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Intuitionistic Fuzzy Sets: Theory and Applications

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بسم الله الرحمن الرحيم

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Dedication

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I dedicate this work to my family and especially to my parents who always encouraged and giving me needed support to my studies.

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Contents

Introduction

Ever since the concept of fuzzy sets (FSs, for short) was introduced by L.A. Zadeh [21, 22], in 1965, it gained immense interest from the researchers in the subject field. Additionally, two years later, J. Goguen [11] proposed an extension to the fuzzy sets, known as L-fuzzy.

In 1983, K.T. Attanassov [3] introduced, the notion of intuitionistic fuzzy set (for short, IFS) as a generalization of the concept of fuzzy set. In fuzzy set theory, the non-membership degree of an element x of the universe is defined as $\nu_A(x) = 1 - \mu_A(x)$, which is fixed, but in intuitionistic fuzzy set theory, the non-membership degree not equal to $1 - \mu_A(x)$ the only condition in this case is $\nu_A(x) + \mu_A(x) \leq 1$. Absolutly, fuzzy sets are intuitionistic fuzzy sets by setting $\nu_A(x) = 1 - \mu_A(x)$.

Intuitionistic fuzzy relation concept is an extension to this direction and this notion generalise the notion of fuzzy relations which introduced by Burillo and Bustince [7, 6]. The concept of an intuitionistic fuzzy lattice and the notion of intuitionistic fuzzy ideal (resp. filter) on a lattice was introduced by Thomas and Nair [20].

The main objective of this memory is to study some of the properties of the intuitionistic fuzzy sets. It also aims at emphasizing the importance of the later for that it presents different applications in different fields (e.g. electoral system). One of the major mathematical applications of this set is Łukasiewicz-Moisil algebras representation, which the present study intends to further explain it.

The current memory is divided into three chapters:

- The first chapter is an overview of the concept of fuzzy sets. It provides generalities on fuzzy subsets. It also briefly presents triangular norms and conorms as well as the L-fuzzy sets.

- The second chapter presents the fundamental concepts of the intuitionistic fuzzy sets theory, its properties, operations on IFSs, the α-level of IFSs, the operators of IFSs, as well as the relations over IFSs.

- The third chapter is concerned with the application of intuitionistic fuzzy sets in the theory of Łukasiewicz-Moisil algebra.

CHAPTER 1

GENERALITIES ON FUZZY SUBSETS

In this chapter, we recall the basic definitions and properties of binary relations, posets, lattices. Next, we recall some basic notions and properties of fuzzy sets, triangular norms, conorms and L-fuzzy sets. For more details please refer to [12, 11, 13, 21].

1.1 Classical Sets

Definition 1.1.1. *Let X be a set and A be a subset of X $(A \subseteq X)$. Then the function*

$$\chi_A(x) = \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad x \notin A \end{cases}$$

is called the characteristic function of the set A in X.

Remark 1.1.1. *Clearly, the empty set $\emptyset$ has a null characteristic function and the universe set X a unity characteristic function.*

There are several operations defined on classical sets and the following are considered to be basic ones:

Definition 1.1.2 (Operations on classical sets). *Let X be a universal set, let A and B are a subsets of X.*

(i) ***Inclusion*** *: $A \subset B$ if and only if for all $x \in X$, $(x \in A) \Rightarrow (x \in B), i.e, \chi_A(x) \leq \chi_B(x)$.*

(ii) ***Equality*** *: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$ i.e., $(\chi_A(x) = \chi_B(x))$.*

(iii) ***Complement*** *: $A^c = \{x \in X \mid x \notin A\}$ i.e., $\chi_{A^c}(x) = 1 - \chi_A(x)$.*

(iv) ***Intersection*** *: $A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}$i.e., $\chi_{A\cap B}(x) = \min(\chi_A(x), \chi_B(x))$.*

(v) ***Union*** *: $A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}$ i.e., $\chi_{A\cup B}(x) = \max(\chi_A(x), \chi_B(x))$.*

(vi) ***Relative complement*** *: $A\backslash B = A - B = A \cap B^c = \{x \in X \mid x \in A \text{ and } x \notin B\}$ i.e., $\chi_{A-B}(x) = \chi_{A\cap B^c}(x) = \min(\chi_A(x), \chi_{B^c}(x))$.*

Example 1.1.1. *Let $X = \{x, y, z, t, w\}$ be a set, let A and B be two subsets of X such that $A = \{x, y, w\}$ and $B = \{x, y, z\}$. Then*

$$\begin{aligned} A^c &= \{z, t\}; \\ B^c &= \{t, w\}; \\ A \cap B &= \{x, y\}; \\ A \cup B &= \{x, y, z, w\}; \\ A \setminus B &= \{w\}; \\ B \setminus A &= \{z\}. \end{aligned}$$

1.2 Crisp relations

Definition 1.2.1 (Cartesian products). *[17] Let X and Y be two non-empty sets. If $x \in X$ and $y \in Y$, then (x, y) denotes the ordered pair of x with y. The Cartesian product of two sets X and Y is generally defined as the set*

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

Remark 1.2.1. *At times we write X^2 for $X \times X$. In fact, for $n \in N,\ n \geq 2$, X^n denote the set of all ordered n-tuples of elements from X.*

Definition 1.2.2 (Binary relations). *[13] Let X and Y be two non-empty sets. A binary relation R between two sets X and Y is a subset of the cartesian product $X \times Y$.*
For $(x,\ y) \in R \subseteq X \times Y$, we write $(x, y) \in R$ as xRy.

Definition 1.2.3 (Properties of binary relations on a set). *[19] A binary relation R on a set X is called:*

(i) ***reflexive****, if, for all $x \in X$, it holds that xRx;*

(ii) ***irreflexive****, if, for all $x \in X$, it holds hat $xR^c x$;*

(iii) ***symmetric****, if, for any $x,\ y \in X$, it holds that xRy implies that yRx;*

(iv) ***asymmetric****, if, for all $x,\ y \in X$, it holds that xRy implies that $yR^c x$;*

(v) ***antisymmetric****, if, for all $x,\ y \in X$, it holds that xRy and yRx imply that $x = y$;*

(vi) ***transitive****, if, for all $x,\ y,\ z \in X$, it holds that xRy and yRz imply that xRz;*

(vii) ***complete****, if, for all $x,\ y \in X$, either xRy or yRx holds.*

Example 1.2.1.

(i) On $\mathbb{Z}$ the relation $<$ is irreflexive, antisymmetric and transitive.

(ii) For any non-empty set X, ***the equality relation*** *$\Delta = \{(x, x) | \ x \in X\}$ is reflexive, symmetric, antisymmetric and transitive.*

Definition 1.2.4. *[19] A binary relation R on a set X is called:*

(i) a pseudo-order relation, if it is reflexive and antisymmetric;

(ii) a strict order, if it is irreflexive and transitive;

(iii) an order relation, if it is reflexive, antisymmetric and transitive;

(iv) a total order relation, if it is reflexive, antisymmetric, transitive and complete;

(v) an equivalence relation, if, it is reflexive, symmetric and transitive.

Definition 1.2.5. *[10]* ***A partial order*** *(order, for short) is a binary relation $\leq$ over a set X which is reflexive ($a \leq a$ for any $x \in X$), antisymmetric ($x \leq y$ and $y \leq x$ imply $x = y$ for any $x,\ y \in X$) and transitive ($x \leq y$ and $y \leq z$ imply $x \leq z$ for any $x,\ y,\ z \in X$).*
A set with an order relation is called a ***partially ordered set*** *(or, a poset) and we will denote it by $(X, \leq)$.*

Example 1.2.2. *The relation of divisibility is a partial order on* $\mathbb{N}^*$*, then the couple* $(\mathbb{N}^*, |)$ *is a poset.*

Definition 1.2.6 (Chains and antichains)**.** *[10] Let* $(X, \leq)$ *be an ordered set.* X *is a* ***chain*** *if, for all* $x,\ y \in X$*, either* $x \leq y$ *or* $y \leq x$ *(that is, if any two elements of* X *are comparable). Alternative names for a chain are linearly ordered set and totally ordered set. At the opposite extreme from a chain is an antichain. The ordered set* X *is an* ***antichain*** *if* $x \leq y$ *in* X*, this implies* $x = y$ *(that is, if any two elements of* X *are incomparable).*

Example 1.2.3. *(i) The sets* $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ *equipped with the usual order* $\leq$ *are a linearly ordered sets (totally ordered sets).*

(ii) The set D(6)={1,2,3,6} of all divisors of the integer 6 equipped with the relation divide | is a chain.

1.3 Classical lattices

Two of the most important classes of ordered sets defined in this way are lattices and complete lattices. Here we recall some basic definitions and notions concerning lattice.

Lattice

Definition 1.3.1. *[10] Let* X *be a non-empty ordered set.*

(i) $(X, \leq)$ *is called a* ***lattice*** *if* $x \vee y$ *and* $x \wedge y$ *exist, for all* $x,\ y \in X$*;*

(ii) $(X, \leq)$ *is called a* ***complete lattice*** *if* $\sup A$ *and* $\inf A$ *exist, for all* $A \subseteq X$*.*

Proposition 1.3.1. *[10] Let* L *be a lattice. Then the following are equivalent:*

(i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$*, for any* $x,\ y,\ z \in L$*;*

(ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$*, for any* $x,\ y,\ z \in L$*.*

Definition 1.3.2. *[10] Let* L *be a lattice.* L *is said to be* ***distrubitive*** *if it satisfies the distrubitive law,*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),\ for\ any\ x,\ y,\ z \in L.$$

Definition 1.3.3 (Sublattice)**.** *Let* L *be a lattice and let* M *be a non-empty subset of* L*.* M *is a* ***sublattice*** *of* L *if for all* $x,\ y \in L,\ x,\ y \in M$ *implies* $x \vee y \in M$ *and* $x \wedge y \in M$*.*

Ideals and Filters in a Lattice

Now, we recall the notion of ideal and filter of a lattice L as follows :

Definition 1.3.4. *[10] Let* L *be a lattice. A non-empty subset* I *of* L *is called an* ***ideal*** *if for all* $x, y \in L$*:*

(i) *If* $y \in I$ *and* $x \leq y$*, then* $x \in I$*,*

(ii) $x,\ y \in I$ *implies* $x \vee y \in I$*.*

Using the duality principle we define a filter as:

Definition 1.3.5. *[10] Let L be a lattice. A non-empty subset F of L is called a **filter** if for all $x,\ y \in L$:*

(i) *If $y \in F$ and $y \leq x$, then $x \in F$,*

(ii) *$x,\ y \in F$ implies $x \wedge y \in F$.*

Remark 1.3.1. *An ideal and a filter is called **proper** if it does not coincide with L. It is easy to show that an ideal I of a lattice L is proper if and only if $1 \notin I$, and dually, a filter F of L is proper if and only if $0 \notin F$.*

Definition 1.3.6. *[10] Let L be a lattice. A proper ideal I of L is **prime** if and only if $x,\ y \in L$ and $x \wedge y \in I$ imply that $x \in I$ or $y \in I$.*

Dually, a proper filter F of L is **prime** if and only if $x,\ y \in L$ and $x \vee y \in F$ imply that $x \in F$ or $y \in F$.

1.4 Fuzzy Sets

A fuzzy set is a set containing elements that have varying degrees of membership in the set. According to Zadeh [21], fuzzy sets are formally defined as follows:

Definition 1.4.1. *[21] **A fuzzy subset** A in X is characterized by an application*

$$\mu_A\colon X \longrightarrow [0,1],$$

where $[0,1]$ means real numbers between 0 and 1. If x an element of X, $\mu_A(x)$ is the degree of membership of x. The fuzzy set A in X may be represented as a set of ordered pairs of generic element $x \in X$ and its grade of membership, i.e.,

$$A = \{\langle x, \mu_A(x)\rangle |\ x \in X\}.$$

Example 1.4.1. *Let $X = \{a, b, c, d\}$ be a universal set. $A_1 = \{\langle a, 0.2\rangle, \langle b, 0.5\rangle \langle c, 0.8\rangle \langle d, 0.1\rangle\}$, $A_2 = \{\langle a, 0.4\rangle, \langle b, 0.1\rangle, \langle c, 0.0\rangle, \langle d, 0.3\rangle\}$ and $A_3 = \{\langle a, 0.7\rangle, \langle b, 0.9\rangle, \langle c, 0.6\rangle, \langle d, 1.0\rangle\}$ be a fuzzy subsets in X.*

Notation 1.4.1. *Let $\mathcal{F}(X)$ denoted the set of all fuzzy subsets of X.*

Operations on Fuzzy subsets

As we know that the fuzzy set theory was generalized from classical set theory so, fuzzy set also allows operations like equality, inclusion, intersection, union and complement of a fuzzy set, addition and multiplication of two fuzzy subsets. The notion related to the operations on fuzzy sets are defined as follows (see,[13, 21]).

Definition 1.4.2 (Equality and inclusion of fuzzy sets)**.** *Let X be a non-empty set and let A and B two fuzzy subsets*

1. *we say A and B are equal. That is,*

$$A = B \text{ if and only if } \mu_A(x) = \mu_B(x), \text{ for all } x \text{ in } X.$$

2. *We say that A is included in B if and only if $\mu_A(x) \leq \mu_B(x)$ for all x in X. In symbols $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$, for all x in X.*

When a fuzzy set A is included in B, then A is called a fuzzy subset of B.

Definition 1.4.3 (Union and intersection of fuzzy sets)**.** *Let X be a non-empty set and let A and B two fuzzy subsets. For all $x \in X$, the union and the intersection are defined as follows:*

$$\begin{array}{rrcl} Union: & \mu_{A\cup B}(x) & = & \max\{\mu_A(x), \mu_B(x)\} \\ & & = & \mu_A(x) \vee \mu_B(x); \\ Intersection: & \mu_{A\cap B}(x) & = & \min\{\mu_A(x), \mu_B(x)\} \\ & & = & \mu_A(x) \wedge \mu_B(x). \end{array}$$

Definition 1.4.4 ((Complement of a fuzzy set)**.** *The complement of a fuzzy set A is denoted by A^c and is defined by :*

$$\mu_{A^c}(x) = 1 - \mu_A(x), \ for \ all \ x \ in \ X.$$

Proposition 1.4.1. *[5] Let X be a non-empty set and let A, B and $C \in \mathcal{F}(X)$, the following properties hold:*

***Complementary** :*

$$(A^c)^c = A.$$

***Commutativity** :*

$$A \cap B = B \cap A$$
$$A \cup B = B \cup A.$$

***Associativity** :*

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C).$$

***Distributivity** :*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

***De Morgan's laws** :*

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c.$$

***Absorption** :*

$$A \cap (A \cup B) = A$$
$$A \cup (A \cap B) = A.$$

Definition 1.4.5 (Addition). *Let X be a non-empty set and let A and B two fuzzy subsets, the addition defined by for all $x \in X$*

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x).$$

Definition 1.4.6 (Multiplication). *Let X be a non-empty set and let A and B two fuzzy subsets, the multiplication defined by for all $x \in X$*

$$\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x).$$

Characteristics of Fuzzy subset

We can characterize fuzzy sets in more detail by referring to the features used in characterizing the membership functions.

Definition 1.4.7. *[16] Let A be a fuzzy subset on a set X. The **support** of A is the crisp subset on X given by:*

$$supp(A) = \{x \in X |\, \mu_A(x) > 0\}.$$

Definition 1.4.8. *[16] Let A be a fuzzy set on a set X. **The kernel** of A is the crisp subset on X given by :*

$$ker(A) = \{x \in X |\, \mu_A(x) = 1\}.$$

Definition 1.4.9. *[16] Let A be a fuzzy subset on a set X. **The height** of A is the highest value taken by its membership function given by:*

$$H(A) = \sup_{x \in X} \mu_A(x).$$

Example 1.4.2. *Let $X = \{a, b, c, d\}$ and let A be a fuzzy subset on a set X, defined by:*

$$A = \{\langle a, 0.3\rangle, \langle b, 0.6\rangle, \langle c, 0.8\rangle, \langle d, 0.0\rangle\}$$

then :
$supp(A) = \{a, b, c\}$,

$ker(A) = \emptyset$,

$H(A) = 0.8$.

Proposition 1.4.2. *[14] The support and kernel of a fuzzy set A satisfied the following :*

(i) $Supp(A^c) = X - Ker(A)$;

(ii) $Ker(A^c) = X - Supp(A)$.

Proof.

(i)

$$\begin{aligned} Supp(A^c) &= \{x \in X \mid \mu_{A^c}(x) \neq 0\} \\ &= \{x \in X \mid 1 - \mu_A(x) \neq 0\} \\ &= \{x \in X \mid \mu_A(x) \neq 1\} \\ &= \{x \in X \mid x \notin Ker(A)\} \\ &= X - Ker(A). \end{aligned}$$

(ii)

$$\begin{aligned} Ker(A^c) &= \{x \in X \mid \mu_{A^c}(x) = 1\} \\ &= \{x \in X \mid 1 - \mu_A(x) = 1\} \\ &= \{x \in X \mid \mu_A(x) = 0\} \\ &= \{x \in X \mid x \notin Supp(A)\} \\ &= X - Supp(A). \end{aligned}$$

The α-cut of fuzzy sets

Definition 1.4.10. *[14] Let A be a fuzzy set in X and let $\alpha \in]0,1]$. **The α-cut of** A, denoted by A_α, we mean all elements of X that belong to A with a degree at least α. That is, A_α is a classical set defined by*

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}.$$

If $\alpha \leq \beta$, then $A_\beta \subseteq A_\alpha$.

Example 1.4.3. *Let $A = \{< 1, 0.6 >, < 2, 0.8 >, < 3, 0.8 >, < 4, 0.1 >, < 5, 0.1 >\}$*
$A_{0.8} = \{x \in X \mid \mu_A(x) \geq 0.8\}$, then $A_{0.8} = \{< 2, 0.8 >, < 3, 0.8 >\}$.

Remark 1.4.1. *The characteristic function χ_{A_α} is such that : $\chi_{A_\alpha}(x) = 1$ if and only if $\mu_A(x) \geq \alpha$.*

Proposition 1.4.3. *[14] Let X be a non-empty set and let $A, B \in F(X)$, the α-cuts of fuzzy sets verify:*

1. $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$.
2. $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$.
3. $A \subset B \Longleftrightarrow A_\alpha \subset B_\alpha$, *for all* $\alpha \in]0,1]$.
4. $A_1 = Ker(A)$.
5. $A_0 = X$.

Cartesian Product on Fuzzy Set

Definition 1.4.11 (Cartesian product on fuzzy set). *[13] The cartesian product applied to n fuzzy sets can be defined as follows:*
Let $\mu_{A_1}, \mu_{A_2}, ..., \mu_{A_n}$ be membership functions of $A_1, A_2, ..., A_n$. Then, the membership degree of $(x_1, x_2, ..., x_n) \in X_1 \times X_2 \times ... \times X_n$ on the fuzzy set $A_1, A_2, ..., A_n$ is,

$$\mu_{A_1 \times A_2 \times ... \times A_n}(x_1, x_2, ..., x_n) = \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)\}.$$

Example 1.4.4. *Let $X_1 = \{\alpha, \beta\}, X_2 = \{a, b, c\}$ and let $A,\ B$ be two fuzzy subsets defined on $X_1,\ X_2$ respectively given by:*

$$\begin{aligned} A &= \{\langle \alpha, 0.2\rangle, \langle \beta, 0.7\rangle\}, \\ B &= \{\langle a, 0.5\rangle, \langle b, 0.9\rangle, \langle c, 1.0\rangle\}. \end{aligned}$$

then we have:

$$A \times B = \left\{ \begin{array}{c} \langle(\alpha, a), 0.2, \rangle, \langle(\alpha, b), 0.2\rangle, \langle(\alpha, c), 0.2\rangle, \\ \langle(\beta, a), 0.5\rangle, \langle(\beta, b), 0.7\rangle, \langle(\beta, c), 0.7\rangle \end{array} \right\}$$

Fuzzy Relations and Composition

Definition 1.4.12. *[24] Let X and Y be two non-empty sets.* ***A fuzzy relation*** *on $X \times Y$, denoted by R, is defined as the fuzzy set*

$$R = \{((x,y), \mu_R(x,y)) \mid (x,y) \in X \times Y\},$$

where the function μ_R: $X \times Y \to [0,1]$ is called a membership function. It gives the degree of membership of the ordered pair (x,y) in R associating with each pair (x,y) in $X \times Y$ a real number in interval $[0,1]$.

Remark 1.4.2. *A fuzzy relation R is a mapping from cartesian space $X \times Y$ to the interval $[0,1]$. If $X = Y$ we say that R is a binary fuzzy relation in X.*

Example 1.4.5. *Let $X = \{x,y,z\}$ and R a binary fuzzy relation defined in X as:*

μ_R	x	y	z
x	0.3	0.7	0.2
y	0.5	0.8	0.5
z	0.1	0.4	0.5

Definition 1.4.13. *The α-cut sets and support of fuzzy relations is defined as in fuzzy sets, i.e.,* ***The α-cut*** *of a fuzzy relation R: $X \times Y \to [0,1]$ is defined as, for all $x \in X$ and $y \in Y$,*

$$R_\alpha = \{(x,y) \in X \times Y \mid R(x,y) \geq \alpha\}.$$

In the same way, we define ***the support*** *of a fuzzy relation $S(R)$ as*

$$S(R) = \{(x,y) \in X \times Y \mid R(x,y) > 0\}.$$

Definition 1.4.14. (***Composition of Fuzzy Relations***) *[13]*
Let R and S two fuzzy relations are defined on sets X, Y and Z. That is, $R \subseteq X \times Y$, $S \subseteq Y \times Z$. The composition $S \circ R$ of two relations R and S is expressed by the relation from X to Z and this composition is defined by the following.
For $(x,y) \in X \times Y$, $(y,z) \in Y \times Z$,

$$\mu_{S \circ R}(x,z) = \max_y[\min\{\mu_R(x,y), \mu_S(y,z)\}]$$
$$= \vee_y[\mu_R(x,y) \wedge \mu_S(y,z)].$$

Example 1.4.6. *Let $X = \{1,2,3,4\}$, $Y = \{a,b\}$ and $Z = \{\alpha,\beta\}$. Consider fuzzy relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, defined by its following tables.*

R	a	b
1	0.1	0.3
2	1.0	0.4
3	*0.6*	*0.9*
4	0.0	0.7

S	α	β
a	0.2	0.4
b	0.9	0.5

the composition of two relations R and S is as follows

$R \circ S$	α	β
1	0.3	0.3
2	0.4	0.4
3	0.9	0.5
4	0.7	0.5

Types of fuzzy Relations

Definition 1.4.15. *[13] Let R be a fuzzy relation and X, Y two non-empty sets we will say that R is:*

(i) ***Reflexive*** *if for any $x \in X$,*

$$\mu_R(x, x) = 1.$$

(ii) ***Symmetric*** *if for all $(x, y) \in X \times Y$,*

$$\mu_R(x, y) = \mu_R(y, x).$$

(iii) ***Antisymmetric*** *if for all $(x, y) \in X \times Y$,*

$$\begin{cases} \mu_R(x, y) > 0 \\ \text{and} \\ \mu_R(y, x) > 0 \end{cases} \implies x = y.$$

(iv) ***Transitive*** *if for all $x, y \in X$,*

$$R \circ R \subseteq R, \ i.e., \ \mu_{R \circ R}(x, y) \leq \mu_R(x, y).$$

The fuzzy reflexivity, symmetry, antisymmetry and transitivity notion were first defined by Zedeh [24].

Definition 1.4.16. *Let X be a non-empty set.*
A binary fuzzy relation R: $X \times X \longrightarrow [0, 1]$ is called a ***fuzzy equivalence relation*** *in X if R is reflexive, transitive and symmetric.*

Definition 1.4.17. *[9] Let X be a non-empty set.*
A binary fuzzy relation R: $X \times X \longrightarrow [0, 1]$ is called :

(i) ***Fuzzy pre-order relation*** *in X if it is a fuzzy reflexive and transitive relation.*

(ii) ***Fuzzy partial order relation*** *if R is fuzzy reflexive, antisymmetric and transitive.*

(iii) ***Fuzzy total order relation*** *if and only if either $R(x, y) > 0$ or $R(y, x) > 0$ for all $x, \ y \in X$.*

If R is a fuzzy partial order relation on a set X, then (X, R) is called a ***fuzzy partially ordered set*** *or* ***fuzzy poset****. If R is a fuzzy total order relation in a set X, then (X, R) is called a* ***fuzzy totally ordered set*** *or a* ***fuzzy chain.***

1.5 Triangular Norms and Conorms

In this section, we will remind some basic definitions and notions concerning triangular norms and conorms.

Definition 1.5.1. *[12] A **triangular norm** (t-norm for short) is a binary operation T on the unit interval $[0,1]$, i.e., it is a function $T\colon [0,1]^2 \longrightarrow [0,1]$ such that for all $x,\ y,\ z \in [0,1]$ the following four axioms are satisfied :*

(t_1) Commutativity i.e., $T(x,y) = T(y,x)$;

(t_2) Associativity i.e., $T(x,T(y,z)) = T(T(x,y),z)$;

(t_3) Monotonicity i.e., $T(x,y) \leq T(x,z)$ whenever $y \leq z$;

(t_4) Boundary condition i.e., $T(x,1) = x$.

Proposition 1.5.1. *[5] Any t-norm T satisfies $T(0,x) = T(x,0) = 0$, for all $x \in [0,1]$.*

Proof: From (t_4) we have $T(0,1) = 0$. Then from (t_3) it follows that

$$T(0,x) \leq T(0,1) = 0,\ \ for\ all\ x \in [0,1],$$

i.e., $T(0,x) = 0$. Then from (t_1) we get $T(x,0) = 0$.

Example 1.5.1. *The following are the four t-norms $T_M T_p, T_D$ and T_L given respectively by :*

$T_M(x,y) = \min(x,y)$, (*minimum*)

$T_p(x,y) = x.y$, (*product*)

$T_L(x,y) = \max(x+y-1,0)$, (*Łukasiewicz t-norm*)

$T_D(x,y) = \begin{cases} 0 & if \quad (x,y) \in\ [0,1[^2, \\ \min(x,y) & Otherwise. \end{cases}$ (*drastic product*)

As a formal construction, triangular conorms are dual to triangular norms.

Definition 1.5.2. *[12] **A triangular conorm** (t-conorm for short) is a binary operation S on the unit interval $[0,1]$, i.e., it is a function S: $[0,1]^2 \longrightarrow [0,1]$ such that for all $x,\ y,\ z \in [0,1]$ the following four axioms are satisfied :*

(s_1) Commutativity : $S(x,y) = S(y,x)$;

(s_2) Associativity : $S(x,S(y,z)) = S(S(x,y),z)$;

(s_3) Monotonicity : $S(x,y) \leq S(x,z)$ whenever $y \leq z$;

(s_4) Boundary condition : $S(x,0) = x$.

Proposition 1.5.2. *[5] Any t-conorm S satisfies $S(1,x) = S(x,1) = 1$, for all $x \in [0,1]$.*

Proof.

From (s_4) we have $S(1,0)=1$ so, from (s_3) it follows that

$$1 = S(1,0) \leq S(1,x),$$

then

$$S(1,x) = 1 = S(x,1), \, for \; all \; x \in [0,1].$$

Example 1.5.2. *The following are the four basic t-conorms S_M, S_p, S_L, and S_D given respectively by:*

$S_M(x,y) = \max(x,y),$ (*maximum*)

$S_p(x,y) = x + y - x.y,$ (*probabilistic sum*)

$S_L(x,y) = \min(x+y,1),$ (*Łukasiewicz t-conorm, bounded sum*)

$S_D(x,y) = \begin{cases} 1 & if \quad (x,y) \in]0,1]^2, \\ \max(x,y) & Otherwise. \end{cases}$ (*drastic sum*)

Proposition 1.5.3. *[5] We have $T(x,y) \leq x \wedge y$ and $S(x,y) \geq x \vee y$ for any t-norm T, t-conorm S, and any $x,\ y \in [0,1]$.*

Proposition 1.5.4. *[6] Let T be a triangular norm, the mapping defines as :*

$$S\colon [0,1] \times [0,1] \longrightarrow [0,1]$$

$$S(x,y) \equiv 1 - T(1-x, 1-y)$$

will be called dual t-conorm of T.

Remark 1.5.1. *All t-norm is an intersection operator i.e., we can define $A \cap_T B$ by its membership function in the following way :*

$$\mu_{A \cap_T B}(x) = T(\mu_A(x), \mu_B(x)), \; for \; all \; x \; in \; X.$$

All t-conorm is a union operator i.e., we can define $A \cup_S B$ by its membership function as follow :

$$\mu_{A \cup_S B}(x) = S(\mu_A(x), \mu_B(x)), \; for \; all \; x \; in \; X.$$

Example 1.5.3. *Let $X = \{x,y,z\}$, let A and B be two fuzzy subsets on X given by:*

$A = \{\langle x, 0.5\rangle, \langle y, 0.9\rangle, \langle z, 1.0\rangle\},$

$B = \{\langle x, 0.2\rangle, \langle y, 0.4\rangle, \langle z, 0.1\rangle\}.$

We define the intersection and union respectively :

(i) $\mu_{A \cap_T B}(x) = T(\mu_A(x), \mu_B(x)) = \max(\mu_A(x) + \mu_B(x) - 1, 0), for$ *all x in* X;

(ii) $\mu_{A \cup_S B}(x) = S(\mu_A(x), \mu_B(x)) = \min(\mu_A(x) + \mu_B(x), 1), \; for \; all \; x \; in \; X.$

So, we get

(i) $A \cap_T B = \{\langle x, 0.0\rangle, \langle y, 0.3\rangle, \langle z, 0.1\rangle\}$.

(ii) $A \cup_S B = \{\langle x, 0.7\rangle, \langle y, 1.0\rangle, \langle z, 1.0\rangle\}$.

Definition 1.5.3. *[5] A function $N\colon [0,1] \to [0,1]$ is called **negation** if :*

(i) N is non-increasing i.e., $x \le y \Rightarrow N(y) \le N(x)$, for all $x,\ y$ in $[0,1]$;

(ii) $N(0) = 1$ *and* $N(1) = 0$.

*The negation $N\colon [0,1] \to [0,1]$ is called **strict negation** if :*

(i) N is a continuos function;

(ii) N is a strictly decreasing function i.e., $x < y \implies N(x) > N(y)$.

*The strict negation $N\colon [0,1] \to [0,1]$ is called **strong negation** if is a involution that is,*

(i) $N(N(x)) = x$ *for x in* $[0,1]$.

Definition 1.5.4. *[5] A t-norm T and a t-conorm S are said to be dual for strict negation if they satisfy the following formulas for all $x,\ y \in [0,1]$:*

$$S(x,y) = N(T(N(x), N(y))).$$

$$T(x,y) = N(S(N(x), N(y))).$$

1.6 Lattice Valued Fuzzy Sets (L-Fuzzy sets)

J. Goguen [11] proposed an extension to the fuzzy sets, known as L-fuzzy. The definition and notation can be found in [5, 11].

Definition 1.6.1. *Let X be a non-empty set and let L be a complete lattice. **An L-fuzzy subset**, on X, is a mapping :*

$$A\colon X \longrightarrow L$$

That is, the family of all L-fuzzy subset, of X, is just L^X consisting of all mappings from X to L.

Definition 1.6.2 (Operations on L-fuzzy sets)**.** *The basic connectives of L-fussy sets are:*

$$\begin{aligned} A \wedge B(x) &= \inf(A(x), B(x)), \\ A \vee B(x) &= \sup(A(x), B(x)), \\ N(A)(x) &= N(A(x)), \end{aligned}$$

for any $x \in X$.

Example 1.6.1. $([0,1], \le)$ *is a completely distributive lattice. Then classical fuzzy sets can be seen as L-fuzzy sets with $L = [0,1]$. We observe that $[0,1]$ is complete and completely distributive lattice. We observe that the L-fuzzy sets with $L = [0,1]$ coincide with the fuzzy sets $\mathcal{F}(X)$.*

CHAPTER 2

GENERALITIES ON INTUITIONISTIC FUZZY SUBSETS

In this chapter, we recall the definition of intuitionistic fuzzy set, basic notions of intuitionistic fuzzy set which suggested by Atanassov [3] in 1983, and some properties of intuitionistic fuzzy relations and intuitionistic fuzzy lattices.

2.1 Intuitionistic Fuzzy Sets

Definition 2.1.1. *[3, 4] Let X be a non-empty set.* ***An intuitionistic fuzzy set*** *(IFS, for short) A on X is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\},$$

where the functions:

$$\mu_A : X \longrightarrow [0, 1]$$

and

$$\nu_A : X \longrightarrow [0, 1]$$

define the degree of membership and the degree of non-membership, of the element $x \in X$, respectively, and for every $x \in X$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Definition 2.1.2. *[3] The value of*

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x),$$

is called the degree of non-determinacy (or uncertainty) of the element x to the intuitionistic fuzzy set A. Clearly, in the case of ordinary fuzzy sets, $\pi_A(x) = 0$, for every $x \in X$.

Example 2.1.1. *[3] Let X be the set of all countries with elective governments. Assume that we know for every country $x \in X$ the percentage of the electorate that has voted for the corresponding government. Denote it by $M(x)$ and let $\mu(x) = \frac{M(x)}{100}$ (degree of membership, validity, etc.). Let $\nu(x) = 1 - \mu(x)$. This number corresponds to the part of the electorate who have not voted for the government. Using only the fuzzy set theory, we cannot consider this value in more detail. However, if we define $\nu(x)$ (degree of non-membership, non-validity, etc.) as the number of votes given to parties or persons outside the government, then we can show the part of electorate who have not voted at all or who have given bad voting-paper and the corresponding number will be $\pi(x) = 1 - \mu(x) - \nu(x)$ (degree of indeterminacy, uncertainty, etc.). Thus, we can construct the set $\{\langle x, \mu(x), \upsilon(x)\rangle | x \in X\}$ and obviously,*

$$0 \leq \mu(x) + \nu(x) \leq 1.$$

Example 2.1.2. *Let $X = \{a, b\}, I = \{0, \frac{1}{2}, 1\} \subset [0, 1]$ we will find the intuitionistic fuzzy subsets of X. for all $x, y \in I$ we define the sets:*
$\wp = \{\phi : X \longrightarrow I^2\}$ and $\tilde{\wp} = \{\wp : X \longrightarrow I^2 \mid x + y \leq 1\}$, the set of all intuitionistic fuzzy subsets of X.
So, $|\tilde{\wp}| = (|I^2|)^{|X|} - |\{(x, y) \mid x + y > 1\}$. Consequently we need to find 36 intuitionistic fuzzy subsets.

$A_1 = \{\langle a, 0, 0\rangle, \langle b, 0, 0\rangle\}$	$A_7 = \{\langle a, 0, \frac{1}{2}\rangle, \langle b, 0, 0\rangle\}$	$A_{13} = \{\langle a, \frac{1}{2}, \frac{1}{2}\rangle, \langle b, 0, 0\rangle\}$
$A_2 = \{\langle a, 0, 0\rangle, \langle b, 0, \frac{1}{2}\rangle\}$	$A_8 = \{\langle a, 0, \frac{1}{2}\rangle, \langle b, 0, \frac{1}{2}\rangle\}$	$A_{14} = \{\langle a, \frac{1}{2}, \frac{1}{2}\rangle, \langle b, 0, 1/2\rangle\}$
$A_3 = \{\langle a, 0, 0\rangle, \langle b, 0, 1\rangle\}$	$A_9 = \{\langle a, 0, \frac{1}{2}\rangle, \langle b, 0, 1\rangle\}$	$A_{15} = \{\langle a, \frac{1}{2}, \frac{1}{2}\rangle, \langle b, 0, 1\rangle\}$
$A_4 = \{\langle a, 0, 0\rangle, \langle b, \frac{1}{2}, 0\rangle\}$	$A_{10} = \{\langle a, 0, \frac{1}{2}\rangle, \langle b, \frac{1}{2}, 0\rangle\}$	$A_{16} = \{\langle a, \frac{1}{2}, \frac{1}{2}\rangle, \langle b, \frac{1}{2}, 0\rangle\}$
$A_5 = \{\langle a, 0, 0\rangle, \langle b, \frac{1}{2}, \frac{1}{2}\rangle\}$	$A_{11} = \{\langle a, 0, \frac{1}{2}\rangle, \langle b, \frac{1}{2}, \frac{1}{2}\rangle\}$	$A_{17} = \{\langle a, \frac{1}{2}, \frac{1}{2}\rangle, \langle b, \frac{1}{2}, \frac{1}{2}\rangle\}$
$A_6 = \{\langle a, 0, 0\rangle, \langle b, 1, 0\rangle\}$	$A_{12} = \{\langle a, 0, \frac{1}{2}\rangle, \langle b, 1, 0\rangle\}$	$A_{18} = \{\langle a, \frac{1}{2}, \frac{1}{2}\rangle, \langle b, 1, 0\rangle\}$

$A_{19} = \{\langle a, 0, 1\rangle, \langle b, 0, 0\rangle\}$	$A_{25} = \{\langle a, \frac{1}{2}, 0\rangle, \langle b, 0, 0\rangle\}$	$A_{31} = \{\langle a, 1, 0\rangle, \langle b, 0, 0\rangle\}$
$A_{20} = \{\langle a, 0, 1\rangle, \langle b, 0, \frac{1}{2}\rangle\}$	$A_{26} = \{\langle a, \frac{1}{2}, 0\rangle, \langle b, 0, 1/2\rangle\}$	$A_{32} = \{\langle a, 1, 0\rangle, \langle b, 0, \frac{1}{2}\rangle\}$
$A_{21} = \{\langle a, 0, 1\rangle, \langle b, 0, 1\rangle\}$	$A_{27} = \{\langle a, \frac{1}{2}, 0\rangle, \langle b, 0, 1\rangle\}$	$A_{33} = \{\langle a, 1, 0\rangle, \langle b, 0, 1\rangle\}$
$A_{22} = \{\langle a, 0, 1\rangle, \langle b, \frac{1}{2}, 0\rangle\}$	$A_{28} = \{\langle a, \frac{1}{2}, 0\rangle, \langle b, \frac{1}{2}, 0\rangle\}$	$A_{34} = \{\langle a, 1, 0\rangle, \langle b, \frac{1}{2}, 0\rangle\}$
$A_{23} = \{\langle a, 0, 1\rangle, \langle b, \frac{1}{2}, \frac{1}{2}\rangle\}$	$A_{29} = \{\langle a, \frac{1}{2}, 0\rangle, \langle b, \frac{1}{2}, \frac{1}{2}\rangle\}$	$A_{35} = \{\langle a, 1, 0\rangle, \langle b, \frac{1}{2}, \frac{1}{2}\rangle\}$
$A_{24} = \{\langle a, 0, 1\rangle, \langle b, 1, 0\rangle\}$	$A_{30} = \{\langle a, \frac{1}{2}, 0\rangle, \langle b, 1, 0\rangle\}$	$A_{36} = \{\langle a, 1, 0\rangle, \langle b, 1, 0\rangle\}$

Definition 2.1.3. *[5] Intuitionistic fuzzy sets can be seen as L-fuzzy sets by considering the lattice $L \subseteq [0, 1]^2$,*

$$L = \{(x, y) \in [0, 1]^2 \mid x + y \leq 1\}$$

where the inequality relation generating the lattice structure is defined by

$$(x, y) \leq_L (z, t) \Leftrightarrow [x \leq z \text{ and } y \geq t].$$

Moreover, L in this case is a complete, completely distributive lattice.

Operations on Intuitionistic Fuzzy sets

For two intuitionistic fuzzy sets A and B on a set X, several operations are defined in the following way(see [3, 4]) :

- **Inclusion**
  $$A \subseteq B \text{ if } \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x), \text{ for any } x \in X.$$
- **Equality**
  $$A = B \text{ if } \mu_A(x) = \mu_B(x) \text{ and } \nu_A(x) = \nu_B(x), \text{ for any } x \in X.$$
- **Complement**
  $$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in X\}.$$
- **Empty set**
  $$A = \emptyset \text{ if } \mu_A(x) = 0 \text{ and } \nu_A(x) = 1.$$
- **Intersection**
  $$A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \ \nu_A(x) \vee \nu_B(x)\rangle | \ x \in X\}.$$

- **Union**
$$A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle | \; x \in X\}.$$
- **Addition**
$$A \oplus B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), \nu_A(x).\nu_B(x)\rangle | \; x \in X\}.$$
- **Multiplication**
$$A \otimes B = \{\langle x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x).\nu_B(x)\rangle | x \in X\}.$$

Example 2.1.3. *Let $X = \{a, b, c\}$ and Let $A,\ B$ be two intuitionistic fuzzy subsets given by:*

$$A = \{\langle a, 0.3, 0.5\rangle, \langle b, 0.6, 0.4\rangle, \langle c, 0.9, 0.0\rangle\};$$
$$B = \{\langle a, 0.7, 0.1\rangle, \langle b, 0.8, 0.1\rangle, \langle c, 1.0, 0.0\rangle\}.$$

Then, we have:

$$\begin{aligned} A \cap B &= \{\langle a, 0.3, 0.5\rangle, \langle b, 0.6, 0.4\rangle, \langle c, 0.9, 0.0\rangle\}; \\ A \cup B &= \{\langle a, 0.7, 0.1\rangle, \langle b, 0.8, 0.1\rangle, \langle c, 1.0, 0.0\rangle\}; \\ A^c &= \{\langle a, 0.5, 0.3\rangle, \langle b, 0.4, 0.6\rangle, \langle c, 0.0, 0.9\rangle\}; \\ B^c &= \{\langle a, 0.1, 0.7\rangle, \langle b, 0.1, 0.8\rangle, \langle c, 0.0, 1.0\rangle\}; \\ A \oplus B &= \{\langle a, 0.79, 0.05\rangle, \langle b, 0.92, 0.04\rangle, \langle c, 1.0, 0.0\rangle\}; \\ A \otimes B &= \{\langle a, 0.21, 0.55\rangle, \langle b, 0.48, 0.46\rangle, \langle c, 0.9, 0.0\rangle\}. \end{aligned}$$

Definition 2.1.4. *[18] Let A be an intuitionistic fuzzy set on universe X. **The support of** A is the crisp subset of X given by:*

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } \nu_A(x) < 1)\}.$$

Definition 2.1.5. *[18] Let A be an intuitionistic fuzzy set on universe X. **The kernel of** A is the crisp subset of X given by:*

$$Ker(A) = \{x \in X \mid \mu_A(x) = 1 \text{ and } \nu_A(x) = 0\}.$$

Example 2.1.4. *From Example 2.1.3 we have :*

$Ker(A) = \emptyset$;

$Supp(A) = \{a, b, c\}$.

Intuitionistic Fuzzy Sets of a Certain Level

This concept follows that of a fuzzy set of level α, where $\alpha \in [0, 1]$. Formally, we have the following definition.

Definition 2.1.6. *[3, 4] Let A be an intuitionistic fuzzy set on a set X, and $\alpha,\ \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. A set of (α, β)-level generated by an IFS A, is defined as:*

$$N_{\alpha,\beta}(A) = \{x \in X \mid \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\}.$$

We will also define two other sets related to an IFS, $\alpha \in [0,1]$ is a fixed number.

Definition 2.1.7. *[3, 4] The set*

$$N_\alpha(A) = \{x \in X \mid \mu_A(x) \geq \alpha\}$$

is called a set of level of membership α generated by A.

And

$$N^\alpha(A) = \{x \in X \mid \nu_A(x) \leq \alpha\}$$

is called a set of level of non-membership α generated by A.

Example 2.1.5. *Let A be an intuitionistic fuzzy subset an a set $X = \{a,b,c,d\}$, defined by:*

$$A = \{\langle a, 0.3, 0.5\rangle, \langle b, 0.7, 0.1\rangle, \langle c, 0.1, 0.8\rangle, \langle d, 0.0, 1.0\rangle\},$$

then

$N_{0.2,0.4}(A) = \{b\}$;

$N_{0.2}(A) = \{a,b\}$;

$N^{0.4}(A) = \{b\}$.

Theorem 2.1.1. *[4] For every IFS A and for every $\alpha, \beta \in [0,1]$, there holds*

$$N_{\alpha,\beta}(A) = N_\alpha(A) \cap N^\beta(A).$$

Proof. Easily from definition 2.1.6.

Operators over Intuitionistic Fuzzy Sets

In 1986, K.T. Atanassov established different ways of changing an intuitionistic fuzzy set into a fuzzy set and defined the following operators:

"Necessity" and "Possibility" Operators

Definition 2.1.8. *[3, 4] Let A be an intuitionistic fuzzy subset on a set X.*

(i) ***The necessity operator*** *is defined as :*

$$[A] = \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle \mid x \in X\}.$$

(ii) ***The possibility operator*** *is defined as :*

$$\langle A\rangle = \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle \mid x \in X\}.$$

Example 2.1.6. *Let $X = \{x, y, z\}$ and let A be intuitionistic fuzzy set given by:*

$$A = \{\langle x, 0.2, 0.3\rangle, \langle y, 0.4, 0.4\rangle, \langle z, 0.6, 0.2\rangle\}$$

then, the necessity and possibility operators have the forms :

$$[A] = \{\langle x, 0.2, 0.8\rangle, \langle y, 0.4, 0.6\rangle, \langle z, 0.6, 0.4\rangle\};$$

$$\langle A\rangle = \{\langle x, 0.7, 0.3\rangle, \langle y, 0.6, 0.4\rangle, \langle z, 0.8, 0.2\rangle\}.$$

Remark 2.1.1. *Obviously, if A is an ordinary fuzzy set, then*

$$[A] = A = \langle A\rangle.$$

Operators D_α and $F_{\alpha,\beta}$

Definition 2.1.9. *[7] Let A be an intuitionistic fuzzy subset.* ***An operator*** *D_p is defined as follows:*

$$D_p(A) = \{\langle x, \mu_A(x) + p.\pi_A(x), 1 - \mu_A(x) - p.\pi_A(x)\rangle | \, x \in X\},$$

with $p \in [0, 1]$ be a fixed number.

Remark 2.1.2. *From the above definition, it follows that $D_p(A)$ is a fuzzy set, because*

$$\mu_A(x) + p.\pi_A(x) + 1 - \mu_A(x) - p.\pi_A(x) = 1.$$

Example 2.1.7. *Let $X = \{x, y, z\}$ and $A_1,\ A_2,\ A_3$ be an intuitionistic fuzzy subsets given by:*

$$A_1 = \{\langle x, 0.3, 0.4\rangle, \langle y, 0.2, 0.6\rangle, \langle z, 0.5, 0.0\rangle\};$$

$$A_2 = \{\langle x, 0.7, 0.3\rangle, \langle y, 0.2, 0.2\rangle, \langle z, 0.6, 0.1\rangle\};$$

$$A_3 = \{\langle x, 0.2, 0.5\rangle, \langle y, 0.0, 0.9\rangle, \langle z, 0.1, 0.4\rangle\}.$$

Then

$$\pi_{A_1} = \{\langle x, 0.3\rangle, \langle y, 0.2\rangle, \langle z, 0.5\rangle\};$$

$$\pi_{A_2} = \{\langle x, 0.0\rangle, \langle y, 0.6\rangle, \langle z, 0.3\rangle\};$$

$$\pi_{A_3} = \{\langle x, 0.3\rangle, \langle y, 0.1\rangle, \langle z, 0.5\rangle\}.$$

Therefore, we have:

$$D_{0.2}(A_1) = \{\langle x, 0.36, 0.64\rangle, \langle y, 0.24, 0.76\rangle, \langle z, 0.6, 0.4\rangle\};$$

$$D_{0.2}(A_2) = \{\langle x, 0.7, 0.3\rangle, \langle y, 0.72, 0.28\rangle, \langle z, 0.66, 0.34\rangle\};$$

$$D_{0.2}(A_3) = \{\langle x, 0.26, 0.74\rangle, \langle y, 0.02, 0.98\rangle, \langle z, 0.2, 0.8\rangle\}.$$

Proposition 2.1.1. *[3, 4] For every intuitionistic fuzzy set A and for every $\alpha,\ \beta \in [0,1]$ we have:*

(i) *If $\alpha \leq \beta$, then $D_\alpha(A) \leq D_\beta(A)$,*

(ii) $D_0(A) = [A]$,

(iii) $D_1(A) = \langle A \rangle$.

Proof.

1. We have $\alpha \leq \beta$, then $\alpha.\pi_A(x) \leq \beta.\pi_A(x)$
therfore $\mu_A(x) + \alpha.\pi_A(x) \leq \mu_A(x) + \beta.\pi_A(x)$
on the other hand we have

$$\alpha \leq \beta \text{ , then } -\mu_A(x) - \alpha.\pi_A(x) \geq -\mu_A(x) - \beta.\pi_A(x)$$

then

$$1 - \mu_A(x) - \alpha.\pi_A(x) \geq 1 - \mu_A(x) - \beta.\pi_A(x)$$

Hence, $D_\alpha(A) \leq D_\beta(A)$.

the proof of (ii) and (iii) easily from definition 2.1.9

Definition 2.1.10. *[3, 4] Let $\alpha,\ \beta \in [0,1]$ and $\alpha + \beta \leq 1$.* ***The operator*** *$F_{\alpha,\beta}$, for the intuitionistic fuzzy set A is defined as :*

$$F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha \cdot \pi_A(x), \nu_A(x) + \beta \cdot \pi_A(x)\rangle |\ x \in X\}.$$

Theorem 2.1.2. *[3, 4] For every intuitionistic fuzzy set A and for every $\alpha,\ \beta,\ \gamma \in [0,1]$, such that $\alpha + \beta \leq 1$:*

(i) *$F_{\alpha,\beta}(A)$ is an IFS,*

(ii) *If $0 \leq \gamma \leq \alpha$, then $F_{\gamma,\beta}(A) \subset F_{\alpha,\beta}(A)$,*

(iii) *If $0 \leq \gamma \leq \beta$, then $F_{\alpha,\beta}(A) \subset F_{\alpha,\gamma}(A)$,*

(iv) $D_\alpha(A) = F_{\alpha,1-\alpha}(A)$,

(v) $[A] = F_{0,1}(A)$,

(vi) $\langle A \rangle = F_{1,0}(A)$.

Proof. Easily from the definition of the operator $F_{\alpha,\beta}$.

Cartesian Product on Intuitionistic Fuzzy Set

Definition 2.1.11. *The cartesian product applied to n intuitonistic fuzzy sets can be defined as follows: Let $\mu_{A_1}, \mu_{A_2}, ..., \mu_{A_n}$ be membership functions of $A_1, A_2, ..., A_n$. Then the membership degree of $(x_1, x_2, ..., x_n) \in X_1 \times X_2 \times ... \times X_n$ on the intuitonistic fuzzy set $A_1, A_2, ..., A_n$ is,*

$$\begin{aligned}\mu_{A_1 \times A_2 \times ... \times A_n}(x_1, x_2, ..., x_n) &= \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)\} \\ &= \mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2) \wedge ... \wedge \mu_{A_n}(x_n)\end{aligned}$$

and the non-membership degree is,

$$\begin{aligned}\nu_{A_1 \times A_2 \times ... \times A_n}(x_1, x_2, ..., x_n) &= \max\{\nu_{A_1}(x_1), \nu_{A_2}(x_2), ..., \nu_{A_n}(x_n)\} \\ &= \nu_{A_1}(x_1) \vee \nu_{A_2}(x_2) \vee ... \vee \nu_{A_n}(x_n).\end{aligned}$$

Example 2.1.8. *Let $X_1 = \{\alpha, \beta\}$, $X_2 = \{a, b, c\}$, and let A_1, A_2 be two intuitionistic fuzzy subsets defined on X_1, X_2 respectively given by:*

$$\begin{aligned}A_1 = &\ \{\langle \alpha, 0.2, 0.6\rangle, \langle \beta, 0.7, 0.2\rangle\}; \\ A_2 = &\ \{\langle a, 0.5, 0.4\rangle, \langle b, 0.9, 0.1\rangle, \langle c, 1.0, 0.0\rangle\}.\end{aligned}$$

Then, we have:

$$A_1 \times A_2 = \left\{ \begin{array}{c} \langle (\alpha, a), 0.2, 0.6, \rangle, \langle (\alpha, b), 0.2, 0.6\rangle, \langle (\alpha, c), 0.2, 0.6\rangle, \\ \langle (\beta, a), 0.5, 0.4\rangle, \langle (\beta, b), 0.7, 0.2\rangle, \langle (\beta, c), 0.7, 0.2\rangle \end{array} \right\}.$$

2.2 Intuitionistic Fuzzy Relations

This section contains the basic definitions and properties of intuitionistic fuzzy relations which introduced by Burillo and Bustince [7, 6] as a natural generalization of fuzzy relation.

Definition 2.2.1. *[8, 6] Let X and Y be two non-empty sets.* ***An intuitionistic fuzzy binary relation*** *from X to Y (IFR, for short) is an intuitionistic fuzzy subset of $X \times Y$, i.e., is an expression R given by*

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle | (x, y) \in X \times Y\}$$

where

$$\mu_R : X \times Y \longrightarrow [0, 1]$$

and

$$\nu_R : X \times Y \longrightarrow [0, 1]$$

satisfy the condition $0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1$, for every x, $y \in X \times Y$. The value $\mu_R(x, y)$ is called the degree of membership of (x, y) in R and $\nu_R(x, y)$ is called the degree of non-membership of (x, y) in R .

Notation 2.2.1. *We will denote by $IFR(X \times Y)$ the set of all the intuitionistic fuzzy subsets in $X \times Y$.*

Example 2.2.1. *Let $X = \{x, y, z\}$ and R be intuitionistic fuzzy binary relation on X given by:*

μ_R	x	y	z
x	0.2	0.5	0.1
y	0	0.4	0.7
z	1	0.6	0.6

ν_R	x	y	z
x	0.6	0.4	0.3
y	0.7	0.2	0.1
z	0	0.3	0.2

Remark 2.2.1.

1. *If* $\mu_R(x,y) \neq 1$, *then* $\nu_R(x,y) \neq 0$.
2. *This definition of intuitionistic fuzzy relation includes, as a particular case, binary relation and fuzzy relation in a way similar intuitionistic fuzzy relation.*

Definition 2.2.2. *[6] Given a binary intuitionistic fuzzy relation between* X *and* Y, *we can define* R^{-1} *between* Y *and* X *by means of*

$$\mu_{R^{-1}}(y,x) = \mu_R(x,y)$$
$$\nu_{R^{-1}}(y,x) = \nu_R(x,y),\ \forall (x,y) \in X \times Y$$

*to which we will call **inverse relation** of* R.

Operations on Intuitionistic Fuzzy Relations

Let R and P be two intuitionistic fuzzy relations from a universe X to a universe Y, for every $(x,y) \in X \times Y$, we can define (see [6]).

1. $R \subseteq P \Leftrightarrow \mu_R(x,y) \leq \mu_P(x,y)$ and $\nu_R(x,y) \geq \nu_P(x,y)$;
2. $R \cup P = \{\langle (x,y), \mu_R(x,y) \vee \mu_P(x,y), \nu_R(x,y) \wedge \nu_P(x,y) \rangle |\, x \in X, y \in Y\}$;
3. $R \cap P = \{\langle (x,y), \mu_R(x,y) \wedge \mu_P(x,y), \nu_R(x,y) \vee \nu_P(x,y) \rangle |\, x \in X, y \in Y\}$;
4. $R_c = \{\langle (x,y), \nu_R(x,y), \mu_R(x,y) \rangle |\, x \in X, y \in Y\}$.

Properties 2.2.1. *[6] Let* R, P *and* Q *be three intuitionistic fuzzy relations from a universe* X *to a universe* Y.

(i) *If* $R \subseteq P$, *then* $R^{-1} \subseteq P^{-1}$;

(ii) $(R \cup P)^{-1} = R^{-1} \cup P^{-1}$;

(iii) $(R \cap P)^{-1} = R^{-1} \cap P^{-1}$;

(iv) $(R^{-1})^{-1} = R$;

(v) $R \cap (P \cup Q) = (R \cap P) \cup (R \cap Q)$ *and* $R \cup (P \cap Q) = (R \cup P) \cap (R \cup Q)$;

(vi) $R \subseteq (R \cup P)$, $P \subseteq (R \cup P)$, $R \cap P \subseteq R$ *and* $R \cap P \subseteq P$;

(vii) *If* $P \subseteq R$ *and* $Q \subseteq R$, *then* $P \cup Q \subseteq R$;

(viii) *If* $R \subseteq P$ *and* $R \subseteq Q$, *then* $R \subseteq P \cap Q$.

Proof.

(i) If $R \subseteq P$, then $\mu_{R^{-1}}(y,x) = \mu_R(x,y) \leq \mu_P(x,y) = \mu_{P^{-1}}(y,x)$ for every (x,y) of $X \times Y$, analogously

$$\nu_{R^{-1}}(y,x) = \nu_R(x,y) \geq \nu_P(x,y) = \nu_{P^{-1}}(y,x)$$

for every (x,y) of $X \times Y$.

(ii)

$$\begin{aligned}\mu_{(R\cup P)^{-1}}(y,x) &= \mu_{R\cup P}(x,y)\\ &= \mu_R(x,y) \vee \mu_P(x,y)\\ &= \mu_{R^{-1}}(y,x) \vee \mu_{P^{-1}}(y,x)\\ &= \mu_{R^{-1}\cup P^{-1}}(y,x).\end{aligned}$$

the proof for $\nu_{(R\cup P)^{-1}}(y,x) = \nu_{R^{-1}\cup P^{-1}}(y,x)$ is done in a similar way.

(v) We will use the fact that $([0,1], \leq, \wedge, \vee)$ is a distrubitive lattice

$$\begin{aligned}\mu_{(R\cap(P\cup Q))}(x,y) &= \mu_R(x,y) \wedge \{\mu_P(x,y) \vee \mu_Q(x,y)\}\\ &= \{\mu_R(x,y) \wedge \mu_P(x,y)\} \vee \{\mu_R(x,y) \wedge \mu_Q(x,y)\}\\ &= \mu_{(R\cap P)}(x,y) \vee \mu_{(R\cap Q)}(x,y)\\ &= \mu_{(R\cap P)\cup(R\cap Q)}(x,y).\end{aligned}$$

The proof is analogous to the previous one, in the case of

$$\nu_{R\cap(P\cup Q)}\,(x,y) = \nu_{(R\cap P)\cup(R\cap Q)}(x,y).$$

The rest of the items are proved in a way similar to the previous ones.

Composition of Intuitionistic Fuzzy Relations

Definition 2.2.3. *[6] Let $\alpha,\ \beta,\ \lambda$ and ρ be t-norms or t-conorms not necessarily dual two-two, $R \in IFR\,(X \times Y)$ and $P \in IFR\,(Y \times Z)$. We will call **composed relation** $P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R \in IFR\,(X \times Z)$ to the one defined by*

$$P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R = \left\{ \langle (x,z), \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z), \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) \rangle \,|\, x \in X, z \in Z \right\}$$

where

$$\mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) = \underset{y}{\alpha}\left\{\beta\left[\mu_R(x,y), \mu_P(y,z)\right]\right\}$$

$$\nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) = \underset{y}{\lambda}\left\{\rho\left[\nu_R(x,y), \nu_P(y,z)\right]\right\}$$

whenever

$$0 \leq \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) + \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) \leq 1, \ \textit{for all } (x,z) \in X \times Z.$$

Proposition 2.2.1. *[6] In the conditions of the Definition 2.2.3, if λ^* and ρ^* are respectively, the dual forms of λ and ρ and $\alpha \leq \lambda^*$, $\beta \leq \rho^*$, then*

$$0 \leq \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) + \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) \leq 1, \ \textit{for all } (x,z) \in X \times Z.$$

Proof. We know that

$$\mu_R(x,y) \leq 1 - \nu_R(x,y), \text{ for all}(x,y) \in X \times Y$$

and

$$\mu_P(y,z) \leq 1 - \nu_P(y,z), \text{ for all}(y,z) \in Y \times Z$$

taking as hypothesis $\alpha \leq \lambda^*$ and $\beta \leq \rho^*$, we have:

$$\beta\left[\mu_R(x,y), \mu_P(y,z)\right] \leq \rho^*\left[1 - \nu_R(x,y), 1 - \nu_P(y,z)\right];$$

$$\begin{aligned}\underset{y}{\alpha}\left\{\beta\left[\mu_R(x,y), \mu_P(y,z)\right]\right\} &\leq \underset{y}{\lambda^*}\left\{\rho^*\left[1 - \nu_R(x,y), 1 - \nu_P(y,z)\right]\right\} \\ &= 1 - \underset{y}{\lambda}\left\{1 - \rho^*\left[1 - \nu_R(x,y), 1 - \nu_P(y,z)\right]\right\} \\ &= 1 - \underset{y}{\lambda}\{\rho[\nu_R(x,y), \nu_P(y,z)]\}\end{aligned}$$

therefore

$$\underset{y}{\alpha}\{\beta[\mu_R(x,y), \mu_P(y,z)]\} + \underset{y}{\lambda}\{\rho[\nu_R(x,y), \nu_P(x,y)]\},$$

i.e.,

$$0 \leq \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) + \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) \leq 1, \text{ for all } (x,z) \in X \times Z.$$

Intuitionistic Fuzzy Binary Relations

We will study now the properties of the binary intuitionistic fuzzy relations in a set X.

Identity relation

Definition 2.2.4. *[6]*

*(i) The relation $\Delta \in IFR(X \times X)$ is called **the relation of identity** if,*

$$\mu_\Delta(x,y) = \begin{cases} 1 & if \quad x = y \\ 0 & if \quad x \neq y, \end{cases}$$

$$\nu_\Delta(x,y) = \begin{cases} 0 & if \quad x = y \\ 1 & if \quad x \neq y \end{cases}$$

for all $x,\ y \in X \times X$.

(ii) The complementary relation Δ_c defined by

$$\mu_{\Delta_c}(x,y) = \begin{cases} 0 & if \quad x = y \\ 1 & if \quad x \neq y, \end{cases}$$

$$\nu_\Delta(x,y) = \begin{cases} 1 & if \quad x = y \\ 0 & if \quad x \neq y \end{cases}$$

for all $x,\ y \in X \times X$.

Remark 2.2.2. *Is evident $\Delta = \Delta^{-1}$.*

Reflexivity and Antireflexivity

Definition 2.2.5. *[8, 6] Let R be an intuitionistic fuzzy relation on X, we will say that R is :*

1. ***Reflexive**, if for every $x \in X,\ \mu_R(x,x) = 1$. Just notice that for every $x \in X,\ \nu_R(x,x) = 0$;*

2. ***Antireflexive**, if for every $x \in X,$ then $\begin{cases} \mu_R(x,x) &= 0 \\ \nu_R(x,x) &= 1, \end{cases}$ that is to say, if its complementary R_c is reflexive.*

Symmetry and Antisymmetry

Definition 2.2.6. *[8, 6] Let R be an intuitionistic fuzzy relation on X, we will say that R is :*

1. ***Symmetric**, if $R = R^{-1}$, that is, for every (x,y) in $X \times X$,*

$$\begin{cases} \mu_R(x,y) &= \mu_R(y,x) \\ \nu_R(x,y) &= \nu_R(y,x) \end{cases}$$

in a contrary manner we will say that it ia asymmetric.

2. ***Antisymmetrical intuitionistic** if,*

for every (x,y) of $X \times X$, $x \neq y$ then

$$\begin{cases} \mu_R(x,y) &\neq \mu_R(y,x) \\ \nu_R(x,y) &\neq \nu_R(y,x) \\ \pi_R(x,y) &= \pi_R(y,x), \end{cases}$$

where $\pi_R(x,y) = 1 - \mu_R(x,y) - \nu_R(x,y)$.

Theorem 2.2.1. *[6] Let R be an element of $FR(X \times X)$. R is antisymmetrical fuzzy intuitionistic if and only if:*

for every $(x,y) \in X \times X,\ x \neq y$ then $\mu_R(x,y) \neq \mu_R(y,x)$.

Proof. As $\nu_R(x,y) = 1 - \mu_R(x,y)$ and $\pi_R(x,y) = 0$ for every $(x,y) \in X \times X$, then

$$\mu_R(x,y) \neq \mu_R(y,x) \text{ if and only if } \begin{cases} \mu_R(x,y) \neq \mu_R(y,x) \\ \nu_R(x,y) \neq \nu_R(y,x) \\ \pi_R(x,y) = \pi_R(y,x) \end{cases}$$

Definition 2.2.7. *[6] Let R be an intuitionistic fuzzy relation on X, we will say that R is **Perfect antisymmetrical fuzzy intuitionistic** relation if, for every $x,\ y \in X$ with $x \neq y$ and*

$$\begin{cases} \mu_R(x,y) > 0 \\ or \\ \mu_R(x,y) = 0 \quad and \quad \nu_R(x,y) < 1, \end{cases}$$

then

$$\begin{cases} \mu_R(y,x) = 0 \\ and \\ \nu_R(y,x) = 1. \end{cases}$$

Example 2.2.2. *Let $X = \{x, y, z\}$, let R be the intuitionistic fuzzy relation given by:*

μ_R	x	y	z
x	0.4	0.3	0.1
y	0.0	0.5	0.0
z	0.0	0.0	0.1

ν_R	x	y	z
x	0.5	0.7	0.4
y	1	0.3	0.6
z	1	1	0.7

R is a perfect antisymmetrical intuitionistic fuzzy relation.

Transitivity and c-transitivity

Definition 2.2.8. *[8, 6] Let's take α t-conorme, β t-norme, λ t-norme and ρ t-conorme.*

- *We will say that $R \in IFR(X \times X)$ is **transitive** if $R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R \subseteq R$.*
- *We will say that $R \in IFR(X \times X)$ is **c-transitive** if $R \subseteq R \overset{\lambda,\beta}{\underset{\alpha,\rho}{\circ}} R$.*

Definition 2.2.9. *[23] Let X be a non-empty crisp set and*

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle |\, x, y \in X\}$$

*be an intuitionistic fuzzy relation on X. Then R is called **an intuitionistic fuzzy ordering** or a partial intuitionistic fuzzy order if it is reflexive, transitive and perfect antisymmetrical fuzzy intuitionistic.*

Example 2.2.3. *Let $X = \{a, b, c, d\}$. Then the intuitionistic fuzzy relation R defined on X by*

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle |\, x, y \in X\},$$

where μ_R and ν_R given by the following tables :

μ_R	a	b	c	d
a	1.0	0.0	0.0	0.45
b	0.0	1.0	0	0.2
c	0.0	0.0	1.0	0.0
d	0.0	0.0	0.0	1.0

ν_R	a	b	c	d
a	0.0	1.0	0.25	0.3
b	0.2	0.0	0.5	0.2
c	1.0	1.0	0.0	0.6
d	1.0	1.0	1.0	0.0

is intuitionistic fuzzy ordering on X .

Example 2.2.4. *Let $m, n \in X = \{1, 2, 3, 4\}$ and let μ_R, ν_R defined as folow :*

$$\mu_R(m, n) = \begin{cases} 1,\ if & m = n \\ 1 - \frac{m}{n},\ if & m < n \\ 0,\ if & m > n, \end{cases} \quad and \quad \nu_R(m, n) = \begin{cases} 0,\ if & m = n \\ \frac{m}{2n},\ if & m < n \\ 1,\ if & m > n \end{cases}$$

The intuitionistic fuzzy relation R on X is an intuitionistic fuzzy ordering.

Now, we recall the notion of linear or total intuitionistic fuzzy ordering as follows:

Definition 2.2.10. *[23] An intuitionistic fuzzy ordering R is **linear** (or total) on X if for every $x, y \in X$, we have:*

$$\mu_R(x, y) > 0 \textit{ and } \nu_R(x, y) = 0$$

or

$$\mu_R(y, x) > 0 \textit{ and } \nu_R(x, y) = 0$$

Definition 2.2.11. *[23] An intuitionistic fuzzy ordered set (X, μ_R, ν_R) in which R is linear is called a **linearly intuitionistic fuzzy ordered set** or **an intuitionistic fuzzy chain**.*

Conversly, we obtain the following definition of incomparable elements.

Definition 2.2.12. *[23] Let (X, μ_R, ν_R) be a non-empty intuitionistic fuzzy ordered set and let $a,\ b$ be two elements of X. We say that a and b are incomparable in (X, μ_R, ν_R) if*

$$\mu_R(a, b) = 0 \textit{ or } \nu_R(a, b) > 0$$

and

$$\mu_R(b, a) = 0 \textit{ or } \nu_R(b, a) > 0.$$

2.3 Intuitionistic Fuzzy lattices

In this section, we recall the basic definitions and properties of intuitionistic fuzzy lattices, intuitionistic fuzzy ideals (resp. filter) and characterize the prime intuitionistic fuzzy ideals (resp. intuitionistic fuzzy filters).

Definition 2.3.1. *[2, 20] Let L be a lattice and $A = \{x, \mu_A(x), \nu_A(x) \mid x \in L\}$ be an intuitionistic fuzzy subset of L. Then A is called an intuitionistic fuzzy sublattice of L (IFL, for short), if the following conditions are satisfied for all $x,\ y \in L$:*

(i) $\mu_A(x \sqcup y) \geq \min\{\mu_A(x), \mu_A(y)\}$;

(ii) $\mu_A(x \sqcap y) \geq \min\{\mu_A(x), \mu_A(y)\}$;

(iii) $\nu_A(x \sqcup y) \leq \max\{\nu_A(x), \nu_A(y)\}$;

(iv) $\nu_A(x \sqcap y) \leq \max\{\nu_A(x), \nu_A(y)\}$.

Example 2.3.1. *Consider the lattice L of divisors of 15. That is $L = \{1, 3, 5, 15\}$. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in L\}$ be given by:*

$$A = \{\langle 1, 0.5, 0.1\rangle, \langle 3, 0.4, 0.5\rangle, \langle 5, 0.4, 0.3\rangle, \langle 15, 0.7, 0.3\rangle\}.$$

Then A is an IFL of L.

Intuitionistic Fuzzy Ideals and Filters on a Lattice

This subsection contains the definitions of intuitionistic fuzzy ideal (resp. filter) on a lattice was first introduced by Thomas and Nair [20], and a basic characterization of these concepts.

Definition 2.3.2. *[20, 15] An intuitionistic fuzzy subset* I *of* L *is called an intuitionistic fuzzy ideal of* L *(IF-ideal, for short), if the following conditions are satisfied for all* $x,\ y \in L$*:*

(i) $\mu_I(x \sqcup y) \geq \min\{\mu_I(x), \mu_I(y)\}$;

(ii) $\mu_I(x \sqcap y) \geq \max\{\mu_I(x), \mu_I(y)\}$;

(iii) $\nu_I(x \sqcup y) \leq \max\{\nu_I(x), \nu_I(y)\}$;

(iv) $\nu_I(x \sqcap y) \leq \min\{\nu_I(x), \nu_I(y)\}$.

Example 2.3.2. *Consider the lattice* $L = \{1, 2, 3, 4, 6, 12\}$ *of divisors of 12. We define* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|\, x \in L\}$ *by:*

$$A = \{\langle 1, 0.7, 0.2\rangle, \langle 2, 0.5, 0.5\rangle, \langle 3, 0.6, 0.3\rangle, \langle 4, 0.4, 0.5\rangle, \langle 6, 0.5, 0.5\rangle, \langle 12, 0.4, 0.5\rangle\}.$$

Then it can be easily verified that A is an IF-ideal of L.

Remark 2.3.1. *Every IF-ideal of L is an IF-lattice, but the converse in general is not true. Consider the lattice* $L = \{1, 2, 5, 10\}$*. The intuitionistic fuzzy set* A *of* L *defined by*

$$A = \{\langle 1, 0.5, 0.1\rangle, \langle 2, 0.4, 0.3\rangle, \langle 5, 0.4, 0.5\rangle, \langle 10, 0.7, 0.3\rangle\}.$$

Here A *is an IF-lattice but not an IF-ideal, because*

$$\mu_A(2) = \mu_A(2 \sqcap 10) = 0.4 \ngeq \max\{\mu_A(2), \mu_A(10)\} = 0.7.$$

Definition 2.3.3. *[20, 15] Let* L *be a lattice and* $F = \{x, \mu_F(x), \nu_F(x) \mid x \in L\}$ *be an IFS on* L*. Then* F *is called an intuitionistic fuzzy filter on* L *(IF-filter, for short), if for all* $x,\ y \in L$*, the following conditions are satisfied:*

(i) $\mu_F(x \sqcup y) \geq \max\{\mu_F(x), \mu_F(y)\}$;

(ii) $\mu_F(x \sqcap y) \geq \min\{\mu_F(x), \mu_F(y)\}$;

(iii) $\nu_F(x \sqcup y) \leq \min\{\nu_F(x), \nu_F(y)\}$;

(iv) $\nu_F(x \sqcap y) \leq \max\{\nu_F(x), \nu_F(y)\}$.

Example 2.3.3. *Consider the lattice* $L = \{1, 2, 3, 4\}$*, with* $\leq$ *is the useal order in* $\mathbb{N}$*. The intuitionistic fuzzy set* F *on* L *defined by:*

$$F = \{\langle 1, 0.1, 0.9\rangle, \langle 2, 0.3, 0.5\rangle, \langle 3, 0.4, 0.3\rangle, \langle 4, 0.7, 0.1\rangle\}.$$

Then it can verified that F *is an IF-filter of* L*.*

Proposition 2.3.1. *[15] Let* L *be a lattice,* L^d *be its order-dual lattice and* A *is an intuitionistic fuzzy set on* L*. Then it holds that* A *is an IF-ideal on* L *if and only if* A *is an IF-filter on* L^d *and conversely.*

Proposition 2.3.2. *[20, 15] Let L be a lattice, A and B are two intuitionistic fuzzy sets on L. Then it holds that*

(i) if A and B are two IF-ideals on L, then $A \cap B$ is an IF-ideal on L;

(ii) if A and B are two IF-filters on L, then $A \cap B$ is an IF-filter on L.

Remark 2.3.2. *The union of two IF-ideals need not be IF-ideal. Consider the lattice L given in exemple 2.3.2. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | \, x \in L\}$ is defined by:*

$$A = \{\langle 1, 0.7, 0.2\rangle, \langle 2, 0.5, 0.5\rangle, \langle 3, 0.6, 0.3\rangle, \langle 4, 0.4, 0.5\rangle, \langle 6, 0.5, 0.5\rangle, \langle 12, 0.4, 0.5\rangle\},$$

and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle | \, x \in L\}$ is given by:

$$B = \{\langle 1, 0.6, 0.2\rangle, \langle 2, 0.6, 0.4\rangle, \langle 3, 0.5, 0.5\rangle, \langle 4, 0.5, 0.4\rangle, \langle 6, 0.4, 0.5\rangle, \langle 12, 0.5, 0.5\rangle\}.$$

Here A and B are IF-ideals of L. Also $A \cup B = \{\langle x, \mu_{A\cup B}(x), \nu_{A\cup B}(x)\rangle | \, x \in L\}$ is

$$\{\langle 1, 0.7, 0.2\rangle, \langle 2, 0.6, 0.4\rangle, \langle 3, 0.6, 0.3\rangle, \langle 4, 0.5, 0.4\rangle, \langle 6, 0.5, 0.5\rangle, \langle 12, 0.5, 0.5\rangle\}.$$

Here $\nu_{A\cup B}(12) = \nu_{A\cup B}(3 \sqcup 4) = 0.5 \nleq \max\{\nu_{A\cup B}(3), \nu_{A\cup B}(4)\}$. Hence, $A \cup B$ is not an IF-ideal.

In the following theorem, we provide a basic characterization of IF-ideals on a lattice.

Theorem 2.3.1. *[15, 19] Let L be a lattice and I is an intuitionistic fuzzy set on L. Then it holds that I is an IF-ideal on L if and only if the following two conditions are satisfied:*

(i) $\mu_{\mathrm{I}}(x \sqcup y) = \min\{\mu_{\mathrm{I}}(x), \mu_{\mathrm{I}}(y)\}, \; for \; x, \; y \in L;$

(ii) $\nu_{\mathrm{I}}(x \sqcup y) = \max\{\nu_{\mathrm{I}}(x), \nu_{\mathrm{I}}(y)\}, \; for \; x, \; y \in L.$

In the same manner, the following theorem provides a basic characterization of IF-filters on a lattice.

Theorem 2.3.2. *[15, 19] Let L be a lattice and F is an intuitionistic fuzzy set on L. Then it holds that F is an intuitionistic fuzzy filter on L if and only if the following conditions are satisfied:*

(i) $\mu_{\mathrm{F}}(x \sqcap y) = \min\{\mu_{\mathrm{F}}(x), \mu_{\mathrm{F}}(y)\}, \; for \; x, \; y \in L;$

(ii) $\nu_{\mathrm{F}}(x \sqcap y) = \max\{\nu_{\mathrm{F}}(x), \nu_{\mathrm{F}}(y)\}, \; for \; x, \; y \in L.$

Proposition 2.3.3. *[15, 19] Let L be a lattice and A is an intuitionistic fuzzy set on L. Then it holds that:*

(i) if A is IF-ideal, then its support $Supp(A)$ is an ideal on L;

(ii) if A is IF-filter, then its support $Supp(A)$ is a filter on L.

Proof. (i) Suppose that A is an IF-ideal. We show that $Supp(A)$ is an ideal on L,
we have:

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0 \; or \; (\mu_A(x) = 0 \text{ and } \nu_A(x) < 1)\}.$$

a) Let $x \in Supp(A)$ and $y \leq x$. We show that $y \in Supp(A)$.
There are two cases to consider :

case 1 $(y \le x$ and $\mu_A(x) > 0)$
Since $y \le x$, then $x \sqcup y = x$. This implies that $\mu_A(x \sqcup y) = \mu_A(x) > 0$ and from Theorem 2.3.1 $\mu_A(x \sqcup y) = \min\{\mu_A(x), \mu_A(y)\}$. Hence $\mu_A(y) > 0$. Thus, $y \in Supp(A)$.

case 2 $(y \le x$ and $(\mu_A(x) = 0$ and $\nu_A(x) < 1)$
Since $y \le x$, then $x \sqcup y = x$. then $\mu_A(x \sqcup y) = \mu_A(x) = 0$ and $\nu_A(x \sqcup y) = \nu_A(x) < 1$. From Theorem 2.3.1 $\mu_A(x \sqcup y) = \min\{\mu_A(x), \mu_A(y)\} = 0$ and $\nu_A(x \sqcup y) = \max\{\nu_A(x), \nu_A(y)\} < 1$. Hence, $\mu_A(y) > 0$ or $(\mu_A(y) = 0$ and $\nu_A(y) < 1)$. Thus, $y \in Supp(A)$.

b) Let $x, y \in Supp(A)$. We show that $x \sqcup y \in Supp(A)$.
There are four cases to consider:

case 1 $\mu_A(x) > 0$ and $\mu_A(y) > 0$

We have A is an IF-ideal, then from Theorem2.3.1, then $\mu_A(x \sqcup y) = \min\{\mu_A(x), \mu_A(y)\} > 0$. Hence, $x \sqcup y \in Supp(A)$.

case 2 $(\mu_A(x) = 0$ and $\nu_A(x) < 1)$ and $(\mu_A(y) = 0$ and $\nu_A(y) < 1)$
From Theorem2.3.1, it follows that $\mu_A(x \sqcup y) = 0$ and $\nu_A(x \sqcup y) < 1$. Then, $x \sqcup y \in Supp(A)$.

case 3 $\mu_A(x) > 0$ and $(\mu_A(y) = 0$ and $\nu_A(y) < 1)$
From Theorem2.3.1, it follows that $\mu_A(x \sqcup y) = 0$ and $\nu_A(x \sqcup y) < 1$. Then, $x \sqcup y \in Supp(A)$.

case 4 $(\mu_A(x) = 0$ and $\nu_A(x) < 1)$ and $\nu_A(y) > 0$.
is analogous to the third case. Hence, $x \sqcup y \in Supp(A)$.

ii) Follows from Proposition2.3.1 and (i).

Prime Intuitionistic Fuzzy Ideals (resp. Filters) on a Lattice

In this subsection, we introduce the notion of a prime IF-ideals (resp. IF-filters) on a lattice

Definition 2.3.4. *[15, 19] An IF-ideal* I *on a lattice L is called a prime IF-ideal if, for any $x,\ y \in L$,*

$$\mu_I(x \sqcap y) \le \max\{\mu_I(x), \mu_I(y)\}$$

and

$$\nu_I(x \sqcap y) \ge \min\{\nu_I(x), \nu_I(y)\}.$$

Definition 2.3.5. *[15, 19] An IF-filter* F *on a lattice L is called a prime IF-filter if for $x,\ y \in L$,*

$$\mu_F(x \sqcup y) \le \max\{\mu_F(x), \mu_F(y)\}$$

and

$$\nu_F(x \sqcup y) \ge \min\{\nu_F(x), \nu_F(y)\}$$

A combination of Theorem 2.3.1 and Definition 2.3.2 leads to the following characterization of prime IF-ideals.

Proposition 2.3.4. *[15, 19] Let L be a lattice and* I *is an intuitionistic fuzzy set on L. Then, it holds that* I *is a prime IF-ideal on L if and only if the following conditions hold:*

(i) $\mu_{\mathrm{I}}(x \sqcup y) = \min\{\mu_{\mathrm{I}}(x), \mu_{\mathrm{I}}(y)\},$ *for any* $x, y \in L$;

(ii) $\mu_{\mathrm{I}}(x \sqcap y) = \max\{\mu_{\mathrm{I}}(x), \mu_{\mathrm{I}}(y)\},$ *for any* $x, y \in L$;

(iii) $\nu_{\mathrm{I}}(x \sqcup y) = \max\{\nu_{\mathrm{I}}(x), \nu_{\mathrm{I}}(y)\},$ *for any* $x, y \in L$;

(iv) $\nu_{\mathrm{I}}(x \sqcap y) = \min\{\nu_{\mathrm{I}}(x), \nu_{\mathrm{I}}(y)\},$ *for any* $x, y \in L$.

Similarly, Theorem 2.3.2 and Definition 2.3.3 lead to the following characterization of prime IF-filters.

Proposition 2.3.5. *[15, 19] Let L be a lattice and F is an intuitionistic fuzzy set on L. Then, it holds that F is a prime IF-filter on L if and only if the following conditions hold:*

(i) $\mu_{\mathrm{F}}(x \sqcup y) = \max\{\mu_{\mathrm{F}}(x), \mu_{\mathrm{F}}(y)\},$ *for any* $x, y \in L$;

(ii) $\mu_{\mathrm{F}}(x \sqcap y) = \min\{\mu_{\mathrm{F}}(x), \mu_{\mathrm{F}}(y)\},$ *for any* $x, y \in L$;

(iii) $\nu_{\mathrm{F}}(x \sqcup y) = \min\{\nu_{\mathrm{F}}(x), \nu_{\mathrm{F}}(y)\},$ *for any* $x, y \in L$;

(iv) $\nu_{\mathrm{F}}(x \sqcap y) = \max\{\nu_{\mathrm{F}}(x), \nu_{\mathrm{F}}(y)\},$ *for any* $x,\ y \in L$.

Proposition 2.3.6. *[15, 19] Let L be a lattice and A is an intuitionistic fuzzy set on L. Then it holds that:*

(i) if A is a prime IF-ideal, then its support $Supp(A)$ is a prime ideal on L;

(ii) if A is a prime IF-filter, then its support $Supp(A)$ is a prime filter on L.

CHAPTER 3

MANY-VALUED LOGIC AND INTUITIONISTIC FUZZY SETS

In this chapter, we will study the representation theory of involutive interval-valued Łukasiewicz-Moisil Algebras by using the notion of intuitionistic fuzzy sets.

Representation of Łukasiewicz Algebra by Intuitionistic Fuzzy sets

Notation 3.0.1. *We note by J the unit interval $[0,1]$ and $J^0 = J - \{0\},\ J^1 = J - \{1\}$ and $J^{0,1} = J - \{0,1\}$.*

3.1 Łukasiewicz-MoisiL algebra

Definition 3.1.1. *[1] A many-valued Łukasiewicz-Moisil algebra is a structure $(L, J^0, (\varphi_\alpha)_{\alpha\in J^0})$ verifying the following axioms:*

1. *L is a bounded distributive lattice,*
2. *For all x of L, $\varphi_\alpha(x)$ belongs to the Boolean algebra,*
3. *$(\varphi_\alpha)_{\alpha\in J^0}$ is a family of morphisms: $L \longrightarrow C(L)$ preserving 0 and 1, where $C(L)$ is the complemented elements of L,*
4. *if $\alpha \leq \beta$, then $\varphi_\beta \leq \varphi_\alpha$,*
5. *for any $\alpha,\ \beta$ of J^0, $\varphi_\beta\varphi_\alpha = \varphi_\alpha$,*
6. *if $\varphi_\alpha(x) = \varphi_\alpha(y)$ for all α of J^0, then $x = y$ (principal Moisil determination).*

Definition 3.1.2 (Homomorphism of two interval-valued Łukasiewicz-Moisil algebras)**.** *[1] Two interval-valued Łukasiewicz-Moisil algebras $(L, J^0, (\varphi_\alpha)_{\alpha\in J^0}), (L^{'}, J^0, (\varphi^{'}_\alpha)_{\alpha\in J^0})$ are said to be homomorphic if there exists a morphism $f : L \longrightarrow L^{'}$ satisfying*

$$f \circ \varphi_\alpha = \varphi^{'}_\alpha \circ f,\ \text{for all } \alpha \in J^0.$$

3.2 Łukasiewicz-Moisil algebra with involution

Definition 3.2.1. *[1]An interval-valued Łukasiewicz-Moisil algebra with* ***involution*** *is the structure* $(L, J^0, (\varphi_\alpha)_{\alpha\in J^0}, (\psi_\alpha)_{\alpha\in J^1}, n, N)$ *verifying the following axioms:*

1. $(L, J^0, (\varphi_\alpha)_{\alpha\in J^0})$ *is an interval-valued Łukasiewicz-Moisil,*
2. n *is a decreasing involution on* J*,*
3. N *is a decreasing involution on* L *such that if* $x \in C(L)$*, then* $N(x)$ *is the complement of* x*,*
4. $(\psi_\alpha)_{\alpha\in J^1}$ *is a family of applications of* L *in* $C(L)$ *verifying*
   $\alpha \leq \beta \implies \psi_\beta \leq \psi_\alpha$,
   for all $\alpha \in J^0$, $\varphi_\alpha N = N\psi_{n\alpha}$*,*
   (we can show that ψ_α *is an endomorphism* $L \longrightarrow C(L)$ *and* $\psi_\alpha \circ \psi_\beta = \psi_\beta$*).*

Definition 3.2.2 (Homomorphism of two involutive interval-valued Łukasiewicz-Moisil algebras)**.** *[1] Two involutive interval-valued Łukasiewicz-Moisil algebras*

$$(L, J^0, (\varphi_\alpha)_{\alpha\in J^0}, (\psi_\alpha)_{\alpha\in J^1}, n, N),$$

$$(L^{'}, J^0, (\varphi^{'}_\alpha)_{\alpha\in J^0}, (\psi^{'}_\alpha)_{\alpha\in J^1}, n, N^{'})$$

are said to be homomorphic if there exists a morphism $f : L \longrightarrow L^{'}$ *satisfying*

(i) $f \circ \varphi_\alpha = \varphi^{'}_\alpha \circ f$*, for all* $\alpha \in J^0$*,*

(ii) $f \circ N = N^{'} \circ f$*.*

In order to show that the set $IF(X)$ of all intuitionistic fuzzy subsets over X admits a structure of an involutive interval-valued Łukasiewicz-Moisil algebra, we define the intuitionistic weak α-cut and the intuitionistic strong α-cut as follow:

1. for each $\alpha \in J^0$, **the intuitionistic weak α-cut** as a mapping N_α: $IF(X) \longrightarrow P(X)$ defined by
$$N_\alpha(A) = \{x \in X \mid \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq n\alpha\}$$
2. for each $\alpha \in J^1$, **the intuitionistic strong α-cut** as a mapping $N^{'}_\alpha$: $IF(X) \longrightarrow P(X)$ defined by
$$N^{'}_\alpha(A) = \{x \in X \mid \mu_A(x) > \alpha \text{ and } \nu_A(x) < n\alpha\}$$

with n is the unary, involutive, order-reversing operator on J.
The intuitionistic fuzzy complementation is the order-reversing involution $\tilde{n}$ on $IF(X)$ defined, for all $A \in IF(X)$ by

$$\begin{aligned}\tilde{n}(A) &= \{\langle x, \mu_{\tilde{n}A}(x), \nu_{\tilde{n}A}(x)\rangle \mid x \in X\}\\ &= \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}.\end{aligned}$$

3.3 Representation theorems

In this section, we will study the relationships between the involutive Łukasiewicz-Moisil algebra and the algebra of intuitionistic fuzzy sets.

Lemma 3.3.1. *[1] The families of maps* $(N_\alpha \,|\, \alpha \in J^0)$ *and* $(N'_\alpha \,|\, \alpha \in J^1)$ *satisfy the following conditions:*

1. $N_\alpha(\emptyset) = \emptyset$, $N_\alpha(X) = X$,
2. *if* $\alpha \leq \beta$, *then* $N_\beta \leq N_\alpha$,
3. $N_\alpha \circ N_\beta = N_\beta$,
4. *if* $A, B \in IF(X)$ *and for every* $\alpha \in J^0$, $N_\alpha(A) = N_\alpha(B)$, *then* $A = B$ *(Moisil's determination principle),*
5. $N'_\alpha \leq N_\alpha$, *for all* $\alpha \in J^{0,1}$,
6. $N_\alpha \tilde{n} = \tilde{n} N'_{n\alpha}$, *for all* $\alpha \in J^0$.

Proof.

1. For all $\alpha \in J^{0,1}$, we have

$$\begin{aligned} N_\alpha(\emptyset) &= \{x \in X \mid \mu_\emptyset(x) \geq \alpha \text{ and } \nu_\emptyset(x) \leq n\alpha\} \\ &= \{x \in X \mid 0 \geq \alpha \text{ and } 1 \leq n\alpha\} \\ &= \emptyset. \end{aligned}$$

$$\begin{aligned} N_\alpha(X) &= \{x \in X \mid \mu_X(x) \geq \alpha \text{ and } \nu_X(x) \leq n\alpha\} \\ &= \{x \in X \mid 1 \geq \alpha \text{ and } 0 \leq n\alpha\} \\ &= X. \end{aligned}$$

2. Let $\alpha,\ \beta \in J^0$ such that $\alpha \leq \beta$. Then we have

$$\begin{aligned} N_\beta(A) &= \{x \in X \mid \mu_A(x) \geq \beta \text{ and } \nu_A(x) \leq n\beta\} \\ &\subseteq \{x \in X \mid \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq n\alpha\} \\ &= N_\alpha(A). \end{aligned}$$

   Hence, $N_\beta \leq N_\alpha$.

3. We show that $N_\alpha \circ N_\beta = N_\beta$,
   we have $N_\alpha \circ N_\beta(A) = N_\alpha(N_\beta(A)) = N_\beta(A)$, because $N_\beta(A) \in P(X)$.

4. Suppose that $N_\alpha(A) = N_\alpha(B)$ for all $\alpha \in J^0$ and we show that $A = B$.
   Let

$$N_\alpha(A) = \{x \in X \mid \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq n\alpha\}$$

   and

$$N_\alpha(B) = \{x \in X \mid \mu_B(x) \geq \alpha \text{ and } \nu_B(x) \leq n\alpha\},$$

   for all $\alpha \in J^0$, $N_\alpha(A) = N_\alpha(B)$, then

$$\{x \in X \mid \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq n\alpha\} = \{x \in X \mid \mu_B(x) \geq \alpha \text{ and } \nu_B(x) \leq n\alpha\}$$

   Hence, $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ for all $x \in X$
   Thus, $A = B$.

5. For all $\alpha \in J^{0,1}$, we have

$$\begin{aligned} N'_\alpha(A) &= \{x \in X \mid \mu_A(x) > \alpha \text{ and } \nu_A(x) < n\alpha\} \\ &\subseteq \{x \in X \mid \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq n\alpha\} \\ &= N_\alpha(A). \end{aligned}$$

   Hence, $N'_\alpha \leq N_\alpha$, for every $\alpha \in J^{0,1}$.

6. We show that $N_\alpha \tilde{n} = \tilde{n} N'_\alpha$, for all $\alpha \in J^0$
   for all $\alpha \in J^0$, we have

$$\begin{aligned} N_\alpha \tilde{n}(A) &= \{x \in X \mid \mu_{\tilde{n}A}(x) \geq \alpha \text{ and } \nu_{\tilde{n}A}(x) \leq n\alpha\} \\ &= \{x \in X \mid n\mu_A(x) \geq \alpha \text{ and } n\nu_A(x) \leq n\alpha\} \\ &= \{x \in X \mid \nu_A(x) \geq \alpha \text{ and } \mu_A(x) \leq n\alpha\} \end{aligned}$$

   Let $\tilde{n}(A) = \bar{A}$ the complement of A, then

$$\begin{aligned} \tilde{n} N'_{n\alpha}(A) &= \tilde{n}\{x \in X \mid \mu_A(x) > n\alpha \text{ and } \nu_A(x) < nn\alpha\} \\ &= \tilde{n}\{x \in X \mid \mu_A(x) > n\alpha \text{ and } \nu_A(x) < \alpha\} \\ &= \{x \in X \mid \mu_A(x) \leq n\alpha \text{ and } \nu_A(x) \geq \alpha\} \\ &= N_\alpha \tilde{n}(A), \end{aligned}$$

   for every $\alpha \in J^0$.

The structure $(IF(X), \cup, \cap, \emptyset, X)$ of all intuitionistic fuzzy subsets is a distributive lattice. For any sublattice F of $IF(X)$, containing $\emptyset$ and X, closed under the intuitionistic fuzzy complementation $\tilde{n}$, closed under the intuitionistic weak α-cut N_α, for all $\alpha \in J^0$, and under the intuitionistic strong α-cut N'_α , for all $\alpha \in J^1$, the sequence $(F, J, (N_\alpha)_{\alpha \in J^0}, (N'_\alpha)_{\alpha \in J^1}, \tilde{n})$ is called an involutive interval-valued intuitionistic fuzzy algebra.

Lemma 3.3.2. *[1] Let $(L, J, (\varphi_\alpha)_{\alpha \in J^0}, (\psi_\alpha)_{\alpha \in J^1}, n, N)$ be an involutive interval-valued Łukasiewicz-Moisil algebra and U be an ultrafilter of $C(L)$.Then*

$$\sup\{\alpha \in J^{0,1} \mid \varphi_\alpha(x) \in U\} = \sup\{\alpha \in J^{0,1} \mid \psi_\alpha(x) \in U\}.$$

Theorem 3.3.1. *[1] An involutive interval-valued intuitionistic fuzzy algebra*

$$(F, J, (N_\alpha)_{\alpha \in J^0}, (N'_\alpha)_{\alpha \in J^1}, n, \tilde{n})$$

is an involutive interval-valued Łukasiewicz-Moisil algebra.

proof.

1. $(IF(X), J^0, (N_\alpha)_{\alpha \in J^0})$ is an interval-valued Łukasiewicz-Moisil algebra,

2. n is a decreasing involution on J (for all $\alpha \in J$, $nn\alpha = 1 - (1 - \alpha) = \alpha$ and for any $\alpha,\ \beta \in J,\ \alpha \leq \beta$ we have $n\alpha \geq n\beta$),

3. We show that $\tilde{n}$ is a decreasing involution on $IF(X)$
let $A \in IF(X)$, then

$$\begin{aligned}\tilde{n}\tilde{n}(A) &= \tilde{n}\{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\} \\ &= \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\} \\ &= A.\end{aligned}$$

Let $A, B \in IF(X)$, if $A \subseteq B$, then we have $\mu_A \leq \mu_B$ and $\nu_A \leq \nu_B$
Hence, $n\mu_B \leq n\mu_A$ and $n\nu_B \leq n\nu_A$. Thus, $\tilde{n}(B) \subseteq \tilde{n}(A)$(decreasing).

4. Easily follows from lemma3.3.1

Theorem 3.3.2. *[1] Any involutive Łukasiewicz-Moisil algebra*

$$(L, J, (\varphi_\alpha)_{\alpha\in J^0}, (\psi_\alpha)_{\alpha\in J^1}, n, N)$$

can be embedded in the Łukasiewicz-Moisil of an involutive interval-valued intuitionistic fuzzy algebra

$$(IF(X), J, (N_\alpha)_{\alpha\in J^0}, (N'_\alpha)_{\alpha\in J^1}, n, \tilde{n}),$$

where X is the set of all ultrafilters of $C(L)$.

Proof.

We show that f is a monomorphism of involutive Łukasiewicz-Moisil algebras. Firstly, we prove that f is a morphisme of lattice. Secondly, we show the following two conditions are verified:

(i) $f \circ \varphi_\alpha = N_\alpha \circ f$, for every $\alpha \in J^0$,

(ii) $f \circ N = \tilde{n} \circ f$.

Let U be an ultrafilter of $C(L)$ and f be a map from L to $IF(X)$ defined by:

$$f(x) = \{U \in X \mid \langle U, \mu_{f(x)}(U), \nu_{f(x)}(U)\rangle\},$$

where

$$\mu_{f(x)}(U) = \sup\{\alpha \in J^0 \mid \varphi_\alpha(x) \in U\},$$

and

$$\nu_{f(x)}(U) = \sup\{\alpha \in J^0 \mid \psi_{n\alpha}(x) \notin U\}.$$

1. we have $f(x) = \{U \in X \mid \langle U, \mu_{f(x)}(U), \nu_{f(x)}(U)\rangle\}$, then

   (a) $f(0) = \{U \in X \mid \langle U, \mu_{f(0)}(U), \nu_{f(0)}(U)\rangle\}$

   with

$$\begin{aligned}\mu_{f(0)}(U) &= \sup\{\alpha \in J^0 \mid \varphi_\alpha(0) \in U\} \\ &= \sup\{\alpha \in J^0 \mid 0 \in U\} \text{ (because } \varphi_\alpha \text{ is a morphisme of } L \longrightarrow C(L)) \\ &= \sup \emptyset \\ &= 0.\end{aligned}$$

and

$$\begin{aligned}\nu_{f(0)}(U) &= \sup\{\alpha \in J^0 \mid \psi_{n\alpha}(0) \notin U\}\\ &= \sup\{\alpha \in J^0 \mid 0 \notin U\} \text{ (because } \psi_\alpha \text{ is an endomorphisme of } L \longrightarrow C(L))\\ &= \sup J^0\\ &= 1.\end{aligned}$$

Hence, $f(0) = \{U \in X \mid \langle U, 0, 1\rangle\} = \emptyset$

(b) $f(1) = \{U \in X \mid \langle U, \mu_{f(1)}(U), \nu_{f(1)}(U)\rangle\}$ with

$$\begin{aligned}\mu_{f(1)}(U) &= \sup\{\alpha \in J^0 \mid \varphi_\alpha(1) \in U\}\\ &= \sup\{\alpha \in J^0 \mid 1 \in U\}\\ &= \sup J^0\\ &= 1.\end{aligned}$$

and

$$\begin{aligned}\nu_{f(1)}(U) &= \sup\{\alpha \in J^0 \mid \psi_{n\alpha}(1) \notin U\}\\ &= \sup\{\alpha \in J^0 \mid 1 \notin U\}\\ &= \sup\emptyset\\ &= 0.\end{aligned}$$

Hence, $f(1) = \{U \in X \mid \langle U, 1, 0\rangle\} = X$.

2. (a) We have $f(x \vee y) = \{U \in X \mid \langle U, \mu_{f(x\vee y)}(U), \nu_{f(x\vee y)}(U)\rangle\}$

where

$$\begin{aligned}\mu_{f(x\vee y)}(U) &= \sup\{\alpha \in J^0 \mid \varphi_\alpha(x \vee y) \in U\}\\ &= \sup\{\alpha \in J^0 \mid \varphi_\alpha(x) \vee \varphi_\alpha(y) \in U\} \text{ (because } \varphi_\alpha \text{ is a morphisme of } L \longrightarrow C(L))\\ &= \sup\{\alpha \in J^0 \mid \varphi_\alpha(x) \in U \vee \varphi_\alpha(y) \in U\}\\ &= \sup\{\alpha \in J^0 \mid \varphi_\alpha(x) \in U\} \vee \sup\{\alpha \in J^0 \mid \varphi_\alpha(y) \in U\}\\ &= \mu_{f(x)}(U) \vee \mu_{f(y)}(U),\end{aligned}$$

and

$$\begin{aligned}\nu_{f(x\vee y)}(U) &= \sup\{\alpha \in J^0 \mid \psi_{n\alpha}(x \vee y) \notin U\}\\ &= \sup\{\alpha \in J^0 \mid \psi_{n\alpha}(x) \vee \psi_{n\alpha}(y) \notin U\} \text{ (because } \psi_\alpha \text{ is an endomorphisme of } L \longrightarrow C(L\\ &= \sup\{\alpha \in J^0 \mid \psi_{n\alpha}(x) \notin U \wedge \psi_{n\alpha}(y) \notin U\}\\ &= \sup\{\alpha \in J^0 \mid \psi_{n\alpha}(x) \notin U\} \wedge \sup\{\alpha \in J^0 \mid \psi_{n\alpha}(y) \notin U\}\\ &= \nu_{f(x)}(U) \wedge \nu_{f(y)}(U).\end{aligned}$$

Then, $f(x \vee y) = f(x) \cup f(y)$

(b) We have

$$f(x \wedge y) = \{U \in X \mid \langle U, \mu_{f(x\wedge y)}(U), \nu_{f(x\wedge y)}(U)\rangle\},$$

where

$$\begin{aligned}\mu_{f(x\wedge y)}(U) &= \sup\{\alpha \in J^0 \mid \varphi_\alpha(x \wedge y) \in U\}\\ &= \sup\{\alpha \in J^0 \mid \varphi_\alpha(x) \wedge \varphi_\alpha(y) \in U\}\\ &= \sup\{\alpha \in J^0 \mid \varphi_\alpha(x) \in U \wedge \varphi_\alpha(y) \in U\}\\ &= \sup\{\alpha \in J^0 \mid \varphi_\alpha(x) \in U\} \wedge \sup\{\alpha \in J^0 \mid \varphi_\alpha(y) \in U\}\\ &= \mu_{f(x)}(U) \wedge \mu_{f(y)}(U),\end{aligned}$$

and

$$\begin{aligned}\nu_{f(x\wedge y)}(U) &= \sup\{\alpha \in J^0 \mid \psi_{n\alpha}(x \wedge y) \notin U\}\\ &= \sup\{\alpha \in J^0 \mid \psi_{n\alpha}(x) \wedge \psi_{n\alpha}(y) \notin U\}\\ &= \sup\{\alpha \in J^0 \mid \psi_{n\alpha}(x) \notin U \vee \psi_{n\alpha}(y) \notin U\}\\ &= \sup\{\alpha \in J^0 \mid \psi_{n\alpha}(x) \notin U\} \vee \sup\{\alpha \in J^0 \mid \psi_{n\alpha}(y) \notin U\}\\ &= \nu_{f(x)}(U) \vee \nu_{f(y)}(U).\end{aligned}$$

Then, $f(x \wedge y) = f(x) \cap f(y)$.

Hence, f is a morhpism of lattice.

3. Now, we show that f is injective
for $x,\ y \in L$, we have $f(x) = f(y)$ implies that

$$\{U \in X \mid \langle U, \mu_{f(x)}(U), \nu_{f(x)}(U)\rangle\} = \{U \in X \mid \langle U, \mu_{f(y)}(U), \nu_{f(y)}(U)\rangle\}.$$

From the definition of equal intuitionistic fuzzy subsets we obtain

$$\mu_{f(x)}(U) = \mu_{f(y)}(U)$$

and

$$\nu_{f(x)}(U) = \nu_{f(y)}(U)$$

for every $U \in X$.
Thus,

$$\sup\{\alpha \in J^0 \mid \varphi_\alpha(x) \in U\} = \sup\{\alpha \in J^0 \mid \varphi_\alpha(y) \in U\}.$$

This implies that:

$$\{\alpha \in J^0 \mid \varphi_\alpha(x) \in U\} = \{\alpha \in J^0 \mid \varphi_\alpha(y) \in U\}.$$

Hence, for all $\alpha \in J^0$, $\varphi_\alpha(x) \in U$ if and only if $\varphi_\alpha(y) \in U$,
from the Moisil's determination principle, we obtain $\varphi_\alpha(x) = \varphi_\alpha(y)$ for all $\alpha \in J^0$, and then $x = y$, thus f is injective.

Hence, f is a monomorphism of lattice.

The second step is to show that

1. $f \circ \varphi_\alpha = N_\alpha \circ f$, for every $\alpha \in J^0$,

2. $f \circ N = \tilde{n} \circ f$.
(1) $f \circ \varphi_\alpha = N_\alpha \circ f$

$$\begin{aligned} f \circ \varphi_\alpha(x) &= f(\varphi_\alpha(x)) \\ &= \{U \in X \mid \langle U, \mu_{f(\varphi_\alpha(x))}(U), \nu_{f(\varphi_\alpha(x))}(U)\rangle\}, \text{ for all } \alpha \in J^0. \end{aligned}$$

with

$$\begin{aligned} \mu_{f(\varphi_\alpha(x))}(U) &= \sup\{\beta \in J^0 \mid \varphi_\beta(\varphi_\alpha(x)) \in U\} \\ &= \sup\{\beta \in J^0 \mid \varphi_\alpha(x) \in U\} \text{ (because } \varphi_\beta(\varphi_\alpha(x)) = \varphi_\alpha(x)) \\ &= \begin{cases} 0 & if \quad \varphi_\alpha(x) \notin U \\ 1 & if \quad \varphi_\alpha(x) \in U. \end{cases} \\ &= \sigma(\varphi_\alpha(x)), \end{aligned}$$

where σ is the stone monomorphisme. Moreover, we have

$$\begin{aligned} \nu_{f(\varphi_\alpha(x))}(U) &= \sup\{\beta \in J^0 \mid \psi_{n\beta}(\varphi_\alpha(x)) \notin U\} \\ &= \sup\{\beta \in J^0 \mid \varphi_\alpha(x) \notin U\} \\ &= \begin{cases} 1 & if \quad \varphi_\alpha(x) \notin U \\ 0 & if \quad \varphi_\alpha(x) \in U. \end{cases} \\ &= n\sigma(\varphi_\alpha(x)), \end{aligned}$$

and

$$f \circ \varphi_\alpha(x) = \{U \in X \mid \langle U, \sigma(\varphi_\alpha(x)), n\sigma(\varphi_\alpha(x))\rangle\}, \text{ for all } \alpha \in J^0.$$

Therefore,

$$\begin{aligned} N_\alpha \circ f(x) &= N_\alpha(f(x)) \\ &= \{U \in X \mid \mu_{f(x)}(U) \geq \alpha \text{ and } \nu_{f(x)}(U) \leq n\alpha\} \\ &= \{U \in X \mid \alpha \in \{\beta \in J^0 \mid \varphi_\beta(x) \in U\} \text{ and } \alpha \notin \{\beta \in J^0 \mid \psi_{n\beta}(x) \notin U\}\} \\ &= \{U \in X \mid \langle U, \sigma(\varphi_\alpha(x)), n\sigma(\varphi_\alpha(x))\rangle\} \\ &= f \circ \varphi_\alpha(x). \end{aligned}$$

Thus, $f \circ \varphi_\alpha = N_\alpha \circ f$.
(2) We show that $f \circ N = \tilde{n} \circ f$,

we have

$$\begin{aligned} f \circ N(x) &= f(N(x)) \\ &= \{U \in X \mid \langle U, \mu_{f(Nx)}(U), \nu_{f(Nx)}(U)\rangle\}, \end{aligned}$$

with

$$\begin{aligned} \mu_{f(Nx)}(U) &= \sup\{\alpha \in J^0 \mid \varphi_\alpha(Nx) \in U\} \\ &= \sup\{\alpha \in J^0 \mid N\psi_{n\alpha}(x) \in U\} \text{(we have } \varphi_\alpha(Nx) = N\psi_{n\alpha}(x)) \\ &= \sup\{\alpha \in J^0 \mid \psi_{n\alpha}(x) \notin U\} \\ &= \nu_{f(x)}(U), \end{aligned}$$

and

$$\begin{aligned}\nu_{f(Nx)}(U) &= \sup\{\alpha \in J^0 \mid \psi_{n\alpha}(Nx) \notin U\} \\ &= \sup\{\alpha \in J^0 \mid N\varphi_\alpha(x) \notin U\}(\text{ we have } \psi_{n\alpha}(Nx) = N\varphi_\alpha(x)) \\ &= \sup\{\alpha \in J^0 \mid \varphi_\alpha(x) \in U\} \\ &= \mu_{f(x)}(U).\end{aligned}$$

Then, $f(N(x)) = \{U \in X \mid \langle U, \nu_{f(x)}(U), \mu_{f(x)}(U)\rangle\}$

and we have

$$\begin{aligned}\tilde{n} \circ f(x) &= \tilde{n}(f(x)) \\ &= \tilde{n}\{U \in X \mid \langle U, \mu_{f(x)}(U), \nu_{f(x)}(U)\rangle\} \\ &= \{U \in X \mid \langle U, \nu_{f(x)}(U), \mu_{f(x)}(U)\rangle\} \\ &= f(N(x)).\end{aligned}$$

Hence, $f \circ N = \tilde{n} \circ f$

Thus, we have f is a monomorphism of involutive Łukasiewicz-Moisil algebras.

Conclusion

In this memory, we have viewed the concept of intuitionistic fuzzy sets as a generalization of the notion of fuzzy sets. We have managed to study some of its properties, operators and relations over intuitionistic fuzzy sets, intuitionistic fuzzy lattices, intuitionistic fuzzy ideals and intuitionistic fuzzy filters. Moreover, we have presented one of the applications of this set on the theory of the Łukasiewicz-Moisil algebras.

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