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THEORETICAL PHYSICS
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by: khelifi cheref eddine
Titled

Noncommutative U(1) extension to the standard model $su(3) \otimes su(2) \otimes u(1) \otimes u(1)$ and dark mater

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Abstract

In this memo, we will compile some ideas from various research studies with the aim of examining the impact of treating the new, extension as an interaction mediator between dark matter particles, which interact with real matter indirectly due to the mixing term with the ordinary photon, on the mass of this new boson. On the other hand, we will also present a new mathematical framework, namely non-commutative geometry, in a simplified manner.

Résumé

Dans ce mémo, nous compilerons des idées provenant de diverses études de recherche dans le but d'examiner l'impact du traitement de la nouvelle extension... en tant que médiateur d'interaction entre les particules de matière noire, qui interagissent indirectement avec la matière réelle en raison du terme de mélange avec le photon ordinaire, sur la masse de ce nouveau boson. D'autre part, nous présenterons également un nouveau cadre mathématique, à savoir la géométrie non commutative, de manière simplifiée.

ملخص

في هذه المذكرة، سنجمع بعض الأفكار من دراسات بحثية مختلفة بهدف دراسة تأثير معالجة الامتداد الجديد... كوسيط تفاعل بين جزيئات المادة المظلمة، التي تتفاعل مع المادة الحقيقية بشكل غير مباشر بسبب مصطلح الخلط مع الفوتون العادي، على كتلة هذا البوزون الجديد. من ناحية أخرى، سنقدم أيضاً إطاراً رياضياً جديداً، وهو الهندسة غير التبادلية، بطريقة مبسطة.

Dedication and thanks

I dedicate this thesis to my dear family, who provided the right environment for me to continue my higher education. I also extend my heartfelt thanks to them for the immense support they have given me throughout this academic journey.
I would also like to express my deep gratitude to Professor Mounir Bousahl for opening new pathways in my scientific pursuits through this thesis, as well as for his continuous guidance and academic efforts.
I also extend my dedication and thanks to the examination committee for their valuable direction and evaluation of this modest work.
Additionally, I extend my sincere thanks to all my colleagues in the field of physics and the professors who encouraged me to persist in this field, as well as for their role in making this experience more enjoyable and exciting.

Contents

Chapter 1

GENERAL INTRODUCTION

1.1 Introduction

The field theory has provided a solid foundation for physicists to delve into the study of elementary particle physics. This research culminated in the construction of the Standard Model (SM). For a significant period, the Standard Model has been the most accurate tool for describing interactions at the quantum level and unifying the three fundamental forces (strong nuclear, weak nuclear, and electromagnetic) within a unified mathematical framework. However, gravity remains far from being integrated with the other forces into a unified theory. It is not clear whether treating gravity in the same manner as the other forces would yield any satisfactory or stable results.

On the other hand, the current challenges in modern physics are not limited to unifying the four fundamental forces. Advances in research technology in particle physics have opened the door to other issues. The Standard Model no longer provides the comprehensive picture of elementary particles that we once believed, and it doesn't seem to be improving with time. Currently, the model fails to explain why neutrinos have mass, which contradicts its predictions, nor does it provide any insight into their strange behavior, such as neutrino oscillation. Additionally, it completely fails to account for the existence of dark matter and dark energy in the universe.

Fortunately for the Standard Model, alternative models are not performing well either. Some offer fundamental solutions to the aforementioned problems and even propose the unification of the four forces, but they continue to fail in other areas and sometimes yield predictions that do not align with current experiments. This leads many to wonder if the solution lies outside the Standard Model entirely. In this context, there are proposals and attempts to modify the model itself, One such idea (which will be part of our research topic in this thesis) involves studying the results obtained by adding a new gauge boson (Z') within the framework of a standard gauge symmetry (U(1)), This new gauge boson could be considered as an additional ordinary gauge boson in the Standard Model, and it could also act as a force carrier between dark matter particles, potentially interacting with normal matter through a kinetic mixing term with the ordinary photon. Thus, its effects on normal matter could be studied.

Additionally, we will explore another mathematical framework that will help us develop our mathematical tools related to particle physics, which is Alain Connes' noncommutative geometry (NCG). We will use it in our study related to the Standard Model. This thesis will also provide a good opportunity to highlight noncommutative geometry. In general, NCG offers a solid foundation for studying spaces and geometric shapes using algebra rather than traditional geometric rules. This will help us obtain a good description of spacetime at the quantum level, especially those requiring additional dimensions such as string theory. NCG is also an excellent tool for studying topological spaces and shapes within the framework of noncommutative algebra, such as the algebra of operators in Hilbert space. Ultimately, this might lead us to develop mathematics capable of describing gravity alongside the other fundamental forces of nature, giving a strong push towards finding a comprehensive unified theory in the future.

1.2 The dark matter

In 1933, Fritz Zwicky presented the results of his study on the motion of galaxies in the Coma Cluster, which confirmed that the galaxies were moving faster than expected, indicating the presence of a greater mass than what was observed (Figure 1.1). On the other hand, during the 1970s, Vera Rubin's attempts to study the motion of stars within galaxies led to some unexpected results. The stars far from the center of the galaxy were moving faster than the laws of physics predicted. These results also required the existence of additional mass in the universe. Following these findings, many astronomical studies supported this conclusion, such as the study of the distribution of galaxies in the universe, the analysis of cosmic microwave background radiation, and the gravitational lensing effect, which predicts unseen mass concentrations.
Some scientists proposed that this mass could be explained by an unfamiliar type of matter existing alongside ordinary matter. This additional matter is expected to have a mass five times greater than the visible mass to account for the required gravitational effects observed in the studies. Moreover, this matter is invisible and does not interact with any force other than gravity. These characteristics are the main and only properties by which we currently identify what is known as dark matter. Dark matter is considered one of the greatest mysteries in the universe today and is a major focus of scientific research worldwide.

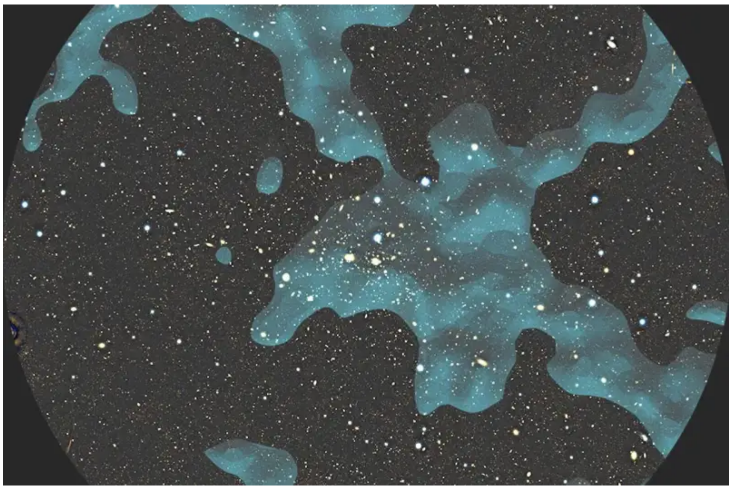

Figure 1.1: A modified image shows filaments of dark matter stretching across light-years within the Coma Cluster.[captured by the Subaru Telescope]

1.3 Why U(1)

General relativity has made several accurate predictions regarding the motion of celestial bodies, planets, stars, and more, providing a physical explanation for the force of gravity. However, its most mysterious predictions, which remain unclear, are dark energy and dark matter. Dark energy accounts for approximately 68% of energy density of the univers, while dark matter constitutes 27%. Ordinary matter makes up the remaining percentage. Regardless of their nature, their effects are evident and measurable. For dark matter, the most common logical explanation is that it may consist of other particles that exist alongside ordinary matter particles but possess properties that make them undetectable. Therefore, there are attempts to expand the Standard Model by adding new particles that could explain dark matter. The extension U(1) represents the simplest possible addition, which we will consider here.

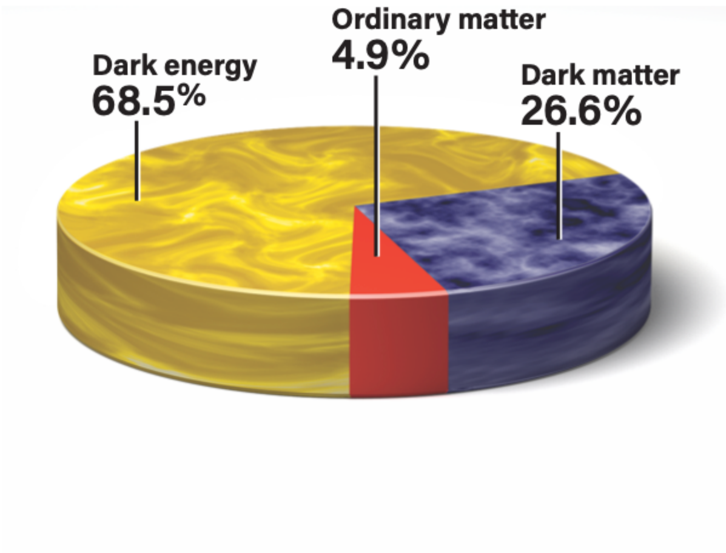

Figure 1.2: The distribution ratio of ordinary matter, dark matter and dark energy in the universe [https://www.astronomy.com]

1.4 Why noncommutative geometry ?

The unification of the four fundamental forces of the universe is the dream of any physicist. Currently, there are two theories: General Relativity, which explains gravitational force, and Quantum Field Theory, from which the Standard Model emerges, explaining the quantum interactions between quantum particles and unifying the strong, weak, and electromagnetic forces. Merging these two theories is the greatest challenge at present. Several attempts have been made, including Quantum Gravity and Cosmological Models, which provide some explanations in certain areas but fail miserably in others. Both approaches rely on quantizing gravity, but the strange behavior of gravity at the quantum level prevents this, as the equations continuously fall into singularities repeatedly, even with renormalization. Another proposed approach for unification currently being pursued is extending noncommutative algebra of quantum operators within Riemannian geometry. Within this approach, noncommutative geometry emerges.

1.5 A reminder of the most significant results of the Standard Model

Before delving into the details, we must first go through a brief reminder. The Standard Model is currently the most precise tool we have for describing particle physics. Although it is not perfect, it is the best framework available for testing the validity of any proposed theory.

1.5.1. Introduction

The Standard Model is an application of quantum field theory within the framework of Yang-Mills theory, with a gauge group $SU(3) \otimes SU(2) \otimes U(1)$. Generally, the model integrates two separate theories: the electroweak theory within the gauge group $SU(2) \otimes U(1)$, and quantum chromodynamics (QCD) within the gauge group $SU(3)$. The overarching aim of the theory is to unify the three fundamental forces (strong, weak, and electromagnetic).

1.5.2. Particles of the Standard Model

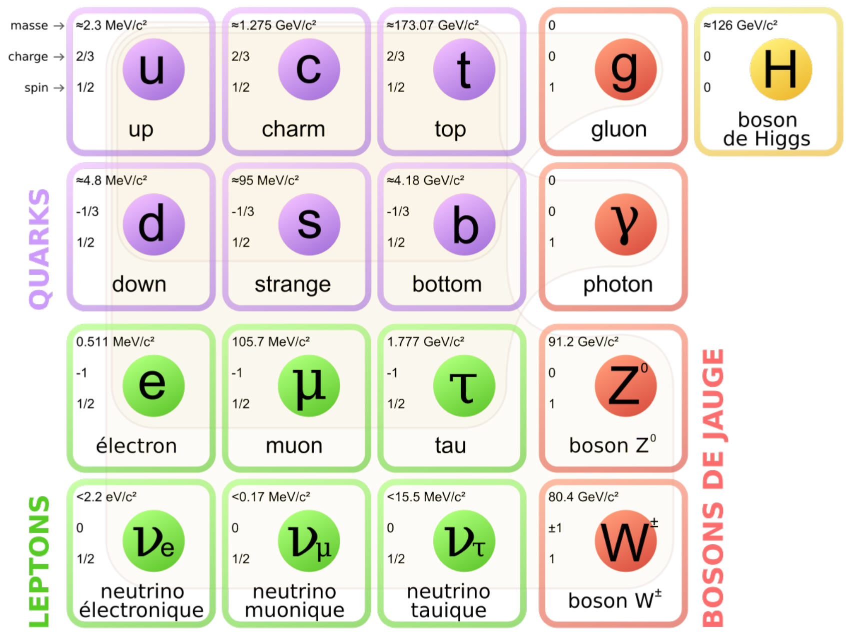

Figure 1.3: Particles of the Standard Model

The Standard Model proposes 12 different fermions (with spin $s = 1/2$) divided into two families (quarks and leptons). Each family is split into three different generations. Each fermion has an antiparticle that carries the same mass but an opposite charges. Furthermore, each fermion can also exhibit two types of chiral polarization, right-handed

and left-handed (except for neutrinos, which exhibit only left-handed polarization, while their antiparticles exhibit right-handed polarization).
On the other hand, there are also 12 gauge bosons (with spin $s = 1$) responsible for mediating the fundamental forces between fermions. The strong force is mediated by 8 different bosons called gluons. The weak force is mediated by 3 bosons: W^+, W^-, and Z. The electromagnetic force is mediated by a single boson, the photon (γ). Lastly, there is a single boson (with spin $s = 0$), represented by the Higgs field, which is responsible for giving mass to all particles in the Standard Model through the mechanism of spontaneous symmetry breaking. Most information (electric charge, color charge, isospin, etc.) about each particle can be summarized in Figure (1.3).

1.5.3. Lagrangian of the Standard Model

The Standard Model Lagrangian consists of several parts: the kinetic terms for the fermions and gauge bosons, the interaction terms, and the Higgs field terms. It can be expressed as:

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{fermion}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}} \tag{1.1}$$
$$\tag{1.2}$$

Gauge Part: The gauge part of the Lagrangian includes the kinetic terms for the gauge fields:

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} - \frac{1}{4}W^i_{\mu\nu}W^{i\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{1.3}$$

where:

- $G^a_{\mu\nu}$ is the field strength tensor for the gluon fields (QCD),$SU(2)$,($a \in \{1, 2, ..., 8\}$)
- $W^i_{\mu\nu}$ is the field strength tensor for the weak isospin fields (electroweak),$SU(2)$,($i \in \{1, 2, 3\}$)
- $B_{\mu\nu}$ is the field strength tensor for the weak hypercharge field (electroweak).

Fermion Part: The fermion part of the Lagrangian describes the kinetic terms for the fermions and their interaction with the gauge fields:

$$\mathcal{L}_{\text{fermion}} = \sum_{\psi} \bar{\psi}\left(i\gamma^\mu D_\mu\right)\psi \tag{1.4}$$

whith $\Psi \in \{\Psi^L_q \;,\; \Psi^L_l \;,\; \Psi^R_q \;,\; \Psi^R_l\}$ and D_μ is the covariant derivative that includes the interactions with the gauge fields.

$$\Psi^L_q = \left\{\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L\right\} \quad , \quad \Psi^L_l = \left\{\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L\right\}$$

$$\Psi^R_q = \{u_R, \, c_R, \, t_R, \, d_R, \, s_R, \, b_R\} \quad , \quad \Psi^R_l = \{e_R, \, \mu_R, \, \tau_R\}$$

Higgs Part: The Higgs part includes the kinetic term for the Higgs field and the potential term that gives rise to spontaneous symmetry breaking:

$$\mathcal{L}_{\text{Higgs}} = (D^\mu \Phi)^\dagger (D_\mu \Phi) - V(\Phi) \tag{1.5}$$

with the Higgs potential given by:

$$V(\Phi) = \mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2 \tag{1.6}$$

for the SM we use a one complex doublet higgs field

$$\Phi = \begin{pmatrix} \Phi^+ \\ \Phi^0 \end{pmatrix}$$

and higgs in the vacuum state

$$\langle \Phi \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

Yukawa Interactions The Yukawa interactions describe the interaction between the Higgs field and the fermions, which generates the fermion masses after spontaneous symmetry breaking:

$$\mathcal{L}_{\text{Yukawa}} = -y_u \bar{\Psi}_q^L \tilde{\Phi} u_i^R - y_d \bar{\Psi}_q^L \Phi d_i^R - y_e \bar{\Psi}_l^L \Phi \Psi_l^R + \text{h.c.} \tag{1.7}$$

where: y_u, y_d, and y_e are the Yukawa coupling constants for the up-type quarks, down-type quarks, and charged leptons respectively,

$$u_i^R = \{u_R, c_R, t_R, \} \quad , \quad d_i^R = \{d_R, s_R, b_R\}$$

Spontaneous Symmetry Breaking

In $su(3)$, we identify the eight gluons. Be aware that the $U(1)$ represents hypercharge and not electric charge. Electric charge is the result of a linear combination of hypercharge and weak isospin, which is determined by the weak mixing angle θ_w, which will be discussed later. In order to charge the W bosons with electricity, this mixing is required. While Z_0 and the photon are (orthogonal) mixes of the third isospin generator and hypercharge, W^+ and W^- are pure isospin states. Due to the bills' high degree of reducibility, there are numerous coins, including 27 intricate Yukawa couplings. Not every one of them has a tangible purpose. They can be transformed into three gauge couplings, eighteen physically meaningful positive numbers [4] .

$$g_3 = 1.218 \pm 0.026, g_2 = 0.6567 \pm 0.0007, g_1 = 0.3575 \pm 0.0001,$$

The Higgs couplings are related to the boson masses:

$$m_W = \frac{1}{2} g_2 v = 80.33 \pm 0.15\,\text{GeV}$$

$$m_Z = \frac{1}{2}\sqrt{g_1^2 + g_2^2}\, v = m_W / \cos\theta_W = 91.187 \pm 0.007\,\text{GeV}$$

$$m_H \approx 2\sqrt{2\lambda} v > 65\,\text{GeV},$$

with the vacuum expectation value $v := \frac{1}{2}\mu/\sqrt{\lambda}$
and the weak mixing angle θ_w defined by

$$sin^2\theta_w := g_22/(g_22 + g_21) = 0.2315 \pm 0.0005$$

The masses of the fundamental fermions are given by:

$$m_f = y_f \frac{v}{\sqrt{2}}$$

where y_f is the Yukawa coupling for the fermion f

$m_e = 0.51099906 \pm 0.00000015\text{MeV}$	$m_u = 5 \pm 3\text{MeV}$	$m_\mu = 0.105658389 \pm 0.000000034\text{GeV}$
$m_\tau = 1.7771 \pm 0.0005\text{GeV}$	$m_c = 1.3 \pm 0.3\text{GeV}$	$m_t = 175 \pm 6\text{GeV}$
$m_d = 10 \pm 5\text{MeV}$	$m_s = 0.2 \pm 0.1\text{GeV}$	$m_b = 4.3 \pm 0.2\text{GeV}$

Since neutrinos have no mass, mixing only happens for quarks, and the Cabibbo-Kobayashi-Maskawa matrix is a unitary matrix that describes this mixing.

$$C_{KM} := \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

For physical purposes it can be parameterized by three angles $\theta_12, \theta_23, \theta_13$ and one CP violating phase δ:

$$C_{KM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

with $c_{kl} := \cos\theta_{kl}$, $s_{kl} := \sin\theta_{kl}$. The absolute values of the matrix elements are:

$$\begin{pmatrix} 0.9755 \pm 0.0006 & 0.221 \pm 0.003 & 0.004 \pm 0.002 \\ 0.221 \pm 0.003 & 0.9745 \pm 0.0007 & 0.040 \pm 0.008 \\ 0.010 \pm 0.006 & 0.039 \pm 0.009 & 0.9991 \pm 0.0004 \end{pmatrix}$$

Chapter 2

INTRODUCTION OF NONCOMMUTATIVE GEOMETRY

2.1 Mathematics tools for the noncommutative geometry

In this section, we will introduce a set of new terms that we will need later on

2.1.1. Vector fields

First,let's define an open subset$U$ of $\mathbb{R}^n$,The Vector fields will be the family of derivable Vectors in the space $\mathbb{R}^n$, where the components of these Vectors are elements of U , Also, note that the Vectors $v(x)$ do not necessarily have to be within the subset U, unlike the points x.In Cartesian coordinates (y^μ,$\mu = 1, 2, , , n$), any Vector field can be expressed as:

$$v = \sum_{\mu=1}^{n} v^\mu(x) \frac{\partial}{\partial y^\mu} \tag{2.1}$$

where $\frac{\partial}{\partial y^\mu}$ are the vector fields with Cartesian components (0,..., 0, 1, 0,..., 0). The one is the μth entry. In this case, $\frac{\partial}{\partial y^\mu}$ is not a differential operator but merely a notation that helps us in transformations from one coordinate system x^μ to another $x^{\mu\prime}$, according to the following relationship:

$$\frac{\partial}{\partial x^\mu}(x) = \sum_\nu \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \tag{2.2}$$

$\frac{\partial y^\nu}{\partial x^\mu}$ is the Jacobian matrix.

2.1.2. Differential forms

Define the p-form ϕas a differentiable family of maps or functions ϕ_x ,in the space of $\mathbb{R}^n$, where its inputs are p vectors, while its outputs are just real numbers

$$\varphi_x : \mathbb{R}^n \cdots \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$(v_1(x), \cdots, v_p(x)) \longmapsto \varphi_x (v_1(x), \cdots, v_p(x))$$

Each map ϕ_x is required to be multilinear (with respect to the real numbers) and alternating, i.e.

$$\varphi(\cdots, v_i, \cdots, v_j, \cdots) = -\varphi(\cdots, v_j, \cdots, v_i \cdots)$$

We frequently suppress the point x for convenience. The set of all p-forms on U is denoted by $\Omega^p U$. Keep in mind that this collection only includes the zero member if $p > n$. The set of all (differentiable) functions from U into the real numbers is defined as $\Omega^0 U$ for p = 0.

2.1.3. Wedge product

We define the wedge product between p-form and q-form as follows.

$$\wedge : \Omega^p U \Omega^q U \longrightarrow \Omega^{p+q} U$$

$$(\varphi, \psi) \longmapsto \varphi \wedge \psi$$

$$(\varphi \wedge \psi)(v, \ldots, v_{p+q}) := \frac{1}{p!q!} \sum_{\pi \in S_{p+q}} \operatorname{sig} \pi \varphi \left(v_{\pi(1)}, \ldots, v_{\pi(p)}\right) \psi \left(v_{\pi(p+1)}, \cdots, v_{\pi(p+q)}\right) \quad (2.3)$$

where the sum is over all permutations of p + q objects and $sig\pi$ is the sign of the permutation π. The wedge product is bilinear, associative and graded commutative, i.e.

$$\varphi \wedge \psi = (-1)^{pq} \psi \wedge \varphi. \quad (2.4)$$

the wedge product gives us a new way to express a p-form , in the form of a product for p 1-forms map, according to the following relationship

$$\varphi = \sum_{\mu_1, \cdots, \mu_p} \varphi_{\mu_1, \cdots \mu_p}(x) \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \quad (2.5)$$

When a p-form is evaluated on p vector fields, it corresponds to a complete contraction, and the wedge product corresponds to the antisymmetrized tensor product of the covariant tensors.A set of vector space $\Omega^p U$, where (p=0,1,. . . ,n), along with a bilinear, associative, and graded-commutative product denoted by $\wedge$, is also known as the exterior algebra or Grassmann algebra.

2.1.4. Exterior derivative

We define the Exterior derivative of a p-form as follow:

$$\mathrm{d} : \Omega^p U \longrightarrow \Omega^{p+1} U$$

$$\varphi \longmapsto \mathrm{d}\varphi$$

$$\mathrm{d}\varphi := \sum_{\mu_1, \cdots, \mu_p, \nu} \left(\frac{\partial}{\partial x^\nu} \varphi_{\mu_1 \cdots \mu_p} \right) \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p} \quad (2.6)$$

The exterior derivative is a linear first order differential operator. It obeys the Leibniz rule

$$\mathrm{d}(\varphi \wedge \psi) = (\mathrm{d}\varphi) \wedge \psi + (-1)^P \varphi \wedge \mathrm{d}\psi \quad (2.7)$$

Another property of the exterior derivative is

$$d^2 = 0$$

2.1.5. Integral

for a m-form ϕ in subset U from $\mathbb{R}^{\ltimes}$

$$\varphi = f_{\mu_1,\cdots\mu_p}(x) \ \ \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \tag{2.8}$$

We can simply define the integral of ϕ as

$$\int\limits_S \varphi = \int\limits_S f_{\mu_1,\cdots\mu_p}(x) dx^1 dx^2..dx^n \tag{2.9}$$

where S is a p–dimensional sufficiently regular piece of U,and $\int\limits_S$ jest a regular integral of the function φ on S

2.1.6. Hodg star

The Hodge star is a map turning a p-form into an (n - p)-form. We define it as:

$$* : \Omega^p U \longrightarrow \Omega^{n-p} U$$

$$\varphi \longmapsto *\varphi$$

$$*\varphi := \frac{1}{(n-p)!} \sum_{\mu_{p+1}\cdots\mu_n} \left[\frac{1}{p!} \sum_{\mu_1 \cdot \mu_p} \epsilon_{\mu_1\cdots\mu_n} \sqrt{|\det gn|} \sum_{\nu_1\cdots\nu_p} \varphi_{\nu_1\cdots\nu_p} g^{\mu_1\nu_1} \cdots g^{\mu_p\nu_p} \right] dI \tag{2.10}$$

where $dI = \mathrm{d}x^{\mu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}$
$\epsilon_{\mu_1\cdots\mu_n}$ is the completely antisymmetric tensor with

$$\epsilon_{1\cdots n} = 1$$

It should be noted that while this definition is independent of the specific coordinate system employed, it does necessitate selecting an orientation in R n. Similar to the wedge product, the Hodge star operation is exclusively algebraic. Its square is plus or minus the identity, and it is linear.

$$** \varphi = (-1)^{p(n-1)+s} \varphi \tag{2.11}$$

s is the number of minus signs in the metric

2.1.7. Vector valued differential forms

Since all operations introduced so far are linear, we can generalize the values of differential forms from the real numbers to vectors in W

$$\Phi_x : \mathbb{R}^n \cdots \mathbb{R}^n \to W$$

W is a finite dimensional real vector space

$\Omega^p(U,W)$ is the set of p–forms on U with values in W, later We will use W as a Lie algebra or a vector space carrying a linear representation of some symmetry group. With a basisT_a, for any element $w \in W$ we ca written it as:

$$w = \sum_{a=1}^{dimW} w^a T_a \tag{2.12}$$

w^a is a real numbers.
for any p–form Φ with values in W it can to be writing as

$$\Phi = \sum_{a=1} \Phi^a T_a \tag{2.13}$$

Ψ^a is a real valued differential forms on U.

we define the the commutator of a p-form and a q-form from W, both with values in the Lie algebra, by

$$[\Phi, \Psi](v_1, \ldots, v_{p+q}) = \frac{1}{p!q!} \sum_{\pi \ell S_{p+q}} \operatorname{sig} \pi \left[\Phi\left(v_{\pi(1)}, \ldots, v_{\pi(p)}\right), \Psi\left(v_{\pi(p+1)}, \ldots, v_{\pi(p+q)}\right)\right] \tag{2.14}$$

or with respect to a basis T_a

$$\Phi = \sum_a \phi_a T^a \qquad , \qquad \Phi = \sum_a \psi_b T^b$$

$$[\Phi, \Psi] = \sum_{a,b} \phi_a \wedge \psi_b [T^a, T^b]$$

The commutator of forms is graded commutative

$$[\Phi, \Psi] = -(-1)^{pq}[\Phi, \Psi] \tag{2.15}$$

2.2 The Yang-Mills-Higgs theory

2.2.1. Spectral triples

A is the algebra of operators endowed with the representation ρ on the Hilbert space H of fermions. D is the Dirac operator, the (A,H,D) called the spectral triple. We now define $C^\infty(\mathcal{M})$, which is the commutative algebra of smooth functions with complex values on spacetime M, and $\mathcal{L}^2(S,M)$ is the Hilbert space of square-integrable Dirac spinors. We define the spectral triple

$$\mathcal{A}_t \equiv C^\infty(\mathcal{M}) \otimes \mathcal{A}$$
$$\mathcal{H}_t \equiv L^2(S,\mathcal{M}) \otimes \mathcal{H}$$
$$D_t \equiv i\gamma^\mu \partial_\mu \otimes I + \gamma^5 \otimes D$$

To generalize Riemannian geometry to noncommutative spacetime, the spectral triple $(\mathcal{A}_t, \mathcal{H}_t, D_t)$ must satisfy certain axioms derived from the commutative case , We define the chirality operator χ, which will serve as the defining tool for right- and left-handed polarization, and the real structure operator J and the charge conjugation C, for matter and antimatter, respectively, where

$$\chi = \begin{pmatrix} -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \quad , \quad J = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix} C$$

We define the representation ρ and D.

$$\rho = \begin{pmatrix} \rho_L & 0 & 0 & 0 \\ 0 & \rho_R & 0 & 0 \\ 0 & 0 & \rho_L^c & 0 \\ 0 & 0 & 0 & \rho_R^c \end{pmatrix} \quad , \quad D = \begin{pmatrix} 0 & M & 0 & 0 \\ M^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{M} \\ 0 & 0 & \bar{M}^* & 0 \end{pmatrix}$$

Finally, we define the following axioms[9]

- $[\rho(a), J\rho(\tilde{a})J^{-1}] = 0$
- $\mathcal{D}\chi = -\chi\mathcal{D}$
- $\mathcal{D}J = +J\mathcal{D}$
- $[\mathcal{D}, \rho(a)]$is bounded for all a in $\mathcal{A}$
- $[[\mathcal{D}, \rho(a)], J\rho(\tilde{a})J^{-1}] = 0 \quad , \quad a, \tilde{a} \in \mathcal{A}$

2.2.2. The Geometrical and Physical constraints

The choice of a spectral triple $(\mathcal{A}_t, \mathcal{H}_t, \mathcal{D}_t)$ is not arbitrary and should satisfy the set of NCG axioms, As already pointed out, the axioms of NCG reduce in our case to reality, Poincaré duality, and orientability axioms since in the case of a product of usual Riemannian geometry with a S^0-real spectral triple all other axioms are obviously satisfied. The reality is equivalent to the following two relations:

$$[\rho(x), J\rho(x')J] = 0, \quad [[\mathcal{D}, \rho(x)], J\rho(x')J] = 0, \quad x, x' \in \mathcal{A}. \tag{2.16}$$

The Poincaré duality

The Poincaré duality[8] means the non-degeneracy of the intersection form which is given by the matrix $\cap$:

$$\cap_{ij} = Tr(\chi\rho(p_i)J\rho(p_j)J) \tag{2.17}$$

Poincare duality holds if and only if the intersection form is non-degenerate, $det\cap \neq 0$ where p_i are the minimal Hermitian projections of our algebra.

The orientability

The orientability axiom means that the chirality can be written in the following manner:

$$\chi = \sum_i \rho(a_i)J\rho(b_i)J, \quad a_i, b_i \in \mathcal{A} \tag{2.18}$$

The unimodularity

we defind the unimodularity condition as

$$Tr_\rho[\rho(x) + J_\rho(x)J] = 0 \quad , \quad x \in g \tag{2.19}$$

Cancellation of gauge anomalies

$$Tr_p[\chi(\rho(x) + J\rho(x)J)^3] = 0 \quad , \quad x \in ig \tag{2.20}$$

The electric charge

For any model, it is necessary to set the appropriate hypercharge to obtain the charge Q in accordance with the experiment,so,we choose $x \in ig$ that it satisfies the following equation:

$$Q = \tilde{\rho}(x) + J\tilde{\rho}(x)J \tag{2.21}$$

2.2.3. The higgs field

In our setup, the field manifests as a special kind of mathematical object known as an antihermitian 1-form. This object describes how the Higgs field and gauge fields are distributed across the product space. Specifically, this 1-form separates into two components: the gauge fields and the Higgs field. This separation occurs due to the unique structure of the mathematical operator D_t.

$$H \equiv \sum_i \rho(x_{0i})[D, \rho(x_{1i})]; x_{0i}, x_{1i} \in \mathbf{A} \tag{2.22}$$

We can express this separation using a purely noncommutative 1-form, assuming it depends implicitly on space and time. The curvature of this 1-form, denoted by C, is defined by a mathematical operation called δ, which represents the differentiation within our internal algebra.

$$C \equiv \delta H + H^2 \tag{2.23}$$

Next, we introduce the preliminary Higgs potential.

$$V_0(H) \equiv Tr(zC^2) \tag{2.24}$$

which is represented by a positive definite matrix denoted by z. This matrix commutes with certain mathematical operators, including $\rho(A)$, $J\rho(A)J$, and D.

$$(w, w') \equiv Tr(zw * w') \tag{2.25}$$

this is a scalar product between two forms w and w' of the same degree. The Higgs field and the Higgs potential have a geometric interpretation in NCG because the Higgs field can be thought of as a gauge field on the internal space and the Higgs potential is the square norm of its curvature. is a scalar product between two forms w and w'of the same degree.
with C denoting the complex conjugation. Finally, the noncommutative gauge coupling z has the form

$$z \equiv diag(x/3I_{3N}, I_2 \otimes y, x/I_{3N}, y, \widetilde{x}/I_{3N}, I_2 \otimes \widetilde{y}, x/I_{3N}, \widetilde{y}) \tag{2.26}$$

where x and $\tilde{x}$ are strictly positive numbers and y and $\tilde{y}$ positive definite diagonal matrices of size NN.

2.2.4. The differential forms on spacetime

We'll start by constructing differential forms based on a spectral triple. In the commutative case, our aim is to produce Rham's differential forms, denoted as ΩM.
Firstly, we define $\Omega\mathcal{A}$, which is the differential algebra of the algebra A, where $\Omega^0\mathcal{A} := \mathcal{A}$ and $\Omega^1\mathcal{A}$ is a 1-form generated using differentiation δa,a $\in \mathcal{A}$,so we can generalize the relation for p-form

$$\Omega^{\mathrm{p}}\mathcal{A} = \sum_{\mathrm{i}} \mathrm{a}_0^{\mathrm{i}}\delta\mathrm{a}_1^{\mathrm{i}}\delta\mathrm{a}_2^{\mathrm{i}}\ldots\ldots\delta\mathrm{a}_{\mathrm{p}}^{\mathrm{i}}, \mathrm{a}_{\mathrm{q}}^{\mathrm{j}} \in \mathcal{A} \tag{2.27}$$

On the other hand, $\delta(\delta a) = 0$, so:

$$\delta(\mathrm{a}_0\delta\mathrm{a}_1\delta\mathrm{a}_2\ldots\ldots\delta\mathrm{a}_{\mathrm{P}}) = \delta\mathrm{a}_0\delta\mathrm{a}_1\delta\mathrm{a}_2\ldots\ldots\delta\mathrm{a}_{\mathrm{P}} \tag{2.28}$$

The involution $*$ is extended from the algebra A to$\Omega^1\mathcal{A}$ by putting $(\delta a)^* := \delta(a^*) =: \delta a^*$ and to the entire differential envelope by $(\kappa\psi)^* = \psi^*\kappa^*$

on othre hand ,The representation ρ from algebra $\mathcal{A}$ must then be extended to its envelope $\Omega\mathcal{A}$. This addon needs a different name lets use π for that.

$$\pi : \Omega\mathcal{A} \rightarrow \oplus_{\mathrm{p}}(End(\mathcal{H})) \tag{2.29}$$

$$\pi(\mathrm{a}_0\delta\mathrm{a}_1\ldots\delta\mathrm{a}_p) = (-i)^p\rho(\mathrm{a}_0)[\mathcal{D}, \rho(\mathrm{a}_1)]\ldots[\mathcal{D}, \rho(\mathrm{a}_{\mathrm{p}})] \tag{2.30}$$

π is a representation of $\Omega\mathcal{A}$ as graded involution algebra .

for this to be achieved, the differentiation operator δ on $\Omega\mathcal{A}$ should also satisfy the relation $\delta(\pi(\phi)) = \pi(\delta\phi)$, However, there are certain elements $\phi \in \Omega\mathcal{A}$ with $\pi(\phi) = 0$ but $\pi(\Omega\phi) \neq 0$, By dividing out these forms, we get differential algebra $\Omega_{\mathcal{D}}\mathcal{A}$ that we want

$$\Omega_D\mathcal{A} = \frac{\pi(\Omega\mathcal{A})}{\mathcal{J}} \tag{2.31}$$

with

$$\mathcal{J} = \pi(\delta ker\pi) = \oplus_p\mathcal{J}^p \tag{2.32}$$

$\mathcal{J}$ is the junk , for 0-form and 1-form ,the differential algebra $\Omega_{\mathcal{D}}\mathcal{A}$ is

$$\Omega^0_D\mathcal{A} = \rho(\mathcal{A}) \tag{2.33}$$

$$\Omega^1_D\mathcal{A} = \pi(\Omega^1\mathcal{A}) \tag{2.34}$$

for $p \geq 2$:

$$\Omega^p_D\mathcal{A} = \frac{\pi(\Omega^p\mathcal{A})}{\pi(\delta(ker\pi)^{p-1})} \tag{2.35}$$

2.2.5. The scalar product

In addition to defining differential forms, we must also define their scalar product ,in the context of noncommutative geometry,The scalar product allows us to define differential forms $\Omega_{\mathbf{D}}\mathcal{A}$ as operators on $\mathcal{H}$ space. For shapes of finite dimensions, we will consider the scalar product between tow operators ϕ and κ as $<\kappa, \phi>= Re\ tr(\kappa^*\phi)$. However, for shapes of infinite dimensions such as the Hilbert space $\mathcal{L}^2(S)$, we will require a Dirac operator for this purpose.
We define the Dixmier tr_w [7] for any positive operator $\mathcal{Q}$ on $\mathcal{H}$ a shape as follows:

$$tr_w(\mathcal{Q}|\mathcal{D}^{-dim}|) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \lambda_n \tag{2.36}$$

Given that λ_n are the eigenvalues of$\mathcal{Q}|\mathcal{D}^{-dim}|$, for the case of finite dimensions, we define The scalar product of$\pi(\Omega\mathbf{A})$ as follows.

$$<\kappa, \phi>= Retr_w(\kappa^*\phi|\mathcal{D}^{-dim}|) \quad , \quad \kappa, \phi \in \pi(\Omega\mathbf{A}) \tag{2.37}$$

In the commutative case, for four-dimensional spacetime M, we define the scalar product as

$$<\kappa, \phi>= \frac{1}{32\pi^2} Re \int_M tr_4[\kappa^*\phi] d^4x \tag{2.38}$$

Considering that $\Omega_{\mathbf{D}}\mathcal{A}$ is a subset of $\pi(\Omega\mathbf{A})$, it inherits The scalar product from it, and from there.

$$(\kappa, \phi) =<\kappa, \phi> \quad , \quad \kappa, \phi \in \Omega_{\mathcal{D}}\mathcal{A} \tag{2.39}$$

In the four-dimensional case, this scalar product disappears, and due to the symmetry between $\Omega^p M$ and $\Omega_\partial\mathcal{A}$, the scalar product of differential forms

$$\Omega_\partial\mathcal{A}$$

is defined as follows.

$$(\kappa, \phi) = \frac{1}{8\pi^2} Re \int_M \kappa^* * \phi \quad , \quad \Omega^p M \tag{2.40}$$

2.3 Standard Model in noncommutative geometry

In this section, we will review the standard model within the mathematical framework previously established[6][8][17][10]. Firstly, we will define the mathematical algebra to be adopted here.

$$\mathcal{A} = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$$

as well as the group.

$$SU(2) \otimes U(1) \otimes SU(3)$$

On the other hand, we will consider the effect of the chiral property. Thus, the Hilbert space adopted will be as follows:

$$\mathcal{H}_L = (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C})$$

$$\mathcal{H}_R = ((\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C})$$

The first part will represent the isospin in the weak interaction, either doublet or singlet. The second will represent the number of generations (N=3). The third part will represent the color charge, either singlet or triplet. After this, we will elaborate on the particles of the standard model in detail

the representation ρ acts on$\mathcal{H}$ by:

$$\rho(a,b,c) = \begin{pmatrix} \rho_{wL}(a) & 0 & 0 & 0 \\ 0 & \rho_{wR}(b) & 0 & 0 \\ 0 & 0 & \bar{\rho}_{sR}(b,c) & 0 \\ 0 & 0 & 0 & \bar{\rho}_{sL}(b,c) \end{pmatrix} \tag{2.41}$$

with

$$\begin{aligned} \rho_{wL}(a) &\equiv diag\,(\mathrm{a} \otimes \mathrm{I_{3N}}, \mathrm{a} \otimes \mathrm{I_N}) \\ \rho_{wR}(b) &\equiv diag\left(\mathrm{b_{3N}}, \overline{\mathrm{b}}\mathrm{I_{3N}}, \overline{\mathrm{b}}\mathrm{I_N}\right) \\ \overline{\rho}_{sL}(b,c) &\equiv diag\left(\mathrm{I_{2N}} \otimes \mathrm{c}, \overline{\mathrm{b}}\mathrm{I_{2N}}\right) \\ \bar{\rho}_{sL}(b,c) &\equiv diag\left(\mathrm{I_{2N}} \otimes \mathrm{c}, \overline{\mathrm{b}}\mathrm{I_N}\right) \end{aligned}$$

The selected representation ρ accounts for strong interactions$\rho_s(b,c)$, $c \in M_3(C)$, c for color and weak interactions $\rho_w(a,b)$,$a \in \mathcal{H}, b \in C$. Leptons, or color singlets, and quarks, or color triplets, are distinguished by this selection. For $\rho(a,b,c)$ to be a representation of A, $b \in C$ must play a key part in both weak interactions $\rho_w(a,b)$ and strong interactions $\rho_s(b,c)$. This is especially true for weak hypercharge computations. Particles and anti-particles appear to be asymmetrical,
the former are vulnerable to weak interactions while the latter are subject to strong ones
The third item in the spectral triple is the Dirac operator

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.42}$$

The mass matrix M is given by:

$$\mathcal{M} = \begin{pmatrix} \begin{pmatrix} M_u \otimes I_3 & 0 \\ 0 & M_d \otimes I_3 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 \\ M_e \end{pmatrix} \end{pmatrix} \tag{2.43}$$

and :

$$M_u \equiv diag(m_u, m_c, m_t) \quad , \quad M_d \equiv V_{CKM} diag(m_d, m_s, m_b) \quad , \quad M_e \equiv diag(m_e, m_\mu, m_\tau)$$

where m_p stands for the mass of particle p and V_{CKM} is the Cabibbo-Kobayashi-Maskawa mixing matrix.
The chirality χ and the charge conjugation J are given by

$$\chi \equiv \begin{pmatrix} -I_{8N} & 0 & 0 & 0 \\ 0 & I_{7N} & 0 & 0 \\ 0 & 0 & -I_{8N} & 0 \\ 0 & 0 & 0 & I_{7N} \end{pmatrix} \tag{2.44}$$

$$J \equiv \begin{pmatrix} 0 & I_{15N} \\ I_{15N} & 0 \end{pmatrix} \tag{2.45}$$

The Higgs being an anti-Hermitian 1-form $H \in \Omega^1_{\mathcal{D}}\mathcal{A}$

$$H = i \begin{pmatrix} 0 & \rho_{wL}(h)\mathcal{M} & 0 & 0 \\ \mathcal{M}^* \rho_{wL}(h^*) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad h = \begin{pmatrix} h_1 & -\overline{h}_2 \\ h_2 & \overline{h}_1 \end{pmatrix} \in \mathbb{H} \tag{2.46}$$

The internal junk in degree two is again is :

$$\mathcal{J} = i \begin{pmatrix} j \otimes \Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad j \in \mathbb{H} \tag{2.47}$$

with:

$$\Delta = \frac{1}{2} \begin{pmatrix} (M_u M_u^* - M_d M_d) \otimes M_3 & \\ & -M_e M_e^* \end{pmatrix} \tag{2.48}$$

Now, The homogeneous scalar variable is:

$$H = i \begin{pmatrix} 0 & \rho_{wL}(\phi)\mathcal{M} & 0 & 0 \\ \mathcal{M}^* \rho_{wL}(\phi^*) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad h = \begin{pmatrix} \phi_1 & -\overline{\phi}_2 \\ \phi_2 & \overline{\phi}_1 \end{pmatrix} \in \mathbb{H} \tag{2.49}$$

the internal field strength is:

$$C = \delta H + H^2 = (1 - |\phi|^2) \begin{pmatrix} I_2 \otimes \Sigma & 0 & 0 & 0 \\ 0 & \mathcal{M}^*\mathcal{M} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.50}$$

$$\Sigma = \frac{1}{2} \begin{pmatrix} (M_u M_u^* + M_d M_d) \otimes M_3 & 0 \\ 0 & M_e M_e^* \end{pmatrix} \tag{2.51}$$

αC dosn't have no junk component so:

$$\alpha C = (1 - |\phi|^2)\rho(\alpha I_2, \beta, \gamma I_3) \tag{2.52}$$

to calculate the real numbers α,β, and γ,we ignore all fermion masses in order except the top quark mass, This estimate is accurate up to$m_b^2/m_t^2 = 0.0006$, and the three linear equations are available.

$$4N\alpha + N \quad \beta + 3N\gamma = \frac{3}{2} m_t^2$$

$$2N\alpha + 12N\beta + 6N\gamma = 3m_t^2$$
$$3N\alpha + 3N\ \beta + 6N\gamma = 3m_t^2$$

the solution is

$$\alpha = 0 \quad , \quad \beta = 0 \quad , \quad \gamma = \frac{1}{2N}m_t^2$$

The Higgs and Yukawa couplings:

$$\mu^2 = (\frac{3}{2} - \frac{1}{N})m_t^2 \tag{2.53}$$
$$\lambda = \frac{\pi^2}{6z}(\frac{3}{2} - \frac{1}{N}) \tag{2.54}$$
$$g_t^2 = \frac{m_t^2}{v^2} = \frac{2\pi^2}{3z} \tag{2.55}$$

and by unimodularity condition

$$tr[P(\rho(a,b,c) + J\rho(a,b,c)J^{-1}] = 0 \tag{2.56}$$

where P is the projection on $\mathcal{H}_L \oplus \mathcal{H}_R$, the space of particles,we get the gauge couplings as:

$$g_3^2 = \frac{2\pi^2}{Nz} \quad , \quad g_2^2 = \frac{2\pi^2}{Nz} \quad , \quad g_1^2 = \frac{6\pi^2}{5Nz}$$

In particular we have

$$sin^2\theta_w = 3/8 \quad , \quad \lambda\frac{3N-2}{24}g_2^2 \quad , \quad g_t^2 = \frac{N}{3}g_2^2$$

Finally, we make a simple addition related to adding the right-handed neutrino to the Standard Model, using the Poincar´e duality condition and for the three minimal projectors

$$p_1 = (I_2, 0, 0) \quad , \quad p_2 = (0, 1, 0) \quad , \quad p_3 = \left(0, 0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

in the normale case of SM we get

$$\cap = -2N \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

is non-degenerate,but if we add right-handed neutrinos to the standard model, massive or not,we get

$$\cap = -2N \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

Poincar´e duality fails.

Chapter 3

U(1) EXTENSION AND THE DARK MATTER

3.1 U(1) extension in Standard Model

[1]In this section, we will incorporate the U(1) gauge field directly into the Standard Model without the previous condition related to dark matter. Ignoring interactions with other Standard Model particles will be challenging, making traditional studies more difficult. Without a specific Higgs model and an appropriate mixing term each time, and with no guarantee that the model won't fundamentally collapse, it would be impractical to study all possible patterns individually. This prevents us from deriving a general result for all potential scenarios of the model. Therefore, we will use noncommutative geometry to address this issue.

We will construct the algebra representation by employing geometric constraints and associated physical properties. We will determine the possible values of the generalized hypercharges and the types of Higgs fields.

3.1.1. Modifications

In order to introduce the required modifications to the standard model, we will only modify the representation $\rho(a, b, c)$ on the algebra $\mathcal{A}$ to take the form $\rho(a, b, b', c)$,where $(a, b, b', c) \in \mathbb{H} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$, without changing the part related to the strong and weak forces. Then, the representation($\rho_L, \rho_L^c, \rho_R^c$) will remain the same, while ρ_L will be written as follows:

$$\rho_R = diag(\alpha I_{3N}, \beta I_{3N}, \gamma I_N)$$

where : $\alpha, \beta, \gamma \in b, \overline{b}, b', \overline{b'}$

it easy to Proof that the orientability condition(2.18) is verified,on other hand , the Poincaré conditions(2.17),give us some zeros for certain possible distributions in the representations in the $C \oplus C$ algebra part. By removing these distributions, we obtain 40 possible distributions according to the following table(3.1).

It should be noted that the intersection form does not discriminate between a representation and its complex conjugate because it contains hermitian projections. Despite

α	β	γ	$det\cap$
$b,\overline{b}$	$b,\overline{b}$	$b,\overline{b}$	$=0$
$b,\overline{b}$	$b,\overline{b}$	$b',\overline{b}'$	$\neq 0$
$b,\overline{b}$	$b',\overline{b}'$	$b,\overline{b}$	$\neq 0$
$b',\overline{b}'$	$b,\overline{b}$	$b,\overline{b}$	$\neq 0$
$b,\overline{b}$	$b',\overline{b}'$	$b',\overline{b}'$	$=0$
$b',\overline{b}'$	$b',\overline{b}'$	$b,\overline{b}$	$\neq 0$
$b',\overline{b}'$	$b,\overline{b}$	$b',\overline{b}'$	$=0$
$b',\overline{b}'$	$b',\overline{b}'$	$b',\overline{b}'$	$\neq 0$

Table 3.1: A table showing the possible distributions of $\alpha\beta\gamma$ [1]

the fact that these 40 spectral triples meet all geometrical axioms, We must check the physical constraints next

first ,we defind the unimodularity condition (3.1) that cancels a U(1) factor is applied to the group of unitary elements of the algebra A, yielding the gauge group G.

$$Tr_\rho[\rho(a,b,b',c)+J_\rho(a,b,b',c)J]=0 \tag{3.1}$$

with $i(a,b,b',c)\in su(2)\otimes i\mathbb{R}\otimes u(3)$ We can reexpress the $u(1)$ Lie algebra resulting from the decomposition $u(3)=u(1)\otimes su(3)$ of the Lie algebra of the unitaries of $M_3(\mathbb{C})$ in terms of the two U(1) emerging from the two $\mathbb{C}$ summands of A . We now observe that $g=su(2)\otimes i\mathbb{R}\otimes i\mathbb{R}\otimes su(3)$ is the Lie algebra of the group of unitary elements of A (after the unimodularity condition is applied), whose representation on the Hilbert space of particles and antiparticles is:

$$\tilde{\rho}=diag(\tilde{\rho}_L,\tilde{\rho}_R,\tilde{\rho}^c_L,\tilde{\rho}^c_R) \tag{3.2}$$

$$\tilde{\rho}_L(a)=\mathrm{diag}\left(\mathrm{a}\otimes\mathrm{I}_{3\ \mathrm{N}},\mathrm{a}\otimes\mathrm{I}_\mathrm{N}\right),$$
$$\tilde{\rho}_R\left(b,b'\right)=\mathrm{diag}\left(\left(\mathrm{y_u b}+\mathrm{y'_u b'}\right)\mathrm{I}_{3\ \mathrm{N}},\left(\mathrm{y_d b}+\mathrm{y'_d b'}\right)\mathrm{I}_{3\ \mathrm{N}},\left(\mathrm{y_e b}+\mathrm{y'_e b'}\right)\mathrm{I}_\mathrm{N},\right)$$
$$\tilde{\rho}^c_L\left(b,b',c\right)=\mathrm{diag}\left(\mathrm{I}_{2\ \mathrm{N}}\otimes\left(\mathrm{c}+\mathrm{ub}_3+\mathrm{u'b'I}_3\right),-\mathrm{bI}_{2\ \mathrm{N}}\right),$$
$$\tilde{\rho}^c_R\left(b,b',c\right)=\mathrm{diag}\left(\mathrm{I}_{2\ \mathrm{N}}\otimes\left(\mathrm{c}+\mathrm{ubI}_3+\mathrm{u'b'I}_3\right),-\mathrm{bI}_\mathrm{N}\right)$$

where where $(a,b,b,c)\in ig$

y, y'$\in 1,0,+1$ (the generalized hypercharges)and u and u' are rational linear combinations of the generalized hypercharges.

Now we will check the anomalies cancellation condition,In the Standard Model, the condition for anomalies cancellation is equivalent to the condition for unimodularity. However, this is not the case in our situation since every extension of the Standard Model is actually anomalous.

with the condition of cancellation of gauge anomalies (2.20),and taking into account the electric charge(2.21) ,This will place us in front of 14 possible hypercharge configurations, corresponding to four type of Higgs,in table (3.1.1)

y_u	y_u'	y_d	y_d'	y_e	y_e'	u	u'	Type
0	1	0	-1	0	-1	1/4	1/12	1
0	-1	0	1	0	1	1/4	-1/12	1
1	0	-1	0	0	1	1/4	-1/12	2
0	1	0	-1	-1	0	1/3	0	2
1	0	-1	0	0	-1	1/3	0	2
1	0	-1	0	0	-1	1/4	1/12	2
0	-1	-1	0	0	1	1/2	1/6	3
1	0	0	-1	-1	0	1/12	1/4	3
1	0	0	1	-1	0	1/12	-1/4	3
0	1	-1	0	0	-1	1/2	-1/6	3
1	0	0	1	0	1	0	-1/3	4
0	1	-1	0	-1	0	7/12	-1/4	4
0	-1	-1	0	-1	0	7/12	1/4	4
1	0	0	-1	0	-1	0	1/3	4

Table 3.2: A table showing the possible distributions of hypercharge [1]

3.1.2. Higgs type

we give the higgs field in the NCG

$$H \equiv \sum_i \rho(x_{0i})[D, \rho(x_{1i})]; x_{0i}, x_{1i} \in \mathbf{A} \tag{3.3}$$

According to Poincare duality, the commutator $[\mathcal{D}, \rho(A)]$ vanishes on antiparticle space, so we should use the change of variable $\Phi = H - iD$,To recover the genuine Higgs field of a YMH theory ,now,Using the new variable, the curvature C defined by

$$C = \Phi^2 + D^2 + \theta$$

where θ is an element of the junk in degree two J^2,that is given by

$$\theta = \sum [D, \rho(x_i^0)][D, \rho(x_i^1)] \tag{3.4}$$

with the condition

$$\sum_i \rho(x_{0i})[D, \rho(x_{1i})] = 0 \tag{3.5}$$

where $x_i^0, x_i^1 \in \mathcal{A}.$, the junk in degree two is

$$j^2 = \left\{ \begin{pmatrix} ih \otimes \Delta_q & 0 & 0 & 0 \\ 0 & ih \otimes \Delta_l & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad h \in \mathbb{H} \right\} \tag{3.6}$$

where: $\Delta_q = \frac{1}{2}(M_u M_u^* - M_d M_d^*)$, $\Delta_l = -\frac{1}{2} M_e M_e^*$ Then, we can compute the Higgs field Φ and its curvature C, and we give the results in our four types of models, using the following notations[1]:

$$\Phi = i\begin{pmatrix} 0 & 0 & \Phi_q & 0 \\ 0 & 0 & 0 & \Phi_l \\ \Phi_q^* & 0 & 0 & 0 \\ 0 & \Phi_l^* & 0 & 0 \end{pmatrix} \quad , \quad C = \begin{pmatrix} C_q^L & 0 & 0 & 0 \\ 0 & C_l^L & 0 & 0 \\ 0 & 0 & C_q^R & 0 \\ 0 & 0 & 0 & C_l^R \end{pmatrix}$$

- **Type 1 models, one scalar doublet:** ϕ

$$\Phi_q = (\phi \otimes I_N \otimes I_3)M_q, \quad \Phi_l = (\phi \otimes I_N)M_l$$

$$C_q^L = (1 - \phi\phi^\dagger)\Sigma_q, \quad C_l^L = (1 - \phi\phi^\dagger)\Sigma_l$$
$$C_q^R = M_q^t(1 - \phi\phi^\dagger)M_q, \quad C_l^R = M_l^t(1 - \phi\phi^\dagger)M_l$$

- **Type 2 models, two scalar doublets:** ϕ_l and $\phi - q$

$$\Phi_q = (\phi_q \otimes I_N \otimes I_3)M_q, \quad \Phi_l = (\phi_l \otimes I_N)M_l$$

$$C_q^L = (1 - \phi_q\phi_q^\dagger)\Sigma_q + (\phi_q\sigma_3\phi_l^\dagger - \phi_q\sigma_3\phi_l^\dagger) \otimes \Delta_q',$$
$$C_l^L = (1 - \phi_l\phi_l^\dagger)\Sigma_l + (\phi_q\sigma_3\phi_l^\dagger - \phi_q\sigma_3\phi_l^\dagger) \otimes \Delta_l'$$
$$C_q^R = M_q^t(1 - \phi_q\phi_q^\dagger)M_q, \quad C_l^R = M_l^t(1 - \phi_l\phi_l^\dagger)M_l$$

- **Type 3 ($\epsilon = -1$) and Type 4 ($\epsilon = +1$), two scalar doublets:** ϕ_l and ϕ_q

$$\Phi_q = (\phi_1 + \phi_2\sigma_3) \otimes I_N \otimes I_3)M_q, \quad \Phi_l = (\phi_1 + \phi_2\sigma_3) \otimes I_N)M_l$$

$$C_q^L = (1 - \phi_1\phi_1^\dagger - \phi_2\phi_2^\dagger)\Sigma_q - (\epsilon\phi_1\sigma_3\phi_2^\dagger + \phi_2\phi_1^\dagger\sigma_3) \otimes \Delta_q,$$
$$C_l^L = (1 - \phi_1\phi_1^\dagger - \phi_2\phi_2^\dagger)\Sigma_l - (\epsilon\phi_1\sigma_3\phi_2^\dagger + \phi_2\phi_1^\dagger\sigma_3) \otimes \Delta_l$$
$$C_q^R = M_q^t(1 - \phi_1\phi_1^\dagger - \phi_2\phi_2^\dagger - \sigma_3\phi_2\phi_2^\dagger)M_q,$$
$$C_l^R = M_l^t(1 - \phi_1\phi_1^\dagger - \phi_2\phi_2^\dagger - \epsilon\sigma_3\phi_2\phi_2^\dagger)M_l$$

All the ϕ's are quaternions that parametrize the Higgs field and ϵ is an integer taking the value -1 (type 3 models) or $+1$ (type 4 models), We use here the notations:

$$\Sigma_q' = \Sigma_q - \frac{\mathrm{Tr}\,(x\Sigma_q\Delta_q) + \epsilon\mathrm{Tr}\,(y\Sigma_1\Delta_1)}{\mathrm{Tr}\left(xA_q^2\right) + \mathrm{Tr}\,(yA_1^2)}\Delta_q, \quad \Sigma_l' = \Sigma_l - \frac{\mathrm{Tr}\,(x\Sigma_q\Delta_q) + \epsilon\mathrm{Tr}\,(y\Sigma_1\Delta_1)}{\mathrm{Tr}\left(xA_q^2\right) + \mathrm{Tr}\,(yA_1^2)}\Delta_1$$

$$\Sigma_q = \frac{1}{2}\left(M_{4u}M_u^* + M_{4d}M_d^*\right), \quad \Sigma_l = \frac{1}{2}M_{4e}M_e^*$$

$$\Delta_q' = \frac{\mathrm{Tr}\,(yA_1^2)}{\mathrm{Tr}\left(xA_q^2\right) + \mathrm{Tr}\,(yA_1^2)}\Delta_q, \quad \Delta_1' = \frac{\mathrm{Tr}\left(xA_q^2\right)}{\mathrm{Tr}\left(xA_q^2\right) + \mathrm{Tr}\,(yA_1^2)}\Delta_1$$

3.1.3. Masses of the gauge bosons

In NCG, the bosonic part (Higgs boson and gauge bosons) of the action is given by the following expression

$$S[X,\Phi] = \int_M (\mathrm{Tr}(z\rho(Y^*)\wedge *\rho(Y)) + \mathrm{Tr}(zD\Phi^*\wedge *D\Phi) + *V(\Phi)) \tag{3.7}$$

The covariant derivative and the Higgs potential are given by the following expressions

$$V(\Phi) = \mathrm{Tr}\left[(C-\rho(x)-\theta)^*(C-\rho(x)-\theta)\right] \tag{3.8}$$

$$D\varphi = d\varphi + [\tilde{\rho}(X),\varphi] \tag{3.9}$$

Using the orthogonality condition between $C-\rho(x)-\theta \text{and} \rho(A)+j^2$, we can determine both x and θ. To obtain the gauge boson masses, we use the usual spontaneous symmetry breaking approach, by finding the expression φ that minimizes the potential $V(\varphi)$, which corresponds to C=0. Using the relation (2.23), we can deduce that C=0 is satisfied only for H=0, and hence the minimum of the Higgs field is $\varphi = iD$. Using the general expression for the curvature C and substituting φ with iD, we can extract the specific conditions for the gauge boson masses. Since we are looking for the general condition for the gauge boson masses, we will not focus on a specific form of the Higgs, thus we will only consider an approximate limit. We define X as follows

$$X_\mu = \frac{1}{2}\left(g_2\sigma_a A^a_\mu,\ g_1\cos\theta B_\mu - g_1'\sin\theta B'_\mu,\ g_1\sin\theta B_\mu + g_1'\cos\theta B'_\mu,\ g_3\lambda^c C^c_\mu\right) \tag{3.10}$$

Due to the unimodularity condition, the two u(1) summands of the Lie algebra g are no longer orthogonal, so we have to introduce a mixing angle θ. The coupling constants (g_1, g_1', g_2, g_3) and the mixing angle θ are defined such that the part corresponding to (s=1) of the action takes the following form

$$\int_{\mathcal{M}} \mathrm{Tr}(z\rho(Y)^*\wedge *\rho(Y)) = \int_{\mathcal{M}} \left(\frac{1}{4}F^i_{\mu\nu}F^{\mu\nu}_i + \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{1}{4}B'_{\mu\nu}B'^{\mu\nu} + \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a\right) d^4x \tag{3.11}$$

$F^i_{\mu\nu}$, $B_{\mu\nu}$, $B'_{\mu\nu}$and $G^a_{\mu\nu}$ represent the curvature corresponding to the gauge bosons G^a_μ,B_μ , B'_μ, and A^i_μ. By substituting φ with iD, we obtain the mass part of the gauge bosons. The charged gauge boson A^1_μ and A^2_μ are orthogonal to the neutral gauge bosons A'^3_μ, B_μ, and B'_μ, and in the vector space spanned by the three neutral gauge bosons, this quadratic form is

$$\begin{pmatrix} A^3_\mu & B_\mu & B'_\mu \end{pmatrix} \begin{pmatrix} M^2_{AA} & M^2_{AB} & M^2_{AB'} \\ M^2_{AB} & M^2_{BB} & 0 \\ M^2_{AB'} & 0 & M^2_{B'B'} \end{pmatrix} \begin{pmatrix} A^3_\mu \\ B_\mu \\ B'_\mu \end{pmatrix} \tag{3.12}$$

where,if we using

$$\alpha = \mathrm{Tr}(xM_uM_u^\dagger) \ \ , \ \ \beta = \mathrm{Tr}(xM_dM_d^\dagger) \ \ , \ \ \mathrm{Tr}(yM_eM_e^\dagger) \tag{3.13}$$

we can writing M_{AA}^2 , M_{BB}^2 , $M_{B'B'}^2$ as

$$M_{AA}^2 = \frac{1}{2}g_2^2\left(\alpha+\beta+\gamma\right) \tag{3.14}$$

$$M_{BB}^2 = \frac{1}{2}g_1^2\left(cos^2\theta(\alpha y_u+\beta y_d+\gamma y_e)+sin^2\theta(\alpha y_u'+\beta y_d'+\gamma y_e')\right) \tag{3.15}$$

$$M_{B'B'}^2 = \frac{1}{2}g_1^{'2}\left(cos^2\theta(\alpha y_u'+\beta y_d'+\gamma y_e')+sin^2\theta(\alpha y_u+\beta y_d+\gamma y_e)\right) \tag{3.16}$$

The coupling constants and the angle θ are defined as

$$g_2^{-2} = Nx + Tr(y),$$

$$g_1^{-2} = \frac{1}{2}\cos^2\theta\left(y_u^2Nx+y_d^2Nx+y_e^2Tr(y)+4N\tilde{x}u^2+3Tr(\tilde{y})\right) + \frac{1}{2}\sin^2\theta\left(y_u^2Nx+y_d^2Nx+y_e^2Tr(y)+4N\tilde{x}u^2\right) - 4N\tilde{x}uu'\cos\theta\sin\theta,$$

$$g_1^{'-2} = \frac{1}{2}\sin^2\theta\left(y_u^2Nx+y_d^2Nx+y_e^2Tr(y)+4N\tilde{x}u^2+3Tr(\tilde{y})\right) + \frac{1}{2}\cos^2\theta\left(y_u^2Nx+y_d^2Nx+y_e^2Tr(y)+4N\tilde{x}u^2\right) + 4N\tilde{x}uu'\cos\theta\sin\theta,$$

$$\tan 2\theta = \frac{8N\tilde{x}uu'}{2Nx\left(y_u^2+y_d^2-1\right)+Tr(y)\left(2y_e^2-1\right)+4N\tilde{x}\left(u^2-u'^2\right)+6Tr(\tilde{y})}. \tag{3.17}$$

We do not need the specific form of M_{AB} and M'_{AB} to determine the masses of the three bosons,we can directly take the trace of the matrix to obtain

$$(1/2)m_Z^2 + (1/2)m_Z^{2'} = M_{AA}^2 + M_{BB}^2 + M_{B'B'}^2 \tag{3.18}$$

The right-hand side heavily depends on the variables $(x, y, \tilde{x}, \tilde{y})$. We observe that the variables $(\tilde{x}, \tilde{y})$ appear only in the coupling constants, and considering them as non-zero positive constants, by ignoring them we obtain the following inequalities

$$g_1^2 < \frac{2}{\cos^2\theta\left(y_u^2Nx+y_d^2Nx+y_e^2Tr(y)\right)+\sin^2\theta\left(y_u'^2Nx+y_d'^2Nx+y_e'^2Tr(y)\right)} \tag{3.19}$$

$$g_1^{2'} < \frac{2}{\sin^2\theta\left(y_u^2Nx+y_d^2Nx+\gamma^2Tr(y)\right)+\cos^2\theta\left(y_u'^2Nx+y_d'^2Nx+y_e'^2Tr(y)\right)} \tag{3.20}$$

We substitute the inequalities into equation (3.18), to simplify and eliminate the remaining variables (x, y), we will use another inequality, of the form

$$\frac{\sum_i^n a_i x_i}{\sum_i^n b_i x_i} < \sum_i^n \frac{a_i}{b_i} \tag{3.21}$$

The previous inequality is only valid for positive values of (x, y). In the end, we reach the final inequality

$$(1/2)m_Z^2 + (1/2)m_Z^{2'} < (3/2N)(Tr(M_uM_u^*) + Tr(M_uM_d^*)) + 3Tr(M_eM_e^*)^2 \tag{3.22}$$

We neglect the masses of the other fermions compared to the top quark mass to obtain

$$m_Z^2 + m_Z^{2\prime} < m_t^2 \tag{3.23}$$

This result tells us that for all possible cases that can be applied within this model[2], it is never possible to obtain a gauge boson z' whose mass exceeds the mass of the top quark.

3.2 U(1) extension as a dark matter interaction mediator

3.2.1. Mixing Term

First, we will define the bosonic part of the Lagrangian as follows:

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}F^1_{\mu\nu}F^{1\mu\nu} - \frac{1}{4}F^2_{\mu\nu}F^{2\mu\nu} - 2cF^1_{\mu\nu}F^{2\mu\nu} \tag{3.24}$$

$$F^r_{\mu\nu} = \partial_\mu A^r_\nu - \partial_\nu A^r_\mu \quad (r = 1, 2)$$

The fields $-\frac{1}{4}F^r_{\mu\nu}$ kinitice term of the gauge fields A_r. Fermions interact with the gauge fields as follows:

$$\mathcal{L}_{\text{int}} = g_1 j^1_\mu A^{1\mu} + g_2 j^2_\mu A^{2\mu} \tag{3.25}$$

The interaction can be written in matrix form as follows:

$$\mathcal{L}_{\text{int}} = \begin{pmatrix} j^1_\mu & j^2_\mu \end{pmatrix} \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \begin{pmatrix} A^{1\mu} \\ A^{2\mu} \end{pmatrix} \tag{3.26}$$

where j^1_μ and j^2_μ are the fermionic currents.

$$j^\mu_r = q^f_r \bar{\Psi}\gamma^\mu\Psi \tag{3.27}$$

Now, to eliminate the mixing term in equation (3.24), we will perform a rotational transformation using the angle θ, resulting in the following:

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}G^1_{\mu\nu}G^{1\mu\nu} - \frac{1}{4}G^2_{\mu\nu}G^{2\mu\nu} \tag{3.28}$$

$$G^r_{\mu\nu} = \partial_\mu B^r_\nu - \partial_\nu B^r_\mu \quad (r = 1, 2)$$

The fields B^1_μ and B^2_μ will undergo the following transformation[16]:

$$\begin{pmatrix} A^1_\mu \\ A^2_\mu \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{1}{\lambda_1}\cos\theta & -\frac{1}{\lambda_2}\sin\theta \\ \frac{1}{\sqrt{\lambda_1}}\sin\theta & \frac{1}{\sqrt{\lambda_2}}\cos\theta \end{pmatrix} \begin{pmatrix} B^1_\mu \\ B^2_\mu \end{pmatrix} \tag{3.29}$$

We define the variables λ_1 and λ_2 by the following relations:

$$\lambda_1 = \frac{1}{4} + 2c\cos\theta\sin\theta \quad , \quad \lambda_2 = \frac{1}{4} - 2c\cos\theta\sin\theta$$

$$\lambda_1 + \lambda_2 = 1/2 \tag{3.30}$$

(3.31)

After redefining the gauge fields A^1_μ and A^2_μ, we need to adjust the coupling matrix between fermions and gauge bosons. This includes changes to the fermionic currents j^1_μ and j^2_μ, as well as the coupling constants g_1 and g_2. The results are as follows:

$$\mathcal{L}_{\text{int}} = \begin{pmatrix} J^\mu_1 & J^\mu_2 \end{pmatrix} \begin{pmatrix} \tilde{g}_1 & 0 \\ 0 & \tilde{g}_2 \end{pmatrix} \begin{pmatrix} B^1_\mu \\ B^2_\mu \end{pmatrix} \tag{3.32}$$

where

$$\begin{pmatrix} J^\mu_1 \\ J^\mu_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\cos\varphi & \sin\varphi \end{pmatrix} \begin{pmatrix} j^\mu_1 \\ j^\mu_2 \end{pmatrix} \tag{3.33}$$

and

$$\cos\varphi = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}, \quad \sin\varphi = \frac{g_2}{\sqrt{g_1^2 + g_2^2}} \quad , \quad \tilde{g}_1 = \sqrt{\frac{g_1^2 + g_2^2}{2\lambda_1}}, \quad \tilde{g}_2 = \sqrt{\frac{g_1^2 + g_2^2}{2\lambda_2}}$$

3.2.2. Spontaneous Symmetry Breaking

For the gauge field ϕ, we can write the covariant derivative as:

$$D_\mu\phi = (\partial_\mu + ig_1q_1A^1_\mu + ig_2q_2A^2_\mu)\phi = (\partial_\mu + i\tilde{g}_1Q_1B^1_\mu + i\tilde{g}_2Q_2B^2_\mu)\phi \tag{3.34}$$

where the charge q is associated with the new gauge fields A^1_μ and A^2_μ. the charge q_1, q_2, carry the same transformation matrix of j^μ_1 and j^μ_1.

$$\begin{pmatrix} Q^\mu_1 \\ Q^\mu_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\cos\varphi & \sin\varphi \end{pmatrix} \begin{pmatrix} q^\mu_1 \\ q^\mu_2 \end{pmatrix} \tag{3.35}$$

where

For the vacuum state $< \phi_0 >= \frac{\nu^2}{\sqrt{2}} \neq 0$, the mass matrix takes the following form:

$$M^2 = \frac{\nu^2}{2} \begin{pmatrix} (\tilde{g}_1Q_1)^2 & (\tilde{g}_1Q_1)^2(\tilde{g}_2Q_2)^2 \\ (\tilde{g}_1Q_1)^2(\tilde{g}_2Q_2)^2 & (\tilde{g}_2Q_2)^2 \end{pmatrix} \tag{3.36}$$

In more detail

$$M^2 = \frac{\nu^2}{2} \begin{pmatrix} \left(\frac{\sqrt{g_1^2+g_2^2}}{2\sqrt{\frac{1}{4}+2c\sin\theta\cos\theta}}Q_1\right)^2 & \left(\frac{\sqrt{g_1^2+g_2^2}}{2\sqrt{\frac{1}{4}+2c\sin\theta\cos\theta}}Q_1\right)\left(\frac{\sqrt{g_1^2+g_2^2}}{2\sqrt{\frac{1}{4}-2c\sin\theta\cos\theta}}Q_2\right) \\ \left(\frac{\sqrt{g_1^2+g_2^2}}{2\sqrt{\frac{1}{4}+2c\sin\theta\cos\theta}}Q_1\right)\left(\frac{\sqrt{g_1^2+g_2^2}}{2\sqrt{\frac{1}{4}-2c\sin\theta\cos\theta}}Q_2\right) & \left(\frac{\sqrt{g_1^2+g_2^2}}{2\sqrt{\frac{1}{4}-2c\sin\theta\cos\theta}}Q_2\right)^2 \end{pmatrix} \tag{3.37}$$

We must consider that the ordinary photon is actually massless. On the other hand, the dark photon should have a mass according to the previously proposed hypothesis. Therefore, we must introduce a new transformation that ensures this:

$$\begin{pmatrix} \chi_1^\mu \\ \chi_2^\mu \end{pmatrix} = \begin{pmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} B_1^\mu \\ B_2^\mu \end{pmatrix} \tag{3.38}$$

where

$$\cos\chi = \frac{1}{N}|\tilde{g}_2 Q_2| \quad , \quad \sin\chi = \frac{1}{N}|\tilde{g}_2 Q_1|$$

and

$$N^2 = (\tilde{g}_2 Q_2)^2 + (\tilde{g}_1 Q_1)^2 \tag{3.39}$$

$$N^2 = \left(\frac{\sqrt{g_1^2 + g_2^2}}{2\sqrt{\frac{1}{4} + 2c\sin\theta\cos\theta}} Q_1 \right)^2 + \left(\frac{\sqrt{g_1^2 + g_2^2}}{2\sqrt{\frac{1}{4} - 2c\sin\theta\cos\theta}} Q_2 \right)^2 \tag{3.40}$$

Thus, we obtain the gauge masses for the dark photon m_2 and the ordinary photon m_1 as follows:

$$m_1^2 = 0 \quad , \quad m_2^2 = N^2 v^2$$

Thus, the intrection lagrangian become

$$\mathcal{L}_{\text{int}} = \begin{pmatrix} J_1^\mu & J_2^\mu \end{pmatrix} \begin{pmatrix} \tilde{g}_1 & 0 \\ 0 & \tilde{g}_2 \end{pmatrix} \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} \chi_1^\mu \\ \chi_2^\mu \end{pmatrix} \tag{3.41}$$

By definition α_1 and α_2 as :

$$\alpha_1 = \frac{Q_2}{\sqrt{Q_1^2 + Q_2^2}} \quad , \quad \alpha_2 = \frac{Q_1}{\sqrt{Q_1^2 + Q_2^2}}$$

we can write the mass of the dark photon

$$m_2^2 = \frac{(g_1^2 + g_2^2)(Q_1^2 + Q_2^2)}{8} \left(\frac{a_1^2}{\lambda_1} + \frac{a_2^2}{\lambda_2} \right) v^2 \tag{3.42}$$

3.2.3. The coupling constants

In this section, we will study the coupling constants of the ordinary photon and the dark photon with the rest of the fermions. We recall the interaction Lagrangian for the ordinary photon and the dark photon:

$$\mathcal{L}_{\text{int}} = \sum_{i,j} g_{ij} J_\mu^i A_i^\mu \tag{3.43}$$

we rewrite equation (3.41) as follows:

$$\begin{aligned} \mathcal{L}_{\text{int}} &= \left[(\tilde{g}_1 J_1^\mu \cos\chi - \tilde{g}_2 J_2^\mu \sin\chi)\, X_{1\mu} + (\tilde{g}_1 J_1^\mu \sin\chi + \tilde{g}_2 J_2^\mu \cos\chi)\, X_{2\mu} \right] \\ &= \left[\left(\frac{\sqrt{g_1^2 + g_2^2}}{2} J_1^\mu \cos\chi - \frac{\sqrt{g_1^2 + g_2^2}}{2\left(\frac{1}{4} + 2c\sin\theta\cos\theta\right)} J_2^\mu \sin\chi \right) X_{1\mu} \right. \\ &\quad \left. + \left(\frac{\sqrt{g_1^2 + g_2^2}}{2} J_1^\mu \sin\chi + \frac{\sqrt{g_1^2 + g_2^2}}{2\left(\frac{1}{4} - 2c\sin\theta\cos\theta\right)} J_2^\mu \cos\chi \right) X_{2\mu} \right] \end{aligned} \tag{3.44}$$

We define the fermionic currents $(\hat{J}^1_\mu)$ and $(\hat{J}^1_\mu)$ as follows:

$$\hat{J}^1_\mu = Q^f \bar{\Psi}\gamma^\mu\Psi = J^\mu_1\alpha_1 + J^\mu_2\alpha_2 \quad , \qquad \hat{J}^2_\mu = Q'^f \bar{\Psi}\gamma^\mu\Psi = -J^\mu_1\alpha_2 + J^\mu_2\alpha_1$$

By substituting into the equation(3.44),we get:

$$\mathcal{L}_{\text{int}} = \mathcal{L}_{X_1} + \mathcal{L}_{X_2} \tag{3.45}$$

where

$$\mathcal{L}_{X_1} = \frac{\tilde{g}_1\tilde{g}_2}{\sqrt{\tilde{g}_1^2\alpha_2^2 + \tilde{g}_2^2\alpha_1^2}}\hat{J}^1_\mu X_{1\mu} = g_{11}\hat{J}^1_\mu X_{1\mu} \tag{3.46}$$

$$\mathcal{L}_{X_2} = \frac{1}{\sqrt{\tilde{g}_1^2\alpha_2^2 + \tilde{g}_2^2\alpha_1^2}}\left(-\alpha_1\alpha_2(\tilde{g}_1^2 - \tilde{g}_2^2)\hat{J}^1_\mu + (\alpha_2^2\tilde{g}_1^2 + \alpha_1^2\tilde{g}_2^2)\hat{J}^2_\mu\right)X_{2\mu} = g_{21}\hat{J}^1_\mu X_{1\mu} + g_{22}\hat{J}^1_\mu X_{1\mu} \tag{3.47}$$

Here, we can distinguish the coupling constants ,g_{11}, g_{12},g_{21} andg_{22} as follows:

$$g_{11} = \frac{\tilde{g}_1\tilde{g}_2}{\sqrt{\tilde{g}_1^2\alpha_2^2 + \tilde{g}_2^2\alpha_1^2}} \quad , \quad g_{21} = 0 \tag{3.48}$$

$$g_{12} = \frac{\alpha_1\alpha_2(\tilde{g}_1^2 - g_2^1)}{\sqrt{\tilde{g}_1^2\alpha_2^2 + \tilde{g}_2^2\alpha_1^2}} \quad , \quad g_{22} = \sqrt{\tilde{g}_1^2\alpha_2^2 + \tilde{g}_2^2\alpha_1^2} \tag{3.49}$$

3.2.4. Numerical Study of the Mass of Boson Z'

Using numerical applications, we can propose some expected results for the mass of the dark photon by studying the ratio as a function $\beta = \frac{m_d}{m_t}$,where m_t here is the mass of top the quark(172 Gev) of the real variable c, for different values of the angle θ and the variable xi, where

$$\xi = \lambda_1\alpha_1^2 + \lambda_2\alpha_2^2 \tag{3.50}$$

and we know that $g_{11} = e$, (e is Electromagnetic Coupling Constant) so we can also write

$$\xi = \frac{1}{8}\left(\frac{g_1^2 + g_2^2}{e^2}\right) \tag{3.51}$$

by those equations we can write (3.42) as

$$m_2^2 = \frac{e^2\xi^2(Q_1^2 + Q_2^2)}{\lambda_1\lambda_2}\nu^2 \tag{3.52}$$

where $e = \sqrt{4\pi\alpha_{EM}}$ and $\lambda_1\lambda_2 = (\frac{1}{16} - 4c^2\sin\theta\cos\theta)$

$$\lambda_1\lambda_2 = (\frac{1}{16} - 4c^2\sin^2\theta\cos^2\theta) > 0$$

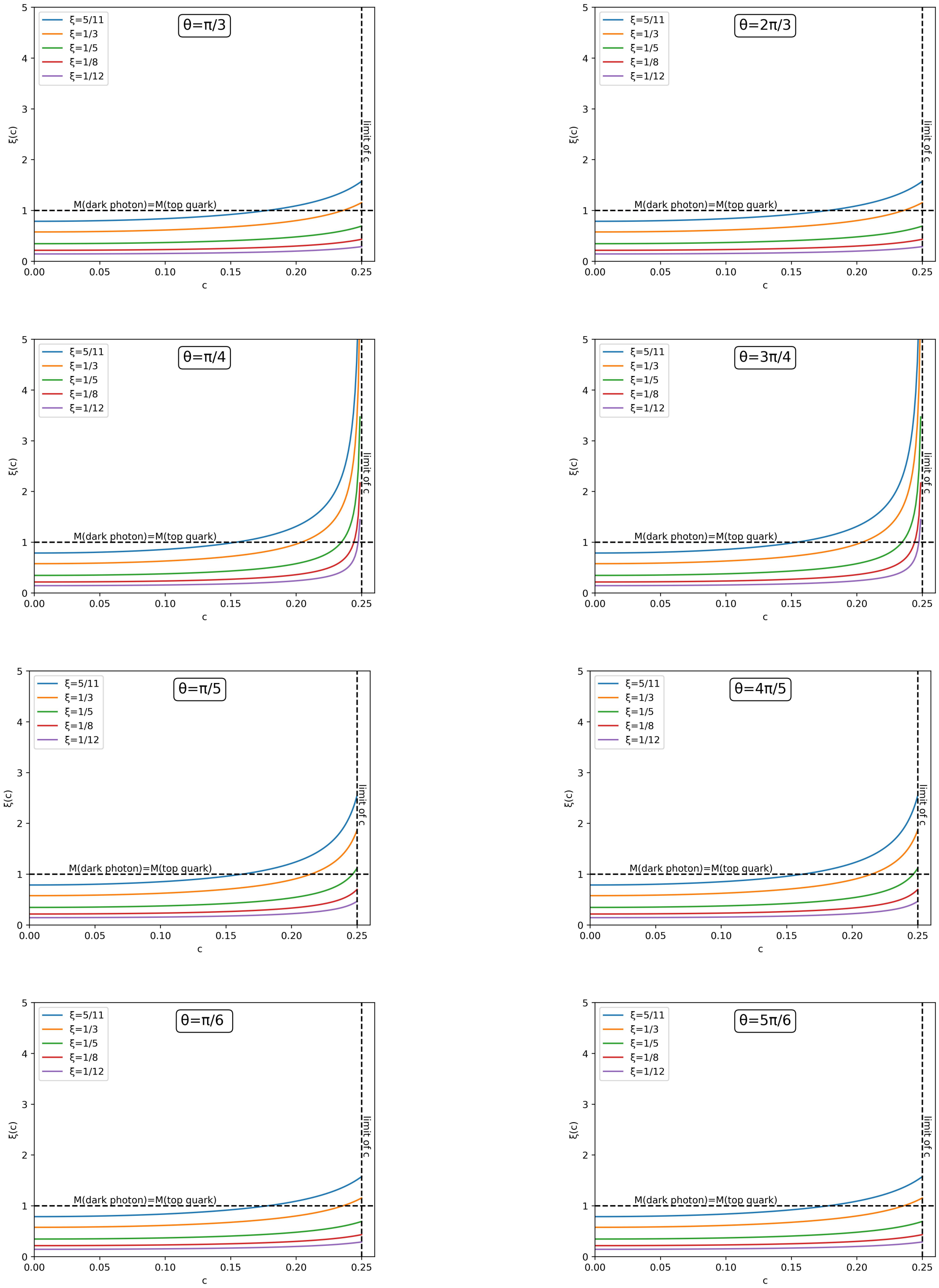

Figure 3.1: Illustrative curves of the Higgs mass as a function of ξ at possible values of θ and ξ.[generated by Jupyter Notebook]

$$c < \left| \frac{1}{8 \sin\theta \cos\theta} \right|$$

On the other hand, we also will choose different values for ξ,taking into account the allowed conditions and values, which we will present as follows:

$$0 < \xi < \lambda_1 + \lambda_2 = 1/2$$

and $Q_1^2 + Q_2^2 = 1$
we will get the results in Figure (3.1)

From the results obtained in the curves, it can be observed that, unlike what was previously obtained, the mass of the new boson can exceed the mass of the top quark. The angle θ plays an important role in determining the possible mass values over the allowed range for the variable c. The function diverges towards infinity at an angle of $\theta_{\text{max}} = n\frac{\pi}{4}$, allowing the Z' boson to achieve a large mass as c approaches $\frac{1}{4}$. Angles close to the angle θ_{max} also allow for relatively large masses, but there is a ceiling for the mass, and this ceiling decreases as we try values further towards angles of $\theta_{\text{min}} = n\frac{\pi}{2}$, where the mass becomes dependent only on the variable ξ, as can be predicted from the equation.

Conclusion

In this thesis, we reviewed the results of adding a new gauge boson to the Standard Model directly within its general framework. As a clear outcome, it is impossible to obtain a gauge boson with a mass exceeding that of the top quark in any manner. However, this can be achieved by considering the new boson as a mediator for dark matter interactions, which indirectly interacts with ordinary matter particles due to the mixing term between it and the photon. On one hand, this memo provides a brief overview of the use of non-commutative geometry in the study of elementary particles, which offers a flexible and more general approach to additions and modifications to the Standard Model compared to the direct method. On the other hand, it paves the way for the unification of the four fundamental forces into a single model,However, there are some issues associated with it, including mathematical complexity and the difficulty in finding an experimental approach to validate it. Additionally, there are discrepancies in some of the results it predicts, such as the mass of the Higgs boson.

Bibliography

[1] Thomas Krajewski, Igor Pris,Towards a Z' Gauge Boson in Noncommutative Geometry,arXiv:hep-th/9607005

[2] C. Alvarez-Luna and J.A.R.Cembranos,"Dark photon searches with atomic transitions",Published for SISSA by Springer, JHEP07(2019)110, https://doi.org/10.1007/JHEP07(2019)110

[3] TAHMI Ouahiba, Boussahel Mounir,Extension U(1) du modèle standard et masse du dark photon $<$ cas d'un seul doublet de Higgs $>$

[4] The Particle Data Group Review of Particle Properties, Phys. Rev. D54 (1996) 1

[5] E. Elbaz, De F électromagnétisme à F électrofaible, Ellipses, 1989

[6] A. Connes, Essay on physics and noncommutative geometry, in The Interface of Mathematics and Particle Physics, eds.: D. G. Quillen et al., Clarendon Press (1990) A. Connes , J. Lott,The metric aspect of noncommutative geometry, in the proceedings of the 1991 Cargese Summer Conference, eds.: J. Frohlich and all ,Plenum Press (1992)

[7] A. Connes, The action functional in non-commutative geometry, Comm. Math. Phys. 117 (1988) 673

[8] Connes A.: Noncommutative geometry and reality, J. Math. Phys. 36 (1995).

[9] Connes A.: Gravity coupled with matter and the foundation of non commutative geometry, hep-th/9603053 (1996).

[10] A. Connes, Noncommutative Geometry, Academic Press (1994)

[11] T.SCHUCKER"Geometries and Forces"(1997)

[12] Biswajoy Brahmachari, Amitava Raychaudhuri," Kinetic mixing and symmetry breaking dependent interactions of the dark photon", Nuclear Physics B, Volume 887, 2014, pp 441- 455.http://dx.doi.org/10.1016/j.nuclphysb.2014.08.015

[13] S. Weinberg, "A Model of Leptons," Phys. Rev. Lett. 19, 1264 - 1266 (1967).

[14] H. Georgi and S. L. Glashow, "Gauge Theory of Weak and Electromagnetic Interactions with Han-Nambu Quarks," Phys. Rev. D7, 561 - 563 (1973).

[15] H. Georgi and S. L. Glashow, "Gauge Theory of Weak and Electromagnetic Interactions with HanNambu Quarks," Phys. Rev. D7, 561 - 563 (1973)

[16] Berkane, Amina, and Mounir Boussahel."Dark Photon As an Extra U (1) Extension to the Standard Model With general Rotation in Kinetic Mixing" arXiv preprint arXiv:2101.03689 (2021).

[17] D. Kastler, A detailed account of Alain Connes' version of the standard model in non-commutative geometry, I and II, Rev. Math. Phys. 5 (1993) 477
D. Kastler, A detailed account of Alain Connes' version of the standard model in non-commutative geometry, III, Rev. Math. Phys. 8 (1996) 103
D. Kastler, T. Sch¨ucker, A detailed account of Alain Connes' version of the standard model in non-commutative geometry, IV, Rev. Math. Phys., 8 (1996) 205
D. Kastler, M. Mebkhout, Lectures on Non-Commutative Differential Geometry, World Scientific, to be published
J. C. V´arilly , J. M. Gracia-Bond´ıa, Connes' noncommutative differential geometry and the standard model, J. Geom. Phys. 12 (1993) 223
C. P. Martın, J. M. Gracia-Bondıa , J. C. Varilly, The standard model as a noncommutative geometry: the low mass regime, hep-th/9605001, Phys. Rep., to appear
D. Kastler , T. Sch¨ucker, The standard model a la Connes-Lott, hep-th/9412185, J. Geom. Phys. 388 (1996) 1
L. Carminati, B. Iochum ,T. Sch¨ucker, The noncommutative constraints on the standard model 'a la Connes, hep-th/9604169, J. Math. Phys. 38 (1997) 1269
L. Carminati, B. Iochum , T. Schucker, Noncommutative Yang-Mills and noncommutative relativity: A bridge over troubled water, hep-th/9706105

[18] Biswajoy Brahmachari, Amitava Raychaudhuri," Kinetic mixing and symmetry breaking dependent interactions of the dark photon", Nuclear Physics B, Volume 887, 2014, pp 441- 455.http://dx.doi.org/10.1016/j.nuclphysb.2014.08.01

[19] Ghaoui, Jade, et al." Brisure de symétrie et mécanisme de Higgs." (2013). Margarete Muhlleitner, Marco O. P. Sampaio, Rui Santos, Jonas Wittbrodit, The N2HDM under Theoretical and Experimental Scrutiny, KA-TP-39-2016.

[20] Del Aguila, F., M. Masip, and M. Perez-Victoria. "Physical parameters and renormalization of$U_a(1) \otimes U_b(1)$ models." Nuclear Physics B 456.3 (1995): 531-549

[21] A. Salam and J. C. Ward, "Gauge Theory of Elementary Interactions," Phys. Rev. 136, B763 - B768 (1964)

[22] W.N. COTTINGHAM and D.A.GREEWOOD, An introduction to the standard model of particle physics, GRAMBRIDGE, 2007.

[23] Iochum B. and Sch¨ucker T.: Yang-Mills-Higgs versus Connes-Lott, Comm. Math. Phys. 178 (1996), 1-26.

[24] Pris I. and Schucker T.: Non-commutative geometry beyond the Standard Model, preprint CPT-96/P3335, hep-th/9604115 (1996).

[25] Krajewski T.: Classification of finite spectral triples, (in preparation)

[26] Sch¨ucker T. and Zylinski J.-M.; Connes' model building kit, J. Geom. Phys. 16 (1994)

[27] Lizzi F., Mangano G., Miele G. and Sparano G.: Constraints on Unified Gauge Theories from Noncommutative Geometry, hep-th/9603095 (1996).