Journal of Computational and Applied Mathematics 377 (2020) 112894

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Efficient spectral-collocation methods for a class of linear Fredholm integro-differential equations on the half-line

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ARTICLE INFO

Article history:
Received 14 November 2019
Received in revised form 19 February 2020

MSC:
45J05
65R20
65N35

Keywords:
Fredholm integro-differential equations
Mapped Legendre
Half-line
Function approximation

ABSTRACT

In this paper, an extension of the Legendre spectral collocation method has been proposed for the numerical solution of a class of linear Fredholm integro-differential equation on the half-line. The properties of mapped Legendre functions are first presented. These properties together with the Legendre–Gauss points are then utilized to reform the Fredholm integro-differential equation in semi-infinite interval into a singular equation in finite interval and to reduce it to the solution of a simple matrix equation. Besides, in order to show the efficiency and accuracy of the proposed method, some numerical examples are considered and solved through a survey of three approaches, namely: Exponential, rational and logarithmic Legendre functions collocation methods. Furthermore, a comparison of the results, shows that using exponential functions, leads to more accurate results and faster convergence.

1. Introduction

Integro-differential equation (IDE) has a great deal of application in different branches of sciences and engineering. It arises naturally in a variety of models from applied mathematics, physics, and other disciplines, such as elasticity theory, electromagnetic, electrodynamics, fluid dynamics, radiative equilibrium, heat and mass transfer, oscillating magnetic field, etc. However, in practice, the IDEs are too difficult to solve analytically.

The present paper focuses on the numerical solution of linear Fredholm-type integro-differential equation of the form

$$\sum_{k=0}^{2} \nu_k(x)u^{(k)}(x) - \int_0^{\infty} k(x,t)u(t)dt = f(x), \quad x \in [0,\infty), \tag{1}$$

under the boundary conditions

$$u(0) = \alpha, \qquad u(\infty) = \beta, \tag{2}$$

In Eq. (1), $k(x,t)$, $f(x)$ and $\nu_k(x)$, $k = 0, 1, 2$ are given continuous functions and $u(x)$ is unknown function. Furthermore, the integral part is assumed to be an improper Riemann integral and its value is assumed to exist.

Various spectral methods are available in the literature to solve problems in semi-infinite domains, $x \in [0,\infty)$. They can be essentially classified in four approaches. The classical approach is to expand the underling solution of the problem in Laguerre polynomials or functions [1]. The second approach consists to solve the problem on a large but finite

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https://doi.org/10.1016/j.cam.2020.112894

interval, $x \in [0, L]$ using Chebyshev polynomials with an appropriate argument-transform. This method is named domain truncation [2]. A third approach is to reformulate the original problem in semi-infinite domain into a singular problem in bounded domain by variable transformation and then using the Jacobi polynomials to approximate the resulting equation [3]. A fourth approach is based on rational orthogonal functions [4].

Baharifard et al. [5] applied rational and exponential Legendre Tau method on the semi-infinite interval for solving the steady flow of a third grade fluid in a porous half space, Coulaud et al. [6], Maday et al. [7], Rahmoune [8] and Shen [9] considered Laguerre spectral approximations for linear problems on the half line. Guo and Shen [10] proposed Laguerre–Galerkin method for non linear partial differential equations on a semi-infinite interval. Guo [11,12] developed a Jacobi spectral method to differential equations on the half-line.

The main objective of the present work is to extend the Legendre spectral method to a class of Fredholm integro-differential equations on the half-line. The key idea is to transform integro-differential equation on the half-line into a singular equation on the finite interval $[-1, 1]$, by means of a suitable family of one-to-one mappings, and then apply the Legendre spectral collocation method to solve the resulting equation. In Section 2, we introduce some properties of mapped Legendre functions. In Section 3, the Legendre collocation method is presented for the resulting equation in which the differential and integral parts are replaced by their operational matrix representations with collocation points. In Section 4, some numerical results are given to clarify and demonstrate the feasibility of the proposed method. The following abbreviations are used throughout the present article, whenever convenient: ELC — Exponential Legendre collocation. RLC — Rational Legendre collocation. LLC — Logarithmic Legendre collocation.

2. Mapped Legendre functions for the semi-infinite interval

In this section, we introduce mapped Legendre functions and express some of their basic properties. Moreover, we present function approximations using orthogonal mapped Legendre functions basis in some weighted $L^2_{\rho_s}[0, \infty)$ space.

2.1. Mapped Legendre functions

The well-known Legendre polynomials are orthogonal in the interval $I = [-1, 1]$ with respect to the uniform weight function. They can be determined with the help of the following recurrence formula [1]:

$$P_{n+1}(y) = \frac{2n+1}{n+1} y P_n(y) - \frac{n}{n+1} P_{n-1}(y), \; n \geq 1, \tag{3}$$

$$(2n+1)P_n(y) = \partial_y P_{n+1}(y) - \partial_y P_{n-1}(y), \; n \geq 1. \tag{4}$$

Besides

$$P_0(y) = 1, \qquad P_1(y) = y, \qquad P_n(1) = 1, \qquad P_n(-1) = (-1)^n. \tag{5}$$

The set of Legendre polynomials forms an $L^2(I)$-orthogonal system, namely,

$$\int_{-1}^{1} P_n(y) P_m(y) dy = \frac{2}{2n+1} \delta_{n,m}, \tag{6}$$

where $\delta_{n,m}$ is the Kronecker delta function. Furthermore, for any function $U \in L^2(I)$, we write

$$U(y) = \sum_{j=0}^{\infty} c_j P_j(y) \quad \text{with } c_j = \frac{2j+1}{2} \int_{-1}^{1} U(y) P_j(y) dy.$$

Next, let us consider a family of one-to-one mappings between $y \in I = [-1, 1]$ and $x \in \Lambda = [0, \infty)$ of the form

$$x = \varphi_s(y), \quad y \in I, \quad s > 0, \tag{7}$$

such that

$$\frac{dx}{dy} = \varphi_s'(y) > 0, \quad \varphi_s(-1) = 0, \quad \varphi_s(1) = \infty, \tag{8}$$

and s is a positive scaling factor. In an adaptive procedure, s would be chosen to minimize some measure of the error. Without loss of generality, we assume that the mapping (7) is explicitly invertible and denote

$$y = \varphi_s^{-1}(x), \quad x \in \Lambda, \quad s > 0. \tag{9}$$

Several interesting mappings have been proposed and implemented in practice. However, the most frequently used are exponential [13], rational and logarithmic functions [2,3].

For a given mapping $y = \varphi_s^{-1}(x)$, we define the so-called mapped Legendre functions by

$$L_{s,n}(x) = P_n(\varphi_s^{-1}(x)), \quad n = 0, 1, 2, \dots. \tag{10}$$

They are orthogonal in the interval $[0, \infty)$ with respect to the weight function

$$\rho_s(x) = \frac{dy}{dx}, \tag{11}$$

with the orthogonality property:

$$\int_0^\infty L_{s,n}(x)L_{s,m}(x)\rho_s(x)dx = \frac{2}{2n+1}\delta_{n,m}. \tag{12}$$

2.2. Function approximation

Let us define

$$L^2_{\rho_s}(\Lambda) := \left\{u : \Lambda \to \mathbb{R} \mid u \text{ is measurable and } \|u\|_{\rho_s} < \infty\right\},$$

where

$$\|u\|^2_{\rho_s} := \int_0^\infty |u(x)|^2 \rho_s(x)dx,$$

is the norm induced by the inner product of the space $L^2_{\rho_s}(\Lambda)$,

$$\langle u, v\rangle_{\rho_s} := \int_0^\infty u(x)v(x)\rho_s(x)dx. \tag{13}$$

Thus, $\{L_{s,j}\}_{j=0}^\infty$ denotes a system which is mutually orthogonal under the inner product given by (13), i.e.,

$$\langle L_{s,n}, L_{s,m}\rangle_{\rho_s} = \frac{2}{2n+1}\delta_{n,m}.$$

It is not hard to show that $\{L_{s,j}\}_{j=0}^\infty$ forms a complete basis in $L^2_{\rho_s}(\Lambda)$. Thus, for any function $u \in L^2_{\rho_s}(\Lambda)$, the following expansion holds

$$u(x) = \sum_{j=0}^\infty \hat{u}_{s,j}L_{s,j}(x) \text{ with } \hat{u}_{s,j} = \frac{2j+1}{2}\int_0^\infty u(x)L_{s,j}(x)\rho_s(x)dx. \tag{14}$$

If $u(x)$ is truncated up to the $N+1$ terms, then it can be written as

$$u(x) \simeq u_N(x) = \sum_{j=0}^N \hat{u}_{s,j}L_{s,j}(x). \tag{15}$$

Now, to provide an upper bound of the truncation error in the weighted L^2-norm, we quote the following theorem from Baharifard et al. [5]. At first, we denote

$$e_N := \|u - u_N\|^2_{\rho_s}, \qquad U_s(y) = u(x) := u(\varphi_s(y)). \tag{16}$$

Theorem 1. *Let $u(x) \in L^2_{\rho_s}(\Lambda)$, $s > 0$, and let $u_N(x)$ be the truncated series obtained by the approximation (15). Further, assume $U_s(y) = u(\varphi_s(y))$ is analytic on $[-1, 1]$. Then an error bound for this approximation can be presented as follows:*

$$e_N \le \sum_{i=N+1}^\infty \frac{4^{i+1}M_i^2 i!^2}{2(2i)!(2i+1)!}, \tag{17}$$

where $M_i := \max|U_s^{(i)}(y)|, y \in [-1, 1]$. Moreover, if $M \ge M_i$, then the error is superlinearly convergent to zero. Fig.2

3. The solution method

By applying a mapping (7), Eq. (1) can be reformed as

$$\sum_{k=0}^2 H_{s,k}(y)U_s^{(k)}(y) - \int_{-1}^1 K_s(y,z)U_s(z)\varphi_s'(z)dz = F_s(y), \quad s > 0, \quad -1 \le y < 1, \tag{18}$$

under the boundary conditions

$$U_s(-1) = \alpha, \qquad U_s(1) = \beta, \tag{19}$$

where $K_s(y,z) = k(\varphi_s(y), \varphi_s(z))$, $F_s(y) = f(\varphi_s(y))$ and $H_{s,k}(y)$ are functions defined in the interval $-1 \le y, z < 1$ with

$$H_{s,0}(y) = \nu_0(\varphi_s(y)), \qquad H_{s,1}(y) = \frac{\nu_1(\varphi_s(y))}{\varphi_s'(y)} - \frac{\varphi_s''(y)}{(\varphi_s'(y))^3}\nu_2(\varphi_s(y)), \qquad H_{s,2}(y) = \frac{\nu_2(\varphi_s(y))}{(\varphi_s'(y))^2}. \tag{20}$$

Let $\mathcal{I}_N : \mathcal{C}(I) \to \mathcal{P}_N$ be the Legendre–Gauss interpolation operator. The Legendre spectral method consists in finding $U_{s,N} \in \mathcal{P}_N$ such that

$$\sum_{k=0}^{2} H_{s,k}(y)U_{s,N}^{(k)}(y) - \int_{-1}^{1} K_s(y,z)U_{s,N}(z)\varphi_s'(z)dz = \mathcal{I}_N F_s(y), \quad -1 \le y < 1. \tag{21}$$

If $U_{s,N}(y) = \sum_{j=0}^{N} c_{s,j}P_j(y)$ is an approximate solution of Eq. (18) using Legendre spectral method, then $u_N(x) = U_{s,N}(y)$ with $y = \varphi_s^{-1}(x)$ an approximate solution of Eq. (1) using mapped Legendre spectral method. Indeed, it holds that

$$\hat{u}_{s,j} = \frac{2j+1}{j}\int_0^{\infty} u(x)L_{s,j}(x)\rho_s(x)dx = \frac{2j+1}{j}\int_{-1}^{1} U_s(y)P_j(y)dy := c_{s,j},$$

hence using (10), we obtain

$$u_N(x) = \sum_{j=0}^{N} \hat{u}_{s,j}L_{s,j}(x) = \sum_{j=0}^{N} c_{s,j}P_j(\varphi_s^{-1}(x)) := U_{s,N}(y). \tag{22}$$

Clearly, $u_N(x)$ is a combination of the mapping $y = \varphi_s^{-1}(x)$, for $y \in I$, with $U_{s,N}(y)$.

3.1. Fundamental matrix relations

First, let us show Eq. (21) in the form

$$D_{s,N}(y) - I_{s,N}(y) = F_{s,N}(y), \tag{23}$$

where

$$D_{s,N}(y) = \sum_{k=0}^{2} H_{s,k}(y)U_{s,N}^{(k)}(y), \qquad I_{s,N}(y) = \int_{-1}^{1} K_s(y,z)U_{s,N}(z)\varphi_s'(z)dz, \qquad F_{s,N}(y) = \mathcal{I}_N F_s(y), \tag{24}$$

and

$$U_{s,N}(y) = \sum_{j=0}^{N} c_{s,j}P_j(y) = \mathrm{P}(y)\mathrm{C}, \quad -1 \le y \le 1, \tag{25}$$

with

$$\begin{aligned} &\mathrm{C} = [c_{s,0}, c_{s,1}, \ldots, c_{s,N}]^{\mathrm{T}}, \\ &\mathrm{P}(y) = [P_0(y), P_1(y), \ldots, P_N(y)]. \end{aligned} \tag{26}$$

The derivative of the solution expressed by (25) can be written in the matrix form as follows

$$[U_{s,N}^{(k)}(y)] = \mathrm{P}^{(k)}(y)\mathrm{C}. \tag{27}$$

First, we can write $\mathrm{P}(y)$ in the matrix form as follows

$$\mathrm{P}(y) = \mathrm{Y}(y)\Pi, \tag{28}$$

where

$$\mathrm{Y}(y) = [1, y, y^2, \ldots, y^N], \tag{29}$$

and for odd values of N,

$$\Pi=\begin{pmatrix}
\frac{1}{2^{0}0!0!} & 0 & \frac{-2!}{2^{2}1!1!} & \cdots & \frac{(-1)^{\frac{N-3}{2}}(N-3)!}{2^{N-3}(\frac{N-3}{2})!(\frac{N-3}{2})!} & 0 & \frac{(-1)^{\frac{N-1}{2}}(N-1)!}{2^{N-1}(\frac{N-1}{2})!(\frac{N-1}{2})!} & 0 \\
0 & \frac{2!}{2^{1}0!1!} & 0 & & & \frac{(-1)^{\frac{N-3}{2}}(N-1)!}{2^{N-2}(\frac{N-3}{2})!(\frac{N-1}{2})!} & 0 & \frac{(-1)^{\frac{N-1}{2}}(N+1)!}{2^{N}(\frac{N-1}{2})!(\frac{N+1}{2})!} \\
0 & 0 & \frac{4!}{2^{2}0!2!} & & & & \frac{(-1)^{\frac{N-3}{2}}(N+1)!}{2^{N-1}(\frac{N-3}{2})!(\frac{N+1}{2})!} & 0 \\
0 & 0 & 0 & \ddots & & & & \frac{(-1)^{\frac{N-3}{2}}(N+3)!}{2^{N}(\frac{N-3}{2})!(\frac{N+3}{2})!} \\
\vdots & \vdots & & & & & & \vdots \\
\vdots & 0 & & & & & 0 & \vdots \\
0 & 0 & & & & & \frac{-(2N-4)!}{2^{N-1}1!(N-2)!} & \\
0 & 0 & & & & \ddots & 0 & \frac{-(2N-2)!}{2^{N}1!(N-1)!} \\
0 & 0 & & & \cdots & 0 & \frac{(2N-2)!}{2^{N-1}0!(N-1)!} & 0 \\
0 & 0 & & & \cdots & 0 & 0 & \frac{(2N)!}{2^{N}0!N!}
\end{pmatrix}, \tag{30}$$

for even values of N

$$\Pi=\begin{pmatrix}
\frac{1}{2^{0}0!0!} & 0 & \frac{-2!}{2^{2}1!1!} & \cdots & \cdots & \frac{(-1)^{\frac{N}{2}-1}(N-2)!}{2^{N-2}(\frac{N}{2}-1)!(\frac{N}{2}-1)!} & 0 & \frac{(-1)^{\frac{N}{2}}N!}{2^{N}\frac{N}{2}!\frac{N}{2}!} \\
0 & \frac{2!}{2^{1}0!1!} & 0 & & & 0 & \frac{(-1)^{\frac{N}{2}-1}(N)!}{2^{N-1}(\frac{N}{2}-1)!(\frac{N}{2})!} & 0 \\
0 & 0 & \frac{4!}{2^{2}0!2!} & & & & 0 & \frac{(-1)^{\frac{N}{2}-1}(N+2)!}{2^{N}(\frac{N}{2}-1)!(\frac{N}{2}+1)!} \\
0 & 0 & 0 & \ddots & & & & 0 \\
\vdots & \vdots & & & & & & \vdots \\
\vdots & 0 & & & & & 0 & \vdots \\
0 & 0 & & & & \ddots & \frac{-(2N-4)!}{2^{N-1}1!(N-2)!} & \\
0 & 0 & & & & & 0 & \frac{-(2N-2)!}{2^{N}1!(N-1)!} \\
0 & 0 & & & \cdots & 0 & \frac{(2N-2)!}{2^{N-1}0!(N-1)!} & 0 \\
0 & 0 & & & \cdots & 0 & 0 & \frac{(2N)!}{2^{N}0!N!}
\end{pmatrix}. \tag{31}$$

On the other hand, we can write

$$\mathrm{P}^{(k)}(y)=\mathrm{Y}^{(k)}(y)\Pi, \tag{32}$$

and

$$\mathrm{Y}^{(1)}(y)=\mathrm{Y}(y)\mathrm{B}, \tag{33}$$

with

$$\mathrm{B}=\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 2 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 3 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & N-1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & N \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{34}$$

Also, from (33), we obtain

$$\begin{aligned}
\mathrm{Y}^{(1)}(y) &= \mathrm{Y}(y)\mathrm{B} \\
\mathrm{Y}^{(2)}(y) &= \mathrm{Y}^{(1)}(y)\mathrm{B}=\mathrm{Y}(y)\mathrm{B}^{2} \\
&\ \vdots \\
\mathrm{Y}^{(k)}(y) &= \mathrm{Y}^{(k-1)}(y)\mathrm{B}=\mathrm{Y}(y)\mathrm{B}^{k}.
\end{aligned}$$

Then

$$\mathrm{P}^{(k)}(y) = \mathrm{Y}(y)\mathrm{B}^k \Pi. \tag{35}$$

Consequently, by substituting the matrix relation (35) into Eq. (27), we obtain the matrix relation for $U_{s,N}^{(k)}(y)$ as follows

$$[U_{s,N}^{(k)}(y)] = \mathrm{Y}(y)\mathrm{B}^k \Pi \mathrm{C}. \tag{36}$$

Next, let $\{\sigma_i\}_{i=0}^N$, be the Legendre–Gauss points, which are zeros of Legendre polynomial $P_{N+1}(y)$. By [14], there exists a unique set of Christoffel numbers $\{\omega_i\}_{i=0}^N$ such that

$$\int_{-1}^{1} \phi(y)dy = \sum_{i=0}^{N} \phi(\sigma_i)\omega_i, \quad \forall \phi \in \mathcal{P}_{2N+1}. \tag{37}$$

In fact, we have

$$\omega_i = \frac{2}{(1-\sigma_i^2)[P'_{N+1}(\sigma_i)]^2}, \quad 0 \leq i \leq N. \tag{38}$$

The fundamental matrix relation for the differential part $D_{s,N}(y)$ based on collocation points is given by

$$\mathrm{D} = \sum_{k=0}^{2} \mathrm{H}_k \mathrm{Y} \mathrm{B}^k \Pi \mathrm{C}. \tag{39}$$

where

$$\mathrm{Y} = \begin{pmatrix} \mathrm{Y}(\sigma_0) \\ \mathrm{Y}(\sigma_1) \\ \vdots \\ \mathrm{Y}(\sigma_N) \end{pmatrix} = \begin{pmatrix} 1 & \sigma_0 & \cdots & \sigma_0^N \\ 1 & \sigma_1 & \cdots & \sigma_1^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \sigma_N & \cdots & \sigma_N^N \end{pmatrix}, \qquad \mathrm{H}_k = \mathrm{diag}(H_{s,k}(\sigma_j))_{0\leq j \leq N}. \tag{40}$$

For the Fredholm integral part, we have

$$I_{s,N}(y) = \sum_{j=0}^{N} c_{s,j} \int_{-1}^{1} K_s(y,z)P_j(z)\varphi_s'(z)dz \simeq \sum_{j=0}^{N} c_{s,j} \sum_{i=0}^{N} K_s(y,\sigma_i)P_j(\sigma_i)\varphi_s'(\sigma_i)\omega_i. \tag{41}$$

By setting

$$\nu_i = \varphi_s'(\sigma_i)\omega_i,$$

the fundamental matrix relation for the integral part $I_{s,N}(y)$ based on collocation points is given by

$$\mathrm{I}_f = \mathrm{M}\Phi\mathrm{Y}\Pi\mathrm{C}, \tag{42}$$

with

$$\mathrm{M} = \left(K_s(\sigma_j,\sigma_i)\right)_{0\leq j,i \leq N}, \qquad \Phi = \mathrm{diag}((\nu_i)_{0\leq i \leq N}). \tag{43}$$

Using relations (39) and (42), Eq. (18) is reduced to the following system

$$\mathrm{W}\mathrm{C} = \mathrm{F}, \tag{44}$$

where

$$\begin{aligned} \mathrm{W} = [w_{ij}] &= \sum_{k=0}^{2} \mathrm{H}_k \mathrm{Y} \mathrm{B}^k \Pi - \mathrm{M}\Phi\mathrm{Y}\Pi, \\ \mathrm{F} &= [F_s(\sigma_0), F_s(\sigma_1), \ldots, F_s(\sigma_N)]^T. \end{aligned} \tag{45}$$

On the other hand, the fundamental matrix for the boundary conditions can be written as

$$\mathrm{V}_0\mathrm{C} = \alpha, \tag{46}$$

$$\mathrm{V}_1\mathrm{C} = \beta, \tag{47}$$

with

$$\mathrm{V}_0 = [v_{00}, v_{01}, \ldots, v_{0N}] = [1, -1, \ldots, (-1)^N], \tag{48}$$

$$\mathrm{V}_1 = [v_{10}, v_{11}, \ldots, v_{1N}] = [1, 1, \ldots, 1]. \tag{49}$$

Table 1
Exponential, rational and logarithmic mapping functions, their inverse, and derivative.

Name	Function $x = \varphi_s(y)$	Inverse $y = \varphi_s^{-1}(x)$	Derivative $\varphi_s'(y) = \frac{dx}{dy}$	Deriv. Mapped $\rho_s(x) = \frac{dy}{dx}$
Exponential	$x = s\ln\left(\frac{2}{1-y}\right)$	$1 - 2\exp(-x/s)$	$\frac{s}{1-y}$	$\frac{2}{s}\exp(-x/s)$
Rational	$\frac{s(1+y)}{1-y}$	$\frac{x-s}{x+s}$	$\frac{2s}{(1-y)^2}$	$\frac{2s}{(x+s)^2}$
Logarithmic	$\frac{s}{2}\ln\left(\frac{3+y}{1-y}\right)$	$2\tanh(x/s) - 1$	$\frac{2s}{(1-y)(3+y)}$	$\frac{2}{s}(1 - \tanh^2(x/s))$

In order to satisfy the boundary conditions in the collocation method we add the above equations to the final system given in (44), then we obtain

$$\tilde{\mathrm{W}}\mathrm{C} = \tilde{\mathrm{F}}, \tag{50}$$

so that the new augmented matrix is of the form

$$\left[\tilde{\mathrm{W}}|\tilde{\mathrm{F}}\right] = \left[\begin{array}{ccccc|c} w_{00} & w_{01} & w_{02} & \cdots & w_{0N} & F_s(\sigma_0) \\ w_{10} & w_{11} & w_{12} & \cdots & w_{1N} & F_s(\sigma_1) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ w_{N,0} & w_{N,1} & w_{N,2} & \cdots & w_{N,N} & F_s(\sigma_N) \\ v_{00} & v_{01} & v_{02} & \cdots & v_{0N} & \alpha \\ v_{10} & v_{11} & v_{12} & \cdots & v_{1N} & \beta \end{array}\right]. \tag{51}$$

Finally, we have an over-determined system with $(N+3)$ linear equations which can be readily solved by using least square method.

4. Numerical illustrations

In this section, the numerical results of the following examples are obtained by considering three practical mapping functions, that are given with some of their properties in Table 1. All computations were carried out by MATLAB R2009b. For the most part, the error of the approximations is measured as the discrete L^2-norm with respect to the mapped Legendre weight $\rho_s(x)$, of the difference between approximate and exact solutions, given by

$$\mathbf{e}_N = \left(\sum_{i=0}^{N} [u(\zeta_i) - u_N(\zeta_i)]^2 w_i\right)^{1/2}, \tag{52}$$

where

$$\zeta_i = \varphi_s(\sigma_i), \quad w_i = \omega_i, \quad 0 \le i \le N, \tag{53}$$

are the mapped Legendre–Gauss points and weights.

Remark 1. Note that, according to Theorem 1, any function defined in $L^2_{\rho_s}(\Lambda)$, whose transformations under aforementioned mappings (see Table 1) are analytic on $[-1, 1]$, can be approximated by a truncated series of the form (15) with the superlinear convergence.

Example 1. Let us first consider the problem

$$u'(x) - \int_0^\infty xte^{-2t}\sin tu(t)dt = -\frac{3}{50}x - e^{-x}, \quad x \in [0, \infty), \tag{54}$$

with $u(0) = 1$, whose exact solution is $u(x) = e^{-x}$. By applying exponential mapping transformation with $s = 2$, we obtain

$$U_2(y) := u(\varphi_2(y)) = e^{-2\ln\left(\frac{2}{1-y}\right)} = 0.25 - 0.5y + 0.25y^2, \quad y \in [-1, 1]. \tag{55}$$

Clearly, $U_2(y)$ is a quadratic, analytic function on $[-1, 1]$. Then by Theorem 1, $u(x)$ can be approximated by a truncated series of the form (15) with superlinear convergence. Now, it may be acceptable to seek an approximate solution for the given problem. Let $N = 5$, then ELC method consists in finding $u_5(x) \in L^2_{\rho_2}(\Lambda)$ such that

$$u_5(x) = U_{2,5}(y) = \sum_{j=0}^{5} c_{2,j}P_j(y), \quad y = 1 - 2\exp(-x/2), \quad y \in I, x \in \Lambda.$$

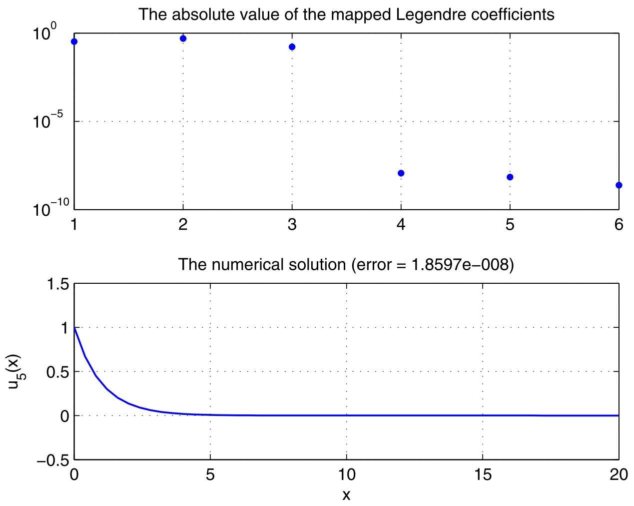


Fig. 1. Numerical results of ELC method for Example 1.

The augmented matrix based on the conditions from Eq. (51) is calculated as

$$[\widetilde{\mathrm{W}}|\widetilde{\mathrm{F}}] = \left[\begin{array}{rrrrrr|r} -0.0110 & 0.9683 & -2.6995 & 4.8503 & -6.9531 & 8.5218 & -0.9377 \\ -0.0594 & 0.8419 & -1.6287 & 1.4699 & -0.0877 & -1.7219 & -0.7122 \\ -0.1532 & 0.6484 & -0.3945 & -0.6845 & 0.9487 & 0.3192 & -0.4410 \\ -0.3089 & 0.4393 & 0.3709 & -0.4487 & -0.6158 & 0.2026 & -0.2608 \\ -0.5678 & 0.2772 & 0.5168 & 0.2274 & -0.0291 & -0.3346 & -0.2418 \\ -1.0836 & 0.2396 & 0.4395 & 0.0284 & 0.1551 & 0.3301 & -0.4077 \\ 1.0000 & -1.0000 & 1.0000 & -1.0000 & 1.0000 & -1.0000 & 1.0000 \end{array}\right].$$

By solving this system, the Legendre coefficients matrix is gained as

$$\mathrm{C} = [1/3 \quad -1/2 \quad 1/6 \quad -1/87540636 \quad -1/145575192 \quad -1/418815286]^T,$$

and the approximate solution of the integro-differential equation (18) is given by

$$\begin{aligned} U_{2,5}(y) &= 0.333333334481310P_0(y) - 0.500000012393087P_1(y) + 0.166666651635238P_2(y) \\ &\quad - 0.000000011423266P_3(y) - 0.000000006869302P_4(y) - 0.000000002387687P_5(y), \end{aligned}$$

or in the form

$$U_{2,5}(y) = 0.25 - 0.5y + 0.25y^2 - (7.6659\text{e-}09)y^3 - (3.0053\text{e-}08)y^4 - (1.8803\text{e-}08)y^5.$$

From (55), it follows

$$U_2(y) = U_{2,5}(y) + E(y), \qquad E(y) = (7.6659\text{e-}09)y^3 + (3.0053\text{e-}08)y^4 + (1.8803\text{e-}08)y^5.$$

Thus, we get

$$\|u - u_5\|^2_{\rho_2} = \int_0^{\infty} |u(x) - u_5(x)|^2 \rho_s(x)dx = \int_{-1}^{1} |E(y)|^2 dy = 3.4584\text{e}{-16}.$$

Fig. 1 shows the obtained numerical results. From this figure, it is clear that with an appropriate scaling factor $s > 0$, the ELC method gives us results with higher degree of accuracy with small values of N.

Example 2. Consider the problem

$$u'(x) - \int_0^{\infty} \sqrt{x}te^{-t}u(t)dt = e^{-x} - \frac{7}{4}\sqrt{x}, \quad x \in [0, \infty), \tag{56}$$

with $u(0) = 1$, whose exact solution is $u(x) = 2 - e^{-x}$. Table 2 shows the numerical errors obtained by using ELC, RLC and LLC methods with $s = 8$ and different values of N.

Example 3 ([15]). Consider the problem

$$u''(x) - 2u'(x) - 8u(x) - \int_0^{\infty} x(t^2 + 1)u(t)dt = -\frac{3}{4}x, \quad x \in [0, \infty), \tag{57}$$

Table 2
Discrete L^2 errors by ELC, RLC and LLC methods for Example 2.

N	ELC	RLC	LLC
6	7.8662e−03	7.6029e−03	2.6636e−02
8	8.6465e−10	6.8591e−04	2.4486e−03
10	3.6415e−11	8.3579e−05	1.7688e−04
12	2.6318e−12	1.5909e−05	1.0957e−05
14	2.7790e−13	4.1988e−06	6.1064e−07
16	1.3974e−14	1.1358e−06	3.2104e−08
18	5.2350e−16	2.7400e−07	1.9096e−09
20	2.5363e−15	5.1813e−08	2.6364e−10
24	5.4519e−15	1.4717e−09	5.1830e−11
32	7.3918e−14	1.4672e−11	6.0497e−12
40	2.3759e−14	5.3125e−12	1.1109e−12

Table 3
Comparison of the maximum absolute errors for Example 3.

N	ELC		RLC		LLC		[15]
	$s=2$	$s=3$	$s=2$	$s=7/2$	$s=3/2$	$s=2$	$m=0, m'=1$
4	2.09e−07	1.96e−02	1.55e−02	1.44e−02	1.92e−02	4.45e−02	9.27e−02
6	1.42e−08	6.47e−11	2.46e−03	2.46e−03	2.32e−04	4.35e−04	2.30e−02
8	1.86e−09	2.92e−12	4.41e−04	4.09e−04	1.90e−05	7.18e−05	3.20e−03

Table 4
Discrete L^2 errors by RLC method for Example 4.

N	$s=1$	$s=2$	$s=3$
08	5.4381e−05	2.0199e−05	9.0576e−04
10	9.6498e−07	1.4783e−06	1.1360e−04
12	1.1506e−06	1.2708e−07	1.2084e−05
14	2.8205e−07	1.3798e−08	1.1902e−06
16	5.3692e−08	2.5737e−09	1.1103e−07
18	8.3869e−09	2.9531e−10	1.0021e−08
24	1.0922e−10	1.8303e−13	6.3295e−12
30	2.4630e−12	5.3664e−15	1.0460e−14
36	1.5953e−14	5.3071e−14	5.5598e−14
42	6.6118e−15	3.2543e−14	7.4442e−14

Table 5
Comparison of the maximum absolute errors for Example 4.

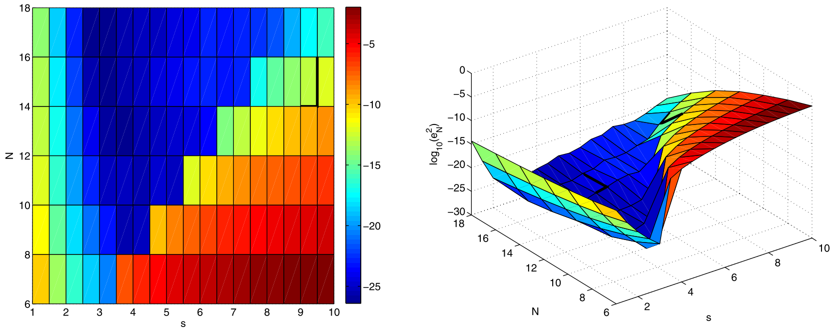
N	RLC	[15]
4	5.7945e−04	6.5671e−04
6	1.4138e−04	7.1255e−04
8	2.1271e−06	2.8510e−04

with $u(0)=1, u(\infty)=0$, whose exact solution is $u(x)=e^{-2x}$. Sub-figures (a), (b), (c), (d), (e) and (f) of Fig. 2 show the 2D contour plots and corresponding 3D surface plots of the Logarithmic $\mathbf{e}_N^2$ for ELC, RLC and LLC methods with different values of s and N. These sub-figures illustrate the interval that we can choose the s-parameter to provide a good estimate of the truncation error for the given problem. As we can see, using ELC method gives more accurate results and faster convergence compared to RLC and LLC methods. However, LLC method is less sensitive to the scale factor s. In Table 3, comparison between the values of $\|U_s - U_{s,N}\|_\infty$ (computed as the maximum of the absolute difference between the exact solution of the resulting Eq. (18) and the approximated ones) shows that the numerical results based on all the three methods are in good agreement with those obtained from [15].

Example 4 (*[15]*)**.** Consider the problem

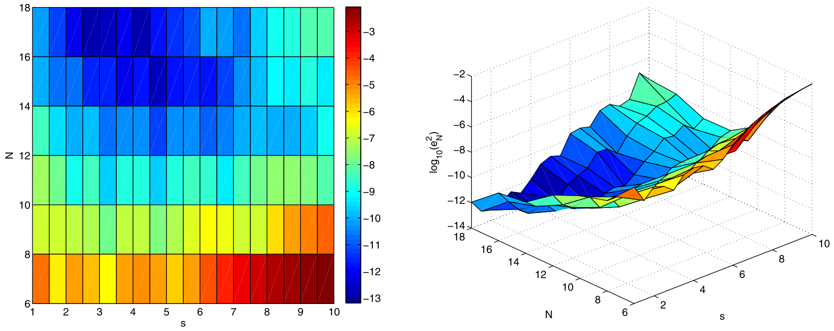
$$(1+x)^2u''(x)-(1+x)u'(x)+u(x)-\int_0^\infty x(t^2-1)e^{-3t}u(t)dt=\frac{4}{1+x}+\frac{2}{9}x, \quad x\in[0,\infty), \tag{58}$$

with $u(0)=1, u(\infty)=0$. The exact solution of this problem is $u(x)=\frac{1}{1+x}$, which is a smooth function and decays very slowly at infinity. Numerical errors obtained by using RLC method with different values of N and s are reported in Table 4. Once again, Table 5 shows that RLC method with $s=3/2$ performs significantly better than [15].



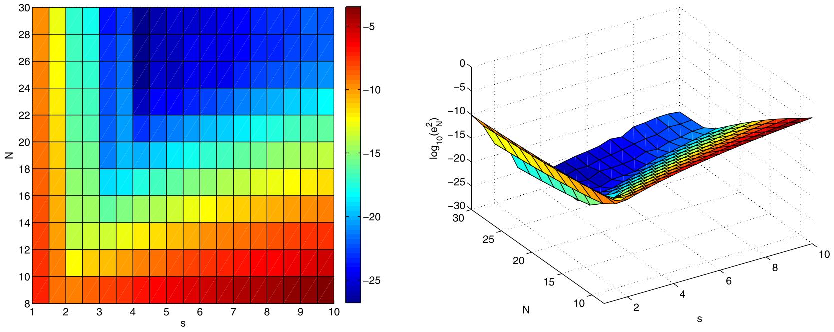
(a) ELC : 2D view.

(b) ELC : 3D view.



(c) RLC : 2D view.

(d) RLC : 3D view.



(e) LLC : 2D view.

(f) LLC : 3D view.

Fig. 2. 2D contour plots and corresponding 3D surface plots of the Logarithmic $\mathbf{e}_N^2$ by ELC, RLC and LLC methods for Example 3.

5. Conclusion

In this paper, we use a suitable family of variable transformations to reform Fredholm integro-differential equation on the half-line, and then fit the solution of the resulting singular equation on the finite interval $[-1, 1]$ by the Legendre polynomials. One of the main advantages of this method is that the numerical solution of the problem can be converted into a system of algebraic equations using the operational matrix. Superlinear convergence rates of the mapped Legendre function approximations can be achieved under sufficient conditions (see Theorem 1). Therefore, a careful choice of the mappings is required to obtain a superlinear convergence rate of the proposed scheme. This has been clearly illustrated among the typical numerical tests discussed in Section 4, where the superlinear convergence was achieved for our three proposed methods. Moreover, it is worth noticing that the convergence can be greatly accelerated by using ElC method with an appropriate scaling parameter $s > 0$. The obtained results clearly show that the present method is a powerful mathematical tool for finding the numerical solutions of integro-differential equations on the half line. Although we have not yet actually done so, it is also possible to extend this method to the high-order integro-differential equations, and this represents on near future work.

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