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Relationship between T and ∇T for other types of summability

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Dedication

I dedicate this work to :

Myself, the perseverant and resilient.

The first ones who encouraged me to pursue this specialization, my mother and father, may God bless you for me.

My family and my friends.

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Finally, I extend my thanks to my parents and my entire family.

Contents

List of Symbols

$\mathbb{K}$	The field of real or complex numbers($\mathbb{R}$ ou $\mathbb{C}$)
p^*	The conjugate of the number p $(1 \leq p \leq \infty)$, that is $\frac{1}{p} + \frac{1}{p^*} = 1$
B_X	The closed unit ball of X
X^*	The topological dual of X
$Lip_0(X, Y)$	The set of all Lipschitz operators between X and Y that vanish at 0
$X^\# = Lip_0(X, \mathbb{R})$	The Lipschitz dual of the pointed metric space X
$(X, \\| \cdot \\|)$	Normed vector space
L_p	Lebesgue spaces
$C(K)$	The space of continuous functions from $\mathbb{K}$ to $\mathbb{R}$
$l_p(X)$, (resp).$l_p^n(X)$	The space of sequences (resp. finite) weakly p-summing
$L(X, Y)$	The set of all linear operators
$\mathcal{B}(X, Y)$	The set of all continuous linear operators

$SL(X,Y)$	The set of all sublinear operators
$\mathcal{SB}(X,Y)$	The set of all bounded sublinear operators
Π_p	The class of all sublinear p-summing operators $(1 \leq p \leq \infty)$
Π_p^+	The class of all positive p-summing sublinear operators $(1 \leq p \leq \infty)$
Π_p^L	The class of all Lipschitz p-summing sublinear operators $(1 \leq p \leq \infty)$
Π_p^{Ls}	The class of all super Lipschitz p-summing sublinear operators $(1 \leq p \leq \infty)$
D_p^L	The set of all Lipschitz p-dominated operators $(1 \leq p \leq \infty)$
$D_{st.p}^L$	The set of all Lipschitz strongly p-summing operators $(1 \leq p \leq \infty)$
D_p	The set of all strongly p-summing operators $(1 \leq p \leq \infty)$
D_p^+	The set of all positive strongly p-summing operators $(1 \leq p \leq \infty)$

Introduction

Sublinear operators have been a heavily studied mathematical object , particularly in the fields of interpolation and factorizations in all their forms. Grafakos, Defant, Garcia Cureva, and others have made significant contributions to this notion. In this memory, we will focus on sublinear operators T and linear operators $\leq T$. We will attempt to introduce, compare, and generalize classical or universal results from linear to sublinear settings.

In this study, we explore the collection of bounded sublinear operators, which forms a positive cone within $\mathrm{Lip}_0(X, Y)$. We focus on the the following situation where X represents a Banach space, Y is a complete Banach lattice, and T denotes a bounded sublinear operator from X to Y (i.e., it is positively homogeneous and subadditive). We denote ∇T as the set of u in $\mathcal{B}(X, Y)$ (the Banach space comprising all bounded linear operators from X to Y) such that $u \leq T$. Our objective is to examine the relationship between T and its subdifferential ∇T concerning the concept of Lipschitz p-summing and other classes of summability.

This work is divided into three chapters .

In Chapter 1, we provide a general overview of Banach lattices, sublinear operators and some useful properties. We also present some recent results related to this class of operators.

In Chapter 2, we introduce the concept of p-summing sublinear operators for $1 < p \leq \infty$, the Lipschitz p-summing sublinear operators and we introduce the class of Lipschitz super p-summing sublinear operators. Conclude this chapter by presenting a relationship between these summabilities.

In Chapter 3, we study the relational problem between p-summing sublinear operators from a Banach space X to a Banach lattice space Y and u in ∇T . In other words,

$$T \text{ is } p\text{-summing } \iff \forall\, u \text{ in } \nabla T, u \text{ is } p\text{-summing } ?.$$

It is shown that the first implication holds true easily and without difficulties by Theorem 3.1.1 and we answer negatively regarding the inverse implication .We look at the other types of summability. We are interested to the notion of Lipschitz p-dominated operators . We show that if T is Lipschitz p-dominated sublinear operator, then $\nabla T \subset \Pi_p(X, Y)$. We prove that if T is in $\mathcal{D}_p^L(X, Y)$ (the space of Lipschitz strongly p-summing operators) for $1 < p \leq \infty$, then u is positive strongly p-summing for all u in ∇T and hence u^* is positive p^*-summing with $\pi_{p^*}^+(u^*) \leq 2d_p^L(T)$. We will conclude this chapter by studying in section four some relationships between the strongly p-summing sublinear operators T and the linear operators $u \in \nabla T$.

CHAPTER 1

BANACH LATTICES AND SUBLINEAR OPERATORS

We are going to introduce some terminology concerning Banach lattices. For more details, the interested reader can consult the references [8, 10].

1.1 Banach lattices

Definition 1.1.1. A partially ordered vector space (abbreviated (POVS)) is a vector space X equipped with a partial order that is compatible with its vector structure in the sense that

1. $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in X$.

2. $x \geq 0$ implies $\alpha x \geq 0$ for any $x \in X$ and $\alpha \geq 0$.

We denote by the set $X^+ = \{x \in X; x \geq 0\}$ is referred to as the positive cone of X.

Definition 1.1.2. A convex cone in a vector space X is a set C characterized by the properties.

1. $C + C \subset C$.

2. $\alpha C \subset C, \ \ $ for any $\alpha \geq 0$.

3. $C \cap (-C) = \{0\}$.

Definition 1.1.3. (Lattice space)[15]. A space E is a lattice space if it is a POVS and, at the same time, a lattice under its ordering.

Proposition 1.1.4. *For an element x in a lattice space X, we can define its positive and negative part, and its absolute value, respectively, by*

$$x^+ = \sup\{x, 0\}, \quad x^- = \sup\{-x, 0\}, \quad |x| = \sup\{x, -x\}.$$

Proposition 1.1.5. *For arbitrary elements a, b, c of a lattice space E, the following identities hold.*

1. $a + b = \sup\{a, b\} + \inf\{a, b\}$,
2. $a + \sup\{b, c\} = \sup\{a + b, a + c\}$ *and* $a + \inf\{b, c\} = \inf\{a + b, a + c\}$,
3. $\sup\{a, b\} = -\inf\{-a, -b\}$ *and* $\inf\{a, b\} = -\sup\{-a, -b\}$,
4. $\alpha \sup\{a, b\} = \sup\{\alpha a, \alpha b\}$ *and* $\alpha \inf\{a, b\} = \inf\{\alpha a, \alpha b\}$ *for* $\alpha \geq 0$.

Proposition 1.1.6. *If x is an element of a lattice space E, then*

$$x = x_+ - x_-, \quad |x| = x_+ + x_-$$

Thus, in particular the positive cone C in a lattice space is generating,i, e, $E = C - C$

Proof. By Proposition 1.1.5 (1) and (3), we have

$$x = x + 0 = \sup\{x, 0\} + \inf\{x, 0\} = \sup\{x, 0\} - \sup\{-x, 0\} = x_+ - x_-.$$

Furthermore, from Theorem 1.1.5(2) and (4), and the previous result, we get

$$\begin{aligned} |x| = \sup\{x, -x\} &= \sup\{2x, 0\} - x \\ &= 2\sup\{x, 0\} - x = 2x_+ - x \\ &= 2x_+ - (x_+ - x_-) \\ &= x_+ + x_-. \end{aligned}$$

This completes the proof. □

Definition 1.1.7. A Banach lattice X is a Banach space with a lattice order, such that for all $x, y \in X$, the implication holds

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\| \tag{1.1}$$

.

The space X is a complete Banach lattice if By 1.1, we have the important identity

$$\|x\| = \||x|\|, x \in X. \tag{1.2}$$

Indeed, this follows as taking first x and $y = |x|$, we have $|x| \leq |(|x|)|$ and hence $\|x\| \leq \||x|\|$. On taking $|x|$ and $y = x$, we also have $|(|x|)| \leq |x|$ and hence $\||x|\| \leq \|x\|$.

Remarks 1.1.8. [8, Proposition 1.a.2]

(1) The dual X^* of a complete Banach lattice with the natural order

$$x_1^* \leq x_2^* \iff \langle x_1^*, x\rangle \leq \langle x_2^*, x\rangle, \forall x \in X_+. \tag{1.3}$$

(2) Every reflexive Banach lattice space is reflexive a complete Banach lattice space.

Example 1.1.9. We provide some examples of Banach Lattices.

(1) The spaces $L^p(1 \leq p \leq \infty)$ are complete Banach lattices .

(2) The $C(K)$ is a Banach lattice,is a complete Banach lattice if K is stonia

1.2 Sublinear operators

We will now study the class of sublinear operators that are positively homogeneous and subadditive and briefly giving some properties.For more details on sublinear operators, see [3, **?**]. Subsequently, we are only interested in Banach lattices.

Definition 1.2.1. A mapping T from a Banach space X into a Banach lattice Y is said to be sublinear if for all $x, y \in X$ and $\alpha \in \mathbb{R}^+$, we have

1. $T(\alpha x) = \alpha T(x)$ (i.e., positively homogeneous),

2. $T(x + y) \leq T(x) + T(y)$ (i.e., subadditive).

Note that the sum of two sublinear operators is a sublinear operator, and the multiplication by a positive number is also a sublinear operator. Let us denote by

$$SL(X;Y) = \{\text{sublinear operators } T : X \to Y\}.$$

We denote by

$$L(X;Y) = \{\text{linear mappings } u : X \to Y\}$$

if X is a Banach lattice space then we say that u is positive if $u(x) \geq 0$,$\forall x \geq 0$

$$SL(X;Y) = \{\text{sublinear operators } T : X \to Y\}$$

and we equip it with the natural order induced by Y

$$T_1 \leq T_2 \iff T_1(x) \leq T_2(x), \forall x \in X \tag{1.4}$$

also by

$$\nabla T = \{u \in L(X,Y) : u \leq T \text{ (i.e., } \forall x \in X, u(x) \leq T(x))\}$$

As an immediate consequence

$$u \leq T \quad \iff \quad -T(-x) \leq u(x) \leq T(x), \quad \forall x \in X \tag{1.5}$$

because

$$\begin{aligned} u \leq T \iff & \quad u(x) \leq T(x), \quad \forall x \in X, \\ \iff & u(-x) \leq T(-x), \forall x \in X, \\ \iff & -u(x) \leq T(-x), \forall x \in X, \\ \iff & u(x) \geq -T(-x), \forall x \in X. \end{aligned}$$

Proposition 1.2.2. *Let $T : X \to Y$ be a sublinear operator. We denote by then, for every x in X, there exists u_x in ∇T such that*

$$T(x) = u_x(x) \quad (\textit{i.e}, T(x) = \sup_{u \in \nabla T} u(x)).$$

Definition 1.2.3. Let $T \in \mathrm{SL}(X;Y)$. We say that T is symmetric if for every x in X, $T(x) = T(-x)$, and if X is a lattice, T is increasing if for every x, y in X^+

$$x \leq y \Rightarrow T(x) \leq T(y).$$

Remark 1.2.4. 1. Let T be a symmetric sublinear operator between Banach lattices spaces X and Y. Then, T is positive in the sense of (1.2). Indeed, for any $x \in X$, we have

$$\begin{aligned} 0 &= T(x-x) \\ &\leq T(x) + T(-x) \\ &\leq T(x) + T(x) \\ &\leq 2T(x). \end{aligned}$$

The converse is false even if T is increasing.

2. It is also known that symmetry does not imply monotonicity.

1.2.1 Essential properties

If u is linear and positive, then u is increasing and (conversely) . Indeed, let u be a linear and positive operator for $x, y \in X$ such that $x \geq y$, we have

$$x - y \geq 0 \implies u(x-y) \geq 0$$

Because

$$u(x-y) = u(x) - u(y) \geq 0 \implies u(x) \geq u(y).$$

So, u is increasing.

Proposition 1.2.5. *Let X, Y ,Z be Banach spaces such that Y, Z are Banach lattices.*

1. *For all $T \in SL(X;Y)$ and for all $u \in L(Y;Z)$ positive $\Longrightarrow u \circ T \in SL(X;Z)$.*

2. *For all $u \in L(X;Y)$ and for all $T \in SL(Y;Z) \Longrightarrow T \circ u \in SL(X;Z)$.*

Proof. 1. Let x, y be in X. Then

$$\begin{aligned} u \circ T(x+y) &= u(T(x+y)) \\ &\leq u(T(x)+T(y)) \quad \text{(because } u \text{ is positive)} \\ &\leq u \circ T(x) + u \circ T(y). \end{aligned}$$

Let $\lambda \in \mathbb{R}^+$ and $x \in X$

$$\begin{aligned} uoT(\lambda x) = u(T(\lambda x)) &= u(\lambda T(x)) \\ &= \lambda (uoT)(x) \end{aligned}$$

So

$$uoT \in SL(X, Z).$$

2. Let x, y be in X

$$\begin{aligned} Tou(x+y) &= T(u(x+y)) \\ &= T(u(x)+u(y)) \\ &\leq Tou(x) + Tou(y) \end{aligned}$$

Let $\lambda \in \mathbb{R}^+$ and $x \in X$

$$Tou(\lambda x) = T(\lambda u(x)) = \lambda Tou(x).$$

So

$$Tou \in SL(X, Z).$$

This completes the proof. □

Proposition 1.2.6. *Let T be a sublinear from a Banach space X into a Banach lattice Y. Then, the following properties are equivalent.*

1. *The operator T is continuous on X,*

2. *The operator T is continuous in 0,*

3. *There is a constant $C > 0$ such that for all $x \in X$, $\|T(x)\| \leq C\|x\|$.*

In this case, we say that T is bounded and we define

$$\|T\| = \sup\{\|T(x)\| : x \in B_X \text{ ou } \|x\| = 1\}.$$

Proof. The proof is like the linear case.

1. For (1) implies (2) (obvious)

2. For (2) implies (3)

   Let T be a continuous sublinear operator in 0, therefore

   $$\forall \varepsilon > 0 : \forall (x \neq 0) \in X; \|x\| \leq \varepsilon \Longrightarrow \|T(x)\| \leq 1.$$

   We define $y = \frac{x}{\|x\|}\varepsilon$,then

   $$\|T(y)\| \leq 1$$

   and therefore

   $$\|T(x)\| \leq \frac{1}{\varepsilon}\|x\|.$$

   so (3) .

3. For (3) implies (1)

   Let's suppose that there exists $C > 0$ such that

   $$\forall x \in X : \|T(x)\| \leq C\|x\|.$$

   Let x_0 be in X. We have

   $$T(x) = T(x + x_0 - x_0) \leq T(x - x_0) + T(x_0) \Longrightarrow T(x) - T(x_0) \leq T(x - x_0)$$

   and

   $$T(x_0) = T(x_0 + x - x) \leq T(x_0 - x) + T(x)) \Longrightarrow T(x_0) - T(x) \leq T(x_0 - x)$$

   So for every x in X we have by taking sup

   $$\begin{aligned} |T(x) - T(x_0)| &\leq \sup\{T(x - x_0), T(x_0 - x)\} \\ &\leq |T(x - x_0)| + |T(x_0 - x)| \end{aligned}$$

   and hence

   $$\begin{aligned} \||T(x) - T(x_0)|\| &= \|T(x) - T(x_0)\| \\ &\leq \|T(x - x_0)\| + \|T(x_0 - x)\| \\ &\leq 2C\|x - x_0\|. \end{aligned} \tag{1.6}$$

So T is lipshitzienne and hence continuous. □

1.2.2 Relationships between linear operators and sublinear operators

The following theorem establishes a direct link between linear operators and sublinear operators.

Theorem 1.2.7. *[?] Let X and Y be two Banach spaces with Y being completely lattice , and let $T : X \to Y$ be a continuous sublinear operator. Then,*

$(\mathbf{a})\quad \forall x \in X,\ \|T(x)\| \leq\ \sup_{u\in\nabla T}\ \|u(x)\| \leq \|T(x)\| + \|T(-x)\|\ .$

$(\mathbf{b})\quad \|T\| \leq \sup_{u\in\nabla T} \|u\| \leq 2\|T\|.$

Proof. (a) Let x be $\in X$.On one hand ,

$$\|T(x)\| = \|u_x(x)\| \leq \sup_{u\in\nabla T} \|u(x)\|.$$

And on the other hand,

$$\begin{aligned} |u(x)| &\leqslant \sup\{|T(x)|, |T(-x)|\} \\ &\leqslant |T(x)| + |T(-x)|. \end{aligned}$$

This implies

$$\begin{aligned} \|u(x)\| &\leq \||T(x)|\| + \||T(-x)|\| \\ &\leq \|T(x)\| + \|T(-x)\|. \end{aligned} \tag{1.7}$$

Hence

$$\sup_{u\in\nabla T}\ \|u(x)\| \leq \|T(x)\| + \|T(-x)\|\ .$$

(b) Using (a) , we find

$$\begin{aligned} \|T\| &\leq \sup_{u\in\nabla T}\ \|u\| \\ &\leq \|T(x)\| + \|T(-x)\| \\ &\leq \|T(x)\| + \|T(x)\| \\ &\leq 2\|T\| \end{aligned}$$

and this preuves this theorem. □

Corollary 1.2.8. *As an immediate consequence .*

1. *T is continuous $\Leftrightarrow$ For all $u \in \nabla T$ u, is continuous.*

2. *If T is symmetric, we will have*

$$\begin{cases} (\mathbf{1}) \qquad \|T(x)\| &= \sup_{u\in\nabla T} \|u(x)\|, \forall x \in X. \\ (\mathbf{2}) \qquad \|T\| = & \sup_{u\in\nabla T} \|u\|. \end{cases}$$

1.3 Lipschitz spaces

1.3.1 Lipschitz functions

Definition 1.3.1. A map $f : (X, \mathrm{d}_X) \longrightarrow (Y, \mathrm{d}_Y)$ between two metric spaces is called Lipschitz if there is a positive constant C such that

$$\forall x, y \in X, \quad \mathrm{d}_Y(f(x), f(y)) \leq C\mathrm{d}_X(x, y).$$

$If\ C = 1$, the map is called nonexpansive (and contraction $if\ C < 1$) .

For a Lipschitz map f, we define its Lipschitz constant by

$$\|f\|_{\mathrm{Lip}} = \mathrm{Lip}\,(f) := \sup_{x\neq y} \frac{\mathrm{d}_Y(f(x), f(y))}{\mathrm{d}_X(x, y)} =$$
$$\inf\{C : C \text{ verifying the above inequality}\}$$

Let $(X, \mathrm{e}_X, \mathrm{d}_X), (Y, \mathrm{e}_Y, \mathrm{d}_Y)$ be pointed metric spaces. We say a map $f : (X, \mathrm{e}_X, \mathrm{d}_X) \longrightarrow (Y, \mathrm{e}_Y, \mathrm{d}_Y)$ preserves distinguished point if $f(\mathrm{e}_X) = \mathrm{e}_Y$.

Proposition 1.3.2. *Let X, Y, and Z be metric spaces and let $f : (X, d_X) \to (Y, d_Y)$, $g : (Y, d_Y) \to (Z, d_Z)$ be Lipschitz maps. Then $g \circ f : (X, d_X) \to (Z, d_Z)$ is Lipschitz, and Lip$(g \circ f) \leq$ Lip$(g) \cdot$ Lip(f).*

Proof. For $x, y \in X$, we have

$$\begin{aligned} d_Z(g \circ f(x), g \circ f(y)) &\leq \mathrm{Lip}(g)\, d_Y(f(x), f(y)) \\ &\leq \mathrm{Lip}(g)\, \mathrm{Lip}(f)\, d_X(x, y), \end{aligned}$$

and this shows the proposition. □

Proposition 1.3.3. *Let X_0 and Y_0 be metric spaces and let X and Y be their completions. Consider a Lipschitz map $f_0 : X_0 \to Y_0$. Then f_0 has a unique Lipschitz extension $f : X \to Y$, and furthermore Lip$(f) =$ Lip(f_0).*

1.3.2 Lipschitz spaces

Definition 1.3.4. (a) Let (X, d) be a metric space. Then $Lip(X)$ is the space of all bounded scalar valued Lipschitz functions on X with the norm

$$\|f\|_L = \max\{\|f\|_\infty, \mathrm{Lip}(f)\}.$$

(b) We denote by $Lip_0(X)$ the space of all bounded scalar valued Lipschitz mappings on X, vanishing at e with the norm

$$\mathrm{Lip}(f) := \sup_{x\neq y} \frac{\mathrm{d}_Y(f(x), f(y))}{\mathrm{d}_X(x, y)}.$$

If E is a Banach space, $\mathrm{Lip}_0(X; E)$ is a Banach space under the Lipschitz norm given by

$$\mathrm{Lip}(f) = \sup\left\{\frac{\|f(x) - f(y)\|}{d(x, y)}, x \neq y\right\}.$$

For $E = \mathbb{R}$, we designate $\mathrm{Lip}_0(X, \mathbb{R}) = \mathrm{Lip}_0(X) = X^{\#}$. The Banach space $X^{\#}$ is called also Lipschitz dual of X. It has been used by various mathematicians as a framework to extend results from linear functional analysis to the nonlinear case.

Remark 1.3.5. Designed by $B_{X^{\#}}$ the unit ball of $X^{\#}$. Then $B_{X^{\#}}$ is a compact Hausdorff space in the topology of pointwise convergence on X (see [Wea99, Page 39]).

1.3.3 Relationships between sublinear operators and lipschitz functions

Definition 1.3.6. We will denote by $\mathcal{SB}(X, Y)$ the set of all bounded sublinear operators from X into Y.The set $\mathcal{SB}(X, Y)$ is a pointed positive convex cone of $\mathrm{Lip}_0(X, Y)$ but not salient because

$$\mathcal{SB}(X, Y) \cap (-\mathcal{SB}(X, Y)) = \mathcal{B}(X, Y).$$

Remark 1.3.7. We have by Proposition 1.2.6, for $T \in \mathcal{SB}(X, Y)$

$$\|T\| \leq \mathrm{Lip}(T) \leq 2\|T\|.$$

In addition if T is symmetric (i.e., $T(x) = T(-x)$ for all x in X), then

$$\|T\| = \mathrm{Lip}(T).$$

Remark 1.3.8. we can see $\mathcal{SB}(X,Y)$ as a cone in $(\mathrm{Lip}_0(X,Y), \mathrm{Lip}(.))$. We denote by

$$\triangle\mathcal{SB}(X,Y) = \mathcal{SB}(X,Y) - \mathcal{SB}(X,Y)$$

In the sequel, we can see $\mathcal{SB}(X,Y)$ as a cone in $(\mathrm{Lip}_0(X,Y), \mathrm{Lip}(.))$. We denote by $\triangle\mathcal{SB}(X,Y)$ the subspace of $\mathrm{Lip}_0(X,Y)$ spanned by $\mathcal{SB}(X,Y)$, i.e.

$$\triangle\mathcal{SB}(X,Y) = \{T_1 - T_2 : T_1, T_2 \in \mathcal{SB}(X,Y)\}.$$

CHAPTER 2

ON DIFFERENT SUMMABILITY OF OPERATORS

2.1 Banach space of sequences

Let's begin this section with some necessary preliminaries on the definitions of p-summing sequence spaces,which are $l_p(x), X(l_p^w), l_{p,w}(X)$. For more informations in this chapter , we can see [1] .

Definition 2.1.1. Let X be a Banach lattice and let $1 \leq p \leq \infty$. We denote by $l_p(X)$ (resp. $l_p^n(X)$) the Banach lattice of sequences $(x_i)_{i=1}^{\infty} \subset \mathbb{R}$ (resp. $(x_i)_{i=1}^{n}$) in X that are absolutely p-summing, equipped with the norm

$$\|(x_i)_{i=1}^{\infty}\|_p = \begin{cases} \left(\sum_{i=1}^{\infty} \|x_i\|^p\right)^{\frac{1}{p}}, \text{ if } 1 \leq p < +\infty, \\ \\ \sup_{i \geq 1} \|x_i\|, \text{ if } p = +\infty. \end{cases}$$

resp.

$$\|(x_i)_{i=1}^{n}\|_p = \begin{cases} \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{\frac{1}{p}}, \text{ if } 1 \leq p < +\infty, \\ \\ \sup_{1 \leq i \leq n} \|x_i\|, \text{ if } p = +\infty. \end{cases}$$

Definition 2.1.2. We denote by $X(l_p^n)$ (resp. $X(l_p^n)$), the space of sequences $(x_i)_{i=1}^{\infty} \subset \mathbb{R}$ (resp.

$(x_i)_{i=1}^n$) of elements in X such that

$$\|(x_i)_{i=1}^{\infty}\|_{X(l_p)} = \begin{cases} \left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\|, \text{ if } 1 \leq p < +\infty, \\ \\ \left\| \sup_{1\leq i\leq n} |x_i| \right\|, \text{ if } p = +\infty. \end{cases}$$

resp.

$$\|(x_i)_{i=1}^{n}\|_{X(l_p^n)} = \begin{cases} \left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\|, \text{ if } 1 \leq p < +\infty, \\ \\ \left\| \sup_{1\leq i\leq n} |x_i| \right\|, \text{ if } p = +\infty. \end{cases}$$

The space $X(l_p^n)$ equipped with the natural order

$$x \leq y \Longleftrightarrow x_i \leq y_i, \forall\, 1 \leq i \leq n,$$

is Banach lattice.

Definition 2.1.3. We denote by $l_{p,w}(X)$ (resp. $l_{p,w}^n(X)$) the space of weakly p-summing sequences $(x_i)_{i=1}^{\infty}$ (resp. $(x_i)_{i=1}^{n}$) in X, equipped with the norm

$$\|(x_i)_{i=1}^{\infty}\|_{p,w} = \begin{cases} \sup_{\xi\in B_{X^*}} \left(\sum_{i=1}^{+\infty} |\langle x_i, \xi\rangle|^p \right)^{\frac{1}{p}}, \text{ if } 1 \leq p < +\infty, \\ \\ \sup_{\xi\in B_{X^*}} \sup_{i\geq 1} |\langle x_i, \xi\rangle|, \text{ if } p = +\infty. \end{cases}$$

$$resp.\ \|(x_i)_{i=1}^{n}\|_{p,w} = \begin{cases} \sup_{\xi\in B_{X^*}} \left(\sum_{i=1}^{n} |\langle x_i, \xi\rangle|^p \right)^{\frac{1}{p}}, \text{ if } 1 \leq p < +\infty, \\ \\ \sup_{\varphi\in B_{X^*}} \sup_{1\leq i\leq n} |\langle x_i, \xi\rangle|, \text{ if } p = +\infty. \end{cases}$$

where X^* is the (topological) dual of X. The closed unit ball of X^* is denoted by B_{X^*}.

Proposition 2.1.4. (1) *For $1 \leq p < \infty$, we have $l_{p,w}(X) \subseteq l_p(X)$, and*

$$\|(x_i)_{i=1}^{\infty}\|_p \leq \|(x_i)_{i=1}^{\infty}\|_{p,w} .$$

(2) *If $p = \infty$, we have $\ell_\infty(X) = l_{\infty,w}(X)$, and*

$$\|(x_i)_{i=1}^{\infty}\|_\infty = \|(x_i)_{i=1}^{\infty}\|_{\infty,w} .$$

(3) *$l_p(X) = l_{p,w}(X)$ for all $1 \leq p < \infty$ if and only if dim (X) is finite.*

2.2 p-summing sublinear operators

In this paragraph, we will generalize the concept of p-summing operators introduced by Pietsch [11] to sublinear operators and present its famous dominations /factorizations theorem along with some fundamental properties.

Definition 2.2.1. Let $1 \leq p < \infty$. Let X and Y be two Banach spaces with Y being lattice. A sublinear operator $T : X \to Y$ is called p-summing if there exists a $C > 0$ such that given any $x_1, x_2, \ldots, x_n \in X$ for $1 \leq p < \infty$, we have

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{\xi \in B_X^*} \left(\sum_{i=1}^{n} |\langle x_i, \xi\rangle|^p\right)^{\frac{1}{p}} \tag{2.1}$$

We denote by $\Pi_p(X, Y)$ the class of all p-summing sublinear operators from X into Y and $\pi_p(T) = \inf\{C \mid \text{verifying the inequality (2.1)}\}$. For the definition of positive p-summing, we replace X by X_+ and $\pi_p(T)$ by $\pi_p^+(T)$.

Example 2.2.2. Let K be a compact set and μ a probability measure on K. The canonical operator

$$\begin{array}{rcccl} T & : & C(K) & \longrightarrow & L_p(\mu) \\ & & f & \longmapsto & |f| \end{array}$$

is p-summing and $\pi_p(T) = \mu(K)^{\frac{1}{p}}$.

Proposition 2.2.3. *(Ideal property). Let $T \in \Pi_p(X;Y)$, $u : E \to X$ be a continuous linear map, and $v : Y \to F$ be a continuous positive linear map (where E and F are any two Banach spaces with F being lattice). Then, vTu is p-summing.*

$$\pi_p(vTu) \leq \|v\|\pi_p(T)\|u\| \tag{2.2}$$

Proof. We have that vTu is sublinear according to Proposition.1.2.5. We also have

$$\|wTv(x)\| \leq \|w\| \, \|T(v(x))\| \, , \quad \forall x \in E.$$

Let $n \in \mathbb{N}$ and $\{x_1, ..., x_n\}$ in E then $\{u(x_1), ..., u(x_n)\} \subset X$,

So

$$\begin{aligned}(\sum_1^n \|T(u(x_i))\|^p)^{1/p} \leq & \quad \pi_p(T) \sup_{\xi \in B_{X^*}} (\sum_1^n |\langle u(x_i), \xi\rangle|^p)^{\frac{1}{p}}, \\ \leq & \quad \pi_p(T) \sup_{\xi \in B_{X^*}} (\sum_1^n |\langle x_i, u^*(\xi)\rangle|^p)^{\frac{1}{p}}.\end{aligned}$$

We define $\eta = \frac{v^\star(\xi)}{\|v\|} \in B_{E^\star}$, so

$$(\sum_1^n \|T(u(x_i))\|^p)^{1/p} \leq \pi_p(T)\|u\| \sup_{\eta \in B_{E^*}} (\sum_1^n |\langle x_i, \eta\rangle|^p)^{\frac{1}{p}}$$

Hence

$$(\sum_1^n \|vTu(x_i)\|^p)^{1/p} \leq \|v\| \, \pi_p(T) \, \|u\| \sup_{\eta \in B_{E^*}} (\sum_1^n |\langle x_i, \eta\rangle|^p)^{\frac{1}{p}}$$

and this concludes the proof. □

Corollary 2.2.4. *Let X_0 be a subspace of a Banach space X, and let T be a p-summing sublinear operator from X to Y .then*

$$T/_{X_0} \in \Pi_p (X_0, Y) \text{ and } \pi_p (T/_{X_0}) \leq \pi_p (T) \, .$$

Theorem 2.2.5. *(Inclusion theorem). Let $T \in L(X,Y)$ and $(1 \leq p \leq q < \infty)$, we have*

$$\pi_p(X;Y) \subset \pi_q(X,Y)$$

and

$$\pi_q(T) \leq \pi_p(T).$$

Proof. Let $n \in \mathbb{N}$ and $\{x_1, \ldots, x_n\} \subset X$. If we set

$$z_i = \|T(x_i)\|^{(\frac{q}{p})-1},$$

We will have

$$\|T(x_i)\|^q = \|T(z_i x_i)\|^p.$$

If T is p-summing, we have

$$\begin{aligned} \left(\sum_1^n \|T\left(x_i\right)\|^q\right)^{\frac{1}{p}} &= \left(\sum_1^n \|T\left(z_i x_i\right)\|^p\right)^{\frac{1}{p}}, \\ &\leq \pi_p\left(T\right) \sup_{\xi \in B_{X^*}} \left(\sum_1^n z_i^p \left|< x_i, \xi >\right|^p\right)^{\frac{1}{p}}. \end{aligned}$$

since $p \leq q$ and according to $H\ddot{o}lder's$ inequality ($\frac{1}{(q/p)} + \frac{1}{(q/q-p)} = 1$) , we obtain

$$\begin{aligned} \left(\sum_1^n \|T\left(x_i\right)\|^q\right)^{\frac{1}{p}} &\leq \pi_p\left(T\right) \left(\sum_1^n z_i^{\frac{pq}{q-p}}\right)^{\frac{q-p}{pq}} \sup_{\xi \in B_{X^*}} \left(\sum_1^n \left| < x_i, \xi > \right|^q\right)^{\frac{1}{q}}, \\ &\leq \pi_p\left(T\right) \left(\sum_1^n \|T\left(x_i\right)\|^q\right)^{\frac{1}{p}-\frac{1}{q}} . \sup_{\xi \in B_{X^*}} \left(\sum_1^n \left|< x_i, \xi >\right|^q\right)^{\frac{1}{q}}. \end{aligned}$$

This implies that

$$\left(\sum_1^n \|T\left(x_i\right)\|^q\right)^{\frac{1}{q}} \leq \pi_p\left(T\right) \sup_{\xi \in B_{X^*}} \left(\sum_1^n \left|< x_i, \xi >\right|^q\right)^{\frac{1}{q}}$$

and consequently, T is q-summing and $\pi_q(T) \leq \pi_p(T)$. □

Proposition 2.2.6. *Suppose that $1 \leq p < \infty$. Let X be a complete Banach lattice space and Y be a Banach lattice. Then, the following properties of the constant C and the linear operator $T : X \to Y$ are equivalent.*

1. *The operator $T \in \pi_p(X, Y)$ and $\pi_p(T) \leq C$.*

2. *For all $n \in \mathbb{N}$ and all $v : \ell_{p^*}^n \to X$, we have $\pi_p(Tv) \leq C\|v\|$.*

Proof. (1)$\Rightarrow$(2). Suppose $T \in \Pi_p(X, Y)$ and $\pi_p(T) \leq C$. Let $n \in \mathbb{N}$ and $v : l_{p^*}^n \longrightarrow X$, according to the ideal property, we have

$$Tv \in \Pi_p\left(l_{p^*}^n, Y\right) \text{ et } \pi_p(Tv) \leq C \|v\|$$

(2)$\Rightarrow$(1). We define v by $v(e_i) = x_i, i \in \{1, ..., n\}$ and $x_i \in X$ where (e_i) is the canonical basis of $l_{p^*}^n$We have

$$\|u\| = \sup_{\xi \in B_{X^*}} \left(\sum_1^n | < x_i, \xi > |^p\right)^{\frac{1}{p}}.$$

Then

$$\begin{aligned}
(\sum_1^n \|T(x_i)\|^p)^{1/p} = & \quad (\sum_1^n \|T(u(e_i))\|^p)^{1/p}, \\
\leq & \ C\|u\|, \\
\leq & \quad C \sup_{\xi \in B_{X^*}} \left(\sum_1^n | < x_i, \xi > |^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence $T \in \Pi_p(X; Y)$ and $\pi_p(T) \leq C$. □

Remark 2.2.7. The operator T is p-summing , if there exists a positive constant C such that for every n in $\mathbb{N}$ the mappings

$$\begin{aligned}
T_n : \quad l_{p,\omega}^n(X) & \longrightarrow \quad l_p^n(Y) \\
(x_i)_{1 \leq i \leq n} & \longmapsto \quad (T(x_i))_{1 \leq i \leq n}
\end{aligned}$$

are uniformly bounded by C. We put in this case

$$\pi_p(T) = \sup_n \|T_n\|.$$

We give now Pietsch domination theorem for p-summing sublinear operators. The proof, to which we refer the reader to the references [11, 12, 16], will proceed exactly as for linear operators.

Theorem 2.2.8. *(Pietsch domination theorem). Let $T : X \to Y$ be a sublinear operator that is p-summing, where $0 \leq p < \infty$. Then, there exists a Radon probability measure μ on $(B_{X^*}; \sigma(X^*, X))$ such that*

$$\|T(x)\| \leq \pi_p(T) \left(\int_{B_{X^*}} |\langle x, \xi \rangle|^p \, d\mu(\xi) \right)^{\frac{1}{p}}, \ for\ all\ x \in X. \tag{2.3}$$

Conversely, if there exists a Radon probability measure μ on $(B_{X^*}; \sigma(X^*, X))$ and $C > 0$ such that the above formula holds, then T is p-summing and $\pi_p(T) \leq C$.

Proposition 2.2.9. *Let X, Y, and Z be three Banach spaces, with Y being reflexive. Let C be a positive constant, and let T be a continuous sublinear operator from X to Y. Let $v : X \longrightarrow Z$ be linear operator such that*

$$\|T(x)\| \leq C\|v(x)\|.$$

$Then\ ,there\ exists\ \widetilde{T} : \overline{v(X)} \longrightarrow Y\ \ a\ continuous\ sublinear\ operator$

$$T = \tilde{T} o v$$

and

$$\left\| \tilde{T} \right\| \leq C.$$

Remark 2.2.10. the inequality 2.3 of Pietsch implies, according to Proposition 2.2.9, that there exists $\widetilde{T} : S \longrightarrow Y$ such that the following diagram commutes

$$\begin{array}{ccc} X & \overset{T}{\to} & Y \\ & & \\ i \downarrow & & \uparrow \tilde{T} \\ S_\infty & \overset{j/S}{\longrightarrow} & S_p \\ & & \\ \cap & & \cap \\ C(K) & \overset{j}{\to} & L_p(K, \mu) \end{array}$$

with $T = \tilde{T} \circ j/S \circ i$ and $\left\|\tilde{T}\right\| = \pi_p(T)$, where

$K = (B_{X^*},\ \sigma(X^*, X))$,

$C(K) = \{$ continuous functions on K$\}$,

$i : X \longrightarrow S_\infty, i(x)$ which is an injective isometry, with $< i(x), \xi >=< i(x), \xi >$.

S_∞ which is a closed subspace of $C(K)$,

$j : C(K) \longrightarrow L_p(K, \mu)$ is the natural injection, and $\pi_p(j) = 1$ (j/S is j restricted to S), $S = \overline{j(\overline{S})}^{L_p(K,\mu)}$.

Corollary 2.2.11. *If T is 2-summing then T factors through $L_\infty(K;\mu)$ and $L_2(K;\mu)$.*

Proof. Let's reconsider the diagram from Remark 2.2.10 for p = 2.

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ i\downarrow & & \uparrow \tilde{T} \\ S_\infty & \xrightarrow{j/S} & S \\ \cap & & \cap \uparrow P \\ C(K) & \xrightarrow{j} & L_2(K, \mu) \end{array}$$

P is the projection from $L^2(K;\mu)$ onto S of norm ≤ 1. □

2.3 Lipschitz p-summing sublinear operators

In this paragraph, we will retrict the concept of Lipschitz p-summing operators introduced by Farmer and Johnson [6] to sublinear operators and present its famous factorization theorem along with some fundamental properties.

Definition 2.3.1. A sublinear operator between a Banach space X and a Banach lattice Y is Lipschitz p-summing ($1 \leq p < \infty$) if and only if there exists a positive constant C such that for all $(x_i)_{1\leq i\leq n}, (y_i)_{1\leq i\leq n} \subset X$ and all $(a_i)_{1\leq i\leq n} \subset \mathbb{R}^+$, we have

$$\sum_{i=1}^{n} a_i \|T(x_i) - T(y_i)\|_p \leq C^p \sup_{f\in B_{X^\#}} \sum_{i=1}^{n} a_i |f(x_i) - f(y_i)|^p.$$

The least constant C for which this inequality always holds is denoted by $\pi_p^L(T)$. We shall write $\Pi_p^L(X,Y)$ for the set of all Lipschitz p-summing sublinear operators from X to Y.

Proposition 2.3.2. *Let X_0 be a subspace of a Banach space X and T be a mapping in $\Pi_p^L(X,Y)$, then*

$$T/X_0 \in \Pi_p^L(X_0,Y) \text{ and } \pi_p^L(T/X_0) \leq \pi_p^L(T).$$

Proposition 2.3.3. *If $j : Y \to Z$ is an isometry, then*

$$T \in \Pi_p^L(X,Y) \implies jT \in \Pi_p^L(X,Z).$$

In this case, we have

$$\pi_p^L(T) = \pi_p^L(jT).$$

Proposition 2.3.4. *(Ideal property). Let X , Z and E be Banach spaces and F be a Banach lattice. Let $R : Z \to X$, $S : E \to F$ be Lipschitz functions, and $T : X \to E$ be a Lipschitz p-summing sublinear operator. Then STR is a Lipschitz p-summing sublinear operator, and $\pi_p^L(STR) \leq Lip(S)\pi_p^L(T)Lip(R)$.*

Proof. we have

$$\begin{aligned}
&\sum_{i=1}^{n} \|STR(z_i) - STR(z_i')\|^p \\
&\quad \leq \text{Lip}(S)^p \sum_{i=1}^{n} \|TR(z_i) - TR(z_i')\|^p \\
&\quad \leq \text{Lip}(S)^p \pi_p^L(T)^P \sup_{f\in B_{X^\#}} \sum_{i=1}^{n} |f(R(z_i)) - f(R(z_i'))|^p \\
&\quad \leq \text{Lip}(S)^p \pi_p^L(T)^p \text{Lip}(R)^p \sup_{f\in B_{X^\#}} \sum_{i=1}^{n} \left| \frac{f(R(z_i))}{\text{Lip}(R)} - \frac{f(R(z_i'))}{\text{Lip}(R)} \right|^p \\
&\quad \leq \text{Lip}(S)\pi_p^L(T)\text{Lip}(R) \sup_{g\in B_{Y^\#}} \sum_{i=1}^{n} |g(z_i) - g(z_i')|^p
\end{aligned}$$

$\Rightarrow$ STR is Lipschitz p-sommant sublinear operator and $\pi_p^L(STR) \leq \text{Lip}(S)\pi_p^L(T)\text{Lip}(R)$. ☐

Theorem 2.3.5. *(Pietsch domination theorem). Suppose that $1 \leq p < \infty$. Let X , be Banach a space and Y be a Banach lattice.The following properties are equivalent for a mapping*

$T : X \to Y$ and a positive constant C.

(a) The mapping T is Lipschitz p-summing sublinear and $\pi_p^L(T) \leq C$.

(b) There is a probability measure μ on $B_{X^\#}$ such that

$$\|T(x) - T(y)\| \leq C \left(\int_{B_{X^\#}} |f(x) - f(y)|^p \, d\mu(f) \right)^{\frac{1}{p}}.$$

(c) For any isometric embedding j of Y into a 1-injective space Z, the following diagram commutes:

$$\begin{array}{lcl} L_\infty(\mathcal{B}_{X^\#}, \mu) & \stackrel{i_p}{\rightarrow} & L_p(\mathcal{B}_{X^\#}, \mu) \\ i \uparrow & & \downarrow \widetilde{T} \\ X \stackrel{T}{\rightarrow} Y & \stackrel{j}{\rightarrow} & Z \end{array}$$

with Lip($\tilde{T}$) $\leq C$.

Proof. The proof, to which we refer the reader to the references [6], will proceed exactly as for linear operators. □

2.4 Super Lipschitz p-summing operators

We introduce the class of Lipschitz super p-summing sublinear operators. We characterize this type of operators by giving a domination theorem. Also, we give some properties concerning this class. (see [9]).

Definition 2.4.1. Let X be a Banach space and Y be a Banach lattice. Let a map T in $\Delta\mathcal{SB}(X, Y)$. T is Lipschitz super p-summing $(1 \leq p < \infty)$, if and only if there exists a positive constant $c \geq 0$ such that for all $(x_i)_{1\leq i\leq n}, (y_i)_{1\leq i\leq n} \subset X$ and all $(a_i)_{1\leq i\leq n} \subset \mathbb{R}^+$, we have

$$\sum_{i=1}^{n} a_i \|T(x_i - y_i)\|_p \leq C^p \sup_{f\in B_{X^\#}} \sum_{i=1}^{n} a_i |f(x_i) - f(y_i)|^p. \tag{2.4}$$

We denote by $\Pi_p^{Ls}(X, Y)$ the space of the Lipschitz super p-summing $(1 \leq p < \infty)$ in $\Delta\mathcal{SB}(X, Y)$ and $\pi_p^{Ls}(T)$ its norm

$$\pi_p^{Ls}(T) = \inf\{C \mid \text{verifying the inequality } (2.4)\}$$

Corollary 2.4.2. *If $i : Y_0 \to Y$ is an isometry, then*

$$T \in \Pi_p^{Ls}(X, Y_0) \iff iT \in \Pi_p^{Ls}(X, Y)$$

In this case, we have $\pi_p^{Ls}(T) = \pi_p^{Ls}(iT)$.

Corollary 2.4.3. *Let X_0 be a subspace of a Banach space X and T be a mapping in $\Pi_p^{Ls}(X, Y)$, then*

$$T/X_0 \in \Pi_p^{Ls}(X_0, Y) \text{ and } \pi_p^{Ls}(T/X_0) \leq \pi_p^{Ls}(T).$$

Proposition 2.4.4. *(Ideal property). Let $T : X \longrightarrow Y$ be Lipschitz super p-summing, $W : E \longrightarrow X$ be a bounded linear function.Then, $WT : E \longrightarrow Y$ is Lipschitz super p-summing and*

$$\pi_p^{Ls}(T \circ W) \leq \pi_p^{Ls}(T) \|W\|$$

Proof. Let x y be in X. Then

$$\begin{aligned} \|(T \circ W)(x-y)\| = \; & \|T(W(x) - W(y))\| \\ & \leq \pi_p^{Ls}(T) \left(\int_{\mathcal{B}_{X^\#}} |f(W(x)) - f(W(y))|^p \, d\mu(f) \right)^{\frac{1}{p}} \\ & \leq \pi_p^{Ls}(T) \|W\| \left(\int_{\mathcal{B}_{E^\#}} |g(x) - g(y)|^p \, d\mu(g) \right)^{\frac{1}{p}}. \end{aligned}$$

Where $g(x) = \dfrac{f(W(x))}{\|W\|}$. Therefore, $T \circ W$ is Lipschitz super p-summing and

$$\pi_p^{Ls}(T \circ W) \leq \pi_p^{Ls}(T) \|W\|.$$

This ends the proof. □

Corollary 2.4.5. *If T is Lipschitz super p-summing $(1 \leq p < \infty)$, then for all $u \in \nabla T$, u is p-summing.*

Theorem 2.4.6. *(Pietsch domination theorem). Let $1 \leq p < \infty$. Let X be a Banach space and Y be a Banach lattice. The following properties are equivalent for T in $\Delta\mathcal{SB}(X, Y)$ and a positive constant C.*

(a) The mapping T is Lipschitz super p-summing and $\pi_p^{Ls}(T) \leq C$.

(b) There is a probability μ on $\mathcal{B}_{X^\#}$ such that

$$\|T(x-y)\| \leq C\left(\int_{\mathcal{B}_{X^{\#}}} |f(x)-f(y)|^{p}\, d\mu(f)\right)^{\frac{1}{p}} \tag{2.5}$$

for all x, y in X.

Proof. The proof is the same than that used in the Lipschitz p-summing case. □

Remark 2.4.7. If T is in $\Pi_p^{Ls}(X,Y)$ then T is in $\Pi_p^{L}(X,Y)$ and $\pi_p^{L}(T) \leq 2\pi_p^{Ls}(T)$.

Proof. Since T is Lipschitz super p-summing, there exists a Radon probability measure μ on $(B_{X^*}, \mathcal{B}(X^*, X))$ such that for all $x \in X$,

$$\|T(x-y)\| \leq C\left(\int_{\mathcal{B}_{X^{\#}}} |f(x)-f(y)|^{p}\, d\mu(f)\right)^{\frac{1}{p}}$$

For all x in X, we have

$$\|T(x)-T(y)\| \leq \|T(x-y)\| + \|T(y-x)\| \leq 2\pi_p^{Ls}\left(\int_{\mathcal{B}_{X^{\#}}} |f(x)-f(y)|^{p}\, d\mu(f)\right)^{\frac{1}{p}}.$$

Therefore, T is in $\Pi_p^{L}(X,Y)$ and $\pi_p^{L}(T) \leq 2\pi_p^{Ls}(T)$. □

Proposition 2.4.8. *(Inclusion theorem). Let $1 \leq p < q < \infty$. If $T : X \longrightarrow Y$ is Lipschitz super p-summing, then T is Lipschitz super q-summing and $\pi_q^{Ls}(T) \leq \pi_p^{Ls}(T)$.*

Proof. T is Lipschitz super p-summing, there exists a Radon probability measure μ on $(B_{X^*}, \mathcal{B}(X^*, X))$ such that for all $x \in X$,

$$\begin{aligned}\|T(x-y)\| &\leq C\left(\int_{\mathcal{B}_{X^{\#}}} |f(x)-f(y)|^{p}\, d\mu(f)\right)^{\frac{1}{p}} \\ &\leq C\left(\int_{\mathcal{B}_{X^{\#}}} |f(x)-f(y)|^{q}\, d\mu(f)\right)^{\frac{1}{q}}.\end{aligned}$$

Therefore, T is is Lipschitz super q-summing and $\pi_q^{Ls}(T) \leq \pi_p^{Ls}(T)$. □

2.5 Relationship between $\Pi_p^{Ls}(X,Y)$,$\Pi_p(X;Y)$,$\Pi_p^{L}(X,Y)$

Theorem 2.5.1. *Suppose $1 \leq p < \infty$. Let $u : X \longrightarrow Y$ be a bounded linear operator between Banach space X and Banach lattice Y. Then, the notions of Lipschitz super p-summing, Lipschitz p-summing* *and p-summing coincide and*

$$\pi_p(u) = \pi_p^L(u) = \pi_p^{Ls}(u).$$

Proof. (a) We prove that $\pi_p(u) = \pi_p^L(u)$.

Suppose u is p-summing . Then regardless of the natural number n and the choice of $x_1, ..., x_n$ in X, we have

$$\sum_{i=1}^{n} \|u(x_i)\|^p \leq \pi_p(u)^p \cdot \sup_{f\in B_{X^*}} \left(\sum_{i=1}^{n} |f(x_i)|^p \right).$$

Since T is linear then for $\{z_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ in X with $x_i = z_i - y_i$ and taking $a_i = 1$ for each i, we hav

$$\begin{aligned}\sum_{i=1}^{n} \|u(z_i) - u(y_i)\|^p &= \sum_{i=1}^{n} \|u(z_i - y_i)\|^p \\ &\leq (\pi_p(u))^p \cdot \sup_{\xi\in B_{X^*}} \left(\sum_{i=1}^{n} |\xi(z_i - y_i)|^p \right) \\ &= (\pi_p(u))^p \cdot \sup_{f\in B_{X^*}} \left(\sum_{i=1}^{n} |\xi(z_i) - \xi(y_i)|^p \right) \\ &\leq (\pi_p(u))^p \cdot \sup_{f\in B_{X^\sharp}} \left(\sum_{i=1}^{n} |f(z_i) - f(y_i)|^p \right).\end{aligned}$$

Hence

$$\sum_{i=1}^{n} \|u(z_i) - u(y_i)\|^p \leq (\pi_p(u))^p \cdot \sup_{f\in B_{X^\sharp}} \left(\sum_{i=1}^{n} |f(z_i) - f(y_i)|^p \right).$$

This shows that u is Lipschitz p-summing and

$$\pi_p^L(u) \leq \pi_p(u). \tag{2.6}$$

Conversely, if T is Lipschitz p-summing, then it is p-summing in [**?**, Theorem 2] with

$$\pi_p(u) \leq \pi_p^L(u) \tag{2.7}$$

Combining 2.6 and 2.7 , we have

$$\pi_p^L(u) = \pi_p(u).$$

(b) $\pi_p^L(u) = \pi_p^{Ls}(u)$ because u is linear. This ends the proof.

□

We give now the weak Dvoretzky-Rogers. For the proof, we can consult [16, theorem 2.18]

Theorem 2.5.2. *(Weak Dvoretzky-Rogers). Let $1 \leq p < \infty$. Every infinite dimensional Banach space X contains a weakly p-summable sequence which fails to be strongly p-summable .*

Remark 2.5.3. If T is linear, then from the closed graph theorem, it is p-summing if, and only if, it satisfies that for every infinite sequence $(x_n)_{n\in\mathbb{N}}$ in X, we have

$$\left(\sum_{n\in\mathbb{N}} |\langle \xi, x_n\rangle|^p < \infty, \quad \forall \xi \in X^*\right) \Longrightarrow \sum_{n\in\mathbb{N}} \|T(x_n)\|^p < \infty. \tag{2.8}$$

Proposition 2.5.4. *Let X be a Banach lattice. Then, the following properties are equivalent.*

(1) *The identity is in $\Pi_p^{Ls}(X,X)$.*

(2) *The identity is in $\Pi_p^{L}(X,X)$.*

(3) *The identity is in $\Pi_p(X,X)$.*

(4) *The space X is of finite dimensional.*

Proof. To prove the equivalence of these properties, let's consider each implication

$(1) \Rightarrow (2)$ Since $\Pi_p^{Ls}(X,X) \subset \Pi_p^{L}(X,X)$, if the identity is in $\Pi_p^{Ls}(X,X)$, it must also be in $\Pi_p^{L}(X,X)$.

$(2) \Rightarrow (3)$ This follows trivially because $\Pi_p^{L}(X,X) \subset \Pi_p(X,X)$.

$(3) \Rightarrow (4)$ If the identity is in $\Pi_p(X,X)$, it implies that X is of finite dimension (weak Dvoretzky-Rogers).

$(4) \Rightarrow (1)$ If X is of finite dimension, then the identity is in $\Pi_p^{Ls}(X,X)$(weak Dvoretzky-Rogers).

So, we've shown that all four properties are equivalent. □

Chapter 3

Relationship between T and $u \in \nabla T$ for other types of summability

We study the relationship between T and u in ∇T concerning this type of summability and we look at the other types of summability .

3.1 Relationship between T and $u \in \nabla T$ for other types of summability

a- Let X be a set and $\mathcal{F}\ell$ a non-empty collection of subsets of $X(\mathcal{F}\ell \subset X)$

$\mathcal{F}\ell$ is a filter if :

$$1) - \emptyset \notin \mathcal{F}\ell.$$

$$2) - \text{if } F_1, F_2 \in \mathcal{F}\ell \Longrightarrow F_1 \cap F_2 \in \mathcal{F}\ell.$$

$$3) - \text{if } A \supset F, F \in F\ell \Longrightarrow A \in F\ell.$$

b-If B is a non-empty collection of subsets of X it is a filter base if:

$$1) - B \subset F\ell$$

$$2) - If\ F \in \mathcal{F}\ell\ then\ \exists B \in \mathcal{B}\ such\ that\ B \subset F$$

Theorem 3.1.1. *Let X be a Banach space and Y be a Banach lattice. Let $T : X \to Y$ a continuous sublinear operator. If*

$$T \in \pi_p(X, Y)$$

then

$$\textit{for all } u \in \nabla T,\ u \in \pi_p(X, Y)$$

.

Proof. Since T is p-summing, there exists a Radon probability measure μ on $(B_{X^*}, \mathcal{B}(X^*, X))$ such that for all $x \in X$,

$$\|T(x)\| \leq \pi_p(T) \left(\int_{B_{X^*}} |\langle x, \xi \rangle|^p \, d\mu(\xi) \right)^{\frac{1}{p}}.$$

For all x in X and all u in ∇T, we have

$$\|u(x)\| \leq \|T(x)\| + \|T(-x)\| \leq 2\pi_p(T) \left(\int_{B_{X^*}} |\langle x, \xi \rangle|^p \, d\mu(\xi) \right)^{\frac{1}{p}}.$$

Therefore, u is p-summing and $\pi_p(u) \leq 2\pi_p(T)$. □

Theorem 3.1.2. *[?] Let T be a continuous sublinear operator from a Banach space X into a Banach lattice Y and ($1 \leq p < \infty$). Suppose there exists $C > 0$ and operators filter $\{u_i\}_{i \in I} \subset \nabla T$ such that,*

$$\begin{cases} \forall i \in \mathbf{I},\ \pi_p(u_i) \leq C \ and \\ \forall x \in X,\ \|u_i(x)\| \longrightarrow \|T(x)\|. \end{cases}$$

Then

$$T \in \pi_p(X, Y) \ et \ \pi_p(T) \leq C.$$

Proof. Since u_i est is p-summing, there exists a probability λ_i on $B_X^{\cdot}$ such that

$$\forall x \in X,\ \|u_i(x)\| \leq C(\int_{B_{X^*}} | < x, \xi > |^p \, d\lambda_i(\xi)))^{\frac{1}{p}}.$$

As we have for all $x \in X$

$$\|u_i(x)\| \longrightarrow \|T(x)\|$$

So

$$\forall x \in X,\ \|T(x)\| \leq C \lim_{i \in I} (\int_{B_{X^*}} |< x, \xi >|^p \, d\lambda_i(\xi))^{\frac{1}{p}}.$$

The unit ball B_{X^*} is weakly compact, so λ_i weakly converges to a probability measure λ on B_{X^*} and consequently

$$\forall x \in X, \|T(x)\| \leq C(\int_{B_{X^*}} | < x, \xi > |^p d\lambda(\xi))^{\frac{1}{p}}.$$

This implies that T is p-summing and $\pi_p(T) \leq C$. □

We prove that the reciprocal of the theorem 3.1.1 is false by the following example .

Example 3.1.3. Let X be a Banach space of infinite dimensional and consider the map $T\colon X \longrightarrow \mathbb{R}$ defined by $T(x) = \|x\|$. The operator T is a bounded sublinear operator. The set

$$\nabla T = \{u \in \mathcal{B}(X, \mathbb{R}) \subset \mathcal{B}_{X^*}, u \leq T\}$$

is in $\Pi_p(X, \mathbb{R})$ and $\pi_p(u) \leq \|T\|$. But T is not p-summing.

Proof. Let $T\colon X \longrightarrow \mathbb{R}$ be defined as $T(x) = \|x\|$

(a) Homogeneity: for any $x \in X$ and any $\alpha \geq 0$, we have

$$T(\alpha x) = \|\alpha x\| = \alpha\|x\| = \alpha T(x).$$

(a) Subadditivity: for any $x, y \in X$, we have

$$T(x + y) = \|x + y\| \leq \|x\| + \|y\| = T(x) + T(y).$$

Since T satisfies both the properties of homogeneity and subadditivity, it is a sublinear operator. We prove that T is bounded. We need to find a constant $C > 0$ such that $|T(x)| \leq C\|x\|$ for all $x \in X$. Since $|x| \leq \|x\|$ for all $x \in X$, we can choose $C = 1$. Then for any $x \in X$, we have:

$$|T(x)| = |x| \leq \|x\| = 1 \cdot \|x\| = C\|x\|,$$

which shows that T is bounded with $C = 1$.

We prove that $T\colon X \longrightarrow R$ is not p-summing.

For a contradiction, let's assume that T is p-summing. This implies the existence of a positive constant C such that for all n in $\mathbb{N}$ and for all sequences $(x_i)_{1\leq i\leq n}$ in X, the inequality

$$\left(\sum_{i=1}^{n} \|x_i\|^p\right) \leq C^p \sup_{\xi \in \mathcal{B}_{X^*}} \left(\sum_{i=1}^{n} |\xi(x_i)|^p\right)$$

holds. This would imply that the identity operator Id_X is p-summing. it follows that for every infinite sequence $(x_i)_{1\leq i\leq n}$ in X, if the series

$$\sum_{n\in\mathbb{N}} |\langle \xi, x_n\rangle|^p$$

converges for all $\xi \in X^*$, then the series

$$\sum_{n\in\mathbb{N}} \|T(x_n)\|^p$$

also converges.

However, this contradicts the weak Dvoretzky-Rogers Theorem. Therefore, we conclude that T can not be p-summing. □

Remark 3.1.4. Let X be a Banach space and Y be a Banach lattice. The notions of p-summing and Lipschitz p-summing do not coincide on $\mathcal{SB}(X,Y)$. The opposite makes our problem trivial.

Remark 3.1.5. Let X be a Banach space of infinite dimensional. The sublinear operator T defined on X by $T(x) = \|x\|$ is not p-summing by Example 3.1.3 but is Lipschitz p-summing.

Proof. For $(x_i)_{1\leq i\leq n}$, $(y_i)_{1\leq i\leq n}$in X, we have

$$\begin{aligned}
& \left(\sum_{i=1}^{n} |T(x_i) - T(y_i)|^p\right)^{\frac{1}{p}} \\
= & \left(\sum_{i=1}^{n} |\|x_i\| - \|y_i\||^p\right)^{\frac{1}{p}} \\
= & \left(\sum_{i=1}^{n} |f_0(x_i) - f_0(y_i)|^p\right)^{\frac{1}{p}} \\
& \text{(where } f_0(.) = \|.\|\text{; which is a Lipschitz function and } \mathrm{Lip}(f_0) \leq 1) \\
\leq & \sup_{f\in\mathcal{B}_{X^\#}} \left(\sum_{i=1}^{n} |f(x_i) - f(y_i)|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

This concluds the proof. □

Remark 3.1.6. We can add that the sublinear operator T is not super Lipschitz p-summing. Indeed, suppose that there exists $C > 0$ such that for every $\{x_i\}_{1\leq i\leq n}$, $\{y_i\}_{1\leq i\leq n}$in X, we have

$$
\begin{aligned}
& \left(\sum_{i=1}^{n}\left|T\left(x_i-y_i\right)\right|^p\right)^{\frac{1}{p}} \\
= & \left(\sum_{i=1}^{n}\left\|x_i-y_i\right\|^p\right)^{\frac{1}{p}} \\
\leq & C \sup_{\mathcal{B}_{X^\#}}\left(\sum_{i=1}^{n}\left|f\left(x_i\right)-f\left(y_i\right)\right|^p\right)^{\frac{1}{p}} .
\end{aligned}
$$

This implies that Id_X is Lipschitz p-summing and consequently by proposition 2.5.4 is p-summing. This contradicts the weak Dvoretzky-Rogers.

The following proposition explores the connection between $u \in \nabla T$ and T, in the context of T being Lipschitz super p-summing.

Proposition 3.1.7. *Let X be any Banach space and Y be a complete Banach lattice. Suppose T is an operator in $\mathcal{SB}(X,Y)$. If T belongs to $\Pi_p^{Ls}(X,Y)$ for $1 \leq p < \infty$, then $\nabla T \subset \Pi_p^{Ls}(X,Y)$. However, the converse does not hold in general.*

Proof. Since T is Lipschitz super p-summing, according to Theorem 2.4.6, there exists a Radon probability measure μ on $\mathcal{B}_{X^\#}$ such that

$$
\forall x, y \in X \ \ \|T(x-y)\| \leq C\left(\int_{\mathcal{B}_{X^\#}}|f(x)-f(y)|^p \, d\mu(f)\right)^{\frac{1}{p}}
$$

According to Theorem 1.2.7 for every x,y in X and every u in ∇T

$$
\begin{aligned}
\|u(x-y)\| & \leq \|T(x-y)\|+\|T(y-x)\| \\
& \leq 2\pi_p^{Ls}(T)\left(\int_{\mathcal{B}_{X^\#}}|f(x)-f(y)|^p \, d\mu(f)\right)^{\frac{1}{p}} .
\end{aligned}
$$

Hence u is $\Pi_p^{Ls}(X,Y)=\Pi_p(X,Y)$ and $\pi_p(u) \leq 2\pi_p^{Ls}(T)$. According to the previous result the opposite is false.

Consider $T\colon X \longrightarrow \mathbb{R}$ defined $T(x)=\|x\|$. Then T is a bounded sublinear operators. We have

$$
\nabla T=\left\{\begin{array}{c} u \in \mathcal{B}(X, \mathbb{R}) \subset \\ X^*, \quad u \leq T \end{array}\right\} \subset \Pi_p(X, \mathbb{R}) \subset \Pi_p^{Ls}(X, \mathbb{R}).
$$

Thus $\nabla T \subset \Pi_p^{Ls}(X,\mathbb{R})$ but T is not Lipschitz super p-summing. Suppose that T is Lipschitz super p-summing, then there is a positive constant C such that for every n in $\mathbb{N}$, $(x_i)_{1\leq i\leq n}, (y_i)_{1\leq i\leq n}$ in X, we have

$$\begin{aligned} \left(\sum_{i=1}^{n} |T(x_i - y_i)|^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^{n} \|x_i - y_i\|^p\right)^{\frac{1}{p}} \\ &\leq \sup_{f\in\mathcal{B}_{X^\#}} \|(f(x_i)-f(y_i))_i\|_p \,. \end{aligned}$$

This implies that $\dim(X)$ is finite and hence contradiction. □

3.2 Other types of summability

3.2.1 Lipschitz p-dominated operators

We give now the notion of Lipschitz p-dominated operators introduced by D. Chen and B. Zheng in [5].

Definition 3.2.1. A Lipschitz mapping $T : X \to Y$ between Banach spaces X and Y is Lipschitz p-dominated ($1 \leq p < \infty$) if there exists a Banach space Z and an operator $L \in \Pi_p(X,Z)$ such that

$$\|Tx - Ty\| \leq \|Lx - Ly\|, \quad \forall x,y \in X \tag{3.1}$$

We denote by $D_p^L(X,Y)$ the Banach space of all Lipschitz p-dominated operators . For $T \in D_p^L(X,Y)$, we set $d_p^L(T)$ to be the infimum of $\pi_p(L)$, the infimum being taken over all the above Z and L.

Remark 3.2.2. If $T(0) = 0$ and we replace in (3.1) x by $x-y$ and y by 0, we will have

$$\| T(x-y) \| \leq \| L(x-y) \|, \quad \forall x,y \in X.$$

The converse is true for sublinear operators, by (1.6) we have

$$\| T(x) - T(y) \| \leq \| 2L(x-y) \|, \quad \text{for all } x,y \in X. \tag{3.2}$$

Theorem 3.2.3. *For a Lipschitz mapping $T : X \to Y$ the following are equivalent:*

(i) $T \in D_p^L(X,Y)$.

(ii) There exists a constant $C > 0$ such that for any $\{x_i\}_{i=1}^n and \{y_i\}_{i=1}^n in X$, we have

$$\left(\sum_{i=1}^{n} \|Tx_i - Ty_i\|^p\right)^{1/p} \leq C \sup_{\xi \in B_{X^*}} \left(\sum_{i=1}^{n} |\langle \xi, x_i - y_i\rangle|^p\right)^{1/p}.$$

(iii) there exist a constant $C > 0$ and a probability measure μ on B_{X^*} such that for all $x, y \in X$, we have $\|Tx - Ty\| \leq C(\int_{B_{X^*}} |\langle x^*, x - y\rangle|^p \, d\mu(x^*))^{1/p}$. In that case $d_p^L(T) = \inf\{C : C$ as in $(ii)\} = \inf\{C : C, \mu$ as in $(iii)\}$.

Proposition 3.2.4. *Let X be an arbitrary Banach space and Y be a complete Banach lattice. Consider T in $\triangle\mathcal{SB}(X,Y)$. If T is in $\mathcal{D}_p^L(X,Y)$ for $1 \leq p < \infty$ then, T is p-summing in the sense of Definition 2.2.7 and hence T is Lipschitz super p-summing, which implies that T is Lipschitz p-summing.*

Proof. By Theorem 3.2.3, there exist a probability measure μ on $\mathcal{B}_{X^*}$ such that for all $x, y \in X$, we have

$$\|Tx - Ty\| \leq d_p^L(T)(\int_{B_{X^*}} |\langle x^*, x - y\rangle|^p \, d\mu(x^*))^{1/p}.$$

If we take $y = 0$, we obtain $\|T(x)\| \leq \mathrm{d}_p^L(T) \left(\int_{\mathcal{B}_{X^*}} |x^*(x)|^p \, d\mu(x^*)\right)^{\frac{1}{p}}$. Replace in the precedent inequality x by $x - y$, we get

$$\begin{aligned} \|T(x-y)\| &\leq \mathrm{d}_p^{L+}(T) \left(\int_{\mathcal{B}_{X^*}} |x^*(x-y))|^p \, d\mu(x^*)\right)^{\frac{1}{p}}, \\ &\leq \mathrm{d}_p^{L+}(T) \left(\int_{\mathcal{B}_{X^*}} |x^*(x) - x^*(y)|^p \, d\mu(x^*)\right)^{\frac{1}{p}}. \end{aligned}$$

By using Remark 2.4.7, we finish the last implication. □

Proposition 3.2.5. *Let X be an arbitrary Banach space and Y be a complete Banach lattice. Consider T in $\mathcal{SB}(X,Y)$. Then, if T is in $\mathcal{D}_p^L(X,Y)$ for $1 \leq p < \infty$, we have $\nabla T \subset \mathcal{D}_p^L(X,Y)$ and consequently $\nabla T \subset \Pi_p^L(X,Y)$.*

Proof. By (1.7), we have

$$\|u(x)\| \leq \|T(x)\| + \|T(-x)\|$$

If we replace x by $x-y$ and y by 0 we will have

$$\|u(x-y)\| \leq \|T(x-y)\| + \|T(y-x)\|$$

Thus, by Theorem 3.2.3

$$\begin{aligned}
\|u(x-y)\| &\leq 2\mathrm{d}_p^L(T)\left(\int_{\mathcal{B}_{X^*}} |x^*(x-y)|^p \, d\mu(x^*)\right)^{\frac{1}{p}}, \\
\text{(because } T(0)=0\text{)} &\leq 2\mathrm{d}_p^L(T)\left(\int_{\mathcal{B}_{X^*}} |x^*(x)-x^*(y)|^p \, d\mu(x^*)\right)^{\frac{1}{p}}.
\end{aligned}$$

This implies that u is p-summing and consequently is Lipschitz p-summing. □

3.2.2 Lipschitz strongly p-summing operators

We present a generalization about the class of Lipschitz strongly p-summing operators introduced independently by [13] and [14]. We deduce in the same spirit from that used in, [7] the following definition.

Definition 3.2.6. Let X be a Banach space and Y be a Banach lattice. Let $T: X \to Y$ be a Lipschitz map. T is Lipschitz strongly p-summing $(1 < p \leq \infty)$ if there is a constant $C > 0$ such that for any $n \in \mathbb{N}^*$, $(x_i)_i$, $(x'_i)_i$ in X; $(y_i^*)_i$ in Y^*, and $(\lambda_i)_i$ in $\mathbb{R}_+^*$ $(1 \leq i \leq n)$, we have

$$\sum_{i=1}^{n} \lambda_i \left|\left\langle T(x_i) - T(x'_i), y_i^* \right\rangle\right| \leq C \left(\sum_{i=1}^{n} \lambda_i \|x_i - x'_i\|^p\right)^{\frac{1}{p}} \sup_{y^{**} \in \mathcal{B}_{Y^{**}}} \|(\langle y_i^*, y^{**}\rangle)\|_{l_{p^*}^n}. \tag{3.3}$$

If $T(0) = 0$, then this definition is equivalent to

$$\sum_{i=1}^{n} \lambda_i \left|\left\langle T(x_i - x'_i), y_i^* \right\rangle\right| \leq C \left(\sum_{i=1}^{n} \lambda_i \|x_i - x'_i\|^p\right)^{\frac{1}{p}} \sup_{y^{**} \in \mathcal{B}_{Y^{**}}} \|(\langle y_i^*, y^{**}\rangle)\|_{l_{p^*}^n}.$$

We denote by $D_{st.p}^L(X;Y)$ the Banach space of all laticially Lipschitz strongly p-summing operators, and $d_{st.p}^L(T)$ its norm defined as

$$d_{st.p}^L(T) = \inf\{C > 0 \,|\, C \text{ verifying } 3.3\}$$

As in the linear case, if $p = 1$, we have $D_{st.p}^L(X;Y) = \mathrm{Lip}_0(X;Y)$.

Proposition 3.2.7. *Let u be a bounded linear operator between two Banach spaces E and F and $1 < p \leq \infty$, then*

$$d_p(u) = d_{st.p}^L(u).$$

3.2.3 Strongly p-summing operators

We introduce the following generalization of the class of strongly p-summing linear operators to sublinear operators . It was generalized by D. Achour, L. Mezrag, and A. Tiaiba . For further details, see [4] .

Definition 3.2.8. Let X be a Banach space and Y be a Banach lattice. A sublinear operator $T : X \to Y$ is strongly p-summing $(1 < p < \infty)$, if there is a positive constant C such that for any $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $y_1^*, y_2^*, \ldots, y_n^* \in Y^*$, we have

$$\sum_{i=1}^{n} |\langle T(x_i), y_i^* \rangle| \leq C \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}} \sup_{y^{**} \in \mathcal{B}_{Y^{**}}} \left(\sum_{i=1}^{n} \langle y_i^*, y^{**} \rangle^{p^*} \right)^{\frac{1}{p^*}}. \tag{3.4}$$

We denote by $\mathcal{D}_p(X, Y)$ the class of all strongly p-summing sublinear operators from X into Y, and $d_p(T)$ the smallest constant C such that the inequality 3.4 holds. For the definition of strongly positive p-summing, we replace Y^* by Y_+^* and $d_p(T)$ by $d_p^+(T)$.

Proposition 3.2.9. *(Propety ideal). Let X be a Banach space and Y, Z be two Banach lattices. Let $T \in SB(X, Y)$, R be a positive operator in $B(Y, Z)$, and $S \in B(E, X)$.*

(a) If T is a strongly p-summing sublinear operator, then $R \circ T$ is a strongly p-summing sublinear operator and $d_p(R \circ T) \leq \|R\| d_p(T)$.

(b) If T is a strongly p-summing sublinear operator, then $T \circ S$ is a strongly p-summing sublinear operator and $d_p(T \circ S) \leq \|S\| d_p(T)$.

Theorem 3.2.10. *(Domination theorem) Let X be a Banach space and Y be a Banach lattice. An operator $T \in \mathcal{SB}(X, Y)$ is strongly p-summing $(1 < p < \infty)$ if there exist a positive constant $C > 0$ and a Radon probability measure μ on $B_{Y^{**}}$ such that for all $x \in X$, we have*

$$|\langle T(x), y^* \rangle| \leq C \|x\| \left(\int_{\mathcal{B}_{Y^{**}}^{+}} ||y^*| (y^{**})|^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}}. \tag{3.5}$$

Moreover, in this case

$$\mathrm{d}_p(u) = \inf\{C, \textit{satisfying } 3.5)\}.$$

Corollary 3.2.11. *Consider $1 < p_1, p_2 < \infty$ such that $p_1 \leq p_2$. if $T \in D_{p_2}(X,Y)$, then $T \in D_{p_1}(X,Y)$ and*

$$d_{p_1}(T) \leq d_{p_2}(T).$$

Corollary 3.2.12. *Let X be a Banach space, Y be a Banach lattice space, and $T_1 \in D_p(X,Y)$ $(1 < p < \infty)$ be a sublinear operator. If the sublinear operator $T_2 \leq T_1$, then $T_2 \in D_p^+(X,Y)$.*

We conclude this chapter by presenting some relationships between T and $u \in \nabla T$ in this summability.

Proposition 3.2.13. *[2] Let X be a Banach space and E be a Banach lattice. And let $T \in \mathcal{SB}(E,X)$, then T is positive p-summing if and only if there is a constant $C > 0$ such that the inequality*

$$\|T(x)\| \leq C \left(\int\limits_{B^+_{E^*}} \langle |x|, x^* \rangle^p \, d\mu(x^*) \right)^{\frac{1}{p}}$$

holds for any finite sequence $(x_i)_{i=1}^n$ in E.

Proposition 3.2.14. *[9] Let X be an arbitrary Banach space and Y be a complete Banach lattice. Consider T in $\mathcal{SB}(X,Y)$. Then, if T is in $\mathcal{D}^L_{st.p}(X,Y)$ for $1 < p \leq \infty$, we have u positive strongly p-summing for all u in ∇T and hence u^* is positive p^*-summing and $\pi^+_{p^*}(u^*) \leq 2d_L^{st.p}(T)$.*

Proof. According to 1.3, for every x in X and y^* in Y^*_+

$$\langle u(x), y^* \rangle \leq \langle T(x), y^* \rangle .$$

And therefore

$$-\langle u(x), y^* \rangle \leq \langle T(-x), y^* \rangle$$

which implies that

$$\begin{aligned} |\langle u(x), y^* \rangle| \leq & \quad \sup \{\langle T(x), y^* \rangle, \langle T(-x), y^* \rangle\} \\ \leq & \quad \sup \{|\langle T(x), y^* \rangle|, |\langle T(-x), y^* \rangle|\} \\ \leq & \quad |\langle T(x), y^* \rangle| + |\langle T(-x), y^* \rangle|. \end{aligned}$$

And hence by using (3.5)

$$|\langle u(x), y^*\rangle| \leq 2d^L_{st.p}(T)\|x\| \left(\int\limits_{\mathcal{B}_{Y^{**}}} |y^*(y^{**})|^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}}.$$

for all y^* in Y^*_+. Consider $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$. By using Hölder inequality, we have

$$\begin{aligned}
&\sum_{i=1}^n |\langle u(x_i), y_i^*\rangle| \\
\leq\ & 2d^L_{st.p}(T) \sum_{i=1}^n \|x_i\| \left(\int\limits_{\mathcal{B}_{Y^{**}}} |y_i^*(y^{**})|^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}} \\
\leq\ & 2d^L_{st.p}(T) \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n \int\limits_{\mathcal{B}_{Y^{**}}} |y_i^*(y^{**})|^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}} \\
\leq\ & 2d^L_{st.p}(T) \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \left(\sup_{y^{**}\in\mathcal{B}_{Y^{**}}} \sum_{i=1}^n \int\limits_{\mathcal{B}_{Y^{**}}} |y_i^*(y^{**})|^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}} \\
\leq\ & 2d^L_{st.p}(T) \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \left(\sup_{y^{**}\in\mathcal{B}_{Y^{**}}} \sum_{i=1}^n |y_i^*(y^{**})|^{p^*} \right)^{\frac{1}{p^*}} \\
\leq\ & 2d^L_{st.p}(T) \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \left(\sup_{y^{**}\in\mathcal{B}^+_{Y^{**}}} \sum_{i=1}^n (y_i^*(y^{**}))^{p^*} \right)^{\frac{1}{p^*}}.
\end{aligned}$$

We immediately have u positive strongly p-summing from theorem (3.2.10) and therefore $\mathrm{d}_p^+(u) \leq 2d^L_{st.p}(T)$. The characterization (3.5) of positive strongly p-summing linear operators yields for all $x \in X$ and $y^* \in Y^*$ that

$$|\langle u(x), y^*\rangle| \leq 2d^L_{st.p}(T)\|x\| \left(\int\limits_{\mathcal{B}^+_{Y^{**}}} (|y^*|(y^{**}))^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}}.$$

and thus

$$|\langle x, u^*(y^*)\rangle| \leq 2d^L_{st.p}(T)\|x\| \left(\int\limits_{\mathcal{B}^+_{Y^{**}}} (|y^*|(y^{**}))^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}}.$$

Hence

$$\|u^*(y^*)\| \leq 2d^L_{st.p}(T) \left(\int\limits_{\mathcal{B}^+_{Y^{**}}} (|y^*|(y^{**}))^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}}.$$

This implies by proposition 3.2.13 that u^* is positive p^*-summing and $\pi^+_{p^*}(u^*) \leq 2d^L_{st.p}(T)$. □

Proposition 3.2.15. *Let X be an arbitary Banach space and Y a completely lattice Banach space. Let T be a bounded sublinear operator from X to Y. Suppose that T is strongly p-summing ($1 < p < \infty$). Then, for every $u \in \nabla T$, u is strongly positively p-summing, and u^* is positively p^*-summing.*

Proof. According to 1.3, for any x in X and y^* in Y_+^*, we have:

$$\langle u(x), y^* \rangle \leq \langle T(x), y^* \rangle$$

and consequently

$$-\langle u(x), y^* \rangle \leq \langle T(-x), y^* \rangle.$$

This implies that

$$|\langle u(x), y^* \rangle| \leq \sup\{\langle T(x), y^* \rangle, \langle T(-x), y^* \rangle\}$$

$$\leq \sup\{|\langle T(x), y^* \rangle|, |\langle T(-x), y^* \rangle|\}$$

$$\leq |\langle T(x), y^* \rangle| + |\langle T(-x), y^* \rangle|$$

and according to (3.2.10), we have

$$|\langle u(x), y^* \rangle| \leq 2d_p(T)\|x\| \left(\int_{B_{Y^*}} |y^*(y)|^p d\mu(y^*) \right)^{\frac{1}{p}}.$$

This leads to

$$\|u^*(y^*)\| \leq 2d_p(T) \left(\int_{B_{Y^*}} |y^*(y)|^{p^*} d\mu(y^*) \right)^{\frac{1}{p^*}}.$$

Thus, the operator u^* is positively p^*-summing and $\pi_+(u^*) \leq 2d_p(T)$. □

We continue by studying the inverse of the proposition 3.2.15 .

Proposition 3.2.16. *Let X be Banach space and Y a completely lattice Banach space. Let T in $\mathcal{SB}(X, Y)$. Suppose there exists a finite positive constant $C > 0$, a set I, an ultrafilter U on I, and $\{u_i\}_{i \in I} \subset \nabla T$ such that for every x in X,*

$$|\langle u_i(x), y' \rangle| \xrightarrow{u} |\langle T(x), y' \rangle|$$

and $d_p(u_i) \leq C$ uniformly. Then,

$$T \in D_p(X, Y) \quad \textit{and} \quad d_p(T) \leq C.$$

Proof. For all $i \in I$ let us take u_i strongly p-summing. According to Theorem (3.2.10) there exists a Radon probability measure μ_i on $B_{Y^{**}}$ such that for all $x \in X$, we have

$$|\langle u_i(x), y^{**}\rangle| \leq d_p(u_i)\|x\| \left(\int_{B_{Y^{**}}} |\langle y^*(y^{**})\rangle|^{p^*} \, d\mu_i(y^{**})\right)^{\frac{1}{p^*}}.$$

As we have for all x in X and y^* in Y^*,

$$|\langle u_i(x), y^*\rangle| \underset{u}{\rightarrow} |\langle T(x), y^*\rangle|$$

we obtain that for all x in X and y^* in Y^*,

$$|\langle T(x), y^*\rangle| \leq \lim_u d_p(u_i)\|x\| \left(\int_{B_{Y^{**}}} |\langle y^*(y^{**})\rangle|^{p^*} \, d\mu_i(y^{**})\right)^{\frac{1}{p^*}}.$$

The unit ball $B_{Y^{**}}$ is weakly compact, thus μ_i weakly converges towards a probability μ on $B_{Y^{**}}$. Consequently,

$$|\langle T(x), y^{**}\rangle| \leq C\|x\| \left(\int_{B_{Y^{**}}} |\langle y^*(y^{**})\rangle|^{p^*} \, d\mu(y^{**})\right)^{\frac{1}{p^*}}$$

for all x in X and y^* in Y^*. This implies that $d_p(T) \leq C$. □

Conclusion

We conclude this memory by posing some questions related to this work. These two problems are posed in [9] .

Problem 1. For which spaces X, Y we have $\Pi_p^{Ls}(X, Y) = \Pi_p^{L}(X, Y)$?

Problem 2. Let X be a Banach lattice (for example $X = L_q$). Is the Lipschitz operator

$$\begin{array}{rccl} T: & X & \longrightarrow & X \\ & x & \longmapsto & T(x) = |x| \end{array}$$

Lipschitz p-summing for some p?

In conclusion, this research explored the relationship between the operator T and its differential ∇T in the context of the summability of sublinear operators. The results demonstrated that there are specific properties and interactions that can be analyzed and characterized through the presented theories. We hope this research has added valuable knowledge to the field of functional analysis and opened new horizons for future research in pure and applied mathematics. We look forward to this study being a starting point for further research that addresses other mathematical operators and their interactions in different spaces .

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ملخص

في هذه المذكرة، نقدم وندرس مفاهيم التجميع في الحالة شبه الخطية. نثبت بعض التوصيفات من خلال مبرهنة الهيمنة وبعض خصائص هذه المفاهيم. نحن مهتمون بدراسة العلاقة بين T وتحت التفاضل الخاص به ∇T (مجموعة جميع المؤثرات الخطية المحدودة $u : X \longrightarrow Y$ بحيث أن $u(x) \leq T(x)$ لكل x في X) فيما يتعلق ببعض مفاهيم التجميع من نوع ليبشيتز.

كلمات مفتاحية

مشبك بناخ ، مؤثر ليبشيتزي p-مهيمن، مؤثر ليبشيتزي p-مجموع، مؤثر p-مجموع، مؤثر تحت-خطي.

Abstract

In this memory, we introduce and study the notions of summability in the sublinear case. We prove some characterizations in terms of a domination theorem and some properties of this notions.we are interested in studying the relationship between T and its subdifferential ∇T (the set of all bounded linear operators $u : X \longrightarrow Y$ such that $u(x) \leq T(x)$ for all x in X) ; concerning certain notions of Lipschitz summability.

Key words

Banach lattice, Lipschitz p-dominated operator, Lipschitz p-summing operator, p-summing operator, sublinear operator.

Résumé

Dans ce mémoire, nous introduisons et étudions les notions de sommabilité dans le cas sous-linéaire. Nous prouvons quelques caractérisations en termes d'un théorème de domination et certaines propriétés de ces notions. Nous nous intéressons à l'étude de la relation entre T et son sous-différentiel ∇T (l'ensemble de tous les opérateurs linéaires bornés $u : X \longrightarrow Y$ tels que $u(x) \leq T(x)$ pour tout x dans X), en ce qui concerne certaines notions de sommabilité de Lipschitz. .

Mot-clés

Banach réticulé ,opérateur Lipschitzien p-dominé ,opérateur Lipschitzien p-sommant, opérateur p-sommant, opérateur sous-linéaire.