mg\,\vec{k}$$

$$-\left(\frac{dE_P}{dx}\vec{\imath} + \frac{dE_P}{dy}\vec{\jmath} + \frac{dE_P}{dz}\vec{k}\right) = -mg\,\vec{k}$$

$$\frac{dE_P}{dz} = mg \Rightarrow dE_P = mg\,dz \Rightarrow E_P = \int mg\,dz = mgz + Constant$$

or

$$W_{\vec{r_i}\to\vec{r_f}}(\vec{P}) = \int_{\vec{r_i}}^{\vec{r_f}} dW_i = \int_{\vec{r_i}}^{\vec{r_f}} \vec{P}\,.\,\vec{dr} \quad \text{where} \begin{cases} \vec{dr} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k} \\ \vec{P} = -mg\,\vec{k} \end{cases}$$

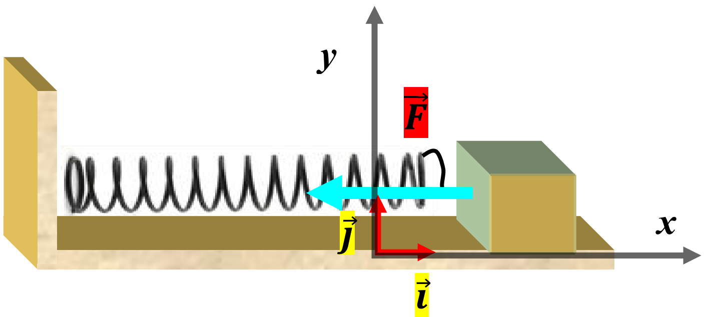
$$\vec{r_f} = x_i\,\vec{i} + y_i\,\vec{j} + z_i\,\vec{k} \quad \text{and} \quad \vec{r_f} = x_f\,\vec{i} + y_f\,\vec{j} + z_f\,\vec{k}$$

$$\Rightarrow W_{\vec{r_i}\to\vec{r_f}}(\vec{P}) = \int_{z_i}^{z_f} -mg\,dz = -mg(z_f - z_i)$$

$$\Delta E_P = E_{P_f} - E_{P_i} = -W_{\vec{r_i}\to\vec{r_f}}(\vec{P}) = mg(z_f - z_i)$$

6-2) The Elastic Potential Energy

To calculate the spring potential energy of an object, we write:



$$\vec{F} = -\vec{\nabla}E_P = -\left(\frac{dE_P}{dx}\vec{i} + \frac{dE_P}{dy}\vec{j} + \frac{dE_P}{dz}\vec{k}\right) \text{ and } \vec{F} = -kx\,\vec{i}$$

$$-\left(\frac{dE_P}{dx}\vec{i} + \frac{dE_P}{dy}\vec{j} + \frac{dE_P}{dz}\vec{k}\right) = -kx\,\vec{i}$$

$$\frac{dE_P}{dx} = k\,x \Rightarrow dE_P = k\,x\,dx \Rightarrow E_P = \int k\,x\,dx = \frac{1}{2}\,k\,x^2 + Constant$$

7) Mechanical energy

The total mechanical energy of an object $\boldsymbol{E_T}$ *is defined as the sum of all of the kinetic energies* $\boldsymbol{E_K}$ *of the objects within the system plus all of the potential energies* $\boldsymbol{E_P}$*.*

$$\boldsymbol{E_T = E_K + E_P}$$

According to the kinetic energy theory, which states that the change in kinetic energy between two initial and final positions is the result of all the work of the external conservative and non-conservative forces acting on the system.

$$W_{\vec{r_i}\to\vec{r_f}}\left(\vec{F}_{external}\right) = \Delta E_K = W_{\vec{r_i}\to\vec{r_f}}\left(\vec{F}_{CON}\right) + W_{\vec{r_i}\to\vec{r_f}}\left(\vec{F}_{NCON}\right)$$

$$E_K\left(\vec{r_f}\right) - E_K\left(\vec{r_i}\right) = W_{\vec{r_i}\to\vec{r_f}}\left(\vec{F}_{CON}\right) + W_{\vec{r_i}\to\vec{r_f}}\left(\vec{F}_{NCON}\right)$$

$$W_{\vec{r_i}\to\vec{r_f}}\left(\vec{F}_{CON}\right) = -\Delta E_P = -\left(E_P\left(\vec{r_f}\right) - E_P\left(\vec{r_i}\right)\right)$$

$$E_K\left(\vec{r_f}\right) - E_K\left(\vec{r_i}\right) + \left(E_P\left(\vec{r_f}\right) - E_P\left(\vec{r_i}\right)\right) = W_{\vec{r_i}\to\vec{r_f}}\left(\vec{F}_{NCON}\right)$$

$$\left(E_K\left(\vec{r_f}\right) + E_P\left(\vec{r_f}\right)\right) - \left(E_K\left(\vec{r_i}\right) + E_P\left(\vec{r_i}\right)\right) = W_{\vec{r_i}\to\vec{r_f}}\left(\vec{F}_{NCON}\right)$$

According to the definition of mechanical energy, we can write

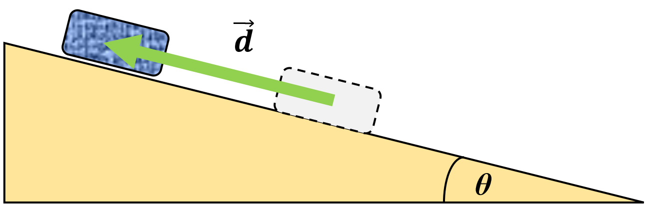
$$\begin{cases} E_K(\vec{r_i}) + E_P(\vec{r_i}) = E_T(\vec{r_i}) \\ E_K\left(\vec{r_f}\right) + E_P\left(\vec{r_f}\right) = E_T\left(\vec{r_f}\right) \end{cases} \Rightarrow \Delta E_T = E_T\left(\vec{r_f}\right) - E_T(\vec{r_i}) = W_{\vec{r_i}\to\vec{r_f}}\left(\vec{F}_{NCON}\right)$$

Therefore, the change in the mechanical energy of the system is equal to the total work of the non-conservative forces.

8) solved exercises

EXERCISE 01

A block of mass m=0.2 kg is pushed up a rough inclined plane with an angle $\theta = \frac{\pi}{12}\, rad$ *using a constant force* $\|\vec{F}\| = 10\, N$*, which is parallel to the slope, as shown in the figure below. The mass reaches the top of the incline after traveling a distance* ***d*** *of 3 meters.*



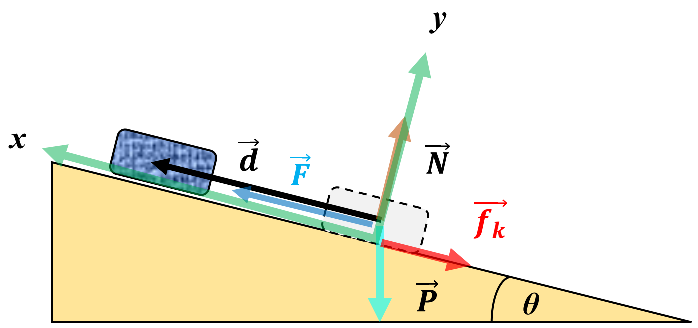
a) Represent the forces acting on the mass and write the analytical expressions for each of them.

b) Calculate the work done by each force, where g=9.8m/s^2 μ_k =0.2

c) Determine the initial velocity at which the mass was pushed.

SOLUTION

a) Representing forces acting on the block: $\vec{F}$*: the driving force ;* $\vec{P}$*: the weight force ;* $\vec{N}$*: the normal force;* $\vec{f}_k$*: the kinetic friction force*



- ✓ $\vec{F}$: *the driving force* $\vec{F} = F\,\vec{\imath} = 10\,\vec{\imath}$
- ✓ $\vec{P}$: *the weight force* $\vec{P} = -m\,g\sin\theta\;\vec{\imath} - m\,g\cos\theta\;\vec{\jmath}$
- ✓ $\vec{N}$: *the normal force* $\vec{N} = N\;\vec{\jmath}$

$$\sum \overrightarrow{F_{ext}} = m\,\vec{a} \quad \Rightarrow \begin{cases} on\ (ox) \Rightarrow -\mu_k\,N\;\vec{\imath} - m\,g\sin\theta\;\vec{\imath} + F\,\vec{\imath} = m\,a\,\vec{\imath} \\ on\ (oy) \Rightarrow -m\,g\cos\theta\;\vec{\jmath} + N\;\vec{\jmath} = 0\;\vec{\jmath} \Rightarrow m\,g\cos\theta\; = N \end{cases}$$

$$\vec{N} = N\;\vec{\jmath} = m\,g\cos\theta\;\;\vec{\jmath}$$

- ✓ $\vec{f_k}$: *the kinetic friction force* $\vec{f_k} = -\,\mu_k\,m\,g\cos\theta\;\;\vec{\imath}$

b) Calculate the work done by each force.

the displacement vector $\vec{d} = d\,\vec{\imath} = 3\vec{\imath}$

- ✓ $\vec{F}$: *the driving force*

$$W\left(\vec{F}\right) = \vec{F}\,.\vec{d} = F\,\vec{\imath}\,.d\,\vec{\imath} = Fd = 30\,J$$

- ✓ $\vec{P}$: *the weight force*

$$W\left(\vec{P}\right) = \vec{P}\,.\vec{d} = (-\,m\,g\sin\theta\;\;\vec{\imath} - m\,g\cos\theta\;\;\vec{\jmath}).\,d\,\vec{\imath} = -m\,g\sin\theta\;\,d = -1.52\,J$$

- ✓ $\vec{N}$: *the normal force*

$$W\left(\vec{N}\right) = \vec{N}\,.\vec{d} = m\,g\cos\theta\ \ \vec{j}\,.d\ \vec{i} = 0$$

$\vec{f_k}$*: the kinetic friction force* $\vec{f_k} = -\,\mu_k\,m\,g\cos\theta\ \ \vec{i}$

$$W\left(\vec{f_k}\right) = \vec{f_k}\,.\vec{d} = -\,\mu_k\,m\,g\cos\theta\ \ \vec{i}\,.d\,\vec{i} = -\,\mu_k\,m\,g\cos\theta\ \ d = -1.14\,J$$

c) Determining the initial velocity at which the mass was pushed.

$$W_{\vec{r_i}\to\vec{r_f}}\left(\vec{F}_{external}\right) = \Delta E_K = E_K\left(\vec{r_f}\right) - E_K\left(\vec{r_i}\right)$$

$$E_K\left(\vec{r_f}\right) = 0 \ and\ \ E_K\left(\vec{r_i}\right) = -\frac{1}{2}\,m\,v^2{}_i = W_{\vec{r_i}\to\vec{r_f}}\left(\vec{F}_{external}\right)$$

$$\frac{1}{2}\,m\,v^2{}_i = W_{\vec{r_i}\to\vec{r_f}}\left(\vec{F}_{external}\right) = W\left(\vec{P}\right) + W\left(\vec{F}\right) + W\left(\vec{N}\right) + W\left(\vec{f}_k\right)$$

$$\frac{1}{2}\,m\,v^2{}_i = -1.14 - 1.52 + 30 = 27.34\,J$$

$$v_i = \sqrt{2 \times \frac{27.34}{0.2}} = 16.53\ m/s$$

EXERCISE 02

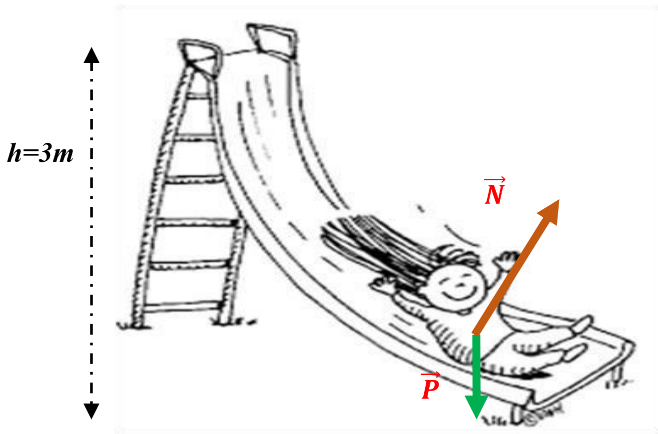
A child weighing 25 kg slides on a curved track without friction and without initial velocity as shown in the figure.

1- Represent the forces acting on the child while he is sliding.

A- Calculate the final velocity at which he reaches the ground..

SOLUTION

A- Representing the forces acting on the child while he is sliding.



B- Calculating the final velocity at which he reaches the ground.

$$W_{\vec{r_i} \to \vec{r_f}}\left(\vec{F}_{external}\right) = \Delta E_K = E_K(\vec{r_f}) - E_K(\vec{r_i})$$

$$E_K(\vec{r_f}) = \frac{1}{2} m\, v^2{}_f \text{ and } E_K(\vec{r_i}) = 0 \Rightarrow \frac{1}{2} m\, v^2{}_f = W_{\vec{r_i} \to \vec{r_f}}\left(\vec{F}_{external}\right)$$

$$\frac{1}{2} m\, v^2{}_f = W_{\vec{r_i} \to \vec{r_f}}\left(\vec{F}_{external}\right) = W(\vec{P}) + W(\vec{N})$$

$$\frac{1}{2} m\, v^2{}_f = m\, g\, h$$

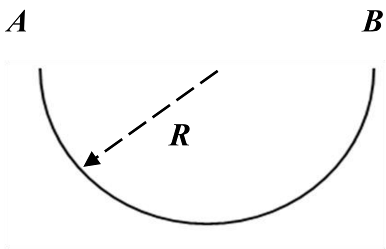
$$v_f = \sqrt{2\, g\, h} = \mathbf{7.67}\ \text{m/s}$$

EXERCISE 03

A small object with a mass of m=10 g is released from point A without any initial velocity. The guide is a fixed hemispherical cylinder with a radius R, lying horizontally in an inertial reference frame (Earth frame).

When the object reaches point B, the lowest point of the guide, for the first time, its velocity is v_B=4m/s. Given: R=0.4 and g=10m/s^2.

1. *Calculate the work done by the friction force.*
2. *Determine the constant value of this friction force*



EXERCISE 04

A particle moves under the effect of the force $\vec{F} = (x^2+y^2)\vec{\imath} + (xy)\vec{\jmath}$ + starting from the the point A to point C. Calculate the work done by this force for the following two paths:

Path A→B→C

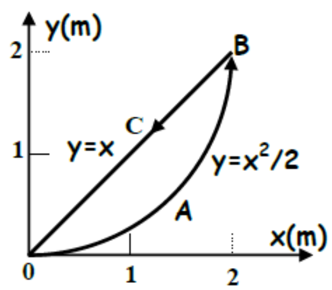
- *A(0,0)   B (1, 0)   C(1,1)*
- *A(0,0)   B (1/2, 1/2)   C(1,1)*

EXERCISE 05

A body of mass m, subjected to a force $\overrightarrow{F_n}$, moves along a closed path OABCO consisting of a parabolic arc and a straight line segment, following the direction indicated by the arrow. Compute the work done by $\overrightarrow{F_n}$, in the following cases:

a) $\vec{F}_1 = -y\vec{\imath} + x\vec{\jmath}$ *b)* $\vec{F}_2 = x\vec{\imath} + y\vec{\jmath}$

1. *What conclusions can be drawn in each case?*



References

1. *Radi, H. A., & Rasmussen, J. O. (2013). Principles of physics. Springer, Berlin Heidelberg, New Delhi,.*
2. *Alrasheed, Salma. Principles of mechanics: Fundamental university physics. Springer Nature, 2019.*
3. *P. SYLVIE, B.YVES. MÉCANIQUE GÉNÉRALE Cours et exercices corrigés.*
4. *H. MICHEL et D. NICOLAS, MINI MANUEL DE MECANIQUE DU POINT, édition Dunod (2008).*
5. *R Spiegel, Murray. "Theory and problems of theoretical mechanics (schaums outline)." (2021).*
6. *Bellanger, Eric, et al. Physique MPSI-PCSI-PTSI: Programme 2013. Pearson Education France, 2013.*
7. *Fazely, Ali R. "Foundation of Physics for Scientists and Engineers: Volume I: Mechanics, Heat and Sound-eBooks and textbooks from bookboon. com." (2015).*
8. *Gebreselasie, Daniel. "Mechanics, Thermodynamics, Oscillation and waves: College physics I Notes and exercises." (2015).*