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To my dear parents

To my brothers

To my family

To my friends

To anyone interested in this work ...

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List of Papers

Part I

Chapter 1

A. Limam, B. Benabderrahmane, and Y. Boukhatem. On a coupled system of viscoelastic wave equation of infinite memory with acoustic boundary conditions. *Studia Universitatis Babes-Bolyai Mathematica*, In Production, 2021.

Chapter 2

A. Limam, B. Benabderrahmane, and Y. Boukhatem. General decay for a coupled system of viscoelastic wave equation of infinite memory with acoustic boundary conditions. *2021 International Conference on Recent Advances in Mathematics and Informatics (ICRAMI)*, pp. 1-9, 2021. DOI: https://doi.org/10.1109/ICRAMI52622.2021.9585955.

Part II

A. Limam, Y. Boukhatem, and B. Benabderrahmane. New general stability for a variable coefficient thermo-viscoelastic-coupled system of second sound with acoustic boundary conditions. *Computational and Applied Mathematics*, 40(3):88, 2021. DOI: https://doi.org/10.1007/s40314-021-01459-w.

Part III

Chapter 5

A. Limam, Y. Boukhatem, and B. Benabderrahmane. General decay result for a type III thermoelastic coupled system with acoustic boundary conditions in the presence of distributed time delay. *Journal of Mathematical Physics, Analysis, Geometry*, 17(2):175–200, 2021. DOI: https://doi.org/10.15407/mag17.02.175.

Chapter 6

A. Limam, B. Benabderrahmane, and Y. Boukhatem. Global solvability and decay estimates for a type III thermo-viscoelastic coupled system with infinite memory and boundary interaction feedback. *Mathematische Nachrichten*, In Production, 2021. DOI: https://doi.org/10.1002/mana.202100111.

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List of Symbols and Notations

Ω	is a bounded domain of $\mathbb{R}^n$, $n \geq 1$ an integer.
$\Gamma = \partial\Omega$	is the smooth boundary of Ω.
Γ_0, Γ_1	are disjoint closed subsets of $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, with positive measure of Γ_0.
ν	represents the unit outward normal to Γ.
$\mathbf{A}$, $\mathbf{B}$	are elliptic operators of second order defined as follows $\mathbf{A}u = -\mathrm{div}(\mathrm{A}\nabla u) = -\sum\limits_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j})$, $\mathbf{B}u = -\mathrm{div}(\mathrm{B}\nabla u) = -\sum\limits_{i,j=1}^{n} \frac{\partial}{\partial x_i}(b_{ij}(x)\frac{\partial u}{\partial x_j})$.
$\frac{\partial u}{\partial \nu_{\mathrm{A}}}$	denotes the outward normal derivative of u, i. e., $\frac{\partial u}{\partial \nu_{\mathrm{A}}} = (\mathrm{A}\nabla u).\nu = \sum\limits_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\nu_i$.
$(.,.)$, $\Vert . \Vert_2$	are L^2-inner product and L^2-norm on $\mathrm{L}^2(\Omega)$, respectively.
$\langle .,. \rangle_\Gamma$, $\Vert . \Vert_{2,\Gamma}$	are L^2-inner product and L^2-norm on $\mathrm{L}^2(\Gamma)$, respectively.
$a(.,.)$, $b(.,.)$	are bilinear forms defined by $a(u,v) = (\mathrm{A}\nabla u, \nabla v) = \sum\limits_{i,j=1}^{n} \int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\mathrm{d}x$, $b(u,v) = (\mathrm{B}\nabla u, \nabla v) = \sum\limits_{i,j=1}^{n} \int_\Omega b_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\mathrm{d}x$, equipped with the norm equivalent to the usual norm in $\mathrm{H}_0^1(\Omega)$.
$\mathrm{H}(\mathrm{div},\Omega)$	is the Hilbert space where $\mathrm{H}(\mathrm{div},\Omega) = \{\varphi \in (\mathrm{L}^2(\Omega))^n; \mathrm{div}(\varphi) \in \mathrm{L}^2(\Omega)\}$, equipped with the norm $\Vert\varphi\Vert_{\mathrm{H}(\mathrm{div},\Omega)} = \left(\Vert\varphi\Vert^2_{(\mathrm{L}^2(\Omega))^n} + \Vert\mathrm{div}(\varphi)\Vert_2^2\right)^{1/2}$.
$\mathcal{L} \sim E$	$\mathcal{L}$ is equivalent to E.

Introduction

This thesis is intended to present some viscoelastic problems for a strongly elliptic operator of second order with variable coefficients in bounded domains. The main aim is to outline an overview of the local and global existence, and asymptotic behavior of solutions. In this regard, we investigate several coupled systems with mixed Dirichlet-Neumann boundary conditions. The coupling is via the acoustic boundary conditions on a portion of the boundary. A review of the recent studies on the generalized thermoelasticity theories and their associated modified models is also presented. Our objective goal is designed to establish exponential, polynomial, and general decay rate results.

Before the thesis statement, we need to present some motivations that permit us to formulate our different problems. So that is why we start by deriving keywords related to the research that will be included in this work. Our results description is presented at the end of the introduction.

History and motivation

Dissipative systems

Dissipative systems are of particular interest in engineering, physics, and mathematical modeling. It is considered a thermodynamically open system that is operating out of, and often far from, thermodynamic equilibrium in an environment with which it exchanges energy and matter as results in a fundamental constraint on their dynamic behavior. Typical examples of these systems are electrical networks in which part of the electrical energy is dissipated in the resistors in the form of heat, viscoelastic systems in which viscous friction is responsible for a similar loss in energy, and thermodynamic systems for which the second law postulates a form of dissipation leading to an increase in entropy. Otherwise, a conservative system is a dynamical system that stands in contrast to a dissipative system. Roughly speaking, such systems have no friction or other mechanism to dissipate the dynamics, and thus, their phase space does not shrink over time.

The theory of dissipative systems, since its introduction by Willems [89], has become a central tool in modeling systems storing or dissipating energy and motivated some researchers to adjust the asymptotic behavior of solutions of the damped semilinear wave equation

$$\begin{cases} u_{tt} - \Delta u + \alpha u_t = f(u) & \text{in} \quad \Omega \times \mathbb{R}_+ \\ u = 0 & \text{on} \quad \Gamma \times \mathbb{R}_+ \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) & \text{for} \quad x \in \Omega. \end{cases} \tag{1}$$

There is a wide literature on Problem (1). In the case of no damping ($\alpha = 0$), Sattinger [86] developed the so-called potential well theory. Subsequently, equations with linear damping term ($\alpha > 0$) have been considered by many authors and proved global existence and decay estimates under various assumptions on the source term f, one can refer to some of the most important papers [17, 45, 54, 57] using deeper techniques.

The nonlinear damping, owing to the difficulty in the analysis of dynamical systems, has been either produced a lack of regularity and/or obtained a source of instability which was intrinsically rated of infinite-dimensional in the literature. This effect has attracted considerable attention many researchers by exploiting strong internal resonance in a nonlinearly damped as a sort of regularization, see [24, 46] and the references therein. For investigations on several semi-linear models of the wave equation dealing with nonlinear boundary conditions, Lasiecka and Tataru [60] proved, under some hypotheses on nonlinear damping term F and nonlinear functions f_0 and f_1, global existence of solution and established a uniform decay rate for the solution of the following system

$$\begin{cases} u_{tt} - \Delta u = f_0(u) & \text{in} \quad \Omega \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu} + F(u_t) = f_1(u_t) & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ u = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}_+ \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) & \text{for} \quad x \in \Omega. \end{cases} \tag{2}$$

Problems like (2) were studied existence of global weak solutions by, when $f_0 = 0$, Cavalcanti et al. [30] by using nonlinear semigroup theory arguments inspired in the work of Chueshov et al.[37], and Vitillaro [88] with the Schauder fixed point arguments. In the case of $f_0 = f_1 = 0$, The existence was proved by means of the Faedo-Galerkin method, see [28, 32]. For the Neumann damped condition overfull the boundary in (2), Cavalcanti et al. [33] showed general existence and stability results by applying essentially the same technique as in [60] that are based on a natural tool for the monotone operator theory.

Time delay phenomenon is still resistant to many physical, chemical, biological, and thermal phenomena. The inclusion of delays explicitly in the equations is often a simplification or idealization that is introduced because a detailed description of the underlying processes is too complicated to be modeled mathematically, or because some of the details are unknown. Delay effects destabilize the dissipative system, which can be disastrous in the long term. In these cases, it may be necessary to choose between a model with discrete-time or time-varying delays and a model with distributed or continuous delay. A question of great importance is how does the qualitative behavior depend on the form and magnitude of the delays?. However, a big interest has been directed to achieve tight distributed control of partial differential equation (PDE) systems (maybe a source of instability and/or ill-posedness due to the time delay) in recent years. As a result, Datko et al. [40] proved that for a small delay in the boundary of the wave equation, the system becomes unstable. The inquiries related to the stability of wave equations with boundary delay have attracted considerable attention in recent years and many researchers were interested in the connection between the weight of the delay and the damping memory term or the frictional damping term of the following

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ u = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu} = -\mu_1 u_t - \mu_2 u_t(.,. - \tau) & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) & \text{for} \quad x \in \Omega \\ u(x,t) = f_0(x,t) & \text{for} \quad (x,t) \in \Omega \times (-\tau, 0), \end{cases} \tag{3}$$

where $\tau > 0$ is a constant delay parameter, $\mu_1, \mu_2 \geq 0$ and the initial data (u_0, u_1, f_0) belong to a suitable space. They can also think of the term $\mu_1 u_t + \mu_2 u_t(.,. - \tau)$ in the first equation of (3) as an internal feedback law where the second term represents the delay. It is well known that the energy of Problem (3) is exponentially decaying to zero in the absence of delay ($\mu_2 = 0$), in this sense, see [57]. In the presence of both damping and delay terms, Nicaise and Pignotti [76] proved exponential stability of the solution of (3) if the delay factor is less than the damping factor ($\mu_2 < \mu_1$). The result is obtained by introducing suitable energies and by using some observability inequalities. If $\mu_1 = \mu_2$, they further showed that there exist a sequence of arbitrary small (and large) delays such that instabilities occur. In the case $\mu_2 > \mu_1$, they also obtained delays that destabilize the system. We refer the reader to, Xu [92] in one space dimension, Ammari et al. [13] with interior delay, Fridman et al. [43] with time-varying delay and Nicaise-Pignotti [77] with distributed delay.

Thermoelasticity

Thermoelastic damping is a source of intrinsic material damping due to thermoelasticity present in almost all materials. As the name thermoelastic suggests, it describes the coupling between the elastic field in the structure caused by deformation and the temperature field. The earliest study of thermoelastic damping can be found in Zener's classical work, [93] in 1937/1938, in which he studied thermoelastic damping in beams undergoing flexural vibrations. Flexural vibrations cause alternating tensile and compressive strains to build up on opposite sides of the neutral axis leading to a thermal imbalance. Irreversible heat flow which is driven by the temperature gradient causes vibrational energy to be dissipated. This process is governed by the Fourier law of heat conduction, which is the constitutive relation between the heat flux vector q and the temperature gradient $\nabla\theta$. This theory formulates a linear relationship that is given by

$$q = -\kappa\nabla\theta, \tag{4}$$

where κ is called the heat conductivity.

In thermoelasticity, the behaviors of parameters in the Cattaneo heat flux model are qualitatively similar to those in Fourier's law. The classical heat conduction (Fourier's law), or as we call it the type I thermoelasticity, predicts an infinite speed of heat propagation. Actually, the experiments reveal that fluid temperature has an inverse relationship with the thermal relaxation time. The purpose of Cattaneo's law is to overcome this paradox and disturbances. One of them, developed by Lord-Shulman [63], suggests that Fourier's law (4) be replaced, for the relaxation time $\tau_0 > 0$, by

$$\tau_0 q_t + q + \kappa\nabla\theta = 0, \tag{5}$$

which can predict the effects on the boundary layer. This thermal effect is a quantum mechanical phenomenon in which it involves a hyperbolic-form transport correlation and is motivated by experiments illustrating more accurately the wave-form heat transfer, rather than by the more usual mechanism of diffusion. A wave-like thermal disturbance is referred to as second sound and a nonclassical theory predicting the occurrence of such disturbances is known as thermoelasticity with finite wave speeds (type II) or second sound thermoelasticity.

Green and Naghdi [48] postulated a new concept in thermoelasticity theories and proposed three models. They rated the heat conduction of type III in which has been of dissipative nature, where the heat flux is a combination of type I and II as limiting cases. In addition, the thermoelasticity of type III allows the constitutive functions for free energy, stress tensor, entropy, and heat flux to depend on the strain tensor, the time derivative of the thermal displacement, the gradient of thermal displacement, and the time derivative of the gradient of thermal displacement. This theory allows the dissipation of energy, but the heat flux is partially determined from the Helmholtz free energy potential. Both, types II and III, overcome the unnatural property of Fourier's law of infinite propagation speed and imply a finite wave propagation.

Many researchers have formulated such theories in different fields and analyze various problems of existence and asymptotic behavior, presenting characteristic properties of these theories, see [62, 70, 87] for examples. Although the majority of examples involve only one space dimension, but in [84, 95], the authors get into a huddle to extend the result in higher space dimensions.

Viscoelastic materials

In materials science and continuum mechanics, viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation. The term viscous implies that they deform slowly when exposed to an external force. The term elastic implies that once a deforming force (or the stress) has been removed the material will immediately return to its original configuration. According to viscous materials, like water, resist shear flow and strain linearly with time when stress is applied. On the other hand, Hooke's law states that the displacement of elastic materials is linearly proportional to the applied load and the effects of multiple load systems can be computed by simple linear superposition. However, there are materials with properties that are intermediate between elasticity and viscosity.

The mechanical properties of materials are usually examined by means of stress-strain (or load-deformation) behavior. Its mathematical expressions are either in shear mode or extensional mode, both forms being acceptable for the development of mathematical equations of transient deformation phenomena. For purely elastic materials, loading and unloading stress versus strain curves (lines) are superimposed. For viscoelastic ones, they form a hysteresis loop. The area within the loop represents the energy lost which dissipates as heat. This energy absorption behavior in part explains why viscoelastic materials are good shock absorbers. A further important property of viscoelastic materials is that their mechanical properties depend on the rate at which they are deformed.

The problems related to viscoelasticity (see Christensen [36]) have been available for a much longer time since 1874. There were several early contributors, such as Maxwell, Kelvin, and Voigt, Boltzmann [21] apparently supplied the first formulation of a three-dimensional theory of isotropic viscoelasticity. Many advances in the studies of well posedness and stability were made by the work of Dafermos [39]. The primary discussion touched upon by several authors is to use the integral term of memory instead the frictional damping term. In this setting, we are concerned with the following viscoelastic wave equation

$$\begin{cases} u_{tt} + \mathbf{A}u - \int_0^t g(t-s)\mathbf{A}u(s)\mathrm{d}s = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ u = 0 & \text{on} \quad \Gamma \times \mathbb{R}_+ \\ u(x,0) = u_0(x);\ u_t(x,0) = u_1(x) & \text{for} \quad x \in \Omega. \end{cases} \tag{6}$$

where $\mathbf{A}$ is an elliptic operator of second order and the integral term is the finite memory responsible for the viscoelastic damping, where g is called the relaxation function. It follows in both cases finite and infinite memories as convolution kernels from distribution theory that

$$\mathrm{H}_g \diamond \mathrm{H}_{\mathbf{A}u}(t) = \int_0^t g(t-s)\mathbf{A}u(s)\mathrm{d}s,$$

and

$$\mathrm{H}_g \diamond \mathbf{A}u(t) = \int_0^{+\infty} g(s)\mathbf{A}u(t-s)\mathrm{d}s = \int_{-\infty}^t g(t-s)\mathbf{A}u(s)\mathrm{d}s$$

where $\mathrm{H}_g = \mathrm{H}(g)$ and H is the Heaviside function.

The question that has been focused their attention as an important work is the viscoelastic damping of memory effect should be strong enough to procreate the decay of the system. It is by now well known that the decay of the solutions of (6) strongly depends on the decay of the relaxation function g. As highlighted by some papers, there have been encouraging advances toward obtaining the asymptotic behavior in both cases of finite and infinite memories.

Finite memory

First, when the relaxation function (kernel) take the form exponentially ($g(s) = e^{-s}$), Hrusa [55] studied a one-dimensional nonlinear viscoelastic wave equation and proved several global existence results and an exponential decay result. Muñoz Rivera [74] proved that the energy solution of linear isotropic homogeneous viscoelastic solids in bounded domains is exponentially decay to zero. Moreover, the decay is polynomial when the body occupies the whole space $\mathbb{R}^n$. Many authors, see for example [42, 75], have established the exponential decay of solutions for system (6) under specific condition on g

$$\exists \xi > 0 \quad : \quad g' \leq -\xi g(t), \quad \forall t \geq 0. \tag{7}$$

Messaoudi and Tatar [67] considered the following equation, for $\rho > 0$ and $\gamma \geq 0$

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s - \gamma \Delta u_t = b|u|u^{p-2} \quad \text{in} \ \ \Omega \times \mathbb{R}_+, \tag{8}$$

and established exponential and polynomial decay results in the absence, as well as in the presence, of a source term, where $\gamma = 0$ the kernel g satisfies

$$\exists \xi_1, \xi_2 > 0 \quad : \quad \xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad \forall t \geq 0. \tag{9}$$

For a nonincreasing function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ and g satisfying

$$g' \leq -\xi(t) g(t), \quad \forall t \geq 0, \tag{10}$$

Messaoudi [64] looked at (6) with $\mathbf{A} = -\Delta$ and proved a more general decay result. The same stability estimate has been established by Said-Houari et al. [85] for coupled two semilinear viscoelastic wave equations. For more results in this domain, see [53, 78, 91] and references therein. In [56], Kirane and Said-Houari established the energy decay result for system (6) with $\mathbf{A} = -\Delta$ in the presence of delay.

In the case of

$$g'(t) \leq -\xi(t) g^p(t), \ \forall t \geq 0, \tag{11}$$

Messaoudi and Al-Khulaifi [65] considered (8) with $\gamma = b = 0$ and proved a general decay rate from which the exponential decay and the polynomial decay are only special cases, where the kernel satisfy (11) when $1 \leq p < 3/2$. The same results were extended by Messaoudi-Al-Khulaifi [66] with boundary feedback and Mustafa [72] for an abstract viscoelastic problem as (6) for $1 \leq p < 2$.

Recently, Alabau-Boussouira et al. [10] introduced the following general assumption of the relaxation function

$$g'(t) \leq -\mathcal{H}(g(t)), \quad \forall t \geq 0, \tag{12}$$

where $\mathcal{H}$ is positive, strictly increasing and strictly $\mathcal{C}^2$ convex near the origin with $\mathcal{H}(0) = \mathcal{H}'(0) = 0$. Under some additional conditions, an explicit rate of decay is given. We refer to some of the excellent reviews about explicit general decay [31, 49, 59] and [73] with $\mathcal{H} \in \mathcal{C}^1(\mathbb{R})$ and $\mathcal{H}(0) = 0$.

More recently for a larger type of relaxation functions, Mustafa [71] introduced the following general latest assumption

$$g'(t) \leq -\xi(t)\mathcal{H}(g(t)), \quad \forall t \geq 0, \tag{13}$$

where $\mathcal{H}$ is an increasing and convex function near the origin and ξ is a nonincreasing function. He established the best decay rates of (6) with $\mathbf{A} = -\Delta$ that address both the optimality and generality. The assumption (13) has wildly attracted considerable attention many researchers in the last few years. For example, Al-Gharabli et al. [1] extend and combine the result with nonlinear boundary feedback stabilization, as well as for a nonlinear internal damping has been obtained by Belhannache et al. [20]. Al-Mahdi et al. [5] for abstract evolution equation with time-dependent nonlinear dissipation. The arguments of Mustafa [71] were used in Chellaoua-Boukhatem [35] to prove the stability for the following second-order abstract viscoelastic equation in Hilbert spaces with time-varying delay $\tau(t) > 0$

$$u_{tt} + \mathbf{A}u - \int_0^t g(t-s)\mathbf{B}u(s)\mathrm{d}s + \mu_1 u_t + \mu_2 u_t(t-\tau(t)) = 0, \quad \text{in } \ \Omega \times \mathbb{R}_+,$$

where **A** and **B** are self-adjoint linear positive operators.

Infinite memory

The viscoelastic problems with infinite memory terms have been studied by several authors. Giorgi et al. [47] considered the following semilinear hyperbolic equation in a bounded domain $\Omega \in \mathbb{R}^3$

$$u_{tt} - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)ds + g(u) = f \quad \text{in } \Omega \times \mathbb{R}_+^*,$$

with $K(0), K(\infty) > 0$ and $K' \leq 0$, and proved the existence of global attractors for the solutions. Conti and Pata [38] considered the following semilinear hyperbolic equation

$$u_{tt} + \alpha u_t - K(0)\Delta u + \int_0^{+\infty} K'(s)\Delta u(t-s)\mathrm{d}s + g(u) = f \quad \text{in } \Omega \times \mathbb{R}_+,$$

where K and g are functions satisfying specific conditions. They proved the existence of a regular global attractor where the kernel is a convex decreasing smooth function. Pata [80] discussed the decay properties of the semigroup generated by the following abstract equation

$$u_{tt} + \alpha\mathbf{A}u + \beta u_t - \int_0^{+\infty} g(s)\mathbf{A}u(t-s)\mathrm{d}s = 0, \quad \forall t \in \mathbb{R}_+^*$$

where **A** is a strictly positive self-adjoint linear operator and $\alpha > 0$, $\beta \geq 0$ and the memory kernel g is a decreasing function satisfying some conditions. He established the necessary as well as the sufficient conditions for exponential stability.

Under a class of infinite history kernels that satisfies the following condition

$$\int_0^{+\infty} \frac{g(s)}{\mathcal{H}^{-1}(-g'(s))}\mathrm{d}s + \sup_{s\in\mathbb{R}_+} \frac{g(s)}{\mathcal{H}^{-1}(-g'(s))} < +\infty, \tag{14}$$

where $\mathcal{H} : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing strictly convex function of class $\mathcal{C}^1(\mathbb{R}_+) \cup \mathcal{C}^2(]0,\ \infty[)$ satisfying

$$\mathcal{H}(0) = \mathcal{H}'(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} \mathcal{H}'(t) = \infty, \tag{15}$$

Guesmia [49] established general decay result for the following second-order abstract viscoelastic equation

$$u_{tt} + \mathbf{A}u - \int_0^{+\infty} g(s)\mathbf{B}u(t-s)\mathrm{d}s = 0, \quad \forall t > 0, \tag{16}$$

where **A** and **B** are two positive self-adjoint operators that satisfy some conditions. Guesmia and Tatar [52] obtained the stability result under appropriate assumptions on the operators **A** and **B** and by combining two kernels g and f with arbitrary decay for the following abstract hyperbolic equations

$$u_{tt} + \mathbf{A}u - \int_0^{+\infty} g(s)\mathbf{B}u(t-s)\mathrm{d}s + \int_0^{+\infty} f(s)u_t(t-s)\mathrm{d}s = 0, \quad \forall t > 0.$$

For a much larger class of relaxation functions satisfying (13), Guesmia [50] established the general stability results of two viscoelastic wave equations (16) with ($\mathbf{A} = -\Delta$, $\mathbf{B} = -\Delta$ or $-\mathrm{I}$) in a bounded domain $\Omega \subset \mathrm{I\!R}^n$. Al-Mahdi [2] obtained the same result of the following plate problem

$$u_{tt} - \sigma \Delta u_{tt} + \Delta^2 u - \int_0^{+\infty} g(s)\Delta^2 u(t-s)\mathrm{d}s = 0 \quad \text{in } \Omega \times \mathrm{I\!R}_+,$$

where σ is a positive constant. The arguments of Guesmia [50] were recently used in Chellaoua-Boukhatem [34] to prove the stability of the abstract evolution equation (16) with time delay. For a viscoelastic wave equation with nonlinear boundary feedback, Al-Mahdi and Al-Gharabli [3] proved a more general stability result without imposing any restrictive growth assumption on the damping term. From the results that are exponentially or polynomially decay to zero when $\mathcal{H}(s) = s^p$ and p covers the full admissible range $[1, 2)$, Al-Mahdi et al. [6] showed the general decay rate to Problem (16) with ($\mathbf{A} = \mathbf{B} = -\Delta$) and source term ($|u|^{p-1}u$).

For a coupled system of viscoelastic wave equations acting two kernals, Al-Mahdi et al. [4] treated the following

$$\begin{cases} u_{tt} - \Delta u - \int_0^{+\infty} g_1(s)\Delta u(t-s)ds + f_1(u,v) = 0 \\ v_{tt} - \Delta v - \int_0^{+\infty} g_2(s)\Delta v(t-s)ds + f_2(u,v) = 0 \end{cases}$$

in $\Omega \times \mathrm{I\!R}_+$, where f_1 and f_2 are nonlinear functions describing the interacation between the two waves. They established a relation between the decay rate of the solutions and the growth of $g_i; (i = 1, 2)$ at infinity satisfying (13) with specific properties of ξ and $\mathcal{H}_i$.

In one dimensional Timoshenko system, Al-Mahdi et al. [8] treated the following

$$\begin{cases} \rho_1 \varphi_{tt} + K(\varphi_x + \psi)_x = 0 \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \int_0^{+\infty} g(s)\psi_{xx}(t-s)\mathrm{d}s = 0 \end{cases}$$

where $(x,t) \in (0,\ L) \times \mathrm{I\!R}_+$, L, b, K, ρ_1, and ρ_2 are positive physical constants. Under appropriate conditions on ξ and $\mathcal{H}$ satisfying (13), the authors established some new decay results in the equal speeds of propagation case, as well as the nonequal-speed case. For the thermoelastic porous system, we mention the stability result of Al-Mahdi et al. [7]. They obtained a relation between the decay rate of the solutions and the growth of g at infinity satisfying (13) with specific properties of ξ and $\mathcal{H}$. These results generalize and improve many earlier results in the literature such as [49, 51] and delete some assumptions on the boundedness of initial data.

One of important motivations to studying exponential stability of the associated semigroup comes from the spectral analysis. This purpose recalls the related results given by the Gearhart-Prüss theorem (see [62, 82]). It is shown all eigenvalues approach a line that parallel to the imaginary axis. Moreover, the resolvent operator is bounded for all eigenvalues of the generator associated. The proof is the combination of the contradiction argument with a PDE technique. Let us mention some papers on weakly dissipative coupled systems. In Komornik-Rao [58], the exponential decay is established for each of the wave equations that have been damped on the boundary. Prüss [83] gave the optimal results to characterize polynomial as well as exponential decay rates for viscoelastic materials. Apalara et al. [14] studied the exponential stability of laminated beams when the frictional damping acts on the effective rotation angle. For weak damping acting only one equation, the following coupled wave equation

$$\begin{cases} u_{tt} - \Delta u + \int_0^{+\infty} g(s)\Delta u(s)\mathrm{d}s + \alpha v = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ v_{tt} - \Delta v + \alpha u = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ u = v = 0 & \text{on} \quad \Gamma \times \mathbb{R}_+ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{for} \quad x \in \Omega \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x) & \text{for} \quad x \in \Omega \end{cases} \tag{17}$$

has been considered by Almeida-Santos [12]. In [12], they proved the lack of exponential decay to system (17). The authors obtained the optimal polynomial decay by using the recent results due to Borichev-Tomilov [22]. The method used in this contexts introduced by Alabau-Boussouira [9] and developed by Alabau-Boussouira et al. [11].

Acoustic boundary conditions

In the nineteenth century since the studies of vibration and sound first appeared, the science of acoustics has expanded in many directions. Products of the jet age have added economic incentive to the solution of problems related to the generation and transmission of noise control. The noise of sound propagates through a certain acoustic medium. New mathematical techniques for the solution of problems involving coupled acoustic systems have been developed and the correspondingly great improvement in acoustical measurement techniques has made the theoretical solutions of more than academic interest. The phenomena of acoustics have taken on new importance and significance, both scientifically and technologically.

The PDEs coupled system of a viscoelastic wave equation with acoustic boundary conditions was first introduced by Morse-Ingard [68] and developed by Beale-Rosencrans [19, 18]. Moreover, the impedance models provide additional constraints to these conditions that are imposed simultaneously on the excess pressure and the displacement. The additions could be, for example, a known vibration velocity, a sound hard wall, or a symmetry plane. Imposing such constraints, in this case, Beale and Rosencrans [19] derived the equations

$$u_{tt} - \Delta u = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+ \tag{18}$$

$$\frac{\partial u}{\partial \nu} = z_t \quad \text{on} \quad \Gamma \times \mathbb{R}_+ \tag{19}$$

$$h z_{tt} + f z_t + m z = -\rho_0 u_t \quad \text{on} \quad \Gamma \times \mathbb{R}_+ \tag{20}$$

where $u = u(x,t)$ represents the velocity potential, $z = z(x,t)$ is the displacement, and ρ_0 is the unperturbed density of the gas. The nonnegative functions on the boundary h, f and m are the mass per unit area, the resistivity, and the spring, respectively. The authors proved the global existence and regularity of solutions. In [18], the problem was formulated as an initial value problem in a Hilbert space and semigroup methods were used to solve it. The loss of decay has obtained by [18] where the term z_{tt} was included. Result concerning existence and asymptotic behavior of smooth, as well as weak solution of wave equation with acoustic boundary conditions have been established by many authors, we recall to see [16, 44, 69, 79, 90].

The above model would be to describe the motion of fluid particles from rest in the domain Ω into the surface at a given point $x \in \Gamma$, which can be expressed by the pressure at that point. The relationship between the velocity potential at a point on the surface and the normal displacement is proportional to the pressure. It is called acoustic impedance. This impedance may be complex in the case of the velocity potential was not in phase with the pressure. In addition, the acoustic boundary layer can be used to describe a vibrating impermeable wall. It is well known that the fluid particles follow the wall motion. Moreover, the acoustic pressure was combined with the displacement at the actual position in order to characterize the impenetrability condition (20), which is obtained from the continuity of the velocity on the boundary. For example, through the air, in a room which is characterized by a bounded domain Ω and whose walls, ceiling and floor are described by the boundary conditions. This is the description of Jieqiong Wu in [90].

The first work bringing an analysis involving Dirichlet, feedback and acoustic boundary conditions to a thermoelastic system has been investigated by Braz e Silva et al. [28]. More precisely, they treated the nonlinear coupled system of thermoelastic type with acoustic boundary conditions

$$\begin{cases} u_{tt} - \alpha(t)\Delta u + \lambda|u|^p u + (a.\nabla)\theta = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ \theta_t - \beta\left(\int_\Omega \theta \mathrm{d}x\right)\Delta\theta + (a.\nabla)u_t = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ u_t + h(x)z_{tt} + f(x)z_t + m(x)z = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu} - z_t + \eta(., u_t) = 0 & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ u = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}_+ \\ \theta = 0 & \text{on} \quad \Gamma \times \mathbb{R}_+, \end{cases} \tag{21}$$

where α and β are given functions, λ and p are positive real constants, a is a constant known vector of $\mathbb{R}^n$ and the feedback function $\eta(., u_t)$ models a frictional damping on Γ_1. They proved the global existence and uniqueness of solution of (21), and therefore they obtained an exponential decay rate of the energy associated under the geometrical conditions on the boundary Γ and suitable assumptions on the feedback η.

Recently, Boukhatem and Benabderrahmane [26] considered the following variable coefficient viscoelastic wave equation with acoustic boundary conditions and a time-varying delay in the boundary feedback

$$\begin{cases} u_{tt} + \mathbf{A}u - \int_0^t g(t-s)\mathbf{A}u(s)ds = |u|^{p-2}u & \text{in} \quad \Omega \times \mathbb{R}_+ \\ u = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu_{\mathrm{A}}} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu_{\mathrm{A}}}(s)ds & \\ +\mu_1 k_1(u_t(x,t)) + \mu_2 k_2(u_t(x, t-\tau(t))) = h(x)z_t & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ u_t + f(x)z_t + m(x)z = 0 & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in} \quad \Omega \\ z(x,0) = z_0(x) & \text{in} \quad \Gamma_1, \\ u_t(x, t-\tau(0)) = j_0(x, t-\tau(0)) & \text{in} \quad \Gamma_1 \times (0,\ \tau(0)), \end{cases} \tag{22}$$

where $\mathbf{A}$ is an elliptic operator of second order. Under suitable assumptions, they established general decay results of the energy solution of (22) via suitable Lyapunov functionals and some properties of the convex functions. The same authors in [23] studied the global existence and exponential decay of solution of system (22) when $\mu_1 = \mu_2 = 0$. This result generalizes the result of Park in [79], in the absence of the term source and when $\mathbf{A} = -\Delta$, where the authors proved some decay rates under the assumption that the kernel g is assumed to decay like (10) and $\ell = 1 - \int_0^{+\infty} g(s)ds < 1/2$. The absence of the second derivative z_{tt} in $(22)_4$ brings some difficulties in the study because of the abnormality of the system. It can not apply directly to the semigroups or Faedo-Galerkin's theories. They added in the arguments the term εz_{tt} when $\varepsilon \to 0$ to overcome the difficulty. In the case of infinite memory, they obtained, in [27], two arbitrary decay results of system (22) without term source. The first stability result is given with a relation between the damping term and relaxation function and the second result is given without imposing any restrictive growth assumption on the damping term and the kernel function g. The polynomial decay and the blow-up of solutions of system (22), with $\mu_1 = \mu_2 = 0$ where the damping term $\alpha(x)u_t$ was added, are showed in Boukhatem-Benabderrahmane [25]. The estimates for the lifespan of solutions were also given.

Results description

This thesis is divided into three separate parts composed of six related chapters.

The first part deals with a coupled system of viscoelastic wave equation of infinite memory where the acoustic boundary condition satisfying (20). The same problem studied by [23] in absence of the second derivative z_{tt}. In **Chapter 1**, we show the global existence of energy-associated solutions using semigroup theory in the complex spectral distribution. Motivated by the mentioned works above concerning Gearhart-Prüss' theorem, the exponential stability of the corresponding semigroup is concluded. **Chapter 2** is dedicated to studying the general stability result under a very wider class of relaxation functions. The decay rate is improved explicitly by some properties on the convex function and without the boundedness condition of initial data. We end the chapter by giving an example to illustrate our result. This work substantially improves the earlier results in cases of acoustic boundary conditions.

In **Part II**, we consider a thermo-viscoelastic-coupled system of second sound where the heat conduction is given by Cattaneo's law. We introduce a new model that generalizes (21) through this heat conduction law. As far as we know, there are no results of our system. More precisely in **Chapter 3**, we reformulate a variational initial-boundary problem in absence of the second derivative z_{tt}, and then we can use the Faedo-Galerkin approach and the compactness method to deal with the issue of local or global existence. Under a more class of generality of the relaxation function, we establish a uniform decay rate for the energy as time goes to infinity in **Chapter 4**. The advantage here is that from our general estimate we can derive the exponential and polynomial decay rate. Our result extends the various decay results obtained for problems with or without thermo-viscoelasticity.

The last part is devoted to present a type III thermo-viscoelastic coupled system with distributed delay in both cases of finite and infinite memories. The interaction feedback between the nonlinear damping and the acoustic conditions is reacted on a portion of the boundary. This part extends the results of [72] under a general assumption on the nonlinear damping term which was first considered in [60]. Our work suggested here applies the system (21) studied in [28] with the thermoelasticity of type III. In **Chapter 5**, further use of the recent papers concerning different mechanisms of damping allows one to obtain the best decay estimates that address both the optimality and generality. This chapter studies the case of finite memory of the viscoelastic problem. Finally, in **Chapter 6**, once given the semigroup method combined with Schauder's fixed point theorem, it is known that a solution exists for all time and an interesting question is to ask about the asymptotic behavior of the solution as time goes to infinity. We can construct a suitable Lyapunov functional to establish the decay of solution. We also give some examples to illustrate our result. This work generalizes the composite stability between infinite memory and nonlinear damping.

Part I

A Coupled System of Viscoelastic Wave Equation of Infinite Memory with Acoustic Boundary Conditions

Chapter 1

Well Posedness and Exponential Stability in the Complex Plane

Our main results, in this chapter, are dedicated to studying well posedness and exponential decay for a coupled system of viscoelastic wave equation of infinite memory, in which we analyze the spectral distribution in the complex plane. The semigroup theory is used to show the described results of energy-associated solution which is decreasing over time.

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1.1 Problem statement

Consider the following viscoelastic wave equation with mixed boundary conditions

$$\begin{cases} u_{tt} - \mathrm{div}(\mathrm{A}\nabla u) + \int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla u(t-s))\mathrm{d}s = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ u = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu_{\mathrm{A}}} - \int_0^{+\infty} g(t-s)\frac{\partial u}{\partial \nu_{\mathrm{A}}}(s)\mathrm{d}s = z_t & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ hz_{tt} + fz_t + mz + u_t = 0 & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{for} \quad x \in \Omega \\ z(x,0) = z_0(x),\ z_t(x,0) = z_1(x) & \text{for} \quad x \in \Gamma_1, \end{cases} \tag{1.1}$$

where the functions $h, f, m : \Gamma_1 \to \mathbb{R}^+$ are essentially bounded. The initial conditions $u_0, u_1 : \Omega \to \mathbb{R}(\text{or } \mathbb{C})$, $z_0, z_1 : \Gamma_1 \to \mathbb{R}(\text{or } \mathbb{C})$ are given functions.

Following the idea of Dafermos [39], we introduce a new variable η such as

$$\eta(x,s,t) = u(x,t) - u(x,t-s), \quad \forall x \in \Omega,\ \forall t,s \in \mathbb{R}_+.$$

Then, the system (1.1) becomes

$$\begin{cases} u_{tt} - \ell\mathrm{div}(\mathrm{A}\nabla u) - \int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla\eta(s))\mathrm{d}s = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ \eta_t + \eta_s - u_t = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \\ hz_{tt} + fz_t + mz + u_t = 0 & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ u = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}_+ \\ \ell\frac{\partial u}{\partial \nu_{\mathrm{A}}} + \int_0^{+\infty} g(s)\frac{\partial \eta}{\partial \nu_{\mathrm{A}}}(s)\mathrm{d}s = z_t & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ u(0) = u_0,\ u_t(0) = u_1 & \text{in} \quad \Omega \\ \eta(x,s,0) = u_0(x) - u(x,-s) = \eta_0(x,s) & \text{for} \quad x \in \Omega, s \in \mathbb{R}_+ \\ z(0) = z_0,\ z_t(0) = z_1 & \text{in} \quad \Gamma_1. \end{cases} \tag{1.2}$$

1.2 Assumptions

To study our problems and to formulate existence and stability theorems, we will need some assumptions in all parts of this thesis.

(H.1) The coefficients $a_{ij} \in \mathcal{C}^1(\overline{\Omega})$ are symmetric and there exists a constants $a_0 > 0$ such that

$$\sum_{i,j=1} Problema_{ij}(x)\zeta_i\zeta_j \geq a_0|\zeta|^2, \quad \forall x \in \overline{\Omega},\ \forall \zeta \in \mathbb{R}^n(\text{or } \mathbb{C}^n). \tag{1.3}$$

(H.2) The relaxation function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded $\mathcal{C}^1$ nonincreasing function satisfying, for some $\beta_0 > 0$

$$g'(t) \geq -\beta_0 g(t), \quad g(0) > 0, \quad 1 - \int_0^\infty g(s)\mathrm{d}s = \ell > 0, \tag{1.4}$$

and there exists a constant $\beta > 0$ such that

$$g'(t) \leq -\beta g(t), \quad \forall t \geq 0. \tag{1.5}$$

(**H.3**) There exist three positive constants f_0, m_0, and h_0 such that

$$f(x) \geq f_0, \ m(x) \geq m_0, \text{ and } h(x) \geq h_0,$$

for almost everywhere x in Γ_1.

We denote the closed subspace of $\mathrm{H}^1(\Omega)$ by

$$\mathrm{H}^1_{\Gamma_0}(\Omega) = \Big\{ u \in \mathrm{H}^1(\Omega) \mid \ u = 0 \text{ on } \Gamma_0 \Big\}$$

equipped with the norm equivalent to the usual norm in $\mathrm{H}^1(\Omega)$.

Remark 1.2.1. The Poincaré inequality holds on $\mathrm{H}^1_{\Gamma_0}(\Omega)$, *i.e.*,

$$\exists C_* > 0 \quad \text{such that} \quad \|u(t)\|_2 \leq C_* \|\nabla u(t)\|_2, \quad \forall u \in \mathrm{H}^1_{\Gamma_0}(\Omega). \tag{1.6}$$

Moreover,

$$\exists \bar{C}_* > 0 \quad \text{such that} \quad \|u(t)\|_{2,\Gamma_1} \leq \bar{C}_* \|\nabla u(t)\|_2, \quad \forall u \in \mathrm{H}^1_{\Gamma_0}(\Omega). \tag{1.7}$$

Furthermore, we notice, for $t \geq 0$

$$a(u(t), v(t)) = (\mathrm{A}u(t), v(t)) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial u(t)}{\partial x_i} \overline{\frac{\partial v(t)}{\partial x_i}} \mathrm{d}x.$$

Remark 1.2.2. Using the hypothesis (**H.1**), it is easy to verify that the bilinear (sesquilinear) form $a(.,.) : \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_{\Gamma_0}(\Omega) \to \mathbb{R}(\text{or } \mathbb{C})$ is continuous, and by (1.3), we deduce that a is coercive.

1.3 The well posedness

In this section, we will show the well posedness and regularity of our system.

In order to give a reformulation as a first-order evolution system, we denote by

$$U = (u, v, \eta, z, \delta)^\intercal \quad \text{with} \quad v = u_t \quad \text{and} \quad \delta = z_t.$$

We consider the product Hilbert spaces

$$\mathbb{H} = \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{L}^2(\Omega) \times \mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega)) \times \mathrm{L}^2(\Gamma_1) \times \mathrm{L}^2(\Gamma_1),$$

endowed with the following inner product

$$\begin{aligned} \left\langle U(t), \tilde{U}(t) \right\rangle_{\mathbb{H}} \ &= \ \ell a(u(t), \tilde{u}(t)) + \int_\Omega v(t)\overline{\tilde{v}(t)} \mathrm{d}x + \langle \eta(t), \tilde{\eta}(t) \rangle_{\mathrm{L}^2_g} \\ &\quad + \langle m(x) z(t), \tilde{z}(t) \rangle_{\Gamma_1} + \left\langle h(x)\delta(t), \tilde{\delta}(t) \right\rangle_{\Gamma_1}, \end{aligned} \tag{1.8}$$

where $\mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega))$ denotes the Hilbert space of $\mathrm{H}^1_{\Gamma_0}(\Omega)-$valued functions on $\mathbb{R}_+$, endowed with the inner product

$$\langle \eta(t), \tilde{\eta}(t) \rangle_{\mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega))} = \int_0^{+\infty} g(s) a(\eta(s,t), \tilde{\eta}(s,t)) \mathrm{d}s, \tag{1.9}$$

for every $U = (u, v, \eta, z, \delta)^\intercal$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\eta}, \tilde{z}, \tilde{\delta})^\intercal$ in $\mathbb{H}$.

Thus, the system (1.2) can be rewritten in the following

$$\begin{cases} U_t(t) = \mathcal{A}U(t), \quad \forall t \geq 0 \\ U(0) = U_0 = (u_0, u_1, \eta_0, z_0, z_1)^\intercal \in \mathbb{H} \end{cases} \tag{1.10}$$

where the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H}$ is defined by

$$\mathcal{A}U(t) = \begin{pmatrix} v(t) \\ \ell \mathrm{div}(\mathrm{A}\nabla u(t)) + \int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla\eta(t,s))\mathrm{d}s \\ v(t) - \eta_s(t,s) \\ \delta(t) \\ \frac{1}{h(x)}\left(-v(t) - m(x)z(t) - f(x)\delta(t)\right) \end{pmatrix} \tag{1.11}$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ U \;\middle|\; \begin{array}{l} \mathrm{A}\nabla u \in \mathrm{H}(\mathrm{div},\Omega);\; v \in \mathrm{H}^1_{\Gamma_0}(\Omega); \\ \eta \in \mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega));\; z, \delta \in \mathrm{L}^2(\Gamma_1); \\ \ell \frac{\partial u}{\partial \nu_\mathrm{A}} + \int_0^{+\infty} g(s)\frac{\partial \eta}{\partial \nu_\mathrm{A}}(s)\mathrm{d}s = \delta \;\text{ on } \Gamma_1 \end{array} \right\} \tag{1.12}$$

Set the energy functional E of the system (1.2)

$$E(t) = \frac{1}{2}\left\langle U(t), U(t)\right\rangle_{\mathbb{H}}. \tag{1.13}$$

Lemma 1.3.1. *The energy functional (1.13), along the solution of (1.2), is a nonincreasing function satisfying, for all* $t \geq 0$

$$E'(t) = \frac{1}{2}\int_0^{+\infty} g'(s)a(\eta(s,t),\eta(s,t))\mathrm{d}s - \|f^{1/2}\delta(t)\|^2_{2,\Gamma_1}. \tag{1.14}$$

Proof. Taking the scalar product of $(1.2)_1$ with u_t and $(1.2)_3$ with z_t in $\mathrm{L}^2(\Omega)$ and $\mathrm{L}^2(\Gamma_1)$, respectively, then adding it to the inner product (1.9) of $(1.2)_2$ with η. Using Green's formula and taking its real part, we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t(t)\|_2^2 + \ell a(u(t),u(t)) + \|\eta(t)\|^2_{\mathrm{L}^2_g} + \|m^{1/2}z(t)\|^2_{2,\Gamma_1} + \|h^{1/2}\delta(t)\|^2_{2,\Gamma_1}\right) \\ = &-\int_0^{+\infty} g(s)a(\eta_s(t,s),\eta(t,s))\mathrm{d}s - \|f^{1/2}\delta(t)\|^2_{2,\Gamma_1}. \end{aligned} \tag{1.15}$$

Using (1.4) and the properties of η, we have

$$\int_0^{+\infty} g(s)a(\eta(s),\eta_s(s))\mathrm{d}s = -\frac{1}{2}\int_0^{+\infty} g'(s)a(\eta(s),\eta(s))\mathrm{d}s. \tag{1.16}$$

Combining (1.16) and (1.15), we get (1.14). ■

Our aim is ensured by the following theorem

Theorem 1.3.2. *The operator* $\mathcal{A}$ *is the infinitesimal generator of a* $\mathcal{C}_0-$*semigroup of contractions over the Hilbert space* $\mathbb{H}$. *Thus, for any initial data* $U_0 \in \mathbb{H}$, *the problem (1.10) has a unique weak solution* $U \in \mathcal{C}(\mathbb{R}_+; \mathbb{H})$. *Moreover, if* $U_0 \in D(\mathcal{A})$, *then the solution* $U \in \mathcal{C}(\mathbb{R}_+; D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+; \mathbb{H})$.

Proof. We will use the Hille-Yosida theorem. For this purpose, $\mathcal{A}$ is dissipative. Indeed, using (1.10) and (1.13), we have, for $U \in D(\mathcal{A})$

$$E'(t) = \Re \langle U_t(t), U(t)\rangle_{\mathbb{H}} = \Re \langle \mathcal{A}U(t), U(t)\rangle_{\mathbb{H}}.$$

Therefore, we deduce from Lemma 1.3.1 that

$$\Re \langle \mathcal{A}U(t), U(t)\rangle_{\mathbb{H}} = \frac{1}{2}\int_0^{+\infty} g'(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s - \|f^{1/2}\delta(t)\|_{\Gamma_1} \leq 0. \tag{1.17}$$

Next, $\mathrm{I} - \mathcal{A}$ is surjective. Indeed, for each $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5)^\intercal \in \mathbb{H}$, we show that there exists $U \in D(\mathcal{A})$ such that

$$(\mathrm{I} - \mathcal{A})U = \mathcal{F}.$$

Then, the previous equation reads

$$\begin{aligned} u - v &= f_1 & (1.18)\\ v - \ell \mathrm{div}(\mathrm{A}\nabla u) - \int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla \eta(s))\mathrm{d}s &= f_2 & (1.19)\\ \eta + \eta_s - v &= f_3 & (1.20)\\ z - \delta &= f_4 & (1.21)\\ h(x)\delta + v + f(x)\delta + m(x)z &= h(x)f_5. & (1.22) \end{aligned}$$

Suppose (u, z) are found in $\mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{L}^2(\Gamma_1)$. Thus, (1.18) and (1.21) yield

$$\begin{cases} v = u - f_1, \\ \delta = z - f_4, \end{cases} \tag{1.23}$$

Then,

$$v \in \mathrm{H}^1_{\Gamma_0}(\Omega), \quad \text{and} \quad \delta \in \mathrm{L}^2(\Gamma_1).$$

From (1.20), we can determine

$$\eta(s) = -ve^{-s} + v + \int_0^s f_3(\tau)e^{\tau - s}\mathrm{d}\tau, \quad \forall s \in \mathbb{R}_+, \tag{1.24}$$

that is $\eta(0) = 0$. According (1.24) with $(1.23)_1$, we have

$$\eta(s) = -ue^{-s} + u + \eta_1(s), \quad \forall s \in \mathbb{R}_+, \tag{1.25}$$

with $\eta_1 \in \mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega))$ defined by

$$\eta_1(s) = f_1 e^{-s} - f_1 + \int_0^s f_3(\tau)e^{\tau - s}\mathrm{d}\tau.$$

Then, $(1.2)_5$ becomes

$$\ell_g \frac{\partial u}{\partial \nu_{\mathrm{A}}} + \int_0^{+\infty} g(s)\frac{\partial \eta_1}{\partial \nu_{\mathrm{A}}}(s)\mathrm{d}s = z - f_4 \quad \text{on } \Gamma_1 \times \mathbb{R}_+,$$

where $\ell_g = \left(\ell + \int_0^{+\infty} g(s)(1 - e^{-s})\mathrm{d}s\right) > 0$. Inserting (1.23) and (1.25) into (1.19) and (1.22), adding the results, we get

$$u - \ell_g \mathrm{div}(\mathrm{A}\nabla u) = f_1 + f_2 + \int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla \eta_1(s))\mathrm{d}s, \tag{1.26}$$

$$(1+m(x)+f(x))z+u=h(x)f_5+f_1+(1+f(x))f_4 \tag{1.27}$$

Taking the inner product of (1.26) with $\tilde{u}$ in $\mathrm{L}^2(\Omega)$, then adding it to the complex conjugate of the inner product of (1.27) with $\tilde{z}$ in $\mathrm{L}^2(\Gamma_1)$. Using Green's formula, we obtain the sesquilinear from

$$\mathfrak{B}:(\mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1))\times(\mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1))\to\mathbb{C}$$

defined by

$$\mathfrak{B}((u,z),(\tilde{u},\tilde{z}))=(u,\tilde{u})+\ell_g a(u,\tilde{u})-\langle z,\tilde{u}\rangle_{\Gamma_1}+\langle u,\tilde{z}\rangle_{\Gamma_1}+(1+m(x)+f(x))\,\langle z,\tilde{z}\rangle_{\Gamma_1},$$

for every $(u,z),(\tilde{u},\tilde{z})\in \mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1)$, and the conjugate-linear from

$$\mathcal{G}:\mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1)\to\mathbb{C}$$

defined by

$$\mathcal{G}(\tilde{u},\tilde{z})=(f_1+f_2,\tilde{u})-\int_0^{+\infty} g(s)a(\eta_1(s),\tilde{u})\mathrm{d}s+\langle h(x)f_5+f_1+f(x)f_4,\tilde{z}\rangle_{\Gamma_1},$$

for every $(\tilde{u},\tilde{z})\in \mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1)$.

It's easy to see that $\mathfrak{B}$ is a continuous sesquilinear form and coercive on $(\mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1))\times(\mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1))$ and $\mathcal{G}$ is a continuous conjugate-linear form on $\mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1)$. Using complex Lax-Milgram's theorem, then there exists a unique solution $(u,z)\in \mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1)$, satisfying, for all $(\tilde{u},\tilde{z})\in \mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1)$

$$\mathfrak{B}((u,z),(\tilde{u},\tilde{z}))=\mathcal{G}(\tilde{u},\tilde{z}). \tag{1.28}$$

Additionally, we proceed to get more regularity.
Taking $\tilde{z}=0$ in (1.28). Since $\mathcal{D}(\Omega)$ is dense in $\mathrm{H}^1_{\Gamma_0}(\Omega)$, we deduce that

$$\ell_g\left(\mathrm{div}(\mathrm{A}\nabla u),\tilde{u}\right)=\left(\int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla\eta_1(s))\mathrm{d}s,\tilde{u}\right)+\langle z,\tilde{u}\rangle_{\Gamma_1}-(u,\tilde{u}),$$

for every $\tilde{u}\in \mathrm{H}^1_{\Gamma_0}(\Omega)$. Hence, $\mathrm{A}\nabla u\in \mathrm{H}(\mathrm{div},\Omega)\cap \mathrm{H}^1_{\Gamma_0}(\Omega)$.

Then $U\in D(\mathcal{A})$. Consequently, Lumper-Phillips' theorem guarantees the generator $\mathcal{A}$ of a $\mathcal{C}_0-$semigroup on $\mathbb{H}$. ■

1.4 Exponential stability

Here we will show the exponential stability of (1.10). The method that we shall use is based on the Gearhart-Prüss theorem [62] to complex value dissipative systems.

Our starting point is to show that the semigroup associated with (1.10), generated by $\mathcal{A}$, is exponentially stable. The following theorem gives our main result, that is to verify the conditions (*i*) and (*ii*) of Theorem A.2.10.

Theorem 1.4.1. *Assume that* ***(H.1)****-****(H.3)*** *holds. Then,* $e^{\mathcal{A}t}$ *generated by* $\mathcal{A}$ *is exponentially stable.*

Proof. We first show that the resolvent of the system (1.10) is located on the imaginary axes. Note that the resolvent equation $(\mathrm{i}\lambda\mathrm{I}-\mathcal{A})U=\mathcal{F}\in\mathbb{H}$ is given by

$$\mathrm{i}\lambda u-v = f_1 \tag{1.29}$$

$$\mathrm{i}\lambda v-\ell\mathrm{div}(\mathrm{A}\nabla u)-\int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla\eta(s))\mathrm{d}s = f_2 \tag{1.30}$$

$$\mathrm{i}\lambda\eta+\eta_s-v = f_3 \tag{1.31}$$

$$\mathrm{i}\lambda z-\delta = f_4 \tag{1.32}$$

$$\mathrm{i}\lambda\delta+f(x)\delta+m(x)z+v = f(x)f_5. \tag{1.33}$$

It’s means to show that $\mathrm{i}\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset$, where $\sigma(\mathcal{A})$ is the spectrum of $\mathcal{A}$.
Using contradiction arguments. Let us suppose that $\mathcal{A}$ has an imaginary eigenvalue. Then, we have

$$\mathcal{A}U = i\lambda U, \qquad \lambda \in \mathbb{R}. \tag{1.34}$$

Thus, $\mathcal{F} \equiv 0$ in (1.29)-(1.33). From (1.17) and (1.34), we can get

$$0 = \Re \langle \mathcal{A}U(t), U(t)\rangle_{\mathbb{H}} \leq \frac{1}{2}\int_0^{+\infty} g'(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s - \|f^{1/2}\delta(t)\|_{\Gamma_1} \leq 0.$$

It follows that $\delta = 0$, and from the hypothesis of g that $\nabla\eta = 0$. Using the fact $u = 0$ in $\Gamma \times \mathbb{R}_+$ that $\eta = 0$. This implies by (1.29) and (1.24) that $u = v = 0$. From equation (1.33), we conclude that $z = 0$. Hence, $U \equiv 0$. We obtain a contradiction.

We now prove (ii) by contradiction arguments again. Suppose that (ii) is not true. Then there exist a sequence λ_k with $|\lambda_k| \to +\infty$ and a sequence of functions

$$U_k = (u_k, v_k, \eta_k, z_k, \delta_k)^\intercal \in D(\mathcal{A}) \qquad \text{with} \qquad \|U_k\|_{\mathbb{H}} = 1, \tag{1.35}$$

such that, as $n \to +\infty$;

$$(\mathrm{i}\lambda_k \mathrm{I} - \mathcal{A})U_k \to 0 \quad \text{in} \quad \mathbb{H} \tag{1.36}$$

i.e.,

$$\mathrm{i}\lambda_k u_k - v_k \to 0 \quad \text{in} \quad \mathrm{H}^1_{\Gamma_0}(\Omega) \tag{1.37}$$

$$\mathrm{i}\lambda_k v_k - \ell\mathrm{div}(\mathrm{A}\nabla u_k) - \int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla\eta_k(s))\mathrm{d}s \to 0 \quad \text{in} \quad \mathrm{L}^2(\Omega) \tag{1.38}$$

$$\mathrm{i}\lambda_k\eta_k + \partial_s\eta_k - v_k \to 0 \quad \text{in} \quad \mathrm{L}^2_g \tag{1.39}$$

$$\mathrm{i}\lambda_k z_k - \delta_k \to 0 \quad \text{in} \quad \mathrm{L}^2(\Gamma_1) \tag{1.40}$$

$$\mathrm{i}\lambda_k\delta_k + f(x)\delta_k + m(x)z_k + v_k \to 0 \quad \text{in} \quad \mathrm{L}^2(\Gamma_1) \tag{1.41}$$

Taking the inner product (1.8) of (1.36) with U_k and then taking its real part yields

$$-\Re\langle(\mathrm{i}\lambda_k\mathrm{I} - \mathcal{A})U_k, U_k\rangle_{\mathbb{H}} = -\frac{1}{2}\int_0^{+\infty} g'(s)a(\eta_k(s),\eta_k(s))\mathrm{d}s + \|f^{1/2}\delta_k\|^2_{2,\Gamma_1} \to 0. \tag{1.42}$$

Using (1.5) and (**H.3**), we find that

$$\eta_k \to 0 \qquad \text{in } \mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega)), \tag{1.43}$$

$$\delta_k \to 0 \qquad \text{in } \mathrm{L}^2(\Gamma_1). \tag{1.44}$$

On the other hand, taking the complex conjugate of the inner product of (1.37) with ℓu_k in $\mathrm{H}^1_0(\Omega)$, then adding it to the inner product of (1.38) with v_k in $\mathrm{L}^2(\Omega)$ and using Green’s formula, we get

$$\mathrm{i}\left(-\ell a(u_k,u_k) + \|v_k\|_2^2\right) - \frac{1}{\lambda_k}\langle\delta_k, v_k\rangle_{\Gamma_1} + \frac{1}{\lambda_k}\int_0^{+\infty} g(s)a(\eta_k(s), v_k)\mathrm{d}s \to 0. \tag{1.45}$$

We can deduce from (1.37) that $\frac{1}{\lambda_k}\|\nabla v_k\|_2^2$ is uniformly bounded. By using (1.7), (1.43) and (1.44), the last two terms in (1.45) converge to zero. Hence,

$$\ell a(u_k,u_k) - \|v_k\|_2^2 \to 0. \tag{1.46}$$

Adding the complex conjugate of the inner product of (1.40) with $m(x)\delta_k$ to the inner product of (1.41) with z_k in $\mathrm{L}^2(\Gamma_1)$, we have

$$\mathrm{i}(\|m^{1/2}z_k\|^2_{2,\Gamma_1} + \|\delta_k\|^2_{2,\Gamma_1}) + \frac{1}{\lambda_k}\|f^{1/2}\delta_k\|^2_{2,\Gamma_1} + \frac{1}{\lambda_k}\langle v_k, \delta_k\rangle_{\Gamma_1} \to 0.$$

By using (1.38) and the fact that $\frac{1}{\lambda_k}\|\nabla v_k\|_2^2$ and $\|\delta_k\|^2_{2,\Gamma_1}$ are uniformly bounded, we obtain

$$\|m^{1/2} z_k\|^2_{2,\Gamma_1} \to 0. \tag{1.47}$$

Combining (1.35) with (1.43), (1.44) and (1.47). Then, using (1.46), we find that

$$a(u_k, u_k) \to \frac{1}{2}, \tag{1.48}$$

$$\|v_k\|_2^2 \to \frac{1}{2}. \tag{1.49}$$

It's easy to see that $\frac{1}{\lambda_k} v_k \in \mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega))$. Then, taking the inner product (1.9) of (1.39) with $\frac{1}{\lambda_k} v_k$, we have

$$\frac{1}{\lambda_k}\langle \eta_k(s), v_k\rangle_{\mathrm{L}^2_g} + \frac{1}{\lambda_k^2}\langle \partial_s \eta_k(s), v_k\rangle_{\mathrm{L}^2_g} - \frac{1}{\lambda_k^2}\langle v_k, v_k\rangle_{\mathrm{L}^2_g} \to 0. \tag{1.50}$$

Using again the fact that $\frac{v_k}{\lambda_k}$ is bounded in $\mathrm{H}^1_{\Gamma_0}(\Omega)$ and by using (1.43), we get that the first term of (1.50) converges to zero. This yields

$$\frac{(1-\ell)}{\lambda_k^2} a(v_k, v_k) - \underbrace{\frac{1}{\lambda_k^2}\int_0^{+\infty} g(s) a\left(\partial_s \eta_k(s), v_k\right) \mathrm{d}s}_{I_1} \to 0. \tag{1.51}$$

The second term (I_1) in (1.51) converges to zero. Indeed, from (1.5) and by using again that $\frac{v_k}{\lambda_k}$ is bounded in $\mathrm{H}^1_{\Gamma_0}(\Omega)$, we have

$$\begin{aligned} |I_1| &= \frac{1}{|\lambda_k|}\left|\int_0^{+\infty} g'(s) a(\eta_k(s), \frac{v_k}{\lambda_k})\mathrm{d}s\right| \\ &\leq \frac{\beta a_1}{|\lambda_k|}\left\|\frac{\nabla v_k}{\lambda_k}\right\|_2 \left(\frac{(1-\ell)}{a_0}\int_0^{+\infty} g(s) a(\eta_k(s), \eta_k(s))\mathrm{d}s\right)^{1/2} \to 0, \end{aligned}$$

where $a_1 = \max\limits_{1\leq j\leq n}\left(\sum\limits_{i=1}^{n} \|a_{ij}\|_\infty^2\right)$. This with (1.51), leads to

$$\frac{v_k}{\lambda_k} \to 0 \quad \text{in } \mathrm{H}^1_{\Gamma_0}(\Omega). \tag{1.52}$$

Taking the inner product of (1.37) with ℓu_k in $\mathrm{H}^1_0(\Omega)$. Since u_k is bounded in $\mathrm{H}^1_{\Gamma_0}(\Omega)$. By using (1.52), we obtain

$$a(u_k, u_k) \to 0.$$

This contradicts (1.48). Therefore, the proof is completed. ■

Chapter 2

General Decay of Energy-associated Solution

This chapter is devoted to the investigation of the general decay rate using the energy method. This result is obtained without the boundedness condition of initial data assumed in many earlier papers in the literature.

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2.1 Modified Lyapunov functional

We are going to construct modified Lyapunov functional $\mathcal{L}$ as follows

$$\mathcal{L}(t) = NE(t) + N_1\varphi(t) + N_2\psi(t), \tag{2.1}$$

where N and $N_i,\ i = 1, 2$; are positive constants to be fixed later, the energy E defined by (1.13), and

$$\begin{aligned}(g \diamond u)(t) &= \|\eta(t)\|_{\mathrm{L}^2_g(\mathbb{R}_+;\mathrm{H}^1\Gamma_0(\Omega))} = \int_0^{+\infty} g(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s,\\ \varphi(t) &= \int_\Omega u_t(t)u(t)\mathrm{d}x + \int_{\Gamma_1} h(x)z_t(t)z(t)\mathrm{d}\Gamma + \int_{\Gamma_1} z(t)u(t)\mathrm{d}\Gamma + \frac{1}{2}\|f^{1/2}z(t)\|^2_{2,\Gamma_1}, \quad (2.2)\\ \psi(t) &= -\int_\Omega u_t(t)\int_0^{+\infty} g(s)\eta(t,s)\mathrm{d}s\mathrm{d}x. \quad (2.3)\end{aligned}$$

Let us define

$$C_\alpha = \int_0^{+\infty} \frac{g^2(s)}{\alpha g(s) - g'(s)}\mathrm{d}s, \quad k(t) = \alpha g(t) - g'(t), \quad \text{and} \quad k_1 = \int_0^{+\infty} k(s)\mathrm{d}s. \tag{2.4}$$

Remark 2.1.1. As

$$\frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} < g(s),$$

using the Lebesgue dominated convergence theorem to show

$$\alpha C_\alpha = \int_0^\infty \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)}\mathrm{d}s \to 0 \quad \text{as} \quad \alpha \to 0.$$

Lemma 2.1.2. *The functional φ defined by* (2.2) *satisfies, for any $0 < \alpha < 1$, for all $t \geq 0$, the estimate*

$$\varphi'(t) \leq \|u_t(t)\|_2^2 - \ell a(u(t),u(t)) + C_1C_\alpha(k\diamond u)(t) - \|m^{1/2}z(t)\|^2_{2,\Gamma_1} + C_2\|h^{1/2}z_t(t)\|^2_{2,\Gamma_1}, \tag{2.5}$$

where

$$a_1 = \max_{1\leq j\leq n}\left(\sum_{i=1}^n \|a_{ij}\|^2_\infty\right), \quad C_1 = \frac{na_1}{4a_0C_*^2}, \quad \textit{and} \quad C_2 = 1 + \frac{\|h\|_\infty}{h_0^2}.$$

Proof. We differentiate (2.2) with respect to t, we arrive by using Green's formula and the boundary conditions $(1.2)_{4,5}$ at

$$\begin{aligned}\varphi'(t) &= -a(u(t),u(t)) + \int_0^{+\infty} g(s)a(u(t-s),u(t))\mathrm{d}s + \|h^{1/2}z_t(t)\|^2_{2,\Gamma_1}\\ &\quad + \|u_t(t)\|_2^2 + 2\int_{\Gamma_1} z_t(t)u(t)\mathrm{d}\Gamma - \|m^{1/2}z(t)\|^2_{2,\Gamma_1}. \quad (2.6)\end{aligned}$$

Now, thanks to Cauchy-Schwarz's inequality

$$\begin{aligned}\int_\Omega\left(\int_0^{+\infty} g(s)\nabla\eta(t,s)\mathrm{d}s\right)^2\mathrm{d}x &= \int_\Omega\left(\int_0^{+\infty}\frac{g(s)}{\sqrt{k(s)}}\sqrt{k(s)}\nabla\eta(t,s)\mathrm{d}s\right)^2\mathrm{d}x\\ &\leq \frac{C_\alpha}{a_0}(k\diamond u)(t). \quad (2.7)\end{aligned}$$

From this, Young's inequality, (1.3), and (1.4), we obtain

$$\int_0^{+\infty} g(s)a(u(t-s),u(t))\mathrm{d}s \leq \left((1-\ell)+\delta\frac{a_1}{a_0}\right)a(u(t),u(t))+\frac{nC_\alpha}{4a_0\delta}(k\diamond u)(t), \tag{2.8}$$

for some $\delta > 0$ and

$$2\int_{\Gamma_1} z_t(t)u(t)\mathrm{d}\Gamma \leq \frac{\|h\|_\infty}{h_0^2}\|h^{1/2}z_t(t)\|_{2,\Gamma_1}^2+\frac{\overline{C}_*^2}{a_0}a(u(t),u(t)). \tag{2.9}$$

Substituting (2.8)-(2.9) into (2.6). By choosing $\delta = \dfrac{C_*^2}{a_1}$, we find (2.5). ■

Lemma 2.1.3. *The functional* ψ *defined by* (2.3) *satisfies, for* $\delta > 0$, *for all* $t \geq 0$, *the estimate*

$$\psi'(t) \leq -(1-\ell-\delta)\|u_t(t)\|_2^2+\delta\frac{a_1}{a_0}a(u(t),u(t))+C_3\|h^{1/2}z_t(t)\|_{2,\Gamma_1}^2+C_4C_\alpha(k\diamond u)(t), \tag{2.10}$$

where

$$C_3=\frac{\overline{C}_*^2\|h\|_\infty}{h_0^2} \quad \text{and} \quad C_4=\frac{1}{4a_0}\max\left\{\frac{2\overline{C}_*^2k_1}{\delta},1+4\sqrt{a_1n}+\frac{n+2\alpha^2C_*^2}{\delta}\right\}.$$

Proof. By exploiting $(1.2)_1$, we have

$$\begin{aligned}\psi'(t) \quad = \quad & -\int_{\Gamma_1} z_t(t)\int_0^{+\infty} g(s)\eta(t,s)\mathrm{d}s\mathrm{d}\Gamma-\int_\Omega u_t(t)\int_0^{+\infty} g'(s)\eta(t,s)\mathrm{d}s\mathrm{d}x\\ & -(1-\ell)\|u_t(t)\|_2^2+\ell\int_\Omega \mathrm{A}\nabla u(t)\int_0^{+\infty} g(s)\nabla\eta(t,s)\mathrm{d}s\mathrm{d}x\\ & +\int_\Omega\left(\int_0^{+\infty} g(s)\mathrm{A}\nabla\eta(s)\mathrm{d}s\right)\left(\int_0^{+\infty} g(s)\nabla\eta(s)\mathrm{d}s\right)\mathrm{d}x.\end{aligned} \tag{2.11}$$

Similar calculations in (2.8) and (2.9), we obtain

$$\int_\Omega \mathrm{A}\nabla u(t)\int_0^{+\infty} g(s)\nabla\eta(t,s)\mathrm{d}s\mathrm{d}x \leq \delta\frac{a_1}{a_0}a(u(t),u(t))+\frac{nC_\alpha}{4a_0\delta}(k\diamond u)(t), \tag{2.12}$$

for $\delta > 0$

$$-\int_{\Gamma_1} z_t(t)\int_0^{+\infty} g(s)\eta(t,s)\mathrm{d}s\mathrm{d}\Gamma \leq \frac{C_\alpha}{4a_0}(k\diamond u)(t)+\frac{\overline{C}_*^2\|h\|_\infty}{h_0^2}\|h^{1/2}z_t(t)\|_{2,\Gamma_1}^2, \tag{2.13}$$

and

$$-\int_\Omega u_t(t)\int_0^{+\infty} g'(s)\eta(t,s)\mathrm{d}s\mathrm{d}x \leq \delta_1\|u_t(t)\|_2^2+\frac{(\alpha^2C_\alpha+k_1)C_*^2}{2\delta_1a_0}(k\diamond u)(t), \tag{2.14}$$

for $\delta_1 > 0$. We arrive to estimate

$$\int\limits_\Omega\left(\int\limits_0^{+\infty} g(s)\mathrm{A}\nabla\eta(s)\mathrm{d}s\right)\left(\int\limits_0^{+\infty} g(s)\nabla\eta(s)\mathrm{d}s\right)\mathrm{d}x \leq \frac{\sqrt{a_1n}C_\alpha}{a_0}(k\diamond u)(t). \tag{2.15}$$

Inserting (2.12)-(2.15) into (2.11), we obtain (2.10). ■

Lemma 2.1.4. *For N large enough, the modified Lyapunov functional* (2.1) *is equivalent to the energy functional* (1.13)*, and the inequality*

$$\kappa_1 E(t) \leq \mathcal{L}(t) \leq \kappa_2 E(t), \tag{2.16}$$

holds for two positive constants κ_1 and κ_2.

Proof. We can estimate each terms of (2.1). Indeed, we have

$$|\mathcal{L}(t) - NE(t)| \leq N_1|\varphi(t)| + N_2|\psi(t)|. \tag{2.17}$$

Using the inequalities of Cauchy-Schwarz and Young, from (1.6) and (1.7), we obtain

$$\int_{\Gamma_1} z(t)u(t)\mathrm{d}\Gamma \leq \frac{\|m\|_\infty}{2m_0^2}\|m^{1/2}z(t)\|_{2,\Gamma_1}^2 + \frac{\overline{C}_*^2}{2a_0}a(u(t),u(t)),$$

$$\int_{\Gamma_1} h(x)z_t(t)z(t)\mathrm{d}\Gamma \leq \frac{\|h\|_\infty}{2}\|h^{1/2}z_t(t)\|_{2,\Gamma_1}^2 + \frac{\|m\|_\infty}{2m_0^2}\|m^{1/2}z(t)\|_{2,\Gamma_1}^2,$$

and

$$\frac{1}{2}\|f^{1/2}z(t)\|_{2,\Gamma_1}^2 \leq \frac{\|f\|_\infty}{2m_0}\|m^{1/2}z(t)\|_{2,\Gamma_1}^2.$$

Then,

$$\begin{aligned} |\varphi(t)| \quad \leq \quad & \frac{1}{2}\|u_t(t)\|_2^2 + \frac{C_*^2 + \overline{C}_*^2}{2a_0}a(u(t),u(t)) + \frac{\|h\|_\infty}{2}\|h^{1/2}z_t(t)\|_{2,\Gamma_1}^2 \\ & + \frac{1}{2m_0}\left(\frac{2\|m\|_\infty}{m_0} + \|f\|_\infty\right)\|m^{1/2}z(t)\|_{2,\Gamma_1}^2. \end{aligned}$$

Also,

$$|\psi(t)| \leq \frac{1}{2}\|u_t(t)\|_2^2 + \frac{C_*^2(1-\ell)}{2a_0}(g \diamond u)(t).$$

From (2.17), we have

$$|\mathcal{L}(t) - NE(t)| \leq CE(t),$$

where $C \geq 0$. By choosing N large enough, we conclude (2.16). ■

Lemma 2.1.5. *For each $i = 1, 2$; N_i large enough while N so large that the functional $\mathcal{L}$ defined by* (2.1) *satisfies, for all $t \geq 0$*

$$\begin{aligned} \mathcal{L}'(t) \quad \leq \quad & -\|u_t(t)\|_2^2 - (\varrho + 1)(1-\ell)a(u(t),u(t)) + \frac{1}{4}(g \diamond u)(t) \\ & -\|m^{1/2}z(t)\|_{2,\Gamma_1}^2 - \|h^{1/2}z_t(t)\|_{2,\Gamma_1}^2, \end{aligned} \tag{2.18}$$

where $\varrho = \dfrac{2a_1 n}{a_0^2} + 1$ is a positive constant.

Proof. By Combining (1.14), (2.1), (2.5), and (2.10), taking $\delta = \ell a_0/(a_1 N_2)$, we obtain

$$\begin{aligned} \mathcal{L}'(t) \quad \leq \quad & -\left((1-\ell)N_2 - N_1 - \frac{\ell a_0}{a_1}\right)\|u_t(t)\|_2^2 - \|m^{1/2}z(t)\|_{2,\Gamma_1}^2 + \frac{\alpha N}{2}(g \diamond u)(t) \\ & - (\ell N_1 - \ell)\, a(u(t),u(t)) - \left(\frac{N}{2} - C_\alpha\left(C_1 N_1 + C_4 N_2\right)\right)(k \diamond u)(t) \\ & - \left(\frac{\|f\|_\infty}{h_0^2}N - C_2 N_1 - C_3 N_2\right)\|h^{1/2}z_t(t)\|_{2,\Gamma_1}^2, \end{aligned}$$

We choose N_1 large enough, so

$$\ell N_1 - \ell > (\varrho + 1)(1 - \ell),$$

while N_2 large enough such that

$$(1-\ell)N_2 - N_1 - \frac{\ell a_0}{a_1} > 1.$$

Let us choose N so large, that's to say

$$\frac{\|f\|_\infty}{h_0^2} N - C_2 N_1 - C_3 N_2 > 1.$$

From Remark 2.1.1, there is $0 < \alpha_0 < 1$ such that if $\alpha < \alpha_0$, hence

$$\alpha C_\alpha < \frac{1}{C_1 N_1 + C_4 N_2} \qquad \text{and} \qquad \alpha = \frac{1}{2N} < \alpha_0,$$

which means

$$\frac{N}{2} - C_\alpha (C_1 N_1 + C_4 N_2) > 0.$$

So, we conclude (4.21). ■

Lemma 2.1.6. *There exists a positive constant* M_0 *such that*

$$\int_t^{+\infty} g(s) a(u(t) - u(t-s), u(t) - u(t-s)) \mathrm{d}s \le M_0 h_0(t), \tag{2.19}$$

where $h_0(t) = \int_t^{+\infty} g(t+s) \left(1 + \|\nabla u_0(s)\|_2^2\right) \mathrm{d}s$.

Proof. As in [2, 50]. Using (1.3) and (1.13), we obtain, for all $t \ge 0$

$$\begin{aligned}
& \int_t^{+\infty} g(s) a(u(t) - u(t-s), u(t) - u(t-s)) \mathrm{d}s \\
\le\; & 2C \|\nabla u(t)\|_2^2 \int_t^{+\infty} g(s) \mathrm{d}s + 2C \int_t^{+\infty} g(s) \|\nabla u(t-s)\|_2^2 \mathrm{d}s \\
\le\; & 2C \sup_{s \ge 0} \|\nabla u(s)\|_2^2 \int_0^{+\infty} g(t+s) \mathrm{d}s + 2C \int_0^{+\infty} g(t+s) \|\nabla u(-s)\|_2^2 \mathrm{d}s \\
\le\; & \frac{4C}{a_0 \ell} E(0) \int_0^{+\infty} g(t+s) \mathrm{d}s + 2C \int_0^{+\infty} g(t+s) \|\nabla u_0(s)\|_2^2 \mathrm{d}s,
\end{aligned}$$

which gives (2.19) where $M_0 = C \max\left\{ \frac{4E(0)}{a_0 \ell}, 2 \right\}$. ■

Remark 2.1.7 ([50]). The function h_0 is of class $\mathcal{C}^1$ on $\mathbb{R}_+$. Indeed, since $\eta_0 \in \mathrm{L}_g^2(\mathbb{R}_+; \mathrm{H}_{\Gamma_0}^1(\Omega))$, so, for any $t \ge 0$

$$\begin{aligned}
h_0(t) &= \int_0^{+\infty} g(t+s) \left(1 + \|\nabla u_0(s)\|_2^2\right) \mathrm{d}s \\
&\le (1+\ell) + \int_0^{+\infty} g(t+s) \|\nabla u_0(0) - \nabla \eta_0(s)\|_2^2 \mathrm{d}s \\
&\le (1-\ell) \left(1 + 2\|\nabla u_0(0)\|_2^2\right) + \frac{2}{a_0^2} \int_0^{+\infty} g(s) a(\eta_0(s), \eta_0(s)) \mathrm{d}s = M_1,
\end{aligned}$$

and, using the left assumption in (1.4)

$$\begin{aligned}|h_0'(t)| &= -\int_0^{+\infty} g'(t+s)\left(1+\|\nabla u_0(s)\|_2^2\right)\mathrm{d}s\\ &\le \beta_0\int_0^{+\infty} g(t+s)\left(1+\|\nabla u_0(s)\|_2^2\right)\mathrm{d}s=\beta_0 h_0(t)\le \beta_0 M_1.\end{aligned}$$

Next, we use the functional

$$\Phi(t)=\int_0^t p(t-s)a(u(s),u(s))\mathrm{d}s,$$

where $p(t)=\int_t^{+\infty} g(s)\mathrm{d}s$.

Lemma 2.1.8. *The functional* Φ *satisfies the estimate*

$$\Phi'(t)\le -\frac{1}{2}(g\diamond u)(t)+\rho(1-\ell)a(u(t),u(t))+\frac{M_0}{2}h_0(t). \tag{2.20}$$

Proof. The proof of (2.20) is based on similar proceedings used in the one of Lemma 3.4 in [8] with suitable changes. In fact, we have

$$p'(t)=-g(t),$$

and

$$\begin{aligned}\int_0^t g(t-s)\mathrm{d}s &= \int_0^t g(s)\mathrm{d}s\\ &= \int_0^{+\infty} g(s)\mathrm{d}s-\int_t^{+\infty} g(s)\mathrm{d}s=p(0)-p(t).\end{aligned}$$

Now, differentiating Φ, we get

$$\begin{aligned}\Phi'(t) &= p(0)a(u(t),u(t))-\int_0^t g(t-s)a(u(s),u(s))\mathrm{d}s\\ &= p(0)a(u(t),u(t))-\int_0^t g(s)a(\eta(t,s)-u(t),\eta(t,s)-u(t))\mathrm{d}s\\ &= -\int_0^t g(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s+2\int_0^t g(s)a(\eta(t,s),u(t))\mathrm{d}s\\ &\quad+\left(p(0)-\int_0^t g(s)\mathrm{d}s\right)a(u(t),u(t))\\ &= p(t)a(u(t),u(t))+\int_t^{+\infty} g(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s\\ &\quad-(g\diamond u)(t)+2\int_0^t g(s)a(\eta(t,s),u(t))\mathrm{d}s.\end{aligned}$$

Using Young's inequality for some $\delta>0$, the last term can be estimated as follows

$$\begin{aligned}\int_0^t g(t-s)a(\eta(t,s),u(t))\mathrm{d}s &= \int_0^t g(s)\int_\Omega \mathrm{A}\left(\nabla u(t)-\nabla u(t-s)\right).\nabla u(t)\mathrm{d}x\mathrm{d}s\\ &\le n\delta\int_0^t g(s)\mathrm{d}s\|\nabla u(t)\|_2^2+\frac{a_1}{4\delta}\int_0^t g(s)\|\nabla\eta(t,s)\|_2^2\mathrm{d}s\\ &\le \delta\frac{n\int_0^t g(s)\mathrm{d}s}{a_0^2}a(u(t),u(t))+\frac{a_1}{4a_0^2\delta}(g\diamond u)(t)\\ &\quad-\frac{a_1}{4a_0^2\delta}\int_t^{+\infty} g(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s.\end{aligned}$$

By choosing $\delta = \dfrac{a_1(1-\ell)}{a_0^2 \int_0^t g(s)\mathrm{d}s}$. Then, as $p(t) \le p(0) = (1-\ell)$ and $\int_0^t g(s)\mathrm{d}s \le (1-\ell)$, from (2.19), we conclude (2.20). ■

2.2 Stability result

Here, we shall use a very wider class of relaxation functions and minimal conditions on this context, namely

(**H.4**) There exists a function $\mathcal{H} : \mathbb{R}_+ \to \mathbb{R}_+$ which is strictly increasing and strictly convex $\mathcal{C}^2$ function on $(0, r]$, for a positive constant $r \le g(0)$, with $\mathcal{H}(0) = \mathcal{H}'(0) = 0$ such that

$$g'(t) \le -\xi(t)\mathcal{H}(g(t)), \quad \forall t \ge 0, \tag{2.21}$$

where $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing differentiable function.

Following this assumption, we will formulate some corollaries and provide some remarks

Remark 2.2.1. Let $\mathcal{H}$ be a strictly convex on $(0, r]$ with $\mathcal{H}(0) = 0$, then

$$\mathcal{H}(\gamma x) \le \gamma \mathcal{H}(x), \quad \forall \gamma \in [0,1], \quad \forall x \in (0, r]. \tag{2.22}$$

Remark 2.2.2 ([71]). If (**H.4**) holds. Then $\overline{\mathcal{H}}$ is an extension of $\mathcal{H}$, which is a strictly increasing and a strictly convex $\mathcal{C}^2$ function on $\mathbb{R}_+$. For example, setting $\mathcal{H}(r) = \alpha$, $\mathcal{H}'(r) = \beta$, and $\mathcal{H}''(r) = \gamma$, thus $\overline{\mathcal{H}}$ defined by, for all $t > r$,

$$\overline{\mathcal{H}}(t) = \frac{\gamma}{2}t^2 + (\beta - \gamma r)t + \left(\alpha + \frac{\gamma}{2}r^2 - \beta r\right). \tag{2.23}$$

Thanks to the argument given in [15, pp. 61–64] to present the following corollary

Corollary 2.2.3. *Let $\overline{\mathcal{H}}^*$ be the convex conjugate of $\overline{\mathcal{H}}$ defined by*

$$\overline{\mathcal{H}}^*(s) = \sup_{t \in \mathbb{R}_+} \{st - \overline{\mathcal{H}}(t)\}.$$

Then, we have

$$\overline{\mathcal{H}}^*(s) = s(\overline{\mathcal{H}}')^{-1}(s) - \overline{\mathcal{H}}\left((\overline{\mathcal{H}}')^{-1}(s)\right) \le s(\overline{\mathcal{H}}')^{-1}(s), \quad \forall s > 0 \tag{2.24}$$

Remark 2.2.4. Here, the generalized Young inequality holds, i.e.,

$$AB \le \overline{\mathcal{H}}^*(A) + \overline{\mathcal{H}}(B). \tag{2.25}$$

At this Position, we introduce the class D of functions $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying, for some constants $\alpha_1, \alpha_2 > 0$

$$\vartheta \in \mathcal{C}^1(\mathbb{R}_+), \quad \vartheta \le 1, \quad \vartheta' \le 0, \tag{2.26}$$

and

$$\alpha_2 \mathcal{H}\left(M_0 h_1(t) h_0(t)\right) \le \alpha_1 \left(\mathcal{H}_2\left(\frac{\mathcal{H}_3(t)}{\vartheta(t)}\right) - \frac{\mathcal{H}_2\left(\mathcal{H}_3(t)\right)}{\vartheta(t)}\right), \tag{2.27}$$

at which

$$\mathcal{H}_1(t) = \int_t^r \frac{\mathrm{d}s}{s\mathcal{H}'(s)}, \quad \mathcal{H}_2(t) = t\mathcal{H}'(\varepsilon_0 t), \quad \text{and} \quad \mathcal{H}_3(t) = \mathcal{H}_1^{-1}\left(\alpha_1 \int_0^t \xi(s)\mathrm{d}s\right), \tag{2.28}$$

where ε_0 is positive constant to be fixed later. The functions h_0 and h_1 are defined in (2.19) and (2.31)(below), respectively.

Remark 2.2.5.

1. $\mathcal{H}_1$ is strictly decreasing and convex on $(0, r]$, with $\lim\limits_{t\to 0} \mathcal{H}_1(t) = +\infty$.

2. $\mathcal{H}_2$ is convex increasing on $\mathbb{R}_+$ and defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$.

3. $\mathcal{H}_3$ is of class $\mathcal{C}_1$ on $\mathbb{R}_+$.

4. The set D contains at least the function $\vartheta(s) = \varepsilon \mathcal{H}_3(s)$ for any $0 < \varepsilon \leq 1$ small enough, then it is not empty. According to $\mathcal{H}_1$ and $\mathcal{H}_3$, (2.26) is satisfied. On the other hand, we have $h_1(s)h_0(s)$ is nonincreasing, $0 < \mathcal{H}_3 \leq 1$, and $\mathcal{H}'$ is increasing, then (2.27) is satisfied if

$$\alpha_2 \mathcal{H}\left(M_0 h_1(t) h_0(t)\right) \leq \frac{\alpha_1}{\varepsilon}\left(\mathcal{H}'\left(\frac{1}{\varepsilon}\right) - \mathcal{H}'(1)\right),$$

which holds, for $0 < \varepsilon \leq 1$ small enough, since $\lim\limits_{t\to+\infty} \mathcal{H}'(t) = +\infty$.

The following theorem gives us the stability result

Theorem 2.2.6. *Suppose that **(H.1)**-**(H.4)** hold. Then, there exists a constant* $C > 0$ *such that*

$$E(t) \leq C \frac{\mathcal{H}_3(t)}{\vartheta(t) h_1(t)}, \tag{2.29}$$

for all $t \geq 0$.

Proof. We start by using (2.18) and (2.20) to obtain

$$\mathcal{L}_1'(t) \leq -\beta_1 E(t) + \frac{M_0}{2} h_0(t),$$

where β_1 is positive constant and $\mathcal{L}_1(t) = \mathcal{L}(t) + \Phi(t)$. Therefore, for all $t \geq 0$

$$\beta_1 \int_0^t E(s) \mathrm{d}s \leq \mathcal{L}_1(0) + \frac{M_0}{2} \int_0^t h_0(s) \mathrm{d}s,$$

which means

$$\int_0^t E(s) \mathrm{d}s \leq \beta_2 \left(1 + \int_0^t h_0(s) \mathrm{d}s\right), \tag{2.30}$$

where $\beta_2 = \max\left\{\frac{\mathcal{L}_1(0)}{\beta_1}, \frac{M_0}{2\beta_1}\right\}$. From (2.30), we have, for a constant $C > 0$

$$\begin{aligned}
\int_0^t a(\eta(t,s), \eta(t,s)) \mathrm{d}s &\leq 2C \int_0^t \left(||\nabla u(t)||_2^2 + ||\nabla u(t-s)||_2^2\right) \mathrm{d}s \\
&\leq \frac{8C}{a_0 \ell} \int_0^t E(t) \mathrm{d}s \leq \frac{8\beta_2 C}{a_0 \ell}\left(1 + \int_0^t h_0(s) \mathrm{d}s\right).
\end{aligned}$$

Then, for $0 < c_2 < \min\left\{1, \frac{a_0 \ell}{8\beta_2 C}, \frac{r}{M_1}\right\}$, we obtain, for all $t \geq 0$

$$h_1(t) = \frac{c_2}{1 + \int_0^t h_0(s) \mathrm{d}s} < 1 \quad \text{and} \quad h_1(t) \int_0^t a(\eta(t,s), \eta(t,s)) \mathrm{d}s < 1. \tag{2.31}$$

Now, we define the function, for a constant $\beta_3 > 0$

$$\lambda(t) = -\int_0^t g'(s) a(\eta(t,s), \eta(t,s)) \mathrm{d}s \leq -\beta_3 E'(t). \tag{2.32}$$

From the hypothesis (**H.4**), the using of (2.22), (2.31), and Jensen's inequality (A.2) leads to

$$\begin{aligned}\lambda(t) &= \frac{1}{h_1(t)}\int_0^t h_1(t)(-g'(s))a(\eta(t,s),\eta(t,s))\mathrm{d}s\\ &\geq \frac{1}{h_1(t)}\int_0^t h_1(t)\xi(s)\mathcal{H}(g(s))a(\eta(t,s),\eta(t,s))\mathrm{d}s\\ &\geq \frac{\xi(t)}{h_1(t)}\mathcal{H}\left(\int_0^t h_1(t)g(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s\right)\\ &\geq \frac{\xi(t)}{h_1(t)}\overline{\mathcal{H}}\left(h_1(t)\int_0^t g(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s\right). \end{aligned}\tag{2.33}$$

This implies that

$$\int_0^t g(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s \leq \frac{1}{h_1(t)}\overline{\mathcal{H}}^{-1}\left(\frac{h_1(t)\lambda(t)}{\xi(t)}\right), \tag{2.34}$$

which (2.18) gives

$$\mathcal{L}'(t) \leq -\beta_1 E(t) + \frac{\beta_4}{h_1(t)}\overline{\mathcal{H}}^{-1}\left(\frac{h_1(t)\lambda(t)}{\xi(t)}\right) + \beta_4 h_0(t), \tag{2.35}$$

where $\beta_4 = \max\{\frac{1}{4}, M_0\}$.

Now, for $\varepsilon_0 > 0$, we define the functional

$$\mathcal{F}(t) = \overline{\mathcal{H}}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right)\mathcal{L}(t).$$

From (2.35) and the fact that $E' < 0$, $\overline{\mathcal{H}}' > 0$, $\overline{\mathcal{H}}'' > 0$, and $h_1' < 0$, we conclude that $\mathcal{F} \sim E$ and

$$\begin{aligned}\mathcal{F}'(t) \leq\; & -\beta_1 E(t)\overline{\mathcal{H}}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right) + \frac{\beta_4}{h_1(t)}\overline{\mathcal{H}}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right)\overline{\mathcal{H}}^{-1}\left(\frac{h_1(t)\lambda(t)}{\xi(t)}\right)\\ & +\beta_4 h_0(t)\overline{\mathcal{H}}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right). \end{aligned}\tag{2.36}$$

In view of (2.24) and (2.25) with

$$A = \overline{\mathcal{H}}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right) \quad \text{and} \quad B = \overline{\mathcal{H}}^{-1}\left(\frac{h_1(t)\lambda(t)}{\xi(t)}\right)$$

thus, (2.36) gives

$$\mathcal{F}'(t) \leq -(\beta_1 E(0) - \beta_4\varepsilon_0)\frac{E(t)}{E(0)}\overline{\mathcal{H}}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right) + \beta_4\frac{\lambda(t)}{\xi(t)} + \beta_4 h_0(t)\overline{\mathcal{H}}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right). \tag{2.37}$$

So, multiplying (2.37) by $\xi(t)$. From (2.32) and the fact that

$$\varepsilon_0\frac{h_1(t)E(t)}{E(0)} < r,\ \overline{\mathcal{H}}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right) = \mathcal{H}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right),$$

we find

$$\begin{aligned}\xi(t)\mathcal{F}'(t) \quad &\leq \quad -(\beta_1 E(0)-\beta_4\varepsilon_0)\xi(t)\frac{E(t)}{E(0)}\mathcal{H}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right)-\beta_4\beta_3E'(t)\\ &\qquad +\beta_4\frac{\xi(t)}{h_1(t)}h_1(t)h_0(t)\overline{\mathcal{H}}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right).\end{aligned}$$

As above, with $A=\overline{\mathcal{H}}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right)$ and $B=h_1(t)h_0(t)$, in the fact that $B<r$ and $\overline{\mathcal{H}}(B)=\mathcal{H}(B)$, we obtain

$$\begin{aligned}\xi(t)\mathcal{F}'(t) \quad &\leq \quad -\left(\beta_1 E(0)-2\beta_4\varepsilon_0\right)\xi(t)\frac{E(t)}{E(0)}\mathcal{H}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right)-\beta_4\beta_3E'(t)\\ &\qquad +\beta_4\frac{\xi(t)}{h_1(t)}\mathcal{H}\left(h_1(t)h_0(t)\right).\end{aligned}$$

Let us choose ε_0 small enough such that $\beta_5=\beta_1E(0)-2\beta_4\varepsilon_0>0$. Then, we obtain, for all $t\geq 0$

$$\mathcal{F}_1'(t)\leq -\beta_5\frac{\xi(t)}{h_1(t)}\mathcal{H}_2\left(\frac{h_1(t)E(t)}{E(0)}\right)+\beta_4\frac{\xi(t)}{h_1(t)}\mathcal{H}\left(h_1(t)h_0(t)\right), \tag{2.38}$$

where $\mathcal{F}_1=\xi\mathcal{F}+\beta_4\beta_3E\sim E$ and satisfies, for some constants $\kappa_3,\kappa_4>0$

$$\kappa_3\mathcal{F}_1(t)\leq E(t)\leq \kappa_4\mathcal{F}_1(t). \tag{2.39}$$

Since $\mathcal{H}_2'(t)=\mathcal{H}'(\varepsilon_0t)+\varepsilon_0t\mathcal{H}''(\varepsilon_0t)$. Using the strict convexity of $\mathcal{H}$ on $(0,r]$, we find that $\mathcal{H}_2,\mathcal{H}_2'>0$ on $[0,1)$. Hence, using (2.39) with $c_1=\dfrac{\kappa_3}{E(0)}>0$, we get

$$\mathcal{H}_2\left(\frac{h_1(t)E(t)}{E(0)}\right)\geq \mathcal{H}_2(c_1\mathcal{F}_1(t)h_1(t)). \tag{2.40}$$

Let $\mathcal{F}_2(t)=c_1\mathcal{F}_1(t)h_1(t)$. As $c_1h_1(t)$ is nonincreasing and from (2.40) for some constants $\alpha_1=c_1\beta_5>0$ and $\alpha_2=c_1>0$, we arrive at

$$\mathcal{F}_2'(t)\leq -\alpha_1\xi(t)\mathcal{H}_2\left(\mathcal{F}_2(t)\right)+\alpha_2\xi(t)\mathcal{H}\left(M_0h_1(t)h_0(t)\right). \tag{2.41}$$

Taking into account (2.39), thus, there exists a constant $c_2>0$ such that $\mathcal{F}_2(t)\geq c_2h_1(t)E(t)$. Let $\vartheta(t)$ satisfying (2.26) and (2.27). If

$$c_2h_1(t)E(t)\leq 2\frac{\mathcal{H}_3(t)}{\vartheta(t)},$$

then

$$E(t)\leq \frac{2}{c_2}\frac{\mathcal{H}_3(t)}{h_1(t)\vartheta(t)}. \tag{2.42}$$

Else if

$$c_2h_1(t)E(t)> 2\frac{\mathcal{H}_3(t)}{\vartheta(t)}.$$

According to $(h_1E)'(t)\leq 0$, then we have, for $0\leq s\leq t$

$$\mathcal{F}_2(s)\geq c_2h_1(s)E(s)>2\frac{\mathcal{H}_3(t)}{\vartheta(t)}. \tag{2.43}$$

For $0<\varepsilon_1<1$, using (2.22), $0<\vartheta\le 1$, and the fact that $\mathcal{H}_2$ is convex, we obtain

$$\begin{aligned}
&\mathcal{H}_2\left(\varepsilon_1\vartheta(s)\mathcal{F}_2(s)-\varepsilon_1\mathcal{H}_3(s)\right)=\mathcal{H}_2\left(\varepsilon_1\vartheta(s)\mathcal{F}_2(s)-\frac{\varepsilon_1\vartheta(s)\mathcal{H}_3(s)}{\vartheta(s)}\right)\\
\le\;& \varepsilon_1\vartheta(s)\mathcal{H}_2\left(\mathcal{F}_2(s)-\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right)\\
\le\;& \varepsilon_1\vartheta(s)\left(\mathcal{F}_2(s)-\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right)\mathcal{H}'\left(\varepsilon_0\mathcal{F}_2(s)-\varepsilon_0\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right)\\
\le\;& \varepsilon_1\vartheta(s)\mathcal{F}_2(s)\mathcal{H}'\left(\varepsilon_0\mathcal{F}_2(s)-\varepsilon_0\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right)-\varepsilon_1\vartheta(s)\frac{\mathcal{H}_3(s)}{\vartheta(s)}\mathcal{H}'\left(\varepsilon_0\mathcal{F}_2(s)-\varepsilon_0\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right).
\end{aligned}$$

Therefore, using (2.43) and the fact that $\mathcal{H}'$ is increasing, we have, for $0\le s\le t$,

$$\mathcal{H}'\left(\varepsilon_0\mathcal{F}_2(s)-\varepsilon_0\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right)<\mathcal{H}'\left(\varepsilon_0\mathcal{F}_2(s)\right)$$

and

$$\mathcal{H}'\left(\varepsilon_0\mathcal{F}_2(s)-\varepsilon_0\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right)>\mathcal{H}'\left(\varepsilon_0\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right).$$

These imply that

$$\begin{aligned}
&\mathcal{H}_2\left(\varepsilon_1\vartheta(s)\mathcal{F}_2(s)-\varepsilon_1\mathcal{H}_3(s)\right)\\
\le\;& \varepsilon_1\vartheta(s)\mathcal{F}_2(s)\mathcal{H}'\left(\varepsilon_0\mathcal{F}_2(s)\right)-\varepsilon_1\vartheta(s)\frac{\mathcal{H}_3(s)}{\vartheta(s)}\mathcal{H}'\left(\varepsilon_0\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right).
\end{aligned}$$

Now, we let

$$\mathcal{F}_3(s)=\varepsilon_1\vartheta(s)\mathcal{F}_2(s)-\varepsilon_1\mathcal{H}_3(s). \tag{2.44}$$

By choosing ε_1 small enough so that $\mathcal{F}_3(0)\le 1$. Then, using (2.44), we find, for $0\le s\le t$,

$$\mathcal{H}_2\left(\mathcal{F}_3(s)\right)\le\varepsilon_1\vartheta(s)\mathcal{H}_2\left(\mathcal{F}_2(s)\right)-\varepsilon_1\vartheta(s)\mathcal{H}_2\left(\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right). \tag{2.45}$$

As $\vartheta'\le 0$, we have

$$\begin{aligned}
\mathcal{F}_3'(s)&=\varepsilon_1\vartheta'(s)\mathcal{F}_2(s)+\varepsilon_1\vartheta(s)\mathcal{F}_2'(s)-\varepsilon_1\mathcal{H}_3'(s)\\
&\le\varepsilon_1\vartheta(s)\mathcal{F}_2'(s)-\varepsilon_1\mathcal{H}_3'(s).
\end{aligned} \tag{2.46}$$

From (2.41), (2.46), and (2.45), we get, for $0\le s\le t$,

$$\begin{aligned}
\mathcal{F}_3'(s)\le\;& -\alpha_1\xi(s)\mathcal{H}_2(\mathcal{F}_3(s))-\varepsilon_1\alpha_1\vartheta(s)\xi(s)\mathcal{H}_2\left(\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right)\\
&+\varepsilon_1\alpha_2\vartheta(s)\xi(s)\mathcal{H}\left(M_0h_1(s)h_0(s)\right)-\varepsilon_1\mathcal{H}_3'(s).
\end{aligned} \tag{2.47}$$

By the definition of $\mathcal{H}_1$ and $\mathcal{H}_3$, we have

$$\mathcal{H}_1(\mathcal{H}_3(s))=\alpha_1\int_0^s\xi(\varrho)d\varrho,$$

thus, for $0\le s\le t$,

$$\mathcal{H}_3'(s)=-\alpha_1\xi(s)\mathcal{H}_2(\mathcal{H}_3(s)). \tag{2.48}$$

Using (2.48), we obtain

$$\begin{aligned}
&\varepsilon_1\alpha_2\vartheta(s)\xi(s)\mathcal{H}\left(M_0h_1(s)h_0(s)\right)-\varepsilon_1\mathcal{H}_3'(s)-\varepsilon_1\alpha_1\vartheta(s)\xi(s)\mathcal{H}_2\left(\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right)\\
=\;&\varepsilon_1\alpha_2\vartheta(s)\xi(s)\mathcal{H}\left(M_0h_1(s)h_0(s)\right)-\varepsilon_1\alpha_1\vartheta(s)\xi(s)\mathcal{H}_2\left(\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right)+\varepsilon_1\alpha_1\xi(s)\mathcal{H}_2(\mathcal{H}_3(s))\\
=\;&\varepsilon_1\vartheta(s)\xi(s)\left(\alpha_2\mathcal{H}\left(M_0h_1(s)h_0(s)\right)-\alpha_1\mathcal{H}_2\left(\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right)+\alpha_1\frac{\mathcal{H}_2(\mathcal{H}_3(s))}{\vartheta(s)}\right).
\end{aligned}$$

So, by (2.27), we have

$$\varepsilon_1\vartheta(s)\xi(s)\left(\alpha_2\mathcal{H}\left(M_0h_1(s)h_0(s)\right)-\alpha_1\mathcal{H}_2\left(\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right)+\alpha_1\frac{\mathcal{H}_2(\mathcal{H}_3(s))}{\vartheta(s)}\right)\le 0.$$

Consequently, (2.47) leads to

$$\mathcal{F}_3'(s)\le-\alpha_1\xi(s)\mathcal{H}_2(\mathcal{F}_3(s)).$$

Then, by the definition of $\mathcal{H}_1$ and $\mathcal{H}_2$, we get

$$[\mathcal{H}_1(\mathcal{F}_3(s))]'\ge\alpha_1\xi(s). \tag{2.49}$$

By integrating (2.49) over $(0,t)$, we obtain

$$\mathcal{H}_1(\mathcal{F}_3(t))\ge\alpha_1\int_0^t\xi(s)\mathrm{d}s+\mathcal{H}_1(\mathcal{F}_3(0)),$$

Note that $\mathcal{H}_1$ is nonincreasing, $\mathcal{F}_3(0)\le 1$, and $\mathcal{H}_1(1)=0$. So,

$$\mathcal{F}_3(t)\le\mathcal{H}_1^{-1}\left(\alpha_1\int_0^t\xi(s)\mathrm{d}s\right)=\mathcal{H}_3(t).$$

Hence, by the definition of $\mathcal{F}_2$ and $\mathcal{F}_3$, we get

$$\mathcal{F}_2(t)\le\frac{1+\varepsilon_1}{\varepsilon_1}\frac{\mathcal{H}_3(t)}{\vartheta(t)}\quad\text{and}\quad\mathcal{F}_1(t)\le\frac{1+\varepsilon_1}{\varepsilon_1c_1}\frac{\mathcal{H}_3(t)}{\vartheta(t)h_1(t)}.$$

Then, as $\mathcal{F}_1\sim E$, there exists a constant $c_3>0$ such that $E(t)\le c_3\mathcal{F}_1$. Consequently, we deduce that

$$E(t)\le\frac{c_3(1+\varepsilon_1)}{\varepsilon_1c_1}\frac{\mathcal{H}_3(t)}{\vartheta(t)h_1(t)}.$$

From this estimate and (2.42), the estimate (2.29) is established with $C=\max\left\{\frac{2}{c_2},\frac{c_3(1+\varepsilon_1)}{\varepsilon_1c_1}\right\}$. ■

Now, we give an example to illustrate our result

Example 2.2.7 ([50]). Let $g(t)=b(1+t)^{-q}$, where $q>1$ and $0<b<q-1$ so that (**H.2**) is satisfied, for any $\beta_0\ge q$. In this case $\xi(t)=qb^{\frac{-1}{q}}$ and $\mathcal{H}(t)=t^{\frac{q+1}{q}}$. Then, there exist three positive constants $b_i; i=1,\dots,3$ depending only on ε_0, α_1, q, and b such that $\mathcal{H}_1(t)=b_1(t^{\frac{-1}{q}}-1)$, $\mathcal{H}_2(t)=b_2t^{\frac{q+1}{q}}$, and $\mathcal{H}_3(t)=(b_3t+1)^{-q}$.

If

$$m_0(1+t)^k\le 1+\|\nabla u_0(t)\|_2^2\le m_1(1+t)^k \tag{2.50}$$

where $0<k<q-1$ and $m_0,m_1>0$, then, for some positive constants $b_i; i=4,\dots,7$ depending only on q, b, m_0, m_1, and k, the estimates

$$b_4(1+t)^{1+k-q}\le h_0(t)\le b_5(1+t)^{1+k-q}, \tag{2.51}$$

$$\frac{c_2}{h_1(t)} \geq b_6 \begin{cases} 1+\ln(1+t), & q-k=2; \\ 2, & q-k>2; \\ (1+t)^{k+2-q}, & 1<q-k<2, \end{cases} \tag{2.52}$$

and

$$\frac{c_2}{h_1(t)} \leq b_7 \begin{cases} 1+\ln(1+t), & q-k=2; \\ 2, & q-k>2; \\ (1+t)^{k+2-q}, & 1<q-k<2. \end{cases} \tag{2.53}$$

Condition (2.27) is satisfied if

$$(1+t)^q h_0(t) h_1(t) \vartheta(t) \leq b_8 \left(1-\vartheta^{\frac{1}{q}(t)}\right)^{\frac{q}{q+1}}, \tag{2.54}$$

where $b_8 > 0$. Choosing

$$\vartheta(t) = \lambda \begin{cases} (1+t)^{-k-1}, & q-k \geq 2; \\ (1+t)^{1-q}, & 1<q-k<2, \end{cases} \tag{2.55}$$

so (2.26) is valid. Moreover, using (2.51) and (2.52), we see that (2.54) is satisfied if $0 < \lambda \leq 1$ is small enough in (2.55), and then (2.27) is satisfied. Therefore (2.29) and (2.53) imply that, for all $t \geq 0$

$$E(t) \leq C \begin{cases} (1+\ln(1+t))^{1+k-q}, & q-k=2; \\ (1+t)^{1+k-q}, & q-k>2 \text{ or } 1<q-k<2. \end{cases} \tag{2.56}$$

This estimate gives $\lim\limits_{t\to\infty} E(t) = 0$.

On the other hand, if $k = 0$ in (2.50), then (2.56) holds with $k = 0$.

Part II

A Variable Coefficient Thermo-viscoelastic-coupled System of Second Sound with Acoustic Boundary Conditions

Chapter 3

Global Existence and Uniqueness of Solution

In this chapter, we are concerned with global existence and uniqueness of solution of a variable coefficient thermo-viscoelastic-coupled system of second sound with acoustic boundary conditions. The proof is obtained by means of the Faedo-Galerkin techniques and the compactness method.

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3.1 Problem statement

We look into $u, \theta, q : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ and $z : \Gamma_1 \times \mathbb{R}_+ \to \mathbb{R}$ solutions of the following thermo-viscoelastic-coupled system of second sound

$$\begin{aligned} u_{tt} + \mathbf{A}u - \int_0^t g(t-s)\mathbf{A}u(s)\mathrm{d}s + \operatorname{div}(\sigma(x)\theta) &= 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+ && (3.1)\\ \rho(x)\theta_t + (\kappa \cdot \nabla)q + (\sigma(x).\nabla)u_t &= 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+ && (3.2)\\ \tau_0 q_t + q + (\kappa \cdot \nabla)\theta &= 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, && (3.3) \end{aligned}$$

the boundary conditions

$$\begin{aligned} u &= 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_+ && (3.4)\\ \theta &= 0 \quad \text{on} \quad \Gamma \times \mathbb{R}_+, && (3.5) \end{aligned}$$

and the acoustic boundary conditions

$$\begin{aligned} \frac{\partial u}{\partial \nu_{\mathrm{A}}} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu_{\mathrm{A}}}(s)\mathrm{d}s &= h(x)z_t, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+ && (3.6)\\ u_t + f(x)z_t + m(x)z &= 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+. && (3.7) \end{aligned}$$

Here, the function $u = u(x,t)$ describes the velocity, $\theta = \theta(x,t)$ is the temperature difference, $q = q(x,t)$ is the heat flux, and the normal displacement on part of the boundary represented by $z = z(x,t)$. The function ρ is assumed to be $\mathcal{C}(\overline{\Omega})$ and the function vector σ in $\mathcal{C}^1(\overline{\Omega}, \mathbb{R}^n)$. The coefficient $\kappa = (\kappa_1, \ldots, \kappa_n)$ is the vector of nonnegative components called the heat conductivity vector and the parameter τ_0 is a positive constant represents the relaxation time.

In addition, the initial conditions

$$\begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{for} \quad x \in \Omega\\ \theta(x,0) = \theta_0(x),\ q(x,0) = q_0(x) & \text{for} \quad x \in \Omega\\ z(x,0) = z_0(x) & \text{for} \quad x \in \Gamma_1, \end{cases} \tag{3.8}$$

where $u_0, u_1, \theta_0, q_0 : \Omega \to \mathbb{R}$ and $z_0 : \Gamma_1 \to \mathbb{R}$ are given functions.

3.2 Notations and variational formulation

In this section, we will derive a variational (or weak) formulation of the boundary value problem (3.1)-(3.8). We will discuss all fundamental theoretical results that provide a rigorous understanding of how to solve our problem.

Let us introduce the following notation

$$(g \diamond u)(t) = \int_0^t g(t-s)a(u(t)-u(s), u(t)-u(s))\mathrm{d}s.$$

Lemma 3.2.1. *We have*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(g \diamond u)(t) &= (g' \diamond u)(t) - 2\int_0^t g(t-s)a(u(s), u_t(t))\mathrm{d}s\\ &\quad + \frac{\mathrm{d}}{\mathrm{d}t}\left(a(u(t),u(t))\int_0^t g(s)\mathrm{d}s\right) - g(t)a(u(t),u(t)). \end{aligned} \tag{3.9}$$

Proof. Since

$$2a(u(t),u_t(t)) = \frac{\mathrm{d}}{\mathrm{d}t}a(u(t),u(t)).$$

Then

$$\begin{aligned}
&2\int_0^t g(t-s)a(u(s),u_t(t))\mathrm{d}s = 2\int_0^t g(t-s)\mathrm{A}\nabla u(s)\nabla u_t(t)\mathrm{d}s\\
= &-2\int_0^t g(t-s)\mathrm{A}(\nabla u(t)-\nabla u(s))\frac{\mathrm{d}}{\mathrm{d}t}(\nabla u(t)-\nabla u(s))\mathrm{d}s + 2\int_0^t g(t-s)a(u(t),u_t(t))\mathrm{d}s\\
= &-\int_0^t g(t-s)\left(\frac{\mathrm{d}}{\mathrm{d}t}a(u(t)-u(s),u(t)-u(s))\right)\mathrm{d}s + \int_0^t g(t-s)\left(\frac{\mathrm{d}}{\mathrm{d}t}a(u(t),u(t))\right)\mathrm{d}s\\
= &\frac{\mathrm{d}}{\mathrm{d}t}\left(a(u(t),u(t))\int_0^t g(s)\mathrm{d}s\right) + \int_0^t g'(t-s)a(u(t)-u(s),u(t)-u(s))\mathrm{d}s\\
&-g(t)a(u(t),u(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t g(t-s)a(u(t)-u(s),u(t)-u(s))\mathrm{d}s\right)\\
= &\frac{\mathrm{d}}{\mathrm{d}t}\left(a(u(t),u(t))\int_0^t g(s)\mathrm{d}s\right) + (g'\diamond u)(t) - g(t)a(u(t),u(t)) - \frac{\mathrm{d}}{\mathrm{d}t}(g\diamond u)(t),
\end{aligned}$$

which gives (3.9). ■

We are going to transform the boundary value problem (3.1)-(3.8) into an entirely different kind of problem that is amenable to existence and uniqueness theory.

Proposition 3.2.2. *Assume that* $(u,\theta,q,z) \in \mathrm{H}^1_{\Gamma_0}(\Omega)\cap\mathrm{H}^2(\Omega)\times\mathrm{H}^1_0(\Omega)\times\mathrm{H}^1(\Omega)\times\mathrm{L}^2(\Gamma_1)$ *solve the problem* (3.1)-(3.8). *Then, we have, for any* $\varepsilon > 0$

$$\begin{aligned}
0 \;=\; &\int_\Omega [u_{tt} + \operatorname{div}(\sigma(x)\theta)]\,w\mathrm{d}x + a(u,w) - \int_{\Gamma_1} h(x)z_t\gamma_0(w)\mathrm{d}\Gamma\\
&+\int_\Omega [\rho(x)\theta_t + (\kappa\cdot\nabla)q + (\sigma(x)\cdot\nabla)u_t]\,v\mathrm{d}x + \int_\Omega [\tau_0 q_t + q + (\kappa\cdot\nabla)\theta]\,\mu\mathrm{d}x\\
&+\int_{\Gamma_1} [\varepsilon z_{tt} + h(x)f(x)z_t + h(x)m(x)z + h(x)u_t]\,\xi\mathrm{d}\Gamma,
\end{aligned}$$

for all $w \in \mathrm{H}^1_{\Gamma_0}(\Omega)$, $v \in \mathrm{H}^1_0(\Omega)$, $\mu \in \mathrm{H}^1(\Omega)$, and $\xi \in \mathrm{L}^2(\Gamma_1)$.

Proof. Let $\varepsilon > 0$. We take an arbitrary $w \in \mathrm{H}^1_{\Gamma_0}(\Omega)$, $v \in \mathrm{H}^1_0(\Omega)$, $\mu \in \mathrm{H}^1(\Omega)$, and $\xi \in \mathrm{L}^2(\Gamma_1)$. Multiplying the equations (3.1) by w, (3.2) by v, (3.3) by μ, and (3.7) by ξ, respectively,

$$\begin{aligned}
u_{tt}w + \mathbf{A}uw - \int_0^t g(t-s)\mathbf{A}u(s)\mathrm{d}sw + \operatorname{div}(\sigma(x)\theta)w &= 0\\
\rho(x)\theta_t v + (\kappa\cdot\nabla)qv + (\sigma(x).\nabla)u_t v &= 0\\
\tau_0 q_t\mu + q\mu + (\kappa\cdot\nabla)\theta\mu &= 0\\
\varepsilon z_{tt}\xi + h(x)f(x)z_t\xi + h(x)m(x)z\xi + h(x)u_t\xi &= 0.
\end{aligned}$$

Integrating the first three equations over Ω and the laste over Γ_1, and adding the result. Every term is integrable since $\mathrm{A}\nabla u \in \mathrm{H}(\operatorname{div},\Omega)$, $\theta \in \mathrm{H}^1(\Omega)$, $q \in \mathrm{H}^1(\Omega)$, and $\xi \in \mathrm{L}^2(\Gamma_1)$, hence, $\mathbf{A}u \in \mathrm{L}^2(\Omega)$ and $w \in \mathrm{L}^2(\Omega)$ imply $\mathbf{A}uw \in \mathrm{L}^1(\Omega)$. In addition, $q, (\kappa\cdot\nabla)q, (\kappa\cdot\nabla)\theta \in \mathrm{L}^2(\Omega)$ and $v,\mu \in \mathrm{L}^2(\Omega)$ imply $q\mu, (\kappa\cdot\nabla)qv, (\kappa\cdot\nabla)\theta\mu \in \mathrm{L}^1(\Omega)$. Also, $z,\xi \in \mathrm{L}^2(\Gamma_1)$ imply $z\xi, \gamma_0(u_t)\xi \in \mathrm{L}^1(\Gamma_1)$.

Using Green's formula and the impenetrability boundary condition (3.6), according to which

$$\begin{aligned}
&\int_\Omega \left(\mathbf{A}u - \int_0^t g(t-s)\mathbf{A}u(s)\mathrm{d}s\right) w\mathrm{d}x \\
= \;& a(u,w) - \int_{\Gamma_0} \left(\frac{\partial u}{\partial \nu_{\mathrm{A}}} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu_{\mathrm{A}}}(s)\mathrm{d}s\right)\gamma_0(w)\mathrm{d}\Gamma \\
& - \int_0^t g(t-s)a(u(s),w)\mathrm{d}s - \int_{\Gamma_1} h(x)z_t\gamma_0(w)\mathrm{d}\Gamma, \qquad (3.10)
\end{aligned}$$

and we conclude since $w \in \mathrm{H}^1_{\Gamma_0}(\Omega)$ is equivalent to $\gamma_0(w) = 0$ on Γ_0.

In order to recovry the impenetrability boundary condition (3.6), taking $w \in \mathcal{D}(\overline{\Omega})$ in (3.10) and using Green's formula.

Similarly, $v \in \mathrm{H}^1_0(\Omega)$

$$\begin{aligned}
\int_\Omega (\kappa\cdot\nabla)qv\mathrm{d}x + \int_\Omega (\sigma(x)\cdot\nabla)u_t v\mathrm{d}x \;=\;& \int_\Gamma (\sigma(x)\cdot\nu)\gamma_0(u_t)\gamma_0(v)\mathrm{d}\Gamma - \int_\Omega u_t(\sigma(x)\cdot\nabla)v\mathrm{d}x \\
& + \int_\Gamma (\kappa\cdot\nu)\gamma_0(q)\gamma_0(v)\mathrm{d}\Gamma - \int_\Omega q(\kappa\cdot\nabla)v\mathrm{d}x,
\end{aligned}$$

is equivalent to $\gamma_0(v) = 0$ on Γ.

Also, we have

$$\int_\Omega (\kappa\cdot\nabla)\theta\mu\mathrm{d}x = \int_\Gamma (\kappa\cdot\nu)\gamma_0(\theta)\gamma_0(\mu)\mathrm{d}\Gamma - \int_\Omega \theta(\kappa\cdot\nabla)\mu\mathrm{d}x. \qquad (3.11)$$

It follows that $\mu \in \mathrm{H}^1(\Omega)$. ■

Actually, the variational formulation is not entirely complete, since we have not yet decided in which space to look for (u,θ). In fact, the reasonable way to impose the Dirichlet boundary condition is to require that $(u,\theta) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_0(\Omega)$.

Let us rewrite the variational formulation in a standard abstract form. We let (the functional space) $(u,\theta,q,z) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1(\Omega) \times \mathrm{L}^2(\Gamma_1)$. The variational formulation then reads

$$\begin{aligned}
0 \;=\;& (u_{tt},w) + a(u,w) - \int_0^t g(t-s)a(u(s),w)\mathrm{d}s + (\mathrm{div}(\sigma\theta),w) - \langle hz_t, w\rangle_{\Gamma_1} + (\rho\theta_t, v) \\
& + ((\kappa\cdot\nabla)q, v) + ((\sigma\cdot\nabla)u_t, v) + \tau_0\,(q_t,\mu) + (q,\mu) + ((\kappa\cdot\nabla)\theta,\mu) \\
& + \langle \varepsilon z_{tt},\xi\rangle_{\Gamma_1} + \langle hfz_t,\xi\rangle_{\Gamma_1} + \langle hmz,\xi\rangle_{\Gamma_1} + \langle hu_t,\xi\rangle_{\Gamma_1}\,, \qquad (3.12)
\end{aligned}$$

for all $(w,v,\mu,\xi) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1(\Omega) \times \mathrm{L}^2(\Gamma_1)$.

3.3 Existence and uniqueness

The global existence and uniqueness of solution of system (3.1)-(3.8) is established here.

Definition 3.3.1. A regular weak solution for the initial–boundary value problem (3.1)-(3.8) is a quadruple of real functions $\{u,\theta,q,z\}$ such that, for each fixed $T > 0$, it satisfies

$$\begin{aligned}
& u, u_t \in \mathrm{L}^\infty(0,T;\mathrm{H}^1_{\Gamma_0}(\Omega)), \quad u_{tt} \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)), \quad \mathrm{A}\nabla u \in \mathrm{H}(\mathrm{div},\Omega); \\
& \theta \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)), \quad \theta_t \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)); \\
& q \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)), \quad q_t \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)); \\
& z, z_t \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Gamma_1));
\end{aligned}$$

and the initial conditions (3.8).

The following theorem paves our result

Theorem 3.3.2. *Let* $(u_0, u_1, \theta_0, q_0, z_0) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{L}^2(\Omega) \times \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1(\Omega) \times \mathrm{L}^2(\Gamma_1)$ *be given. Suppose that* ***(H.1)-(H.3)*** *hold. Then, the problem (3.1)-(3.8) has a unique regular global weak solution*

$$\begin{aligned} &u \in \mathcal{C}(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega)) \cap \mathcal{C}^1(\mathbb{R}_+; \mathrm{L}^2(\Omega));\\ &\theta \in \mathcal{C}(\mathbb{R}_+; \mathrm{L}^2(\Omega)) \cap \mathcal{C}^1(\mathbb{R}_+; \mathrm{L}^2(\Omega));\\ &q \in \mathcal{C}(\mathbb{R}_+; \mathrm{L}^2(\Omega)) \cap \mathcal{C}^1(\mathbb{R}_+; \mathrm{L}^2(\Omega));\\ &z \in \mathcal{C}(\mathbb{R}_+; \mathrm{L}^2(\Gamma_1)). \end{aligned}$$

Moreover, if $u_0 \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_{\Gamma_0}(\Omega)$*,* $u_1 \in \mathrm{H}^1_{\Gamma_0}(\Omega)$*, and* $\theta_0 \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$*, then the solution* (u, θ) *satisfies*

$$\begin{aligned} &u \in \mathcal{C}^1(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega)) \cap \mathcal{C}^2(\mathbb{R}_+; \mathrm{L}^2(\Omega)), \quad \mathrm{A}\nabla u \in \mathrm{H}(\mathrm{div}, \Omega) \cap \mathrm{H}^1_{\Gamma_0}(\Omega);\\ &\theta \in \mathcal{C}(\mathbb{R}_+; \mathrm{H}^1_0(\Omega)) \cap \mathcal{C}^1(\mathbb{R}_+; \mathrm{L}^2(\Omega)). \end{aligned}$$

The proof of this theorem will be given in Section 3.4.

3.4 Proof of Theorem 3.3.2

We want to show that problem (3.1)-(3.8) has exactly one weak solution (u, θ, q, z), which depends continuously on the data in a suitable norm. We will focus on the Faedo-Galerkin approach and the compactness method. Let us describe the main strategy which divides into three steps

- Solution of the approximate problem by using the Faedo-Galerkin method.
- A priori estimates for this approximated solution are derived.
- Passage to limits by the compactness method, derive the regularity of solution and justify the initial condition.

For each $\varepsilon \in (0, 1)$ and $k \in \mathbb{N}$. Let $T_k > 0$ a real number and denote by $\{\alpha_{jk}, \beta_{jk}, \gamma_{jk}, \delta_{jk}\}$ a real functions defined on $[0, T_k]$, which define the approximated solution $(u_{k\varepsilon}, \theta_{k\varepsilon}, q_{k\varepsilon}, z_{k\varepsilon})$

$$u_{k\varepsilon}(x,t) = \sum_{j=1}^{k} \alpha_{jk}(t) w_j(x), \quad \theta_{k\varepsilon}(x,t) = \sum_{j=1}^{k} \beta_{jk}(t) v_j(x), \quad \text{and} \quad q_{k\varepsilon}(x,t) = \sum_{j=1}^{k} \gamma_{jk}(t) \mu_j(x);$$

for every $x \in \Omega$, and

$$z_{k\varepsilon}(x,t) = \sum_{j=1}^{k} \delta_{jk}(t) \xi_j(x);$$

for every $x \in \Gamma_1$, where

- $\{w_j\}_{1 \le j \le k}$ be an orthonormal bases of $\mathrm{H}^1_{\Gamma_0}(\Omega)$ which be an orthogonal and complete system of eigenfunctions (λ_j) of $\mathbf{A}$ in $\mathrm{H}^1_{\Gamma_0}(\Omega) \cap \mathrm{H}^2(\Omega)$; that's generate the space $W_k = \mathrm{Span}\{w_1, w_2, \dots, w_k\}$.
- $\{v_j\}_{1 \le j \le k}$ be an orthonormal bases of $\mathrm{H}^1_0(\Omega)$ which be an orthogonal and complete system in $\mathrm{H}^1_0(\Omega)$; that's generate the space $V_k = \mathrm{Span}\{v_1, v_2, \dots, v_k\}$.
- $\{\mu_j\}_{1 \le j \le k}$ be an orthonormal bases of $\mathrm{H}^1(\Omega)$; that's generate the space $\Delta_k = \mathrm{Span}\{\mu_1, \mu_2, \dots, \mu_k\}$.

- $\{\xi_j\}_{1\leq j\leq k}$ be an orthonormal bases of $\mathrm{L}^2(\Gamma_1)$; that's generate the space $\Lambda_k = \mathrm{Span}\{\xi_1, \xi_2, \dots, \xi_k\}$.

The approximate perturbed system of (3.1)-(3.7) is given by

$$\begin{aligned}&\int_\Omega \Big[u''_{k\varepsilon}(t) + \mathrm{div}(\sigma(x)\theta_{k\varepsilon}(t))\Big]w\mathrm{d}x + a(u_{k\varepsilon}(t), w)\\ = \;& \int_{\Gamma_1} h(x)z'_{k\varepsilon}(t)w\mathrm{d}\Gamma + \int_0^t g(t-s)a(u_{k\varepsilon}(s), w)\mathrm{d}s, \quad \forall w \in W_k;\end{aligned} \tag{3.13}$$

$$\int_\Omega \Big[\rho(x)\theta'_{k\varepsilon}(t) + (\kappa\cdot\nabla)q_{k\varepsilon}(t) + (\sigma(x).\nabla)u'_{k\varepsilon}(t)\Big]v\mathrm{d}x = 0, \quad \forall v \in V_k; \tag{3.14}$$

$$\int_\Omega \Big[\tau_0 q'_{k\varepsilon}(t)\mathrm{d}x + q_{k\varepsilon}(t) + (\kappa\cdot\nabla)\theta_{k\varepsilon}(t)\Big]\mu\mathrm{d}x = 0, \quad \forall \mu \in \Delta_k; \tag{3.15}$$

$$\int_{\Gamma_1} \Big[\varepsilon z''_{k\varepsilon}(t) + h(x)u'_{k\varepsilon}(t) + h(x)f(x)z'_{k\varepsilon}(t) + h(x)m(x)z_{k\varepsilon}(t)\Big]\xi\mathrm{d}\Gamma = 0, \quad \forall \xi \in \Lambda_k. \tag{3.16}$$

The initial conditions

$$\begin{cases} u_{k\varepsilon}(0) = u_{0k} = \sum_{j=1}^k < u_0, w_j > w_j\\ u'_{k\varepsilon}(0) = u_{1k} = \sum_{j=1}^k < u_1, w_j > w_j\\ \theta_{k\varepsilon}(0) = \theta_{0k} = \sum_{j=1}^k < \theta_0, v_j > v_j\\ q_{k\varepsilon}(0) = q_{0k} = \sum_{j=1}^k < q_0, \mu_j > \mu_j\\ z_{k\varepsilon}(0) = z_{0k} = \sum_{j=1}^k < z_0, \xi_j > \xi_j\\ z'_{k\varepsilon}(0) = z_{1k} = -\left(\dfrac{u_{1k} + mz_{0k}}{f}\right). \end{cases} \tag{3.17}$$

Obviously,

$$\begin{aligned} u_{0k} &\to u_0 && \text{strongly in} && \mathrm{H}^1_{\Gamma_0}(\Omega)\cap\mathrm{H}^2(\Omega),\\ u_{1k} &\to u_1 && \text{strongly in} && \mathrm{H}^1_{\Gamma_0}(\Omega),\\ \theta_{0k} &\to \theta_0 && \text{strongly in} && \mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega),\\ q_{0k} &\to q_0 && \text{strongly in} && \mathrm{H}^1(\Omega),\\ z_{0k} &\to z_0 && \text{strongly in} && \mathrm{L}^2(\Gamma_1). \end{aligned} \tag{3.18}$$

Since the system (3.13)-(3.17) is a normal system of ordinary differential equations, then there exists a unique local solution (u, θ, q, z) which can be extended globally at time. This solution will be obtained as the limit of $(u_{k\varepsilon}, \theta_{k\varepsilon}, q_{k\varepsilon}, z_{k\varepsilon})$ as k to infinity and ε to zero.

In order to facilitate a compactness procedure for passing to the limit, it suffices to use some a priori estimate for the approximate solution.

In the following, we need to show that the solution can be extended for all $t > 0$.

First estimate

Taking $w = 2u'_{k\varepsilon}$ in (3.13), $v = 2\theta_{k\varepsilon}$ in (3.14), $\mu = 2q_{k\varepsilon}$ in (3.15), and $\xi = 2z'_{k\varepsilon}$ in (3.16). Since $\theta_{k\varepsilon} \in V_k \subset \mathrm{H}^1_0(\Omega)$, the Gauss divergence theorem yields

$$2\int_\Omega \mathrm{div}(\sigma(x)\theta_{k\varepsilon}(t))u'_{k\varepsilon}(t)\mathrm{d}x + 2\int_\Omega (\sigma(x).\nabla)u'_{k\varepsilon}(t)\theta_{k\varepsilon}(t)\mathrm{d}x = 0$$

and

$$2\int_\Omega (\kappa\cdot\nabla)q_{k\varepsilon}(t)\theta_{k\varepsilon}(t)\mathrm{d}x + 2\int_\Omega (\kappa\cdot\nabla)\theta_{k\varepsilon}(t)q_{k\varepsilon}(t)\mathrm{d}x = 0.$$

From (3.9), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u'_{k\varepsilon}(t)\|_2^2+\left(1-\int_0^t g(s)\mathrm{d}s\right)a(u_{k\varepsilon}(t),u_{k\varepsilon}(t))+\int_\Omega \rho(x)|\theta_{k\varepsilon}(t)|^2\mathrm{d}x\right.\\
&\left.+\tau_0\|q_{k\varepsilon}(t)\|_2^2+\varepsilon\|z'_{k\varepsilon}(t)\|_{2,\Gamma_1}^2+\int_{\Gamma_1} m(x)|h^{1/2}z_{k\varepsilon}(t)|^2\mathrm{d}\Gamma\right)+2\int_\Omega f(x)|h^{1/2}z'_{k\varepsilon}(t)|^2\mathrm{d}\Gamma\\
=\;&(g'\diamond u_{k\varepsilon})(t)-\frac{\mathrm{d}}{\mathrm{d}t}(g\diamond u_{k\varepsilon})(t)-g(t)a(u_{k\varepsilon}(t),u_{k\varepsilon}(t))-2\|q_{k\varepsilon}(t)\|_2^2. \qquad (3.19)
\end{aligned}$$

Integrating over $(0,t)$ and observing that

$$(g\diamond u_{k\varepsilon})(t)-\int_0^t(g'\diamond u_{k\varepsilon})(s)\mathrm{d}s+\int_0^t g(s)a(u_{k\varepsilon}(s),u_{k\varepsilon}(s))\mathrm{d}s\geq 0, \qquad (3.20)$$

which (3.19) becomes

$$\begin{aligned}
&\|u'_{k\varepsilon}(t)\|_2^2+a_0\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u_{k\varepsilon}(t)\|_2^2+\rho_0\|\theta_{k\varepsilon}(t)\|_2^2+\tau_0\|q_{k\varepsilon}(t)\|_2^2\\
&+\varepsilon\|z'_{k\varepsilon}(t)\|_{2,\Gamma_1}^2+m_0\|h^{1/2}z_{k\varepsilon}(t)\|_{2,\Gamma_1}^2+2f_0\int_0^t\|h^{1/2}z'_{k\varepsilon}(s)\|_{2,\Gamma_1}^2\mathrm{d}s\\
\leq\;&\|u_{1k}\|_2^2+a_0\|\nabla u_{0k}\|_2^2+\|\rho\|_\infty\|\theta_{0k}\|_2^2+\tau_0\|q_{0k}\|_2^2+\varepsilon\|z_{1k}\|_{2,\Gamma_1}^2+\|h\|_\infty\|m\|_\infty\|z_{0k}\|_{2,\Gamma_1}^2.
\end{aligned}$$

According to (3.18), we deduce that there exists a constant K independent of k, ε, and t, such that, for all $t>0$

$$\begin{aligned}
&\|u'_{k\varepsilon}(t)\|_2^2+a_0\ell\|\nabla u_{k\varepsilon}(t)\|_2^2+\rho_0\|\theta_{k\varepsilon}(t)\|_2^2+\tau_0\|q_{k\varepsilon}(t)\|_2^2+\varepsilon\|z'_{k\varepsilon}(t)\|_{2,\Gamma_1}^2\\
&+m_0\|h^{1/2}z_{k\varepsilon}(t)\|_{2,\Gamma_1}^2+2f_0\int_0^t\|h^{1/2}z'_{k\varepsilon}(s)\|_{2,\Gamma_1}^2\mathrm{d}s\leq K. \qquad (3.21)
\end{aligned}$$

These estimates imply that the solution $(u_{k\varepsilon},\theta_{k\varepsilon},q_{k\varepsilon},z_{k\varepsilon})$ of the system (3.13)-(3.17) exists globally at time.

Second estimate

First of all, we have to estimate $\|u''_{k\varepsilon}(0)\|_2^2$, $\|\theta'_{k\varepsilon}(0)\|_2^2$, $\|q'_{k\varepsilon}(0)\|_2^2$, and $\|z''_{k\varepsilon}(0)\|_{2,\Gamma_1}^2$. Taking $w=u''_{k\varepsilon}(t)$ in (3.13), $v=\theta'_{k\varepsilon}(t)$ in (3.14), $\mu=q'_{k\varepsilon}(t)$ in (3.15), and $\xi=z''_{k\varepsilon}(t)$ in (3.16), then considering $t=0$ to find

$$\|u''_{k\varepsilon}(0)\|_2\leq\|\mathbf{A}u_{0k}\|_2+\|\sigma\|\|\nabla\theta_{0k}\|_2,$$

$$\rho_0\|\theta'_{k\varepsilon}(0)\|_2\leq\|\kappa\|_\infty\|\nabla q_{0k}\|_2+\|\sigma\|\|\nabla u_{1k}\|_2,$$

$$\tau_0\|q'_{k\varepsilon}(0)\|_2\leq\|q_{0k}\|_2+\|\kappa\|_\infty\|\nabla\theta_{0k}\|_2,$$

and

$$\varepsilon\|z''_{k\varepsilon}(0)\|_{2,\Gamma_1}\leq\max_{x\in\Gamma_1}\{|h|,|hf|,|hm|\}\left(\|u_{1k}\|_2+\|z_{1k}\|_{2,\Gamma_1}+\|z_{0k}\|_{2,\Gamma_1}\right).$$

Consequently, we conclude that there exists $C>0$ independent of k and ε due to (3.18) satisfying

$$\|u''_{k\varepsilon}(0)\|_2^2+\|\theta'_{k\varepsilon}(0)\|_2^2+\|q'_{k\varepsilon}(0)\|_2^2\leq C\quad\text{and}\quad\|z''_{k\varepsilon}(0)\|_{2,\Gamma_1}^2=0. \qquad (3.22)$$

Next, differentiating (3.13)-(3.16) with respect to t, setting $w = 2u''_{k\varepsilon}(t)$, $v = 2\theta'_{k\varepsilon}(t)$, $\mu = 2q'_{k\varepsilon}(t)$, and $\xi = 2z''_{k\varepsilon}(t)$, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\|u''_{k\varepsilon}(t)\|_2^2 + a(u'_{k\varepsilon}(t), u'_{k\varepsilon}(t)) + \int_\Omega \rho(x)|\theta'_{k\varepsilon}(t)|^2\mathrm{d}x + \tau_0\|q'_{k\varepsilon}(t)\|_2^2 \\ &+\varepsilon\|z''_{k\varepsilon}(t)\|_{2,\Gamma_1}^2 + \int_{\Gamma_1} m|h^{1/2}z'_{k\varepsilon}(t)|^2\mathrm{d}\Gamma\Bigg) + 2\int_\Omega f|h^{1/2}z''_{k\varepsilon}(t)|^2\mathrm{d}\Gamma \\ = \;& -2\|q'_{k\varepsilon}(t)\|_2^2 + 2\int_\Omega a_{ij}(x)\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t g(t-s)\frac{\partial u_{k\varepsilon}(s)}{\partial x_j}\mathrm{d}s\right)\frac{\partial u''_{k\varepsilon}(t)}{\partial x_i}\mathrm{d}x. \end{aligned} \tag{3.23}$$

Whereas

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t g(t-s)\frac{\partial u_{k\varepsilon}(s)}{\partial x_j}\mathrm{d}s = g(t)\frac{\partial u_{k\varepsilon}(0)}{\partial x_j} + \int_0^t g(t-s)\frac{\partial u'_{k\varepsilon}(s)}{\partial x_j}\mathrm{d}s,$$

then, the last term of the right side in (3.23) gives

$$\begin{aligned} &\int_\Omega a_{ij}(x)\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t g(t-s)\frac{\partial u_{k\varepsilon}(s)}{\partial x_j}\mathrm{d}s\right)\frac{\partial u''_{k\varepsilon}(t)}{\partial x_i}\mathrm{d}x \\ = \;& \int_0^t g(t-s)a(u'_{k\varepsilon}(s), u''_{k\varepsilon}(t))\mathrm{d}s + \int_\Omega \mathrm{A}\nabla u_{0k}\left(\frac{\mathrm{d}}{\mathrm{d}t}(g(t)\nabla u'_{k\varepsilon}(t)) - g'(t)\nabla u'_{k\varepsilon}(t)\right)\mathrm{d}x. \end{aligned}$$

It follows together with the above equality that

$$\begin{aligned} &\|u''_{k\varepsilon}(t)\|_2^2 + a_0\ell\|\nabla u'_{k\varepsilon}(t)\|_2^2 + \rho_0\|\theta'_{k\varepsilon}(t)\|_2^2 + \tau_0\|q'_{k\varepsilon}(t)\|_2^2 + \varepsilon\|z''_{k\varepsilon}(t)\|_{2,\Gamma_1}^2 \\ &+m_0\|h^{1/2}z'_{k\varepsilon}(t)\|_{2,\Gamma_1}^2 + 2f_0\int_0^t \|h^{1/2}z''_{k\varepsilon}(s)\|_{2,\Gamma_1}^2\mathrm{d}s \\ \leq \;& \|u''_{k\varepsilon}(0)\|_2^2 + a_0\|\nabla u_{1k}\|_2^2 + \|\rho\|_\infty\|\theta'_{k\varepsilon}(0)\|_2^2 + \tau_0\|q'_{k\varepsilon}(0)\|_2^2 \\ &+\|hm\|_\infty\|z_{1k}\|_{2,\Gamma_1}^2 + \varepsilon\|z''_{k\varepsilon}(0)\|_{2,\Gamma_1}^2 - (g\diamond u'_{k\varepsilon})(t) + \int_0^t (g'\diamond u'_{k\varepsilon})(s)\mathrm{d}s \\ &-\int_0^t g(s)a(u'_{k\varepsilon}(s), u'_{k\varepsilon}(s))\mathrm{d}s + 2\int_\Omega g(t)\mathrm{A}\nabla u_{0k}\nabla u'_{k\varepsilon}(t)\mathrm{d}x \\ &-2\int_0^t\int_\Omega g'(s)\mathrm{A}\nabla u_{0k}\nabla u'_{k\varepsilon}(s)\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{3.24}$$

As $g \in \mathcal{C}^1(\mathrm{I\!R}^+)$, by using Young's inequality ($ab \leq \frac{1}{2\delta}a^2 + 2\delta b^2$), we get

$$\begin{aligned} \int_\Omega g(t)\mathrm{A}\nabla u_{0k}\nabla u'_{k\varepsilon}(t)\mathrm{d}x \;&\leq\; \|g\|_\infty\sum_{i,j=1}^n\left(\frac{1}{2\delta}\int_\Omega\left|\frac{\partial u_{0k}}{\partial x_j}\right|^2\mathrm{d}x + 2\delta\int_\Omega\left|a_{ij}(x)\frac{\partial u'_{k\varepsilon}(t)}{\partial x_i}\right|^2\mathrm{d}x\right) \\ &\leq\; \|g\|_\infty\frac{n}{2\delta}\|\nabla u_{0k}\|_2^2 + 2\delta\|g\|_\infty a_1\|\nabla u'_{k\varepsilon}(t)\|_2^2, \end{aligned}$$

and

$$\begin{aligned} &-\int_0^t\int_\Omega g'(s)\mathrm{A}\nabla u_{0k}\nabla u'_{k\varepsilon}(s)\mathrm{d}x\mathrm{d}s \\ \leq \;& \sum_{i,j=1}^n\left(\int_\Omega\left|\frac{\partial u_{0k}}{\partial x_j}\right|^2\mathrm{d}x + \int_\Omega\left(\int_0^t \beta g(s)a_{ij}(x)\frac{\partial u'_{k\varepsilon}(s)}{\partial x_i}\mathrm{d}s\right)^2\mathrm{d}x\right) \\ \leq \;& n\|\nabla u_{0k}\|_2^2 + a_1\beta^2(1-l)^2\int_0^t\|\nabla u'_{k\varepsilon}(s)\|_2^2\mathrm{d}s, \end{aligned}$$

Substituting these last two inequalities into (3.24) and observing that

$$(g\diamond u'_{k\varepsilon})(t) - \int_0^t(g'\diamond u'_{k\varepsilon})(s)\mathrm{d}s + \int_0^t g(s)a(u'_{k\varepsilon}(s), u'_{k\varepsilon}(s))\mathrm{d}s \geq 0,$$

by choosing $\delta < a_0 l/(2a_1\|g\|_\infty)$, we can apply Gronwall's inequality to get a constant $K_1 > 0$, independent of k, ε, and t, such that

$$\begin{aligned}&\|u''_{k\varepsilon}(t)\|_2^2 + \|\nabla u'_{k\varepsilon}(t)\|_2^2 + \rho_0\|\theta'_{k\varepsilon}(t)\|_2^2 + \tau_0\|q'_{k\varepsilon}(t)\|_2^2 + \varepsilon\|z''_{k\varepsilon}(t)\|_{2,\Gamma_1}^2 \\ &+\|h^{1/2}z'_{k\varepsilon}(t)\|_{2,\Gamma_1}^2 + 2\int_0^t \|h^{1/2}z''_{k\varepsilon}(s)\|_{2,\Gamma_1}^2 \mathrm{d}s \le K_1. \end{aligned} \tag{3.25}$$

Passing to the limit

The estimates (3.21) and (3.25) are enough to get solutions in the sense of Definition 3.3.1. In fact, for all $T > 0$ fixed, we get, among other estimates, the following one

$$\begin{array}{lll}
(u_{k\varepsilon}), (u'_{k\varepsilon}) & \text{are bounded in} & \mathrm{L}^\infty(0,T;\mathrm{H}^1_{\Gamma_0}(\Omega)),\\
(u''_{k\varepsilon}) & \text{is bounded in} & \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)),\\
(\theta_{k\varepsilon}), (\theta'_{k\varepsilon}) & \text{are bounded in} & \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)),\\
(q_{k\varepsilon}), (q'_{k\varepsilon}) & \text{are bounded in} & \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)),\\
(h^{1/2}z_{k\varepsilon}), (h^{1/2}z'_{k\varepsilon}) & \text{are bounded in} & \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Gamma_1)),\\
(h^{1/2}z'_{k\varepsilon}), (h^{1/2}z''_{k\varepsilon}) & \text{are bounded in} & \mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma_1)).
\end{array}$$

Consequently, we can extract subsequences of approximations $(u_{k\varepsilon})_{k\in\mathbb{N}}$, $(\theta_{k\varepsilon})_{k\in\mathbb{N}}$, $(q_{k\varepsilon})_{k\in\mathbb{N}}$, and $(z_{k\varepsilon})_{k\in\mathbb{N}}$, which still denote by $(u_{k\varepsilon})_{k\in\mathbb{N}}$, $(\theta_{k\varepsilon})_{k\in\mathbb{N}}$, $(q_{k\varepsilon})_{k\in\mathbb{N}}$, and $(z_{k\varepsilon})_{k\in\mathbb{N}}$ (independent of ε), respectively, such that

$$\begin{array}{lll}
u_{k\varepsilon} \rightharpoonup u & \text{weakly star in} & \mathrm{L}^\infty(0,T;\mathrm{H}^1_{\Gamma_0}(\Omega)),\\
u'_{k\varepsilon} \rightharpoonup u' & \text{weakly star in} & \mathrm{L}^\infty(0,T;\mathrm{H}^1_{\Gamma_0}(\Omega)),\\
u''_{k\varepsilon} \rightharpoonup u'' & \text{weakly star in} & \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)),\\
\theta_{k\varepsilon} \rightharpoonup \theta & \text{weakly star in} & \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)),\\
\theta'_{k\varepsilon} \rightharpoonup \theta' & \text{weakly star in} & \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)),\\
q_{k\varepsilon} \rightharpoonup q & \text{weakly star in} & \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)),\\
q'_{k\varepsilon} \rightharpoonup q' & \text{weakly star in} & \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)),\\
h^{1/2}z_{k\varepsilon} \rightharpoonup h^{1/2}z & \text{weakly star in} & \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Gamma_1)),\\
h^{1/2}z'_{k\varepsilon} \rightharpoonup h^{1/2}z' & \text{weakly star in} & \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Gamma_1)),\\
h^{1/2}z''_{k\varepsilon} \rightharpoonup h^{1/2}z'' & \text{weakly in} & \mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma_1)),\\
\lim_{k\to+\infty,\varepsilon\to 0}\|z''_{k\varepsilon}\|_{2,\Gamma_1} = 0 & \text{a.e. in} & [0,T].
\end{array} \tag{3.26}$$

Thanks to Green's formula in the fact $\theta_{k\varepsilon}(t) \in \mathrm{H}^1_0(\Omega)$, from the convergence $(3.26)_{4,6}$, we are able to show the convergence

$$\int_\Omega (\kappa\cdot\nabla)\theta_{k\varepsilon}(t)\mu\mathrm{d}x \to -\int_\Omega \theta(t)(\kappa\cdot\nabla)\mu\mathrm{d}x = \int_\Omega (\kappa\cdot\nabla)\theta(t)\mu\mathrm{d}x, \tag{3.27}$$

for all $\mu \in \Delta_k \subset \mathrm{H}^1(\Omega)$, and

$$\int_\Omega (\kappa\cdot\nabla)q_{k\varepsilon}(t)v\mathrm{d}x \to -\int_\Omega q(t)(\kappa\cdot\nabla)v\mathrm{d}x = \int_\Omega (\kappa\cdot\nabla)q(t)v\mathrm{d}x, \tag{3.28}$$

for all $v \in V_k \subset \mathrm{H}^1_0(\Omega)$ as $k \to \infty$. Every convergence in (3.26)-(3.28) allows us to pass to the limit in (3.13)-(3.16).

Uniqueness

Let $(u_1, \theta_1, q_1, z_1)$, $(u_2, \theta_2, q_2, z_2)$ be two quadruple of solutions to (3.1)-(3.8). Define $\bar{u} = u_1 - u_2$, $\bar{\theta} = \theta_1 - \theta_2$, $\bar{q} = q_1 - q_2$, and $\bar{z} = z_1 - z_2$, then $(\bar{u}, \bar{\theta}, \bar{q}, \bar{z})$ satisfy this system

$$\begin{cases} \bar{u}'' + \mathbf{A}\bar{u} - \int_0^t g(t-s)\mathbf{A}\bar{u}(s)\mathrm{d}s + \operatorname{div}(\sigma(x)\bar{\theta}) = 0 & \text{in} \quad \Omega \times (0,T) \\ \rho(x)\bar{\theta}' + (\kappa \cdot \nabla)\bar{q} + (\sigma(x).\nabla)\bar{u}' = 0 & \text{in} \quad \Omega \times (0,T) \\ \tau_0 \bar{q}' + \bar{q} + (\kappa \cdot \nabla)\bar{\theta} = 0 & \text{in} \quad \Omega \times (0,T) \\ \frac{\partial \bar{u}}{\partial \nu_{\mathrm{A}}} - \int_0^t g(t-s)\frac{\partial \bar{u}}{\partial \nu_{\mathrm{A}}}(s)\mathrm{d}s = h(x)\bar{z}' & \text{on} \quad \Gamma_1 \times (0,T) \\ \bar{u}' + f(x)\bar{z}' + m(x)\bar{z} = 0 & \text{on} \quad \Gamma_1 \times (0,T) \\ \bar{u} = 0 & \text{on} \quad \Gamma_0 \times (0,T) \\ \bar{\theta} = 0 & \text{on} \quad \Gamma \times (0,T) \\ \bar{u}(x,0) = \bar{u}'(x,0) = \bar{\theta}(x,0) = \bar{q}(x,0) = 0 & \qquad x \in \Omega \\ \bar{z}(x,0) = \bar{z}'(x,0) = 0 & \qquad x \in \Gamma_1. \end{cases} \tag{3.29}$$

Therefore,

$$\begin{aligned} & \int_\Omega \Big[\bar{u}''(t) + \operatorname{div}(\sigma(x)\bar{\theta}(t))\Big] w \mathrm{d}x + a(\bar{u}(t), w) - \int_{\Gamma_1} h(x)\bar{z}'(t) w \mathrm{d}\Gamma \\ = \quad & \int_0^t g(t-s) a(\bar{u}(s), w)\mathrm{d}s \end{aligned} \tag{3.30}$$

$$\int_\Omega \Big[\rho(x)\bar{\theta}'(t) + (\kappa \cdot \nabla)\bar{q}(t) + (\sigma(x).\nabla)\bar{u}'(t)\Big] v \mathrm{d}x = 0 \tag{3.31}$$

$$\int_\Omega \Big[\tau_0 \bar{q}'(t)\mathrm{d}x + \bar{q}(t) + (\kappa \cdot \nabla)\bar{\theta}(t)\Big] \mu \mathrm{d}x = 0 \tag{3.32}$$

$$\int_{\Gamma_1} \Big[\varepsilon \bar{z}''(t) + h(x)v'(t) + h(x)f(x)\bar{z}'(t) + h(x)m(x)\bar{z}(t)\Big] \xi \mathrm{d}\Gamma = 0, \tag{3.33}$$

for all $w \in \mathrm{H}^1_{\Gamma_0}(\Omega)$, $v \in \mathrm{H}^1_0(\Omega)$, $\mu \in \mathrm{H}^1(\Omega)$, and $\xi \in \mathrm{L}^2(\Gamma_1)$. By replacing $w = 2\bar{u}'$ in (3.30), $v = 2\bar{\theta}$ in (3.31), $\mu = 2\bar{q}$ in (3.32), and $\xi = 2\bar{z}'$ in (3.33). By exploiting (3.9), we find

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\|\bar{u}'(t)\|_2^2 + \left(1 - \int_0^t g(s)\mathrm{d}s\right) a(\bar{u}(t), \bar{u}(t)) + \int_\Omega \rho(x)|\bar{\theta}(t)|^2 \mathrm{d}x \\ & + \tau_0\|\bar{q}(t)\|_2^2 + \int_{\Gamma_1} m|h^{1/2}\bar{z}(t)|^2 \mathrm{d}\Gamma\Bigg) + 2\int_\Omega f|h^{1/2}\bar{z}'(t)|^2 \mathrm{d}\Gamma \\ \leq \quad & (g' \diamond \bar{u})(t) - \frac{\mathrm{d}}{\mathrm{d}t}\left[(g \diamond \bar{u})(t)\right] - g(t)a(\bar{u}(t), \bar{u}(t)). \end{aligned}$$

Integrating the above estimate over $(0,t)$ and using (3.20), we get

$$\begin{aligned} 0 \quad \leq \quad & \|\bar{u}'(t)\|_2^2 + a_0\ell\|\nabla\bar{u}(t)\|_2^2 + \rho_0\|\bar{\theta}(t)\|_2^2 + \tau_0\|\bar{q}(t)\|_2^2 \\ & + m_0\|h^{1/2}\bar{z}(t)\|_{2,\Gamma_1}^2 + 2f_0\int_0^t \|h^{1/2}\bar{z}'(s)\|_{2,\Gamma_1}^2 \\ \leq \quad & 0. \end{aligned}$$

This implies that $\bar{u} = \bar{\theta} = \bar{q} = \bar{z} = 0$.

Chapter 4

Optimal and General Decay of Energy-associated Solution

We are interested in studying, in this chapter, the optimal and the general decay rates of the energy associated solution with system (3.1)-(3.8). The proof is based on suitable assumptions of the relaxation function and on some properties of the convex functions, from which we can construct a suitable Lyapunov functional.

Let us introduce the energy functional E associated with system (3.1)-(3.8) by

$$E(t) = E_1(t) + E_2(t), \quad \forall t \geq 0,$$

where the first-order energy defined by

$$\begin{aligned} E_1(t) \quad &= \quad \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s\right) a(u(t), u(t)) + \frac{1}{2}(g \diamond u)(t) \\ &\quad + \frac{1}{2}\int_\Omega \rho(x)|\theta(t)|^2\mathrm{d}x + \frac{\tau_0}{2}\|q(t)\|_2^2 + \frac{1}{2}\int_{\Gamma_1} h(x)m(x)|z(t)|^2\mathrm{d}\Gamma, \end{aligned}$$

and, the second-order energy

$$\begin{aligned} E_2(t) \quad &= \quad \frac{1}{2}\|u_{tt}(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s\right) a(u_t(t), u_t(t)) \\ &\quad + \frac{1}{2}(g \diamond u_t)(t) + \frac{\tau_0}{2}\|q_t(t)\|_2^2 + \frac{1}{2}\int_\Omega \rho(x)|\theta_t(t)|^2\mathrm{d}x \\ &\quad + \frac{1}{2}\int_{\Gamma_1} h(x)m(x)|z_t(t)|^2\mathrm{d}\Gamma - g(t)a(u_0, u_t(t)). \end{aligned}$$

We now define Lyapunov functional $\mathcal{L}$ as follows

$$\mathcal{L}(t) = NE(t) + \epsilon K_1(t) + MK_2(t) + K_3(t), \tag{4.1}$$

where N, ϵ, and M are positive constants to be fixed later.

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4.1 Technical lemmas

Here we will state and prove some technical lemmas needed to establish the desired results.

Lemma 4.1.1. *The energy functional satisfies, along the solution of (3.1)-(3.8), the estimates, for all* $t \geq 0$

$$E_1'(t) = \frac{1}{2}(g' \diamond u)(t) - \frac{1}{2}g(t)a(u(t), u(t)) - \|q(t)\|_2^2 - \int_{\Gamma_1} h(x)f(x)|z_t(t)|^2 \mathrm{d}\Gamma \leq 0, \tag{4.2}$$

$$E_2'(t) \leq -\|q_t(t)\|_2^2 + cg(t)\|\nabla u_0\|_2^2, \tag{4.3}$$

where $c = \dfrac{\beta^2 a_1}{2a_0}$.

Proof. Taking $w = u_t$, $v = \theta$, $\mu = q$, and $\xi = z$ in the variational formulation (3.12). Knowing that $\varepsilon \to 0$, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t(t)\|_2^2 + a(u(t), u(t)) + \int_\Omega \rho(x)|\theta(t)|^2 \mathrm{d}x + \tau_0\|q(t)\|_2^2 + \int_{\Gamma_1} h(x)m(x)|z(t)|^2 \mathrm{d}\Gamma\right) \\
&= -\|q(t)\|_2^2 - \int_{\Gamma_1} h(x)f(x)|z_t(t)|^2 \mathrm{d}\Gamma + \int_0^t g(t-s)a(u(s), u_t(t))\mathrm{d}s.
\end{aligned}$$

This and by using (3.9), we get (4.2).

On the other hand, taking the time derivation of the variational formulation (3.12) and setting $w = u_{tt}$, $v = \theta_t$, $\mu = q_t$, and $\xi = z_t$. According to (3.9) and $\varepsilon \to 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_{tt}(t)\|^2 + \left(1 - \int_0^t g(s)\mathrm{d}s\right) a(u_t(t), u_t(t)) + (g \diamond u_t)(t)\right. \\
&\qquad \left. + \int_\Omega \rho(x)|\theta_t(t)|^2 \mathrm{d}x + \tau_0\|q_t(t)\|_2^2 \mathrm{d}x + \int_{\Gamma_1} h(x)m(x)|z_t(t)|^2 \mathrm{d}\Gamma\right) \\
= \;& -\frac{1}{2}(g' \diamond u_t)(t) - \frac{1}{2}g(t)a(u_t(t), u_t(t)) - \int_{\Gamma_1} h(x)f(x)|z_{tt}(t)|^2 \mathrm{d}\Gamma \\
& -\|q_t(t)\|_2^2 + g(t)a(u_0, u_{tt}(t)).
\end{aligned} \tag{4.4}$$

By using Young's inequality and the fact (1.5), the last term of the right hand side in (4.4) gives, for $\delta > 0$

$$\begin{aligned}
g(t)a(u_0, u_{tt}(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[g(t)a(u_0, u_t(t))\right] - g'(t)a(u_0, u_t(t)) \\
&\leq \frac{\mathrm{d}}{\mathrm{d}t}\left[g(t)a(u_0, u_t(t))\right] + \frac{\beta\delta}{2}g(t)\|\nabla u_0\|_2^2 + \frac{\beta a_1}{2\delta a_0}g(t)a(u_t(t), u_t(t)).
\end{aligned}$$

By choosing $\delta = \dfrac{\beta a_1}{a_0}$. From (4.4), we conclude (4.3). ■

Remark 4.1.2. The second-order energy of system (3.1)-(3.8) is a bounded nondecreasing function satisfying, for all $t \geq 0$

$$E_2(t) \leq E_2(0) + c\|\nabla u_0\|_2^2 \int_0^t g(s)\mathrm{d}s \leq C,$$

where C is a positive constant. Moreover, we conclude that

$$E_1'(t) \leq E'(t). \tag{4.5}$$

Now we are going to construct Lyapunov functional equivalent to the energy functional by resorting to the following functions ($K_i, i = 1, 2, 3$), with which we can show the desired result.

Lemma 4.1.3. *Let (u, θ, q, z) the solution of (3.1)-(3.8). Then, the functional*

$$K_1(t) = \int_\Omega u_t(t)u(t)\mathrm{d}x + \int_{\Gamma_1} h(x)u(t)z(t)\mathrm{d}\Gamma + \frac{1}{2}\int_{\Gamma_1} h(x)f(x)|z(t)|^2\mathrm{d}\Gamma,$$

satisfies the estimate, for any $0 < \alpha < 1$, for all $t \geq 0$

$$\begin{aligned} K_1'(t) \quad &\leq \quad \|u_t(t)\|_2^2 - \frac{\ell}{2}a(u(t), u(t)) + \frac{C_1C_\alpha}{4a_0}(k \diamond u)(t) + C_2\int_\Omega \rho(x)|\theta(t)|^2\mathrm{d}x \\ &\quad + C_3\int_{\Gamma_1} h(x)f(x)|z_t(t)|^2\mathrm{d}\Gamma - \int_{\Gamma_1} h(x)m(x)|z(t)|^2\mathrm{d}\Gamma, \end{aligned} \tag{4.6}$$

where

$$C_1 = \frac{6a_1n}{a_0\ell},\ \|\sigma\| = \sup_{x\in\Omega}\sup_{1\leq i\leq n}|\sigma_i(x)|,\ C_2 = \frac{3\|\sigma\|^2\|\rho\|_\infty}{2a_0\ell\rho_0^2},\ C_3 = \frac{6\bar{C}_*^2\|h\|_\infty\|f\|_\infty}{a_0\ell f_0^2}. \tag{4.7}$$

Proof. Using (3.1) and (3.7) in direct computation of K_1, we get

$$\begin{aligned} K_1'(t) \quad &= \quad \|u_t(t)\|_2^2 - a(u(t), u(t)) + \int_0^t g(t-s)a(u(s), u(t))\mathrm{d}s + \int_\Omega \theta(t)(\sigma\cdot\nabla)u(t)\mathrm{d}x \\ &\quad - \int_{\Gamma_1} h(x)m(x)|z(t)|^2\mathrm{d}\Gamma + 2\int_{\Gamma_1} h(x)u(t)z_t(t)\mathrm{d}\Gamma. \end{aligned} \tag{4.8}$$

As in (2.7), we have

$$\int_\Omega\left(\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))\mathrm{d}s\right)^2\mathrm{d}x \leq \frac{C_\alpha}{a_0}(k\diamond u)(t).$$

Using Young's inequality, (1.3), and (1.4), we obtain, for some constant $\delta > 0$

$$\begin{aligned} \int_0^t g(t-s)a(u(s), u(t))\mathrm{d}s \quad &= \quad \sum_{i,j=1}^n\int_0^t g(t-s)\int_\Omega a_{ij}(x)\left(\frac{\partial u(s)}{\partial x_j} - \frac{\partial u(t)}{\partial x_j}\right)\frac{\partial u(t)}{\partial x_i}\mathrm{d}x\mathrm{d}s \\ &\quad + \left(\int_0^t g(s)\mathrm{d}s\right)a(u(t), u(t)), \\ &\leq \left((1-\ell) + \delta\frac{a_1}{a_0}\right)a(u(t), u(t)) + \frac{nC_\alpha}{4a_0\delta}(k\diamond u)(t). \end{aligned} \tag{4.9}$$

Using Cauchy-Schwarz's and Young's inequalities, from (1.7), we get, for $\delta_1 > 0$

$$\begin{aligned} \left|\int_{\Gamma_1} h(x)u(t)z_t(t)\mathrm{d}\Gamma\right| \quad &= \quad \left|\int_{\Gamma_1}\frac{h(x)f(x)}{f(x)}u(t)z_t(t)\mathrm{d}\Gamma\right| \\ &\leq \frac{\|h\|_\infty^{1/2}\|f\|_\infty^{1/2}}{f_0}\left(\int_{\Gamma_1} h(x)f(x)|z_t(t)|^2\mathrm{d}\Gamma\right)^{1/2}\left(\int_{\Gamma_1}|u(t)|^2\mathrm{d}\Gamma\right)^{1/2} \\ &\leq \delta_1\frac{\bar{C}_*^2}{a_0}a(u(t), u(t)) + \frac{\|h\|_\infty\|f\|_\infty}{4\delta_1 f_0^2}\int_{\Gamma_1} h(x)f(x)|z_t(t)|^2\mathrm{d}\Gamma, \end{aligned} \tag{4.10}$$

and, for $\delta_2 > 0$

$$\left|\int_\Omega(\sigma\cdot\nabla)u(t)\theta(t)\mathrm{d}x\right| \leq \frac{\|\sigma\|^2\|\rho\|_\infty}{4\delta_2\rho_0^2}\int_\Omega \rho(x)|\theta(t)|^2\mathrm{d}x + \frac{\delta_2}{a_0}a(u(t), u(t)), \tag{4.11}$$

Substituting (4.9)-(4.11) into (4.8), we find that

$$\begin{aligned} K_1'(t) &\leq \|u_t(t)\|_2^2 + \left(-\ell + \delta\frac{a_1}{a_0} + 2\delta_1\frac{\bar{C}_*^2}{a_0} + \frac{\delta_2}{a_0}\right) a(u(t),u(t)) \\ &\quad + \frac{nC_\alpha}{4a_0\delta}(k\diamond u)(t) + \frac{\|h\|_\infty\|f\|_\infty}{2\delta_1 f_0^2}\int_{\Gamma_1} h(x)f(x)|z_t(t)|^2\mathrm{d}\Gamma \\ &\quad + \frac{\|\sigma\|^2\|\rho\|_\infty}{4\delta_2\rho_0^2}\int_\Omega \rho(x)|\theta(t)|^2\mathrm{d}x - \int_{\Gamma_1} h(x)m(x)|z(t)|^2\mathrm{d}\Gamma. \end{aligned}$$

Let us choose $\delta = \frac{a_0\ell}{6a_1}$, $\delta_1 = \frac{a_0\ell}{12\bar{C}_*^2}$, and $\delta_2 = \frac{a_0\ell}{6}$. Then, we get (4.6). ■

Lemma 4.1.4. *Let (u,θ,q,z) the solution of (3.1)-(3.8). Then, the functional*

$$K_2(t) = -\int_\Omega u_t(t)\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x,$$

satisfies the estimate, for $\delta > 0$, for all $t \geq 0$

$$\begin{aligned} K_2'(t) &\leq C_4(\delta)a(u(t),u(t)) + \left(\frac{C_\alpha C_5(\delta)+C_6(\delta)}{4a_0}\right)(k\diamond u)(t) + C_7\int_\Omega \rho(x)|\theta(t)|^2\mathrm{d}x \\ &\quad -\left(\int_0^t g(s)\mathrm{d}s - \delta\right)\|u_t(t)\|_2^2 + C_8\int_{\Gamma_1} h(x)f(x)|z_t(t)|^2\mathrm{d}\Gamma, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} &C_4(\delta) = \delta\frac{a_1}{a_0}, \quad C_5(\delta) = 1 + 4\sqrt{na_1} + \frac{2\alpha^2C_*^2+n}{\delta}, \\ &C_6(\delta) = \frac{2k_1C_*^2}{\delta}, \quad C_7 = \frac{2\|\rho\|_\infty\|\sigma\|^2}{\rho_0^2}, \quad C_8 = \frac{2\bar{C}_*^2\|h\|_\infty\|f\|_\infty}{f_0^2}. \end{aligned}$$

Proof. By exploiting (3.1) and using Green's formula, we have

$$\begin{aligned} K_2'(t) &= \left(1 - \int_0^t g(s)\mathrm{d}s\right)\int_\Omega A\nabla u(t)\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\mathrm{d}s\mathrm{d}x \\ &\quad + \int_\Omega\int_0^t g(t-s)A\nabla(u(t)-u(s))\mathrm{d}s\int_0^t g(t-s)\nabla(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\ &\quad - \int_\Omega u_t(t)\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x - \left(\int_0^t g(s)\mathrm{d}s\right)\|u_t(t)\|_2^2 \\ &\quad - \int_\Omega \sigma(x)\theta(t)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\mathrm{d}x \\ &\quad - \int_{\Gamma_1} h(x)z_t(t)\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}\Gamma. \end{aligned} \tag{4.13}$$

As result of (4.9), we obtain, for $\delta > 0$

$$\int_\Omega \mathrm{A}\nabla u(t)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\mathrm{d}x \leq \delta\frac{a_1}{a_0}a(u(t),u(t)) + \frac{nC_\alpha}{4a_0\delta}(k\diamond u)(t), \tag{4.14}$$

and, for $\delta_1 > 0$

$$\begin{aligned} &-\int_\Omega \sigma(x)\theta(t)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\mathrm{d}x \\ &\leq \delta_1\frac{\|\rho\|_\infty\|\sigma\|^2}{\rho_0^2}\int_\Omega \rho(x)|\theta(t)|^2\mathrm{d}x + \frac{C_\alpha}{4a_0\delta_1}(k\diamond u)(t). \end{aligned} \tag{4.15}$$

By repeating the same arguments of (4.9) and (4.10), we get, for $\delta_2 > 0$

$$\begin{aligned} & \int_{\Gamma_1} h(x)z_t(t)\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}\Gamma \\ \leq \; & \frac{\bar{C}_*^2 C_\alpha}{4\delta_2 a_0}(k\diamond u)(t)+\delta_2\frac{\|h\|_\infty\|f\|_\infty}{f_0^2}\int_{\Gamma_1} h(x)f(x)|z_t(t)|^2\mathrm{d}\Gamma. \end{aligned} \tag{4.16}$$

We arrive to estimate

$$\int_\Omega\int_0^t g(t-s)A\nabla(u(t)-u(s))\mathrm{d}s\int_0^t g(t-s)\nabla(u(t)-u(s))\mathrm{d}s\mathrm{d}x \leq \frac{\sqrt{a_1 n}C_\alpha}{a_0}(k\diamond u)(t), \tag{4.17}$$

and, for $\delta_3 > 0$

$$-\int_\Omega u_t(t)\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \leq \delta_3\|u_t(t)\|_2^2+\frac{\alpha^2[C_\alpha+k_1]C_*^2}{2\delta_3 a_0}(k\diamond u)(t). \tag{4.18}$$

Inserting (4.14)-(4.18) into (4.13), we obtain

$$\begin{aligned} K_2'(t) \;\leq\; & -\left(\int_0^t g(s)\mathrm{d}s-\delta_3\right)\|u_t(t)\|_2^2+\delta\frac{a_1}{a_0}a(u(t),u(t))+\frac{2k_1C_*^2}{4\delta_3 a_0}(k\diamond u)(t) \\ & +\frac{C_\alpha}{4a_0}\left[\frac{n}{\delta}+\frac{1}{\delta_1}+\frac{\bar{C}_*^2}{\delta_2}+4\sqrt{a_1 n}+\frac{2\alpha^2C_*^2}{\delta_3}\right](k\diamond u)(t) \\ & +\delta_1\frac{\|\rho\|_\infty\|\sigma\|^2}{\rho_0^2}\int_\Omega\rho(x)|\theta(t)|^2\mathrm{d}x+\delta_2\frac{\|h\|_\infty\|f\|_\infty}{f_0^2}\int_{\Gamma_1}h(x)f(x)|z_t(t)|^2\mathrm{d}\Gamma. \end{aligned}$$

By choosing $\delta_1 = 2$ and $\delta_2 = 2\bar{C}_*^2$. Then, we get the estimation (4.12). ■

Lemma 4.1.5. *Let (u,θ,q,z) the solution of (3.1)-(3.8). Then, the functional*

$$K_3(t) = -\tau_0\int_\Omega q(t)\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x,$$

satisfies the estimate, for all $t \geq 0$

$$K_3'(t) \leq \|u_t(t)\|_2^2+\frac{C_\alpha+C_9}{4a_0}(k\diamond u)(t)+C_{10}\int_\Omega\rho(x)|\theta(t)|^2\mathrm{d}x+C_{11}\|q(t)\|_2^2, \tag{4.19}$$

where

$$C_9 = \frac{\tau_0^2 k_1}{2(\tau_0\alpha-1)^2}, \quad C_{10} = \frac{2\|\kappa\|_\infty\|\rho\|_\infty}{\rho_0^2}, \quad C_{11} = \frac{1}{4}\tau_0^2(1-\ell)^2+4(\tau_0\alpha-1)^2C_*^2.$$

Proof. By differentiating K_3, using (3.3), Green's formula, and recalling the boundary conditions, we obtain

$$\begin{aligned} K_3'(t) \;=\; & -\tau_0\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega q(t)u_t(t)\mathrm{d}x-\int_\Omega\theta(t)\int_0^t g(t-s)(\kappa\cdot\nabla)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\ & -\int_\Omega q(t)\int_0^t(\tau_0 g'(t-s)-g(t-s))(u(t)-u(s))\mathrm{d}s\mathrm{d}x. \end{aligned}$$

Applying the similar procedure of (4.9) and (4.10) to get, for $\delta, \delta_1, \delta_2 > 0$

$$\begin{aligned} K_3'(t) &\leq \delta_1\|u_t(t)\|_2^2 + \frac{C_\alpha}{4a_0}\left(\frac{\|\kappa\|_\infty}{\delta} + \frac{2(\tau_0\alpha-1)^2C_*^2}{\delta_2}\right)(k\diamond u)(t) \\ &\quad + \frac{2\tau_0^2k_1C_*^2}{4\delta_2 a_0}(k\diamond u)(t) + \delta\frac{\|\rho\|_\infty}{\rho_0^2}\int_\Omega \rho(x)|\theta(t)|^2\mathrm{d}x \\ &\quad + \left(\frac{\tau_0^2(1-\ell)^2}{4\delta_1} + \delta_2\right)\|q(t)\|_2^2. \end{aligned}$$

Let us choose $\delta = 2\|\kappa\|_\infty$, $\delta_1 = 1$, and $\delta_2 = 4(\tau_0\alpha-1)^2C_*^2$. Then, we find (4.19). ■

Next, we use the functional

$$K_4(t) = \int_0^t p(t-s)a(u(s),u(s))\mathrm{d}s, \quad \forall t \geq 0,$$

where $p(t) = \int_t^{+\infty} g(s)\mathrm{d}s$.

Lemma 4.1.6. *Assume that (1.3) and (1.4) hold. The functional K_4 satisfies, along the solution of (3.1)-(3.8), the estimate*

$$K_4'(t) \leq -\frac{1}{2}(g\diamond u)(t) + \varrho(1-\ell)a(u(t),u(t)). \tag{4.20}$$

Proof. By Young's inequality and the fact $p'(t) = -g(t)$, we see that

$$\begin{aligned} K_4'(t) &= p(0)a(u(t),u(t)) - \int_0^t g(t-s)a(u(s),u(s))\mathrm{d}s \\ &= -(g\diamond u)(t) + 2\int_0^t g(t-s)a(u(t)-u(s),u(t))\mathrm{d}s + p(t)a(u(t),u(t)). \end{aligned}$$

But

$$\int_0^t g(t-s)a(u(t)-u(s),u(t))\mathrm{d}s \leq \frac{a_1 n}{a_0^2}(1-\ell)a(u(t),u(t)) + \frac{\int_0^t g(s)\mathrm{d}s}{4(1-\ell)}(g\diamond u)(t).$$

Then, as $p(t) \leq p(0) = (1-\ell)$ and $\int_0^t g(s)\mathrm{d}s \leq (1-\ell)$, we get (4.20). ■

Lemma 4.1.7. *For each ϵ and M large enough while N so large that there exist two positive constants α_1 and α_2 such that, for all $t \geq 0$*

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t),$$

and the functional $\mathcal{L}$ defined by (4.1) satisfies

$$\begin{aligned} \mathcal{L}'(t) &\leq -\|u_t(t)\|_2^2 - (\varrho+1)(1-\ell)a(u(t),u(t)) + \frac{1}{4}(g\diamond u)(t) \\ &\quad -\epsilon\int_{\Gamma_1} h(x)m(x)|z(t)|^2\mathrm{d}\Gamma - (C_2\epsilon + C_7M + C_{10})\int_\Omega \rho(x)|\theta(t)|^2\mathrm{d}x \\ &\quad -\tau_0\|q(t)\|_2^2 + cNg(t)\|\nabla u_0\|_2^2. \end{aligned} \tag{4.21}$$

Proof. First, we are going to estimate each terms of the following

$$|\mathcal{L}(t) - NE(t)| \leq \epsilon|K_1(t)| + M|K_2(t)| + |K_3(t)|. \tag{4.22}$$

It follows from (1.3), Cauchy-Schwarz's, Young's, and Poincaré's inequalities that

$$\left|\int_\Omega u_t(t)u(t)\mathrm{d}x\right| \le \frac{1}{2}\|u_t(t)\|_2^2 + \frac{C_*^2}{2a_0}a(u(t),u(t)),$$

and, by using (1.7)

$$\left|\int_{\Gamma_1} h(x)u(t)z(t)\mathrm{d}\Gamma\right| \le \frac{\|h\|_\infty\|m\|_\infty}{2m_0^2}\int_{\Gamma_1} h(x)m(x)|z(t)|^2\mathrm{d}\Gamma + \frac{\bar{C}_*^2}{2a_0}a(u(t),u(t)).$$

Hence, we get

$$\begin{aligned} |K_1(t)| &\le \frac{1}{2}\|u_t(t)\|_2^2 + \frac{C_*^2+\bar{C}_*^2}{2a_0}a(u(t),u(t)) \\ &\quad + \frac{1}{2m_0}\left(\frac{\|h\|_\infty\|m\|_\infty}{m_0} + \|f\|_\infty\right)\int_{\Gamma_1} h(x)m(x)|z(t)|^2\mathrm{d}\Gamma, \end{aligned} \tag{4.23}$$

$$\begin{aligned} |K_2(t)| &\le \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\int_\Omega\left(\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\right)^2\mathrm{d}x \\ &\le \frac{1}{2}\|u_t(t)\|_2^2 + \frac{C_*^2(1-\ell)}{2a_0}(g\diamond u)(t), \end{aligned} \tag{4.24}$$

and

$$|K_3(t)| \le \frac{\tau_0^2}{2}\|q(t)\|_2^2 + \frac{C_*^2(1-\ell)}{2a_0}(g\diamond u)(t). \tag{4.25}$$

Inserting (4.23)-(4.25) into (4.22), we obtain

$$\begin{aligned} |\mathcal{L}(t) - NE(t)| &\le \frac{(\epsilon+M)}{2}\|u_t(t)\|_2^2 + \frac{\epsilon(C_*^2+\bar{C}_*^2)}{2a_0}a(u(t),u(t)) \\ &\quad + \frac{C_*^2(1-\ell)}{2a_0}(M+1)(g\diamond u)(t) + \frac{\tau_0^2}{2}\|q(t)\|_2^2 \\ &\quad + \epsilon\left(\frac{\|h\|_\infty\|m\|_\infty}{2m_0^2} + \frac{\|f\|_\infty}{2m_0}\right)\int_{\Gamma_1} h(x)m(x)|z(t)|^2\mathrm{d}\Gamma \\ &\le CE_1(t). \end{aligned}$$

where C is a positive constant. By choosing $N>0$ so large, we conclude that $\mathcal{L}\sim E$.

We turn now to proving (4.21). Let $g_1 = \int_0^{t_0} g(s)\mathrm{d}s$. By combining (4.1), (4.2), (4.3), (4.6), (4.12), and (4.19), taking $\delta = la_0/(4a_1M)$, we obtain

$$\begin{aligned} \mathcal{L}'(t) &\le -\left(g_1M - \frac{la_0}{4a_1} - \epsilon - 1\right)\|u_t(t)\|_2^2 - \left(\frac{\ell}{2}\epsilon - \frac{\ell}{2}\right)a(u(t),u(t)) \\ &\quad + \frac{\alpha}{2}N(g\diamond u)(t) + cNg(t)\|\nabla u_0\|_2^2 - N\|q_t(t)\|_2^2 \\ &\quad - \left(\frac{N}{2} - \frac{C_\alpha}{4a_0}(C_1\epsilon + C_5M + 1) - \frac{1}{4a_0}(C_6M + C_9)\right)(k\diamond u)(t) \\ &\quad - (N - C_3\epsilon - C_8M)\int_{\Gamma_1} h(x)f(x)|z_t(t)|^2\mathrm{d}\Gamma - (N - C_{11})\|q(t)\|_2^2 \\ &\quad - (C_2\epsilon + C_7M + C_{10})\int_\Omega \rho(x)|\theta(t)|^2\mathrm{d}x - \epsilon\int_{\Gamma_1} h(x)m(x)|z(t)|^2\mathrm{d}\Gamma \\ &\quad + 2(C_2\epsilon + C_7M + C_{10})\int_\Omega \rho(x)|\theta(t)|^2\mathrm{d}x. \end{aligned}$$

Then, we can estimate (3.3) to determine

$$2\int_{\Omega}\rho(x)|\theta(t)|^2\mathrm{d}x \leq C_{12}\tau_0^2\|q_t(t)\|_2^2 + C_{12}\|q(t)\|_2^2,$$

where $C_{12} = \frac{4\|\rho\|_\infty \bar{C}_*^2}{\kappa_0^2}$ and $\kappa_0 = \min\limits_{1\leq i\leq n}|\kappa_i|$. Consequently,

$$\begin{aligned}
\mathcal{L}'(t) \quad \leq \quad & \frac{\alpha}{2}N(g\diamond u)(t) - \left(\frac{\ell}{2}\epsilon - \frac{\ell}{2}\right)a(u(t),u(t)) - \left(g_1M - \frac{la_0}{4a_1} - \epsilon - 1\right)\|u_t(t)\|_2^2 \\
& - \left(\frac{N}{2} - \frac{C_\alpha}{4a_0}\left(C_1\epsilon + C_5M + 1\right) - \frac{1}{4a_0}\left(C_6M + C_9\right)\right)(k\diamond u)(t) \\
& - \left(C_2\epsilon + C_7M + C_{10}\right)\int_{\Omega}\rho(x)|\theta(t)|^2\mathrm{d}x - \epsilon\int_{\Gamma_1}h(x)m(x)|z(t)|^2\mathrm{d}\Gamma \\
& - \left(N - C_3\epsilon - C_8M\right)\int_{\Gamma_1}h(x)f(x)|z_t(t)|^2\mathrm{d}\Gamma + cNg(t)\|\nabla u_0\|_2^2 \\
& - \left(N - C_{12}\tau_0^2\left(C_2\epsilon + C_7M + C_{10} + 1\right)\right)\|q_t(t)\|_2^2 \\
& - \left(N - C_{11} - C_{12}C_{10} - C_{12}\left(C_2\epsilon + C_7M + 1\right)\right)\|q(t)\|_2^2.
\end{aligned}$$

At this point, we choose ϵ large enough so that

$$\frac{\ell}{2}\epsilon - \frac{\ell}{2} > (\varrho + 1)(1-\ell), \tag{4.26}$$

and therefore M large enough such that

$$g_1M - \frac{la_0}{4a_1} - \epsilon > 2.$$

We arrive to choose N so large satisfying

$$\begin{aligned}
& N - \frac{1}{2a_0}\left(C_6M + C_9\right) > 0, \\
& N - C_{12}\tau_0^2\left(C_2\epsilon + C_7M + C_{10} + 1\right) > 0, \\
& N - C_3\epsilon - C_8M > 0, \\
& N - C_{12}\left(C_2\epsilon + C_7M + 1\right) - C_{11} - C_{12}C_{10} > \tau_0.
\end{aligned}$$

From Remark 2.1.1, there is $0 < \alpha_0 < 1$ such that if $\alpha < \alpha_0$ then

$$\alpha C_\alpha < \frac{1}{4(C_1\epsilon + C_5M + 1)} \qquad \text{and} \qquad \alpha = \frac{1}{2N} < \alpha_0,$$

which means

$$\frac{N}{2} - \frac{C_\alpha}{4a_0}(C_1\epsilon + C_5M + 1) - \frac{1}{4a_0}(C_6M + C_9) > 0.$$

■

4.2 Stability result

This section is devoted to giving the stability result where the first-order energy solution decays optimality and generality.

For the purpose of (**H.4**), we can reproduce the following lemma

Lemma 4.2.1. *There exists a constant* t_0 *which (2.21) satisfying, for all* $t \leq t_0$

$$g'(t) \leq -\beta g(t), \tag{4.27}$$

where β *is a possitive constant.*

Proof. From (1.4) and (**H.4**), we can easily deduce that there is t_0 large enough while $g(t_0) = r$. Hence, for all $t \leq t_0$

$$\begin{cases} 0 < g(t_0) \leq g(t) \leq g(0); \\ 0 < \xi(t_0) \leq \xi(t) \leq \xi(0); \end{cases}$$

which implies that there are two positive constants ϵ_1 and ϵ_2 such that

$$\epsilon_1 \leq \xi(t)\mathcal{H}(g(t)) \leq \epsilon_2.$$

Therefore,

$$\exists \beta = \frac{\epsilon_1}{g(0)} > 0 \quad \text{such that} \quad g'(t) \leq -\beta g(t), \quad \forall t \leq t_0.$$

■

Our main stability result is the following

Theorem 4.2.2. *Let* (u, θ, q, z) *the solution of (3.1)-(3.8). Assume that (**H.1**)-(**H.4**) hold. Then there exist three positive constants* $k_i, i = 1, 2, 3$; *such that, the first-order energy functional satisfies, for all* $t \geq t_0$

$$E_1(t) \leq k_2 e^{-k_1 \int_{t_0}^t \xi(s)\mathrm{d}s} + k_3 \|\nabla u_0\|_2^2 \int_{t_0}^t \xi(s) g(s) e^{-k_1 \int_s^t \xi(\tau)\mathrm{d}\tau} \mathrm{d}s, \tag{4.28}$$

if $\mathcal{H}$ *is linear. Moreover, if* $\mathcal{H}$ *is nonlinear, then*

$$E_1(t) \leq \mathcal{H}_3(t) \left(k_2 + k_3 \|\nabla u_0\|_2^2 \int_{t_0}^t \frac{\xi(s)\mathcal{H}(g(s))}{\mathcal{H}_3(s)} \mathrm{d}s \right), \tag{4.29}$$

for all $t \geq t_0$. *Here,* $\mathcal{H}_3(t) = \mathcal{H}_1^{-1}\left(k_1 \int_{t_0}^t \xi(s)\mathrm{d}s\right)$.

Remark 4.2.3. In case $\displaystyle\int_0^{+\infty} \xi(s)\mathrm{d}s = +\infty$; Theorem 4.2.2 ensures

$$\lim_{t \to +\infty} E_1(t) = 0.$$

Corollary 4.2.4 (Arbitrary decay). *Assume that* $\mathcal{H}(s) = s^p$, $1 < p < 2$. *By simple calculations in (4.29), we see that the decay rate of* $E_1(t)$ *is given by, for some* $k_1, k_2 > 0$

$$E_1(t) \leq \frac{\left(k_1 + k_2 \|\nabla u_0\|_2^2 \int_0^t \xi(s) g^p(s) \left(1 + \int_0^s \xi(\tau)\mathrm{d}\tau\right)^{1/(p-1)} \mathrm{d}s\right)}{\left(1 + \int_0^t \xi(s)\mathrm{d}s\right)^{1/(p-1)}}.$$

Proof of Theorem 4.2.2. We start using (4.2) and (4.27) to find, for all $t \geq t_0$

$$\int_0^{t_0} g(s)a(u(t)-u(t-s),u(t)-u(t-s))\mathrm{d}s \leq \frac{-1}{\beta}(g' \diamond u)(t) \leq -c_1 E_1'(t), \tag{4.30}$$

where c_1 is a positive constant. Inserting (4.30) into (4.21), we get, for all $t \geq t_0$

$$\mathcal{F}'(t) \leq -\beta_1 E_1(t) + \beta_2 \int_{t_0}^{t} g(s)a(u(t)-u(t-s),u(t)-u(t-s))\mathrm{d}s + \beta_3 g(t)\|\nabla u_0\|_2^2, \tag{4.31}$$

where $\mathcal{F} = (\mathcal{L} + c_1 E_1) \sim E$ and $\beta_i, i = 1,2,3;$ are positive constants. Here $\beta_1 = 2\min\{1, (\varrho+1), \epsilon\} \geq 1$ because of (4.26).

Now, we study the stability under suitable assumption of $\mathcal{H}$.

Case 1: $\mathcal{H}$ is linear. Multiplying (4.31) by $\xi(t)$, using (4.2) and (2.21), we have, for some t_0 small enough

$$\begin{aligned}
\xi(t)\mathcal{F}'(t) &\leq -\beta_1\xi(t)E_1(t) + \beta_2\xi(t)\int_{t_0}^{t} g(s)a(u(t)-u(t-s),u(t)-u(t-s))\mathrm{d}s \\
&\quad +\beta_3\xi(t)g(t)\|\nabla u_0\|_2^2 \\
&\leq -\beta_1\xi(t)E_1(t) + \beta_2\int_{t_0}^{t} \xi(s)g(s)a(u(t)-u(t-s),u(t)-u(t-s))\mathrm{d}s \\
&\quad +\beta_3\xi(t)g(t)\|\nabla u_0\|_2^2 \\
&\leq -\beta_1\xi(t)E_1(t) - \beta_2(g' \diamond u)(t) + \beta_3\xi(t)g(t)\|\nabla u_0\|_2^2 \\
&\leq -\beta_1\xi(t)E_1(t) - \beta_2 E_1'(t) + \beta_3\xi(t)g(t)\|\nabla u_0\|_2^2.
\end{aligned}$$

Since ξ is a positive nonincreasing function, then

$$(\xi\mathcal{F} + \beta_2 E_1)'(t) \leq -\xi(t)E_1(t) + \beta_3\xi(t)g(t)\|\nabla u_0\|_2^2. \tag{4.32}$$

Using the fact that $(\xi\mathcal{F} + \beta_2 E_1) \sim E$ and (4.5), hence, (4.32) yields a linear first-order ordinary differential equation. Consequently, we conclude (4.28).

Case 2: $\mathcal{H}$ is nonlinear. First, we use (4.21) and (4.20) to deduce that

$$\mathcal{L}_1(t) = \mathcal{L}(t) + K_4(t),$$

is nondecreasing function satisfying, for all $t \geq t_0$

$$\mathcal{L}_1'(t) \leq -bE_1(t) + \beta_3 g(t)\|\nabla u_0\|_2^2,$$

where b is a positive constant. This implies that

$$\begin{aligned}
b\int_{t_0}^{t} E_1(s)\mathrm{d}s &\leq \mathcal{L}_1(t_0) - \mathcal{L}_1(t) + \beta_3\|\nabla u_0\|_2^2 \int_{t_0}^{t} g(s)\mathrm{d}s \\
&\leq \mathcal{L}_1(t_0) + \beta_3\|\nabla u_0\|_2^2 \int_0^{\infty} g(s)\mathrm{d}s.
\end{aligned}$$

Therefore,

$$\int_0^{\infty} E_1(s)\mathrm{d}s < \infty. \tag{4.33}$$

We define the function

$$I(t) = c_2\int_{t_0}^{t} a(u(t)-u(t-s),u(t)-u(t-s))\mathrm{d}s, \quad \forall t \geq t_0.$$

Note that, for $c_2 > 0$ and for all $t \geq t_0$, $I(t) > 0$. Otherwise, (4.31) gives (4.28).
From (4.2) and (4.33), we have, for $C > 0$

$$\begin{aligned}\int_{t_0}^{t} a(u(t)-u(t-s), u(t)-u(t-s))\mathrm{d}s &\leq 2C\int_{t_0}^{t}\left(\|\nabla u(t)\|_2^2 + \|\nabla u(t-s)\|_2^2\right)\mathrm{d}s\\ &\leq \frac{8C}{a_0(1-\ell)}\int_{t_0}^{t} E_1(t)\mathrm{d}s < \infty. \end{aligned} \tag{4.34}$$

Then, (4.34) allows for a constant $0 < c_2 < 1$ chosen so that, for all $t \geq t_0$

$$0 < I(t) < 1. \tag{4.35}$$

We define another functional

$$\lambda(t) = -\int_{t_0}^{t} g'(s)a(u(t)-u(t-s), u(t)-u(t-s))\mathrm{d}s, \quad \forall t \geq t_0.$$

Using (4.2), we observe that, for $c_3 > 0$

$$\lambda(t) \leq -c_3 E_1'(t). \tag{4.36}$$

The use of (2.22) in the fact (2.23), hypothesis (**H.4**), (4.35), and Jensen's inequality (A.2) leads to

$$\begin{aligned}\lambda(t) &= \frac{1}{c_2 I(t)}\int_{t_0}^{t} I(t)(-g'(s))c_2 a(u(t)-u(t-s), u(t)-u(t-s))\mathrm{d}s\\ &\geq \frac{1}{c_2 I(t)}\int_{t_0}^{t} I(t)\xi(s)\mathcal{H}(g(s))c_2 a(u(t)-u(t-s), u(t)-u(t-s))\mathrm{d}s\\ &\geq \frac{\xi(t)}{c_2 I(t)}\int_{t_0}^{t} \mathcal{H}(I(t)g(s))c_2 a(u(t)-u(t-s), u(t)-u(t-s))\mathrm{d}s\\ &\geq \frac{\xi(t)}{c_2}\overline{\mathcal{H}}\left(c_2\int_{t_0}^{t} g(s)a(u(t)-u(t-s), u(t)-u(t-s))\mathrm{d}s\right).\end{aligned}$$

This implies that

$$\int_{t_0}^{t} g(s)a(u(t)-u(t-s), u(t)-u(t-s))\mathrm{d}s \leq \frac{1}{c_2}\overline{\mathcal{H}}^{-1}\left(\frac{c_2\lambda(t)}{\xi(t)}\right).$$

Thus, (4.31) becomes

$$\mathcal{F}'(t) \leq -\beta_1 E_1(t) + \frac{\beta_2}{c_2}\overline{\mathcal{H}}^{-1}\left(\frac{c_2\lambda(t)}{\xi(t)}\right) + \beta_3 g(t)\|\nabla u_0\|_2^2. \tag{4.37}$$

Now, for $\varepsilon_0 > 0$ and $c_4 > 0$, we define the functional

$$\mathcal{F}_1(t) = \overline{\mathcal{H}}'\left(\varepsilon_0\frac{E_1(t)}{E_1(0)}\right)\mathcal{F}(t) + c_4 E_1(t),$$

Using (4.37) and the fact that $E_1'<0,\ \overline{\mathcal{H}}'>0,\ \overline{\mathcal{H}}''>0$, we conclude that $\mathcal{F}_1\sim E$ and

$$\begin{aligned}\mathcal{F}_1'(t) &= \varepsilon_0\frac{E_1'(t)}{E_1(0)}\overline{\mathcal{H}}''\left(\varepsilon_0\frac{E_1(t)}{E_1(0)}\right)\mathcal{F}(t)+\overline{\mathcal{H}}'\left(\varepsilon_0\frac{E_1(t)}{E_1(0)}\right)\mathcal{F}'(t)+c_4E_1'(t)\\ &\le -\beta_1E_1(t)\overline{\mathcal{H}}'\left(\varepsilon_0\frac{E_1(t)}{E_1(0)}\right)+\frac{\beta_2}{c_2}\overline{\mathcal{H}}'\left(\varepsilon_0\frac{E_1(t)}{E_1(0)}\right)\overline{\mathcal{H}}^{-1}\left(\frac{c_2\lambda(t)}{\xi(t)}\right)\\ &\quad +\beta_3g(t)\|\nabla u_0\|_2^2\overline{\mathcal{H}}'\left(\varepsilon_0\frac{E_1(t)}{E_1(0)}\right)+c_4E_1'(t).\end{aligned}\tag{4.38}$$

In view of (2.24) and (2.25) with $A=\overline{\mathcal{H}}'\left(\varepsilon_0\frac{E_1(t)}{E_1(0)}\right)$ and $B=\overline{\mathcal{H}}^{-1}\left(\frac{c_2\lambda(t)}{\xi(t)}\right)$, (4.38) gives

$$\begin{aligned}\mathcal{F}_1'(t) &\le -\left(\beta_1E_1(0)-\varepsilon_0\frac{\beta_2}{c_2}\right)\frac{E_1(t)}{E_1(0)}\overline{\mathcal{H}}'\left(\varepsilon_0\frac{E_1(t)}{E_1(0)}\right)+\beta_2\frac{\lambda(t)}{\xi(t)}\\ &\quad +\beta_3g(t)\|\nabla u_0\|_2^2\overline{\mathcal{H}}'\left(\varepsilon_0\frac{E_1(t)}{E_1(0)}\right)+c_4E_1'(t).\end{aligned}$$

As previously way with $A=\overline{\mathcal{H}}'\left(\varepsilon_0\frac{E_1(t)}{E_1(0)}\right)$ and $B=g(t)$, we obtain

$$\begin{aligned}\mathcal{F}_1'(t) &\le -\left(\beta_1E_1(0)-\varepsilon_0\left(\frac{\beta_2}{c_2}+\beta_3\|\nabla u_0\|_2^2\right)\right)\frac{E_1(t)}{E_1(0)}\overline{\mathcal{H}}'\left(\varepsilon_0\frac{E_1(t)}{E_1(0)}\right)\\ &\quad +\beta_2\frac{\lambda(t)}{\xi(t)}+\beta_3\|\nabla u_0\|_2^2\overline{\mathcal{H}}(g(t))+c_4E_1'(t).\end{aligned}$$

Then we multiply by $\xi(t)$, using (4.36) and the fact that, as $\dfrac{\varepsilon_0E_1(t)}{E_1(0)}<r$, $\overline{\mathcal{H}}'\left(\varepsilon_0\frac{E_1(t)}{E_1(0)}\right)=\mathcal{H}'\left(\varepsilon_0\frac{E_1(t)}{E_1(0)}\right)$, we find that

$$\begin{aligned}\mathcal{F}_2'(t) &\le -\left(\beta_1E_1(0)-\varepsilon_0\left(\frac{\beta_2}{c_2}+\beta_3\|\nabla u_0\|_2^2\right)\right)\xi(t)\frac{E_1(t)}{E_1(0)}\mathcal{H}'\left(\varepsilon_0\frac{E_1(t)}{E_1(0)}\right)\\ &\quad +\beta_3\|\nabla u_0\|_2^2\xi(t)\mathcal{H}(g(t))-\left(\beta_2c_3-c_4\xi(0)\right)E_1'(t),\end{aligned}$$

where $\mathcal{F}_2=(\xi\mathcal{F}_1)\sim E$. Let us choose ε_0 and c_4 small enough such that

$$c_5=\left(\beta_1E_1(0)-\varepsilon_0\left(\frac{\beta_2}{c_2}+\beta_3\|\nabla u_0\|_2^2\right)\right)\ge E_1(0)>0$$

and

$$\beta_2c_3-c_4\xi(0)>0,$$

Then,

$$\mathcal{F}_2'(t)\le -c_5\xi(t)\mathcal{H}_2\left(\frac{E_1(t)}{E_1(0)}\right)+\beta_3\|\nabla u_0\|_2^2\xi(t)\mathcal{H}(g(t)).\tag{4.39}$$

Since $\mathcal{H}_2'(t)=\mathcal{H}'(\varepsilon_0t)+\varepsilon_0t\mathcal{H}''(\varepsilon_0t)$, and using the strict convexity of $\mathcal{H}$ on $(0,r]$, we find that $\mathcal{H}_2,\mathcal{H}_2'>0$ on $(0,1]$. Let

$$R(t)=\frac{E_1(t)}{E_1(0)}.$$

Using the fact that (4.5), thus, (4.39) yields

$$R'(t)\le -\frac{c_5}{E_1(0)}\xi(t)\mathcal{H}_2(R(t))+\frac{\beta_3}{E_1(0)}\|\nabla u_0\|_2^2\xi(t)\mathcal{H}(g(t)).\tag{4.40}$$

This is an example of the inhomogeneous first-order nonlinear differential equation. The general solution is written in the form

$$R(t) = R_h(t) + R_p(t),$$

where $R_h(t)$ is a solution of the associated nonlinear and homogeneous equation and $R_p(t)$ is a particular solution of (4.40). A simple integration over (t_0, t), using the fact that $\varepsilon_0 R_h(t_0) < r$, we obtain

$$\mathcal{H}_1(\varepsilon_0 R_h(t)) = \int_{\varepsilon_0 R_h(t)}^{\varepsilon_0 R_h(t_0)} \frac{\mathrm{d}s}{s\mathcal{H}'(s)} \geq \frac{c_5}{E_1(0)} \int_{t_0}^{t} \xi(s)\mathrm{d}s.$$

Then by the properties of $\mathcal{H}_1$

$$R_h(t) \leq \frac{1}{\varepsilon_0} \mathcal{H}_1^{-1} \left(\frac{c_5}{E_1(0)} \int_{t_0}^{t} \xi(s)\mathrm{d}s \right).$$

To determine the particular solution, we use the variation of constants method. So, we set

$$R_p(t) = C(t)\mathcal{H}_1^{-1} \left(\frac{c_5}{E_1(0)} \int_{t_0}^{t} \xi(s)\mathrm{d}s \right), \tag{4.41}$$

where $\varepsilon_0 C(t) \geq 1$, for all $t \geq t_0$. We differentiate using the product and chain rules to find

$$\begin{aligned} & C'(t)\mathcal{H}_1^{-1} \left(\frac{c_5}{E_1(0)} \int_{t_0}^{t} \xi(s)\mathrm{d}s \right) + C(t) \left[\mathcal{H}_1^{-1} \left(\frac{c_5}{E_1(0)} \int_{t_0}^{t} \xi(s)\mathrm{d}s \right) \right]' \\ \leq \quad & -\frac{c_5}{E_1(0)} \xi(t)\mathcal{H}_2(R_p(t)) + \frac{\beta_3}{E_1(0)} \|\nabla u_0\|_2^2 \xi(t)\mathcal{H}(g(t)). \end{aligned}$$

We can estimate from the fact $\xi > 0, \mathcal{H}' > 0, \mathcal{H}'' > 0$ and (4.41) that

$$\begin{aligned} C(t) \left[\mathcal{H}_1^{-1} \left(\frac{c_5}{E_1(0)} \int_{t_0}^{t} \xi(s)\mathrm{d}s \right) \right]' &= C(t)\xi(t) \left[\mathcal{H}_1' \left(\mathcal{H}_1^{-1} \left(\frac{c_5}{E_1(0)} \int_{t_0}^{t} \xi(s)\mathrm{d}s \right) \right) \right]^{-1} \\ &= -\xi(t) R_p(t) \mathcal{H}' \left(\mathcal{H}_1^{-1} \left(\frac{c_5}{E_1(0)} \int_{t_0}^{t} \xi(s)\mathrm{d}s \right) \right) \\ &\geq -\frac{c_5}{E_1(0)} \xi(t)\mathcal{H}_2(R_p(t)). \end{aligned}$$

Thus, (4.40) becomes

$$C'(t)\mathcal{H}_1^{-1} \left(\frac{c_5}{E_1(0)} \int_{t_0}^{t} \xi(s)\mathrm{d}s \right) \leq \frac{\beta_3}{E_1(0)} \|\nabla u_0\|_2^2 \xi(t)\mathcal{H}(g(t)),$$

We find $C(t)$ by integrating this equality over (t_0, t). Consequently,

$$\begin{aligned} E_1(t) \quad \leq \quad & \frac{E_1(0)}{\varepsilon_0} \mathcal{H}_1^{-1} \left(\frac{c_5}{E_1(0)} \int_{t_0}^{t} \xi(s)\mathrm{d}s \right). \\ & + \beta_3 \|\nabla u_0\|_2^2 \int_{t_0}^{t} \xi(s)\mathcal{H}(g(s))\mathcal{H}_1^{-1} \left(\frac{c_5}{E_1(0)} \int_{t_0}^{t} \xi(\tau)\mathrm{d}\tau \right) \\ & \left[\mathcal{H}_1^{-1} \left(\frac{c_5}{E_1(0)} \int_{t_0}^{s} \xi(\tau)\mathrm{d}\tau \right) \right]^{-1} \mathrm{d}s \end{aligned}$$

This finishes the proof. ∎

Part III

A Type III Thermoelastic Coupled System with Boundary Interaction Feedback in the Presence of Distributed Delay

Chapter 5

Existence and Stability Estimates of Energy Solution

The main objective of this chapter is to give the existence of solution, optimal and general stability estimates for a type III thermoelastic coupled system. The decay rates are improved explicitly by using the convexity of the relaxation function g and without imposing any restrictive growth assumption on the damping term. The proof is based on the construction of a suitable Lyapunov functional.

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5.1 Problem statement

Consider the type III thermoelastic coupled system of the form

$$\begin{cases} u_{tt}+\mathbf{A}u-\int_0^t g(t-s)\mathbf{A}u(s)\mathrm{d}s+\operatorname{div}(\sigma\theta)=0 & \text{in} \quad \Omega\times\mathbb{R}_+ \\ \theta_{tt}+\mathbf{B}\theta+\mathbf{B}\theta_t+\int_0^{+\infty}\mu(s)\theta_t(t-s)\mathrm{d}s+(\sigma\cdot\nabla)u_{tt}=0 & \text{in} \quad \Omega\times\mathbb{R}_+ \\ u=0 & \text{on} \quad \Gamma_0\times\mathbb{R}_+ \\ \frac{\partial u}{\partial\nu_{\mathrm{A}}}-\int_0^t g(t-s)\frac{\partial u}{\partial\nu_{\mathrm{A}}}(s)\mathrm{d}s+F(u_t)=hz_t & \text{on} \quad \Gamma_1\times\mathbb{R}_+ \\ u_t+fz_t+mz=0 & \text{on} \quad \Gamma_1\times\mathbb{R}_+ \\ \theta=0 & \text{on} \quad \Gamma\times\mathbb{R}_+ \\ \theta_t(x,-s)=\phi(x,s) & \text{for} \quad x\in\Omega, s\in\mathbb{R}_+ \\ u(0)=u_0,\ u_t(0)=u_1,\ \theta(0)=\theta_0,\ \theta_t(0)=\theta_1 & \text{in} \quad \Omega \\ z(0)=z_0 & \text{in} \quad \Gamma_1, \end{cases} \tag{5.1}$$

where the second integral term $\left(\int_0^{+\infty}\mu(s)\theta_t(t-s)\mathrm{d}s\right)$ represents the infinite distributed delay. The functions μ, F and ϕ are given.

For completeness of the results, we first reformulate our system and present some additional assumptions to contribute to the existence theorem result.

Let us take a new variable φ first introduced in [94]

$$\varphi(x,t)=\int_0^t\theta(x,s)\mathrm{d}s+\Phi(x), \quad \forall x\in\Omega,\ \forall t\geq 0,$$

where $\Phi\in\mathrm{H}_0^1(\Omega)$ solves

$$\begin{cases} \mathbf{B}\Phi=\theta_1+\mathbf{B}\theta_0+\int_0^{+\infty}\mu(s)\theta(-s)\mathrm{d}s+(\sigma\cdot\nabla)u_1 & \text{in} \quad \Omega \\ \Phi=0 & \text{on} \quad \Gamma. \end{cases}$$

As in [76], let us set

$$\omega(x,p,s,t)=\varphi_t(x,t-sp), \quad (x,p,s)\in\mathrm{Q}=\Omega\times(0,1)\times\mathbb{R}_+,\ t\geq 0.$$

Then our system (5.1) leads to

$$\begin{cases} u_{tt}+\mathbf{A}u-\int_0^t g(t-s)\mathbf{A}u(s)\mathrm{d}s+\operatorname{div}(\sigma\varphi_t)=0 & \text{in} \quad \Omega\times\mathbb{R}_+ \\ \varphi_{tt}+\mathbf{B}\varphi+\mathbf{B}\varphi_t+\int_0^{+\infty}\mu(s)\omega(x,1,s,t)\mathrm{d}s+(\sigma\cdot\nabla)u_t=0 & \text{in} \quad \Omega\times\mathbb{R}_+ \\ s\omega_t+\omega_p=0 & \text{in} \quad \mathrm{Q}\times\mathbb{R}_+ \\ fz_t+mz+u_t=0 & \text{on} \quad \Gamma_1\times\mathbb{R}_+ \\ u=0 & \text{on} \quad \Gamma_0\times\mathbb{R}_+ \\ \frac{\partial u}{\partial\nu_{\mathrm{A}}}-\int_0^t g(t-s)\frac{\partial u}{\partial\nu_{\mathrm{A}}}(s)\mathrm{d}s+F(u_t)=hz_t & \text{on} \quad \Gamma_1\times\mathbb{R}_+ \\ \varphi=0 & \text{on} \quad \Gamma\times\mathbb{R}_+ \\ \omega(x,p,s,0)=\omega_0(x,p,s) & \text{for} \quad (x,p,s)\in\mathrm{Q} \\ u(0)=u_0,\ u_t(0)=u_1,\ \varphi(0)=\Phi,\ \varphi_t(0)=\theta_0 & \text{in} \quad \Omega \\ z(0)=z_0 & \text{in} \quad \Gamma_1 \end{cases} \tag{5.2}$$

Remark 5.1.1. In the consequence of Remark 1.2.2, we deduce that the bilinear form $b(\cdot,\cdot) : \mathrm{H}_0^1(\Omega) \times \mathrm{H}_0^1(\Omega) \to \mathbb{R}$ is symmetric, continuous and coercive where the coefficients $b_{ij} \in \mathcal{C}^1(\overline{\Omega})$ satisfying, for a constant $b_0 > 0$

$$\sum_{i,j=1}^{n} b_{ij}(x)\zeta_i\zeta_j \geq b_0|\zeta|^2, \quad \forall x \in \overline{\Omega},\ \forall \zeta \in \mathbb{R}^n. \tag{5.3}$$

5.2 Additional assumptions

Here, we present some assumptions in addition to (**H.1**)-(**H.3**) and (**H.4**) defined on Sect. 1.2 and Sect. 2.2, respectively.

(**H.5**) We assume that $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded nonincreasing function satisfying

$$c_1 = b_0 - C_*^2 \int_0^{+\infty} \mu(s)\mathrm{d}s > 0. \tag{5.4}$$

(**H.6**) $F : \mathbb{R} \to \mathbb{R}$ is an increasing $\mathcal{C}^0$ function such that there exists a strictly increasing function $F_0 \in \mathcal{C}^1(\mathbb{R}_+)$ with $F_0(0) = 0$. Furthermore, there exist three constants $c_1', c_2', c_3' > 0$ such that

$$(F(s) - F(r))\,(s - r) \geq c_1'(s - r)^2 \quad \text{for all} \quad s \neq r, \tag{5.5}$$

$$c_2'|s| \leq |F(s)| \leq c_3'|s| \quad \text{for all} \quad |s| \geq \varepsilon, \tag{5.6}$$

$$F_0(|s|) \leq |F(s)| \leq F_0^{-1}(|s|) \quad \text{for all} \quad |s| \leq \varepsilon. \tag{5.7}$$

In addition, we assume that the function G, defined by $\mathrm{G}(s) = \sqrt{s}F_0(\sqrt{s})$, is a strictly convex $\mathcal{C}^2$ function on $(0, r_1]$; $(r_1 > 0)$. This hypothesis was first considered by [60].

We introduce the following functions

$$\mathrm{G}_1(t) = \int_t^{r_1} \frac{\mathrm{d}s}{s\mathrm{G}'(s)}, \quad \mathrm{G}_2(t) = t\mathrm{G}'(\varepsilon_1 t), \tag{5.8}$$

$$\mathcal{W} = \left(\mathrm{G}^{-1} + \mathcal{H}^{-1}\right)^{-1}, \quad \text{and} \quad \mathcal{W}_1(t) = t\mathcal{W}'(\varepsilon_2 t), \tag{5.9}$$

where ε_1 and ε_2 are positive constants to be fixed later.

Remark 5.2.1. According to the properties of $\mathcal{H}$ and G introduced in (**H.4**) and (**H.6**), respectively,

1. G_1 is strictly decreasing and convex on $(0, r_1]$, with $\lim_{t\to 0} \mathrm{G}_1(t) = +\infty$.
2. G_2 is convex increasing on $\mathbb{R}_+$ and defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$.
3. $\mathcal{W}$ and $\mathcal{W}_1$ are strictly convex and increasing on $(0, r_0]$ where $r_0 = \min\{r, r_1\} > 0$.

We state, in the following theorem, the global existence of solution for the system (5.2)

Theorem 5.2.2. *Let* $u_0 \in \mathrm{H}^2(\Omega) \cap \mathrm{H}_{\Gamma_0}^1(\Omega)$, $u_1 \in \mathrm{H}_{\Gamma_0}^1(\Omega)$, $\Phi \in \mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega)$, $\theta_0 \in \mathrm{H}_0^1(\Omega)$, $\omega_0 \in \mathrm{L}^2(\mathrm{Q})$ *and* $z_0 \in \mathrm{L}^2(\Gamma_1)$ *be given. Assume that* ***(H.1)-(H.3)****, (5.3), and* ***(H.5)-(H.6)*** *hold. Then, the system* (5.2) *has a unique regular global weak solution*

$$\begin{aligned}
&u \in \mathcal{C}^1(\mathbb{R}_+; \mathrm{H}_{\Gamma_0}^1(\Omega)) \cap \mathcal{C}^2(\mathbb{R}_+; \mathrm{L}^2(\Omega)), \quad \mathrm{A}\nabla u \in \mathrm{H}(\mathrm{div}, \Omega) \cap \mathrm{H}_{\Gamma_0}^1(\Omega);\\
&\varphi \in \mathcal{C}^1(\mathbb{R}_+; \mathrm{H}_0^1(\Omega)) \cap \mathcal{C}^2(\mathbb{R}_+; \mathrm{L}^2(\Omega)), \quad \mathrm{B}\nabla\varphi \in \mathrm{H}(\mathrm{div}, \Omega) \cap \mathrm{H}_0^1(\Omega);\\
&\omega \in \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{L}^2(\mathrm{Q}));\\
&z, z_t \in \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{L}^2(\Gamma_1)).
\end{aligned}$$

Proof. This theorem can be established by means of the Faedo-Galerkin method as the proof of Theorem 3.3.2. ■

5.3 Energy functional

Multiply $(5.2)_1$ by u_t and integrate over Ω. Using Green's formula and the boundary conditions, we obtain

$$\begin{aligned} 0 &= \int_\Omega u_{tt}(t)u_t(t)\mathrm{d}x + a(u(t),u_t(t)) - \int_0^t g(t-s)a(u(s),u_t(t))\mathrm{d}s \\ &\quad + \langle F(u_t(t)),u_t(t)\rangle_{\Gamma_1} - \langle hz_t(t),u_t(t)\rangle_{\Gamma_1} - \int_\Omega (\sigma\cdot\nabla)u_t(t)\varphi_t(t)\mathrm{d}x, \end{aligned} \tag{5.10}$$

and multiply $(5.2)_2$ by φ_t, we get

$$\begin{aligned} 0 &= \int_\Omega \varphi_{tt}(t)\varphi_t(t)\mathrm{d}x + b(\varphi(t),\varphi(t)) + b(\varphi(t),\varphi_t(t)) \\ &\quad + \int_0^{+\infty} \mu(s)\omega(x,1,s,t)\mathrm{d}s + \int_\Omega (\sigma\cdot\nabla)u_t(t)\varphi_t(t)\mathrm{d}x. \end{aligned} \tag{5.11}$$

Taking the inner product of $(5.2)_3$ with $s\mu(s)\omega$ in $\mathrm{L}^2(\mathrm{Q})$, we arrive at

$$\int_{\mathrm{Q}} s\mu(s)\omega_t(p,s,t)\omega(p,s,t)\mathrm{d}x\mathrm{d}p\mathrm{d}s + \int_{\mathrm{Q}} \mu(s)\omega_p(p,s,t)\omega(p,s,t)\mathrm{d}x\mathrm{d}p\mathrm{d}s = 0 \tag{5.12}$$

We notice from $(5.2)_4$ that

$$\begin{aligned} -\langle hz_t(t),u_t(t)\rangle_{\Gamma_1} &= \|h^{1/2}f^{1/2}z_t(t)\|_{2,\Gamma_1}^2 + \langle hm(x)z(t),z_t(t)\rangle_{\Gamma_1} \\ &= \|h^{1/2}f^{1/2}z_t(t)\|_{2,\Gamma_1}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h^{1/2}m^{1/2}z(t)\|_{2,\Gamma_1}^2. \end{aligned} \tag{5.13}$$

Adding the results (5.10)-(5.13). From (3.9), we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\|u_t(t)\|_2^2 + \left(1-\int_0^t g(s)\mathrm{d}s\right)a(u(t),u(t)) + (g\diamond u)(t) + \|\varphi_t(t)\|_2^2 \\ &\quad + b(\varphi(t),\varphi(t)) + \|h^{1/2}m^{1/2}z(t)\|_{2,\Gamma_1}^2 + \int_{\mathrm{Q}} s\mu(s)\omega^2(x,p,s,t)\mathrm{d}s\mathrm{d}p\mathrm{d}x\Bigg) \\ = {} & \frac{1}{2}(g'\diamond u)(t) - \frac{1}{2}g(t)a(u(t),u(t)) - \|h^{1/2}f^{1/2}z_t(t)\|_{2,\Gamma_1}^2 \\ &\quad - \langle F(u_t(t)),u_t(t)\rangle_{\Gamma_1} - \frac{1}{2}\int_\Omega\int_0^{+\infty}\mu(s)\left[\omega^2(x,1,s,t)-\omega^2(x,0,s,t)\right]\mathrm{d}s\mathrm{d}x \\ &\quad - b(\varphi_t(t),\varphi_t(t)) - \int_\Omega \varphi_t(t)\int_0^{+\infty}\mu(s)\omega(x,1,s,t)\mathrm{d}s\mathrm{d}x. \end{aligned} \tag{5.14}$$

Set the energy E of system (5.2), for all $t \geq 0$

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)a(u(t),u(t)) + \frac{1}{2}(g\diamond u)(t) \\ &\quad + \frac{1}{2}b(\varphi(t),\varphi(t)) + \frac{1}{2}\|\varphi_t(t)\|_2^2 + \frac{1}{2}\|h^{1/2}m^{1/2}z(t)\|_{2,\Gamma_1}^2 \\ &\quad + \frac{1}{2}\int_{\mathrm{Q}} s\mu(s)\omega^2(x,p,s,t)\mathrm{d}s\mathrm{d}p\mathrm{d}x. \end{aligned} \tag{5.15}$$

Lemma 5.3.1. *The energy functional (5.15), along the solution of* (5.2), *is a nonincreasing function and satisfies the estimate*

$$\begin{aligned} E'(t) &\leq \frac{1}{2}(g'\diamond u)(t) - \frac{1}{2}g(t)a(u(t),u(t)) - \|h^{1/2}f^{1/2}z_t(t)\|_{2,\Gamma_1}^2 \\ &\quad - \langle F(u_t(t)),u_t(t)\rangle_{\Gamma_1} - c_1\|\nabla\varphi_t(t)\|_2^2. \end{aligned} \tag{5.16}$$

Proof. The last two terms of the right side of (5.14) can be estimated as follows From (5.3), we have

$$-b(\varphi_t(t),\varphi_t(t)) \leq -b_0\|\nabla\varphi_t(t)\|_2^2. \tag{5.17}$$

Using Young's and Poincaré's inequalities, we obtain

$$\begin{aligned} &-\int_\Omega \varphi_t(t)\int_0^{+\infty}\mu(s)\omega(x,1,s,t)\mathrm{d}s\mathrm{d}x \\ \leq\ & \frac{C_*^2}{2}\left(\int_0^{+\infty}\mu(s)\mathrm{d}s\right)\|\nabla\varphi_t(t)\|_2^2 + \frac{1}{2}\int_\Omega\int_0^{+\infty}\mu(s)\omega^2(x,1,s,t)\mathrm{d}s\mathrm{d}x. \end{aligned} \tag{5.18}$$

Substituting (5.17)-(5.18) into (5.14), from (5.4) ,we get the estimate (5.16). ■

5.4 Modified Lyapunov functional

We now turn to construct modified Lyapunov functional $\mathcal{L}$, which is defined by

$$\mathcal{L}(t) = NE(t) + N_1K_1(t) + N_2K_2(t) + N_3K_3(t) + N_4K_4(t), \tag{5.19}$$

where $N_i,\ i = 1,2,3,4$; are positive constants to be chosen properly later and

$$K_1(t) = \int_\Omega u_t(t)u(t)\mathrm{d}x + \langle hz(t),u(t)\rangle_{\Gamma_1} + \frac{1}{2}\|h^{1/2}f^{1/2}z(t)\|_{2,\Gamma_1}^2, \tag{5.20}$$

$$K_2(t) = \int_\Omega \varphi_t(t)\varphi(t)\mathrm{d}x + \frac{1}{2}b(\varphi,\varphi) + \int_\Omega(\sigma(x)\cdot\nabla)u(t)\varphi(t)\mathrm{d}x, \tag{5.21}$$

$$K_3(t) = \int_{\mathrm{Q}} se^{-sp}\mu(s)\omega^2(x,p,s,t)\mathrm{d}p\mathrm{d}s\mathrm{d}x, \tag{5.22}$$

$$K_4(t) = -\int_\Omega u_t(t)\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x, \tag{5.23}$$

Lemma 5.4.1. *For N large enough, the inequality*

$$\alpha_1E(t) \leq \mathcal{L}(t) \leq \alpha_2E(t), \tag{5.24}$$

holds for two positive constants α_1 and α_2.

Proof. From (5.19), we have

$$|\mathcal{L}(t) - NE(t)| \leq N_1|K_1(t)| + N_2|K_2(t)| + N_3|K_3(t)| + N_4|K_4(t)|. \tag{5.25}$$

The terms in (5.25) are estimated as follows
As in (4.23), we have

$$\begin{aligned} |K_1(t)| \leq\ & \frac{1}{2}\|u_t(t)\|_2^2 + \frac{C_*^2+\bar{C}_*^2}{2a_0}a(u(t),u(t)) \\ & + \frac{1}{2m_0}\left(\frac{\|h\|_\infty\|m\|_\infty}{m_0} + \|f\|_\infty\right)\|h^{1/2}m^{1/2}z(t)\|_{2,\Gamma_1}^2. \end{aligned} \tag{5.26}$$

Using Cauchy-Schwarz's and Young's inequalities, we obtain

$$|K_2(t)| \leq \frac{1}{2}\|\varphi_t(t)\|_2^2 + \frac{C_*^2}{2a_0}a(u(t),u(t)) + \left(\frac{C_*^2}{2b_0} + \frac{\|\sigma\|_\infty}{2} + \frac{1}{2}\right)b(\varphi(t),\varphi(t)), \tag{5.27}$$

and

$$|K_3(t)| \leq \int_{\mathrm{Q}} s\mu(s)\omega^2(x,p,s,t)\mathrm{d}x\mathrm{d}p\mathrm{d}x. \tag{5.28}$$

It follows from (4.24) that

$$\begin{aligned} |K_4(t)| &\leq \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\right)^2 \mathrm{d}x \\ &\leq \frac{1}{2}\|u_t(t)\|_2^2 + \frac{C_*^2(1-\ell)}{2a_0}(g \diamond u)(t). \end{aligned} \tag{5.29}$$

Substituting (5.26)-(5.29) into (5.25), we obtain

$$\begin{aligned} |\mathcal{L}(t) - NE(t)| &\leq \frac{(N_1+N_4)}{2}\|u_t(t)\|_2^2 + \frac{(N_1+N_2)C_*^2 + N_1\bar{C}_*^2}{2a_0}a(u(t),u(t)) \\ &\quad + \frac{N_4C_*^2(1-\ell)}{2a_0}(g\diamond u)(t) + \frac{N_2}{2}\|\varphi_t(t)\|_2^2 \\ &\quad + N_2\left(\frac{C_*^2}{2b_0} + \frac{\|\sigma\|_\infty}{2} + \frac{1}{2}\right) b(\varphi(t),\varphi(t)) \\ &\quad + N_3\int_{\mathrm{Q}} s\mu(s)\omega^2(x,p,s,t)\mathrm{d}x\mathrm{d}p\mathrm{d}x \\ &\quad + N_1\left(\frac{\|h\|_\infty\|m\|_\infty}{2m_0^2} + \frac{\|f\|_\infty}{2m_0}\right)\|h^{1/2}m^{1/2}z(t)\|_{2,\Gamma_1}^2 \\ &\leq CE(t). \end{aligned}$$

where C is a positive constant. By choosing N so large, we obtain (5.24). ■

Lemma 5.4.2. *Let (u,φ,ω,z) be a solution of (5.2). Then, the functional K_1 defined by (5.20) satisfies the estimate, for any $0<\alpha<1$, for all $t\geq 0$*

$$\begin{aligned} K_1'(t) &\leq \|u_t(t)\|_2^2 - \frac{\ell}{2}a(u(t),u(t)) + \frac{C_1C_\alpha}{4a_0}(k\diamond u)(t) + C_2\int_{\Gamma_1} F^2(u_t(t))\mathrm{d}\Gamma \\ &\quad + C_3\|\nabla\varphi_t(t)\|_2^2 + C_4\|h^{1/2}f^{1/2}z_t(t)\|_{2,\Gamma_1}^2 - \|h^{1/2}m^{1/2}z(t)\|_{2,\Gamma_1}^2, \end{aligned} \tag{5.30}$$

where the constant C_1, and the constant C_α and the function k are defined in (4.7) and (2.4), respectively, and

$$C_2 = \frac{2\overline{C}_*^2}{a_0\ell}, \quad C_3 = 2\frac{C_*^4\|\mathrm{div}(\sigma)\|_\infty^2 + C_*^2\|\sigma\|^2}{a_0\ell}, \quad C_4 = \frac{6\overline{C}_*^2\|h\|_\infty\|f\|_\infty}{a_0\ell f_0^2}.$$

Proof. Direct computations, by using $(5.2)_1$ and $(5.2)_5$, gives

$$\begin{aligned} K_1'(t) &= \|u_t(t)\|_2^2 - a(u(t),u(t)) + \int_0^t g(t-s)a(u(s),u(t))\mathrm{d}s \\ &\quad - \langle F(u_t(t)),u(t)\rangle_{\Gamma_1} - \|h^{1/2}m^{1/2}z(t)\|_{2,\Gamma_1}^2 + 2\langle hz_t(t),u(t)\rangle_{\Gamma_1} \\ &\quad - \int_\Omega \mathrm{div}(\sigma(x)\varphi_t(t))u(t)\mathrm{d}x. \end{aligned} \tag{5.31}$$

As in (4.9), we have, for some constant $\delta > 0$

$$\begin{aligned} & \int_0^t g(t-s)a(u(s),u(t))\mathrm{d}s \\ \leq \quad & \left((1-\ell)+\delta\frac{a_1}{a_0}\right)a(u(t),u(t))+\frac{nC_\alpha}{4a_0\delta}(k\diamond u)(t). \end{aligned} \tag{5.32}$$

Using Cauchy-Schwarz's and Young's inequalities, using (1.6)-(1.7), we get, for $\delta_1 > 0$

$$\begin{aligned} \left|\langle hz_t(t),u(t)\rangle_{\Gamma_1}\right| \quad & \leq \quad \frac{\|h\|_\infty^{1/2}\|f\|_\infty^{1/2}}{f_0}\|h^{1/2}f^{1/2}z_t(t)\|_{2,\Gamma_1}\|u(t)\|_{2,\Gamma_1} \\ & \leq \quad \delta_1\frac{\overline{C}_*^2}{a_0}a(u(t),u(t))+\frac{\|h\|_\infty\|f\|_\infty}{4\delta_1 f_0^2}\|h^{1/2}f^{1/2}z_t(t)\|_{2,\Gamma_1}^2, \end{aligned} \tag{5.33}$$

for $\delta_2 > 0$

$$\left|\int_\Omega \mathrm{div}(\sigma(x)\varphi_t(t))u(t)\mathrm{d}x\right| \quad \leq \quad \frac{C_*^2\|\mathrm{div}(\sigma)\|_\infty^2+\|\sigma\|^2}{4\delta_2}\|\nabla\varphi_t(t)\|_2^2+\frac{\delta_2 C_*^2}{a_0}a(u(t),u(t)), \tag{5.34}$$

and for $\delta_3 > 0$

$$-\langle F(u_t(t)),u(t)\rangle_{\Gamma_1} \leq \frac{1}{4\delta_3}\int_{\Gamma_1}F^2(u_t(t))\mathrm{d}\Gamma+\frac{\delta_3\overline{C}_*^2}{a_0}a(u(t),u(t)). \tag{5.35}$$

Substituting (5.32)-(5.35) into (5.31), we find that

$$\begin{aligned} K_1'(t) \quad \leq \quad & \|u_t(t)\|_2^2+\left[-\ell+\delta\frac{a_1}{a_0}+2\delta_1\frac{\overline{C}_*^2}{a_0}+\frac{\delta_2 C_*^2}{a_0}+\frac{\delta_3\overline{C}_*^2}{a_0}\right]a(u(t),u(t)) \\ & +\frac{nC_\alpha}{4a_0\delta}(k\diamond u)(t)+\frac{C_*^2\|\mathrm{div}(\sigma)\|^2+\|\sigma\|^2}{4\delta_2}\|\nabla\varphi_t(t)\|_2^2-\|h^{1/2}m^{1/2}z(t)\|_{2,\Gamma_1}^2 \\ & +\frac{\|h\|_\infty\|f\|_\infty}{2\delta_1 f_0^2}\|h^{1/2}f^{1/2}z_t(t)\|_{2,\Gamma_1}^2+\frac{1}{4\delta_3}\int_{\Gamma_1}F^2(u_t(t))\mathrm{d}\Gamma. \end{aligned}$$

Let us choose

$$\delta=\frac{a_0\ell}{8a_1},\quad \delta_1=\frac{a_0\ell}{16\overline{C}_*^2},\quad \delta_2=\frac{a_0\ell}{8C_*^2}\quad \text{and}\quad \delta_3=\frac{a_0\ell}{8\overline{C}_*^2}.$$

Then we obtain (5.30). ■

Lemma 5.4.3. *Let (u,φ,ω,z) be a solution of (5.2). Then, the functional K_2 defined by (5.21) satisfies the estimate, for all $t \geq 0$*

$$\begin{aligned} K_2'(t) \quad \leq \quad & \frac{C_*^2}{2a_0}a(u(t),u(t))+C_5\|\nabla\varphi_t\|_2^2-\frac{1}{2}b(\varphi(t),\varphi(t)) \\ & +C_6\int_\Omega\int_0^{+\infty}\mu(s)\omega^2(x,1,s,t)\mathrm{d}s\mathrm{d}x, \end{aligned} \tag{5.36}$$

where $2C_5 = 2C_*^2+\|\sigma\|^2$ and $C_6 = \frac{C_*^2}{2b_0}\left(\int_0^{+\infty}\mu(s)\mathrm{d}s\right)$.

Proof. By exploiting $(5.2)_2$ and using Green's formula, we have

$$K_2'(t) \quad = \quad \|\varphi_t(t)\|_2^2-b(\varphi(t),\varphi(t))-\int_\Omega(\sigma(x)\cdot\nabla)\varphi_t(t)u(t)\mathrm{d}x$$

$$-\int_\Omega \varphi(t)\int_0^{+\infty}\mu(s)\omega(x,1,s,t)\mathrm{d}s\mathrm{d}x. \tag{5.37}$$

We can estimate the last two terms above for similar calculations in (5.32) in the following

$$-\int_\Omega (\sigma(x)\cdot\nabla)\varphi_t(t)u(t)\mathrm{d}x \leq \frac{\|\sigma\|^2}{2}\|\nabla\varphi_t\|_2^2+\frac{C_*^2}{2a_0}a(u(t),u(t)), \tag{5.38}$$

and for $\delta>0$

$$\begin{aligned} &-\int_\Omega \varphi(t)\int_0^{+\infty}\mu(s)\omega(x,1,s,t)\mathrm{d}s\mathrm{d}x \\ \leq\ & \frac{C_*^2}{4\delta b_0}b(\varphi(t),\varphi(t))+\delta\left(\int_0^{+\infty}\mu(s)\mathrm{d}s\right)\int_\Omega\int_0^{+\infty}\mu(s)\omega^2(x,1,s,t)\mathrm{d}s\mathrm{d}x. \end{aligned}\tag{5.39}$$

Let us choose $\delta=\dfrac{C_*^2}{2b_0}$. Inserting (5.38) and (5.39) into (5.37), we get (5.36). ■

Lemma 5.4.4. *Let (u,φ,ω,z) be a solution of* (5.2). *Then, the functional K_3 defined by* (5.22) *satisfies the estimate, for all $t\geq 0$*

$$K_3'(t)\leq C_7\|\nabla\varphi_t(t)\|_2^2-\int_\Omega\int_0^{+\infty}\mu(s)\omega^2(x,1,s,t)\mathrm{d}s\mathrm{d}x-\int_Q s\mu(s)\omega^2(x,p,s,t)\mathrm{d}p\mathrm{d}s\mathrm{d}x, \tag{5.40}$$

where $C_7=C_*^2\displaystyle\int_0^{+\infty}\mu(s)\mathrm{d}s$.

Proof. By differentiating K_3 with respect to t, we obtain

$$\begin{aligned} K_3'(t) &= -2\int_\Omega\int_0^{+\infty}\mu(s)\int_0^1 e^{-sp}\omega(p,s,t)\omega_p(p,s,t)\mathrm{d}p\mathrm{d}s\mathrm{d}x, \\ &= -\int_\Omega\int_0^{+\infty}\mu(s)\left(e^{-s}\omega^2(1,s,t)-\omega^2(0,t)+s\int_0^1 e^{-sp}\omega(p,s,t)\mathrm{d}p\right)\mathrm{d}s\mathrm{d}x, \\ &\leq C_7\|\nabla\varphi_t(t)\|_2^2-\int_\Omega\int_0^{+\infty}\mu(s)\omega^2(1,s,t)\mathrm{d}s\mathrm{d}x-\int_Q s\mu(s)\omega^2(p,s,t)\mathrm{d}p\mathrm{d}s\mathrm{d}x. \end{aligned}$$

■

Lemma 5.4.5. *Let (u,φ,ω,z) be a solution of* (5.2). *Then, the functional k_4 defined by* (5.23) *satisfies the estimate, for $\delta>0$, for all $t\geq 0$*

$$\begin{aligned} K_4'(t) \leq\ & -\left(\int_0^t g(s)\mathrm{d}s-\delta\right)\|u_t(t)\|_2^2+C_8(\delta)a(u(t),u(t)) \\ & +\left(\frac{C_\alpha C_9(\delta)+C_{10}(\delta)}{4a_0}\right)(k\diamond u)(t)+C_{11}\|\nabla\varphi_t(t)\|_2^2 \\ & +C_{12}\|h^{1/2}f^{1/2}z_t(t)\|_{2,\Gamma_1}^2+C_{13}\int_{\Gamma_1}F^2(u_t(t))\mathrm{d}\Gamma, \end{aligned}\tag{5.41}$$

where

$$C_8(\delta)=\delta\frac{a_1}{a_0},\quad C_9(\delta)=1+4\sqrt{na_1}+\frac{2\alpha^2C_*^2+n}{\delta},\quad C_{10}(\delta)=\frac{2k_1C_*^2}{\delta},$$

$$C_{11}=3C_*^2\|\sigma\|^2,\ C_{12}=\frac{3\bar{C}_*^2\|h\|_\infty\|f\|_\infty}{f_0^2},\quad C_{13}=3\bar{C}_*^2.$$

Proof. By exploiting $(3.1)_1$ and using Green's formula, we have

$$\begin{aligned} K_4'(t) &= \left(1-\int_0^t g(s)\mathrm{d}s\right)\int_\Omega \mathrm{A}\nabla u(t)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\mathrm{d}x \\ &\quad +\int_\Omega\int_0^t g(t-s)\mathrm{A}(\nabla u(t)-\nabla u(s))\mathrm{d}s\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\mathrm{d}x \\ &\quad -\left\langle hz_t(t),\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\right\rangle_{\Gamma_1}-\left(\int_0^t g(s)\mathrm{d}s\right)\|u_t(t)\|_2^2 \\ &\quad -\int_\Omega u_t(t)\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\ &\quad -\int_\Omega \sigma(x)\varphi_t(t).\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\mathrm{d}x \\ &\quad +\left\langle F(u_t(t)),\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\right\rangle_{\Gamma_1}. \end{aligned} \tag{5.42}$$

Using the similar calculations in (5.32), we obtain

$$\int_\Omega \mathrm{A}\nabla u(t)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\mathrm{d}x \le \delta\frac{a_1}{a_0}a(u(t),u(t))+\frac{nC_\alpha}{4a_0\delta}(k\diamond u)(t), \tag{5.43}$$

for $\delta>0$ and

$$\begin{aligned} &-\int_\Omega \sigma(x)\varphi_t(t).\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\mathrm{d}x \\ \le\ & \delta_1 C_*^2\|\sigma\|^2\|\nabla\varphi_t(t)\|_2^2+\frac{C_\alpha}{4a_0\delta_1}(k\diamond u)(t), \end{aligned} \tag{5.44}$$

for $\delta_1>0$. By repeating the same arguments of (5.32)-(5.33), we get, for $\delta_2>0$

$$\begin{aligned} &-\left\langle hz_t(t),\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\right\rangle_{\Gamma_1} \\ \le\ & \frac{\bar{C}_*^2C_\alpha}{4\delta_2 a_0}(k\diamond u)(t)+\frac{\delta_2\|h\|_\infty\|f\|_\infty}{f_0^2}\|h^{1/2}f^{1/2}z_t(t)\|_{2,\Gamma_1}^2. \end{aligned} \tag{5.45}$$

We arrive to estimate

$$\begin{aligned} &\int_\Omega\int_0^t g(t-s)\mathrm{A}(\nabla u(t)-\nabla u(s))\mathrm{d}s\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\mathrm{d}x \\ \le\ & \frac{\sqrt{a_1 n}C_\alpha}{a_0}(k\diamond u)(t). \end{aligned} \tag{5.46}$$

Using Cauchy-Schwarz's and Young's inequalities, we obtain, for $\delta_3>0$

$$-\int_\Omega u_t(t)\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \le \delta_3\|u_t(t)\|_2^2+\frac{\alpha^2[C_\alpha+k_1]C_*^2}{2\delta_3 a_0}(k\diamond u)(t), \tag{5.47}$$

and for $\delta_4>0$

$$\left\langle F(u_t(t)),\int_0^t g(t-s)(u(t)-u(s))\right\rangle_{\Gamma_1} \le \delta_4\int_{\Gamma_1}F^2(u_t(t))\mathrm{d}\Gamma+\frac{\bar{C}_*^2C_\alpha}{4\delta_4 a_0}(k\diamond u)(t). \tag{5.48}$$

Inserting (5.43)-(5.48) into (5.42), we obtain

$$\begin{aligned}
K_4'(t) \quad \leq \quad & -\left(\int_0^t g(s)\mathrm{d}s - \delta_4\right)\|u_t(t)\|_2^2 + \delta\frac{a_1}{a_0}a(u(t),u(t)) + \frac{2k_1C_*^2}{4\delta_3 a_0}(k\diamond u)(t) \\
& +\frac{C_\alpha}{4a_0}\left[\frac{n}{\delta} + \frac{1}{\delta_1} + \frac{\bar{C}_*^2}{\delta_2} + 4\sqrt{a_1 n} + \frac{2\alpha^2C_*^2}{\delta_3} + \frac{\bar{C}_*^2}{\delta_4}\right](k\diamond u)(t) \\
& +\delta_1 C_*^2\|\sigma\|^2\|\nabla\varphi_t(t)\|_2^2 + \delta_4\int_{\Gamma_1} F^2(u_t(t))\mathrm{d}\Gamma \\
& +\delta_2\frac{\|h\|_\infty\|f\|_\infty}{f_0^2}\|h^{1/2}f^{1/2}z_t(t)\|_{2,\Gamma_1}^2.
\end{aligned}$$

If we choose $\delta_1 = 3$ and $\delta_2 = \delta_4 = 3\bar{C}_*^2$, then we get (5.41). ■

Recall the predefined functional

$$K_5(t) = \int_0^t p(t-s)a(u(s),u(s))\mathrm{d}s,$$

where $p(t) = \int_t^{+\infty} g(s)\mathrm{d}s$.

Lemma 5.4.6. *Assume that* (1.3) *and* (1.4) *hold. The functional* K_5 *satisfies, along the solution of* (5.2), *the estimate*

$$K_5'(t) \leq -\frac{1}{2}(g\diamond u)(t) + \varrho(1-\ell)a(u(t),u(t)). \tag{5.49}$$

Proof. The proof obtained with the same approach that is given in (4.20). ■

Lemma 5.4.7. *For each* $i = 1,2,3,4;$ N_i *large enough while* N *so large that the functional* $\mathcal{L}$ *defined by* (5.19) *satisfies, for all* $t \geq 0$

$$\begin{aligned}
\mathcal{L}'(t) \quad \leq \quad & -\|u_t(t)\|_2^2 - (\varrho+1)(1-\ell)a(u(t),u(t)) + \frac{1}{4}(g\diamond u)(t) - \frac{N_2}{2}b(\varphi(t),\varphi(t)) \\
& -\frac{Nc_1}{2C_*^2}\|\varphi_t(t)\|_2^2 + (C_2N_1 + C_{13}N_4)\int_{\Gamma_1} F^2(u_t(t))\mathrm{d}\Gamma \\
& -N_1\|h^{1/2}m^{1/2}z(t)\|_{2,\Gamma_1}^2 - N_3\int_{\mathrm{Q}} s\mu(s)\omega^2(x,p,s,t)\mathrm{d}p\mathrm{d}s\mathrm{d}x. \qquad (5.50)
\end{aligned}$$

Proof. By combining (5.16), (5.19), (5.30), (5.36), and (5.40)-(5.41), taking $\delta = \ell a_0/(4a_1N_4)$ and $N_2 = \ell a_0/(2C_*^2)$, we obtain

$$\begin{aligned}
\mathcal{L}'(t) \quad \leq \quad & -\left(g_1N_4 - \frac{\ell a_0}{4a_1} - N_1\right)\|u_t(t)\|_2^2 - \left(\frac{\ell}{2}N_1 - \frac{\ell}{2}\right)a(u(t),u(t)) + \frac{\alpha N}{2}(g\diamond u)(t) \\
& -\frac{\ell a_0}{4C_*^2}b(\varphi(t),\varphi(t)) - \left(\frac{N}{2} - \frac{C_\alpha}{4a_0}(C_1N_1 + C_9N_4) - \frac{C_{10}N_4}{4a_0}\right)(k\diamond u)(t) \\
& -N_3\int_{\mathrm{Q}} s\mu(s)\omega^2(x,p,s,t)\mathrm{d}p\mathrm{d}s\mathrm{d}x + (C_2N_1 + C_{13}N_4)\int_{\Gamma_1} F^2(u_t(t))\mathrm{d}\Gamma \\
& -(N - C_4N_1 - C_{12}N_4)\|h^{1/2}f^{1/2}z_t(t)\|_{2,\Gamma_1}^2 - N_1\|h^{1/2}m^{1/2}z(t)\|_{2,\Gamma_1}^2 \\
& -\left(N_3 - \frac{C_6\ell a_0}{2C_*^2}\right)\int_\Omega\int_0^{+\infty}\mu(s)\omega^2(x,1,s,t)\mathrm{d}s\mathrm{d}x - \frac{Nc_1}{2C_*^2}\|\varphi_t(t)\|_2^2 \\
& -\left(\frac{c_1N}{2} - C_3N_1 - \frac{C_5la_0}{2C_*^2} - C_7N_3 - C_{11}N_4\right)\|\nabla\varphi_t(t)\|_2^2.
\end{aligned}$$

We choose N_1 large enough so that

$$\frac{\ell}{2}N_1 - \frac{\ell}{2} > (\varrho + 1)(1-\ell),$$

and therefore N_4 large enough such that

$$g_1 N_4 - \frac{\ell a_0}{4a_1} - N_1 > 1.$$

By choosing $N_3 > \dfrac{C_6 \ell a_0}{2C_*^2}$, we arrive to choose N so large that $N > N_0$ where

$$N_0 = \max\left\{\frac{C_{10}N_4}{2a_0}, C_4N_1 + C_{12}N_4, \frac{2}{c_1}\left(C_3N_1 + \frac{C_5\ell a_0}{2C_*^2} + C_7N_3 + C_{11}N_4\right)\right\}.$$

From Remark 2.1.1, there is $0 < \alpha_0 < 1$ such that if $\alpha < \alpha_0$, then

$$\alpha C_\alpha < \frac{1}{4(C_1N_1 + C_9N_4)} \quad \text{and} \quad \alpha = \frac{1}{2N} < \alpha_0,$$

which means

$$\frac{N}{2} - \frac{C_\alpha}{4a_0}[C_1N_1 + C_9N_4] - \frac{C_{10}N_4}{4a_0} > 0.$$

Thus (5.50) is proven. ■

5.5 Optimal and general stability

The optimal explicit and general decay results are established here. These theorems are consequently divided according to the nature of the function F_0 that is defined on (**H.6**). In Subsect. 5.5.1, we give the general decay theorem when F_0 is linear. In the case of F_0 being nonlinear, the general stability result is given in Subsect. 5.5.2.

5.5.1 First general theorem

Theorem 5.5.1. *Assume that (**H.1**)-(**H.6**) and (5.3) hold and F_0 is linear. Then, there exist two constants $k_1, k_2 > 0$ such that*

$$E(t) \leq k_1 \mathcal{H}_1^{-1}\left(k_2 \int_{t_0}^{t} \xi(s)\mathrm{d}s\right), \tag{5.51}$$

for all $t \geq t_0$.

Remark 5.5.2. The decay rate of $E(t)$ driven by (5.51) is optimal in the sense that it is consistent with the decay rate of $g(t)$ driven by (2.21). So, we have

$$\int_{g^{-1}(r)}^{t} \frac{-g'(s)}{\mathcal{H}(g(s))}\mathrm{d}s = \int_{g(t)}^{r} \frac{\mathrm{d}s}{\mathcal{H}(s)} = \mathcal{H}_0(g(t)) \geq \int_{g^{-1}(r)}^{t} \xi(s)\mathrm{d}s.$$

Note that $\mathcal{H}_0$ is strictly decreasing and convex on $(0, r]$, with $\lim_{t\to 0} \mathcal{H}_0(t) = +\infty$. Then,

$$g(t) \leq \mathcal{H}_0^{-1}\left(\int_{g^{-1}(r)}^{t} \xi(s)\mathrm{d}s\right), \quad \forall t \geq g^{-1}(r).$$

We notice by the properties of $\mathcal{H}$, $\mathcal{H}_0$, and $\mathcal{H}_1$ that

$$\mathcal{H}_1(t) = \int_{t}^{r} \frac{\mathrm{d}s}{s\mathcal{H}'(s)} \leq \mathcal{H}_0(t) = \int_{t}^{r} \frac{\mathrm{d}s}{\mathcal{H}(s)} = \mathcal{H}_0(t) \Rightarrow \mathcal{H}_1^{-1}(t) \leq \mathcal{H}_0^{-1}(t)$$

This shows that (5.51) provides the best decay rates expected under the very general assumption (**H.4**).

Remark 5.5.3. Since $\mathcal{H}_1^{-1}$ is decreasing. It is easy to deduce that we can start the integration inside at zero whereas if $k_1^* < k_1$ be chosen that $k_1^* \int_{t_0}^{t_1} \mathrm{d}s = k_1 \int_0^{t_1} \mathrm{d}s$, then,

$$E(t) \leq \mathcal{H}_1^{-1}\left(k_1^* \int_{0}^{t} \xi(s)\mathrm{d}s\right), \quad \forall t \geq t_1.$$

Proof of Theorem 5.5.1. From (4.27) and (5.16), we have

$$\int_0^{t_0} g(s)a(u(t) - u(t-s), u(t) - u(t-s))\mathrm{d}s \leq -c_2 E'(t),$$

where c_2 is a positive constant. It follows from (5.50) that, for all $t \geq t_0$

$$\begin{aligned} \mathcal{F}'(t) \quad \leq \quad & -\beta_1 E(t) + \beta_2 \int_{t_0}^{t} g(s)a(u(t) - u(t-s), u(t) - u(t-s))\mathrm{d}s \\ & + \beta_3 \int_{\Gamma_1} F^2(u_t(t))\mathrm{d}\Gamma, \end{aligned} \tag{5.52}$$

where $\beta_i, \ i = 1, 2, 3;$ are positive constants and $\mathcal{F} = (\mathcal{L} + c_2 E) \sim E$.

Under the assumption (2.21)of $\mathcal{H}$, we study the stability effect.

Case 1: $\mathcal{H}$ is linear. Multiply (5.52) by $\xi(t)$, using (2.21) and (5.16), for some t_0 small enough, we obtain

$$\begin{aligned}\xi(t)\mathcal{F}'(t) &\leq -\beta_1\xi(t)E(t)+\beta_2\int_{t_0}^{t}\xi(s)g(s)a(u(t)-u(t-s),u(t)-u(t-s))\mathrm{d}s\\ &\quad +\beta_3\xi(t)\int_{\Gamma_1}F^2(u_t(t))\mathrm{d}\Gamma\\ &\leq -\beta_1\xi(t)E(t)-\beta_2(g'\diamond u)(t)+\beta_3c_2'\xi(t)\left\langle F(u_t(t)),u_t(t)\right\rangle_{\Gamma_1}\\ &\leq -\beta_1\xi(t)E(t)-\beta_4E'(t),\end{aligned}$$

where $\beta_4=(\beta_2-\beta_3c_2'\xi(0))$. Since ξ is a positive nonincreasing function, then

$$(\xi\mathcal{F}+\beta_4E)'(t)\leq-\beta_1\xi(t)E(t).$$

Using the fact that $(\xi\mathcal{F}+\beta_4E)\sim E$, we conclude, for all $t\geq t_0$

$$E(t)\leq k_1e^{-k_2\int_{t_0}^{t}\xi(s)\mathrm{d}s}.$$

Case 2: $\mathcal{H}$ is nonlinear. First, we consider

$$\mathcal{L}_1(t)=\mathcal{L}(t)+K_5(t),$$

to be nonnegative, and it follows from (5.16), (5.49), and (5.50) that

$$\begin{aligned}\mathcal{L}_1'(t) &\leq -c_2E(t)+c^*c_2'\left\langle F(u_t(t)),u_t(t)\right\rangle_{\Gamma_1}\\ &\leq -c_2E(t)-c^*c_2'E'(t),\end{aligned}$$

where c_2 and c^* are positive constants. Therefore, for all $t\geq t_0$

$$c_2\int_{t_0}^{t}E(s)\mathrm{d}s\leq\mathcal{L}_2(t_0)-\mathcal{L}_2(t)\leq\mathcal{L}_2(t_0)<\infty,$$

where $\mathcal{L}_2=(\mathcal{L}_1+c^*c_2'E)\sim E$.

Next, we define the function

$$I(t)=c_3\int_{t_0}^{t}a(u(t)-u(t-s),u(t)-u(t-s))\mathrm{d}s.$$

Note that, for $c_3>0$ and for all $t\geq t_0$, $I(t)>0$. Otherwise, an exponential decay is concluded. After that, for a constant $C>0$, for all $t\geq t_0$, we have

$$\begin{aligned}\int_{t_0}^{t}a(u(t)-u(t-s),u(t)-u(t-s))\mathrm{d}s &\leq 2C\int_{t_0}^{t}\left(\|\nabla u(t)\|_2^2+\|\nabla u(t-s)\|_2^2\right)\mathrm{d}s\\ &\leq \frac{8C}{a_0(1-\ell)}\int_{t_0}^{t}E(t)\mathrm{d}s<\infty,\end{aligned}$$

and therefore for a constant $0<c_3<1$ chosen so that

$$0<I(t)<1. \tag{5.53}$$

We also define the functional

$$\lambda(t)=-\int_{t_0}^{t}g'(s)a(u(t)-u(t-s),u(t)-u(t-s))\mathrm{d}s,$$

for each t_0 small enough. By using (5.16), we observe that for a constant $c_4 > 0$,

$$\lambda(t) \leq -c_4 E'(t). \tag{5.54}$$

From the hypothesis (**H.4**), the using of (2.22), (5.53), and Jensen's inequality (A.2) leads to

$$\begin{aligned}
\lambda(t) &= \frac{1}{c_3 I(t)} \int_{t_0}^{t} I(t)[-g'(s)]c_3 a(u(t)-u(t-s), u(t)-u(t-s))\mathrm{d}s \\
&\geq \frac{1}{c_3 I(t)} \int_{t_0}^{t} I(t)\xi(s)\mathcal{H}(g(s))c_3 a(u(t)-u(t-s), u(t)-u(t-s))\mathrm{d}s \\
&\geq \frac{\xi(t)}{c_3 I(t)} \int_{t_0}^{t} \mathcal{H}(I(t)g(s))c_3 a(u(t)-u(t-s), u(t)-u(t-s))\mathrm{d}s \\
&\geq \frac{\xi(t)}{c_3}\overline{\mathcal{H}}\left(c_3 \int_{t_0}^{t} g(s)a(u(t)-u(t-s), u(t)-u(t-s))\mathrm{d}s\right).
\end{aligned} \tag{5.55}$$

It follows that

$$\int_{t_0}^{t} g(s)a(u(t)-u(t-s), u(t)-u(t-s))\mathrm{d}s \leq \frac{1}{c_3}\overline{\mathcal{H}}^{-1}\left(\frac{c_3\lambda(t)}{\xi(t)}\right).$$

Thus, (5.52) becomes

$$\mathcal{F}_1'(t) \leq -\beta_1 E(t) + \frac{\beta_2}{c_3}\overline{\mathcal{H}}^{-1}\left(\frac{c_3\lambda(t)}{\xi(t)}\right), \tag{5.56}$$

where $\mathcal{F}_1 = (\mathcal{F} - \beta_3 c_2' \xi(0)E) \sim E$.

Now, for $\varepsilon_0 > 0$ and $\beta_5 > 0$, we define the functional

$$\mathcal{F}_2(t) = \overline{\mathcal{H}}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{F}_1(t) + \beta_5 E(t).$$

Using (5.56) and the fact that $E' < 0$, $\overline{\mathcal{H}}' > 0$, $\overline{\mathcal{H}}'' > 0$, we conclude that $\mathcal{F}_2 \sim E$ and

$$\begin{aligned}
\mathcal{F}_2'(t) &= \varepsilon_0 \frac{E'(t)}{E(0)}\overline{\mathcal{H}}''\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{F}_1(t) + \overline{\mathcal{H}}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{F}_1'(t) + \beta_5 E'(t) \\
&\leq -\beta_1 E(t)\overline{\mathcal{H}}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + \frac{\beta_2}{c_3}\overline{\mathcal{H}}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\overline{\mathcal{H}}^{-1}\left(\frac{c_3\lambda(t)}{\xi(t)}\right) + \beta_5 E'(t).
\end{aligned} \tag{5.57}$$

On the other hand, in view of (2.24) and (2.25) with $A = \overline{\mathcal{H}}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$ and $B = \overline{\mathcal{H}}^{-1}\left(\frac{c_3\lambda(t)}{\xi(t)}\right)$, (5.57) gives

$$\mathcal{F}_2'(t) \leq -(\beta_1 E(0) - \varepsilon_0)\frac{E(t)}{E(0)}\overline{\mathcal{H}}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + \beta_2 \frac{\lambda(t)}{\xi(t)} + \beta_5 E'(t).$$

So, multiplying by $\xi(t)$, using (5.54) and the fact that, as

$$\varepsilon_0 \frac{E(t)}{E(0)} < r, \quad \overline{\mathcal{H}}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) = \mathcal{H}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right),$$

we find that

$$\mathcal{F}_3'(t) \leq \xi(t)\mathcal{F}_2'(t) \leq -(\beta_1 E(0) - \varepsilon_0)\xi(t)\frac{E(t)}{E(0)}\overline{\mathcal{H}}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - (\beta_2 c_4 - \beta_5 \xi(0))E'(t).$$

where $\mathcal{F}_3 = (\xi \mathcal{F}_2)$. Let us choose ε_0 and β_5 small enough such that

$$\beta_6 = \beta_1 E(0) - \varepsilon_0 > 0 \quad \text{and} \quad \beta_2 c_4 - \beta_5 \xi(0) > 0.$$

Then, for some constants $\alpha_1, \alpha_2 > 0$

$$\alpha_1 \mathcal{F}_3(t) \le E(t) \le \alpha_2 \mathcal{F}_3(t),$$

and

$$\mathcal{F}_3'(t) \le -\beta_6 \xi(t) \frac{E(t)}{E(0)} \overline{\mathcal{H}}' \left(\varepsilon_0 \frac{E(t)}{E(0)} \right) = -\beta_6 \xi(t) H_2 \left(\frac{E(t)}{E(0)} \right), \tag{5.58}$$

Since $\mathcal{H}_2'(t) = \mathcal{H}'(\varepsilon_0 t) + \varepsilon_0 t \mathcal{H}''(\varepsilon_0 t)$, then, by using the strict convexity of $\mathcal{H}$ on $(0, r]$, we find that $\mathcal{H}_2, \mathcal{H}_2' > 0$ on $(0, 1]$. Let

$$R(t) = \frac{\alpha_1 \mathcal{F}_3(t)}{E(0)}.$$

From the fact that $R \sim E$, thus, (5.58) yields

$$R'(t) \le -\beta_6 \xi(t) \mathcal{H}_2(R(t)).$$

A simple integration over (t_0, t), taking into consideration that $\varepsilon_0 R(t_0) < r$, gives

$$\mathcal{H}_1(\varepsilon_0 R(t)) \ge \int_{\varepsilon_0 R(t)}^{\varepsilon_0 R(t_0)} \frac{\mathrm{d}s}{s \mathcal{H}'(s)} \ge \beta_6 \int_{t_0}^{t} \xi(s) \mathrm{d}s.$$

Using the fact that $\mathcal{H}_1$ is strictly decreasing function on $(0, r]$ and

$$\lim_{t \to 0} \mathcal{H}_1(t) = +\infty,$$

we get (5.51). ■

5.5.2 Second general theorem

Theorem 5.5.4. *Assume that (**H.1**)-(**H.6**) and (5.3) hold and F_0 is nonlinear. There exist two constants $k_1, k_2 > 0$ such that, for all $t \ge t_0$*

$$E(t) \le k_1 \mathrm{G}_1^{-1} \left(k_2 \int_{t_0}^{t} \xi(s) \mathrm{d}s \right), \tag{5.59}$$

if $\mathcal{H}$ is linear. Moreover, if $\mathcal{H}$ is nonlinear, then there exist another two constants $k_3, k_4 > 0$ such that

$$E(t) \le k_3 [t - t_0] \mathcal{W}_1^{-1} \left(\frac{k_4}{[t - t_0] \int_{t_0}^{t} \xi(s) \mathrm{d}s} \right), \tag{5.60}$$

for all $t > t_0$, where G_1 and $\mathcal{W}_1$ defined in (5.8) and (5.9), respectively.

Remark 5.5.5. Suppose that (**H.4**) and (**H.6**) hold. The energy decay in each case that driven by (5.51), (5.59) and (5.60) is missing and satisfies $\lim_{t \to +\infty} E(t) = 0$ provided that $\int_0^{+\infty} \xi(s) \mathrm{d}s = +\infty$.

Proof of Theorem 5.5.4. . First, we assume that $\max\{r_1, F_0(r_1)\} < \varepsilon$; otherwise r_1 small enough. Let $\varepsilon_1 = \min\{r_1, F_0(r_1)\}$. From (5.6)-(5.7), we have

$$\begin{cases} c_1' |s| \le |F(s)| \le c_2' |s| & \text{for all } |s| \ge \varepsilon_1, \\ F_0(|s|) \le |F(s)| \le F_0^{-1}(|s|) & \text{for all } |s| < \varepsilon_1. \end{cases}$$

Then, for all $|s| \le \varepsilon_1$

$$\mathrm{G}(F^2(s)) = |F(s)|F_0(|F(s)|) \le sF(s),$$

which gives

$$F^2(s) \le \mathrm{G}^{-1}(sF(s)), \quad \forall |s| \le \varepsilon_1. \tag{5.61}$$

The following partition was first introduced by Komornik [57]

$$\Gamma_{11} = \{x \in \Gamma_1 \quad : \quad |u_t(t)| \ge \varepsilon_1\}, \quad \Gamma_{12} = \{x \in \Gamma_1 \quad : \quad |u_t(t)| \le \varepsilon_1\}. \tag{5.62}$$

Note that, for a constant $c_5 > 0$

$$J(t) = \frac{1}{|\Gamma_{12}|} \int_{\Gamma_{12}} u_t(t) F(u_t(t)) \mathrm{d}\Gamma \le -c_5 E'(t). \tag{5.63}$$

Then, by using (5.16), (5.61), (5.62) and Jensen's inequality (A.2), we get

$$\int_{\Gamma_1} F^2(u_t(t)) \mathrm{d}\Gamma \le \mathrm{G}^{-1}(J(t)) - c_2' E'(t). \tag{5.64}$$

Case 1: $\mathcal{H}$ is linear. Multiplying (5.52) by $\xi(t)$ and using (5.64), we have

$$\mathcal{F}_4'(t) \le -\beta_1 \xi(t) E(t) + \beta_3 \xi(t) \mathrm{G}^{-1}(J(t)), \tag{5.65}$$

where $\mathcal{F}_4 = (\xi \mathcal{F} + (c_2' + \beta_4)E) \sim E$.

Now, for $0 < \varepsilon_1 < r_1$ and $\beta_7 > 0$, by using (5.65) and the fact that $E' \le 0$, $\mathrm{G}' > 0$, $\mathrm{G}'' > 0$ on $(0, r_1]$, we find that the functional $\mathcal{F}_5$, defined by

$$\mathcal{F}_5(t) = \mathrm{G}'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) \mathcal{F}_4(t) + \beta_7 E(t),$$

satisfies, for some $\alpha_3, \alpha_4 > 0$

$$\alpha_3 \mathcal{F}_5(t) \le E(t) \le \alpha_4 \mathcal{F}_5(t).$$

Using the convex conjugate of G in the sense of Young (2.25), as in (2.25), with $A = \mathrm{G}'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right)$ and $B = \mathrm{G}^{-1}(J(t))$, by using (5.63) and again the fact that $E' \le 0$, $\mathrm{G}' > 0$, $\mathrm{G}'' > 0$, we obtain

$$\begin{aligned}
\mathcal{F}_5'(t) &= \varepsilon_1 \frac{E'(t)}{E(0)} \mathrm{G}''\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) \mathcal{F}_4(t) + \mathrm{G}'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) \mathcal{F}_4'(t) + \beta_7 E'(t) \\
&\le -\beta_1 \xi(t) E(t) \mathrm{G}'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) + \beta_3 \xi(t) \mathrm{G}'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) \mathrm{G}^{-1}(J(t)) + \beta_7 E'(t) \\
&\le -(\beta_1 E(0) - \varepsilon_1) \xi(t) \frac{E(t)}{E(0)} \mathrm{G}'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) + (\beta_7 - c_5) E'(t).
\end{aligned}$$

Consequently, with a suitable choice of ε_1 and β_7, we obtain, for all $t \ge t_0$

$$\mathcal{F}_5'(t) \le -\beta_8 \xi(t) \frac{E(t)}{E(0)} \mathrm{G}'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) = -\beta_8 \xi(t) \mathrm{G}_2\left(\frac{E(t)}{E(0)}\right),$$

where $\beta_8 = \beta_1 E(0) - \varepsilon_1 > 0$.Since $\mathrm{G}_2'(t) = \mathrm{G}'(\varepsilon_1 t) + \varepsilon_1 t \mathrm{G}''(\varepsilon_1 t)$, then, using the strict convexity of G on $(0, r_1]$, we find that $\mathrm{G}_2, \mathrm{G}_2' > 0$ on $(0, 1]$. Let

$$R_1(t) = \alpha_3 \frac{\mathcal{F}_5(t)}{E(0)}.$$

Using the fact that $R_1 \sim E$, we have

$$R_1'(t) \leq -\beta_8 \xi(t) \mathrm{G}_2(R_1(t)), \quad \forall t \geq t_0.$$

This concludes (5.59).
Case 2: $\mathcal{H}$ is nonlinear. First, we define the functional

$$I_1(t) = \frac{c_6}{(t-t_0)} \int_{t_0}^{t} a(u(t)-u(t-s), u(t)-u(t-s)) \mathrm{d}s,$$

and choosing $0 < c_6 < 1$ small enough so that

$$0 < I_1(t) < 1,$$

for all $t > t_0$. In the same way as in (5.55), we obtain

$$\int_{t_0}^{t} g(s) a(u(t)-u(t-s), u(t)-u(t-s)) \mathrm{d}s \leq \frac{(t-t_0)}{c_6} \overline{\mathcal{H}}^{-1} \left(\frac{c_6 \lambda(t)}{(t-t_0)\xi(t)} \right).$$

Hence, we can write (5.52) as follows

$$\mathcal{F}'(t) \leq -\beta_1 E(t) + \frac{(t-t_0)}{c_6} \overline{\mathcal{H}}^{-1} \left(\frac{c_6 \lambda(t)}{(t-t_0)\xi(t)} \right) + \mathrm{G}^{-1}(J(t)) - \beta_4 E'(t).$$

Since

$$\lim_{t \to +\infty} \frac{c_6}{t-t_0} = 0,$$

there exists $t_1 \geq t_0$ such that $\frac{c_6}{t-t_0} < 1$ for all $t > t_1$. Combining this with the strictly increasing and strictly convex properties of $\overline{\mathrm{G}}$, using (2.22), we obtain, for all $t \geq t_1$

$$\mathcal{F}_6'(t) \leq -\beta_1 E(t) + \frac{(t-t_0)}{c_6} \overline{\mathcal{H}}^{-1} \left(\frac{c_6 \lambda(t)}{(t-t_0)\xi(t)} \right) + \frac{(t-t_0)}{c_6} \overline{\mathrm{G}}^{-1} \left(\frac{c_6 J(t)}{(t-t_0)} \right),$$

where $\mathcal{F}_6 = (\mathcal{F} + \beta_4 E) \sim E$.

Let $r_0 = \min\{r, r_1\}$ and $\chi(t) = c_6 \max\left\{ \frac{\lambda(t)}{(t-t_0)\xi(t)}, \frac{J(t)}{(t-t_0)} \right\}$. So,

$$\mathcal{F}_6'(t) \leq -\beta_1 E(t) + \frac{(t-t_0)}{c_6} \mathcal{W}^{-1}(\chi(t)). \tag{5.66}$$

Now, for $0 < \varepsilon_2 < r_0$, using (5.66) and the fact that $E' \leq 0$, $\mathcal{W}' > 0$, $\mathcal{W}'' > 0$ on $(0, r_0]$, we find that the functional $\mathcal{F}_7$, defined by

$$\mathcal{F}_7(t) = \mathcal{W}' \left(\frac{\varepsilon_2}{(t-t_0)} \frac{E(t)}{E(0)} \right) \mathcal{F}_6(t),$$

satisfies, for some $\alpha_5, \alpha_6 > 0$

$$\alpha_5 \mathcal{F}_7(t) \leq E(t) \leq \alpha_6 \mathcal{F}_7(t).$$

As above, using the convex conjugate of $\mathcal{W}$, as in (2.24), we get

$$\begin{aligned}
\mathcal{F}_7'(t) &= \left(\frac{-\varepsilon_2}{(t-t_0)^2}+\frac{\varepsilon_2}{(t-t_0)}\frac{E'(t)}{E(0)}\right)\mathcal{W}''\left(\frac{\varepsilon_2}{(t-t_0)}\frac{E(t)}{E(0)}\right)\mathcal{F}_6(t)\\
&\quad +\mathcal{W}'\left(\frac{\varepsilon_2}{(t-t_0)}\frac{E(t)}{E(0)}\right)\mathcal{F}_6'(t)\\
&\leq -\beta_1 E(t)\mathcal{W}'\left(\frac{\varepsilon_2}{(t-t_0)}\frac{E(t)}{E(0)}\right)+\frac{(t-t_0)}{c_6}\mathcal{W}'\left(\frac{\varepsilon_2}{(t-t_0)}\frac{E(t)}{E(0)}\right)\mathcal{W}^{-1}\left(\chi(t)\right)\\
&\leq -(\beta_1 E(0)-\varepsilon_2)\frac{E(t)}{E(0)}\mathcal{W}'\left(\frac{\varepsilon_2}{(t-t_0)}\frac{E(t)}{E(0)}\right)+\frac{(t-t_0)}{c_6}\chi(t). \qquad (5.67)
\end{aligned}$$

From (5.54) and (5.63), we observe that, for $c_7>0$

$$\frac{(t-t_0)}{c_6}\xi(t)\chi(t)\leq -c_7E'(t).$$

After multiplying (5.67) by $\xi(t)$, from the fact $\varepsilon_2\frac{E(t)}{E(0)}\leq r_0$, it follows that

$$\mathcal{F}_8'(t)\leq -(\beta_1E(0)-\varepsilon_2)\xi(t)\frac{E(t)}{E(0)}\mathcal{W}'\left(\frac{\varepsilon_2}{(t-t_0)}\frac{E(t)}{E(0)}\right),$$

where $\mathcal{F}_8=(\xi\mathcal{F}_7+c_7E)\sim E$. Let us choose ε_2 small enough such that

$$\beta_9=\beta_1E(0)-\varepsilon_2>0$$

Therefore, for all $t\geq t_1$

$$\beta_9\xi(t)\frac{E(t)}{E(0)}\mathcal{W}'\left(\frac{\varepsilon_2}{(t-t_0)}\frac{E(t)}{E(0)}\right)\leq -\mathcal{F}_8'(t). \qquad (5.68)$$

Integration (5.68) and multiplying the result by $\frac{1}{(t-t_0)}$, by using the fact that $\mathcal{W}',\mathcal{W}''>0$ and taking into consideration the nonincreasing property of E, we deduce that, for all $t\geq t_1$

$$\beta_9\mathcal{W}_1\left(\frac{1}{(t-t_0)}\frac{E(t)}{E(0)}\right)\int_{t_1}^t\xi(s)\mathrm{d}s\leq\int_{t_1}^t\beta_9\mathcal{W}_1\left(\frac{1}{(s-t_0)}\frac{E(s)}{E(0)}\right)\xi(s)\mathrm{d}s\leq\frac{\mathcal{F}_8(t_1)}{(t-t_0)}.$$

Consequently, (5.60) is derived as follows

$$E(t)\leq E(0)(t-t_0)\mathcal{W}_1^{-1}\left(\frac{\mathcal{F}_8(t_1)}{\beta_9(t-t_0)\int_{t_1}^t\xi(s)\mathrm{d}s}\right).$$

■

Chapter 6

Well Posedness and General Decay in the Infinite Memory Case

Our main aim in this chapter is the study of global existence and general decay estimates of energy solutions for a thermo-viscoelastic coupled system of type III with nonlinear boundary feedback, infinite memory and distributed delay. The well posedness and regularity is proved in Sect. 6.2 by using semigroup theory combined with Schauder's fixed point theorem. In Sect. 6.3, we establish general decay rates by constructing a suitable Lyapunov functional.

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6.1 Problem statement

We are concerned with system (5.2) in the case of infinite memory where the first equation leads to

$$u_{tt} + \mathbf{A}u - \int_0^{+\infty} g(t)\mathbf{A}u(t-s)\mathrm{d}s + \operatorname{div}(\sigma\varphi_t) = 0 \quad \text{in } \Omega \times \mathbb{R}_+,$$

with the initial condition

$$u(x,-s) = u_0(x,s) \quad \text{for } x \in \Omega,\ s \in \mathbb{R}_+,$$

the fourth equation given by

$$\frac{\partial u}{\partial \nu_{\mathrm{A}}} - \int_0^{+\infty} g(t)\frac{\partial u}{\partial \nu_{\mathrm{A}}}(t-s)\mathrm{d}s + F(u_t) = hz_t \quad \text{on } \Gamma_1 \times \mathbb{R}_+,$$

and the sixth equation by

$$z_{tt} + fz_t + mz + u_t = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}_+.$$

Our new system can be rewritten by

$$\begin{cases} u_{tt} + \ell\mathbf{A}u + \int_0^{+\infty} g(s)\mathbf{A}\eta(s)\mathrm{d}s + \operatorname{div}(\sigma\varphi_t) = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ \eta_t + \eta_s - u_t = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \\ \varphi_{tt} + \mathbf{B}(\varphi + \varphi_t) + \int_0^{+\infty} \mu(s)\omega(1,s)\mathrm{d}s + (\sigma.\nabla)u_t = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ s\omega_t + \omega_p = 0 & \text{in} \quad \mathrm{Q} \times \mathbb{R}_+ \\ z_{tt} + fz_t + mz + u_t = 0 & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ u = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}_+ \\ \ell\frac{\partial u}{\partial \nu_{\mathrm{A}}} + \int_0^{+\infty} g(s)\frac{\partial \eta}{\partial \nu_{\mathrm{A}}}(s)\mathrm{d}s + F(u_t) = hz_t & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ \varphi = 0 & \text{on} \quad \Gamma \times \mathbb{R}_+ \\ u(x,-s) = u_0(x,s) & \text{for} \quad x \in \Omega, s \in \mathbb{R}_+ \\ u_t(0) = u_1,\ \varphi(0) = \Phi,\ \varphi_t(0) = \theta_0 & \text{in} \quad \Omega \\ \omega(x,p,s,0) = \omega_0(x,p,s) & \text{for} \quad (x,p,s) \in \mathrm{Q} \\ \eta(x,s,0) = u_0(x,0) - u_0(x,s) = \eta_0(x,s) & \text{for} \quad x \in \Omega, s \in \mathbb{R}_+ \\ z(0) = z_0,\ z_t(0) = z_1 & \text{in} \quad \Gamma_1. \end{cases} \tag{6.1}$$

6.2 The well posedness

This section concerns the existence and uniqueness of solutions to problem (6.1).

We denote by

$$U=(u,v,\eta,\varphi,\psi,\omega,z,\delta)^{\intercal} \quad \text{with} \quad v=u_t, \quad \psi=\varphi_t \quad \text{and} \quad \delta=z_t.$$

Our first-order evolution system is given by

$$\begin{cases} U_t(t)=\mathcal{A}U(t), \quad \forall t\geq 0 \\ U(0)=U_0=(u_0,u_1,\eta_0,\Phi,\theta_0,\omega_0,z_0,z_1)^{\intercal}\in \mathbb{H}, \end{cases} \tag{6.2}$$

where the operator $\mathcal{A}:D(\mathcal{A})\subset\mathbb{H}\to\mathbb{H}$ given by

$$\mathcal{A}U(t)=\begin{pmatrix} v(t) \\ -\ell\mathbf{A}u(t)-\int_0^{+\infty}g(s)\mathbf{A}\eta(s,t)\mathrm{d}s-\operatorname{div}(\sigma\psi(t)) \\ v(t)-\eta_s(s,t) \\ \psi(t) \\ -(\sigma\cdot\nabla)v(t)-\mathbf{B}\varphi(t)-\mathbf{B}\psi(t)-\int_0^{+\infty}\mu(s)\omega(1,s,t)\mathrm{d}s \\ \frac{-1}{s}\omega_p(p,s,t) \\ \delta(t) \\ -v(t)-m(x)z(t)-f(x)\delta(t) \end{pmatrix}$$

and the domain $D(\mathcal{A})$

$$D(\mathcal{A})=\left\{ \begin{array}{l|l} U\in\mathbb{H} & \mathrm{A}\nabla u, \mathrm{B}\nabla\varphi\in \mathrm{H}(\operatorname{div},\Omega);\ v\in \mathrm{H}^1_{\Gamma_0}(\Omega); \\ & \eta,\eta_s\in \mathrm{L}^2_g(\mathbb{R}_+;\mathrm{H}^1_{\Gamma_0}(\Omega));\ \eta(0)=0; \\ & \psi\in\mathrm{H}^1_0(\Omega);\ \omega(0,.)=\psi;\ z,\delta\in\mathrm{H}^{1/2}(\Gamma_1); \\ & \ell\frac{\partial u}{\partial\nu_{\mathrm{A}}}+\int_0^{+\infty}g(s)\frac{\partial\eta}{\partial\nu_{\mathrm{A}}}(s)\mathrm{d}s+F(v)=h\delta \ \text{ on } \Gamma_1 \end{array} \right\}$$

We set the product Hilbert spaces

$$\mathbb{H}=\mathrm{H}^1_{\Gamma_0}(\Omega)\times\mathrm{L}^2(\Omega)\times\mathrm{L}^2_g(\mathbb{R}_+;\mathrm{H}^1_{\Gamma_0}(\Omega))\times\mathrm{H}^1_0(\Omega)\times\mathrm{L}^2(\Omega)\times\mathrm{L}^2(\mathrm{Q})\times\mathrm{L}^2(\Gamma_1)\times\mathrm{L}^2(\Gamma_1),$$

equipped with the following inner product

$$\begin{aligned} \left\langle U(t),\tilde{U}(t)\right\rangle_{\mathbb{H}} &= \ell a(u(t),\tilde{u}(t))+\int_\Omega v(t)\tilde{v}(t)\mathrm{d}x+\langle\eta(t),\tilde{\eta}(t)\rangle_{\mathrm{L}^2_g}+b(\varphi(t),\tilde{\varphi}(t)) \\ &\quad +\int_\Omega\psi(t)\tilde{\psi}(t)\mathrm{d}x+\int_{\mathrm{Q}}s\mu(s)\omega(x,p,s,t)\tilde{\omega}(x,p,s,t)\mathrm{d}s\mathrm{d}p\mathrm{d}x \\ &\quad +\langle hmz(t),\tilde{z}(t)\rangle_{\Gamma_1}+\langle h\delta(t),\delta(t)\rangle_{\Gamma_1}, \end{aligned} \tag{6.3}$$

for every $U=(u,v,\eta,\varphi,\psi,\omega,z,\delta)^{\intercal}$ and $\tilde{U}=(\tilde{u},\tilde{v},\tilde{\eta},\tilde{\varphi},\tilde{\psi},\tilde{\omega},\tilde{z},\tilde{\delta})^{\intercal}$ in $\mathbb{H}$.

Our main global existence of solution is ensured by the following theorem

Theorem 6.2.1. *Assume that (**H.1**)-(**H.3**), (5.3), and (**H.5**)-(**H.6**) hold. The operator $\mathcal{A}$ is the infinitesimal generator of a $\mathcal{C}_0-$semigroup on $\mathbb{H}$.*

Proof. For the proof, we collect the well-known Lumer-Phillips theorem's and its corollary concerning generation of the $\mathcal{C}_0$-semigroup generated by a dissipative operator (see Pazy [81]).

First, for $U \in D(\mathcal{A})$ and using the inner product (6.3), we get

$$\begin{aligned}
\langle \mathcal{A}U(t), U(t)\rangle_{\mathbb{H}} &= \ell a(u(t), v(t)) + \int_0^{+\infty} g(s) a(\eta(s,t), v(t) - \eta_s(s,t))\mathrm{d}s \\
&\quad + \int_\Omega \left(-\ell \mathbf{A} u(t) - \int_0^{+\infty} g(s)\mathbf{A}\eta(s)\mathrm{d}s - \operatorname{div}(\sigma(x)\psi(t))\right) v(t)\mathrm{d}x \\
&\quad - \int_\Omega \mathbf{B}\varphi(t)\psi(t)\mathrm{d}x + b(\varphi(t), \psi(t)) - \int_\Omega (\sigma(x)\cdot\nabla) v(t)\psi(t)\mathrm{d}x \\
&\quad + \langle hm\delta(t), z(t)\rangle_{\Gamma_1} - \langle hv(t) + hmz(t) + hf\delta(t), \delta(t)\rangle_{\Gamma_1} \\
&\quad + \int_\Omega \left(-\mathbf{B}\psi(t) - \int_0^{+\infty} \mu(s)\omega(x,1,s,t)\mathrm{d}s\right)\psi(t)\mathrm{d}x \\
&\quad - \int_{\mathrm{Q}} \mu(s)\omega(x,p,s,t)\omega_p(x,p,s,t)\mathrm{d}s\mathrm{d}p\mathrm{d}x. \qquad (6.4)
\end{aligned}$$

Using the Green formula. Thus, (6.4) can be written as follows

$$\begin{aligned}
\langle \mathcal{A}U(t), U(t)\rangle &= -\int_0^{+\infty} g(s) a(\eta(s,t), \eta_s(s,t))\mathrm{d}s - \langle F(v(t)), v(t)\rangle_{\Gamma_1} \\
&\quad + \frac{1}{2}\int_\Omega \int_0^{+\infty} \mu(s)\left[\omega^2(x,0,s,t) - \omega^2(x,1,s,t)\right]\mathrm{d}s\mathrm{d}x \\
&\quad - b(\psi(t), \psi(t)) - \|h^{1/2} f^{1/2}\delta(t)\|^2_{2,\Gamma_1} \\
&\quad - \int_\Omega \psi(t)\int_0^{+\infty} \mu(s)\omega(x,1,s,t)\mathrm{d}s\mathrm{d}x. \qquad (6.5)
\end{aligned}$$

From (**H.2**), we have

$$-\int_0^{+\infty} g(s) a(\eta(s,t), \eta_s(s,t))\mathrm{d}s = \frac{1}{2}\int_0^{+\infty} g'(s) a(\eta(s,t), \eta(s,t))\mathrm{d}s \leq 0.$$

From (**H.6**) and (5.62), we get

$$-\langle F(v(t)), v(t)\rangle_{\Gamma_1} \leq -c_2'\|v(t)\|^2_{2,\Gamma_{11}} - \int_{\Gamma_{12}} \mathrm{G}(v^2(t))\mathrm{d}\Gamma \leq 0.$$

Using Young's and Poincaré's inequalities in the last term of the right side of (6.5) can be estimated as follows

$$\begin{aligned}
& -\int_\Omega \psi(t)\int_0^{+\infty} \mu(s)\omega(x,1,s,t)\mathrm{d}s\mathrm{d}x \\
\leq\ & \frac{\mathcal{C}_*^2}{2}\left(\int_0^{+\infty}\mu(s)\mathrm{d}s\right)\|\nabla\psi(t)\|_2^2 + \frac{1}{2}\int_\Omega\int_0^{+\infty}\mu(s)\omega^2(x,1,s,t)\mathrm{d}s\mathrm{d}x.
\end{aligned}$$

Therefore, we conclude from (5.4) that the operator $\mathcal{A}$ is dissipative.

Next, we notice that $\mathrm{I} - \mathcal{A}$ is surjective. Indeed, for each $\mathcal{F} \in \mathbb{H}$, we show that there exists $U \in D(\mathcal{A})$ such that

$$(\mathrm{I} - \mathcal{A})U = \mathcal{F} = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^{\intercal}.$$

The previous equation reads to

$$\begin{aligned}
u - v &= f_1 && (6.6)\\
v + \ell \mathbf{A} u + \int_0^{+\infty} g(s)\mathbf{A}\eta(s)\mathrm{d}s + \operatorname{div}(\sigma\psi) &= f_2 && (6.7)\\
\eta + \eta_s - v &= f_3 && (6.8)\\
\varphi - \psi &= f_4 && (6.9)\\
\psi + (\sigma\cdot\nabla)v + \mathbf{B}\varphi + \mathbf{B}\psi(t) + \int_0^{+\infty} \mu(s)\omega(x,1,s,t)\mathrm{d}s &= f_5 && (6.10)\\
s\omega + \omega_p &= sf_6 && (6.11)\\
z - \delta &= f_7 && (6.12)\\
\delta + f\delta + mz + v &= f_8 && (6.13)
\end{aligned}$$

We can determine from (6.8) that

$$\eta(x,s) = -v(x)e^{-s} + v(x) + \int_0^s f_3(x,\tau)e^{\tau-s}\mathrm{d}\tau, \quad \forall x \in \Omega,\ \forall s \in \mathbb{R}_+, \tag{6.14}$$

and from (6.11) that

$$\omega(x,p,s) = \psi(x)e^{-sp} + se^{-sp}\int_0^p f_6(x,\tau,s)e^{s\tau}\mathrm{d}\tau, \quad \forall (x,p,s) \in \mathrm{Q}. \tag{6.15}$$

So, $\eta(.,0) = 0$ and $\omega(x,0,.) = \psi(x)$. According (6.6) with (6.14), we have

$$\eta(x,s) = u(x)\left(1 - e^{-s}\right) + \eta_1(x,s), \quad \forall x \in \Omega,\ \forall s \in \mathbb{R}_+, \tag{6.16}$$

with $\eta_1 \in \mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega))$ defined by

$$\eta_1(x,s) = -f_1(x)\left(1 - e^{-s}\right) + \int_0^s f_3(x,\tau)e^{\tau-s}\mathrm{d}\tau.$$

In particular, from (6.9), we have

$$\omega(x,1,s) = \varphi(x)e^{-s} + \omega_0(x,s), \quad \forall x \in \Omega,\ \forall s \in \mathbb{R}_+, \tag{6.17}$$

with $\omega_0 \in \mathrm{L}^2(\Omega \times \mathbb{R}_+)$ defined by

$$\omega_0(x,s) = -f_4(x)e^{-s} + se^{-s}\int_0^1 f_6(x,\tau,s)e^{s\tau}\mathrm{d}\tau.$$

Substituting v, ψ and δ from (6.6) and (6.9) into (6.7) and (6.10), respectively, using (6.16) and (6.17), we get

$$u + \ell_g \mathbf{A} u + \operatorname{div}(\sigma\varphi) = f_2 + f_1 + \operatorname{div}(\sigma(x)f_4) - \int_0^{+\infty} g(s)\mathbf{A}\eta_1(s)\mathrm{d}s, \tag{6.18}$$

$$\mu_1\varphi + (\sigma(x)\cdot\nabla)u + 2\mathbf{B}\varphi = (\sigma\cdot\nabla)f_1 + \mathbf{B}f_4 + f_4 + f_5 - \int_0^{+\infty} \mu(s)\omega_0(s)\mathrm{d}s, \tag{6.19}$$

where, $\ell_g = \ell + \int_0^{+\infty} g(s)(1 - e^{-s})\mathrm{d}s$ and $\mu_1 = 1 + \int_0^{+\infty} \mu(s)e^{-s}\mathrm{d}s$ are positive constants due to (**H.2**) and (**H.5**), respectively. Moreover, it can be shown that the right-hand side of (6.18) and (6.19) both are in $\mathrm{H}^{-1}(\Omega)$.

On the other hand, inserting v and z from (6.6) and (6.12) into (6.13). Then, $(6.1)_7$ becomes

$$\ell_g \frac{\partial u}{\partial \nu_{\mathrm{A}}} + \int_0^{+\infty} g(s) \frac{\partial \eta_1}{\partial \nu_{\mathrm{A}}}(s)\mathrm{d}s + F(u - f_1) = \mu_2\left(f_1 - m f_7 + f_8\right) - \mu_2 u \quad \text{on } \Gamma_1,$$

where $\mu_2 = \dfrac{h}{1+f+m}$. From the hypotheses of h, f, and m, we observe that

$$0 \leq \frac{h_0}{1 + \|f\|_\infty + \|m\|_\infty} \leq \mu_2 \leq \frac{\|h\|_\infty}{1 + f_0 + m_0}.$$

Now, multiplying (6.18) by $\tilde{u}$ and (6.19) by $\tilde{\varphi}$, integrating over Ω and using Green's formula, we obtain the bilinear form

$$\mathfrak{B} : (\mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_0(\Omega)) \times (\mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_0(\Omega)) \to \mathbb{R}$$

defined by

$$\begin{aligned} \mathfrak{B}((u,\varphi),(\tilde{u},\tilde{\varphi})) \quad = \quad & \int_\Omega u\tilde{u}\mathrm{d}x + \ell_g a(u,\tilde{u}) + \mu_2 \langle u, \tilde{u}\rangle_{\Gamma_1} + \mu_1 \int_\Omega \varphi\tilde{\varphi}\mathrm{d}x \\ & +2b(\varphi,\tilde{\varphi}) + \int_\Omega [-\varphi(\sigma(x)\cdot\nabla)\tilde{u} + (\sigma(x)\cdot\nabla)u\tilde{\varphi}]\mathrm{d}x, \end{aligned}$$

for every $(u,\varphi),(\tilde{u},\tilde{\varphi}) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_0(\Omega)$, and the linear formm

$$\mathcal{G} : \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_0(\Omega) \to \mathbb{R}$$

defined by

$$\begin{aligned} \mathcal{G}(\tilde{u},\tilde{\varphi}) \quad = \quad & \int_\Omega [f_1 + f_2 + \mathrm{div}(\sigma(x) f_4)]\tilde{u}\mathrm{d}x + \langle \mu_2\left(f_1 - m f_7 + f_8\right), \tilde{u}\rangle_{\Gamma_1} \\ & + \int_\Omega \left[(\sigma(x)\cdot\nabla) f_1 + f_4 + f_5\right]\tilde{\varphi}\mathrm{d}x - \int_0^{+\infty} g(s) a(\eta_1(s),\tilde{u})\mathrm{d}s \\ & + b(f_4,\tilde{\varphi}) - \int_\Omega \left(\int_0^{+\infty} \mu(s)\omega_0(s)\mathrm{d}s\right)\tilde{\varphi}\mathrm{d}x, \end{aligned}$$

for every $(\tilde{u},\tilde{\varphi}) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_0(\Omega)$, which

$$\mathfrak{B}((u,\varphi),(\tilde{u},\tilde{\varphi})) + \langle F(u - f_1), \tilde{u}\rangle_{\Gamma_1} = \mathcal{G}(\tilde{u},\tilde{\varphi}). \tag{6.20}$$

It's easy to see that the bilinear form $\mathfrak{B}$ is continuous and coercive on $(\mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_0(\Omega)) \times (\mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_0(\Omega))$, that is to say, there exists a constant $\beta^* > 0$ such that

$$\mathfrak{B}((u,\varphi),(u,\varphi)) \geq \beta^* \left(\|\nabla u\|_2^2 + \|\nabla\varphi\|_2^2\right), \tag{6.21}$$

for all $(u,\varphi) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_0(\Omega)$. Moreover, the linear form $\mathcal{G}$ is continuous on $\mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_0(\Omega)$. Then, there exists a constant $M > 0$ such that

$$\mathcal{G}(\tilde{u},\tilde{\varphi}) \leq M\left(\|\nabla\tilde{u}\|_2^2 + \|\nabla\tilde{\varphi}\|_2^2\right)^{1/2}, \tag{6.22}$$

forall $(\tilde{u},\tilde{\varphi}) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_0(\Omega)$.

The existence of solution of (6.20) can be proved by Schauder's fixed point theorem (see Edwards [41]).
Let $u^* \in \mathrm{H}^{1/2}(\Gamma_1)$. This implies from the fact (**H.6**) that $F(u^* - f_1) \in \mathrm{H}^{1/2}(\Gamma_1)$. Using Lax-Milgram's theorem, then there exists a unique solution $(u, \varphi) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_0(\Omega)$ satisfying

$$\mathfrak{B}((u,\varphi),(\tilde{u},\tilde{\varphi})) = \mathcal{G}(\tilde{u},\tilde{\varphi}) - \langle F(u^* - f_1), \tilde{u}\rangle_{\Gamma_1}, \tag{6.23}$$

for all $(\tilde{u},\tilde{\varphi}) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_0(\Omega)$.
In the following, we define the map T from $\mathrm{H}^1_{\Gamma_0}(\Omega)$ to $\mathrm{H}^1_{\Gamma_0}(\Omega)$ by $T(u^*) = u$, where u is the solution of problem (6.23). Suppose R be large enough. It suffices to show the map S that sends from $\gamma_0(B_R)$ to B_R, in which $S(\gamma_0(u^*)) = T(u^*) = u$, is continuous and $\{S(u^*) : u^* \in \gamma_0(B_R)\}$ is relatively compact in $\mathrm{H}^1_{\Gamma_0}(\Omega)$ where $B_R = \{v \in \mathrm{H}^1_{\Gamma_0}(\Omega) : \|\nabla v\|_2 \leq R\}$.

- (Estimation) Taking $(\tilde{u},\tilde{\varphi}) = (u, 0)$ in (6.23), by using (1.7), (**H.6**) and taking into account (6.21)-(6.22), we obtain

  $$\beta^* \|\nabla u\|_2 \leq M + \overline{C}_* \|F(u^* - f_1)\|_{\mathrm{H}^{1/2}(\Gamma_1)}.$$

  It follows that

  $$\|u^*\|_{\mathrm{H}^{1/2}(\Gamma_1)} \leq R \Rightarrow \|\nabla u\|_2 \leq R. \tag{6.24}$$

- S is continuous. Indeed, let $(u_k^*)_{k\in\mathbb{N}}$ a sequence of $\mathrm{H}^{1/2}(\Gamma_1)$ and $u^* \in \mathrm{H}^{1/2}(\Gamma_1)$. Suppose $u_k^* \to u^*$ in $\mathrm{H}^{1/2}(\Gamma_1)$ (when $k \to \infty$). Since $F \in \mathcal{C}^0(\mathbb{R})$ and $f_1 \in \mathrm{H}^{1/2}(\Gamma_1)$. Then,

  $$F(u_k^* - f_1) \to F(u^* - f_1) \qquad \text{a.e in } \Gamma_1,$$

  and $\|F(u_k^* - f_1)\|_{\mathrm{H}^{1/2}(\Gamma_1)} \leq cR + \|f_1\|_{\mathrm{H}^{1/2}(\Gamma_1)}$. By using (Lemme 1.3. in Lions [61]), we get

  $$F(u_k^* - f_1) \rightharpoonup F(u^* - f_1) \qquad \text{weakly in } \mathrm{H}^{1/2}(\Gamma_1). \tag{6.25}$$

  From (6.24), we deduce that $(u_k)_{k\in\mathbb{N}} = (S(u_k^*))_{k\in\mathbb{N}}$ is bounded in $\mathrm{H}^1_{\Gamma_0}(\Omega)$ and therefore there exists subsequence of $(u_k)_{k\in\mathbb{N}}$, which we still denote by $(u_k)_{k\in\mathbb{N}}$ such that

  $$u_k \rightharpoonup l_1 \qquad \text{weakly in } \mathrm{H}^1(\Omega).$$

  It follows from Rellich-Kondrachoff's theorem, as $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$ be compactly embedded, that

  $$u_k \to l_1 \qquad \text{strongly in } \mathrm{L}^2(\Omega); \tag{6.26}$$

  $$\nabla u_k \rightharpoonup l_2 \qquad \text{weakly in } \mathrm{L}^2(\Omega). \tag{6.27}$$

  Let $\varphi \in \mathrm{H}^1_0(\Omega)$ be fixed. Since $(u_k)_{k\in\mathbb{N}} = (S(u_k^*))_{k\in\mathbb{N}}$, we have

  $$\mathfrak{B}((u_k,\varphi),(\tilde{u},\varphi)) = \mathcal{G}(\tilde{u},\varphi) - \langle F(u_k^* - f_1), \tilde{u}\rangle_{\Gamma_1} \qquad \forall \tilde{u} \in \mathrm{H}^1_{\Gamma_0}(\Omega).$$

  Passing to the limit when $k \to \infty$ and taking into account (6.25)-(6.26) in the previous equality, we conclude that $u = S(u^*)$ with

  $$u_k \to u \qquad \text{strongly in } \mathrm{L}^2(\Omega).$$

  This implies from distribution theory and (6.27) that $l_2 = \nabla u$ and

  $$\nabla u_k \rightharpoonup \nabla u \qquad \text{weakly in } \mathrm{L}^2(\Omega).$$

  Subsequently,

  $$\limsup_{k\to\infty} \|\nabla u_k\|_2 = \liminf_{k\to\infty} \|\nabla u_k\|_2 = \lim_{k\to\infty} \|\nabla u_k\|_2 = \|\nabla u\|_2.$$

  If we consider $u \in \mathcal{D}(\overline{\Omega})$ in (6.23) when $\varphi \in \mathrm{H}^1_0(\Omega)$, we find that (u, φ) be solution of (6.18) and (6.19) in $\mathcal{D}'(\Omega)$ and thus $\gamma_0(u) = 0$ on Γ_0. Hence,

  $$S(u_k^*) \to S(u^*) \quad \text{strongly in } \mathrm{H}^1_{\Gamma_0}(\Omega).$$

- S is compact. Indeed, let $(u_k^*)_{k\in\mathbb{N}}$ a bounded sequence of $\mathrm{H}^{1/2}(\Gamma_1)$. Since the embedding of $\mathrm{H}^{1/2}(\Gamma_1)$ in $\mathrm{L}^2(\Gamma_1)$ is compact. There exists a subsequence denote again $(u_k^*)_{k\in\mathbb{N}}$, which is strongly converge to l_3 in $\mathrm{L}^2(\Gamma_1)$. It follows that

  $$F(u_k^* - f_1) \to l_4 \quad \text{a.e. in } \Gamma_1$$

  where $l_4 = F(l_3 - f_1) \in \mathrm{L}^2(\Gamma_1)$. Suppose $(u_k)_{k\in\mathbb{N}} = (S(u_k^*))_{k\in\mathbb{N}}$. Then, there exists a unique solution $u \in \mathrm{H}^1_{\Gamma_0}(\Omega)$ satisfies

  $$\mathfrak{B}((u,\varphi),(\tilde{u},\varphi)) = \mathcal{G}(\tilde{u},\varphi) - \langle l_4, \tilde{u}\rangle_{\Gamma_1} \qquad \forall \tilde{u} \in \mathrm{H}^1_{\Gamma_0}(\Omega). \tag{6.28}$$

  Our aim is to prove that the sequence $(u_k)_{k\in\mathbb{N}}$ converge in $\mathrm{H}^1_{\Gamma_0}(\Omega)$ to solution of (6.28), $u = S(u^*)$.
  We well know from (6.24) that $(u_k)_{k\in\mathbb{N}}$ is bounded in $\mathrm{H}^1_{\Gamma_0}(\Omega)$, so there exists a subsequence, which we still denote $(u_k)_{k\in\mathbb{N}}$, that's weakly converge to l_5 in $\mathrm{H}^1(\Omega)$. Hence, we have

  $$\mathfrak{B}((u_k,\varphi),(\tilde{u},\varphi)) = \mathcal{G}(\tilde{u},\varphi) - \langle F(u_k^* - f_1), \tilde{u}\rangle_{\Gamma_1} \qquad \forall \tilde{u} \in \mathrm{H}^1_{\Gamma_0}(\Omega). \tag{6.29}$$

  Passing to the limit, (6.29) tends to

  $$\mathfrak{B}((l_5,\varphi),(\tilde{u},\varphi)) = \mathcal{G}(\tilde{u},\varphi) - \langle l_4, \tilde{u}\rangle_{\Gamma_1} \qquad \forall \tilde{u} \in \mathrm{H}^1_{\Gamma_0}(\Omega).$$

  It means from (6.28) that $u = l_5$. This result gives,

  $$u_k \rightharpoonup u \quad \text{weakly in } \mathrm{H}^1(\Omega).$$

  For completeness our main existence, we are going to state that $(u_k)_{k\in\mathbb{N}}$ be strongly converge to u in $\mathrm{H}^1_{\Gamma_0}(\Omega)$. For this purpose, by using Lebesgue's dominated convergence theorem, we remark that (when $k \to \infty$.)

  $$\mathfrak{B}((u_k,\varphi),(u_k,\varphi)) = \mathcal{G}(u_k,\varphi) - \langle F(u_k^* - f_1), u_k\rangle_{\Gamma_1} \to \mathcal{G}(u,\varphi) - \langle l_4, u\rangle_{\Gamma_1}.$$

  Since u is solution of (6.28), we obtain

  $$\lim_{k\to\infty} \mathfrak{B}((u_k,\varphi),(u_k,\varphi)) = \mathfrak{B}((u,\varphi),(u,\varphi)).$$

  This implies as a consequence of (6.21) that (when $k \to \infty$.)

  $$\beta^* \|\nabla(u_k - u)\|_2^2 \le \mathfrak{B}((u_k - u, 0),(u_k - u, 0)) \to 0;$$

  i.e.

  $$u_k \to u \quad \text{strongly in } \mathrm{H}^1_{\Gamma_0}(\Omega).$$

Therefore, Schauder's theorem allows that there exists a solution $u \in \mathrm{H}^1_{\Gamma_0}(\Omega)$ of (6.20) such that $u = S(u)$.

For uniqueness. Assuming that $u = S(u)$ and $u^* = S(u^*)$ are solutions of (6.20), and we set $W = u - u^*$. It is straightforward to see that W satisfies, for all $\tilde{u} \in \mathrm{H}^1_{\Gamma_0}(\Omega)$

$$\mathfrak{B}((W,\varphi),(\tilde{u},\varphi)) + \langle F(u - f_1) - F(u^* - f_1), \tilde{u}\rangle_{\Gamma_1} = 0. \tag{6.30}$$

Taking $\tilde{u} = W$ in (6.30) and by using (5.5), we obtain

$$\begin{aligned} \mathfrak{B}((W,\varphi),(W,\varphi)) &= -\langle F(u - f_1) - F(u^* - f_1), W\rangle_{\Gamma_1} \\ &\le -c_1' \|W\|^2_{\mathrm{H}^{1/2}(\Gamma)} \qquad\qquad \forall \varphi \in \mathrm{H}^1_0(\Omega). \end{aligned}$$

In particular,

$$\beta^*\|\nabla W\|_2^2 \leq \mathfrak{B}((W,0),(W,0)) \leq 0.$$

Then, we conclude that $u = u^*$. This guarantees the uniqueness of solution of (6.20).

Additionally, from (6.6), (6.9), (6.12) and (6.13), we have $v \in \mathrm{H}^1_{\Gamma_0}(\Omega)$, $\psi \in \mathrm{H}^1_0(\Omega)$ and $z, \delta \in \mathrm{H}^{1/2}(\Gamma_1)$.

Now, from (6.14) and (6.15), we obtain

$$\|\eta\|^2_{\mathrm{L}^2_g(\mathbb{R}_+;\mathrm{H}^1_{\Gamma_0}(\Omega))} \leq C\left(\|\nabla v\|_2^2 + \|f_3\|^2_{\mathrm{L}^2_g(\mathbb{R}_+;\mathrm{H}^1_{\Gamma_0}(\Omega))}\right),$$

and

$$\|\omega\|^2_{\mathrm{L}^2(\mathrm{Q})} \leq C_1\left(\|\psi\|_2^2 + \|f_6\|^2_{\mathrm{L}^2(\mathrm{Q})}\right),$$

from where it follows that $\eta \in \mathrm{L}^2_g(\mathbb{R}_+;\mathrm{H}^1_{\Gamma_0}(\Omega))$ and $\omega \in \mathrm{L}^2(\mathrm{Q})$.

From (6.18) and (6.19), we get

$$\mathrm{A}\nabla u, \mathrm{B}\nabla\varphi \in \mathrm{H}(\mathrm{div}, \Omega).$$

On the other hand, from (6.8), we obtain

$$\|\eta_s\|^2_{\mathrm{L}^2_g(\mathbb{R}_+;\mathrm{H}^1_{\Gamma_0}(\Omega))} \leq C_2\left(\|\nabla v\|_2^2 + \|f_3\|^2_{\mathrm{L}^2_g(\mathbb{R}_+;\mathrm{H}^1_{\Gamma_0}(\Omega))} + \|\eta\|^2_{\mathrm{L}^2_g(\mathbb{R}_+;\mathrm{H}^1_{\Gamma_0}(\Omega))}\right).$$

It follows that $\eta_s \in \mathrm{L}^2_g(\mathbb{R}_+;\mathrm{H}^1_{\Gamma_0}(\Omega))$.

Therefore, $U \in D(\mathcal{A})$. Consequently, Lumper-Phillips' theorem guarantees the generator $\mathcal{A}$ of a $\mathcal{C}_0-$semigroup on $\mathbb{H}$. The proof is completed. ■

6.3 Stability result

To obtain the general decay results, we present some lemmas needed for the proof. We define a Lyapunov functional $\mathcal{L}$ as follows

$$\mathcal{L}(t) = NE(t) + N_1K_1(t) + N_2K_2(t) + N_3K_3(t) + N_4K_4(t), \tag{6.31}$$

where $N_i,\ i = 1, \cdots, 4$; are positive constants to be fixed later and the following functions defined by

$$\begin{aligned}
(g \diamond u)(t) &= \|\eta(t,s)\|^2_{\mathrm{L}_g} = \int_0^{+\infty} g(s)a(u(t)-u(t-s), u(t)-u(t-s))\mathrm{d}s,\\
E(t) &= \frac{1}{2}\langle U(t), U(t)\rangle_{\mathbb{H}} = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{\ell}{2}a(u(t),u(t)) + \frac{1}{2}(g\diamond u)(t)\\
&\quad + \frac{1}{2}b(\varphi(t),\varphi(t)) + \frac{1}{2}\|\varphi_t(t)\|_2^2 + \frac{1}{2}\|h^{1/2}m^{1/2}z(t)\|^2_{2,\Gamma_1}\\
&\quad + \frac{1}{2}\int_{\mathrm{Q}} s\mu(s)\omega^2(x,p,s,t)\mathrm{d}s\mathrm{d}p\mathrm{d}x,\\
K_1(t) &= \int_\Omega u_t(t)u(t)\mathrm{d}x + \langle hz(t), u(t)\rangle_{\Gamma_1} + \langle hz_t(t), z(t)\rangle_{\Gamma_1} + \frac{1}{2}\|h^{1/2}f^{1/2}z(t)\|^2_{2,\Gamma_1},\\
K_2(t) &= \int_\Omega \varphi_t(t)\varphi(t)\mathrm{d}x + \frac{1}{2}b(\varphi,\varphi) + \int_\Omega (\sigma(x)\cdot\nabla)u(t)\varphi(t)\mathrm{d}x,\\
K_3(t) &= \int_{\mathrm{Q}} se^{-sp}\mu(s)\omega^2(x,p,s,t)\mathrm{d}p\mathrm{d}s\mathrm{d}x\\
K_4(t) &= -\int_\Omega u_t(t)\int_0^{+\infty} g(s)(u(t)-u(t-s))\mathrm{d}s\mathrm{d}x.
\end{aligned}$$

Remark 6.3.1. Let N be large enough. According to Lemma 5.4.1, then $\mathcal{L} \sim E$.

Remark 6.3.2. Repeating exactly the same arguments in (5.16), (5.30), (5.36), (5.40) and (5.41). Then, the modified Lyapunov functional $\mathcal{L}$ defined by (6.31) satisfies (5.50) under a suitable choice of N and $N_i,\ i = 1, \cdots, 4$.

Lemma 6.3.3. *The functional*

$$K_5(t) = \int_0^t p(t-s)a(u(s), u(s))\mathrm{d}s,$$

satisfies, along the solution of (6.1), the estimate

$$K_5'(t) \leq -\frac{1}{2}(g \diamond u)(t) + \rho(1-\ell)a(u(t), u(t)) + \frac{M_0}{2}h_0(t), \tag{6.32}$$

where $p(t) = \int_t^{+\infty} g(s)\mathrm{d}s$.

Proof. The proof of (6.32) is the same as the one of Lemma 2.1.8. ■

The following theorem gives us the general stability results according to the nature of the function F_0 defined in (**H.6**)

Theorem 6.3.4. *Assume that (**H.1**)-(**H.6**) and (5.3) hold. Then, there exist two constants* $k_1, k_2 > 0$ *such that the solution of (6.1) satisfies, for all* $t \geq 0$

$$E(t) \leq \frac{E(0)}{h_1(t)}\mathcal{H}_2^{-1}\left(\frac{k_1 + k_2\int_0^t \xi(s)\mathcal{H}(h_0(s)h_1(s))\mathrm{d}s}{\int_0^t \xi(s)\mathrm{d}s}\right), \tag{6.33}$$

if F_0 *is linear. Moreover, if* F_0 *is nonlinear, then there exist other two constants* $k_3, k_4 > 0$ *such that, for all* $t \geq 0$

$$E(t) \leq \frac{E(0)}{h_1(t)}\mathcal{W}_1^{-1}\left(\frac{k_3 + k_4\int_0^t \xi(s)\mathcal{W}(h_0(s)h_1(s))\mathrm{d}s}{\int_0^t \xi(s)\mathrm{d}s}\right), \tag{6.34}$$

where ξ, $\mathcal{H}$, $\mathcal{H}_2$, $\mathcal{W}$, $\mathcal{W}_1$, h_0, *and* h_1 *are functions defined in (**H.4**), (2.28), (5.9), (2.19), and (2.31), respectively.*

Proof. **Case 1: F_0 is linear.** Let $\mathcal{L}_1(t) = \mathcal{L}(t) + K_5(t)$. By combining (5.50) and (6.32), using (5.6) and (5.16), we obtain

$$\begin{aligned}
\mathcal{L}_1'(t) &\leq -\beta_1 E(t) + \beta_2 \langle F(u_t(t)), u_t(t)\rangle_{\Gamma_1} + \frac{M_0}{2}h_0(t), \\
&\leq -\beta_1 E(t) - \beta_2 E'(t) + \frac{M_0}{2}h_0(t),
\end{aligned}$$

where β_1 and β_2 are positive constants. Therefore, for all $t \geq 0$

$$\beta_1 \int_0^t E(s)\mathrm{d}s \leq \mathcal{L}_2(0) + \frac{M_0}{2}\int_0^t h_0(s)\mathrm{d}s,$$

where $\mathcal{L}_2 = (\mathcal{L}_1 + \beta_2 E) \sim E$. It follows that

$$\int_0^t E(s)\mathrm{d}s \leq \beta_3\left(1 + \int_0^t h_0(s)\mathrm{d}s\right), \tag{6.35}$$

where $\beta_3 = \max\left\{\frac{\mathcal{L}_2(0)}{\beta_1}, \frac{M_0}{2\beta_1}\right\}$. From (6.35), we have, for a constant $C > 0$

$$\begin{aligned}\int_0^t a(u(t)-u(t-s), u(t)-u(t-s))\mathrm{d}s &\leq 2C\int_0^t \left(\|\nabla u(t)\|_2^2 + \|\nabla u(t-s)\|_2^2\right)\mathrm{d}s \\ &\leq \frac{8C}{a_0\ell}\int_0^t E(t)\mathrm{d}s \leq \frac{8\beta_3 C}{a_0\ell}\left(1 + \int_0^t h_0(s)\mathrm{d}s\right).\end{aligned}$$

Then, for $0 < c_2 < \min\left\{1, \frac{a_0\ell}{8\beta_3 C}, \frac{r}{M_1}, \frac{r_1}{M_1}\right\}$, we recall (2.31), i.e.,

$$h_1(t) = \frac{c_2}{1 + \int_0^t h_0(s)\mathrm{d}s} < 1 \quad \text{and} \quad h_1(t)\int_0^t a(\eta(t,s), \eta(t,s))\mathrm{d}s < 1. \tag{6.36}$$

In the same way as in (2.32)-(2.38, we have

$$\mathcal{F}_2'(t) \leq -\beta_5\frac{\xi(t)}{h_1(t)}\mathcal{H}_2\left(\frac{h_1(t)E(t)}{E(0)}\right) + \beta_4\frac{\xi(t)}{h_1(t)}\mathcal{H}\left(h_1(t)h_0(t)\right). \tag{6.37}$$

Since $\mathcal{H}_2'(t) = \mathcal{H}'(\varepsilon_0 t) + \varepsilon_0 t\mathcal{H}''(\varepsilon_0 t)$ and by using the strict convexity of $\mathcal{H}$ on $(0, r]$, we conclude that $\mathcal{H}_2, \mathcal{H}_2' > 0$ on $[0,1)$. As $E' < 0$ and $h_1' < 0$, then $\mathcal{H}_2\left(\frac{h_1(t)E(t)}{E(0)}\right)$ is decreasing. Hence, for $0 \leq t \leq T$, we have

$$\mathcal{H}_2\left(\frac{h_1(T)E(T)}{E(0)}\right) \leq \mathcal{H}_2\left(\frac{h_1(t)E(t)}{E(0)}\right). \tag{6.38}$$

Combining (6.37) with (6.38) and multiplying by $h_1(t)$, we get

$$h_1(t)\mathcal{F}_2'(t) + \beta_5\xi(t)\mathcal{H}_2\left(\frac{h_1(T)E(T)}{E(0)}\right) \leq \beta_4\xi(t)\mathcal{H}\left(h_1(t)h_0(t)\right).$$

Since $h_1(t)$ is a nonincreasing function, then for all $0 \leq t \leq T$

$$(h_1\mathcal{F}_2)'(t) + \beta_5\xi(t)\mathcal{H}_2\left(\frac{h_1(T)E(T)}{E(0)}\right) \leq \beta_4\xi(t)\mathcal{H}\left(h_1(t)h_0(t)\right). \tag{6.39}$$

Integrating (6.39) over $[0, T]$, we arrive at

$$\beta_5\mathcal{H}_2\left(\frac{h_1(T)E(T)}{E(0)}\right)\int_0^T \xi(s)\mathrm{d}s \leq (h_1\mathcal{F}_2)(0) + \beta_4\int_0^T \xi(s)\mathcal{H}\left(h_1(s)h_0(s)\right)\mathrm{d}s,$$

and thus

$$\mathcal{H}_2\left(\frac{h_1(T)E(T)}{E(0)}\right) \leq \left(\frac{\frac{(h_1\mathcal{F}_2)(0)}{\beta_5} + \beta_4\int_0^T \xi(s)\mathcal{H}(h_0(s)h_1(s))\mathrm{d}s}{\int_0^T \xi(s)\mathrm{d}s}\right).$$

Then, we obtain

$$E(T) \leq \frac{E(0)}{h_1(T)}\mathcal{H}_2^{-1}\left(\frac{\frac{(h_1\mathcal{F}_2)(0)}{\beta_5} + \beta_4\int_0^T \xi(s)\mathcal{H}(h_0(s)h_1(s))\mathrm{d}s}{\int_0^T \xi(s)\mathrm{d}s}\right).$$

This gives (6.33).

Case 2: F_0 is nonlinear. First, we assume that $\max\{r_1, F_0(r_1)\} < \varepsilon$; otherwise r_1 small enough. Let $\varepsilon_1 = \min\{r_1, F_0(r_1)\}$. From (5.6)-(5.7), we have

$$\begin{cases} c_1'|s| \leq |F(s)| \leq c_2'|s| & \text{for all } |s| \geq \varepsilon_1, \\ F_0(|s|) \leq |F(s)| \leq F_0^{-1}(|s|) & \text{for all } |s| < \varepsilon_1. \end{cases}$$

Then, for all $|s| \leq \varepsilon_1$

$$\mathrm{G}(F^2(s)) = |F(s)|F_0(|F(s)|) \leq sF(s),$$

which gives

$$F^2(s) \leq \mathrm{G}^{-1}(sF(s)), \quad \forall |s| \leq \varepsilon_1. \tag{6.40}$$

Note that, for a constant $c_7 > 0$

$$J(t) = \frac{1}{|\Gamma_{12}|} \int_{\Gamma_{12}} u_t(t) F(u_t(t)) \mathrm{d}\Gamma \leq -c_7 E'(t). \tag{6.41}$$

Then, by using (5.62), (5.16), (6.40) and Jensen's inequality (A.2), we get

$$\int_{\Gamma_1} F^2(u_t(t)) \mathrm{d}\Gamma \leq \mathrm{G}^{-1}\left(J(t)\right) - c_2' E'(t). \tag{6.42}$$

As above in (2.33)-(2.34), we can write (2.35) as follows

$$\mathcal{F}'(t) \leq -\beta_1 E(t) + \frac{\beta_4}{h_1(t)} \overline{\mathcal{H}}^{-1}\left(\frac{h_1(t)\lambda(t)}{\xi(t)}\right) + \beta_2 \overline{\mathrm{G}}^{-1}\left(J(t)\right) + \beta_4 h_0(t).$$

Combining this with the strictly increasing and strictly convex properties of G as in (2.22), using (6.36), we obtain

$$\mathcal{F}'(t) \leq -\beta_1 E(t) + \frac{\beta_4}{h_1(t)} \overline{\mathcal{H}}^{-1}\left(\frac{h_1(t)\lambda(t)}{\xi(t)}\right) + \frac{\beta_2}{h_1(t)} \overline{\mathrm{G}}^{-1}\left(h_1(t)J(t)\right) + \beta_4 h_0(t).$$

Let $r_0 = \min\{r, r_1\}$, $\beta_6 = \max\{\beta_2, \beta_4\}$ and $\chi(t) = \max\left\{\frac{\lambda(t)}{\xi(t)}, J(t)\right\}$. So,

$$\mathcal{F}'(t) \leq -\beta_1 E(t) + \frac{\beta_6}{h_1(t)} \mathcal{W}^{-1}\left(h_1(t)\chi(t)\right) + \beta_4 h_0(t). \tag{6.43}$$

Now, for $0 < \varepsilon_2 < r_0$, using (6.43) and the fact that $E' \leq 0$, $\mathcal{W}' > 0$, $\mathcal{W}'' > 0$ on $(0, r_0]$, we find that the functional $\mathcal{F}_3$, defined by

$$\mathcal{F}_3(t) = \mathcal{W}'\left(\varepsilon_2 \frac{h_1(t)E(t)}{E(0)}\right) \mathcal{F}(t),$$

satisfies, for some $\kappa_3, \kappa_4 > 0$

$$\kappa_3 \mathcal{F}_3(t) \leq E(t) \leq \kappa_4 \mathcal{F}_3(t).$$

We can follow the same above steps used in (2.36) in the sense of the convex function $\mathcal{W}$, then

$$\begin{aligned} \mathcal{F}_3'(t) &= \varepsilon_2 \frac{(h_1 E)'(t)}{E(0)} \mathcal{W}''\left(\varepsilon_2 \frac{h_1(t)E(t)}{E(0)}\right) \mathcal{F}(t) + \mathcal{W}'\left(\varepsilon_2 \frac{h_1(t)E(t)}{E(0)}\right) \mathcal{F}'(t) \\ &\leq -\beta_1 E(t) \mathcal{W}'\left(\varepsilon_2 \frac{h_1(t)E(t)}{E(0)}\right) + \frac{\beta_6}{h_1(t)} \mathcal{W}'\left(\varepsilon_2 \frac{h_1(t)E(t)}{E(0)}\right) \mathcal{W}^{-1}\left(h_1(t)\chi(t)\right) \\ &\quad + \beta_4 h_0(t) \mathcal{W}'\left(\varepsilon_2 \frac{h_1(t)E(t)}{E(0)}\right). \end{aligned} \tag{6.44}$$

So, with $A = \mathcal{W}'\left(\varepsilon_2 \frac{h_1(t)E(t)}{E(0)}\right)$ and $B = \mathcal{W}^{-1}\left(h_1(t)\chi(t)\right)$, using (2.24) and (2.25), thus (6.44) yields

$$\mathcal{F}_3'(t) \leq -(\beta_1 E(0) - \beta_4 \varepsilon_2) \frac{E(t)}{E(0)} \mathcal{W}'\left(\varepsilon_2 \frac{h_1(t)E(t)}{E(0)}\right) + \beta_6 \chi(t) + \beta_4 h_0(t) \mathcal{W}'\left(\varepsilon_2 \frac{h_1(t)E(t)}{E(0)}\right). \tag{6.45}$$

From (2.32) and (6.41), we observe that, for $c_8 = \min\{\beta_3, c_7\} > 0$

$$\xi(t)\chi(t) \leq -c_8 E'(t). \tag{6.46}$$

After multiplying (6.45) by $\xi(t)$, from the fact that

$$\varepsilon_2 \frac{h_1(t)E(t)}{E(0)} < r_0,$$

it follows from (6.46) that

$$\begin{aligned} \xi(t)\mathcal{F}_3'(t) \quad &\leq \quad -(\beta_1 E(0) - \beta_4 \varepsilon_2)\xi(t)\frac{E(t)}{E(0)}\mathcal{W}'\left(\varepsilon_2 \frac{h_1(t)E(t)}{E(0)}\right) \\ &\quad -\beta_6 c_8 E'(t) + \beta_4 \frac{\xi(t)}{h_1(t)} h_1(t)h_0(t)\mathcal{W}'\left(\varepsilon_2 \frac{h_1(t)E(t)}{E(0)}\right). \end{aligned}$$

Again with $A = \mathcal{W}'\left(\varepsilon_2 \frac{h_1(t)E(t)}{E(0)}\right)$ and $B = h_1(t)h_0(t)$ in the fact that $B < r_0$, we obtain

$$\mathcal{F}_4'(t) \leq -\left(\beta_1 E(0) - 2\beta_4\varepsilon_2\right)\xi(t)\frac{E(t)}{E(0)}\mathcal{W}'\left(\varepsilon_2 \frac{h_1(t)E(t)}{E(0)}\right) + \beta_4 \frac{\xi(t)}{h_1(t)}\mathcal{W}\left(h_1(t)h_0(t)\right).$$

where $\mathcal{F}_4 = (\xi\mathcal{F}_3 + \beta_6 c_8 E) \sim E$. Let us choose ε_2 small enough such that $\beta_7 = \beta_1 E(0) - 2\beta_4\varepsilon_2 > 0$. Then, we obtain, for all $t \geq 0$

$$\mathcal{F}_4'(t) \leq -\beta_7 \frac{\xi(t)}{h_1(t)}\mathcal{W}_1\left(\frac{h_1(t)E(t)}{E(0)}\right) + \beta_4 \frac{\xi(t)}{h_1(t)}\mathcal{W}\left(h_1(t)h_0(t)\right). \tag{6.47}$$

Note that $\mathcal{W}_1'(t) = \mathcal{W}'(\varepsilon_2 t) + t\mathcal{W}''(\varepsilon_2 t)$. Using the strict convexity of $\mathcal{W}$ on $(0, r_0]$, we find that $\mathcal{W}_1(t), \mathcal{W}_1'(t) > 0$ on $(0, 1]$. Since $E' < 0$ and $h_1' < 0$, we conclude that $\mathcal{W}_1\left(\frac{h_1(t)E(t)}{E(0)}\right)$ is decreasing function. Then, for $0 \leq t \leq T$, we have

$$\mathcal{W}_1\left(\frac{h_1(T)E(T)}{E(0)}\right) \leq \mathcal{W}_1\left(\frac{h_1(t)E(t)}{E(0)}\right).$$

Combining this with (6.47) and integrating the result over $[0, T]$. By using the fact that $\mathcal{W}', \mathcal{W}'' > 0$, and taking into consideration the nonincreasing properties of E and h_1, we get

$$\begin{aligned} \beta_7 \mathcal{W}_1\left(\frac{h_1(T)E(T)}{E(0)}\right)\int_0^T \xi(s)\mathrm{d}s \quad &\leq \quad \beta_7 \int_0^T \mathcal{W}_1\left(\frac{h_1(s)E(s)}{E(0)}\right)\xi(s)\mathrm{d}s \\ &\leq \quad (h_1\mathcal{F}_4)(0) + \beta_4 \int_0^T \xi(s)\mathcal{W}(h_0(s)h_1(s))\mathrm{d}s, \end{aligned}$$

and thus

$$\mathcal{W}_1\left(\frac{h_1(T)E(T)}{E(0)}\right) \leq \left(\frac{\frac{(h_1\mathcal{F}_4)(0)}{\beta_7} + \beta_4 \int_0^T \xi(s)\mathcal{W}(h_0(s)h_1(s))\mathrm{d}s}{\int_0^T \xi(s)\mathrm{d}s}\right).$$

Then, we obtain

$$E(T) \leq \frac{E(0)}{h_1(T)}\mathcal{W}_1^{-1}\left(\frac{\frac{(h_1\mathcal{F}_4)(0)}{\beta_7} + \beta_4 \int_0^T \xi(s)\mathcal{W}(h_0(s)h_1(s))\mathrm{d}s}{\int_0^T \xi(s)\mathrm{d}s}\right).$$

This gives (6.34). ■

Now, we give an example to illustrate our result.

Example 6.3.5 ([50])**.** Let $g(t) = \alpha(1+t)^{-q}$, where $q > 1$ and $0 < \alpha < q-1$ so that (**H.2**) is satisfied, for any $\beta_0 \geq q$. In this case $\xi(t) = q\alpha^{\frac{-1}{q}}$ and $\mathcal{H}(t) = t^{\frac{q+1}{q}}$. Then, there exists a constants $\alpha_1 > 0$ depending only on ε_0, q and α such that $\mathcal{H}_2(t) = \alpha_1 t^{\frac{q+1}{q}}$.

Case 1: F_0 is linear. If $s \longmapsto \|\nabla u_0(s)\|_2^2$ is not bounded and satisfies

$$m_0(1+t)^k \leq 1 + \|\nabla u_0(t)\|_2^2 \leq m_1(1+t)^k, \tag{6.48}$$

where $0 < k < q-1$ and $m_0, m_1 > 0$, then we have, for some positive constants $\alpha_i,\ i = 2,3,4,5$; depending only on q, α, m_0, m_1 and k, the estimates

$$\alpha_2(1+t)^{1+k-q} \leq h_0(t) \leq \alpha_3(1+t)^{1+k-q}, \tag{6.49}$$

$$\frac{1}{h_1(t)} \geq \alpha_4 \begin{cases} 1+\ln(1+t), & \text{if } \ q-k=2; \\ 1, & \text{if } \ q-k>2; \\ (1+t)^{k+2-q}, & \text{if } \ 1<q-k<2, \end{cases} \tag{6.50}$$

and

$$\frac{1}{h_1(t)} \leq \alpha_5 \begin{cases} 1+\ln(1+t), & \text{if } \ q-k=2; \\ 2, & \text{if } \ q-k>2; \\ (1+t)^{k+2-q}, & \text{if } \ 1<q-k<2. \end{cases} \tag{6.51}$$

Then, using the first assumption in (6.36)

$$h_1(t)h_0(t) \leq c_2\alpha_3\alpha_4 \left\{ \begin{array}{ll} \left[(1+t)(1+\ln(1+t))\right]^{-1}, & \text{if } \ q-k=2; \\ \frac{1}{2}(1+t)^{-(q-k-1)}, & \text{if } \ q-k>2; \\ (1+t)^{-1}, & \text{if } \ 1<q-k<2, \end{array} \right\} \leq r_0, \tag{6.52}$$

where $r_0 = \min\{r, r_1\}$ with $r, r_1 > 0$ defined in (**H.4**), (**H.6**), respectively.

For example, if $1 < q-k < 2$, we have, for all $0 \leq t \leq T$

$$\int_0^T \xi(s)\mathcal{H}(h_0(s)h_1(s))\mathrm{d}s = -q^2\alpha^{-\frac{1}{q}}(1+T)^{-\frac{1}{q}}.$$

It follows that, for some $k_1, k_2, \alpha_6 > 0$

$$\int_0^T \xi(s)\mathcal{H}(h_0(s)h_1(s))\mathrm{d}s < +\infty,$$

$$\mathcal{H}_2^{-1}\left(\frac{k_1 + k_2\int_0^t \xi(s)\mathcal{H}(h_0(s)h_1(s))\mathrm{d}s}{\int_0^t \xi(s)\mathrm{d}s}\right) \leq \alpha_6 T^{-\left(\frac{q}{q+1}\right)}. \tag{6.53}$$

Using (6.49)-(6.53), then there exists a constant $\alpha_7 > 0$ such that, the energy functional (6.33) satisfies

$$E(T) \leq \alpha_7 \begin{cases} (1+\ln(1+T))\,T^{-\left(\frac{q}{q+1}\right)}, & \text{if } \ q-k=2; \\ T^{-\left(\frac{q}{q+1}\right)}, & \text{if } \ q-k>2; \\ (1+T)^{-\left(q-k-2+\frac{q}{q+1}\right)}, & \text{if } \ 1<q-k<2. \end{cases} \tag{6.54}$$

For $q-k \geq 2$ or $1/2\left(k+\sqrt{k^2+4k+8}\right) < q < k+2$, the estimate (6.54) gives $\lim\limits_{T\to\infty} E(T) = 0$.

On the other hand, if $s \longmapsto \|\nabla u_0(s)\|_2^2$ is bounded ($k = 0$ in (6.48)) as it was assumed in [49, 51], then, for $q > \sqrt{2}$, (6.54) holds with $k = 0$. This result generalizes and improves the decay rate $\left((1+t)^{-k},\ \text{for } 0 < k < \frac{q-1}{q+1}\right)$ obtained in [49] .

Case 2: F_0 is nonlinear. Assume that $F_0(t) = ct^{\beta}$, where $\beta > 1$ and $\mathrm{G}(t) = \sqrt{t}F_0(\sqrt{t}) = ct^{\frac{\beta+1}{2}}$. Thus,

$$\mathcal{W}^{-1}(t) = \left(\mathrm{G}^{-1} + \mathcal{H}^{-1}\right)(t) = c\left(t^{\frac{2}{\beta+1}} + t^{\frac{q}{q+1}}\right).$$

If $q = 2$ and $\beta = 5$, we find that $\mathrm{G}(t) = ct^3$ and $\mathcal{H}(t) = t^{\frac{3}{2}}$. Here $g(t) = \alpha(1+t)^{-2}$ and $\xi(t) = \gamma$, where γ is a fixed constant and α is chosen so that assumption (2.21) remains valid. Then, we have, for all $t \geq 0$

$$\mathcal{W}(t) = c\left(\frac{\sqrt{4t+1}-1}{2}\right)^3, \tag{6.55}$$

and

$$\begin{aligned} \mathcal{W}_1(t) = \frac{3ct}{\sqrt{4t+1}}\left(\frac{\sqrt{4t+1}-1}{2}\right)^2 &= \frac{3ct}{2\sqrt{4t+1}} + \frac{3ct^2}{\sqrt{4t+1}} - \frac{3ct}{2} \\ &\leq \frac{3ct}{2} + \frac{3ct^2}{2\sqrt{t}} - \frac{3ct}{2} = ct^{\frac{3}{2}}. \end{aligned} \tag{6.56}$$

Let us consider

$$m_0(1+t)^k \leq 1 + \|\nabla u_0(t)\|_2^2 \leq m_1(1+t)^k, \tag{6.57}$$

where $0 \leq k < 1$ and $m_0, m_1 > 0$, then we have the following, for some $\alpha_i,\ i = 2, 3, 4$;

$$\alpha_2(1+t)^{-1+k} \leq h_0(t) \leq \alpha_3(1+t)^{-1+k}, \tag{6.58}$$

$$\frac{1}{h_1(t)} \geq \alpha_4 \begin{cases} 1 + \ln(1+t), & \text{if } k = 0; \\ (1+t)^k, & \text{if } 0 < k < 1, \end{cases} \tag{6.59}$$

and

$$\frac{1}{h_1(t)} \leq \alpha_5 \begin{cases} 1 + \ln(1+t), & \text{if } k = 0; \\ (1+t)^k, & \text{if } 0 < k < 1. \end{cases} \tag{6.60}$$

It follows from (6.58)-(6.59) that

$$h_1(t)h_0(t) \leq c_2\alpha_3\alpha_4 \begin{cases} \left[(1+t)(1+\ln(1+t))\right]^{-1}, & \text{if } k = 0; \\ (1+t)^{-1}, & \text{if } 0 < k < 1; \end{cases} \tag{6.61}$$

Combining (6.55), (6.56) and (6.61), we get, for some $k_3, k_4, \beta_6 > 0$, for all $0 \leq t \leq T$

$$\int_0^T \xi(s)\mathcal{W}(h_0(s)h_1(s))\mathrm{d}s < +\infty.$$

$$\mathcal{W}_1^{-1}\left(\frac{k_3 + k_4\int_0^t \xi(s)\mathcal{W}(h_0(s)h_1(s))\mathrm{d}s}{\int_0^t \xi(s)\mathrm{d}s}\right) \leq \beta_6 T^{-\frac{2}{3}}.$$

Then there exists a constant $\alpha_7 > 0$ such that, the energy functional (6.34) satisfies

$$E(T) \leq \alpha_7 \begin{cases} (1 + \ln(1+T))\,T^{-\frac{2}{3}}, & \text{if } k = 0; \\ (1+T)^{-\left(\frac{2}{3}-k\right)}, & \text{if } 0 < k < 1. \end{cases} \tag{6.62}$$

Thus, for $0 \leq k < \frac{1}{3}$, the estimate (6.62) gives $\lim\limits_{T\to+\infty} E(T) = 0$.

Appendices

Appendix A

Basic Concepts in Standard Application

Throughout this thesis, we recall a well-known concepts in standard application. We will also present the properties that will interest us, as well as the results known in the case of partial differential equations (For further proofs of the following theorems and lemmas, we refer the reader to references [29, 41, 54, 81]).

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A.1 Unbounded operators

Let $\mathbb{H}$ be a real or complex Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and the induced norm $\| \cdot \|_{\mathbb{H}}$, and $\mathcal{A}$ is a linear mapping from $D(\mathcal{A}) \subset \mathbb{H}$ into $\mathbb{H}$. The subspace $D(\mathcal{A})$ is called the domain of the operator $\mathcal{A}$.

Definition A.1.1. Let $\mathcal{A}$ be a linear (unbounded) operator on $\mathbb{H}$. $\mathcal{A}$ is dissipative if and only if

$$\forall \varphi \in D(\mathcal{A}), \quad \Re \langle \mathcal{A}\varphi, \varphi \rangle_{\mathbb{H}} \leq 0.$$

Remark A.1.2. If $\mathbb{H}$ is a Banach space, a linear operator $\mathcal{A}$, is dissipative if and only if

$$\forall \varphi \in D(\mathcal{A}), \quad \forall \lambda > 0, \quad \|(\lambda \mathrm{I} - \mathcal{A})\varphi\|_{\mathbb{H}} \geq \lambda \|\varphi\|_{\mathbb{H}}.$$

Definition A.1.3. An unbounded linear operator $\mathcal{A}$ on $\mathbb{H}$, is m-dissipative if and only if

- $\mathcal{A}$ is dissipative,
- $\forall f \in \mathbb{H}$, $\forall \lambda > 0$, $\exists \varphi \in D(\mathcal{A})$ such that

$$(\lambda \mathrm{I} - \mathcal{A})\varphi = f.$$

Remark A.1.4. Let $\mathcal{A}$ be a dissipative operator on $\mathbb{H}$. The operator $\mathcal{A}$ is m-dissipative if and only if

$$\exists \lambda_0 > 0, \, \forall f \in \mathbb{H}, \, \exists \varphi \in D(\mathcal{A}) \mid (\lambda_0 \mathrm{I} - \mathcal{A})\varphi = f.$$

Corollary A.1.5. *If $\mathcal{A}$ is a m-dissipative operator then $\mathcal{A}$ is closed and $D(\mathcal{A})$ is dense in $\mathbb{H}$.*

Theorem A.1.6. *If $\mathcal{A}$ is a m-dissipative operator then, for all $\lambda > 0$, the operator $(\lambda \mathrm{I} - \mathcal{A})$ admits an inverse, $(\lambda \mathrm{I} - \mathcal{A})^{-1} f$ belongs to $D(\mathcal{A})$ for all $f \in \mathbb{H}$, and $R(\lambda; \mathcal{A}) = (\lambda \mathrm{I} - \mathcal{A})^{-1}$ is a linear bounded operator on $\mathbb{H}$ satisfying*

$$\|R(\lambda; \mathcal{A})\|_{\mathscr{L}(\mathbb{H})} \leq \frac{1}{\lambda}.$$

Definition A.1.7. The resolvent set $\rho(\mathcal{A})$ of an operator $\mathcal{A}$ is the set

$$\rho(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \lambda \mathrm{I} - \mathcal{A} \text{ is invertible}\}.$$

The spectrum of operator $\mathcal{A}$, denoted $\sigma(\mathcal{A})$, is the complement of the resolvent set $\rho(\mathcal{A})$

$$\sigma(\mathcal{A}) = \mathbb{C} \setminus \rho(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } \mathcal{A}\},$$

where each eigenvalue of operator $\mathcal{A}$ is a complex number λ such that there exists a nonzero $\varphi \in \mathbb{H}$, called eigenvector with property $\mathcal{A}\varphi = \lambda\varphi$, in other words $\varphi \in \ker(\lambda \mathrm{I} - \mathcal{A})$.

Corollary A.1.8.

1. *If $|\lambda| > \|\mathcal{A}\|$ then $\lambda \in \rho(\mathcal{A})$,hence the spectrum is bounded.*

2. *The resolvent set $\rho(\mathcal{A})$ is open, i.e. for any $\lambda \in \rho(\mathcal{A})$ then there exist $\epsilon > 0$ such that all μ with $|\lambda - \mu| < \epsilon$ are also in $\rho(\mathcal{A})$, i.e. the resolvent set is open and the spectrum is closed.*

Both statements together imply that the spectrum is compact.

A.2 Semigroup on a Hilbert space

We are interested in equation

$$\varphi'(t) = \mathcal{A}\varphi(t), \quad \varphi(0) = \varphi_0 \in \mathbb{H}, \qquad \forall t \in \mathbb{R}_+. \tag{A.1}$$

When equation (A.1) does admit a solution in $\mathcal{C}^1(\mathbb{R};\mathbb{H})$ given by

$$\varphi(t) = S(t)\varphi_0, \quad \forall t \geq 0.$$

Definition A.2.1. A family of bounded linear operators $\{S(t),\ t \geq 0\}$ on $\mathbb{H}$ is a strongly continuous semigroup ($\mathcal{C}_0-$semigroup) when the following conditions hold:

(i) $S(0) = \mathrm{I}$,

(ii) $S(t+s) = S(t)S(s),\ \forall t, s \geq 0$,

(iii) $\lim\limits_{t\to 0} \|S(t)\varphi - \varphi\|_{\mathbb{H}} = 0,\ \forall \varphi \in \mathbb{H}$.

Remark A.2.2. For all $\varphi \in \mathbb{H}$, the mapping $t \to S(t)\varphi$ is continuous from $\mathbb{R}_+$ into $\mathbb{H}$.

Definition A.2.3. A semigroup of bounded linear operator, $S(t)$, is uniformly continuous if

$$\lim_{t\to 0} \|S(t) - \mathrm{I}\|_{\mathscr{L}(\mathbb{H})} = 0.$$

Definition A.2.4. The linear operator $\mathcal{A}$ defined by, with domain $D(\mathcal{A})$ consisting of points φ such that the following limit exists

$$\mathcal{A}\varphi = \lim_{t\to 0} \frac{S(t)\varphi - \varphi}{t} = \left[\frac{\mathrm{d}}{\mathrm{d}t} S(t)\varphi\right]_{t=0}, \text{ for } \varphi \in D(\mathcal{A}),$$

is called the infinitesimal generator of the semigroup $S(t)$.

Theorem A.2.5. *Let $S(t)$ be a strongly continuous semigroup on $\mathbb{H}$. Then there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|S(t)\|_{\mathscr{L}(\mathbb{H})} \leq M e^{\omega t}, \quad \forall t \geq 0.$$

Remark A.2.6. A $\mathcal{C}_0-$semigroup $S(t) = e^{\mathcal{A}t}$ generated by $\mathcal{A}$

- is exponentially stable if $M \geq 1$ and $\omega < 0$.
- is uniformly bounded if $M \geq 1$ and $\omega \geq 0$.
- If $M = 1$ it is called a $\mathcal{C}_0-$semigroup of contractions.

Theorem A.2.7. *Let $\mathcal{A}$ be the infinitesimal generator of $S(t)$, a $\mathcal{C}_0-$semigroup on $\mathbb{H}$. For any initial data $\varphi_0 \in \mathbb{H}$, the problem (A.1) has a unique weak solution $\varphi \in \mathcal{C}(\mathbb{R}_+;\mathbb{H})$. Moreover, if $\varphi_0 \in D(\mathcal{A})$, then the solution*

$$\varphi \in \mathcal{C}(\mathbb{R}_+; D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+;\mathbb{H}).$$

Proof. **Existence:** Let $\varphi_0 \in D(\mathcal{A})$ and set $\varphi(t) = S(t)\varphi_0$. We know that

$$\mathcal{A}S(t)\varphi_0 = S(t)\mathcal{A}\varphi_0.$$

Since the mapping $t \to \mathcal{A}S(t)\varphi_0$, is continuous from $\mathbb{R}_+$ into $\mathbb{H}$, then $\varphi \in \mathcal{C}(\mathbb{R}_+; D(\mathcal{A}))$.

Moreover

$$\frac{\mathrm{d}}{\mathrm{d}t}S(t)\varphi_0 = \mathcal{A}S(t)\varphi_0 = S(t)\mathcal{A}\varphi_0.$$

Therefore $\varphi \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{H})$ and $\varphi' = \mathcal{A}\varphi$.
Uniqueness: Let $t > 0$ be arbitrarily fixed. Let $u \in \mathcal{C}(\mathbb{R}_+; D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+;\mathbb{H})$ be an other solution of the problem. Set

$$v(s) = S(t-s)u(s), \quad \text{for} \quad 0 \le s \le t.$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}v(s) = -\mathcal{A}S(t-s)u(s) + S(t-s)\mathcal{A}u(s) = 0.$$

Consequently $v(s) = v(0)$ for all $s \in [0,t]$. In particular $v(t) = u(t)$ and $v(0) = \varphi(t)$. Thus $u(t) = \varphi(t)$. ■

Theorem A.2.8 (Hille-Yosida). *A linear (unbounded) operator $\mathcal{A}$ is the infinitesimal generator of a $\mathcal{C}_0-$semigroup of contractions $S(t)$, if and only if*

(a) *$\mathcal{A}$ is closed and $\overline{D(\mathcal{A})} = \mathbb{H}$,*

(b) *the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ contains $\mathbb{R}_+$ and for every $\lambda > 0$,*

$$\|(\lambda \mathrm{I} - \mathcal{A})^{-1}\|_{\mathscr{L}(\mathbb{H})} \le \frac{1}{\lambda}.$$

Theorem A.2.9 (Lumer-Phillips). *An unbounded linear operator $\mathcal{A}$ in $\mathbb{H}$ is the infinitesimal generator of a semigroup of contractions on $\mathbb{H}$ if and only if $\mathcal{A}$ is m-dissipative in $\mathbb{H}$.*

Theorem A.2.10 (Gearhart-Prüss). *Let $S(t)$ be a $\mathcal{C}_0-$semigroup of contractions on Hilbert space $\mathbb{H}$. Then $S(t)$ is exponentially stable if and only if*

(i) *The resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ contains the imaginary axis ($i\mathbb{R} \subset \rho(\mathcal{A})$),*

(ii) $\limsup\limits_{|\lambda|\to\infty} \|(i\lambda \mathrm{I} - \mathcal{A})^{-1}\|_{\mathscr{L}(\mathbb{H})} < \infty.$

A.3 Principal theorems and inequalities

Notation A.3.1. Let $1 \le p \le +\infty$; we denote by q the conjugate exponent

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Theorem A.3.2 (Hölder's inequality). *Assume that $f \in \mathrm{L}^p(\Omega)$ and $g \in \mathrm{L}^q(\Omega)$ with $1 \le p \le +\infty$. Then $fg \in \mathrm{L}^1(\Omega)$ and*

$$\int_\Omega |f(x)g(x)|\mathrm{d}x \le \|f\|_p \|g\|_q.$$

Corollary A.3.3 (Cauchy-Schwarz's inequality). *By taking $p = q = 2$, we obtain*

$$(f,g) \le \|f\|_2 \|g\|_2.$$

Theorem A.3.4 (Young's inequality). *Let $a, b \ge 0$. Then for any $\delta > 0$,*

$$ab \le \delta a^2 + \frac{b^2}{4\delta}.$$

Lemma A.3.5. *For all $a, b \geq 0$, the following inequality holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Theorem A.3.6 (Jensen's inequality)**.** *Suppose φ is a non-negative measurable function satisfying $\int_\Omega \varphi(x)dx = 1$. If f is any real-valued measurable function and Ψ is convex over the range of f, then Jensen's inequality states that*

$$\Psi\left[\int_\Omega f(x)\varphi(x)dx\right] \leq \int_\Omega \Psi[f(x)]\varphi(x)dx. \tag{A.2}$$

Theorem A.3.7 (Gronwall's inequality)**.** *Assume $\varphi : [0,\ T] \to \mathbb{R}$ is a bounded nonnegative measurable function, $f : [0,\ T] \to \mathbb{R}$ is a nonnegative integrable function and $\alpha \geq 0$ is a constant with the property that*

$$\varphi(t) \leq \alpha + \int_0^t f(\tau)\varphi(\tau)\mathrm{d}\tau, \quad \forall t \in [0,\ T].$$

Then

$$\varphi(t) \leq \alpha \exp\left(\int_0^t f(\tau)\mathrm{d}\tau\right), \quad \forall t \in [0,\ T].$$

Theorem A.3.8 (Schauder fixed point)**.** *A continuous compact mapping on a closed, bounded, convex set in a Banach space has a fixed point. Alternatively, a continuous mapping on a compact, convex, set in a Banach space has a fixed point.*

Theorem A.3.9 (Rellich-Kondrachov)**.** *Suppose that Ω is bounded and of class $\mathcal{C}^1$. Then we have the following compact injections:*

$$\begin{aligned}
&\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}^q(\Omega), \quad \forall q \in [1,\ p^*), \qquad \text{where} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, && \text{if} \quad p < n,\\
&\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}^q(\Omega), \quad \forall q \in [1,\ +\infty), && \text{if} \quad p = n,\\
&\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega}), && \text{if} \quad p > n,
\end{aligned}$$

Corollary A.3.10. *In particular, $\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}^p(\Omega)$ with compact injection for all p (and for all n).*

Definition A.3.11. For any $u \in \mathcal{C}^\infty(\Omega)$, define the trace operator $\gamma_0 : \mathrm{H}^1(\Omega) \to \mathrm{L}^2(\Gamma)$ by

$$\gamma_0(u) = u_{|\Gamma},$$

and the Neumann trace operator $\gamma_1 : \mathrm{H}^2(\Omega) \to \mathrm{L}^2(\Gamma)$ by

$$\gamma_1(u) = \left(\frac{\partial u}{\partial \nu}\right)_\Gamma.$$

Proposition A.3.12. *The most important properties of the trace are the following*

(i) *If $u \in \mathrm{H}^1(\Omega)$, then in fact $\gamma_0(u) \in \mathrm{H}^{1/2}(\Gamma)$ and*

$$\|\gamma_0(u)\|_{\mathrm{H}^{1/2}(\Gamma)} \leq C\|u\|_{\mathrm{H}^1(\Omega)}, \quad \forall u \in \mathrm{H}^1(\Omega).$$

(ii) *The kernel of the trace operator is $\mathrm{H}^1_0(\Omega)$, i.e.,*

$$\mathrm{H}^1_0(\Omega) = \Big\{u \in \mathrm{H}^1(\Omega) \ |\ \ \gamma_0(u) = 0 \text{ on } \Gamma\Big\}.$$

(iii) *Set* $\mathrm{A}\nabla u \in \mathrm{H}(\mathrm{div}, \Omega)$*, then* $\dfrac{\partial u}{\partial \nu_{\mathrm{A}}}$ *has a meaning and belongs to* $\mathrm{H}^{-1/2}(\Gamma)$*. Moreover, there exists a constant* $C > 0$ *such that*

$$\left| \left\langle \frac{\partial u}{\partial \nu_{\mathrm{A}}}, v \right\rangle_{\mathrm{H}^{-1/2}(\Gamma), \mathrm{H}^{1/2}(\Gamma)} \right| \leq C \|\nabla u\|_{\mathrm{H}(\mathrm{div},\Omega)} \|v\|_{\mathrm{H}^{1/2}(\Gamma)} \qquad \forall v \in \mathrm{H}^{1/2}(\Gamma). \tag{A.3}$$

Theorem A.3.13 (Trace theorem)**.** *Let* Ω *be a bounded simply connected Lipschitz domain. Then the trace operator* $u \longmapsto \{\gamma_0(u), \gamma_1(u)\}$ *is bounded, linear, and surjective from* $\mathrm{H}^2(\Omega)$ *onto* $\mathrm{H}^{3/2}(\Gamma) \times \mathrm{H}^{1/2}(\Gamma)$*.*

Theorem A.3.14 (Poincaré's inequality)**.** *Suppose that* $1 \leq p \leq +\infty$ *and* Ω *is a bounded (at least in one direction) open set. Then there exists a constant* C^* *(depending on* Ω *and* p*) such that*

$$\|\varphi\|_p \leq C_* \|\nabla \varphi\|_p, \quad \forall \varphi \in \mathrm{w}_0^{1,p}(\Omega). \tag{A.4}$$

Lemma A.3.15 (Green's formula)**.** *Let* Ω *be a bounded domain of* $\mathbb{R}^n$ *with a smooth boundary* Γ*. For* $u \in \mathrm{H}^2(\Omega)$ *and* $v \in \mathrm{H}^1(\Omega)$*, we have*

$$\int_\Omega \Delta u v \mathrm{d}x = \int_\Gamma \frac{\partial u}{\partial \nu} v \mathrm{d}\Gamma - \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x$$

Remark A.3.16. If $v \in \mathrm{H}_0^1(\Omega)$, then Green's formula is reduced to

$$\int_\Omega \Delta u v \mathrm{d}x = - \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x$$

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الملخص: في هذه الرسالة، نعتبر بعض المسائل المرتبطة باللزوجة المرنة من أجل مؤثر قطعي ناقصي بشدة من الدرجة الثانية ذي معاملات متغيرة على ميادين محدودة. تقييم الدراسات الحديثة حول نظريات المرونة الحرارية المعممة والنماذج المعدلة المرتبطة بها سوف تؤخذ بعين الاعتبار. في هذا الصدد نقوم بدراسة العديد من الأنظمة المقترنة بشروط حدودية مختلطة من نوع دريكلي-نيومان بإضافة الى ردود فعل التخميد على الحدود. الاقتران يتم عبر شروط الحدود الصوتية المعتبرة على جزء من الحافة. باستخدام طرق مهمة في التحليل الرياضي، مثل نظرية نصف الزمرة ونظرية النقطة الثابتة لشودر وتقريبات فاودو-قلاركين وتقديرات التراص، نبرهن الوجود المحلي والإجمالي للحلول المرتبطة بالطاقة. علاوة على ذلك، مع الأخذ في عين الاعتبار نظرية جيرهارت-برايس، نقوم بإثبات الاستقرار الأسي بواسطة نصف الزمرة المرفقة. نتطرق أيضا، من أجل فئة واسعة جدًا من توابع الاسترخاء، إلى العديد من النتائج المرتبطة بالتناقص العام للطاقة باستعمال الطرق الطاقوية.

الكلمات المفتاحية: استقرار أسي، تخميد لزج-مرن، تأثيرات حرارية، تناقص عام، شروط حدية صوتية، وجود إجمالي للحل.

Abstract: In this thesis, we consider some viscoelastic problems for a strongly elliptic operator of second order with variable coefficients in bounded domains. A review of the recent studies on the generalized thermoelasticity theories and their associated modified models is also presented. In this regard, we investigate several coupled systems with mixed Dirichlet-Neumann boundary conditions and boundary interaction feedback. The coupling is via the acoustic boundary conditions on a portion of the boundary. Using some interesting methods of mathematical analysis as, semigroup theory, Schauder fixed point, Faedo-Galerkin approach, and compactness estimates, to show the local and global existence of energy-associated solutions. In addition, taking into account the Gearhart-Prüss theorem, the exponential stability of the corresponding semigroup is concluded. Moreover, under a very wider class of generality of relaxation functions, we establish several general decay results using the energy methods.

Keywords: Acoustic boundary conditions, Exponential stability, General decay, Global existence of solution, Thermoelastic effect, Viscoelastic damping.

Résumé: Dans cette thèse, nous considérons quelques problèmes viscoélastiques pour un opérateur fortement elliptique du second ordre avec des coefficients variables dans des domaines bornés. Une évaluation des études récentes, sur les théories de la thermoélasticité généralisée et leurs modèles modifiés associés, est également présentée. À cet égard, nous étudions plusieurs systèmes couplés avec des conditions aux limites mixtes Dirichlet-Neumann et une rétroaction d'interaction aux limites. Le couplage se fait via les conditions aux limites acoustiques sur une portion de la frontière. En utilisant des méthodes intéressantes d'analyse mathématique comme la théorie des semi-groupes, le point fixe de Schauder, les approximations de Faedo-Galerkin et les estimations de compacité, pour montrer l'existence locale et globale des solutions associées à l'énergie. De plus, en tenant compte du théorème de Gearhart-Prüss, la stabilité exponentielle du semi-groupe correspondant est obtenue. Ensuite, sous une classe de généralité très large des fonctions de relaxation, nous établissons plusieurs résultats généraux de décroissance en utilisant des méthodes énergétiques.

Mots clés: Amortissement viscoélastique, Conditions aux limites acoustiques, Effet thermoélastique, Existence globale de solution, Décroissance générale, Stabilité exponentielle.