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Unbounded linear operators having self-adjoint powers and some related results

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Contents

List of symbols

$\langle \cdot \rangle$	Inner product.
H	Hilbert space.
$B(H)$	The set of all bounded linear operator defined on a Hilbert space H.
T	A linear operator defined on a Hilbert space H.
$D(T)$	The domain of T.
$\overline{T}$	The closure of the closable operator T.
I	The identity operator .
$\|T\|$	The norm of T.
$\ker(T)$	The kernel of T.
$Im(T)$	The image of T.
$G(T)$	The graph of T.
T^*	The adjoint operator of T.
$\sigma(T)$	The spectrum of T.

$\rho(T)$	The resolvent set of T.
T^{-1}	The inverse of T.
U	The partial isometry defined on on a Hilbert space H.
$\lvert T\rvert$	The modulus of T.

Introduction

Self-adjoint operators are the infinite-dimensional analogues of symmetric matrices. They are fundamental objects in both mathematics and quantum physics. According to the spectral theorem, any self-adjoint operator T can be represented as an integral $T = \int \lambda dE(\lambda)$ with respect to some unique spectral measure E.

The spectrum of a self-adjoint operator is always a subset of the real numbers. In quantum physics it is postulated that each observable is corresponds to a self-adjoint operator T. Hence, the spectrum of T is then the set of possible measured values of the observable. All of this necessitates that the operator T must be self-adjoint. For instance, the spectrum of symmetric operator T is no longer a subset of the reals, and it is impossible to obtain the integral representation.

J.Von Neumann showed that if T is a densely defined closed operator, then both TT^* and T^*T are self-adjoint. Recently, Z. Sebestyn and Zs. Tarcsay [15] proved the converse, that if TT^* and T^*T are both self-adjoint, then T must be closed. Then, F. Gesztesy and K. Schmdgen provided in [9] a simpler proof based on a technique using matrices of unbounded operators. Notice that GesztesySchmdgens proof only works for complex Hilbert spaces while the original proof by Z. Sebestyn and Zs. Tarcsay works also for real Hilbert spaces. In our work, we showed that if T is a densely defined closable such that $\sigma(TT^*) \cup \sigma(T^*T) \neq \mathbb{C}$, then T is necessarily closed, which might considered as a generalization of the SebestynTarcsayGesztesySchmdgen reversed von Neumann theorem.

Our work is divided into three chapters:

In the first chapter, we recall some fundamental concepts and outcomes related to closed operators on Hilbert spaces. We have presented the definitions of closed linear operators, closable linear operators, the spectrum of closed operators, and adjoint operators. Additionally, we have introduced several classes of unbounded linear operators such as normal operators, symmetric operators and self-adjoint operators.

In the second chapter, we present the definition and the properties of matrices of operators, then we establish some results related to the closedness of the operator T. Indeed, we studied both the monomials and polynomials of T to ensure its closedness and self-adjointness.

In the last chapter, we studied the powers of hyponormal operators. In fact, we used Bzouts theorem to ensure their self-adjointness. Furthermore, we investigated the roots of quasinormal operators.

Chapter 1

Essential background

In this chapter, we introduce basic definitions and fundamental properties of closed linear operators, which will be utilized in the coming chapters.

1.1 Hilbert Spaces

Definition 1.1.1. *An inner product on a complex vector space E can be defined as a function $\langle \cdot, \cdot \rangle$ defined from $E \times E$ to $\mathbb{C}$, satisfies the following conditions:*

1. $\langle \lambda x_1 + \beta x_2, y \rangle = \lambda \langle x_1, y \rangle + \beta \langle x_2, y \rangle$, *for all* $x_1, x_2, y \in E$ *and all* $\lambda, \beta \in \mathbb{C}$.
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$, $\forall x, y \in E$
3. $\langle x, x \rangle \geqslant 0$, $\forall x \in E$ *and* $\langle x, x \rangle = 0$ *if and only if* $x = 0$.

The above definition deals only with complex vector spaces. For vector spaces defined over $\mathbb{R}$ we only need to substitute $\mathbb{C}$ with $\mathbb{R}$ and consider complex conjugation as an operation that leaves the elements of $\mathbb{R}$ unchanged.

Example 1.1.2. *Let $\mathbb{C}\left([a,b]\right)$ be the space of all complex-valued continuous functions defined on $[a,b]$. An inner product on $\mathbb{C}\left([a,b]\right)$ can be defined as follows:*

$$\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}dx$$

for all $f,g \in \mathbb{C}([a,b])$.

Definition 1.1.3. *Let H be a complex linear space. H is said to be a Hilbert space if H is a complete normed inner product space.*

Example 1.1.4. *Let $x,y \in \mathbb{C}^n$, such that $x = (x_1, x_2, ...; x_n)$ and $y = (y_1, y_2, ...; y_n)$. An inner product on $\mathbb{C}$ can be defined as follows:*

$$\langle x,y\rangle = \sum_1^n x_i\overline{y}_i$$

It's easily to verify that $\mathbb{C}^n$ is a Hilbert space.

1.2 Basics of closed linear operators

Consider a linear operator T defined on a linear subspace $D(T)$ of a Hilbert space H. The subspace $D(T)$ is referred to as the domain of T. We say that T is an extension of a linear operator $T^{'}$ defined on $D(T^{'})$ if $D(T^{'}) \subseteq D(T)$ and $T^{'}(x) = T(x)$ for all $x \in D(T^{'})$, this relationship can be denoted as $T^{'} \subseteq T$.

Definition 1.2.1. *Let S and T be two linear operators defined on H and let $\alpha \in \mathbb{C}$. The operator αT is defned as follows:*

$$\left(\alpha T\right)x = \alpha\left(Tx\right) \text{ for all } x \in D\left(\alpha T\right) = D(T).$$

The domains of the operators ST an $(S+T)$ is defined as follows:

$$D\left(ST\right) = \{x \in D\left(T\right) \text{ and } Tx \in D\left(S\right)\}$$

and

$$D\left(S+T\right) = D\left(S\right) \cap D\left(T\right).$$

Example 1.2.2. *Let M be a measurable subset of $\mathbb{R}^n$. One can define an operator T on $L^2(M)$ by $Tf = xf$ where $x \in M$ and*

$$D(T) = \left\{ f \in L^2(M) \text{ such that } xf \in L^2(M) \right\}.$$

$D(T)$ is a dense subspace of $L^2(M)$ and T is called the maximal operator.

Proposition 1.2.3. *Let R, S and T be linear operators defined on H. We have*

$$(R+S)T = RT + ST \quad \text{and} \quad RT + ST \subseteq (R+S)T.$$

Definition 1.2.4. *Let T be a linear operator defined on H. The image or the range of T is defined as follows:*

$$Im(T) = \{T(x) \in H \text{ such that } x \in D(T)\} = T(D(T)),$$

The kernel of T is defined by:

$$\ker(T) = \{x \in D(T) : T(x) = 0\}.$$

For linear operator T, we say that T is injective if and only if $\ker(T) = \{0\}$ and T is said to be surjetive if and only if $Im(T) = H$. T is said to be bijective if and only if T is both injective and surjective.

Next, we define the graph of linear operators.

Definition 1.2.5. *Let T be a linear operator defined on H. The graph of T is defined as follows:*

$$G(T) = \{(x, Tx) \text{ with } x \in D(T)\}$$

Remark 1.2.6. *Obviously, $G(T)$ is a subspace of $H \times H$.*

Definition 1.2.7. *Let T be a linear operator defined on H. We say that T is closed if its graph $G(T)$ is closed subspace of $H \times H$. We say that T is closable if there exists a closed operator S such that $T \subseteq S$.*

Remark 1.2.8. *Another definition equivalent to the definition of closable operator T is that T is closable if $\overline{G(T)}$ is a graph.*

Let T be a closable operator, the unique operator $\overline{T}$ such that $G(\overline{T}) = \overline{G\,(T)}$ is called the closure of T.

In the next theorem, we will provide sufficient and necessary conditions to ensure the closedness of T.

Theorem 1.2.9 ([14])**.** *Let T be a linear operator defined on $D(T) \subseteq H$. We have:*

1. *T is closed if and only if, for any sequence $(x_n)_{n\in\mathbb{N}}$ in $D(T)$ such that $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} T(x_n) = y$, then we obtain $x \in D(T)$ and $Tx = y$.*
2. *$(D(T), \|\,.\,\|_T)$ is a Banach space, where*

$$\| x \|_T = \| x \| + \| Tx \|$$

3. *T is closable if and only if, for any sequence $(x_n)_{n\in\mathbb{N}}$ in $D(T)$ such that $\lim\limits_{n\to\infty} x_n = 0$ and $\lim\limits_{n\to\infty} T(x_n) = y$, then we obtain $y = 0$.*
4. *If T is closable, then the domain of $\overline{T}$ is defined as follows:*

$$D(\overline{T}) = \{x \in H : \textit{there exists a sequence}\,(x_n) \textit{ in } D(T) \textit{ such that } \lim x_n = x \textit{ and}$$

$$\lim_{n\to\infty} Tx_n = y\}.$$

Theorem 1.2.10 ([6])**.** *Let T be a linear operator defined on $D(T) \subseteq H$. Then*

1. *If $D(T)$ is a closed subspace of H then $T$is bounded if and only if T is closed.*
2. *If T is closed linear operator, then $\ker(T)$is a closed.*

Next, we provide a sufficient condition to ensure the closabilty of T.

Theorem 1.2.11 ([6])**.** *Let T be a linear operator defined on $D(T) \subseteq H$. If there exists a positive number c and a complex number λ such that*

$$\|(T - \lambda I)x\| \geq c||x|| \quad \textit{for all } x \in D(T)$$

then, T is closable and $Im(T - \lambda I)$ is closed.

Example 1.2.12. *1. Let T be the restriction of the identity operator on a dense domain. Then T is a closable but not closed.*

2. Let D be a dense subspace of H and let $z \in H$. Let S be a linear functional on D such that S is not continuous. The operator T defined on D by $T(x) = S(x)z$ is not closable.

1.3 Adjoint operators

Let T be a linear operator with the dense domain $D(T) \subseteq H$. Set

$$D(T^*) = \{y \in H : \text{the functional } x \to \langle Tx, y\rangle \text{ is continous on } D(T)\}.$$

Obviously, $D(T^*)$ is a non-empty subspace of H. By Riesz theorem, we have

$$D(T^*) = \{y \in H : \text{ there exists } z \in H \text{ such that } \langle Tx, y\rangle = \langle x, z\rangle \text{ for all } x \in D(T)\}.$$

The element z is uniquely determined, in fact if

$$\langle x, z\rangle = \left\langle x, z^{'}\right\rangle$$

then $z - z^{'} \in (D(T)^{\perp}) = \{0\}$ for all $x \in D(T)$ and so $z = z^{'}$ as $D(T)$ is dense. Setting $T^*y = z$, we obtain a well-defined linear operator T^* defined on $D(T^*)$ called the adjoint of T.

Example 1.3.1. *Let T be the operator defined as in the second item of Example 1.2.12. Since S is discontinues, then the functional $x \to \langle Tx, y\rangle = S(x)\langle z, y\rangle$ is continuous if and only if $\langle z, y\rangle = 0$. Therefore, $D(T^*) = \{y\}^{\perp}$ and $T^* = 0$ on $D(T^*)$.*

Next, we present some basic properties of the adjoint operators.

Theorem 1.3.2. *Let T be a densely defined on H.*

1. If $\overline{D(T^*)} = H$, then $T \subseteq T^{**}$.

2. $ker(T^*) = Im(T)^{\perp}$.

Proof. 1. Since $\overline{D(T^*)} = H$, then T^{**} is a well-defined linear operator. For all $x \in D(T)$ and $y \in D(T^*)$, we have

$$\langle T^*y, x\rangle = \langle y, T^{**}x\rangle = \langle y, Tx\rangle.$$

This means that $T \subseteq T^{**}$.

2. For $y \in D(T^*)$, We have

$$\langle Tx, y\rangle = \langle x, T^*y\rangle,$$

for all $x \in D(T)$. Then $y \in ker(T^*)$ is equivalent to $y \in Im(T)^\perp$.

□

Theorem 1.3.3 ([14])**.** *Let T and S be two linear operators such that $D(T)$ is dense in H. Then*

1. *T^* is closed.*
2. *T is closable if and only if $D(T^*)$ is dense in H, moreover, $\overline{T} = T^{**}$ and $(\overline{T})^* = T^*$.*
3. *T is closed if and only if $T = T^{**}$.*
4. *if $T \subseteq S$ then $S^* \subseteq T^*$.*
5. *if $\overline{D(T+S)} = H$, then $T^* + S^* \subseteq (T+S)^*$.*
6. *if $\overline{D(S)} = H$, then $T^*S^* \subseteq (TS)^*$.*
7. *If S is bounded, then $(T+S)^* = T^* + S^*$ and $(TS)^* = T^*S^*$.*

Definition 1.3.4. *We say that a linear operator T is invertible if there exists a bounded everywhere defined liner operator T^{-1} such that*

$$TT^{-1} = I \quad and \quad T^{-1}T \subseteq I,$$

such that I is the identity operator on H, or equivalently, T is invertible if and only if T is bijective.

Proposition 1.3.5. *Let T be a densely defined operator. Then, T is invertible if and only if T^* is invertible. Moreover, $(T^*)^{-1} = (T^*)^{-1}$.*

Proof. Since

$$T^{-1}T \subseteq I \text{ and } TT^{-1} = I$$

We obtain

$$(T^{-1})^*T^* \subseteq (TT^{-1})^* = I$$

and

$$T^*(T^{-1})^* = (T^{-1}T)^* = I$$

Therefore $(T^*)^{-1} = (T^{-1})^*$. □

1.4 Spectrum of closed linear operators

First, we start by defining the resolvent and the spectrum of a closed linear operator T.

Definition 1.4.1. *The resolvent set $\rho(T)$ is the set of all complex numbers λ such that $(T - \lambda I)$ is invertible. The spectrum of T is defined by: $\sigma(T) = \mathbb{C}\backslash\rho(T)$.*

Remark 1.4.2. *If T is not closed then $\sigma(T) = \mathbb{C}$.*

Theorem 1.4.3. *Let T be a closed densely defined linear operator on H. Then*

$$\sigma(T^*) = \{\lambda \in \mathbb{C} \text{ such that } \overline{\lambda} \in \sigma(T)\}$$

and

$$\rho(T^*) = \{\lambda \in \mathbb{C} \text{ such that } \overline{\lambda} \in \rho(T)\}.$$

Proof. Let $\overline{\lambda} \in \rho(T)$, then $(T - \overline{\lambda}I)$ is invertible. By Proposition 1.3.5, it follows that $(T^* - \lambda I)$ is invertible. Hence $\lambda \in \rho(T^*)$. □

Definition 1.4.4. *the set of eigenvalues $\sigma_p(T)$ consists of all complex numbers λ such that $Tx = \lambda x$ for some non-null $x \in D(T)$.*

Theorem 1.4.5 ([14])**.** *Let T be a closed densely defined linear operator. Then*

1. $\sigma_p(T) \subseteq \sigma(T)$.

2. $\sigma(T)$ *is a closed subset of* $\mathbb{C}$, *and if* $D(T) = H$ *then* $\sigma(T)$ *is a compact subset of* $\mathbb{C}$.

1.5 Special classes of closed linear operators

1.5.1 Self-adjoint operators

Definition 1.5.1. *Let T be a densely defined linear operator. T is said to be symmetric if $T \subseteq T^*$ or equivalently*

$$\langle Tx, y\rangle = \langle x, Ty\rangle \text{ for all } x, y \in D(T). \tag{1.1}$$

Example 1.5.2. *Let T be the restriction of the null operator on a densely subspace of H. Clearly, T is symmetric operator, and $T \subseteq T^* = 0_H$.*

Proposition 1.5.3. *Let T be a densely defined linear operator. T is symmetric if and only if $\langle Tx, x\rangle$ is a real number for all $x \in D(T)$.*

Proof. If T is symmetric, then

$$\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$$

which means that $\langle Tx, x\rangle$ is real. If$\langle Tx, x\rangle$ is a real for all $x \in D(T)$, the polarization formula implies that T is a symmetric. □

Definition 1.5.4. *Let T be a closed densely defined linear operator. T is said to be self-adjoint if $T = T^*$*

Remark 1.5.5. *Obviously, symmetric operators are self-adjoint but the opposite is not true in general.*

Example 1.5.6. *Let T be a closed densely defined linear operator. We define the operator M as follows:*

$$M = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}$$

easily, we can check that M is self-adjoint.

Definition 1.5.7. *Let T be a closable symmetric operator. If $\overline{T}$ is self-adjoint then we say that T is essentially self-adjoint.*

Proposition 1.5.8. *Let T be a symmetric operator. T is essentially self-adjoint if and only if T^* is self-adjoint.*

Proof. If T is essentially self-adjoint, then

$$T^* = (\overline{T})^* = \overline{T} = T^{**}$$

and so T^* is self-adjoint. Conversely, if T^* is self-adjoint, then

$$\overline{T}^* = T^* = T^{**} = \overline{T}$$

and hence, T is essentially self-adjoint. □

Next, we provide some properties related to symmetric operators.

Theorem 1.5.9. *Let T be a symmetric operator. Then*

1. $\sigma_p(T) \subseteq \mathbb{R}$.

2. $\langle Tx, x\rangle \geqslant 0$ *for all* $x \in D(T)$. *If* $\langle Ty, y\rangle = 0$ *for some* $y \in D(T)$, *then* $Ty = 0$.

Proof.

1. Let $\lambda \in \sigma_p(T)$ and $(T - \lambda I)x = 0$ with $x \neq 0$. Since $\langle Tx, x\rangle$ is a real number and $\langle Tx, x\rangle = \lambda\|x\|^2$ is real, then λ is real.

2. The CauchySchwartz inequality implies that

$$|\langle Ty, x\rangle|^2 \leq \langle Ty, y\rangle\langle Tx, x\rangle = 0,$$

for all $x \in D(T)$. As $D(T)$ is dense, then $Ty = 0$.

□

Now, we present some properties of self-adjoint operators.

Theorem 1.5.10. *Let T be a closable symmetric operator on the Hilbert space H. If there exists a real number λ such that*

$$H = \ker(T - \lambda I) + Im(T - \lambda I),$$

then

$$H = \ker(T - \lambda I) \oplus Im(T - \lambda I),$$

and T must be self-adjoint.

Proof. We have

$$\ker(T - \lambda I) \subseteq \ker(\overline{T} - \lambda I) = Im(T^* - \lambda I)^\perp \subseteq Im(T - \lambda I)^\perp.$$

Hence, $Im(T - \lambda I)$ is closed and

$$H = \ker(T - \lambda I) \oplus Im(T - \lambda I).$$

Let us show that T is self-adjoint, it suffices to prove that $D(T^*) \subseteq D(T)$. We have

$$\ker(T - \lambda I) = Im(T - \lambda I)^\perp,$$

and

$$Im(T^* - \lambda I) = \ker(T - \lambda I)^\perp = Im(T - \lambda I).$$

Let $x \in D(T^*)$ and $y = (T^* - \lambda I)x$. Since $Im(T^* - \lambda I) \subseteq Im(T - \lambda I)$, there exists an $x_0 \in D(T) \subseteq D(T^*)$ such that

$$(T - \lambda I)x_0 = (T^* - \lambda I)x_0 = y = (T^* - \lambda I)x.$$

Hence

$$x - x_0 \in \ker(T^* - \lambda I) = Im(T - \lambda I)^\perp = \ker(T - \lambda I) \subseteq D(T),$$

and $x \in D(T)$. □

Corollary 1.5.11. *If T is a surjective symmetric operator, then T is self-adjoint.*

Theorem 1.5.12 ([14])**.** *Let T be a densely defined linear operator on H. Then*

1. *If T is a symmetric operator and if there exists a complex number λ such that $Im(T-\lambda I) = \overline{Im(T-\overline{\lambda} I)} = H$, then T is self-adjoint.*

2. *If T is symmetric, then T is self-adjoint if and only if $\sigma(T) \subseteq \mathbb{R}$.*

3. *If T is self-adjoint, then $\lambda \in \sigma_p(T)$ if and only if $\overline{Im(T)} \neq H$.*

Definition 1.5.13. *Let T be a densely defined linear operator. We say that T is a positive if and only if $\langle Tx, x\rangle \geq 0$.*

Theorem 1.5.14 ([14])**.** *Let T be a densely defined closed linear operator on a Hilbert space H. Then:*

1. *T^*T is a positive self-adjoint operator.*

2. *There exists a unique positive operator $|T|$ called the square root of T such that $|T|^2 = T^*T$.*

3. *$(I + T^*T)$ is invertible.*

1.5.2 Normal operators

Definition 1.5.15. *Let T be a densely defined operator on a Hilbert space H. We say that T is normal if*

$$D(T) = D(T^*) \quad \text{and} \quad ||Tx|| = ||T^*x|| \quad \text{for all} \quad x \in D(T).$$

Remark 1.5.16. *Normal operators are self-adjoint but the opposite is not true in general.*

Theorem 1.5.17. *Let T be a normal operator on H. Then*

1. *If $T \subseteq N$ where N is normal then $T = N$.*

2. *T is normal if and only if T^* is normal.*

3. *T is normal if and only if T is closed and $T^*T = TT^*$.*

Proof. 1. Let N be a normal operator such that $T \subseteq N$, then

$$D(T) \subseteq D(N) = D(N^*) \subseteq D(T^*) = D(T).$$

Therefore $D(T) = D(N)$, and $T = N$.

2. Since $(T^*)^* = T$. Then T is normal if and only if T^* is normal.

3. Let $x, y \in D(T)$, we have $\|T(x + \tau y)\| = \|T^*(x + \tau y)\|$ for $\tau = 1, -1, i, -i$, and $\langle Tx, Ty\rangle = \langle T^*x, T^*y\rangle$.

   For $y \in D(T^*T)$, we have $y \in D(T)$, and so

$$\langle x, T^*Ty\rangle = \langle Tx, Ty\rangle = \langle T^*x, T^*y\rangle,$$

   for all $x \in D(T)$. Hence, $T^*y \in D(T^{**}) = D(T)$ and $TT^*y = T^*Ty$. This proves that $T^*T \subseteq TT^*$. Since $T = T^{**}$ and T^* is also normal, so we can interchange T and T^*, which yields $TT^* \subseteq T^*T$. Thus, $T^*T = TT^*$.

   Conversely, suppose that T is closed and $T^*T = TT^*$. For $x \in D(T^*T) = D(TT^*)$, we have

$$\|Tx\|^2 = \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle = \|T^*x\|^2,$$

   Hence $\|x\|_T = \|x\|_{T^*}$ and $(D, \|.\|_T) = (D, \|.\|^*)$. This implies $\|Tx\| = \|T^*x\|$ for $x \in D(T) = D(T^*)$.

□

Proposition 1.5.18 ([6])**.** *Let T be a normal operator. For any complex number λ, we have*

1. $Im(T) = Im(T^*)$.
2. $H = \overline{Im(T - \lambda I)} \oplus \ker(T - \lambda I)$.
3. *$\lambda \in \sigma_p(T)$ is an eigenvalue of T if and only if $\overline{\lambda} \in \sigma_p(T^*)$.*

Now, we will present the spectral theorem for normal operators.

Theorem 1.5.19 ([14]). *Let T be a normal operator. There exists a unique spectral measure E such that*

$$T = \int_{\mathbb{C}} \lambda dE_T(\lambda).$$

The next result is an essential tool to prove the main results in our work and it is known as the Fuglede-Putnam theorem.

Theorem 1.5.20 ([14]). *Let B be a bounded linear operator and N and M are not necessarily bounded normal operators, then*

$$BN \subset MB \Rightarrow BN^* \subset M^*B.$$

1.5.3 Quasinormal operators

Definition 1.5.21. *Let T be a closed densely defined operator in H. We say that T is quasinormal if $TT^*T \subseteq T^*TT$.*

It is well-known that normal operators are quasinormal and that the reverse implication does not hold in general.

Theorem 1.5.22 ([7]). *Let T be a closed densely defined operator in H. Then the following assertions are equivalent:*

1. $TT^*T \subseteq T^*TT$.
2. $TT^*T = T^*TT$.
3. $(I + T^*)^{-1}T \subseteq T(I + T^*)^{-1}$.
4. $E_{|T|}T \subseteq TTE_{|T|}$, *where $E_{|T|}$ denotes the spectral measure of T.*

Remark 1.5.23. *The inclusion $T^*TT \subseteq TT^*T$ does not imply the quasinormality of T. It suffices to take a closed densely defined operator T such that $D(T^2) = \{0\}$.*

Now we present a characterization of quasinormal operators.

Theorem 1.5.24 ([7]). *Let T be a closed densely defined operator in H. Then the following statements are equivalent:*

1. *T is quasinormal.*

2. $(T^*)^n T^n = (T^*T)^n$, *for every non negative integer* n.

3. *There exists a unique spectral Borel measure E on $\mathbb{R}^+$ such that*

$$(T^*)^n T^n = \int_{\mathbb{R}^+} x^n E(dx), \quad n \in \mathbb{N}^*.$$

4. *There exists a unique spectral Borel measure E on $\mathbb{R}^+$ such that*

$$(T^*)^n T^n = \int_{\mathbb{R}^+} x^n E(dx), \quad n \in \{1, 2, 3\}.$$

5. $(T^*)^n T^n = (T^*T)^n$, *for* $n \in \{1, 2, 3\}$.

Proposition 1.5.25 ([7])**.** *Let T be a quasinormal operator, then for every $n \in \mathbb{N}^*, T^n$ is a quasinormal operator.*

Chapter 2

Unbounded linear operators with closed powers

In this chapter, we study both monomials and polynomials of closable operator T to ensure its closedness and self-adjointness.

2.1 Matrices of operators

In this section, we present the definition and the properties of matrices of operators.

Definition 2.1.1. *Let $T_{jk} : D(T_{jk}) \subset H \to H$, be linear operators, with $j, k = 1, 2$. Let A be a matrix defined by:*

$$A := \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \tag{2.1}$$

It's referred to a (2×2) block operator matrix on $H \times H$. This matrix defines a linear operator on a subspace of $H \times H$ denoted by $D(A)$ and

$$D(A) := (D(T_{11}) \cap D(T_{21})) \times (D(T_{12}) \cap D(T_{22}))$$

and

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} T_{11}x_1 + T_{12}x_2 \\ T_{21}x_1 + T_{22}x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D(A)$$

Next, we present some basic properties of matrices of operators.

Proposition 2.1.2. *Let $A : D(A) \subset H \times H \to H \times H$ be a linear operator. The following statements are equivalent.*

1. *The operator A has a matrix representation as (2.1).*
2. *$D(A) = P_1 D(A) \times P_2 D(A)$, where*
$$P_1 : H \times H \to H, P_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := x_1$$
and
$$P_2 : H \times H \to H P_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := x_2$$
3. *$Q_1 D(A) \subset D(A)$, or equivalently, $Q_2 D(A) \subset D(A)$, where,*
$$Q_1 : H \times H \to H \times H, Q_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$$
and
$$Q_2 : H \times H \to H \times H, Q_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} 0 \\ x_2 \end{pmatrix}.$$

Next, we present the usual operations on matrices.

Definition 2.1.3. *Let $A = (T_{jk})$ and $B = (S_{jk})$ be block operator matrices on $H \times H$. The formal product of matrices A and B is defined by:*
$$A \times B := (\sum_{k=1}^{2} T_{jk} S_{jk}).$$

Theorem 2.1.4 ([8])**.** *Let A and B be two matrices of operators defined on $H \times H$. Then $A \times B = AB$ if and only if AB has a matrix representation.*

Definition 2.1.5. *Let*
$$A = \begin{pmatrix} T & S \\ R & U \end{pmatrix}$$
be a matrix of operators defined on H with dense domain $D_1 \times D_2$. Then the block operator matrix
$$A^{\times} := \begin{pmatrix} (T|_{D_1})^* & (R|_{D_1})^* \\ (S|_{D_2})^* & (U|_{D_2})^* \end{pmatrix}$$
is denoted the formal adjoint of A.

Theorem 2.1.6 ([8]). *Let A be a matrix of operators defined on $H \times H$ with a dense domain. Then $A^{\times} = A^*$ if and only if A^* has a matrix representation.*

2.2 Polynomials of unbounded linear operators

We begin by establishing some conditions on T to guarantee its closedness.

Theorem 2.2.1 ([5]). *Let p denote a complex polynomial of degree n. Let T be a closable operator defined on a Hilbert space H such that $p(T)$ is densely defined and $\sigma[p(T)] \neq \mathbb{C}$. Then $\sigma(T) \neq \mathbb{C}$ and hence T is closed.*

Proof. Let λ be a complex number such that $\lambda \notin \sigma[p(T)]$. We may also assume that the coefficient at the highest power of T in the polynomial $p(T)$ is 1. By the fundamental theorem of algebra, we know that there are complex numbers $\mu_1, \mu_2, \cdots, \mu_n$ such that

$$p(z) - \lambda = (z - \mu_1)(z - \mu_2) \cdots (z - \mu_n)$$

where $z \in \mathbb{C}$. Hence

$$p(T) - \lambda I = (T - \mu_1 I)(T - \mu_2 I) \cdots (T - \mu_n I).$$

On the other hand, we have

$$D[p(T) - \lambda I] = D[(T - \mu_1 I)(T - \mu_2 I) \cdots (T - \mu_n I)] = D(T^n).$$

Since $p(T) - \lambda I$ is invertible, then there exists a bounded linear operator S such that

$$(T - \mu_1 I)(T - \mu_2 I) \cdots (T - \mu_n I)S = I.$$

Since T is closable, so is $(T - \mu_n I)$ and so $(T - \mu_n I)S$ is closable. Hence $(A - \mu_n I)S \in B(H)$ as $D[(T - \mu_n I)S] = H$.

By using a similar argument, we can show that $(T - \mu_{n-1})(T - \mu_n I)S$ is a bounded operator. By induction, it may then be shown that $(T - \mu_2 I) \cdots (T - \mu_n I)B \in B(H)$. Hence, $T - \mu_1 I$ is right invertible. Since

$$(T - \mu_1 I)(T - \mu_2 I) \cdots (TA - \mu_n I) = (T - \mu_2 I)(T - \mu_3 I) \cdots (T - \mu_n I)(T - \mu_1 I)$$

then, $T - \mu_1 I$ is injective, thus, $T - \mu_1 I$ is bijective. Let $T^{'}$ be the left inverse (not necessarily bounded) of $T - \mu_1 I$. Clearly, $T^{'}$ must be defined on all of H. Moreover, $T^{'}(T - \mu_1 I) \subseteq I$ implies that

$$T^{'}(T - \mu_1 I)(T - \mu_2 I) \cdots (T - \mu_n I)S \subseteq (T - \mu_2 I) \cdots (T - \mu_n I)S$$

and so

$$T^{'} \subseteq (T - \mu_2 I) \cdots (T - \mu_n I)S$$

. Thus, $T^{'} = (T - \mu_2 I) \cdots (T - \mu_n I)S$ as they are both defined on all of H. Hence, $T - \mu_1 I$ is boundedly invertible, and $\sigma(T) \neq \mathbb{C}$. Accordingly, $T - \mu_1 I$ is closed which means that T is closed. □

Corollary 2.2.2. *Let p be a non-constant complex polynomial. Let T be a closable non-closed operator in a Hilbert space such that $p(T)$ is closed. Then*

$$\sigma[p(T)] = \mathbb{C}.$$

Corollary 2.2.3. *Let T be a closable densely defined operator such that $\sigma(A^2) \neq \mathbb{C}$. Then T is closed. Moreover, if T is unclosed linear operator such that T^2 is closed, then*

$$\sigma(T^2) = \mathbb{C}.$$

In Corollary 2.2.3, we can not drop the assumption $\sigma(T^2) \neq \mathbb{C}$, It suffice to consider a closable non-closed operator T such that T^2 is closable but unclosed, then $\sigma(T^2) = \mathbb{C}$. Indeed, let T be the identity operator restricted to some dense subspace D.

Remark 2.2.4. *In Corollary 2.2.3, the closability is indispensable. For instance, there is a non-closable operator T with $D(T) = H$ and $T^2 = 0$ on H. Clearly*

$$\sigma(T^2) = \{0\} \neq \mathbb{C}$$

and yet T is not closable.

Similarly, there is unclosable operator T defined on the whole space with $T^2 = I$ on H, and so $\sigma(T^2) = \{1\}$.

Corollary 2.2.5. *Let T be a closable densely defined operator with T^2 is self-adjoint. Then T must be closed.*

In [16] and [17], the authors proved that if a symmetric operator T have a positive self-adjoint square the T must be self-adjoint. In the next proposition, we will present a different proof of this result.

Proposition 2.2.6. *Let T be a symmetric non-necessarily densely defined operator such that T^2 is self-adjoint. Then T must be self-adjoint.*

Proof. Since T^2 is self-adjoint, then both T^2 and T are densely defined. Thus, since T is symmetric, it becomes closable.Now, Let us show that $\overline{T}$ is self-adjoint.

Since $T \subseteq T^*$, we have

$$\overline{T} \subseteq T^* = \overline{T}^*.$$

Hence

$$T^2 \subseteq \overline{T}^2 \subseteq \overline{T}^*\overline{T}.$$

By the maximality of self-adjoint operators T^2 and $\overline{T}^*\overline{T}$, we deduce that

$$\overline{T}^2 = \overline{T}^*\overline{T}.$$

Theorem 3.2 in [4] implies the self-adjointness of $\overline{T}$. On the other hand Corollary 2.2.5 ensures the closedness of T. Thus, T is self-adjoint. □

In the next result, we will extend the above result to the case of polynomials of T.

Theorem 2.2.7. *Let p be a polynomial of degree n. Assume that T is a symmetric non-necessarily densely defined operator such that $p(T)$ is self-adjoint. Then T is self-adjoint.*

Proof. Since T is symmetric, according to Theorem 1.5.12, it is sufficient to prove that $Im(T - \mu I) = H$ and $\overline{Im(T - \overline{\mu} I)} = H$ for some complex number μ.

Let $\lambda \in \mathbb{C} \setminus \sigma[p(A)]$. Similarly, we can assume that the leading coefficient of $p(A)$ is equal to 1, as stated above. We have

$$p(T) - \lambda I = (T - \mu_1 I)(T - \mu_2 I) \cdots (T - \mu_n I)$$

where $\mu_i, i = 1, \cdots, n$ are complex numbers. As above, we can show that $(T - \mu_1 I)$ is right invertible and so is surjective. Since $p(T) - \lambda I$ is boundedly invertible, so is $[p(T) - \lambda I]^*$. But

$$[p(T) - \lambda I]^* = p(T) - \overline{\lambda} I$$

as $p(T)$ is self-adjoint. Since T is symmetric, we deduce that

$$\begin{aligned}(T - \overline{\mu_1} I)(T - \overline{\mu_2} I) \cdots (T - \overline{\mu_n} I) &\subseteq (T^* - \overline{\mu_1} I)(T^* - \overline{\mu_2} I) \cdots (T^* - \overline{\mu_n} I) \\ &\subseteq [(T - \mu_1 I)(T - \mu_2 I) \cdots (T - \mu_n I)]^* \\ &= p(A) - \overline{\lambda} I.\end{aligned}$$

Thus,

$$(T - \overline{\mu_1} I)(T - \overline{\mu_2} I) \cdots (T - \overline{\mu_n} I) = p(T) - \overline{\lambda} I.$$

Since both sides have the same domain, which is $D(T^n)$. Therefore, $T - \overline{\mu_1} I$ is also surjective. Thus, T is self-adjoint. □

Corollary 2.2.8. *Let p be a polynomial of degree n with real coefficients. If T is a symmetric non-necessarily densely defined operator such that $p(T)$ is normal. Then T is self-adjoint.*

Proof. Since T is symmetric, then $p(T)$ is also symmetric as the coefficients of p are real. Thus

$$p(T) \subseteq p(T^*) \subseteq [p(T)]^*,$$

which mean that $p(T)$ is symmetric. Since $p(T)$ is normal, we conclude that $p(T)$ is self-adjoint and so T is self-adjoint by Theorem 2.2.7. □

M. Uchiyama proved that a symmetric quasinormal operator is self-adjoint (see [19]). Consequently, the next result is presented without proof.

Corollary 2.2.9. *Let p be a polynomial of degree n with real coefficients. If T is a symmetric non-necessarily densely defined operator such that $p(T)$ is quasinormal. Then T is self-adjoint.*

The above results might not extend to functions $f(T)$ even when the latter is well defined. Let us illustrate it with a counterexample.

Example 2.2.10. *Let T be a densely defined symmetric non-closed operator such that T^*T is self-adjoint. Thus $|T|^2$ is self-adjoint while T is not closed. as in [15], we set $S = i\frac{d}{dx}$ be a linear operator defined defined on $H^1(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R})\}$. Then S is self-adjoint . Let $T = T|_{H^2(\mathbb{R})}$ such that $H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : f'' \in L^2(\mathbb{R})\}$. Thus T is not closed and $S^* = T^*$. Since*

$$D(T^*T) = D(S^2) = H^2(\mathbb{R}),$$

it follows that $T^*T = S^2$. Thus, since S^2 is self-adjoint so is T^*T.

2.3 Monomials of unbounded linear operators

In this section, we will investigate the squares of unbounded linear operators and monomials in general.
The following Theorem might considered as a generalization of Proposition 2.2.6.

Theorem 2.3.1 ([5])**.** *Let T be an unbounded linear operator defined on $D(T) \subseteq H$ such that T^2 is self-adjoint and $D(T) \subseteq D(T^*)$. Then T is closed and*

$$D(T) = D(T^*),\ (T^*)^2 = T^2, \text{ and } D(TT^*) = D(T^*T).$$

Proof. Since T is densely defined and $D(T) \subseteq D(T^*)$, it follows that T is closable. Thus Corollary 2.2.5 implies that T is closed. So, we only needs to show that $D(T) = D(T^*)$. We have

$$(T^*)^2 \subseteq (T^2)^* = T^2.$$

Since $D(T) \subseteq D(T^*)$, we obtain

$$D(TT^*) \subseteq D[(T^*)^2] \subseteq D(T^2) \subseteq D(T^*T). \tag{2.2}$$

By Theorem 9.4 in [20], we deduce that $D(\sqrt{TT^*}) \subseteq D(\sqrt{T^*T})$ as TT^* and T^*T are positive self-adjoint operators. Since $\sqrt{TT^*} = |T^*|$ and $\sqrt{T^*T} = |T|$, we finally infer that

$$D(T^*) = D(|T^*|) \subseteq D(|T|) = D(T)$$

thereby $D(T) = D(T^*)$.

Now, let us show that $T^{*2} \subseteq T^2$. Since $T^{*2} \subseteq T^2$ was obtained above, we only need to show that $T^2 \subseteq T^{*2}$. As T^2 is self-adjoint, then $\sigma(T^2) \subseteq \mathbb{R}$ is a nonempty set. Let $\lambda \in \sigma(T^2)$ with $\lambda = \mu^2$ for some $\mu \in \sigma(T)$. Hence $\overline{\mu} \in \sigma(T^*)$. Thereby,

$$\lambda = \overline{\lambda} = \overline{\mu}^2 \in [\sigma(T^*)]^2 = \sigma(T^{*2}).$$

That is, $\rho(T^{*2}) \subseteq \rho(T^2)$. Let $\alpha \in \rho(T^{*2}) \neq \varnothing$ and write

$$T^{*2} - \alpha I \subseteq T^2 - \alpha I.$$

Since $(T^{*2} - \alpha I)$ is surjective and $(T^2 - \alpha I)$ is injective, then Lemma 1.3 in [14] gives us

$$T^{*2} - \alpha I = T^2 - \alpha I$$

and so $T^{*2} = T^2$.

Finally, as $D(T) = D(T^*)$ and $T^{*2} = T^2$, then we can easily conclude that inclusions (2.2) become

$$D(TT^*) = D[T^{*2}] = D(T^2) = D(T^*T),$$

marking the end of the proof. □

Then next result contains sufficient conditions on the operator T to ensure its self-adjointness.

Corollary 2.3.2. *Let T be an unbounded hyponormal operator such that T^2 is positive self-adjoint operator. Then T is self-adjoint.*

Proof. Since T is closable and T^2 is self-adjoint, it follows that T is closed.
In [2], it was shown that closed hyponormal operators with real spectrum are self-adjoint. Let $\lambda \in \sigma(T)$ and hence $\lambda^2 \in \sigma(T^2)$, so $\lambda^2 \geq 0$ because A^2 is positive. Thus λ must be real. Consequently, T is self-adjoint. □

Proposition 2.3.3. *Let T be an unbounded quasinormal operator such that $T^2 = T^{*2}$. Then T is normal.*

Proof. We have

$$|T|^4 = T^*TT^*T = T^{*2}T^2 = T^4$$

and

$$|T^*|^4 = TT^*TT^* = T^*T^2T^* = T^{*4}.$$

Since $T^2 = T^{*2}$, we obtain $T^4 = T^{*4}$, thereby $|T|^4 = |T^*|^4$. By taking the square root , we deduce that $|T|^2 = |T^*|^2$ or $T^*T = TT^*$. Since T is already closed, then it follows that T is normal. □

Remark 2.3.4. *If T is a bounded linear operator, then T^2 is self-adjoint if and only if $T^2 = T^{*2}$. This is not always true when T is densely defined closed operator. Indeed, in [3] the authors presented an example of a closed densely defined operator T such that*

$$D(T^2) = D(T^{*2}) = \{0\}$$

clearly, T^2 cannot be self-adjoint.

A natural question might be raised which is what if $D(T^2)$ was dense subspace of H? This is still not enough. In fact, let S and T be two non strongly commuting self-adjoint operators such that $ST = TS$ on some common core. This clearly is not simple and it is some kind of a Nelson-like counterexample. In order to illustrate this, we need Example 5.5 in [14]). Let

$$A = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}$$

with $D(A) = D(T) \times D(S)$. Then A is closed. Furthermore, $A^* = \begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix}$ with $D(A^*) = D(S) \times D(T)$. Thus

$$A^2 = \begin{pmatrix} ST & 0 \\ 0 & TS \end{pmatrix} = \begin{pmatrix} TS & 0 \\ 0 & ST \end{pmatrix} = A^{*2}$$

as $D(A^2) = D(A^{*2}) = D(ST) \times D(TS)$. If A^2 is self-adjoint, then A and B are strongly commuting, which is impossible.

Next, we define the subnormal operators.

Definition 2.3.5. *Let T be a densely defined linear operator with domain $D(T) \subseteq H$. T is said to be subnormal if there exist a Hilbert space K such that $H \subseteq K$ and a normal operator N with $D(N) \subseteq K$ such that*

$$D(T) \subseteq D(N) \text{ and } Tx = Nx \text{ for all } x \in D(T).$$

The following theorem extends certain known results in the bounded case. Its proof is based on an interesting paper by M. Uchiyama ([19]) about quasinormality of bounded and unbounded operators.

Before presenting the alluded theorem, recall that if T^n is densely defined, then in general $T^{*n} \subsetneq (T^n)^*$ even if T belongs to some special classes of linear operators. For instance, Jabłoński et al [7] gave an example of a quasinormal operator T such that $T^{*n} \subsetneq (T^n)^*$ for all $n \geq 2$.

Theorem 2.3.6 ([5])**.** *Let T be an unbounded quasinormal operator such that T^n is normal for some integer $n \geq 2$. Then T must be normal.*

The next lemma will be needed to prove the above theorem.

Lemma 2.3.7. *Let T be an unbounded quasinormal operator with $D[(T^n)^*] = D(T^n)$. Then*

$$(T^n)^* = T^{*n}.$$

Proof. Since T is quasinormal then T^n is also densely defined (see[7]).
Since $T^{*n} \subseteq (T^n)^*$. We only have to prove that $D[(T^n)^*] \subseteq D(T^{*n})$. Indeed, as T is quasinormal then T can be written as

$$T = U|T| = |T|U$$

where $U|T|$ is the usual polar decomposition of T with U is a partial isometry. Hence

$$T^* = |T|U^* \text{ and } U^*|T| \subseteq |T|U^*.$$

Since T is quasinormal, then $|T^n| = |T|^n$ (see e.g. [7]). Furthermore, according to [18], T^n is closed as T is a closed subnormal operator.

Hence,

$$D(T^{*n}) = D[(|T|U^*)^n] \supset D(U^{*n}|T|^n) = D(|T|^n) = D(|T^n|) = D(T^n),$$

and so

$$D(T^{*n}) = D[(T^n)^*],$$

as required. □

We are now ready to prove Theorem 2.3.6.

Proof of Theorem 2.3.6. By Corollary 3.1 in [19], it is known that if T is quasinormal then

$$T^{*n}T^n = (T^*T)^n \geq (TT^*)^n \geq T^nT^{*n}$$

for all $n \geq 2$.

Note that there are several definitions of the order relations between self-adjoint operators. In our work, for two self-adjoint operators A and B, the relation $A \leq B$ means that $D(\sqrt{B}) \subseteq D(\sqrt{A})$ and $\left\|\sqrt{A}x\right\| \leq \left\|\sqrt{B}x\right\|$ for all $x \in D(\sqrt{B})$

By the normality of T^n, Lemma 2.3.7 and the above inequalities, we have

$$T^n(T^n)^* = (T^n)^*T^n = T^{*n}T^n = (T^*T)^n \geq (TT^*)^n \geq T^nT^{*n} = T^n(T^n)^*.$$

Thereby,

$$(T^*T)^n = (TT^*)^n \ (= T^n(T^n)^* = (T^n)^*T^n).$$

By taking the *nth* positive root, we deduce that $|T|^2 = |T^*|^2$, this implies the normality of T. □

The next result contains a sufficient conditions on subnormal operators to ensure theirs normality.

Corollary 2.3.8. *Let T be a closed subnormal operator such that T^n is normal for some integer $n \geq 2$. Then T must be normal.*

Proof. From Theorem 1.4 in [13],it follows that T is quasinormal. Thus, Theorem 2.3.6 then gives the normality of T.

□

Next, we present a generalization of the Sebestyn-Tarcsay-Gesztesy-Schmüdgen reversed von Neumann theorem. Recall that if TT^* and T^*T are self-adjoint, then TT^* and T^*T are closed and $\sigma(TT^*), \sigma(T^*T) \subseteq \mathbb{R}$.

Theorem 2.3.9. *Let T be a closable densely defined operator such that:*

$$\sigma(TT^*) \cup \sigma(T^*T) \neq \mathbb{C}$$

Then T is closed.

Proof. Let

$$A = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}$$

which is closable. Then

$$A^2 = \begin{pmatrix} TT^* & 0 \\ 0 & T^*T \end{pmatrix}.$$

Since $\sigma(TT^*) \cup \sigma(T^*T) \neq \mathbb{C}$, then $\sigma(TT^*) \neq \mathbb{C}$ and $\sigma(T^*T) \neq \mathbb{C}$. Thus, TT^* and T^*T are closed and so A^2 is also closed. Since $\sigma(A^2) \neq \mathbb{C}$, Theorem 2.2.3 ensures that A is closed which means that T is closed. □

The next consequence is a characterizations of unbounded normal operators.

Corollary 2.3.10. *Let T be a closable densely defined operator. Then the following statements are equivalents.*

1. *T is normal.*

2. *$TT^* = T^*T$ and $\sigma(T^*T) \neq \mathbb{C}$.*

Proof. $1 \Rightarrow 2$. If T is normal, then T is closed and hence T^*T is self-adjoint and $TT^* = T^*T$ and so $\mathbb{C} \neq \sigma(T^*T) \subseteq \mathbb{R}$.

$2 \Rightarrow 1$. Since $\sigma(T^*T) \neq \mathbb{C}$, then T^*T and TT^* are closed. Theorem 2.3.9 then implies the closedness of T as $\sigma(TT^*) \cup \sigma(T^*T) = \sigma(T^*T) \neq \mathbb{C}$ which means that T is normal. □

By invoking Proposition 1.1 in [18], we have the following result.

Corollary 2.3.11. *Let T be a closable densely defined linear operator such that:*

$$\sigma(TT^*) \cup \sigma(T^*T) \neq \mathbb{C}$$

Then at least one of the following statements holds to be true

1. *T is normal.*
2. *$T^*T \not\subseteq TT^*$.*
3. *$TT^* \not\subseteq T^*T$.*

Proof. Since $\sigma(TT^*) \cup \sigma(T^*T) \neq \mathbb{C}$, then T must be closed. Hence by Proposition 1.1 in [18], either T is normal or $TT^* \not\subseteq T^*T$ or $T^*T \not\subseteq TT^*$. □

Before presenting another consequence, we give an example.

Example 2.3.12. *Let*

$$Sf(x) = e^{2x}f(x) \text{ and } Tf(x) = (e^{-x} + 1)f(x),$$

defined on

$$D(S) = \{f \in L^2(\mathbb{R}) : \ e^{2x}f \in L^2(\mathbb{R})\},$$

and

$$D(T) = \{f \in L^2(\mathbb{R}) : \ e^{-2x}f, e^{-x}f \in L^2(\mathbb{R})\}.$$

Clearly, S is self-adjoint and T is closable non-closed operator.
The operator TS is defined by $TSf(x) = (e^{2x} + e^x)f(x)$ on

$$D(ST) = \{f \in L^2(\mathbb{R}) : \ e^{2x}f, e^xf \in L^2(\mathbb{R})\},$$

is self-adjoint. On the other hand ST is not self-adjoint.

Motivated by the above example, we have:

Proposition 2.3.13. *There are no unbounded operators S and T, such that one of them is closable non-closed operator and the other is self-adjoint, such that both ST and TS are self-adjoint.*

Proof. Let T be a closable non-closed operator with domain $D(T)$, and let S be a self-adjoint operator with domain $D(S)$. Let

$$A = \begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix}$$

with the domain $D(A) = D(S) \times D(T)$. Clearly, A is only closable non-closed operator. We have,

$$A^2 = \begin{pmatrix} TS & 0 \\ 0 & ST \end{pmatrix}.$$

If TS and ST are both self-adjoint, then A^2 must be self-adjoint which implies the closability of A and hence by Corollary 2.2.5, it follows the closedness of A. Since this is absurd, then at least ST or TS must be non-self-adjoint. □

A question might be asked which is: can we find a closable (non-closed) densely defined operator T such that T^2 is closed densely defined operator, and obeys $\sigma(T^2) = \mathbb{C}$?

If $D(T^2)$ were not assumed dense, then an examples is already available and the answer is affirmative.

The challenge arises when $D(T^2)$ is dense. There is a counterexample in this case too. But, to answer the question we need a closed densely defined operator A such that $D(A^2) = D(A)$. So, let's provide such an example:

Example 2.3.14. *([10]) Let B be an unbounded closed operator with domain $D(B)$. Define*

$$A = \begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix}$$

with $D(A) = H \times D(B)$. Then T is closed densely defined operator. On the other hand, we have

$$D(A^2) = \{(x, y) \in H \times D(B) : (x + By, 0) \in H \times D(B)\} = D(A),$$

and hence

$$A^2 = \begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix} = A.$$

With this example at hand, we can now create a non-closed closable densely defined operator T such that T^2 is closed and densely defined with $\sigma(T^2) = \mathbb{C}$.

Example 2.3.15. *([10]) Let A be a closed densely defined operator such that $D(A^2) = D(A) \subseteq H$, and let $I_{D(A)}$ be the restriction of the identity operator on H. Let*

$$T = \begin{pmatrix} 0 & A \\ I_{D(A)} & 0 \end{pmatrix}$$

so $D(T) = D(A) \times D(A)$. Clearly, T is closable but unclosed.

Let $0_{D(T)}$ and $T_{D(T^2)}$ denote the restrictions of the zero operator and of A to the subspaces $D(A)$ and $D(A^2)$ respectively. Thus

$$T^2 = \begin{pmatrix} 0 & A \\ I_{D(A)} & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ I_{D(A)} & 0 \end{pmatrix} = \begin{pmatrix} A & 0_{D(A)} \\ 0_{D(A)} & A_{D(A^2)} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Thus, T^2 is closed $D(T) \times D(T)$, as desired.

In the next corollary, we provide the spectrum of a specific operators.

Corollary 2.3.16. *If $A = \begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix}$, then $\sigma(T) = \mathbb{C}$ for any closed non-every where defined operator B.*

Proof. Let T be as in Example 2.3.15, thus

$$T^2 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

thereby $\sigma(T^2) = \sigma(A)$. But, we already have $\sigma(T^2) = \mathbb{C}$, as if not then T becomes closed. Hence $\sigma(A) = \mathbb{C}$, as needed. □

Chapter 3

Roots and powers of some special operators

In this chapter, we study the powers of hyponormal operators. Furthermore, we investigate the roots of quasinormal operators.

3.1 Unbounded linear operators with self-adjoint powers

The aim of this section is to present a generalization of Corollary 2.3.2 to the case T^n where $n \geq 3$? A direct generalization is obviously false. For example, let $T = e^{2\pi i/3}I$, then T is unitary and non-self-adjoint, but $T^3 = I$ is positive. First, we showed that if T non-necessarily bounded hyponormal and if T^n and T^{n+1} are both positive and self-adjoint for some integer n, then T too is positive self-adjoint operator. After having shown this result, we have proved a better version by using the number theory.

Theorem 3.1.1. *Let T be a non-necessarily bounded hyponormal operator. If T^p and T^q are two self-adjoint operators, where p and q are two co-prime numbers, then T must be* *self-adjoint.*

Proof. Clearly T is closable. Since $\sigma(T^p) \subset \mathbb{R}$, Theorem 2.2.1 gives the closedness of T. Let $\lambda \in \sigma(T)$. Thus $\lambda^p \in \mathbb{R}$ and $\lambda^q \in \mathbb{R}$. If λ is null, then $\sigma(T) = \{0\}$, and so T becomes self-adjoint as T is hyponormal and its spectrum is real.

Now, if $\lambda \neq 0$, then Bzout's theorem in arithmetic yields that $ap + bq = 1$ for some integers a and b. Therefore, $\lambda^{ap}, \lambda^{bq} \in \mathbb{R} - \{0\}$, and so $\lambda^{ap+bq} \in \mathbb{R} - \{0\}$. In other words, $\lambda \in \mathbb{R} - \{0\}$. So in all cases, T has a real spectrum, and hence T becomes self-adjoint. □

We might get a similar result by replacing "self-adjoint" by "positive self-adjoint" in both cases of bounded and unbounded linear operators.

Remark 3.1.2. *The assumption of p and q being co-rime numbers is so important. To see that , let* $T = e^{2\pi i/3} I$. *Then both* T^3 *and* T^6 *are positive, but* T *is not self-adjoint.*

It is known that the square roots of normal operators are not be normal in general, even when $\dim H = 2$. Even if we add an extra condition such as theirs square roots are normal still does not implies the normality. The next result might be useful.

Theorem 3.1.3. *Let* T *be an invertible unbounded operator. If* T^p *and* T^q *are normal, where* p *and* q *are co-prime numbers, then* T *is normal.*

Proof. By Bzout's theorem there exists two integers a and b such that $ap + bq = 1$. Necessarily, ap or bq has to be negative, and let us assume it is bq. Since T is invertible, we have $T^{-1}T \subset I$ and $TT^{-1} = I$. Thus

$$T^{bq}T^{ap} = T \subset T^{ap}T^{bq}.$$

Since, $T^{bq} \in B(H)$ and T^p and T^q are also normal, then T^{ap} and T^{bq} are normal and

$$T^{bq}T^{ap} = T,$$

is normal, as required. □

We might obtain a a similar results for self-adjoint and positive self-adjoint operators.

Theorem 3.1.4. *Let T be an invertible operator. If T^p and T^q are self-adjoint (resp. positive self-adjoint), where p and q are two co-prime numbers, then T is self-adjoint (resp. positive self-adjoint).*

Proof. From Theorem 3.1.3 it follows that T is normal, and since T^p and T^q are self-adjoint are self-adjoint, then T becomes self-adjoint. □

Remark 3.1.5. *We can not drop the condition "p and q are co-prime numbers". Indeed, let*

$$T = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$$

be an invertible non-normal matrix with $T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *then*

$$T^2 = T^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and yet T is not normal.

It is known that if T is a positive self-adjoint operator and commutes with some bounded operator S, which means that $ST \subset TS$, then $S\sqrt{T} \subset \sqrt{T}S$.

Is it true for arbitrary roots? The answer is negative even on finite-dimensional spaces. For instance, let

$$T = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$$

and let S be a 2×2 matrix such that S and T do not commute. Then S commutes with $T^2 = I$ and A^4 but S and T do not commute. Therefore, it's essential to carefully select the exponents if we want a positive results.

In the next result, we put some conditions on T to ensure the commutativity of the square roots.

Proposition 3.1.6. *Let T be an invertible non-necessarily bounded operator, and let S be a bounded operator. If S commutes with both T^p and T^q, which means $ST^p \subset T^pS$ and $ST^q \subset T^qS$ for some co-prime numbers p and q, then $ST \subset TS$.*

Proof. Since p and q are co-prime numbers, then there exist two integers a and b such that $ap + bq = 1$ (choose bq to be negative). Since

$$ST^p \subset T^pT \text{ and } ST^q \subset T^qS,$$

then

$$ST^{ap} \subset T^{ap}S \text{ and } ST^{bq} = T^{bq}T$$

for $T^{bq}, S \in B(H)$. Thus

$$ST^{bq} = T^{bq}T \Longrightarrow ST^{bq}T^{ap} = T^{bq}ST^{ap} \subseteq T^{bq}T^{ap}S.$$

As $T^{bq}T^{ap} = T$. It follows that, $ST \subset TA$, as required. □

3.2 Roots of quasinormal operators

In this section, we present some results related quasinormal operators and their roots. First, we present a characterization of bounded quasinormal operators.

Theorem 3.2.1 ([11])**.** *Let T be a bounded operator. Let $k, l \in \mathbb{N}$ such that $k < l$. Then the following statements are equivalent:*

1. *T is quasinormal.*

2. *The operator T satisfies:*

$$T^{*i}T^i = (T^*T)^i, \quad i \in \{k, k+1, l, l+1\}.$$

In [1], the authors asked whether a bounded subnormal operator T such that T^2 being quasinormal implies that T is also quasinormal? In [12], the authors gave an affirmative answer. Indeed they have proved it with two different methods.

Theorem 3.2.2 ([12])**.** *Let T be a bounded subnormal operator such that T^n is quasinormal, with n is a positive integer, then T must be quasinormal.*

The authors in [13], extended Theorem 3.2.2 in two directions. First, they proved it for large class of operators such as hyponormal, then they investigated it for unbounded subnormal operators.

Theorem 3.2.3 ([13]). *Let T be a bounded hyponormal operator such that T^n is quasinormal, with n is a positive integer, then T must be quasinormal.*

Remark 3.2.4. *Theorem 3.2.3 remains true for p-hyponormal or log-hyponormal operators.*

Theorem 3.2.5 ([13]). *Let T be an unbounded closed densely defined subnormal operator such that T^n is quasinormal, with n is a positive integer, then T must be quasinormal.*

Conclusion

In this work, we have studied the powers and the polynomials of closed linear operators. Indeed, we were looking for sufficient conditions to ensure the closedness or the self-adjointness of the operator T. Among the results we showed that If a closed operator T is quasinoraml and its power T^n is normal with $n \geq 2$ then T must be normal. We have also generalized a result related to reversed J.von Neumanns theorem. The nth roots of quasinormal operators also have been studied.

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