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By:

ELhadj DAHIA

Entitled:

On the tensorial representation of multi-linear ideals

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P.V. de délibération de soutenance de thèse de doctorat en Sciences.

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Etudiant : DAHIA El Hadj

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Rapport :

Le sujet est d'actualité et d'une culture rare. Il présente une profondeur mathématique concrète. Le travail est de très haute qualité scientifique malgré sa densité et sa difficulté. Le manuscrit est très bien rédigé et structuré. L'exposé a été d'une clarté très remarquable. Il a maîtrisé formidablement les outils de la nouvelle technologie. Le candidat a répondu à la presque totalité des questions posées par les membres du jury ce qui prouve sa maîtrise et sa compréhension totale de son sujet. Nous encourageons vivement le candidat à poursuivre dans cette direction qui sera de toutes les façons que bénéfique.

Le Jury adresse une félicitation spéciale au candidat et lui décerne le diplôme de doctorat en sciences avec la mention très honorable.

Les membres du jury.

- ACHOUR DAHMANE
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ملخص
من خلال هذا العمل وباستعمال طريقة استقطاب المثاليات تمكنا من إنشاء مثاليات خطية وأخرى متعددة الخطية ثم مثاليات لمؤثرات كثيرات حدود انطلاقا من مثاليات معروفة سابقا و أثبتنا أنها تملك كل خصائص المثاليات الجمعية كنظرية الاحتواء، نظرية الهيمنة لبيتش ونظرية التفكيك ثم قمنا بتمثيل هذه المثاليات بواسطة نظيمات موترية ملائمة وأنهينا العمل بتقديم نظرية التفكيك للمؤثرات كثيرات الحدود باستعمال تقنيات مستوحاة من النظرية العامة لفضاءات بناخ الشبكية.
الكلمات المفتاحية:
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Abstract
We introduce some new linear and multilinear ideals, and to study the representation of these ideals by suitable tensor norms. We show that the results that worked for the case of the multi-ideals involving summability could be extended to the more general setting that is constructed starting from the interpolation method, as inclusion theorems, Pietsch's domination theorem, factorization theorems and characterizations by using the adjoints of m-linear operators. We also introduce the ideal of (p,σ)-continuous polynomials that extends the nowadays well-known ideal to the more general setting of the interpolated ideals of polynomials. We give a factorization theorem for this new ideal which requires new techniques inspired in the theory of Banach lattices.

Ky words: Absolutely continuous operator (polynomials), multilinear operator, tensor norm, tensorial representation, Pietsch domination theorem, factorization.

Résumé
A travers notre étude, nous introduisons quelques nouveaux idéaux linéaires et multilinéaires ainsi que la représentation de ces idéaux par des normes tensorielles convenables. Nous pouvons dire que les résultats qui s'appliquent pour les cas des multi-idéaux impliquant la sommabilité pourraient être prolongés sur le cadre plus général, construit à partir de la méthode d´interpolation tels que les théorèmes d'inclusion, domination, factorisation de Pietsch et des caractérisations, en employant les adjoints des opérateurs m-linéaires. Nous présentons également l'idéal des polynômes (p,σ)-continus qui prolongent l'idéal bien connu à le cadre plus général des idéaux interpolés des polynômes. Nous donnons, par conséquent, un théorème de factorisation pour ce nouvel idéal qui exige de nouvelles techniques inspirées de la théorie des espaces de Banach réticulés.

Mots clés: Opérateur absolument continu (polynôme), opérateur multilinéaire, norme tensorielle, représentation tensorielle, théorème de domination de Pietsch, factorisation.

Contents

0.1 Introduction

The main topic treated in this thesis is the representation of some non-linear ideals by suitable tensor norms, with emphasis mainly on interpolated operator ideals. The theory of operator ideals was introduced by A. Pietsch in the linear case and is nowadays well established. The reader can find a lot of information about in the excellent monograph [61]. Following the original Grothendieck's approach to the theory of operators, A. Defant and K. Floret published the famous book "*Tensor Norms and Operator Ideals* in 1993, centering the attention in the interplay between the theory of operator ideals and the theory of tensor norms. Regarding interpolated operator ideals, H. Jarchow and U. Matter introduced in 1987 a general interpolation procedure for creating a new operator ideal from two given operator ideals (see [39]). Using this technique, U. Matter defined the operator ideal $\Pi_{p,\sigma}$ of the (p, σ)-absolutely continuous linear operators – where $1 \leq p < \infty$ and $0 \leq \sigma < 1$ – by means of this interpolative construction. The resulting space must be understood as an ideal located in between absolutely p-summing linear operators and continuous linear operators, preserving some of the characteristic properties of the first class. In the nineties, J.A. López Molina and E.A. Sánchez Pérez studied the factorization properties and the trace duality for these operators in a series of papers, introducing the class of tensor norms that represent these operator ideals (see [41], [42], and [69]). Let us write explicitly some definitions: a linear operator T between two Banach spaces X and Y is (p, σ)-absolutely continuous for $1 \leq p < \infty, 0 \leq \sigma < 1$ if there exist a Banach space G and a p-summing linear operator $S : X \longrightarrow G$ such that for all $x \in X$, the inequality

$$\|Tx\| \leq \|x\|^{\sigma} \|Sx\|^{1-\sigma}$$

holds. In his thesis, E.A. Sánchez Pérez introduced several interpolated operator ideals namely the (q, ν, p, σ)-dominated, (q, p, σ)-mixing, (p, σ)-nuclear and (p, σ)-integral operators that generalize the well known operator ideals of (q, p)-dominated, (q, p)-mixing, p-nuclear and p-integral operators respectively (see [70]).

The theory of operator ideals has proved to be a strong tool for the investigation and classification of linear operators between Banach spaces. Nowadays, it has become a usual tool for analyzing some classical problems of the Functional Analysis —as summability of series—, but has also become the starting point to understand and solve new problems related non-linear operators. The linear theory has spread to multilinear operators and to polynomials, leading to the notions of multi-ideals and polynomial ideals. A first outline of such a multilinaer theory was given by A. Pietsch in 1983 (see [62]). An axiomatic theory of multi-ideals for Banach spaces-valued m-linear mappings was given by H.A. Braunss in 1984 (see [22]). These new multi-ideals could also be considered to be the starting point for the study of some specific properties of non-linear operators, that can be considered as a new area in non-linear functional analysis.

An ideal of multilinear mappings —also called multi-ideal— $\mathcal{M}$ is a subclass of the class of all continuous multilinear operators between Banach spaces such that for a positive integer m, Banach spaces $X_1, ..., X_m$ and Y, the components

$$\mathcal{M}(X_1, ..., X_m; Y) := \mathcal{L}(X_1, ..., X_m; Y) \cap \mathcal{M},$$

is a vector subspace of $\mathcal{L}(X_1, ..., X_m; Y)$ that is invariant by the composition of a linear operator on the right and m-linear operators on the left and which contains the m-linear mappings of finite type.

The 1989 paper by R. Alencar and M. C. Matos [6] is another cornerstone in this line of thought. This approach turned out to be very successful and a number of operator ideals have been fruitfully generalized to the multilinear setting in recent years by several authors (see [1], [2], [5], [16], [23], [26], [30], [43], [45], [52], [57], [63]....). In certain multi-ideals, the generalization is elementary. For example, this is the case of completely continuous m-linear mappings, weakly compact m-linear mappings and compact m-linear mappings (see [8], [9] [37], [66], [65], [31], [45]). Some properties that hold in the linear case remain true in the multilinear case, and some of them are not true any more. On the other hand, in the case of the p-summing linear operators the generalization is not unique: many definitions were introduced in this direction and many papers were devoted to the concept of summability for the multilinear mapping (see [11], [12], [13], [30], [59], [58]).

Regarding the general construction of multi-ideals starting from a given operator ideal, A. Pietsch shown three different approaches for creating multi-ideals (see [62]). They are the factorization method, the linearization method and the composition method, which have been developed by several authors. For example, the multi-ideal $\mathcal{L}_{d,(p_1,...,p_m)}$ of $(p_1, ..., p_m)$-dominated m-linear mappings is generated by the factorization method from the operator ideals of p_j-summing, $j = 1, ..., m$, (see [35]). In 2007, G. Botelho, D. Pellegrino and P. Rueda published the paper [19], describing the composition method for generating multi-ideals from a given operator ideal. Actually, this procedure was a particular case of the composition technique introduced A. Pietsch in [62]. A new multi-ideal defined following this construction by D. Achour and L. Mezrag in [5] is the class of Cohen strongly p-summing m-linear mappings $\mathcal{D}_p^m$; D. Achour in [1, Theorem 3.6] proved that this multi-ideal is generated by the composition method from the operator ideals of strongly p-summing linear operators introduced by J.S. Cohen in [25].

Concerning the representation of multi-ideals by tensor norms, the procedure of showing it as a dual space of a topological tensor product comes back to Grothendieck's known resume [36]. There he proved that integral bilinear forms match to linear and continuous functionals on the injective tensor product. This idea has worked successfully in many Banach ideals of multilinear mappings. For example, M. C. Matos in [43] introduced and studied a tensor norm such that linear operators on the corresponding tensor product that are continuous with respect to this norm matches exactly with the class of nuclear multilinear mappings. Let us write explicitly the definition. We say that a tensor norm α of order $m + 1$ represents the multi-ideal $\mathcal{M}$ if

$$\mathcal{M}(X_1, ..., X_m; Y^*) = (X_1 \otimes ... \otimes X_m \otimes Y, \alpha)^* \quad \text{isometrically isomorphic.}$$

There is another method for representing a multi-ideal, that is

$$\mathcal{M}(X_1, ..., X_m; Y) = (X_1 \otimes ... \otimes X_m \otimes Y^*, \alpha)^* \quad \text{isometrically isomorphic.}$$

The main goal of this thesis is to introduce some new linear and multilinear ideals, and to study the representation of these ideals by suitable tensor norms, as well as the factorization of the mappings belonging to these ideals. Also, some topological requirements on the spaces involved for obtaining an adequate multilinear generalization of the (p, σ)-absolutely continuous operators will be studied. Our main motivation is to show that the results that worked for the case of the multi-ideals of operators involving summability could be extended to the more general setting that is constructed starting from the interpolation method. As in the linear case, each one of these new ideals of multilinear mappings must be understood as an intermediate ideal in between a given multi-ideal and the whole class of continuous m-linear mappings. Indeed, both classes must be attained for $\sigma = 0$ and $\sigma = 1$, respectively. As far as we know, this is the first attempt in this regard. These interpolated multi-ideals will have many good properties and will extend almost all the ones that are satisfied by the ideals of absolutely p-summing, Cohen strongly p-summing and p-dominated multilinear mappings, as inclusion theorems, Pietsch's domination theorems, factorization theorems, tensor product representations and characterizations by using the adjoints of m-linear operators.
A second purpose of the thesis is to introduce the polynomial analogue of the (p, σ)-absolutely continuous linear operators. One of the most intriguing problems for classes of absolutely summing polynomials is to find a canonical homogeneous polynomial such that any other polynomial of the class factors through. Even for the well-known class of p-dominated polynomials, this problem has proved to be delicated (see [20, 21]) and many technical tools are required. The purpose of this canonical polynomial is to get a Pietsch-type factorization theorem for dominated polynomials. Same earlier factorizations can be found in [44, 49]. In the thesis we succeed in proving such a factorization for dominated (p, σ)-continuous polynomial.

The thesis consists of four chapters. In the preliminaries (Chapter 1) we establish the notation of the thesis. We introduce some important results concerning sequences Banach spaces and we recall the main definitions and properties of the theory of operator ideals that we will use later. Also, we recall the most important results for the classes of the p-summing and strongly p-summing linear mappings. The third section is devoted to the study of m-fold tensor products of Banach spaces with $m > 2$. We study the projective tensor product of Banach spaces $X_1, ..., X_m$; actually, the projective norm on $X_1 \otimes ... \otimes X_m$ is the norm that guaranties the continuity of the unique linearization of an m-linear mapping. After this, we recall the basic concepts on the theory of multi-ideals and their methods of construction. Concretely, the composition methods is described in detail. Finally, we recall explicitly how to represent a given multi-ideal by a suitable tensor norm.

In Chapter 2 of this thesis we introduce the new class of $(p; p_1, ..., p_1; \sigma)$-absolutely continuous multilinear operators, that is defined using a summability property that provides the multilinear version of the (p, σ)-absolutely continuous operators. We give an analogue of the Pietsch domination theorem and a multilinear version of the associated factorization theorem that holds for (p,σ)-absolutely continuous operators, obtaining in this way a rich factorization theory. We present also a tensor norm which represents this multi-ideal. As an application, we show that $(p;p_1,...,p_1;\sigma)$-absolutely continuous multilinear operators are compact under some requirements.

In the next chapter (Chapter 3) we introduce the new ideal of strongly (p,σ)-continuous linear operators in order to study the adjoints of the (p,σ)-absolutely continuous linear operators. Starting from this ideal we build a new multi-ideal by using the composition method. We prove the corresponding Pietsch domination theorem and we present a representation of this multi-ideal by a tensor norm. A factorization theorem characterizing the corresponding multi-ideal —which is also new for the linear case— is given. When applied to the case of the Cohen strongly p-summing operators, this result gives also a new factorization theorem.

In the last chapter (Chapter 4) we introduce and study the ideal of (p,q,σ)-absolutely continuous polynomials which is a polynomial version of the $(p;p_1,...,p_1;\sigma)$-absolutely continuous multilinear mappings. A particular case is given: the dominated (p,σ)-continuous polynomials, that extend the nowadays well known ideal of p-dominated polynomials to the more general setting of the interpolated ideals of polynomials. We give the polynomial version of Pietsch's factorization theorem for this new ideal, showing that the techniques used in [20] can be generalized to other ideals. Although based in [21], our factorization theorem requires new techniques inspired in the theory of Banach lattices.

Chapter 1

Preliminaries

In this chapter we present the concepts and results used throughout the thesis on some Banach sequence spaces, operator ideals, multi-ideals and tensor representations of multi-ideals.
We denote by $\mathbb{K}$ either the field $\mathbb{R}$ of the real numbers or the field $\mathbb{C}$ of the complex numbers. The set of all natural numbers $\{0, 1, ..., \}$ is denoted by $\mathbb{N}$. If $p \in \mathbb{R}$ and $1 \leq p \leq \infty$, we write p^* for the extended real number satisfying that $1/p + 1/p^* = 1$. We write X for a Banach space with the norm $\|.\|$. The closed unit ball of X is denoted by B_X that is the set $\{x \in X : \|x\| \leq 1\}$. If Y is a Banach space, the space $\mathcal{L}(X, Y)$ of all continuous linear mappings is a Banach space with the norm

$$\|u\| = \sup_{x \in B_X} \|u(x)\| .$$

Throughout the thesis, we call operator to a continuous linear mapping.
The set of all *functionals* of a normed space X (that is, the continuous linear mappings from X into the scalars) is a Banach space, denoted by X^* and called the *dual of* X. For $x \in X$, we shall write $\langle x, x^* \rangle$ (or $\langle x^*, x \rangle$) for the action of the functional x^* on x. The norm of $x^* \in X^*$ is given by

$$\|x^*\| = \sup \{|\langle x, x^* \rangle| : x \in B_X\} .$$

A linear operator $u : X \longrightarrow Y$ between two normed spaces X and Y is an *isomorphism* if u is a continuous bijection whose inverse u^{-1} is also continuous. In such case the spaces X and Y are said to be *isomorphic*. Such a mapping u is an *isometric isomorphism* when $\|u(x)\| = \|x\|$ for all $x \in X$.
A linear operator u is an *embedding of* X *into* Y if u is an isomorphism onto its image $u(X)$. In this case we say that X *embeds in* Y. If $u : X \longrightarrow Y$ is an embedding such that $\|u(x)\| = \|x\|$ for all $x \in X$, then u is said to be an *isometric embedding.*
Given the continuous linear operator $u : X \longrightarrow Y$, the continuous linear operator $u^* : Y^* \longrightarrow X^*$ defined as

$$u^*(y^*)(x) = y^*(u(x)),$$

for every $y^* \in Y^*$ and $x \in X$ is called the *adjoint of* u and has the property that $\|u^*\| = \|u\|$.

The $C(K)$ space. If K is a topological space, then by $C(K)$ we mean the space of all scalar valued (i.e. real or complex valued), bounded, continuous functions on K. This is a Banach space with the norm

$$\|f\|_\infty = \sup_{x\in K} |f(x)| .$$

Clearly, if K is a compact space then $C(K)$ consists of all continuous, scalar valued functions.
The dual of the space $C(K)$, K compact, equals the space $M(K)$ of all regular Borel measures (scalar valued, but obviously not necessarily positive) on K. The duality is defined as

$$\langle f, \mu\rangle = \mu(f) = \int_K f d\mu, \quad f \in C(K), \quad \mu \in M(K).$$

1.1 Some Banach sequence spaces

In order to study the behavior of the summability properties of the linear and multilinear mappings several spaces of vector valued sequences are necessary. We introduce them in this section. Let X be a Banach space over $\mathbb{K}$, and $1 \le p \le \infty$. The classical Banach sequence spaces ℓ_p, ℓ_∞ and c_0 are defined by

$$\ell_p = \left\{ (x_i)_{i=1}^\infty \subset \mathbb{K} : \|(x_i)_{i=1}^\infty\|_p = \left(\sum_{i=1}^\infty |x_i|^p \right)^{\frac{1}{p}} < \infty \right\}, \ 1 \le p < \infty,$$
$$\ell_\infty = \left\{ (x_i)_{i=1}^\infty \subset \mathbb{K} : \|(x_i)_{i=1}^\infty\|_\infty = \sup_{i\in\mathbb{N}} |x_i| < \infty \right\}, \ p = \infty,$$
$$c_0 = \left\{ (x_i)_{i=1}^\infty \subset \mathbb{K} : \lim_{i\to+\infty} |x_i| = 0 \right\}.$$

1.1.1 Absolutely and weakly p-summable sequences

Let $\ell_p(X)$ be the Banach space of all absolutely p-summable sequences $(x_i)_{i=1}^\infty$ in X with the norm

$$\|(x_i)_{i=1}^\infty\|_p = (\sum_{i=1}^\infty \|x_i\|^p)^{\frac{1}{p}}.$$

We denote by $\ell_p^\omega(X)$ the Banach space of all weakly p-summable sequences $(x_i)_{i=1}^\infty$ in X with the norm

$$\|(x_i)_{i=1}^\infty\|_{p,\omega} = \sup_{\|\xi\|_{X^*}\le 1} \|(\xi(x_i))_{i=1}^\infty\|_p = \sup_{\|\xi\|_{X^*}\le 1} (\sum_{i=1}^\infty |\xi(x_i)|^p)^{\frac{1}{p}}.$$

Notice that $\ell_p(X)$ is a linear subspace of $\ell_p^\omega(X)$ and

$$\|(x_i)_{i=1}^\infty\|_{p,\omega} \le \|(x_i)_{i=1}^\infty\|_p \text{ for all } (x_i)_{i=1}^\infty \in \ell_p(X).$$

If $p=\infty$ we are restricted to the case of bounded sequences and in $\ell_\infty(X)$ we use the sup norm. Then the spaces $\ell_\infty(X)$ and $\ell_\infty^\omega(X)$ coincide and

$$\|(x_i)_{i=1}^\infty\|_\infty = \|(x_i)_{i=1}^\infty\|_{\infty,\omega} \text{ for } (x_i)_{i=1}^\infty \in \ell_\infty(X).$$

If X is finite dimensional with $\dim X = n$, then $\ell_p(X) = \ell_p^\omega(X)$ and

$$\|(x_i)_{i=1}^\infty\|_{p,\omega} \le \|(x_i)_{i=1}^\infty\|_p \le n^{\frac{1}{p}} \|(x_i)_{i=1}^\infty\|_{p,\omega} \quad \text{for all } (x_i)_{i=1}^\infty \in \ell_p(X). \tag{1.1}$$

If we take $n=1$ in (1.1), or $X=\mathbb{K}$, then the spaces $\ell_p(\mathbb{K})$ and $\ell_p^\omega(\mathbb{K})$ coincide and we denote $\ell_p(\mathbb{K})$ by ℓ_p. In this case we have

$$\|(x_i)_{i=1}^\infty\|_{p,\omega} = \|(x_i)_{i=1}^\infty\|_p \quad \text{for all } (x_i)_{i=1}^\infty \in \ell_p. \tag{1.2}$$

We know (see [7, Theorem 2.1]) that $\left(\ell_p^n(X)\right)^* = \ell_{p^*}^n(X^*)$ isometrically i.e.,

$$\left\|(x_i)_{i=1}^n\right\|_p = \sup\left\{\left|\sum_{i=1}^n \langle x_i, x_i^*\rangle\right| : (x_i^*)_{i=1}^n \subset X^*, \left\|(x_i^*)_{i=1}^n\right\|_{p^*} \le 1\right\}. \tag{1.3}$$

For the particular case $p=1$ and $X=\mathbb{K}$ we have

$$\left\|(x_i)_{i=1}^n\right\|_1 = \sup\left\{\left|\sum_{i=1}^n \lambda_i x_i\right| : (\lambda_i)_{i=1}^n \subset \mathbb{K}, \left\|(\lambda_i)_{i=1}^n\right\|_\infty \le 1\right\}. \tag{1.4}$$

Let $(x_i^*)_{i=1}^n \subset X^*$. Then it is also known (see [51, Page 1] or [52, Lemma 2.1]) that

$$\left\|(x_i^*)_{i=1}^n\right\|_{p,\omega} = \sup_{\beta\in B_{X^{**}}} (\sum_{i=1}^n |\beta(x_i^*)|^p)^{\frac{1}{p}} = \sup_{x\in B_X} \left\|(x_i^*(x))_{i=1}^n\right\|_p. \tag{1.5}$$

1.1.2 Cohen strongly p-summable sequences

The space of Cohen strongly p-summable sequences was introduced by J. S. Cohen in [25] in order to give a characterization of the class of strongly p-summing linear operators.

A sequence $(x_i)_{i=1}^\infty$ in a Banach space X is Cohen strongly p-summable if the series $\sum_{i=1}^{\infty} |\langle x_i, \xi_i\rangle|$ converges for all $(\xi_i)_{i=1}^\infty \in \ell_{p^*}^\omega(X^*)$. We denote by $\ell_p\langle X\rangle$ the space of Cohen strongly p-summable sequences in X which is a Banach space (see [24, Proposition 2.1.8]) with the norm

$$\|(x_i)_{i=1}^\infty\|_{c,p} = \sup_{\left\|(\xi_i)_{i=1}^\infty\right\|_{p^*,\omega}\le 1} \sum_{i=1}^\infty |\langle x_i, \xi_i\rangle|. \tag{1.6}$$

Notice that

$$\ell_p\langle X\rangle \subset \ell_p(X) \subset \ell_p^\omega(X).$$

Moreover, for all $(x_i)_{i=1}^\infty \in \ell_p\langle X\rangle$;

$$\|(x_i)_{i=1}^\infty\|_{p,\omega} \le \|(x_i)_{i=1}^\infty\|_p \le \|(x_i)_{i=1}^\infty\|_{c,p}. \tag{1.7}$$

If $p=1$ we have $\ell_1\langle X\rangle = \ell_1(X)$ with $\|.\|_{c,1} = \|.\|_1$ and if $p=\infty$ then $\ell_{c,\infty}\langle X\rangle = \ell_\infty(X)$ with $\|.\|_{c,\infty} = \|.\|_\infty$.

1.2 Normed operator ideals

1.2.1 Definitions and general properties

A linear operator $u \in \mathcal{L}(X,Y)$ is said to have *finite rank* if $u(X)$ is finite dimensional. The class of all finite rank linear operators between Banach spaces is denoted by $\mathcal{L}_f(X,Y)$. This space is generated by the mappings of the special form

$$x^* \otimes y : x \longmapsto \langle x, x^* \rangle \, y$$

i.e. if $u \in \mathcal{L}_f(X,Y)$ we have

$$u = \sum_{i=1}^{n} x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^n \subset X^*$ and $(y_i)_{i=1}^n \subset Y$ (see [61, page 25]).

Definition 1.2.1. *An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces X and Y its components $\mathcal{I}(X,Y) := \mathcal{L}(X,Y) \cap \mathcal{I}$ satisfy:*

(i) $\mathcal{I}(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$ which contains the finite rank operators.

(ii) The ideal property: if $u \in \mathcal{L}(X,Z)$, $v \in \mathcal{I}(Z,K)$ and $w \in \mathcal{L}(K,Y)$, then the composition $w \circ v \circ u$ is in $\mathcal{I}(X,Y)$.

If $\|.\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ satisfies

(i') $(\mathcal{I}(X,Y), \|.\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces X and Y,

(ii') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$,

(iii') If $u \in \mathcal{L}(X,Z)$, $v \in \mathcal{I}(Z,K)$ and $w \in \mathcal{L}(K,Y)$,
$\|w \circ v \circ u\|_{\mathcal{I}} \leq \|w\| \, \|v\|_{\mathcal{I}} \, \|u\|$,

then $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is called a normed (Banach) operator ideal.

The operator ideal $\mathcal{I}$ is said to be *closed* if each $\mathcal{I}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$ for the sup norm. The ideal $\mathcal{L}_f$ of finite rank linear operators is the smallest operator ideal and $\mathcal{L}$ the largest one [61, Theorem 1.2.2].

Proposition 1.2.2. *[61, Proposition 6.1.4]*
Let $(\mathcal{I}, \|.\|_{\mathcal{I}})$ be a normed operator ideal. Then $\|u\| \leq \|u\|_{\mathcal{I}}$ for all $u \in \mathcal{I}$.

Definition 1.2.3. (injective operator ideal)
An normed operator ideal $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is said to be injective if for every isometric embedding $i : Y \hookrightarrow G$ and every $u \in \mathcal{L}(X,Y)$ it follows from $i \circ u \in \mathcal{I}(X,G)$ that $u \in \mathcal{I}(X,Y)$. Moreover

$$\|i \circ u\|_{\mathcal{I}} = \|u\|_{\mathcal{I}},$$

i.e. the ideal does not depend on the image space.

Definition 1.2.4. (dual of an operator ideal)
Let $\mathcal{I}$ is a normed operator ideal. A linear mapping $u \in \mathcal{L}(X,Y)$ belongs to $\mathcal{I}^{dual}$ if $u^* \in \mathcal{I}(Y^*, X^*)$, where u^* is the adjoint of the operator u. In this case we define

$$\|u\|_{\mathcal{I}^{dual}} = \|u^*\|_{\mathcal{I}}.$$

Proposition 1.2.5. *[27, Page 114]*
If $\mathcal{I}$ is a normed (Banach) operator ideal, then $(\mathcal{I}^{dual}, \|.\|_{\mathcal{I}^{dual}})$ is as well. This normed (Banach) ideal is called the dual of $\mathcal{I}$.

Some examples

1) Compact linear operators.
A linear operator $u \in \mathcal{L}(X,Y)$ is *compact* if $u(B)$ is a precompact subset of Y for every bounded subset B of X.
An equivalent formulation is that u is compact if and only if every bounded sequence $(x_i)_{i=1}^{\infty}$ in X has a subsequence $(x_{i_k})_{k=1}^{\infty}$ such that $(ux_{i_k})_{k=1}^{\infty}$ converges in Y.
We denote by $\mathcal{K}(X,Y)$ the vector space of all compact linear mappings from X into Y.
2) Weakly compact linear operators.
A continuous linear operator $u : X \to Y$ is said to be *weakly compact*, in symbols $u \in \mathcal{W}(X,Y)$, if u maps B_X onto a relatively weakly compact subset of Y. This is equivalent to say that $(u(x_i))_{i=1}^{\infty}$ has a weakly convergent subsequence for every bounded sequence $(x_i)_{i=1}^{\infty}$ in X.
3) Completely continuous linear operators.
A continuous linear operator $u \in \mathcal{L}(X,Y)$ between Banach spaces is said to be *completely continuous* if for every $(x_i)_{i=1}^{\infty}$ weakly convergent to 0 in X it follows $(ux_i)_{i=1}^{\infty}$ is norm convergent to 0 in Y.
We denote by $\mathcal{V}(X,Y)$ the vector space of all completely continuous linear mappings from X into Y.

Proposition 1.2.6. *(see [61], [27] or [15])*
The three clases $\mathcal{K}, \mathcal{W}$ and $\mathcal{V}$ are closed injective Banach operator ideals, where the ideal norm is the operator norm. In addition we have $\mathcal{K} = \mathcal{K}^{dual}$ and $\mathcal{W} = \mathcal{W}^{dual}$ but $\mathcal{V} \neq \mathcal{V}^{dual}$.

1.2.2 The ideal of p-summing linear operators

The theory of p-summing operators is based on a crucial criterion due to A. Pietsch [60]. We mention that the linear p-summing operators are the starting point in the study of summing continuous multilinear mappings.

Let $1 \leq p < \infty$. A linear operator $u : X \longrightarrow Y$ between Banach spaces is said to be *absolutely p-summing* or just *p-summing* if it takes weakly p-summable sequences $(x_i)_{i=1}^{\infty}$ of X to absolutely p-summable sequences $(u(x_i))_{i=1}^{\infty}$ of Y. This means that $\widehat{u} : (x_i)_{i=1}^{\infty} \longmapsto (u(x_i))_{i=1}^{\infty}$ defines a linear mapping from $\ell_p^{\omega}(X)$ into $\ell_p(Y)$ that is bounded in view of the closed graph theorem (see [29, Proposition 2.1]). Hence there exists a constant $C \geq 0$ such that

$$\left(\sum_{i=1}^{n} \|u(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{X^*} \leq 1} \left(\sum_{i=1}^{n} |\xi(x_i)|^p\right)^{\frac{1}{p}}, \tag{1.8}$$

for every finite family $(x_i)_{i=1}^n \subset X$. This inequality characterizes p-summing operators. The set of all p-summing operators, is denoted by Π_p.
The class Π_p is a injective Banach ideal under the ideal norm

$$\pi_p(u) := \inf \{C, \text{ for all } C \text{ verifying the inequality (1.8)}\} = \|\widehat{u}\|.$$

The nowadays classical Pietsch's domination theorem characterizes the p-summability of an operator by means of a norm domination uniform inequality. Concretely, it says that the mapping $u \in \mathcal{L}(X, Y)$ is p-summing if and only if there exist a constant C and a regular Borel probability measure μ on B_{X^*} (with the weak star topology) so that

$$\|u(x)\| \leq C \left(\int_{B_{X^*}} |\langle x, x^* \rangle|^p \, d\mu(x^*)\right)^{\frac{1}{p}}, \quad x \in X. \tag{1.9}$$

In this case, $\pi_p(u)$ is the least of all the constants C such that (1.9) holds. This inequality also provides a factorization of u through the natural mapping $C(B_{X^*}) \to L^p(\mu)$, that allows to prove a lot of important results in the theory of Banach spaces. For example, the following result —that is called the Dvoretzky-Rogers Theorem— can be proved using this factorization for p-summing operators.

Theorem 1.2.7. *[29, page 50]*
If $1 \leq p < \infty$, a Banach space X is finite dimensional if and only if the identity mapping $id_X : X \to X$ is p-summing.

Other easy consequence of the domination of p-summing operators and Hölder's Inequality is the fact that they form a chain, that is, if $1 \leq p \leq q \leq \infty$, then $\Pi_p \subseteq \Pi_q$.

The spaces of p-summing operators can be represented by duality using normed tensor products. The following representation theorem ensures that the space of p-summing linear operators between the Banach spaces X and Y^*, $\Pi_p(X, Y^*)$, is isometrically isomorphic to $\left(X \widehat{\otimes}_{d_{p^*}} Y\right)^*$, where

$$d_p(z) := \inf \left\{ \|(x_i)_{i=1}^n\|_{p^*,\omega} \, \|(y_i)_{i=1}^n\|_p \right\},$$

and the infimum is taken over all representations of $z = \sum_{i=1}^{n} x_i \otimes y_i$ in $X \otimes Y$.

Theorem 1.2.8. *[68, Page 140] Let $1 < p \leq \infty$ and X, Y be Banach spaces. A linear operator $u \in \mathcal{L}(X, Y^*)$ defines a continuous functional of $\left(X \widehat{\otimes}_{d_{p^*}} Y\right)$ if and only if u is p-summing. Moreover the norm of u in $\left(X \widehat{\otimes}_{d_{p^*}} Y\right)^*$ coincides with $\pi_p(u)$.*

1.2.3 The operator ideal of the strongly p-summing linear operators

Some early results of the operator ideal theory that go back to A. Grothendieck deal with the coincidence of the class of all linear and continuous operators between some classical Banach spaces and the class of the p-summing operators between these spaces. The relevant Grothendieck's Theorem establishes that the identity from ℓ_1 into ℓ_2 is 1-absolutely summing, and so 2-summing (see for instance [60, page 338]). However, the adjoint operator is not 2-absolutely summing. This well-known fact motivated the analysis of the concept of strongly p-summing linear operator ($1 < p \leq \infty$). It was introduced by J.S. Cohen in [25] as a characterization of linear operators having absolutely p-summing adjoint.

Definition 1.2.9. *A linear operator u between two Banach spaces X and Y is strongly p-summing for $1 < p \leq \infty$ if there is a positive constant C such that for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $y_1^*, \ldots, y_n^* \in Y^*$ we have*

$$\|(\langle u(x_i), y_i^* \rangle)_{i=1}^n\|_1 \leq C \, \|(x_i)_{i=1}^n\|_p \, \|(y_i^*)_{i=1}^n\|_{p^*,\omega} \, . \tag{1.10}$$

The collection of all strongly p-summing linear operators, denoted by $\mathcal{D}_p$ is a Banach ideal with the ideal norm

$$d_p(u) := \inf \{C > 0 : C \text{ verifying the inequality (1.10)}\} , \qquad u \in \mathcal{D}_p(X,Y).$$

For $p = 1$ we have $\mathcal{D}_1(X,Y) = \mathcal{L}(X,Y)$.

The following result due to J.S. Cohen [25, Theorem 2.2.2].

Theorem 1.2.10. *i) Let $1 \leq p < \infty$. The linear operator u belongs to $\Pi_p(X,Y)$ if and only if the adjoint operator u^* belongs to $\mathcal{D}_{p^*}(Y^*, X^*)$. In this case $\pi_p(u) = d_{p^*}(u^*)$.*

ii) Let $1 < p \leq \infty$. The linear operator u belongs to $\mathcal{D}_p(X,Y)$ if and only if the adjoint operator u^* belongs to $\Pi_{p^*}(Y^*, X^*)$. In this case $d_p(u) = \pi_{p^*}(u^*)$.

Remark 1.2.11. According to (ii) in the previous theorem we obtain $\mathcal{D}_p = \Pi_{p^*}^{dual}$, thus $(\mathcal{D}_p, d_p)$ is Banach operator ideal.

Also, we have a domination theorem for the strongly p-summing linear operators.

Theorem 1.2.12. *[25] A linear operator $u \in \mathcal{L}(X,Y)$ is strongly p-summing if and only if there is a constant $C > 0$ and a regular Borel probability measure μ on $B_{Y^{**}}$,(with the weak star topology) so that for all $x \in X$ and for all $y^* \in Y^*$, the inequality*

$$|\langle u(x), y^* \rangle| \leq C \, \|x\| \left(\int_{B_{Y^{**}}} |\langle y^*, y^{**} \rangle|^{p^*} \, d\mu \right)^{\frac{1}{p^*}} , \tag{1.11}$$

holds.

The following proposition asserts that the strongly p-summing linear operators are a dual space of a tensor product and this implies that the operator ideal $\mathcal{D}_p$ is represented by a tensor norm.

Proposition 1.2.13. *[25]Let $1 < p \leq \infty$ and let X and Y be Banach spaces. A continuous linear operator $T : X \longrightarrow Y$ defines a continuous linear functional on $(X\widehat{\otimes}_{g_p}Y^*)$ i.e.*

$$\mathcal{D}_p(X,Y) = (X\widehat{\otimes}_{g_p}Y^*)^* \quad \textit{isometrically.}$$

Where

$$g_p(z) := \inf\left\{\|(x_i)_{i=1}^n\|_p \,\|(y_i)_{i=1}^n\|_{p^*,\omega}\right\},$$

and the infimum is taken over all representations of $z = \sum\limits_{i=1}^{n} x_i \otimes y_i$ in $X \otimes Y$.

1.3 Tensor products of Banach spaces

In this section we introduce some elements of theory of tensor product of Banach spaces for the order $m > 2$. In particular, the projective norm, the injective norm and some basic results on general crossnorms will be presented.

Continuous multilinear mappings.

Let $m \in \mathbb{N}$ and consider $X_j(j = 1, ..., m)$, Y the normed spaces over $\mathbb{K}$, (either $\mathbb{R}$ or $\mathbb{C}$). A mapping $T : X_1 \times ... \times X_m \longrightarrow Y$ is called multilinear (or m-linear) if the mappings

$$\begin{array}{rccl} T_j : & X_j & \longrightarrow & Y \\ & x^j & \longmapsto & T(x^1, ..., x^j, ..., x^m), \end{array}$$

are linear for each set of fixed $x^k \in X_k, k \neq j$, i.e.

$$T(x^1, ..., \lambda x^j + y^j, ..., x^m) = \lambda T(x^1, ..., x^j, ..., x^m) + T(x^1, ..., y^j, ..., x^m),$$

for all $\lambda \in K$ and $x^j, y^j \in X_j(j = 1, ..., m)$.
The vector space of such mappings is denoted by $L(X_1, ..., X_m; Y)$. If $Y = \mathbb{K}$, we write $L(X_1, ..., X_m)$.

Remark 1.3.1. The set $\mathcal{S}$ of all vectors in Y of the form $T(x^1, ..., x^m)$, $x^j \in X_j(j = 1, ..., m)$ is not in general a vector subspace of Y (see [38, section 1.1]).

Definition 1.3.2. *An m-linear mapping $T : X_1 \times ... \times X_m \longrightarrow Y$ is continuous if it is continuous as a function between two normed spaces.*

As a consequence of this definition, similar to the linear case, we have a result that gives the characterization of the continuous m-linear mapping.

Theorem 1.3.3. *Let $X_1, ..., X_m, Y$ be normed spaces. For $T \in L(X_1, ..., X_m; Y)$ the following assertions are equivalent.*
(1) T is continuous.

(2) T is continuous in $(0,...,0)$.
(3) There is a constant $C \geq 0$ with

$$\left\|T(x^1,...,x^m)\right\| \leq C \left\|x^1\right\| ... \left\|x^m\right\|, \tag{1.12}$$

for all $x^j \in X_j (j=1,...,m)$.
(4) $\|T\| := \sup_{\|x^j\| \leq 1, j=1,...,m} \|T(x^1,...,x^m)\| < \infty$.

We will write $\mathcal{L}(X_1,...,X_m;Y)$ for the vector space of all continuous m-linear mappings. If $Y = \mathbb{K}$, we write $\mathcal{L}(X_1,...,X_m)$.
It is easy to see that

$$\|T\| = \inf \{C \geq 0, \text{ verifying the inequality (1.12)}\},$$

defines a norm on $\mathcal{L}(X_1,...,X_m;Y)$ which is complete norm when $\|.\|_Y$ is complete. For the general theory of multilinear mappings we refer to [50] or [28].

1.3.1 m-fold Tensor products of vector spaces

In the setting of the theory of multilinear operators, the theory of topological tensor product is a necessary tool, since it provides an easy description of how these operators act on products of Banach spaces, and also because it allows the representation of dual spaces of multilinear mappings. More concretely, we can understand how it works as the tool that provides the answers to the following questions.
Is there a vector space V such that $L(V,Y)$ coincides with (is isomorphic to) $L(X_1,...,X_m;Y)$? Rephrasing the question: can we, in some way, linearize multilinear mappings?
The answer is "yes" and the object is we construct the m-fold tensor product $X_1 \otimes ... \otimes X_m$, of $X_1,...,X_m$ will do the job.
The m-fold tensor product $X_1 \otimes ... \otimes X_m$ of the vector spaces $X_1,...,X_m$ can be constructed from the elements of the space $L(X_1,...,X_m)^*$. For $x^j \in X_j (j=1,...,m)$ we define the linear mapping

$$x^1 \otimes ... \otimes x^m : L(X_1,...,X_m) \longrightarrow \mathbb{K},$$

by

$$x^1 \otimes ... \otimes x^m(\phi) := \phi(x^1,...,x^m),$$

for each m-linear form ϕ on $X_1 \times ... \times X_m$. The functional $x^1 \otimes ... \otimes x^m$ is called an *elementary tensor.*

Definition 1.3.4. *The subspace of $L(X_1,...,X_m)^*$ spanned by the collection of elementary tensors*

$$\left\{x^1 \otimes ... \otimes x^m, x^j \in X_j (j=1,...,m)\right\},$$

is called the m-fold tensor product of $X_1,...,X_m$ and will be denoted by $X_1 \otimes ... \otimes X_m$. The elements of this space are called tensors.

So a typical tensor $u \in X_1 \otimes ... \otimes X_m$ has the form

$$u = \sum_{i=1}^{n} \lambda_i x_i^1 \otimes ... \otimes x_i^m, \tag{1.13}$$

where $(\lambda_i)_{i=1}^n \subset \mathbb{K}, (x_i^j)_{i=1}^n \subset X_j (j = 1, ..., m)$ and $n \in \mathbb{N}$ is arbitrary.
Note that the tensor u can always be rewritten, using the properties of the elementary tensors, in the form

$$u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m.$$

There are other, equally natural approaches for define the m-fold tensor product of vector spaces: the elementary tensors can be viewed as multilinear forms. The proof is an adaptation of [68, page 7]).

Proposition 1.3.5. *For all vector spaces $X_1, ..., X_m$ we have a canonical embedding*

$$X_1 \otimes ... \otimes X_m \subset L(X_1^*, ..., X_m^*).$$

By the last canonical embedding, the elementary tensors $x^1 \otimes ... \otimes x^m \in X_1 \otimes ... \otimes X_m$ is identified with the multilinear form that maps $(\varphi^1, ..., \varphi^m) \in X_1^* \otimes ... \otimes X_m^*$ to $\langle x^1, \varphi^1 \rangle ... \langle x^m, \varphi^m \rangle$. In other word we have

$$x^1 \otimes ... \otimes x^m(\varphi^1, ..., \phi^m) = \left\langle x^1, \varphi^1 \right\rangle ... \left\langle x^m, \varphi^m \right\rangle,$$

for all $\varphi^j \in X_j^* (j = 1, ..., m)$.

Universal property of tensor products.

For the vector spaces $X_1, ..., X_m$ we consider the canonical mapping

$$\sigma_m : X_1 \times ... \times X_m \longrightarrow X_1 \otimes ... \otimes X_m,$$

defined by

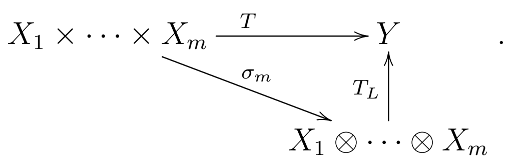
$$\sigma_m(x^1, ..., x^m) = x^1 \otimes ... \otimes x^m.$$

It is clear that the mapping σ_m is multilinear. The proof of the following theorem is an adaptation of [68, Proposition 1.4]).

Theorem 1.3.6. *For every m-linear mapping $T : X_1 \times ... \times X_m \longrightarrow Y$ there exists a unique linear mapping $T_L : X_1 \otimes ... \otimes X_m \longrightarrow Y$ given by*

$$T_L \left(\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \right) = \sum_{i=1}^{n} T\left(x_i^1, ..., x_i^m\right),$$

such that $T = T_L \circ \sigma_m$, i.e. the following diagram commutes



The correspondence $T \longleftrightarrow T_L$ establishes an isomorphism between the vector spaces $L(X_1, ..., X_m; Y)$ and $L(X_1 \otimes ... \otimes X_m, Y)$. The linear operator T_L is called the linearization of the m-linear mapping T.

1.3.2 The projective tensor product

The results related to the projective norm for order $m > 2$ have been treated in some classical works. For example, we can found some details about these concepts in [72]. The proofs of the following results can be found also in [72].
Let $X_1, ..., X_m$ be normed vector spaces. The aim of this subsection is to introduce a norm on $X_1 \otimes ... \otimes X_m$ in order to linearize (continuously) the continuous multilinear mappings defined in $X_1 \times ... \times X_m$.

Definition 1.3.7. *Let $X_1, ..., X_m$ be normed vector spaces over $\mathbb{K}$. For each tensor $u \in X_1 \otimes ... \otimes X_m$, we put*

$$\pi(u) = \inf \sum_{i=1}^{n} \left\|x_i^1\right\| ... \left\|x_i^m\right\|, \tag{1.14}$$

where the infimum is taken over all possible representations of u of the form $\sum\limits_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m$.

Proposition 1.3.8. *Let $X_1, ..., X_m$ be normed vector spaces over $\mathbb{K}$. Then π, as defined in (1.14), is a norm on $X_1 \otimes ... \otimes X_m$. Furthermore:*
1) $\pi(x^1 \otimes ... \otimes x^m) = \|x^1\| ... \|x^m\|$ for all $x^j \in X_j (j = 1, ..., m)$.
2) For each $\varphi^j \in X_j^* (j = 1, ..., m)$ the linear functional $\varphi^1 \otimes ... \otimes \varphi^m \in (X_1 \otimes ... \otimes X_m, \pi)^*$ defined by

$$\varphi^1 \otimes ... \otimes \varphi^m(z) := \sum_{i=1}^{n} \varphi^1(x_i^1) ... \varphi^m(x_i^m), \tag{1.15}$$

for $z = \sum\limits_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \in X_1 \otimes ... \otimes X_m$ is continuous and

$$\left\|\varphi^1 \otimes ... \otimes \varphi^m\right\| = \left\|\varphi^1\right\| ... \left\|\varphi^m\right\|.$$

The norm π is known as the projective norm and we will denote by $X_1 \otimes_\pi ... \otimes_\pi X_m$ the m-fold tensor product $X_1 \otimes ... \otimes X_m$ endowed with the projective norm π. We denote its completion by $X_1 \widehat{\otimes}_\pi ... \widehat{\otimes}_\pi X_m$. The Banach space $X_1 \widehat{\otimes}_\pi ... \widehat{\otimes}_\pi X_m$ is called the *projective tensor product* of Banach spaces $X_1, ..., X_m$. We now present a representation for the elements of a complete projective tensor product, $X_1 \widehat{\otimes}_\pi ... \widehat{\otimes}_\pi X_m$, which complements adequately the representation of the elements of the algebraic tensor product $X_1 \otimes ... \otimes X_m$.

Proposition 1.3.9. *Let $X_1, ..., X_m$ be any Banach spaces. An element*

$$u \in X_1 \widehat{\otimes}_\pi ... \widehat{\otimes}_\pi X_m$$

has the representation

$$u = \sum_{n=1}^{\infty} x_n^1 \otimes ... \otimes x_n^m \quad \textit{with} \quad \sum_{n=1}^{\infty} \|x_n^1\| ... \|x_n^m\| < \infty,$$

and the projective tensor norm of u as

$$\pi(u) = \inf \left\{ \sum_{n=1}^{\infty} \|x_n^1\| \dots \|x_n^m\| : u = \sum_{n=1}^{\infty} x_n^1 \otimes \dots \otimes x_n^m \right\}, \tag{1.16}$$

where the infimum is taken over all possible representations of u as above.

Linearization of continuous multilinear mappings.

Theorem 1.3.10. *Let $X_1, ..., X_m$ and Y be Banach spaces. For every continuous m-linear mapping $T : X_1 \times ... \times X_m \longrightarrow Y$ there exists a unique continuous linear operator $T_L : X_1 \widehat{\otimes}_\pi ... \widehat{\otimes}_\pi X_m \longrightarrow Y$ satisfying*

$$T_L\left(x_i^1 \otimes \dots \otimes x_i^m\right) = T\left(x_i^1, \dots, x_i^m\right),$$

for every $x^j \in X_j (j = 1, ..., m)$.
The Banach space $\mathcal{L}(X_1, ..., X_m; Y)$ is isometrically isomorphic to $\mathcal{L}(X_1 \widehat{\otimes}_\pi ... \widehat{\otimes}_\pi X_m, Y)$ through the correspondence $T \longleftrightarrow T_L$.

The previous theorem gives the canonical identification

$$\mathcal{L}(X_1, ..., X_m; Y) = \mathcal{L}(X_1 \widehat{\otimes}_\pi ... \widehat{\otimes}_\pi X_m, Y). \tag{1.17}$$

If we take $Y = \mathbb{K}$, we obtain a canonical identification of the dual space of the projective tensor product;

$$(X_1 \widehat{\otimes}_\pi ... \widehat{\otimes}_\pi X_m)^* = \mathcal{L}(X_1, ..., X_m). \tag{1.18}$$

With this identification, the action of a continuous form T, as continuous linear functional on $X_1 \widehat{\otimes}_\pi ... \widehat{\otimes}_\pi X_m$, is given by

$$\sum_{i=1}^{n} x_i^1 \otimes \dots \otimes x_i^m \longmapsto \sum_{i=1}^{n} T(x_i^1, \dots, x_i^m).$$

This duality yields a new formula for the projective norm,

$$\pi(u) = \sup \left\{ |T_L(u)|, T \in \mathcal{L}(X_1, ..., X_m) \text{ and } \|T\| \leq 1 \right\}. \tag{1.19}$$

1.3.3 The injective tensor product

Definition 1.3.11. *Let $X_1, ..., X_m$ be normed vector spaces over $\mathbb{K}$. The injective norm on $X_1 \otimes ... \otimes X_m$ is defined by*

$$\epsilon(u) = \sup \left\{ \left| \sum_{i=1}^{n} \varphi^1(x_i^1) \dots \varphi^m(x_i^m) \right|, \varphi^j \in B_{X_j^*} (j = 1, ..., m) \right\}, \tag{1.20}$$

where $\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m$ is any representation of $u \in X_1 \otimes ... \otimes X_m$.

We denote by $X_1 \otimes_\epsilon ... \otimes_\epsilon X_m$ the m-fold tensor product $X_1 \otimes ... \otimes X_m$ with the injective norm. The completion, denoted by $X_1 \widehat{\otimes}_\epsilon ... \widehat{\otimes}_\epsilon X_m$, is called the *injective tensor product* of $X_1, ..., X_m$.
We now recall some properties of the injective norm. The proof is an adaptation of [68, Proposition 3.1]).

Proposition 1.3.12. *Let $X_1, ..., X_m$ be normed vector spaces over $\mathbb{K}$.*
(1) $\epsilon(u) \leq \pi(u)$ for every $u \in X_1 \otimes ... \otimes X_m$.
(2) $\epsilon(x^1 \otimes ... \otimes x^m) = \|x^1\| ... \|x^m\|$ for every $x^j \in X_j (j = 1, ..., m)$
(3) If $\varphi^j \in X_j^* (j = 1, ..., m)$ then $\varphi^1 \otimes ... \otimes \varphi^m$, as defined by (1.15), is a continuous functional on $X_1 \otimes_\epsilon ... \otimes_\epsilon X_m$ with the norm

$$\left\|\varphi^1 \otimes ... \otimes \varphi^m\right\| = \left\|\varphi^1\right\| ... \left\|\varphi^m\right\|.$$

1.3.4 Reasonable crossnorms and tensor norms

The relevant notion of tensor norm for tensor products of Banach spaces has its roots in the work of A. Grothendieck. Indeed, he introduced the notion of reasonable crossnorm, and defined the greatest crossnorm and the least reasonable crossnorm: the *projective* and the *injective* tensor norms, respectively. Although there is not a general comprehensive reference for the theory of tensor norms of order $m > 2$, many results are straightforward generalizations of the case $m = 2$ which, for example, is treated in [68].
We denote by $(X_1 \otimes_\alpha ... \otimes_\alpha X_m)$ the m-fold tensor product $X_1 \otimes ... \otimes X_m$ with a norm α. The completion will be denoted as usual by $(X_1 \widehat{\otimes}_\alpha ... \widehat{\otimes}_\alpha X_m)$.

Definition 1.3.13. *Let $X_j (j = 1, ..., m)$ be normed vector spaces (over the same scalar field $\mathbb{K}$). A norm α on $X_1 \otimes ... \otimes X_m$ will be called a reasonable crossnorm if α satisfies the following requirements:*
1) For any $x^j \in X_j (j = 1, ..., m)$,

$$\alpha(x^1 \otimes ... \otimes x^m) \leq \left\|x^1\right\| ... \left\|x^m\right\|. \tag{1.21}$$

2) For any $\varphi^j \in X_j^* (j = 1, ..., m)$ the linear functional $\varphi^1 \otimes ... \otimes \varphi^m$, as defined by (1.15) on $(X_1 \otimes_\alpha ... \otimes_\alpha X_m)$, is continuous and

$$\left\|\varphi^1 \otimes ... \otimes \varphi^m\right\| \leq \left\|\varphi^1\right\| ... \left\|\varphi^m\right\|. \tag{1.22}$$

Many authors use another definition for reasonable crossnorm of order $m > 2$. This second definition is due to K. Floret and S. Hunfeld, and can be found in [34].

Definition 1.3.14. *A norm α on $X_1 \otimes ... \otimes X_m$ will be called a reasonable crossnorm if*

$$\epsilon(u) \leq \alpha(u) \leq \pi(u), \tag{1.23}$$

for all $u \in X_1 \otimes ... \otimes X_m$.

In the following theorem it is shown that both definitions of reasonable crossnorm are equivalent.

Theorem 1.3.15. *The last two definitions of reasonable crossnorm α are equivalent. In addition every reasonable crossnorm α has the following properties.*
1) For any $x^j \in X_j (j = 1, ..., m)$, we have

$$\alpha(x^1 \otimes ... \otimes x^m) = \left\|x^1\right\| ... \left\|x^m\right\|.$$

2) For any $\varphi^j \in X_j^*(j = 1, ..., m)$ the norm of $\varphi^1 \otimes ... \otimes \varphi^m$ in $(X_1 \otimes_\alpha ... \otimes_\alpha X_m)^*$ satisfies

$$\left\|\varphi^1 \otimes ... \otimes \varphi^m\right\| = \left\|\varphi^1\right\| ... \left\|\varphi^m\right\|.$$

Proof. Suppose that the norm α satisfies the conditions (1.21) and (1.22). Then for every $u = \sum\limits_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \in X_1 \otimes ... \otimes X_m$,

$$\alpha(u) \leq \sum_{i=1}^{n} \alpha(x_i^1 \otimes ... \otimes x_i^m) \leq \sum_{i=1}^{n} \left\|x_i^1\right\| ... \left\|x_i^m\right\|,$$

and it follows that $\alpha(u) \leq \pi(u)$. On the other hand if $\varphi^j \in B_{X_j^*}(j = 1, ..., m)$, then

$$\left\|\varphi^1 \otimes ... \otimes \varphi^m\right\| \leq \left\|\varphi^1\right\| ... \left\|\varphi^m\right\| \leq 1.$$

Then we have

$$\begin{aligned} \epsilon(u) &= \sup\left\{\left|\sum_{i=1}^{n} \varphi^1(x_i^1)...\varphi^m(x_i^m)\right|, \varphi^j \in B_{X_j^*}(j = 1, ..., m)\right\} \\ &= \sup\left\{\left|\left\langle u, \varphi^1 \otimes ... \otimes \varphi^m\right\rangle\right|, \varphi^j \in B_{X_j^*}(j = 1, ..., m)\right\} \\ &\leq \sup\left\{\left|\langle u, \psi\rangle\right|, \psi \in (X_1 \otimes ... \otimes X_m, \alpha)^*, \|\psi\| \leq 1\right\} \\ &= \alpha(u). \end{aligned}$$

Conversely, suppose that α is a norm on $X_1 \otimes ... \otimes X_m$ satisfies the condition (1.23). Then by Proposition 1.3.8 we have

$$\alpha(x^1 \otimes ... \otimes x^m) \leq \pi(x^1 \otimes ... \otimes x^m) = \left\|x^1\right\| ... \left\|x^m\right\|,$$

for any $x^1 \in X_j (j = 1, ..., m)$.
Now, since $\alpha(u) \leq 1$ implies $\epsilon(u) \leq 1$ we have

$$\begin{aligned} \left\|\varphi^1 \otimes ... \otimes \varphi^m\right\| &= \sup\left\{\left|\left\langle u, \varphi^1 \otimes ... \otimes \varphi^m\right\rangle\right|, \alpha(u) \leq 1\right\} \\ &\leq \sup\left\{\left|\left\langle u, \varphi^1 \otimes ... \otimes \varphi^m\right\rangle\right|, \epsilon(u) \leq 1\right\} \\ &= \left\|\varphi^1 \otimes ... \otimes \varphi^m\right\|_{(X_1 \otimes_\epsilon ... \otimes_\epsilon X_m)^*} \\ &= \left\|\varphi^1\right\| ... \left\|\varphi^m\right\|, \end{aligned}$$

for any $\varphi^j \in X_j^*(j = 1, ..., m)$. Therefore α satisfies the conditions (1.21) and (1.22).

If α is a reasonable crossnorm on $X_1 \otimes ... \otimes X_m$ then

$$\epsilon(x^1 \otimes ... \otimes x^m) \leq \alpha(x^1 \otimes ... \otimes x^m) \leq \pi(x^1 \otimes ... \otimes x^m),$$

for any $x^j \in X_j (j = 1, ..., m)$. But by Proposition 1.3.8 and Proposition 1.3.12 we have

$$\epsilon(x^1 \otimes ... \otimes x^m) = \pi(x^1 \otimes ... \otimes x^m) = \left\|x^1\right\| ... \|x^m\|,$$

it follows that

$$\alpha(x^1 \otimes ... \otimes x^m) = \left\|x^1\right\| ... \|x^m\|.$$

Let $\varphi^j \in X_j^* (j = 1, ..., m)$. Then using the fact that $\pi(.) \geq \alpha(.)$,

$$\begin{aligned} \left\|\varphi^1 \otimes ... \otimes \varphi^m\right\| &= \sup\left\{\left|\left\langle u, \varphi^1 \otimes ... \otimes \varphi^m\right\rangle\right|, \alpha(u) \leq 1\right\} \\ &\geq \sup\left\{\left|\left\langle u, \varphi^1 \otimes ... \otimes \varphi^m\right\rangle\right|, \pi(u) \leq 1\right\} \\ &= \left\|\varphi^1 \otimes ... \otimes \varphi^m\right\|_{(X_1 \otimes_\pi ... \otimes_\pi X_m)^*} \\ &= \left\|\varphi^1\right\| ... \|\varphi^m\|, \end{aligned}$$

this becomes $\|\varphi^1 \otimes ... \otimes \varphi^m\| = \|\varphi^1\| ... \|\varphi^m\|$. □

Recall that a norm α on $X_1 \otimes ... \otimes X_m$ and on $Y_1 \otimes ... \otimes Y_m$ satisfies the metric mappings property: if $s_j \in \mathcal{L}(X_j, Y_j), j = 1, ..., m$, then the linear mapping

$$s_1 \otimes ... \otimes s_m : X_1 \otimes_\alpha ... \otimes_\alpha X_m \longrightarrow Y_1 \otimes_\alpha ... \otimes_\alpha Y_m,$$

defined by

$$s_1 \otimes ... \otimes s_m(x^1 \otimes ... \otimes x^m) = s_1(x^1) \otimes ... \otimes s_m(x^m), x^j \in X_j (j = 1, ..., m)$$

is continuous. Moreover we have

$$\|s_1 \otimes ... \otimes s_m\| \leq \|s_1\| ... \|s_m\|.$$

Definition 1.3.16. (Tensor Norm) *We define a tensor norm of order m on $X_1 \otimes ... \otimes X_m$ to be a reasonable crossnorm satisfying the metric mapping property.*

It is clear that the projective norm and the injective norm are tensor norms.

1.4 Ideals of multilinear mappings

A. Pietsch presented his "designs of a theory" for ideals of multilinear forms in the eighties (see [62]). His work provided a general framework from which different lines of investigation developed. The definition of normed ideals of multilinear continuous mappings, was explicitly given by H. A. Braunss for Banach space-valued m-linear mappings in [22].

1.4.1 Basics on the theory of multi-ideals

Let $X_1, ..., X_m, Y$ be Banach spaces. Consider non-zero $\varphi^j \in X_j^*(j = 1, ..., m)$, and $y \in Y$. Define the multilinear mapping

$$\varphi^1 \otimes ... \otimes \varphi^m \otimes y : X_1 \times ... \times X_m \longrightarrow Y,$$

by

$$\varphi^1 \otimes ... \otimes \varphi^m \otimes y(x^1, ..., x^m) := \varphi^1(x^1)...\varphi^m(x^m)y. \tag{1.24}$$

By using Theorem 1.3.3 we obtain $\varphi^1 \otimes ... \otimes \varphi^m \otimes y \in \mathcal{L}(X_1, ..., X_m; Y)$ and

$$\left\|\varphi^1 \otimes ... \otimes \varphi^m \otimes y\right\| = \left\|\varphi^1\right\| ... \left\|\varphi^m\right\| \left\|y\right\|.$$

We denote by $\mathcal{L}_f(X_1, ..., X_m; Y)$, the vector space of finite type, which is generated by the mappings of the form (1.24). All elements T of this space a finite representation of the form

$$T = \sum_{i=1}^{n} \lambda_i \varphi_i^1 \otimes ... \otimes \varphi_i^m \otimes y_i$$

where $(\lambda_i)_{i=1}^n \subset \mathbb{K}$, $(\varphi_i^j)_{i=1}^n \subset X_j^*(j = 1, ..., m)$ and $(y_i)_{i=1}^n \subset Y$.

Definition 1.4.1. *An ideal of multilinear mappings (or multi-ideal) is a subclass $\mathcal{M}$ of all continuous multilinear mappings between Banach spaces such that for all $m \in \mathbb{N}$ and Banach spaces $X_1, \ldots, X_m$ and Y, the components*

$$\mathcal{M}(X_1, \ldots, X_m; Y) := \mathcal{L}(X_1, \ldots, X_m; Y) \cap \mathcal{M},$$

satisfy:

1) $\mathcal{M}(X_1, \ldots, X_m; Y)$ is a linear subspace of $\mathcal{L}(X_1, \ldots, X_m; Y)$ which contains the m-linear mappings of finite type.

2) The ideal property: If $T \in \mathcal{M}(G_1, \ldots, G_m; F)$, $u_j \in \mathcal{L}(X_j, G_j)$ for $j = 1, \ldots, m$ and $v \in \mathcal{L}(F, Y)$, then $v \circ T \circ (u_1, \ldots, u_m)$ is in $\mathcal{M}(X_1, \ldots, X_m; Y)$.

If $\|.\|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}^+$ satisfies:

a) $(\mathcal{M}(X_1, \ldots, X_m; Y), \|.\|_{\mathcal{M}})$ is a normed (Banach) space for all Banach spaces $X_1, \ldots, X_m, Y$ and all m,

b) The m-linear form $T^m : \mathbb{K}^m \to \mathbb{K}$ given by $T^m(x^1, \ldots, x^m) = x^1 \ldots x^m$ satisfies $\|T^m\|_{\mathcal{M}} = 1$ for all m,

c) If $T \in \mathcal{M}(G_1, \ldots, G_m; F)$, $u_j \in \mathcal{L}(X_j, G_j)$ for $j = 1, \ldots, m$ and $v \in \mathcal{L}(F, Y)$, then

$$\|v \circ T \circ (u_1, \ldots, u_m)\|_{\mathcal{M}} \leq \|v\| \|T\|_{\mathcal{M}} \|u_1\| \ldots \|u_m\|,$$

we say that $(\mathcal{M}, \|.\|_{\mathcal{M}})$ is a normed (Banach) multi-ideal.

Of course the Banach spaces considered in this definition are all over the same fixed scalar field.

The multi-ideal $\mathcal{M}$ is said to be *closed* if each $\mathcal{M}(X_1, \ldots, X_m; Y)$ is a closed subspace of $\mathcal{L}(X_1, \ldots, X_m; Y)$ from the sup norm.

Note that $\mathcal{L}$, the class of all continuous multilinear mappings between arbitrary Banach spaces, is the largest multi-ideal.

Proposition 1.4.2. *[35] Let $\mathcal{M}$ be normed multi-ideal, $m \in \mathbb{N}$ and $X_1, \ldots, X_m, Y$ be Banach spaces.*
1) $\|T\| \leq \|T\|_{\mathcal{M}}$ for all $T \in \mathcal{M}(X_1, \ldots, X_m; Y)$.
2) $\|\varphi^1 \otimes ... \otimes \varphi^m \otimes y\|_{\mathcal{M}} = \|\varphi^1\| ... \|\varphi^m\| \|y\|$ for any $\varphi^j \in X_j^*(j = 1, ..., m)$ and $y \in Y$.

In what follows we recall some important ideals of multilinear mappings, and will be used in the sequel.
We say that $T \in \mathcal{L}(X_1 \times ... \times X_m; Y)$ is completely continuous, and we write $T \in \mathcal{L}_{\mathcal{V}}(X_1, ..., X_m; Y)$, if the sequence $(T(x_n^1, ..., x_n^m))_{n=1}^{\infty}$ is norm convergent in Y for every sequences $(x_n^j)_{n=1}^{\infty} \subset X_j(j = 1, ..., m)$ converges weakly.

A multilinear mapping $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is said to be weakly compact, in symbols $T \in \mathcal{L}_{\mathcal{W}}(X_1, ..., X_m; Y)$, if T maps $B_{X_1} \times ... \times B_{X_m}$ onto a relatively weakly compact subset of Y.
This is equivalent to say that $(T(x_n^1, ..., x_n^m))_{n=1}^{\infty}$ has a weakly convergent subsequence for every bounded sequence $(x_n^j)_{n=1}^{\infty} \subset X_j(j = 1, ..., m)$.

A multilinear mapping $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is said to be compact, in symbols $T \in \mathcal{L}_{\mathcal{K}}(X_1, ..., X_m; Y)$, if T maps $B_{X_1} \times ... \times B_{X_m}$ onto a relatively compact subset of Y.
This is equivalent to say that $(T(x_n^1, ..., x_n^m))_{n=1}^{\infty}$ has a convergent subsequence for every bounded sequence $(x_n^j)_{n=1}^{\infty} \subset X_j(j = 1, ..., m)$

Theorem 1.4.3. *[35, 66, 15, 65] The classes of finite type, completely continuous, weakly compact and compact m-linear mappings ($\mathcal{L}_f, \mathcal{L}_{\mathcal{V}}, \mathcal{L}_{\mathcal{W}}$, and $\mathcal{L}_{\mathcal{K}}$ respectively) are multi-ideals.*

1.4.2 Methods of construction

There are different ways of constructing an ideal of multilinear mappings from a given operator ideal. The property enjoyed by the linear operators in this ideal can be generalized to the multilinear case and the resulting classes of multilinear mappings happen to be ideals (that is the case of the ideal of weakly compact operators, for example). The point is that, depending on the operator ideal, it may happen that there is not a unique natural generalization to the multilinear settings. The two methods outlined by A. Pietsch in [62] are described in what follows.

1)The composition method.
Next we recall the procedure of composition. It was firstly presented by G. Botelho et al. in [19], for generating ideals of multilinear mappings starting from an operator ideal $\mathcal{I}$. Actually this procedure is a particular case of a technique introduced by A. Pietsch in [62].
A. Pełczyński proved that a multilinear mapping T between Banach spaces is weakly compact if and only if T can be written as $T = u \circ S$ where S is a continuous multilinear mapping and u is a weakly compact linear operator (see [54]). So, given

an operator ideal $\mathcal{I}$, it is natural to consider the multilinear mappings T which can be written as $T = u \circ S$ with u belonging to $\mathcal{I}$ and S is a continuous m-linear mapping.

Definition 1.4.4. (Composition Ideals) *Let $\mathcal{I}$ be an operator ideal. An m-linear mapping $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ belongs to $\mathcal{I} \circ \mathcal{L}$ if there are a Banach space G, an m-linear mapping $R \in \mathcal{L}(X_1, \ldots, X_m; G)$ and a linear operator $u \in \mathcal{I}(G, Y)$ such that $T = u \circ R$. In this case we write $T \in \mathcal{I} \circ \mathcal{L}(X_1, \ldots, X_m; Y)$.*

To verify if a continuous m-linear mapping T belongs to $\mathcal{I} \circ \mathcal{L}$ or not, by definition is necessary to investigate the existence of a factorization of the form $T = u \circ R$ with $u \in \mathcal{I}$ and $R \in \mathcal{L}$. As it is not always easy (especially to prove that there is not such a factorization), G. Botelho et al. introduced in [19] a criterion for establishing that T belongs to $\mathcal{I} \circ \mathcal{L}$ by using the projective tensor product.

Proposition 1.4.5. *Let $\mathcal{I}$ be an operator ideal. The following are equivalent for $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$.*
1) $T \in \mathcal{I} \circ \mathcal{L}(X_1, \ldots, X_m; Y)$.
2) $T_L \in \mathcal{I}\left(X_1 \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi X_m, Y\right)$.

Proposition 1.4.6. *If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ be a normed (Banach) operator ideal, then $(\mathcal{I} \circ \mathcal{L}, \|.\|_{\mathcal{I} \circ \mathcal{L}})$ is a normed (Banach) multi-ideal.*

Proposition 1.4.7. *Let $(\mathcal{I}, \|.\|_{\mathcal{I}})$ be a normed operator ideal, $m \in \mathbb{N}$ and $X_1, \ldots, X_m, Y$ be Banach spaces. If $T \in \mathcal{I} \circ \mathcal{L}(X_1, \ldots, X_m; Y)$, then*

$$\|T\|_{\mathcal{I} \circ \mathcal{L}} = \|T_L\|_{\mathcal{I}}.$$

In addition, the space $(\mathcal{I} \circ \mathcal{L}(X_1, ..., X_m; Y), \|.\|_{\mathcal{I} \circ \mathcal{L}})$ is isometrically isomorphic to $(\mathcal{I}(X_1 \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi X_m, Y), \|.\|_{\mathcal{I}})$

Proposition 1.4.8. *[19, Proposition 3.5] Let $\mathcal{I}$ be an operator ideal and Y be a Banach space. The following are equivalent:*
(i) $id_Y \in \mathcal{I}(Y, Y)$.
(ii) $\mathcal{I} \circ \mathcal{L}(X_1, ..., X_m; Y) = \mathcal{L}(X_1, ..., X_m; Y)$ for every m and every Banach spaces $X_1, ..., X_m$.
(iii) $\mathcal{I} \circ \mathcal{L}(X_1, ..., X_m; Y) = \mathcal{L}(X_1, ..., X_m; Y)$ for some m and every Banach spaces $X_1, ..., X_m$.

The definition of adjoint of an m-linear mapping is due to M. S. Ramanujan and E. Schock [64]. Recall, if $T \in \mathcal{L}(X_1, ..., X_m; Y)$ we define the adjoint of T by

$$T^* : Y^* \longrightarrow \mathcal{L}(X_1, ..., X_m), \quad y^* \longmapsto T^*(y^*) : X_1 \times ... \times X_m \longrightarrow \mathbb{K},$$

with

$$T^*(y^*)(x^1, ..., x^m) = y^*(T(x^1, ..., x^m)),$$

and has the property that T^* is linear and $\|T^*\| = \|T\|$. It is easy to see that, if $T \in \mathcal{L}(X_1, ..., X_m; Y)$ and $u \in \mathcal{L}(Y, Z)$ we have

$$(u \circ T)^* = T^* \circ u^*.$$

As applications of the composition method we give the following important result.

Corollary 1.4.9. *The m-linear mapping $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is weakly compact if and only if its adjoint T^* is a weakly compact linear operator.*

Proof. Because $\mathcal{L}_{\mathcal{W}} = \mathcal{W} \circ \mathcal{L}$ (see [54]), we have that T belongs to $\mathcal{L}_{\mathcal{W}}(X_1, \ldots, X_m; Y)$ if and only if its linearization

$$T_L : X_1 \widehat{\otimes}_\pi ... \widehat{\otimes}_\pi X_m \longrightarrow Y,$$

is weakly compact linear operator. This is equivalent to

$$(T_L)^* : Y^* \longrightarrow \left(X_1 \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi X_m\right)^*,$$

is weakly compact (see Theorem 1.2.6). On the other hand, we have $(T_L)^* = \Psi \circ T^*$ (then $T^* = \Psi^{-1} \circ (T_L)^*$) where

$$\Psi : \mathcal{L}\left(X_1, \ldots, X_m\right) \longrightarrow (X_1 \widehat{\otimes}_\pi ... \widehat{\otimes}_\pi X_m)^*,$$

is the isomorphic isometry given by $\Psi(\phi) = \phi_L$ (see Theorem 1.3.10). By the ideal property concerning the operator ideal $\mathcal{W}$ we have that $(T_L)^*$ is weakly compact if and only if T^* is too. □

2) The factorization method.
Given the operator ideals $\mathcal{I}_1, ..., \mathcal{I}_m$, an m-linear mapping $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is said to be of type $\mathcal{L}(\mathcal{I}_1, ..., \mathcal{I}_m)$, in symbols $T \in \mathcal{L}(\mathcal{I}_1, ..., \mathcal{I}_m)(X_1, ..., X_m; Y)$, if there are Banach spaces $G_1, ..., G_m$, linear operators $u_j \in \mathcal{I}_j(X_j, G_j)$, $j = 1, ..., m$, and a continuous m-linear mapping $R \in \mathcal{L}(G_1, ..., G_m; Y)$ such that $T = R \circ (u_1, ..., u_m)$. If $\mathcal{I}_1, ..., \mathcal{I}_m$ are normed operator ideals and $T \in \mathcal{L}(\mathcal{I}_1, ..., \mathcal{I}_m)(X_1, ..., X_m)$ we define

$$\|T\|_{\mathcal{L}(\mathcal{I}_1, ..., \mathcal{I}_m)} = \inf \|R\| \, \|u_1\|_{\mathcal{I}_1} \cdots \|u_m\|_{\mathcal{I}_m},$$

where the infimum is taken over all possible factorizations $T = R \circ (u_1, ..., u_m)$ with u_j belonging to $\mathcal{I}_j, (j = 1, ..., m)$ and the continuous m-linear mapping R.
We will see that $\left(\mathcal{L}(\mathcal{I}_1, ..., \mathcal{I}_m), \|.\|_{\mathcal{L}(\mathcal{I}_1, ..., \mathcal{I}_m)}\right)$ is a normed ideal of multilinear mappings. This method of constructing an ideal of multilinear mappings from ideals of linear operators is called the *factorization method*. The proof of the following theorem can be found in [35] or [10].

Theorem 1.4.10. *If $\mathcal{I}_1, ..., \mathcal{I}_m$ are normed (Banach) operators ideals, then*

$$\left(\mathcal{L}(\mathcal{I}_1, ..., \mathcal{I}_m), \|.\|_{\mathcal{L}(\mathcal{I}_1, ..., \mathcal{I}_m)}\right),$$

is a normed (Banach) ideal of multilinear mappings.

1.4.3 Tensor product representation of multi-ideals

Now we recall how represent a multi-ideal by a tensor norm giving some examples of this construction. The next proposition is necessary for defining the tensor representation of a multi-ideal.

Proposition 1.4.11. *[50, Theorem 1.10] Let $X_1, ..., X_m$ and Y be Banach spaces. Then we have the two isometric isomorphism identification.*

$$\mathcal{L}(X_1, ..., X_m, Y) = \mathcal{L}(X_1, ..., X_m; Y^*), \tag{1.25}$$

and

$$\mathcal{L}(X_1, ..., X_m, Y^*) = \mathcal{L}(X_1, ..., X_m; Y). \tag{1.26}$$

By the identifications (1.25) and (1.18) we obtain

$$\mathcal{L}(X_1, ..., X_m; Y^*) = \left(X_1 \widehat{\otimes}_\pi ... \widehat{\otimes}_\pi X_m \widehat{\otimes}_\pi Y\right)^*, \tag{1.27}$$

that is the key to define the concept of tensorial representation of multi-ideals.

Definition 1.4.12. *We say that a tensor norm α of order $m+1$ represents the multi-ideal $\mathcal{M}$ if $\mathcal{M}(X_1, ..., X_m; Y^*)$ and $(X_1 \otimes_\alpha ... \otimes_\alpha X_m \otimes_\alpha Y)^*$ are isometrically isomorphic under the canonical mapping*

$$\Psi : \mathcal{M}(X_1, ..., X_m; Y^*) \longrightarrow (X_1 \otimes_\alpha ... \otimes_\alpha X_m \otimes_\alpha Y)^*,$$

defined by $\Psi(T)(x^1 \otimes ... \otimes x^m \otimes y) := T(x^1, ..., x^m)(y)$, for all $m \in \mathbb{N}$ and all Banach spaces $X_1, ..., X_m, Y$.

Remark 1.4.13. Since $\left(X_1 \widehat{\otimes}_\alpha ... \widehat{\otimes}_\alpha X_m \widehat{\otimes}_\alpha Y\right)$ is the completion of the normed space $(X_1 \otimes_\alpha ... \otimes_\alpha X_m \otimes_\alpha Y)$. Then, in what follows, we can write the isometric identification

$$(X_1 \otimes_\alpha ... \otimes_\alpha X_m \otimes_\alpha Y)^* = \left(X_1 \widehat{\otimes}_\alpha ... \widehat{\otimes}_\alpha X_m \widehat{\otimes}_\alpha Y\right)^*,$$

through the isometry $\varphi \mapsto \overline{\varphi}$, where $\overline{\varphi}$ is the unique extension of φ.

The uniqueness of the representation was proved by G. Botelho et all in [17].

Theorem 1.4.14. *The tensor norm that represents a given multi-ideal, if any, is unique.*

There is another method for representing an ideal of multilinear mappings $\mathcal{M}$, described as follows.

Definition 1.4.15. *We say that a tensor norm α of order $m+1$ represents the multi-ideal $\mathcal{M}$ if $\mathcal{M}(X_1, ..., X_m; Y)$ and $(X_1 \otimes_\alpha ... \otimes_\alpha X_m \otimes_\alpha Y^*)^*$ are isometrically isomorphic.*

Basic examples

1) Tensorial representation of $\mathcal{L}_{d,(p_1,...,p_m)}$.

The definition of absolutely $(p; p_1, ..., p_m)$-summing m-linear functionals is due to A. Pietsch [62]. In [43], M. C. Matos presented a definition for vector-valued mappings.

Definition 1.4.16. *Let $m \in N$ and $1 \leq p, p_1, ..., p_m < \infty$, with $\frac{1}{p} \leq \frac{1}{p_1} + ... + \frac{1}{p_m}$. An m-linear mapping $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is said to be absolutely $(p; p_1, ..., p_m)$-summing if there is a constant $C > 0$ such that for any $x_1^j, ..., x_n^j \in X_j (j = 1, ..., m)$ we have*

$$\left\| \left(T \left(x_i^1, ..., x_i^m \right) \right)_{i=1}^n \right\|_p \leq C \prod_{j=1}^m \left\| \left(x_i^j \right)_{i=1}^n \right\|_{p_j, \omega},$$

for every n.

The vector space of these mappings is indicated by $\mathcal{L}_{as,(p;p_1,...,p_m)} (X_1, ..., X_m; Y)$ and the smallest C satisfying the inequality above, by $\|T\|_{\mathcal{L}_{as,(p;p_1,...,p_m)}}$. This defines a norm on $\mathcal{L}_{as,(p;p_1,...,p_m)} (X_1, ..., X_m; Y)$ and $\left(\mathcal{L}_{as,(p;p_1,...,p_m)}, \|.\|_{\mathcal{L}_{as,(p;p_1,...,p_m)}} \right)$ is a Banach multi-ideal.
This definition is equivalent to say that $(T(x_i^1, ..., x_i^m))_{i=1}^\infty$ belongs to $\ell_p(Y)$ for every $(x_i^j)_{i=1}^\infty \in \ell_{p_j}^\omega(X_j)$.
If $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ we will call T is $(p_1, ..., p_m)$-dominated and we will denote the corresponding vector space and norm by $\mathcal{L}_{d,(p_1,...,p_m)}$ and $\|.\|_{\mathcal{L}_{d,(p_1,...,p_m)}}$, respectively.
In [43], M. C. Matos gives a representation of the multi-ideal $\mathcal{L}_{d,(p_1,...,p_m)}$ by the tensor norm δ_p of order $m + 1$ defined by

$$\delta_p(z) = \inf \|(\lambda_i)_{i=1}^n\|_{p^*} \prod_{j=1}^m \left\| \left(x_i^j \right)_{i=1}^n \right\|_{p_j, \omega} \|(y_i)_{i=1}^n\|_\infty,$$

where $z \in X_1 \otimes ... \otimes X_m \otimes Y$ and the infimum is taken over all representation of z of the form

$$\sum_{i=1}^n \lambda_i x_i^1 \otimes ... \otimes x_i^m \otimes y_i,$$

with $n \in \mathbb{N}, (\lambda_i)_{i=1}^n \subset \mathbb{K}, \left(x_i^j \right)_{i=1}^n \subset X_j (j = 1, ..., m)$ and $(y_i)_{i=1}^n \subset Y$. By this notations we have

$$\mathcal{L}_{d,(p_1,...,p_m)} (X_1, ..., X_m; Y^*) = \left(X_1 \otimes_{\delta_p} ... \otimes_{\delta_p} X_m \otimes_{\delta_p} Y \right)^* \quad \text{isometrically isomorphic.}$$

2) Tensorial representation of $\mathcal{L}_{d,(p_1,...,p_m;r)}$.
The next multilinear generalization of the ideal of $(p; q; r)$-summing linear operators was introduced by D. Achour in [1]. This new multi-ideal is represented by a suitable tensor norm of order $m + 1$.

Definition 1.4.17. *For $1 \leq p, p_1, ..., p_m, r \leq \infty$, with $\frac{1}{p} \leq \frac{1}{p_1} + ... + \frac{1}{p_m} + \frac{1}{r}$, a mapping $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is absolutely $(p; p_1, ..., p_m; r)$-summing if there is a constant $C > 0$ such that for any $x_1^j, ..., x_n^j \in X_j$, $(1 \leq j \leq m)$, and any $y_1^*, ..., y_n^* \in Y^*$, we have*

$$\left\| \left(\left\langle T \left(x_i^1, ..., x_i^m \right), y_i^* \right\rangle \right)_{i=1}^n \right\|_p \leq C \prod_{j=1}^m \left\| \left(x_i^j \right)_{i=1}^n \right\|_{p_j, \omega} \|(y_i^*)_{i=1}^n\|_{r, \omega}. \tag{1.28}$$

The vector space of these mappings is indicated by $\mathcal{L}_{as,(p;p_1,...,p_m;r)}(X_1,...,X_m;Y)$ and the smallest C satisfying the inequality above, by $\|T\|_{\mathcal{L}_{as,(p;p_1,...,p_m;r)}}$. This defines a norm $\mathcal{L}_{as,(p;p_1,...,p_m;r)}(X_1,...,X_m;Y)$ and $\left(\mathcal{L}_{as,(p;p_1,...,p_m;r)}, \|.\|_{\mathcal{L}_{as,(p;p_1,...,p_m;r)}}\right)$ is a Banach multi-ideal.
If $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m} + \frac{1}{r}$ we will call T is $(p_1,...,p_m;r)$-dominated and we will denote the corresponding vector space and norm by $\mathcal{L}_{d,(p_1,...,p_m;r)}$ and $\|.\|_{\mathcal{L}_{d,(p_1,...,p_m;r)}}$, respectively. For $z \in X_1 \otimes ... \otimes X_m \otimes Y$, we consider

$$\mu_{p,r}(z) = \inf \|(\lambda_i)_{i=1}^n\|_{p^*} \prod_{j=1}^{m} \left\|\left(x_i^j\right)_{i=1}^n\right\|_{p_j,\omega} \|(y_i)_{i=1}^n\|_{r,\omega},$$

where the infimum is taken over all representations of z of the form

$$z = \sum_{i=1}^{n} \lambda_i x_i^1 \otimes ... \otimes x_i^m \otimes y_i,$$

with $\lambda_i \in \mathbb{K}, x_i^j \in X_j, y_i \in Y, i = 1,...,n, j = 1,...,m$ and $n, m \in \mathbb{N}$.
The multi-ideal $\mathcal{L}_{d,(p_1,...,p_m;r)}$ is represented by the tensor norm $\mu_{p,r}$ i.e., we have the isometric isomorphic identification

$$\left(\mathcal{L}_{d,(p_1,...,p_m;r)}(X_1,...,X_m;Y^*), \|.\|_{\mathcal{L}_{d,(p_1,...,p_m;r)}}\right) = (X_1 \otimes_{\mu_{p,r}} ... \otimes_{\mu_{p,r}} X_m \otimes_{\mu_{p,r}} Y)^*.$$

3) Tensorial representation of $\mathcal{D}_p^m$
The definition of Cohen strongly p-summing m-linear operators is due to D. Achour and L. Mezrag (see [5]) in order to generalize the concept of strongly p-summing linear operators.

Definition 1.4.18. *For $1 \leq p < \infty$, a mapping $T \in \mathcal{L}(X_1,...,X_m;Y)$ is Cohen strongly p-summing if there is a constant $C > 0$ such that for any $x_1^j,...,x_n^j \in X_j$, $(1 \leq j \leq m)$, and any $y_1^*,...,y_n^* \in Y^*$, we have*

$$\left\|\left(\left\langle T\left(x_i^1,...,x_i^m\right), y_i^*\right\rangle\right)_{i=1}^n\right\|_1 \leq C\left(\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_i^j\right\|^p\right)^{\frac{1}{p}} \|(y_i^*)_{i=1}^n\|_{p^*,\omega}. \qquad (1.29)$$

The vector space of these mappings is indicated by $\mathcal{D}_p^m(X_1,...,X_m;Y)$ and the smallest C satisfying the inequality above, by $\|T\|_{\mathcal{D}_p^m}$. This defines a norm on $\mathcal{D}_p^m(X_1,...,X_m;Y)$ and $\left(\mathcal{D}_p^m, \|.\|_{\mathcal{D}_p^m}\right)$ is a Banach multi-ideal.

In the next result, L. Mezrag and K. Saadi in [48, Theorem 3.1] give a characterization of the Cohen strongly m-linear mappings by using the adjoint operator like that given by J. S. Cohen in the linear case.

Theorem 1.4.19. *Let $1 < p \leq \infty$. Let $T \in \mathcal{L}(X_1,...,X_m;Y)$ and T^* its adjoint. Then T belongs to $\mathcal{D}_p^m(X_1,...,X_m;Y)$, if and only if, T^* belongs to $\Pi_{p^*}(Y^*, \mathcal{L}(X_1,...,X_m))$ and we have $\|T\|_{\mathcal{D}_p^m} = \pi_{p^*}(T^*)$.*

J. A. López Molina in [40] introduced a tensor norm of order $(m+1)$ naturally extending the well-known tensor norm g_p of Chevet-Saphar which described as follows: let $X_j (j = 1, ..., m+1)$ be normed vector spaces, and a fixed natural number k verifying $1 \leq k < m+1$. Consider r and s two fixed finite sequences

$$r := (p_1, ..., p_k) \quad s := (p_{k+1}, ..., p_{m+1}),$$

of positive real numbers such that

$$\frac{1}{p_1} + ... + \frac{1}{p_k} + \frac{1}{p_{k+1}^*} + ... + \frac{1}{p_{m+1}^*} = 1.$$

For every $u \in X_1 \otimes ... \otimes X_m \otimes X_{m+1}$ define the Chevet-Saphar tensor norm of order $(m+1)$ by

$$g_{r,s}(u) := \inf \left(\prod_{j=1}^{k} \left\| \left(x_i^j \right)_{i=1}^n \right\|_{p_j} \right) \left(\prod_{j=k+1}^{m+1} \left\| \left(x_i^j \right)_{i=1}^n \right\|_{p_j^*, \omega} \right), \tag{1.30}$$

taking the infimum over all representations of u of type

$$u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^{m+1},$$

with $(x_i^j)_{i=1}^n \subset X_j (j = 1, ..., m+1)$. Clearly, if $m = 1$, we have the classical tensor norm g_p of Chevet-Saphar [68, Page 135].
The tensor norm $g_{r,s}$ has been built to satisfy the isometric identification mentioned in the following theorem.

Theorem 1.4.20. *[40, Theorem 2] Given the Banach spaces $X_j (j = 1, ..., m+1)$. The space*

$$\left(X_1 \otimes_{g_{r,s}} ... \otimes_{g_{r,s}} X_m \otimes_{g_{r,s}} X_{m+1} \right)^*,$$

is isometrically isomorphic to

$$\mathcal{L}_{d,(p_{k+1}^*, ..., p_{m+1}^*)} \left(X_{k+1}, ..., X_{m+1}; \mathcal{L} \left(X_1, ..., X_k \right) \right).$$

In (1.30), if we take $k = m$, $p_1 = ... = p_m = mp$ and $p_{m+1} = p^*$ we obtain the tensor norm g_p of order $m+1$ defined by

$$g_p(u) = \inf \prod_{j=1}^{m} \left\| \left(x_i^j \right)_{i=1}^n \right\|_{pm} \left\| \left(x_i^{m+1} \right)_{i=1}^n \right\|_{p^*, \omega}, \tag{1.31}$$

and the above identification gives

$$\left(X_1 \otimes_{g_p} ... \otimes_{g_p} X_m \otimes_{g_p} Y^* \right)^* = \Pi_{p^*}(Y^*, \mathcal{L}(X_1, ..., X_m)),$$

for all Banach spaces $X_1, ..., X_m, Y$.
Finally, by Theorem 1.4.19 we obtain the isometric identification

$$\mathcal{D}_p^m \left(X_1, ..., X_m; Y \right) = \left(X_1 \otimes_{g_p} ... \otimes_{g_p} X_m \otimes_{g_p} Y^* \right)^*,$$

which gives the tensorial representation of multi-ideal $\mathcal{D}_p^m$ by the tensor norm g_p.

Chapter 2

Absolutely continuous multilinear operators

This chapter has been the subject of a publication in the Journal of Mathematical Analysis and Applications [26] and is divided in two sections. In Section 1 we present the class of (p, σ)-absolutely continuous linear operators. After defining this concept, we will review all the main results known for this class, along with some new results. Thus, we will show the factorization theorem with a proof that is different than the one given in [47] by U. Matter. As a consequence we show that (p, σ)-absolutely continuous linear operators are compact under some requirements. In the second section we extend to multilinear mappings the concept of (p, σ)-absolutely continuous linear operators, for which the resulting vector space $\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}$ of the $(p; p_1, ..., p_m; \sigma)$-absolutely continuous multilinear operators is a Banach multi-ideal. We establish a domination theorem and a factorization theorem for such operators and we give two examples to prove the difference between our class and the class of absolutely summing multilinear operators. Finally, we present a reasonable crossnorm $\beta_{p,\sigma}$ on $X_1 \otimes ... \otimes X_m \otimes Y$ that satisfies that the topological dual of the corresponding normed tensor product is isometric to the space of Y^*-valued $(p; p_1, ..., p_m; \sigma)$-absolutely continuous multilinear operators on $X_1 \times ... \times X_m$. We generalize in this way the result for the linear case that can be found in [41].

2.1 The ideal of (p, σ)-absolutely continuous linear operators

2.1.1 (p, σ)-weakly summable sequences

The space of (p, σ)-weakly summable sequences was introduced in [41] by J. A. López Molina and E. A. Sánchez Pérez in order to give a characterization of the class of (p, σ)-absolutely continuous linear operators. Now we recall some properties of this space. Let $1 \leq p < \infty$ and $0 \leq \sigma < 1$. Let X be a Banach space and take a sequence $(x_i)_{i=1}^{\infty}$ in it. Define

$$\delta_{p\sigma}((x_i)_{i=1}^{\infty}) = \sup_{\phi \in B_{X^*}} \left(\sum_{i=1}^{\infty} \left(|\phi(x_i)|^{1-\sigma} \|x_i\|^{\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}$$

and

$$H_{p,\sigma}(X) = \{(x_i)_{i=1}^{\infty} \subset X : \delta_{p\sigma}((x_i)_{i=1}^{\infty}) < \infty\} .$$

We have that

$$\|(x_i)_{i=1}^{\infty}\|_{\frac{p}{1-\sigma},\omega} \leq \delta_{p\sigma}((x_i)_{i=1}^{\infty}) \leq \|(x_i)_{i=1}^{\infty}\|_{\frac{p}{1-\sigma}}, \quad (x_i)_{i=1}^{\infty} \in H_{p,\sigma}(X). \tag{2.1}$$

For the extreme cases $\sigma = 1$ and $p = \infty$, we define also for all $0 \leq \tau \leq 1$ and $1 \leq q \leq \infty$

$$\delta_{q1}\left((x_i)_{i=1}^{n}\right) = \delta_{\infty\tau}\left((x_i)_{i=1}^{n}\right) = \sup_{1\leq i\leq n} \|x_i\| = \|(x_i)_{i=1}^{n}\|_{\infty} . \tag{2.2}$$

If μ is a regular Borel probability measure on B_{X^*} (with the weak star topology) and $p = \infty$ or $\sigma = 1$, the expression

$$\left(\int_{B_{X^*}} (|\langle x, x^* \rangle|^{1-\sigma} \|x\|^{\sigma})^{\frac{p}{1-\sigma}} d\mu \right)^{\frac{1-\sigma}{p}}$$

must be understood as $\|x\|$.

Definition 2.1.1. [41, Definition 1.2]
A sequence $(x_i)_{i=1}^{\infty}$ of elements of X is said to be (p,σ)-weakly summable if it belongs to the vector space spanned by $H_{p,\sigma}(X)$.

We denote by $\ell^{p\sigma}(X)$ the vector space of all (p,σ)-weakly summable sequences of X. For $(x_i)_{i=1}^{\infty} \in \ell^{p\sigma}(X)$, we set

$$\|(x_i)_{i=1}^{\infty}\|_{p,\sigma} = \inf \sum_{l=1}^{k} \delta_{p\sigma}\left((x_i^l)_{i=1}^{\infty}\right) \tag{2.3}$$

where the infimum is taken over all representations of $(x_i)_{i=1}^{\infty}$ of the form

$$(x_i)_{i=1}^{\infty} = \sum_{l=1}^{k} \left(x_i^l\right)_{i=1}^{\infty},$$

with $\left(x_i^l\right)_{i=1}^{\infty} \in H_{p,\sigma}(X), k \in \mathbb{N}$.

Proposition 2.1.2. *[41, Proposition 1.3]*
On $\ell^{p\sigma}(X)$, the function $\|.\|_{p,\sigma}$, defined by (2.3), is a norm. In addition, we have the inclusions

$$\ell_{\frac{p}{1-\sigma}}(X) \subset \ell^{p\sigma}(X) \subset \ell_{\frac{p}{1-\sigma},\omega}(X), \tag{2.4}$$

with

$$\|(x_i)_{i=1}^{\infty}\|_{\frac{p}{1-\sigma},\omega} \leq \|(x_i)_{i=1}^{\infty}\|_{p,\sigma} \leq \|(x_i)_{i=1}^{\infty}\|_{\frac{p}{1-\sigma}} \quad \textit{for all } (x_i)_{i=1}^{\infty} \in \ell_{\frac{p}{1-\sigma}}(X).$$

Moreover,

$$\|(x_i)_{i=1}^{\infty}\|_{p,\sigma} \leq \inf \left\{ \sum_{l=1}^{k} \left\|(x_i^l)_{i=1}^{\infty}\right\|_{\frac{p}{1-\sigma}}^{\sigma} \cdot \left\|(x_i^l)_{i=1}^{\infty}\right\|_{\frac{p}{1-\sigma},\omega}^{1-\sigma} \right\}$$

where the infimum is taken over all representations of $(x_i)_{i=1}^{\infty} \in \ell_{\frac{p}{1-\sigma}}(X)$ *of the form*

$$(x_i)_{i=1}^{\infty} = \sum_{l=1}^{k} \left(x_i^l\right)_{i=1}^{\infty},$$

with $\left(x_i^l\right)_{i=1}^{\infty} \in \ell_{\frac{p}{1-\sigma}}(X)$.

We now give a representation for the elements in the completion $\hat{\ell}^{p\sigma}(X)$, which complements nicely the representation of the elements of $\ell^{p\sigma}(X)$. We will continue denoting by $\|.\|_{p,\sigma}$ the norm in $\hat{\ell}^{p\sigma}(X)$.

Proposition 2.1.3. *[41, Proposition 1.4]*
Let $1 \leq p < \infty$ *and let* X *be Banach space. If* $\varphi \in \hat{\ell}^{p\sigma}(X)$, *there exist* $x^n = (x_i^n)_{i=1}^{\infty} \in H_{p,\sigma}(X), n \in \mathbb{N}$ *such that*

$$\sum_{n=1}^{\infty} \delta_{p\sigma}(x^n) < \infty \quad \textit{and} \quad \varphi = \sum_{n=1}^{\infty} x^n \quad \textit{in } \hat{\ell}^{p\sigma}(X)$$

Conversely for $x^n = (x_i^n)_{i=1}^{\infty} \in H_{p,\sigma}(X)$ *there exists a unique* $\varphi \in \hat{\ell}^{p\sigma}(X)$ *such that* $\varphi = \sum_{n=1}^{\infty} x^n$. *In both cases*

$$\|\varphi\|_{p,\sigma} = \inf \sum_{n=1}^{\infty} \delta_{p\sigma}(x^n)$$

where the infimum is taken over all representations of φ *of the appropriate form.*

2.1.2 (p,σ)-absolutely continuous linear operators. Preliminaries

Throughout this subsection, let $1 \leq p < \infty$ and $0 \leq \sigma < 1$.

Definition 2.1.4. [46, Definition 3.1]
We say that $T \in \mathcal{L}(X,Y)$ *is a* (p,σ)*-absolutely continuous operator, in symbols* $T \in \Pi_{p,\sigma}(X,Y)$, *if there exist a Banach space* G *and an operator* $S \in \Pi_p(X,G)$ *such that*

$$\|Tx\| \leq \|x\|^{\sigma} \|Sx\|^{1-\sigma}, \quad x \in X. \tag{2.5}$$

In such case, we put $\pi_{p,\sigma}(T) = \inf \pi_p(S)^{1-\sigma}$, *taking the infimum over all Banach spaces* G *and* $S \in \Pi_p(X,G)$ *such that (2.5) holds.*

Remark 2.1.5. If $T \in \Pi_{p,\sigma}(X,Y)$ then $\|T\| \leq \pi_{p,\sigma}(T)$. In order to see this, there exist a Banach space G and $S \in \Pi_p(X,G)$ such that

$$\|T(x)\| \quad \leq \quad \|S(x)\|^{1-\sigma} \|x\|^{\sigma} \quad \leq \quad \|S\|^{1-\sigma} \|x\|, \text{ for all } x \in X$$

since Π_p is a operator ideal we have $\|S\| \leq \pi_p(S)$ therefore $\|T(x)\| \leq \pi_p(S)^{1-\sigma} \|x\|$. Hence $\|T\| \leq \pi_p(S)^{1-\sigma}$, which implies that $\|T\| \leq \pi_{p,\sigma}(T)$.

Theorem 2.1.6. *[46, Theorem 3.2]*
The class $(\Pi_{p,\sigma}, \pi_{p,\sigma}(.))$ *is an injective Banach operator ideal.*

By the inclusion theorem for the class Π_p we have the result

Proposition 2.1.7. *Let* $1 \leq p \leq q < \infty$ *and* $0 \leq \sigma < 1$. *Then the following holds*

$$\Pi_{p,\sigma}(X,Y) \subset \Pi_{q,\sigma}(X,Y)$$

for all Banach spaces X *and* Y.

The subsequent characterizations of $\Pi_{p,\sigma}$ are derived from the corresponding properties of Π_p. The Pietsch domination theorem concerning the (p,σ)-absolutely continuous linear operators was proved by U. Matter in [46]. As usual, the unit ball B_{X^*}, of X^* is considered as a compact space with respect to the weak star topology.

Theorem 2.1.8. *[46, Theorem 4.1]*
For a linear operator $T \in \mathcal{L}(X,Y)$ *the following statements are equivalent.*
(i) $T \in \Pi_{p,\sigma}(X,Y)$
(ii) There are a constant $C > 0$ *and a regular Borel probability measure* μ *on* B_{X^*}, *such that*

$$\|T(x)\| \leq C \left(\int_{B_{X^*}} \left(|\langle x, x^* \rangle|^{1-\sigma} \|x\|^{\sigma} \right)^{\frac{p}{1-\sigma}} d\mu(x^*) \right)^{\frac{1-\sigma}{p}}, \quad x \in X. \tag{2.6}$$

(iii) There is a constant $C > 0$ *such that for every finite sequence* $(x_i)_{i=1}^n$ *in* X,

$$\left\| (T(x_i))_{i=1}^n \right\|_{\frac{p}{1-\sigma}} \leq C.\delta_{p\sigma}((x_i)_{i=1}^n). \tag{2.7}$$

In addition, $\pi_{p,\sigma}(T)$ *is the smallest number* C *for which (ii) and (iii) hold.*

The main relationship between the absolutely p-summing and the (p,σ)-absolutely continuous linear operators is the following.

Proposition 2.1.9. *[46, Proposition 4.2]*
The inclusion $\Pi_{\frac{p}{1-\sigma}} \subset \Pi_{p,\sigma}$ *holds. Consequently,* $\Pi_p \subset \Pi_{p,\sigma}$.

As in the case of absolutely p-summing linear operators, the natural way of presenting the summability properties of (p,σ)-absolutely continuous linear operators is by defining the corresponding operators between adequate sequence spaces. A linear operator $T \in \mathcal{L}(X,Y)$ induces a linear operator $\widehat{T}$ mapping $\hat{\ell}^{p\sigma}(X)$ into $Y^{\mathbb{N}}$ in the following way: if $(x_i)_{i=1}^{\infty} \in \ell^{p\sigma}(X)$ we have

$$\widehat{T}((x_i)_{i=1}^{\infty}) = (Tx_i)_{i=1}^{\infty};$$

if $\phi \in \hat{\ell}^{p\sigma}(X)$ and $\phi = \lim_{n\to+\infty} (x_i^n)_{i=1}^{\infty}$ in $\hat{\ell}^{p\sigma}(X)$ with $(x_i^n)_{i=1}^{\infty} \in \ell^{p\sigma}(X)$ for each $n \in \mathbb{N}$ we have

$$\widehat{T}(\phi) = (\lim_{n\to+\infty} T(x_i^n))_{i=1}^{\infty}.$$

In this direction, J. A. López Molina and E. A. Sánchez Pérez proved the following result.

Theorem 2.1.10. *[41, Theorem 1.7]*
For $T \in \mathcal{L}(X,Y)$ the following conditions are equivalent:

(a) T is (p,σ)-absolutely continuous.

(b) $\widehat{T}(\hat{\ell}^{p\sigma}(X)) \subset \ell_{\frac{p}{1-\sigma}}(Y)$.

2.1.3 New results for the class $\Pi_{p,\sigma}$

In the following proposition we prove a Dvoretzky-Rogers type theorem for (p,σ)-absolutely continuous linear operators.

Proposition 2.1.11. *Let $1 \le p < \infty$ and $0 \le \sigma < 1$. A Banach space X is finite dimensional if and only if the identity mapping $id_X : X \to X$ is (p,σ)-absolutely continuous.*

Proof. If X is finite dimensional it is clear that $id_X \in \Pi_{p,\sigma}(X,X)$ since it is a finite rank operator.
Conversely, assume that id_X is (p,σ)-absolutely continuous. Then by the domination theorem (see Theorem 2.1.8) there is a constant $C > 0$ and a regular Borel probability measure μ on B_{X^*} such that for all $x \in X$ we have

$$\begin{aligned} \|x\| &\le C\left(\int_{B_{X^*}} (|\phi(x)|^{1-\sigma}\,\|x\|^{\sigma})^{\frac{p}{1-\sigma}} d\mu(\phi)\right)^{\frac{1-\sigma}{p}} \\ &= C\,\|x\|^{\sigma}\left(\int_{B_{X^*}} |\phi(x)|^{p}\, d\mu(\phi)\right)^{\frac{1-\sigma}{p}}. \end{aligned}$$

This implies that

$$\|x\| \le C^{1/1-\sigma}\left(\int_{B_{X^*}} |\phi(x)|^{p}\, d\mu_j(\phi)\right)^{\frac{1}{p}}.$$

Hence id_X is p-summing and the result is obtained by the well-known version of the Dvoretzky-Rogers Theorem involving p-summing mappings. □

Remark 2.1.12. From the previous proposition, Theorem 2.1.10 and the inclusions (2.4) it follows that $\ell_{\frac{p}{1-\sigma}}(X) = \ell^{p\sigma}(X)$ if and only if the Banach space X is finite dimensional.

The family of tensor norms associated to operator ideal of (p,σ)-absolutely continuous linear operators were defined in [41]. They generalize the tensor norms α_{pq} of Lapresté (see [27, page 150]).

Definition 2.1.13. *Let X, Y be Banach spaces and let $1 \leq p, r < \infty, 0 \leq \sigma < 1$ such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. The tensor norm $g_{p,\sigma}$ in $X \otimes Y$ is defined by*

$$g_{p,\sigma}(z) = \inf \left\| (x_i)_{i=1}^n \right\|_r \delta_{p^*\sigma}\left((y_i)_{i=1}^n \right)$$

where the infimum is taken over all representations of the simple tensor z of the form $z = \sum_{i=1}^n x_i \otimes y_i$ with $x_i \in X, y_i \in Y, i = 1, \ldots, n$ and $n \in \mathbb{N}$.

The following proposition provides a representation of the ideal $\Pi_{p,\sigma}$ by the tensor norms $g_{p,\sigma}$.

Proposition 2.1.14. *Let X, Y be Banach spaces and let $1 \leq p, r < \infty, 0 \leq \sigma < 1$ such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. An operator $T \in \mathcal{L}(X, Y^*)$ defines a bounded linear functional on $X \widehat{\otimes}_{g_{p,\sigma}} Y$ if and only if T is (p^*, σ)-absolutely continuous. Furthermore, the norm of T in $\left(X \widehat{\otimes}_{g_{p,\sigma}} Y \right)^*$ coincides with $\pi_{p^*,\sigma}(T)$.*

We recall that every continuous bilinear form B on $X \times Y$ has an extension to a bounded bilinear form $B^\sharp$ on $X^{**} \times Y^{**}$ with the same norm. If $B \in \mathcal{B}(X \times Y)$ is defined by the linear operator $T : X \to Y^*$ by $B(x, y) := \langle y, Tx \rangle$, $x \in X$, $y \in Y$, we may define

$$B^\sharp(x^{**}, y^{**}) := \langle T^* y^{**}, x^{**} \rangle, \quad x^{**} \in X^{**}, \; y^{**} \in Y^{**}.$$

Theorem 2.1.15. *[68, Theorem 6.5] Let α be a tensor norm, let X, Y be Banach spaces and let $B \in \left(X \widehat{\otimes}_\alpha Y \right)^*$. Then the canonical extension of B is a bounded linear functional on $X^{**} \widehat{\otimes}_\alpha Y^{**}$ with the same norm as B.*

Proposition 2.1.16. *Let $T \in \mathcal{L}(X, Y)$. Then T is (p, σ)-absolutely continuous if and only if its second adjoint, $T^{**} \in \mathcal{L}(X^{**}, Y^{**})$, is (p, σ)-absolutely continuous. In this case*

$$\pi_{p,\sigma}(T) = \pi_{p,\sigma}(T^{**}).$$

Proof. The mapping T^{**} extends T. By the ideal property and by the injectivity of $\Pi_{p,\sigma}$, (see Theorem 2.1.6) if $T^{**} : X^{**} \to Y^{**}$ is (p, σ)-absolutely continuous so is T, with $\pi_{p,\sigma}(T) \leq \pi_{p,\sigma}(T^{**})$.
Suppose conversely that $T \in \Pi_{p,\sigma}(X, Y)$. By the ideal property of the (p, σ)-absolutely continuous linear operators, $\Pi_{p,\sigma}(X, Y)$ may be embedded in $\Pi_{p,\sigma}(X, Y^{**})$. Then by Proposition 2.1.14 we can write

$$\Pi_{p,\sigma}(X, Y) \subset \Pi_{p,\sigma}(X, Y^{**}) = \left(X \widehat{\otimes}_{g_{p^*,\sigma}} Y^* \right)^*.$$

Thus we may consider T as an element of $\left(X \widehat{\otimes}_{g_{p^*,\sigma}} Y^* \right)^*$. By Theorem 2.1.15 T^{**} is the canonical extension of T to a bounded linear functional on $X^{**} \widehat{\otimes}_{g_{p^*,\sigma}} Y^{***}$, and the result follows from [27, page 204]. $\square$

Now we give the Pietsch factorization theorem for the (p,σ)-absolutely continuous linear operators. Although this result is essentially already known (it was proved by U. Matter, see [47, Theorem B]), we write a new direct proof that highlights the role of the spaces $C(B_{X^*})$ and $L^p(\eta)$, where η is a regular Borel probability measure on B_{X^*}.
We use the following well-known propositions in the sequel; we write the proofs for the aim of completeness.

Proposition 2.1.17. *Let ν be a semi-norm on the vector space X and let*

$$S = \{x \in X : \nu(x) = 0\}.$$

Then S is a vector subspace of X and the quotient space X/S can be provided with a norm N such that $N([x]) = \nu(z)$ for all $z \in [x]$, where $[x] \in X/S$ is the equivalence class of $x \in X$.

Proof. For all $x, y \in S$ we have

$$\nu(x + \alpha y) \leq \nu(x) + |\alpha| \, \nu(y) = 0$$

and then $x+\alpha y \in S$. Thus S is a vector subspace of X. Consider now the equivalence classes. For all $z \in [x]$ we get $z - x \in S$, i.e. $\nu(z-x) = 0$. This gives $\nu(z) = \nu(x)$ because $|\nu(z) - \nu(x)| \leq \nu(z-x) = 0$. If $N([x])$ is the constant value $\nu(x)$ obtained, let us check that N is a norm on X/S.
i) if $N([x]) = 0$ then $\nu(y) = 0$ for all $y \in [x]$; therefore $[x] = [0]$.
ii) $N(\alpha\,[x]) = N([\alpha x]) = \nu(\alpha x) = |\alpha| \, \nu(x) = |\alpha| \, N([x])$.
iii) $N([x] + [y]) = N([x+y]) = \nu(x+y) \leq \nu(x) + \nu(y) = N([x]) + N([y])$. □

Let X be a Banach space, $p \geq 1$, $0 \leq \sigma < 1$ and let η be a regular Borel probability measure on B_{X^*} (with the weak star topology). We denote by i_X the isometric embedding $X \longrightarrow C(B_{X^*})$ given by $i_X(x) = \langle x, . \rangle$. For $f \in i_X(X) \subset C(B_{X^*})$, we consider

$$\|f\|_{p,\sigma} = \inf \left\{ \sum_{k=1}^{n} \|f_k\|_{i_X(X)}^{\sigma} \cdot \left(\int_{B_{X^*}} |f_k|^p \, d\eta \right)^{\frac{1-\sigma}{p}}, \ f = \sum_{k=1}^{n} f_k, \ (f_k)_{k=1}^n \subset i_X(X) \right\}.$$

Proposition 2.1.18. $\|.\|_{p,\sigma}$ *is a seminorm on $i_X(X)$.*

Proof. Let $f \in i_X(X)$ and $\alpha \in \mathbb{K}, \alpha \neq 0$. If $f = \sum_{k=1}^{n} f_k$ is a representation of f then $\alpha f = \sum_{k=1}^{n} \alpha f_k$, and so we have

$$\begin{aligned} \|\alpha f\|_{p,\sigma} &\leq \sum_{k=1}^{n} \|\alpha f_k\|_{i_X(X)}^{\sigma} \cdot \left(\int_{B_{X^*}} |\alpha f_k|^p \, d\eta \right)^{\frac{1-\sigma}{p}} \\ &= |\alpha| \sum_{k=1}^{n} \|f_k\|_{i_X(X)}^{\sigma} \cdot \left(\int_{B_{X^*}} |f_k|^p \, d\eta \right)^{\frac{1-\sigma}{p}} \end{aligned}$$

i.e.

$$\frac{1}{|\alpha|}\,\|\alpha f\|_{p,\sigma} \quad\leq\quad \sum_{k=1}^{n}\|f_k\|_{i_X(X)}^{\sigma}\cdot\left(\int_{B_{X^*}}|f_k|^p\,d\eta\right)^{\frac{1-\sigma}{p}}.$$

Since this holds for every representation of f, it follows that $\|\alpha f\|_{p,\sigma} \leq |\alpha|\,\|f\|_{p,\sigma}$. In the same way, we have

$$\|f\|_{p,\sigma} \quad=\quad \left\|\tfrac{1}{\alpha}\alpha f\right\|_{p,\sigma} \quad\leq\quad \frac{1}{|\alpha|}\,\|\alpha f\|_{p,\sigma}$$

giving $|\alpha|\,\|f\|_{p,\sigma} \leq \|\alpha f\|_{p,\sigma}$, therefore $\|\alpha f\|_{p,\sigma} = |\alpha|\,\|f\|_{p,\sigma}$. This equality is obvious when $\alpha = 0$.
Let $f, g \in i_X(X)$ and $\varepsilon > 0$ and consider the representations

$$f = \sum_{k=1}^{n'} f_k, (f_k)_{k=1}^{n'} \subset i_X(X) \quad\text{and}\quad g = \sum_{k=1}^{n''} g_k, (g_k)_{k=1}^{n''} \subset i_X(X)$$

of f and g respectively such that

$$\sum_{k=1}^{n'}\|f_k\|_{i_X(X)}^{\sigma}\cdot\left(\int_{B_{X^*}}|f_k|^p\,d\eta\right)^{\frac{1-\sigma}{p}} \leq \|f\|_{p,\sigma} + \frac{\varepsilon}{2} \tag{2.8}$$

and

$$\sum_{k=1}^{n''}\|g_k\|_{i_X(X)}^{\sigma}\cdot\left(\int_{B_{X^*}}|g_k|^p\,d\eta\right)^{\frac{1-\sigma}{p}} \leq \|g\|_{p,\sigma} + \frac{\varepsilon}{2}. \tag{2.9}$$

Let us pose $f + g = \sum_{k=1}^{n'+n''} h_k$ with $\begin{cases} h_k = f_k \text{ if } 1 \leq k \leq n' \\ h_k = g_{k-n'} \text{ if } n'+1 \leq k \leq n'+n'' \end{cases}$. Then we can write

$$\begin{aligned}
&\sum_{k=1}^{n'}\|f_k\|_{i_X(X)}^{\sigma}\cdot\left(\int_{B_{X^*}}|f_k|^p\,d\eta\right)^{\frac{1-\sigma}{p}} + \sum_{k=1}^{n''}\|g_k\|_{i_X(X)}^{\sigma}\cdot\left(\int_{B_{X^*}}|g_k|^p\,d\eta\right)^{\frac{1-\sigma}{p}}\\
=\ &\sum_{k=1}^{n'}\|f_k\|_{i_X(X)}^{\sigma}\cdot\left(\int_{B_{X^*}}|f_k|^p\,d\eta\right)^{\frac{1-\sigma}{p}} + \sum_{k=n'+1}^{n'+n''}\|g_{k-n'}\|_{i_X(X)}^{\sigma}\cdot\left(\int_{B_{X^*}}|g_{k-n'}|^p\,d\eta\right)^{\frac{1-\sigma}{p}}\\
=\ &\sum_{k=1}^{n'+n''}\|h_k\|_{i_X(X)}^{\sigma}\cdot\left(\int_{B_{X^*}}|h_k|^p\,d\eta\right)^{\frac{1-\sigma}{p}}.
\end{aligned}$$

By (2.8) and (2.9) we obtain

$$\sum_{k=1}^{n'+n''}\|h_k\|_{i_X(X)}^{\sigma}\cdot\left(\int_{B_{X^*}}|h_k|^p\,d\eta\right)^{\frac{1-\sigma}{p}} \leq \|f\|_{p,\sigma} + \|g\|_{p,\sigma} + \varepsilon.$$

This proves that $\|f+g\|_{p,\sigma} \leq \|f\|_{p,\sigma} + \|g\|_{p,\sigma}$ and completes the proof. □

Let S be the vector subspace of $i_X(X)$ given by

$$S = \|.\|_{p,\sigma}^{-1}(\{0\}) = \left\{f \in i_X(X), \|f\|_{p,\sigma} = 0\right\}.$$

We write $L_{p,\sigma}(\eta)$ for the completion of the quotient space $i_X(X)/S$ with the norm

$$\|[f]\|_{p,\sigma} = \|f\|_{p,\sigma}, f \in i_X(X).$$

Notice that the value of the norm $\|[.]\|_{p,\sigma}$ is the same for all $g \in i_X(X)$ belonging to the class of f (see Proposition 2.1.17). For the sake of clarity, notice that with this notation in the case $\sigma = 0$ the space $L_{p,0}(\eta)$ do not coincide with $L_p(\eta)$ but with the subspace of $L_p(\eta)$ that allows the factorization theorem for p-summing operators. Let us call $J_{p,\sigma} : i_X(X) \to L_{p,\sigma}(\eta)$ the projection on the quotient.

Lemma 2.1.19. *The canonical mapping $J_{p,\sigma} : i_X(X) \to L_{p,\sigma}(\eta)$ is (p,σ)-absolutely continuous, and $\pi_{p,\sigma}(J_{p,\sigma}) \leq 1$.*

Proof. Let $\delta_\omega : C(B_{X^*}) \to \mathbb{K}, \langle f, \delta_\omega \rangle = f(\omega)$ be the Dirac's delta associated with $\omega \in B_{X^*}$. Since $\|\delta_\omega\| = 1$, we may write, for every $(f_k)_{k=1}^n \subset i_X(X)$

$$\begin{aligned}
\left\|(J_{p,\sigma}(f_k))_{k=1}^n\right\|_{\frac{p}{1-\sigma}} &= \left(\sum_{k=1}^n \|f_k\|_{p,\sigma}^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&\leq \left(\sum_{k=1}^n \|f_k\|^{\frac{\sigma p}{1-\sigma}} \int_{B_{X^*}} |f_k|^p \, d\eta\right)^{\frac{1-\sigma}{p}} \\
&= \left(\int_{B_{X^*}} \sum_{k=1}^n \|f_k\|^{\frac{\sigma p}{1-\sigma}} |f_k|^p \, d\eta\right)^{\frac{1-\sigma}{p}} \\
&\leq \sup_{\omega \in B_{X^*}} \left|\sum_{k=1}^n \|f_k\|^{\frac{\sigma p}{1-\sigma}} . |f_k(\omega)|^p\right|^{\frac{1-\sigma}{p}} \left(\int_{B_{X^*}} d\eta\right)^{\frac{1-\sigma}{p}} \\
&= \sup_{\omega \in B_{X^*}} \left|\sum_{k=1}^n \left(\|f_k\|^\sigma |f_k(\omega)|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\right|^{\frac{1-\sigma}{p}} \\
&= \sup_{\omega \in B_{X^*}} \left|\sum_{k=1}^n \left(\|f_k\|^\sigma |\langle f_k, \delta_\omega\rangle|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\right|^{\frac{1-\sigma}{p}} \\
&\leq \sup_{\omega \in B_{X^*}} \sup_{\substack{\phi \in C(i_X(X))^* \\ \|\phi\| \leq 1}} \left|\sum_{k=1}^n \left(\|f_k\|^\sigma |\langle f_k, \phi\rangle|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\right|^{\frac{1-\sigma}{p}} \\
&= \sup_{\substack{\phi \in C(i_X(X))^* \\ \|\phi\| \leq 1}} \left|\sum_{k=1}^n \left(\|f_k\|^\sigma |\langle f_k, \phi\rangle|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\right|^{\frac{1-\sigma}{p}}.
\end{aligned}$$

Then $J_{p,\sigma} \in \Pi_{p,\sigma}(i_X(X), L_{p,\sigma}(\eta))$ and $\pi_{p,\sigma}(J_{p,\sigma}) \leq 1$.

□

Theorem 2.1.20. *For every linear operator $T : X \to Y$, the following statements are equivalent.*

(i) T is (p,σ)-absolutely continuous.

(ii) There exist a regular Borel probability measure μ on B_{X^*} (with the weak star topology) and a linear continuous operator $\widetilde{T} \in \mathcal{L}(L_{p,\sigma}(\mu), Y)$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & Y \\ {\scriptstyle i_X}\downarrow & & \uparrow{\scriptstyle \widetilde{T}} \\ i_X(X) & \xrightarrow{\quad J_{p,\sigma} \quad} & L_{p,\sigma}(\mu). \end{array}$$

Proof. (i)$\Longrightarrow$(ii) If T is (p,σ)-absolutely continuous, the domination theorem (Theorem 2.1.8) provides a regular Borel probability measure μ on B_{X^*} for which

$$\|Tx\| \le \pi_{p,\sigma}(T).\, \|x\|^{\sigma} . \left(\int_{B_{X^*}} |\langle x, x^* \rangle|^p \, d\mu \right)^{\frac{1-\sigma}{p}} \text{ for all } x \in X.$$

Then, we also obtain

$$\|Tx\| \le \pi_{p,\sigma}(T) \left\| \langle x, . \rangle \right\|_{p,\sigma} = \pi_{p,\sigma}(T) \left\| J_{p,\sigma} \circ i_X(x) \right\|_{p,\sigma} .$$

We denote the range of $J_{p,\sigma} \circ i_X$ by M and then $J_{p,\sigma} \circ i_X : X \to M$. By definition the closure of M is the space $L_{p,\sigma}(\mu)$. Let us consider the operator $T_1 : M \to Y$ with $T_1(J_{p,\sigma} \circ i_X(x)) = Tx$. This linear mapping is well-defined, since by

$$\|T_1(J_{p,\sigma} \circ i_X(x))\| \le \pi_{p,\sigma}(T) \left\| J_{p,\sigma} \circ i_X(x) \right\|_{p,\sigma} \text{for all } x \in X$$

we have

$$J_{p,\sigma} \circ i_X(x) = 0 \text{ implies } T_1(J_{p,\sigma} \circ i_X(x)) = 0$$

Moreover, T_1 is continuous for the $L_{p,\sigma}(\mu)$-topology with norm $\le \pi_{p,\sigma}(T)$. Hence, T_1 can be extended to the continuous linear operator

$$\widetilde{T} : \overline{M} = L_{p,\sigma}(\mu) \to Y$$

such that $\left\| \widetilde{T} \right\| = \|T_1\| \le \pi_{p,\sigma}(T)$.

(ii)$\Longrightarrow$(i) Since $J_{p,\sigma}$ is (p,σ)-absolutely continuous, by the ideal property, it follows that $\widetilde{T} \circ J_{p,\sigma} \circ i_X = T$ is (p,σ)-absolutely continuous and

$$\pi_{p,\sigma}(T) \le \left\| \widetilde{T} \right\| \cdot \pi_{p,\sigma}(J_{p,\sigma}) \cdot \|i_X\| \le \left\| \widetilde{T} \right\| .$$

□

As an application we show that (p,σ)-absolutely continuous linear operators are compact under some requirements. For this we present the next proposition.

Proposition 2.1.21. *Let $0 \le \sigma < 1$, $1 \le p < \infty$ and X be a Banach space. The inclusion/quotient mapping $i : X \to L_{p,\sigma}(\eta)$ defined by $i(x) = [\langle x, . \rangle]$ is completely continuous.*

Proof. Take a sequence (x_n) in X converging weakly to zero. Then for each $x^* \in X^*$ we have that $(\langle x_n, x^*\rangle)_n$ converges to 0. But this means that the functions sequence $(\langle x_n, \cdot\rangle)_n$ converges pointwise to 0. Consider the functions sequence $|\langle x_n, \cdot\rangle|^p \|x_n\|^{\frac{p\sigma}{1-\sigma}}$. Clearly, they converge to 0 too, and its sequence is order bounded in $L^1(\eta)$ by the η-integrable function $\sup_n \|x_n\|^{\frac{p}{1-\sigma}} \chi_{B_{X^*}}$ because for all $x^* \in B_{X^*}$ we have

$$|\langle x_n, x^*\rangle|^p \, \|x_n\|^{\frac{p\sigma}{1-\sigma}} \leq \|x_n\|^p \, \|x^*\|^p \, \|x_n\|^{\frac{p\sigma}{1-\sigma}} \leq \|x_n\|^{\frac{p}{1-\sigma}}$$

and then

$$|\langle x_n, .\rangle|^p \, \|x_n\|^{\frac{p\sigma}{1-\sigma}} \leq \sup_n \|x_n\|^{\frac{p}{1-\sigma}} \, \chi_{B_{X^*}}.$$

Notice that $\|x_n\|$ is bounded because (x_n) converges weakly to 0. The dominated convergence theorem gives that

$$\lim_n \int_{B_{X^*}} |\langle x_n, \cdot\rangle|^p \|x_n\|^{\frac{p\sigma}{1-\sigma}} d\eta = 0.$$

Therefore, since

$$\| \, [\langle x_n, \cdot\rangle] \, \|_{L_{p,\sigma}}^{\frac{p}{1-\sigma}} \leq \int_{B_{X^*}} |\langle x_n, \cdot\rangle|^p \|x_n\|^{\frac{p\sigma}{1-\sigma}} d\eta$$

we obtain that $\| \, [\langle x_n, \cdot\rangle] \, \|_{L_{p,\sigma}} \to_n 0$. The result is proved. □

Corollary 2.1.22. *Let X, Y be Banach space, X in addition reflexive, and let $0 \leq \sigma < 1$, $1 \leq p < \infty$. If $T \in \Pi_{p,\sigma}(X, Y)$, then T is compact.*

Proof. By Theorem 2.1.20 we have $T = \widetilde{T} \circ i$ with $\widetilde{T} \in \mathcal{L}(L_{p,\sigma}(\mu), Y)$ and $i \in \Pi_{p,\sigma}(X, L_{p,\sigma}(\mu))$. This last operator is completely continuous, and then $T = \widetilde{T} \circ i$ is also completely continuous —by the ideal property— and has a reflexive domain. Then T is compact. □

2.2 $(p; p_1, ..., p_m; \sigma)$-Absolutely continuous multilinear mappings

2.2.1 Properties of $(p; p_1, ..., p_m; \sigma)$-absolutely continuous m-linear mappings

In the following we extend the definition of the class of (p, σ)-absolutely continuous linear operators to the case of multilinear mappings. We will show that this new multi-ideal satisfies the inclusion theorem and a characterization by using the corresponding operator between adequate sequence spaces.
The next results, concerning the class of $(p; p_1, ..., p_m)$-summing m-linear mappings, can be found in [35] and [43], and will be used in the sequel.

Proposition 2.2.1. *Let $1 \leq p, p_1, ..., p_m < \infty$, with $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ and $T \in \mathcal{L}(X_1, ..., X_m; Y)$. Then T is $(p_1, ..., p_m)$-dominated if and only if there exist Banach* *spaces* $G_1, ..., G_m$, *an* m*-linear mapping* $S \in \mathcal{L}(G_1, ..., G_m; Y)$ *and linear operators* $u_j \in \Pi_{p_j}(X_j, G_j)$, $j = 1, ..., m$, *such that*

$$T = S \circ (u_1, ..., u_m).$$

Moreover, we have

$$\|T\|_{\mathcal{L}_{d,(p_1,...,p_m)}} = \inf \left\{ \|S\| \prod_{j=1}^{m} \pi_{p_j}(u_j) : T = S \circ (u_1, ..., u_m) \right\}.$$

Proposition 2.2.2. *Let* $1 \leq p \leq q < \infty$ *and* $1 \leq p_j \leq q_j < \infty, j = 1, ..., m$ *be such that*

$$\sum_{j=1}^{m} \frac{1}{p_j} - \frac{1}{p} \leq \sum_{j=1}^{m} \frac{1}{q_j} - \frac{1}{q}.$$

Then

$$\mathcal{L}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y) \subset \mathcal{L}_{as,(q;q_1,...,q_m)}(X_1, ..., X_m; Y).$$

Definition 2.2.3. *Let* $1 \leq p, p_1, ..., p_m < \infty$ *with* $\frac{1}{p} \leq \frac{1}{p_1} + ... + \frac{1}{p_m}$ *and* $0 \leq \sigma < 1$. *A mapping* $T \in \mathcal{L}(X_1, ..., X_m; Y)$ *is* $(p; p_1, ..., p_m; \sigma)$*-absolutely continuous if there is a constant* $C > 0$ *such that for any* $x_1^j, ..., x_n^j \in X_j$, $(1 \leq j \leq m)$ *we have*

$$\left\| \left(T\left(x_i^1, ..., x_i^m\right)\right)_{i=1}^n \right\|_{\frac{p}{1-\sigma}} \leq C \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n). \tag{2.10}$$

The space of all such mappings is denoted by $\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$.
In this case, we define

$$\|T\|_{\mathcal{L}^{\sigma}_{as(p;p_1,...,p_m)}} = \inf\{C > 0 : \ C \text{ satisfies } (2.10)\},$$

and we denote by $\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$ *the class of these mappings.*

For $\sigma = 0$, we have $\mathcal{L}^{0}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y) = \mathcal{L}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$.

As an easy consequence of the definition, we have the following

Remark 2.2.4. If $T \in \mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$. Then

$$\|T\| \leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \tag{2.11}$$

In order to see this, we will write (2.10) for $n = 1$. Thus, for all $x^j \in X_j$, $(1 \leq j \leq m)$, we obtain

$$\begin{aligned}
\|T(x^1, ..., x^m)\| &\leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \sup_{\phi_j \in B_{X_j^*}} |\phi_j(x^j)|^{1-\sigma} \|x^j\|^{\sigma} \\
&\leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \sup_{\phi_j \in B_{X_j^*}} \|\phi_j\| \|x^j\|^{1-\sigma} \|x^j\|^{\sigma} \\
&\leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \|x^j\|
\end{aligned}$$

Hence, $\|T\| \leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}}$.

The next proposition asserts that the class of $(p; p_1, ..., p_m; \sigma)$-absolutely m-linear mappings is a Banach multi-ideal.

Proposition 2.2.5. *The class* $(\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}, \|.\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}})$ *is a Banach ideal of multilinear mappings.*

Proof. (i) It is clear that $T \equiv 0 \in \mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$. By (2.11), if $\|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} = 0$ then $T \equiv 0$.
Let $S, T \in \mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$ and $(x_i^j)_{i=1}^n \subset X_j (j = 1, ..., m)$. Then

$$\begin{aligned} & \left\| \left((S+T)(x_i^1, ..., x_i^m)\right)_{i=1}^n \right\|_{\frac{p}{1-\sigma}} \\ \leq \; & \left\| \left(S(x_i^1, ..., x_i^m)\right)_{i=1}^n \right\|_{\frac{p}{1-\sigma}} + \left\| \left(T(x_i^1, ..., x_i^m)\right)_{i=1}^n \right\|_{\frac{p}{1-\sigma}} \\ \leq \; & \left(\|S\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} + \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \right) \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \end{aligned}$$

which means that $S + T$ is $(p; p_1, ..., p_m; \sigma)$-absolutely continuous and

$$\|S+T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \leq \|S\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} + \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}}$$

By a similar argument, for all $\alpha \in \mathbb{K}(\alpha \neq 0)$ we have

$$\begin{aligned} \left\| (\alpha T(x_i^1, ..., x_i^m))_{i=1}^n \right\|_{\frac{p}{1-\sigma}} &= |\alpha| \left\| (T(x_i^1, ..., x_i^m))_{i=1}^n \right\|_{\frac{p}{1-\sigma}} \\ &\leq |\alpha| \, \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \end{aligned}$$

hence αT is $(p; p_1, ..., p_m; \sigma)$-absolutely continuous and $\|\alpha T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \leq |\alpha| \, \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}}$.
For the reverse inequality, we have $T = \frac{1}{\alpha}.\alpha T$ and then

$$\left\| \frac{1}{\alpha} \alpha T \right\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \leq \left| \frac{1}{\alpha} \right| . \|\alpha T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} .$$

This means

$$\|\alpha T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \geq |\alpha| \, \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} .$$

The case $\alpha = 0$ is evident by (2.11)
Thus, we have shown that $\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$ is a vector subspace of the space $\mathcal{L}(X_1, ..., X_m; Y)$ and that $\|.\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}}$ is a norm on this space.
Now let $0 \neq x_j^* \in X^*$ and $y \in Y$. Then by the general Hölder's inequality and (2.1)

we get

$$\begin{aligned}
&\left\|(x_1^* \otimes ... \otimes x_m^* \otimes y(x_i^1, ..., x_i^m))_{i=1}^n\right\|_{\frac{p}{1-\sigma}} \\
= \ & \|y\| \left(\sum_{i=1}^n |x_1^*(x_i^1)...x_m^*(x_i^m)|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
\leq \ & \|y\| \prod_{j=1}^m \left(\sum_{i=1}^n \left|x_j^*(x_i^j)\right|^{\frac{p_j}{1-\sigma}}\right)^{\frac{1-\sigma}{p_j}} \\
= \ & \|y\| \prod_{j=1}^m \left\|x_j^*\right\| \left(\sum_{i=1}^n \left|\frac{x_j^*}{\|x_j^*\|}(x_i^j)\right|^{\frac{p_j}{1-\sigma}}\right)^{\frac{1-\sigma}{p_j}} \\
\leq \ & \|y\| \prod_{j=1}^m \left\|x_j^*\right\| \left\|(x_i^j)_{i=1}^n\right\|_{\frac{p_j}{1-\sigma},\omega} \\
\leq \ & \|y\| \prod_{j=1}^m \left\|x_j^*\right\| \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n).
\end{aligned}$$

It follows that $x_1^* \otimes ... \otimes x_m^* \otimes y \in \mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$, which gives

$$\mathcal{L}_f(X_1, ..., X_m; Y) \subset \mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y),$$

since the first space is generated by the mappings of the form

$$x_1^* \otimes ... \otimes x_m^* \otimes y : (x^1, ..., x^m) \to x_1^*(x^1) ... x_m^*(x^m)\, y.$$

In order to prove that $\left(\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y), \|.\|_{\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}}\right)$ is a Banach space, we will consider a Cauchy sequence $(T_n)_{n\in\mathbb{N}} \subset \mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$. Hence for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\|T_n - T_k\|_{\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}} < \varepsilon, \quad \text{for all } n, k \geq n_\varepsilon.$$

By Remark (2.2.4), we have

$$\|T_n - T_k\| \leq \|T_n - T_k\|_{\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}} < \varepsilon,$$

which means that $(T_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{L}(X_1, ..., X_m; Y)$. Thus, it exists $T \in \mathcal{L}(X_1, ..., X_m; Y)$ such that $\|T_n - T\| \longrightarrow 0$.
Now, let $(x_i^j)_{i=1}^N \subset X_j (j = 1, ..., m)$. Since $T_n - T_k$ is $(p; p_1, ..., p_m; \sigma)$-absolutely continuous m-linear mapping, it follows that for every $n, k \geq n_\varepsilon$, we have

$$\begin{aligned}
&\left\|(T_n - T_k)(x_i^1, ..., x_i^m))_{i=1}^N\right\|_{\frac{p}{1-\sigma}} \\
\leq \ & \|T_n - T_k\|_{\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}} \prod_{j=1}^m \delta_{p_j\sigma}\left((x_i^j)_{i=1}^N\right) \\
< \ & \varepsilon \prod_{j=1}^m \delta_{p_j\sigma}\left((x_i^j)_{i=1}^N\right).
\end{aligned}$$

Since $T_n(x^1, ..., x^m) \to T(x^1, ..., x^m)$ for all $x^j \in X_j (j = 1, ..., m)$ and after passing to the limit for $k \to +\infty$, we obtain that for every $n \geq n_\varepsilon$,

$$\left\|(T_n - T)(x_i^1, ..., x_i^m))_{i=1}^N\right\|_{\frac{p}{1-\sigma}} < \varepsilon \prod_{j=1}^m \delta_{p_j\sigma}\left((x_i^j)_{i=1}^N\right)$$

which means that $(T_n - T) \in \mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$ and hence,

$$T \in \mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y).$$

In addition,

$$\|T_n - T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} < \varepsilon, \quad \text{for all } n \geq n_\varepsilon$$

i.e. the sequence $(T_n)_{n\in\mathbb{N}}$ is convergent to $T \in \mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$ with respect to the $(p; p_1, ..., p_m; \sigma)$ -absolutely continuous norm, $\|.\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}}$.

(ii) Ideal property: let $T \in \mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$, $v \in \mathcal{L}(Y, F)$ and $u_j \in \mathcal{L}(E_j, X_j)$, $j = 1, ..., m$. For all $(e^j_i)^n_{i=1} \subset E_j (j = 1, ..., m)$ we have

$$\begin{aligned}
& \left\|((v \circ T \circ (u_1, ..., u_m)(e^1_i, ..., e^m_i))^n_{i=1}\right\|_{\frac{p}{1-\sigma}} \\
\leq\ & \|v\| \left\|(T(u_1(e^1_i), ..., u_m(e^m_i)))^n_{i=1}\right\|_{\frac{p}{1-\sigma}} \\
\leq\ & \|v\| \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \sup_{\phi_j \in B_{X^*_j}} \left(\sum_{i=1}^{n} \left(\left|\phi_j(u_j(e^j_i))\right|^{1-\sigma} \left\|u_j(e^j_i)\right\|^{\sigma} \right)^{\frac{p_j}{1-\sigma}} \right)^{\frac{1-\sigma}{p_j}} \\
\leq\ & \|v\| \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \|u_j\| \sup_{\phi_j \in B_{X^*_j}} \left(\sum_{i=1}^{n} \left(\left|\frac{\phi_j \circ u_j}{\|u_j\|}(e^j_i)\right|^{1-\sigma} \left\|e^j_i\right\|^{\sigma} \right)^{\frac{p_j}{1-\sigma}} \right)^{\frac{1-\sigma}{p_j}} \\
\leq\ & \|v\| \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \|u_j\| \prod_{j=1}^{m} \delta_{p_j\sigma}\left((e^j_i)^n_{i=1}\right),
\end{aligned}$$

which means that $v \circ T \circ (u_1, ..., u_m)$ is $(p; p_1, ..., p_m; \sigma)$-absolutely continuous and

$$\|v \circ T \circ (u_1, ..., u_m)\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \leq \|v\| \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \|u_j\|.$$

Let $(\lambda^j_i)^n_{i=1} \subset \mathbb{K}(j = 1, ..., m)$. Again, by using a general form of Hölder's inequality, (1.2) and (2.1) we obtain

$$\begin{aligned}
\left\|(T^m(\lambda^1_i, ..., \lambda^m_i))^n_{i=1}\right\|_{\frac{p}{1-\sigma}} &= \|(\lambda^1_i ... \lambda^m_i)^n_{i=1}\|_{\frac{p}{1-\sigma}} \\
&\leq \prod_{j=1}^{m} \left\|(\lambda^j_i)^n_{i=1}\right\|_{\frac{p_j}{1-\sigma}} \\
&= \prod_{j=1}^{m} \left\|(\lambda^j_i)^n_{i=1}\right\|_{\frac{p_j}{1-\sigma},\omega} \\
&\leq \prod_{j=1}^{m} \delta_{p_j\sigma}\left((\lambda^j_i)^n_{i=1}\right)
\end{aligned}$$

Hence, $T^m \in \mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}({}^m\mathbb{K}; \mathbb{K})$ and $\|T^m\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \leq 1$. For the reverse inequality, we have that

$$1 = \|T^m\| \leq \|T^m\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}},$$

and we obtain $\|T^m\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} = 1$. □

As in the classical cases, the natural way of presenting the summability properties of our m-linear operators is by defining the corresponding m-linear mapping between adequate sequence spaces. An m-linear operator $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ induces an m-linear operator $\widehat{T}$ mapping $\hat{\ell}^{p_1\sigma}(X_1) \times \ldots \times \hat{\ell}^{p_m\sigma}(X_m)$ into $Y^{\mathbb{N}}$ that is given by

$$\widehat{T}\left((x_i^1)_{i=1}^{\infty}, \ldots, (x_i^m)_{i=1}^{\infty}\right) = (T(x_i^1, \ldots, x_i^m))_{i=1}^{\infty},$$

for $(x_i^j)_{i=1}^{\infty} \in \ell^{p_j\sigma}(X_j)$ and

$$\widehat{T}\left(\phi^1, \ldots, \phi^m\right) = (\lim_{n\longrightarrow+\infty} T(x_i^{1,n}, \ldots, x_i^{m,n}))_{i=1}^{\infty},$$

for $\phi^j \in \hat{\ell}^{p_j\sigma}(X_j)$ with $\phi^j = \lim_{n\longrightarrow+\infty}(x_i^{j,n})_{i=1}^{\infty}$ in $\hat{\ell}^{p_j\sigma}(X_j)$ and $(x_i^{j,n})_{i=1}^{\infty} \in \ell^{p_j\sigma}(X_j), j = 1, ..., m$, for each $n \in \mathbb{N}$.

Proposition 2.2.6. *For $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ the following conditions are equivalent:*

(a) T is $(p; p_1, \ldots, p_m; \sigma)$-absolutely continuous.

(b) If $(x_i^j)_{i=1}^{\infty} \in \ell^{p_j\sigma}(X_j)$, for $j = 1, \ldots, m$, then $(T(x_i^1, \ldots, x_i^m))_{i=1}^{\infty} \in \ell_{\frac{p}{1-\sigma}}(Y)$.

(c) The mapping $\widehat{T} : \hat{\ell}^{p_1\sigma}(X_1) \times \ldots \times \hat{\ell}^{p_m\sigma}(X_m) \to \ell_{\frac{p}{1-\sigma}}(Y)$ is well-defined and continuous.

In this case $\|T\|_{\mathcal{L}^{\sigma}_{as(p;p_1,\ldots,p_m)}} = \|\widehat{T}\|$.

Proof. $(c) \Rightarrow (a)$ and $(c) \Rightarrow (b)$. Assume that $\widehat{T}$, already defined, is well-defined and continuous.
Let $(x_i^j)_{i=1}^{n} \subset X_j, j = 1, ..., m$. From the continuity of $\widehat{T}$ and (2.3) we obtain

$$\begin{aligned} \left\|(T(x_i^1, ..., x_i^m))_{i=1}^{n}\right\|_{\frac{p}{1-\sigma}} &= \left\|\widehat{T}\left((x_i^1)_{i=1}^{n}, ..., (x_i^m)_{i=1}^{n}\right)\right\| \\ &\leq \left\|\widehat{T}\right\| \prod_{j=1}^{m} \left\|(x_i^j)_{i=1}^{n}\right\|_{p_j,\sigma} \\ &\leq \left\|\widehat{T}\right\| \prod_{j=1}^{m} \delta_{p_j\sigma}\left(\left(x_i^j\right)_{i=1}^{n}\right). \end{aligned}$$

Hence $T \in \mathcal{L}^{\sigma}_{as,(p;p_1,\ldots,p_m)}(X_1, ..., X_m; Y)$ and $\|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,\ldots,p_m)}} \leq \left\|\widehat{T}\right\|$.
Let $(x_i^j)_{i=1}^{\infty} \in \ell^{p_j\sigma}(X_j), j = 1, ...m$. Again $\widehat{T}$ is continuous; in fact we have

$$\begin{aligned} \left\|(T(x_i^1, ..., x_i^m))_{i=1}^{\infty}\right\|_{\frac{p}{1-\sigma}} &= \left\|\widehat{T}\left((x_i^1)_{i=1}^{\infty}, ..., (x_i^m)_{i=1}^{\infty}\right)\right\| \\ &\leq \left\|\widehat{T}\right\| \prod_{j=1}^{m} \left\|(x_i^j)_{i=1}^{\infty}\right\|_{p_j,\sigma} < \infty. \end{aligned}$$

It follows that $(T(x_i^1, ..., x_i^m))_{i=1}^{\infty} \in \ell_{\frac{p}{1-\sigma}}(Y)$. Thus, (b) is proved.

$(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$. Assume that $T \in \mathcal{L}^{\sigma}_{as(p;p_1,\ldots,p_m)}(X_1,\ldots,X_m;Y)$. Note first that if $(x_i^j)_{i=1}^{\infty} \in H_{p_j,\sigma}(X_j)$, $j=1,\ldots,m$, we have

$$\begin{aligned}\left\|(T(x_i^1,...,x_i^m))_{i=1}^{\infty}\right\|_{\frac{p}{1-\sigma}} &= \sup_n \left\|(T(x_i^1,...,x_i^m))_{i=1}^{n}\right\|_{\frac{p}{1-\sigma}} \\ &\le \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,\ldots,p_m)}} \sup_n \prod_{j=1}^{m} \delta_{p_j\sigma}\left(\left(x_i^j\right)_{i=1}^{n}\right) \\ &= \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,\ldots,p_m)}} \prod_{j=1}^{m} \delta_{p_j\sigma}\left(\left(x_i^j\right)_{i=1}^{\infty}\right).\end{aligned}$$

Now let $(x_i^j)_{i=1}^{\infty} \in \ell^{p_j\sigma}(X_j)$, $j=1,\ldots,m$. For each $j=1,\ldots,m$ and $\varepsilon>0$, there exists $(x_i^{j,l_j})_{i=1}^{\infty} \in H_{p_j,\sigma}(X_j)$ such that

$$(x_i^j)_{i=1}^{\infty} = \sum_{l_j=1}^{k_j}(x_i^{j,l_j})_{i=1}^{\infty} \quad \text{and} \quad \sum_{l_j=1}^{k_j}\delta_{p_j\sigma}\left((x_i^{j,l_j})_{i=1}^{\infty}\right) \le \varepsilon + \left\|(x_i^j)_{i=1}^{\infty}\right\|_{p_j,\sigma}.$$

So we have

$$\begin{aligned}&\left\|\left(T\left(x_i^1,\ldots,x_i^m\right)\right)_{i=1}^{\infty}\right\|_{\frac{p}{1-\sigma}} \le \sum_{l_1=1}^{k_1}\cdots\sum_{l_m=1}^{k_m}\left\|\left(T\left(x_i^{1,l_1},\ldots,x_i^{m,l_m}\right)\right)_{i=1}^{\infty}\right\|_{\frac{p}{1-\sigma}} \\ &\le \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,\ldots,p_m)}} \sum_{l_1=1}^{k_1}\cdots\sum_{l_m=1}^{k_m}\prod_{j=1}^{m}\delta_{p_j\sigma}\left(\left(x_i^{j,l_j}\right)_{i=1}^{\infty}\right) \\ &= \|T\|_{\mathcal{L}^{\sigma}_{as(p;p_1,\ldots,p_m)}}\left(\sum_{l_1=1}^{k_1}\delta_{p_1\sigma}\left((x_i^{1,l_1})_{i=1}^{\infty}\right)\right)\cdots\left(\sum_{l_m=1}^{k_m}\delta_{p_m\sigma}\left((x_i^{m,l_m})_{i=1}^{\infty}\right)\right) \\ &\le \|T\|_{\mathcal{L}^{\sigma}_{as(p;p_1,\ldots,p_m)}}\left(\varepsilon+\left\|(x_i^1)_{i=1}^{\infty}\right\|_{p_1,\sigma}\right)\cdots\left(\varepsilon+\|(x_i^m)_{i=1}^{\infty}\|_{p_m,\sigma}\right).\end{aligned}$$

Since this holds for all $\varepsilon>0$, we obtain

$$\left\|\left(T\left(x_i^1,\ldots,x_i^m\right)\right)_{i=1}^{\infty}\right\|_{\frac{p}{1-\sigma}} \le \|T\|_{\mathcal{L}^{\sigma}_{as(p;p_1,\ldots,p_m)}}\prod_{j=1}^{m}\left\|(x_i^j)_{i=1}^{\infty}\right\|_{p_j,\sigma}.$$

Then it follows that $T:\ell^{p_1\sigma}(X_1)\times\ldots\times\ell^{p_m\sigma}(X_m)\to\ell_{\frac{p}{1-\sigma}}(Y)$ is well-defined and continuous with norm $\le \|T\|_{\mathcal{L}^{\sigma}_{as(p;p_1,\ldots,p_m)}}$. It can shown as in the Theorem 2.2.10 that this mapping is well-defined. Its continuous extension to $\hat{\ell}^{p_1\sigma}(X_1)\times\ldots\times\hat{\ell}^{p_m\sigma}(X_m)$ coincides with the mapping $\widehat{T}:\hat{\ell}^{p_1\sigma}(X_1)\times\ldots\times\hat{\ell}^{p_m\sigma}(X_m)\to\ell_{\frac{p}{1-\sigma}}(Y)$ already defined and we see that (a) implies (b) and (a) implies (c) with

$$\|\widehat{T}\| \le \|T\|_{\mathcal{L}^{\sigma}_{as(p;p_1,\ldots,p_m)}}.$$

$(b) \Rightarrow (c)$. First, for all $\phi^j \in \hat{\ell}^{p_j\sigma}(X_j)$ we have

$$\phi^j = \lim_{n\longrightarrow+\infty} x_i^{j,n}$$

with $(x_i^{j,n})_{i=1}^{\infty} \in \ell^{p_j\sigma}(X_j), j = 1, ..., m$, for all $n \in \mathbb{N}$. Now fix $k \in \{1, ..., m\}$. For all $\phi^j \in \hat{\ell}^{p_k\sigma}(X_j)$, $j \neq k$ we define the linear mapping

$$\widehat{T}_k : \hat{\ell}^{p_k\sigma}(X_k) \to Y^{\mathbb{N}}, \widehat{T}_k(\phi^k) = (\lim_{n \longrightarrow +\infty} T(x_i^{1,n}, \ldots, x_i^{k,n}, ...x_i^{m,n}))_{i=1}^{\infty}.$$

By the hypothesis we have $\widehat{T}_k \left(\hat{\ell}^{p_k\sigma}(X_k)\right) \subset \ell_{\frac{p}{1-\sigma}}(Y)$. Then by the proof of [41, Theorem 1.7 (2⇒1)]) we see $\widehat{T}_k$ is continuous for all $\phi^j \in \hat{\ell}^{p_j\sigma}(X_j), j \in \{1, ..., m\}, j \neq k$. It follows that $\widehat{T}$ is separately continuous between Banach spaces, hence continuous [27, page 8]. □

Proposition 2.2.7. *(Inclusion Theorem).*
Let $p \leq q, p_j \leq q_j (1 \leq j \leq m)$. *If* $\frac{1}{p_1} + ... + \frac{1}{p_m} - \frac{1}{p} \leq \frac{1}{q_1} + ... + \frac{1}{q_m} - \frac{1}{q}$, *then*

$$\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)} (X_1, ..., X_m; Y) \subset \mathcal{L}^{\sigma}_{as,(q;q_1,...,q_m)} (X_1, ..., X_m; Y) .$$

Proof. If $\frac{1}{p_1} + ... + \frac{1}{p_m} - \frac{1}{p} < \frac{1}{q_1} + ... + \frac{1}{q_m} - \frac{1}{q}$, then there exists $t > 0$ such that

$$\frac{1}{p_1} + ... + \frac{1}{p_m} - \frac{1}{p} = \frac{1}{q_1} + ... + \frac{1}{q_m} - \frac{1}{q} - \frac{1}{t}.$$

Considering $\frac{1}{s} = \frac{1}{q} + \frac{1}{t}$, we have

$$\frac{1}{p_1} + ... + \frac{1}{p_m} - \frac{1}{p} = \frac{1}{q_1} + ... + \frac{1}{q_m} - \frac{1}{s}.$$

Considering also $1 \leq r, r_j < \infty$ with $\frac{1}{r} + \frac{1}{s} = \frac{1}{p}$, $\frac{1}{r_j} + \frac{1}{q_j} = \frac{1}{p_j}$ $(1 \leq j \leq m)$ it follows that $\frac{1}{r_1} + ... + \frac{1}{r_m} = \frac{1}{r}$.
Now select a multilinear mapping T in $\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)} (X_1, ..., X_m; Y)$ and $x_1^j, ..., x_n^j \in X_j$, for $j = 1, .., m$. Then, with $\lambda_i^j = \|T(x_i^1, ..., x_i^m)\|^{\frac{s}{r_j}}$, we have

$$\begin{aligned}
\left\|T(\lambda_i^1 x^1, ..., \lambda_i^m x^m)\right\|^{\frac{p}{1-\sigma}} &= \prod_{j=1}^{m} \left\|T(x_i^1, ..., x_i^m)\right\|^{\frac{s}{r_j}\frac{p}{1-\sigma}} \left\|T(x_i^1, ..., x_i^m)\right\|^{\frac{p}{1-\sigma}} \\
&= \left\|T(x_i^1, ..., x_i^m)\right\|^{\frac{ps}{r(1-\sigma)} + \frac{p}{1-\sigma}} \\
&= \left\|T(x_i^1, ..., x_i^m)\right\|^{\frac{s}{1-\sigma}}
\end{aligned}$$

An application of Hölder's inequality reveals that

$$\begin{aligned}
& \left(\sum_{i=1}^{n}\left\|T\left(x_i^1,...,x_i^m\right)\right\|^{\frac{s}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
= & \left(\sum_{i=1}^{n}\left\|T(\lambda_i^1 x_i^1,...,\lambda_i^m x_i^m)\right\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
\leq & \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \delta_{p_j\sigma}\left(\left(\lambda_i^j x_i^j\right)_{i=1}^{n}\right) \\
= & \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \sup_{\phi_j\in B_{X_j^*}} \left(\sum_{i=1}^{n}\left(\lambda_i^j \left|\phi_j(x_i^j)\right|^{1-\sigma}\left\|x_i^j\right\|^{\sigma}\right)^{\frac{p_j}{1-\sigma}}\right)^{\frac{1-\sigma}{p_j}} \\
\leq & \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \sup_{\phi_j\in B_{X_j^*}} \left(\sum_{i=1}^{n}\left|\lambda_i^j\right|^{\frac{r_j}{1-\sigma}}\right)^{\frac{1-\sigma}{r_j}} \left(\sum_{i=1}^{n}\left(\left|\phi_j(x_i^j)\right|^{1-\sigma}\left\|(x_i^j)\right\|^{\sigma}\right)^{\frac{q_j}{1-\sigma}}\right)^{\frac{1-\sigma}{q_j}} \\
= & \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \left(\sum_{i=1}^{n}\left\|T(x_i^1,...,x_i^m)\right\|^{\frac{s}{1-\sigma}}\right)^{\frac{1-\sigma}{r_j}} \delta_{q_j\sigma}\left((x_i^j)_{i=1}^{n}\right) \\
= & \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \left(\sum_{i=1}^{n}\left\|T\left(x_i^1,...,x_i^m\right)\right\|^{\frac{s}{1-\sigma}}\right)^{\frac{1-\sigma}{r}} \prod_{j=1}^{m} \delta_{q_j\sigma}\left(\left(x_i^j\right)_{i=1}^{n}\right).
\end{aligned}$$

Since $\frac{1-\sigma}{p} - \frac{1-\sigma}{r} = \frac{1-\sigma}{s}$, we end up with

$$\left\|\left(T\left(x_i^1,...,x_i^m\right)\right)_{i=1}^{n}\right\|_{\frac{s}{1-\sigma}} \leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \delta_{q_j\sigma}\left(\left(x_i^j\right)_{i=1}^{n}\right).$$

It is clear that $\frac{1}{s} = \frac{1}{q} + \frac{1}{t}$ implies $q > s$, and further

$$\left\|\left(T\left(x_i^1,...,x_i^m\right)\right)_{i=1}^{n}\right\|_{\frac{q}{1-\sigma}} \leq \left\|\left(T\left(x_i^1,...,x_i^m\right)\right)_{i=1}^{n}\right\|_{\frac{s}{1-\sigma}} \leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \delta_{q_j\sigma}\left(\left(x_i^j\right)_{i=1}^{n}\right).$$

Hence $T \in \mathcal{L}^{\sigma}_{as,(q;q_1,...,q_m)}\left(X_1,...,X_m;Y\right)$ and $\|T\|_{\mathcal{L}^{\sigma}_{as,(q;q_1,...,q_m)}} \leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}}$.

□

A relevant special case of $(p;p_1,...,p_m;\sigma)$-absolutely continuous m-linear mappings is when we have $\frac{1}{p} = \frac{1}{p_1}+...+\frac{1}{p_m}$. In this situation —and following the standard notations in similar cases—, we will call the mappings dominated $(p_1,...,p_m;\sigma)$-continuous and we will denote the corresponding Banach space by

$$(\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}(X_1,...,X_m;Y), \|.\|_{\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}}).$$

We can establish the following comparison between the classes of dominated $(p_1,...,p_m;\sigma)$-continuous and $(p_1,...,p_m)$-dominated m-linear mappings.

Proposition 2.2.8. *Let $1 \leq p_j, p < \infty, j = 1,...,m$ such that $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ and $0 \leq \sigma < 1$. Then*

$$\mathcal{L}_{d,(\frac{p_1}{1-\sigma},...,\frac{p_m}{1-\sigma})}\left(X_1,...,X_m;Y\right) \subset \mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}\left(X_1,...,X_m;Y\right).$$

Consequently,

$$\mathcal{L}_{d,(p_1,\dots,p_m)}\left(X_1,\dots,X_m;Y\right) \subset \mathcal{L}^{\sigma}_{d,(p_1,\dots,p_m)}\left(X_1,\dots,X_m;Y\right).$$

Proof. Let $T \in \mathcal{L}_{d,(\frac{p_1}{1-\sigma},\dots,\frac{p_m}{1-\sigma})}(X_1,\dots X_m;Y)$ and let $(x_i^j)_{i=1}^n \subset X_j (1 \leq j \leq m)$. By using the inequality (2.1), we have

$$\begin{aligned} \left\|(T(x_i^1,\dots,x_i^m))_{i=1}^n\right\|_{\frac{p}{1-\sigma}} &\leq \|T\|_{\mathcal{L}_{d,(p_1,\dots,p_m)}} \prod_{j=1}^m \left\|(x_i^j)_{i=1}^n\right\|_{\frac{p_j}{1-\sigma},\omega} \\ &\leq \|T\|_{\mathcal{L}_{d,(p_1,\dots,p_m)}} \prod_{j=1}^m \delta_{p_j\sigma}\left((x_i^j)_{i=1}^n\right). \end{aligned}$$

Then $T \in \mathcal{L}^{\sigma}_{d,(p_1,\dots,p_m)}(X_1,\dots X_m;Y)$, in addition we have

$$\|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,\dots,p_m)}} \leq \|T\|_{\mathcal{L}_{d,(p_1,\dots,p_m)}}.$$

Now we have $p \leq \frac{p}{1-\sigma}$ and $p_j \leq \frac{p_j}{1-\sigma}, (j = 1,\dots,m)$ then by Proposition 2.2.2 we obtain

$$\mathcal{L}_{d,(\frac{p_1}{1-\sigma},\dots,\frac{p_m}{1-\sigma})}(X_1,\dots X_m;Y) \subset \mathcal{L}_{d,(p_1,\dots,p_m)}(X_1,\dots X_m;Y).$$

Consequently,

$$\mathcal{L}_{d,(p_1,\dots,p_m)}(X_1,\dots X_m;Y) \subset \mathcal{L}^{\sigma}_{d,(p_1,\dots,p_m)}(X_1,\dots X_m;Y).$$

□

2.2.2 Domination and factorization theorems

In the case of (p,σ)-absolutely continuous linear mappings it is possible to obtain a Domination Theorem as the one that holds for p-summing linear operators (see Theorem 2.1.8). It can be also extended to the multilinear case. For the proof of this domination theorem we use the full general Pietsch domination theorem recently presented by D. Pellegrino et al. in [56].

Theorem 2.2.9. *Let* $1 \leq p, p_1,\dots,p_m < \infty$ *with* $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ *and* $0 \leq \sigma < 1$. *An* m*-linear mapping* $T \in \mathcal{L}\left(X_1,\dots,X_m;Y\right)$ *is dominated* $(p_1,\dots,p_m;\sigma)$*-continuous if and only if there is a constant* $C > 0$ *and regular Borel probability measures* μ_j *on* $B_{X_j^*}, 1 \leq j \leq m$, *(with the weak star topology) so that for all* $(b^1,\dots,b^m) \in X_1 \times \dots . \times X_m$ *the inequality*

$$\left\|T(b^1,\dots,b^m)\right\| \leq C \prod_{j=1}^m \left(\int_{B_{X_j^*}} \left(\left|\phi(b^j)\right|^{1-\sigma}\left\|b^j\right\|^{\sigma}\right)^{\frac{p_j}{1-\sigma}} d\mu_j(\phi)\right)^{\frac{1-\sigma}{p_j}} \tag{2.12}$$

is valid.
The infimum of all these possible C *is equal to* $\|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,\dots,p_m)}}$.

Proof. Note that by choosing the parameters

$$\begin{cases} t = m \\ E_j = \mathbb{K}, \ j = 1, ..., k \\ G_j = X_j, \ j = 1, ..., m \\ K_j = B_{X_j^*}, \ j = 1, ..., m \\ \mathcal{H} = \mathcal{L}\left(X_1, ..., X_m; Y\right) \\ q = \frac{p}{1-\sigma}, q_j = \frac{p_j}{1-\sigma}, \ j = 1, ..., m \\ S(T, x^1, ..., x^k, b^1, ..., b^m) = \left\| T\left(b^1, ..., b^m\right) \right\| \\ R_j(\phi, x^1, ..., x^k, b^j) = \left|\phi(b^j)\right|^{1-\sigma} \left\| b^j \right\|^{\sigma}, \ j = 1, ..., m \end{cases}$$

It is clear that the mapping

$$(R_j)_{x^1,...,x^m,b} : B_{X_j^*} \to [0, +\infty), (R_j)_{x^1,...,x^m,b}(\phi) = |\phi(b)|^{1-\sigma} \|b\|^{\sigma},$$

is continuous for all $b \in X_j (j = 1, ..., m)$. By the homogeneity of the norm, linearity of $\phi \in B_{X_j^*}$ and the multi-linearity of the mapping T we obtain

$$\begin{aligned} R_j(\phi, x^1, ..., x^m, \eta_j b^j) &= |\phi(\eta_j b)|^{1-\sigma} \|\eta_j b^j\|^{\sigma} \\ &= \eta_j \left|\phi(b^j)\right|^{1-\sigma} \|b^j\|^{\sigma} \\ &= \eta_j R_j(\phi, x^1, ..., x^m, b^j) \end{aligned}$$

and

$$\begin{aligned} S(T, x^1, ..., x^m, \alpha_1 b^1, ..., \alpha_m b^m) &= \left\|T(\alpha_1 b^1, ..., \alpha_m b^m)\right\| \\ &= \alpha_1 ... \alpha_m \left\|T(b^1, ..., b^m)\right\| \\ &= \alpha_1 ... \alpha_m S(T, x^1, ..., x^m, b^1, ..., b^m) \end{aligned}$$

for all $0 \leq \eta_j, \alpha_j \leq 1, b^j \in X_j$ and $\phi \in B_{X_j^*} (j = 1, ..., m)$.
Now we can write

$$\left(\sum_{i=1}^{n} \left(S(T, x^1, ..., x^m, \alpha_1 b^1, ..., \alpha_m b^m)\right)^q\right)^{\frac{1}{q}} = \left(\sum_{i=1}^{n} \|T(b^1, ..., b^m)\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}$$

and

$$\prod_{j=1}^{m} \sup_{\phi \in K_j} \left(\sum_{i=1}^{n} R_j(\phi, x_i^1, ..., x_i^m, b_i^j)^{q_j}\right)^{\frac{1}{q_j}} = \prod_{j=1}^{m} \sup_{\phi \in B_{X_j^*}} \left(\sum_{i=1}^{n} \left(\left|\phi(b_i^j)\right|^{1-\sigma} \left\|b_i^j\right\|^{\sigma}\right)^{\frac{p_j}{1-\sigma}}\right)^{\frac{p_j}{1-\sigma}}.$$

Thus, $T : X_1 \times ... \times X_m \longrightarrow Y$ is dominated $(p_1, ..., p_m; \sigma)$-continuous if and only if T is $R_1, ..., R_m$-S-abstract $(\frac{p_1}{1-\sigma}, ..., \frac{p_m}{1-\sigma})$-summing. The generalized Pietsch domination theorem, [56, Theorem 4.6], tells us that T is $R_1, ..., R_m$-S-abstract $(\frac{p_1}{1-\sigma}, ..., \frac{p_m}{1-\sigma})$-summing (see [56, Definition 4.4]) if and only if there is a $C > 0$ and there are regular Borel probability measures μ_j on $B_{X_j^*}, j = 1, ..., m$, such that

$$S(T, x^1, ..., x^k, b^1, ..., b^t) \leq C \prod_{j=1}^{t} \left(\int_{B_{X_j^*}} R_j(\phi, x^1, ..., x^k, b^j)^{\frac{p_j}{1-\sigma}} d\mu_j(\phi)\right)^{\frac{1-\sigma}{p_j}}$$

i.e;

$$\left\|T(b^1,...,b^m)\right\| \leq C\prod_{j=1}^{m}\left(\int_{B_{X_j^*}}(\left|\phi(b^j)\right|^{1-\sigma}\left\|b^j\right\|^{\sigma})^{\frac{p_j}{1-\sigma}}d\mu_j(\phi)\right)^{\frac{1-\sigma}{p_j}}$$

and we obtain the inequality in the statement of the theorem.

□

It is likely that the class of dominated $(p_1,...,p_m;\sigma)$-continuous m-linear mappings has a nice factorization theorem, like its linear version.

Theorem 2.2.10. *(Multilinear Version)*
Let $1 \leq p, p_1, ..., p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ and $0 \leq \sigma < 1$. Then

$$T \in \mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}(X_1,...,X_m;Y)$$

if and only if there exist Banach spaces $G_1,...,G_m$, (p_j,σ)-absolutely continuous linear operators $u_j \in \mathcal{L}(X_j,G_j)$ and an m-linear mapping $S \in \mathcal{L}(G_1,...,G_m;Y)$ so that

$$T = S \circ (u_1,...,u_m).$$

Moreover,

$$\|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}} = \inf\left\{\|S\|\prod_{j=1}^{m}\pi_{p_j,\sigma}(u_j) : T = So(u_1,...,u_m)\right\}.$$

In other words, we say that the class $\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}$ is the Banach multi-ideal generated by the factorization method from the Banach operator ideals $\Pi_{p_1,\sigma},...,\Pi_{p_m,\sigma}$ (see Theorem 1.4.10) i.e.,

$$\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)} = \mathcal{L}(\Pi_{p_1,\sigma},...,\Pi_{p_m,\sigma}).$$

Proof. For the "if" part, if T has such a factorization, we have

$$\left\|T(x^1,...,x^m)\right\| = \left\|S\left(u_1\left(x^1\right),...,u_m\left(x^m\right)\right)\right\| \leq \|S\|\prod_{j=1}^{m}\left\|u_j\left(x^j\right)\right\|,$$

for all $(x^1,...,x^m) \in X_1 \times .... \times X_m$.
We know that (see Theorem 2.1.8), for each $j = 1,...,m$, there is a regular Borel probability measure μ_j on $B_{X_j^*}$, such that

$$\left\|u_j\left(x^j\right)\right\| \leq \pi_{p_j,\sigma}(u_j)\,(\textstyle\int_{B_{X_j^*}}\left(\left|\langle x^j,\phi\rangle\right|^{1-\sigma}\left\|x^j\right\|^{\sigma}\right)^{\frac{p_j}{1-\sigma}}d\mu_j)^{\frac{1-\sigma}{p_j}}, \quad x^j \in X_j.$$

Now we have

$$\left\|T(x^1,...,x^m)\right\| \leq \|S\| \prod_{j=1}^{m} \pi_{p_j,\sigma}(u_j) \prod_{j=1}^{m} (\int_{B_{X_j^*}} \left(\left|\langle x^j, \phi \rangle\right|^{1-\sigma} \left\|x^j\right\|^{\sigma} \right)^{\frac{p_j}{1-\sigma}} d\mu_j)^{\frac{1-\sigma}{p_j}}.$$

Therefore, by Theorem 2.2.9, T is dominated $(p_1,...,p_m;\sigma)$-continuous and

$$\|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}} \leq \|S\| \prod_{j=1}^{m} \pi_{p_j,\sigma}(u_j).$$

To prove the "only if" part, take $T \in \mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}(X_1,....,X_m;Y)$. Then, by Theorem 2.2.9, there exist regular Borel probability measures μ_j on $B_{X_j^*}$ such that for all $(x^1,...,x^m) \in X_1 \times .... \times X_m$ we have

$$\left\|T(x^1,...,x^m)\right\| \leq \|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}} \prod_{j=1}^{m} (\int_{B_{X_j^*}} \left(\left|\langle x^j, \phi \rangle\right|^{1-\sigma} \left\|x^j\right\|^{\sigma} \right)^{\frac{p_j}{1-\sigma}} d\mu_j(\phi))^{\frac{1-\sigma}{p_j}}.$$

We now consider the linear operator $u_j : X_j \to L_{p_j,\sigma}(\mu_j)$, $u_j(x^j) = [\langle x^j, . \rangle]$. (It is the inclusion/quotient mapping described in Proposition 2.1.21.)
Since $\|\langle x^j, . \rangle\| = \sup_{\phi \in B_{X_j^*}} |\langle x^j, \phi \rangle| = \|x^j\|$ for all $x^j \in X_j$ and $1 \leq j \leq m$, we have

$$\begin{aligned} \left\|u_j(x^j)\right\| &= \inf \left\{ \sum_{k=1}^{n} \|\langle x_k^j, . \rangle\|^{\sigma} \cdot \left(\int_{B_{X_j^*}} \left|\langle x_k^j, \phi \rangle\right|^{p_j} d\mu_j \right)^{\frac{1-\sigma}{p_j}}, \langle x^j, . \rangle = \sum_{k=1}^{n} \langle x_k^j, . \rangle \right\} \\ &\leq \|\langle x^j, . \rangle\|^{\sigma} \cdot \left(\int_{B_{X_j^*}} |\langle x^j, \phi \rangle|^{p_j} d\mu_j \right)^{\frac{1-\sigma}{p_j}} \\ &= \|x^j\|^{\sigma} \cdot \left(\int_{B_{X_j^*}} |\langle x^j, \phi \rangle|^{p_j} d\mu_j \right)^{\frac{1-\sigma}{p_j}}. \end{aligned}$$

Then u_j is (p_j,σ)-absolutely continuous with $\pi_{p_j,\sigma}(u_j) \leq 1$ (see Theorem 2.1.8).
Let S_0 be the m-linear operator defined on $u_1(X_1) \times ... \times u_m(X_m)$ by

$$S_0(u_1(x^1),...,u_m(x^m)) := T(x^1,...,x^m).$$

We prove that the mapping S_0 is well defined and continuous, so we have

$$\left\|S_0\left(u_1\left(x^1\right),...,u_m\left(x^m\right)\right)\right\| \leq \|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}} \prod_{j=1}^{m} (\int_{B_{X_j^*}} \left(\left|\langle x^j, \phi \rangle\right|^{1-\sigma} \left\|x^j\right\|^{\sigma} \right)^{\frac{p_j}{1-\sigma}} d\mu_j)^{\frac{1-\sigma}{p_j}}.$$

Fix $j = 1$ and $\varepsilon > 0$. Then there exists $(x_k^1)_{k=1}^n \subset X_1$ such that $x^1 = \sum_{k=1}^{n} x_k^1$ and

$$\sum_{k=1}^{n} \left\|x_k^1\right\|^{\sigma} \cdot \left(\int_{B_{X_1^*}} \left|\langle x_k^1, \phi \rangle\right|^{p_1} d\mu_1 \right)^{\frac{1-\sigma}{p_1}} \leq \varepsilon + \left\|\left[\langle x^1, . \rangle\right]\right\|_{p_1,\sigma}.$$

We have

$$\begin{aligned}
& \left\| S_0\left(u_1\left(x^1\right), ..., u_m\left(x^m\right)\right)\right\| \\
= & \left\| S_0\left(u_1\left(\sum_{k=1}^{n} x_k^1\right), ..., u_m\left(x^m\right)\right)\right\| \\
\leq & \sum_{k=1}^{n} \left\| S_0\left(u_1\left(x_k^1\right), ..., u_m\left(x^m\right)\right)\right\| \\
\leq & \|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,\ldots,p_m)}} \sum_{k=1}^{n} \|x_k^1\|^{\sigma} \cdot \left(\int_{B_{X_1^*}} |\langle x_k^1, \phi\rangle|^{p_1} \, d\mu_1\right)^{\frac{1-\sigma}{p_1}} \\
& \times \prod_{j=2}^{m} \Big(\int_{B_{X_j^*}} \left(|\langle x^j, \phi\rangle|^{1-\sigma} \|x^j\|^{\sigma}\right)^{\frac{p_j}{1-\sigma}} d\mu_j\Big)^{\frac{1-\sigma}{p_j}} \\
\leq & \|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,\ldots,p_m)}} \left(\varepsilon + \|[\langle x^1, .\rangle]\|_{p_1,\sigma}\right) \prod_{j=2}^{m} \Big(\int_{B_{X_j^*}} \left(|\langle x^j, \phi\rangle|^{1-\sigma} \|x^j\|^{\sigma}\right)^{\frac{p_j}{1-\sigma}} d\mu_j\Big)^{\frac{1-\sigma}{p_j}}.
\end{aligned}$$

We can write the same domination result for $j = 2$, with this new domination, to obtain

$$\begin{aligned}
& \left\| S_0\left(u_1\left(x^1\right), ..., u_m\left(x^m\right)\right)\right\| \\
\leq & \|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,\ldots,p_m)}} \left(\varepsilon + \|[\langle x^1, .\rangle]\|_{p_1,\sigma}\right) \left(\varepsilon + \|[\langle x^2, .\rangle]\|_{p_2,\sigma}\right) \\
& \times \prod_{j=3}^{m} \Big(\int_{B_{X_j^*}} \left(|\langle x^j, \phi\rangle|^{1-\sigma} \|x^j\|^{\sigma}\right)^{\frac{p_j}{1-\sigma}} d\mu_j\Big)^{\frac{1-\sigma}{p_j}}.
\end{aligned}$$

By induction, we get

$$\left\| S_0\left(u_1\left(x^1\right), ..., u_m\left(x^m\right)\right)\right\| \leq \|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,\ldots,p_m)}} \prod_{j=1}^{m} \left(\varepsilon + \left\| \left[\langle x^j, .\rangle\right] \right\|_{p_j,\sigma}\right).$$

Since this is true for all $\varepsilon > 0$, we obtain

$$\left\| S_0\left(u_1\left(x^1\right), ..., u_m\left(x^m\right)\right)\right\| \leq \|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,\ldots,p_m)}} \prod_{j=1}^{m} \left\| \left[\langle x^j, .\rangle\right] \right\|_{p_j,\sigma}. \tag{2.13}$$

For simplicity we show that S_0 is well defined for the case of bilinear mappings $S_0 : u_1(X_1) \times u_2(X_2) \to Y$, but our reasonings extend without further complications to more spaces.
Assume that $(x^1, x^2), (x'^1, x'^2) \in X_1 \times X_2$ satisfy that

$$(u_1(x^1), u_2(x^2)) = (u_1(x'^1), u_2(x'^2)).$$

It follow that $\|u_j(x^j - x'^j)\|_{p_j,\sigma} = \|u_j(x^j) - u_j(x'^j)\|_{p_j,\sigma} = 0$ with $j = 1, 2$.
Let us show that $T(x^1, x^2) = T(x'^1, x'^2)$ (and then the operator S_0 is well-defined).

By the inequality (2.13) we have

$$\begin{aligned} & \|T(x^1,x^2)-T(x'^1,x'^2)\| \\ = & \|T(x^1,x^2)-T(x'^1,x^2)+T(x'^1,x^2)-T(x'^1,x'^2)\| \\ \leq & \|T(x^1-x'^1,x^2)\|+\|T(x'^1,x^2-x'^2)\| \\ \leq & \|T\|_{\mathcal{L}^\sigma_{d,(p_1,p_2)}} \|u_1(x^1-x'^1)\|_{p_1,\sigma} \|u_2(x^2)\|_{p_2,\sigma} \\ & +\|T\|_{\mathcal{L}^\sigma_{d,(p_1,p_2)}} \|u_1(x'^1)\|_{p_1,\sigma} \|u_2(x^2-x'^2)\|_{p_2,\sigma} \\ = & 0. \end{aligned}$$

Again the inequality (2.13) gives the continuity of S_0 on $u_1(X_1)\times ... \times u_m(X_m)$ and has a unique extension S to $\overline{u_1(X_1)}\times ... \times \overline{u_m(X_m)} = G_1\times ... \times G_m$ with

$$G_j := \overline{u_j(X_j)} = L_{p_j,\sigma}(\mu_j), \quad j=1,...,m.$$

Moreover we have

$$\|S\| = \|S_0\| \leq \|T\|_{\mathcal{L}^\sigma_{d,(p_1,...,p_m)}}.$$

Finally, note that $T = S\circ(u_1,...,u_m)$ where $u_j\in \Pi_{p_j,\sigma}(X_j,G_j)$, $(1\leq j\leq m)$, $S\in\mathcal{L}(G_1,...,G_m;Y)$ and

$$\|S\|\prod_{j=1}^m \pi_{p_j,\sigma}(u_j) \leq \|T\|_{\mathcal{L}^\sigma_{d,(p_1,...,p_m)}}.$$

This completes the proof. □

Compactness of multilinear mappings is in general a property that is not easy to characterize, and it is nowadays not very well known. In what follows we prove that under certain summability conditions we can assure that the multilinear mapping is compact. We relax the requirements that are necessary for the case of p-summing multilinear mappings by using Corollary 2.1.22 and the factorization theorem for the class of dominated $(p_1,...,p_m;\sigma)$-continuous multilinear mappings that we have proved (the previous theorem).

Corollary 2.2.11. *Let Y a Banach space, $0\leq\sigma<1$ and $1\leq p,p_1,...,p_m<\infty$ with $\frac{1}{p}=\frac{1}{p_1}+...+\frac{1}{p_m}$ and let $X_1,...,X_m$ be reflexive Banach spaces. If $T\in\mathcal{L}^\sigma_{d,(p_1,...,p_m)}(X_1,...,X_m;Y)$, then T is compact.*

Proof. By the previous theorem there exist Banach spaces G_j, mappings $u_j\in\Pi_{p_j,\sigma}(X_j,G_j)$, $j=1,...,m$ and $S\in\mathcal{L}(G_1,...,G_1;Y)$ such that

$$T = S\circ(u_1,...,u_m).$$

Corollary 2.1.22 asserts that $u_j(j=1,...,m)$ is a compact linear operator. On the other hand,

$$T(B_{X_1}\times ... \times B_{X_m}) = S(u_1(B_{X_1})\times ... \times u_m(B_{X_m})).$$

By the compactness of $u_j(j=1,...,m)$ and the continuity of S the set $T(B_{X_1}\times ... \times B_{X_m})$ is a relatively compact subset of Y. □

D. Ruch in [65, Proposition 8] proved that the bilinear mapping $T \in \mathcal{L}(X_1, X_2; c_0)$ has the form

$$T(x^1, x^2) = \left(b_n(x^1, x^2)\right)_n$$

with $\|b_n\| \longrightarrow 0$, $b_n \in \mathcal{L}(X_1, X_2; \mathbb{K})$, if and only if T is compact. As a consequence of this result we have the following

Corollary 2.2.12. *Let X_1, X_2 be reflexive Banach spaces and $T \in \mathcal{L}^{\sigma}_{d,(p_1,p_2)}(X_1, X_2; c_0)$. Then T can be written as $T(x^1, x^2) = (b_n(x^1, x^2))_n$ for a norm null sequence $(b_n)_n$ of continuous bilinear forms.*

Regarding Proposition 2.2.8 and Theorem 2.2.10, let us show with some examples the difference between absolutely summing and absolutely continuous multilinear mappings.

Example 2.2.13. *Let $m \in \mathbb{N}, m \geq 2, p > 1$ and $0 < \sigma < 1$ such that $p^* < \frac{p}{1-\sigma}$ and $p > m$. Consider the m-linear mapping*

$$S : \ell_{\frac{p}{1-\sigma}} \times ... \times \ell_{\frac{p}{1-\sigma}} \to \ell_{\frac{p}{1-\sigma}}, \quad S\left(x^1, ..., x^m\right) = \left(x_i^1 x_i^2 ... x_i^m\right)_{i=1}^{\infty},$$

where $x^j = (x_i^j)_{i=1}^{\infty} \in \ell_{\frac{p}{1-\sigma}}$, $j = 1, ..., m$. Consider also $u \in \mathcal{L}(\ell_{p^*}, \ell_{\frac{p}{1-\sigma}})$ defined by $u(e_i) = (\frac{1}{i})^{\frac{1}{p}} e_i$, where $(e_i)_{i=1}^{\infty}$ is the unit vector basis of ℓ_{p^*}. The m-linear mapping $T \in \mathcal{L}\left(\ell_{p^*}, ..., \ell_{p^*}; \ell_{\frac{p}{1-\sigma}}\right)$ given by $T = S \circ (u, ..., u)$ is dominated $(p, ..., p; \sigma)$-continuous but it is not $(p, ..., p)$-dominated.

In order to see this, note that by [41, Example 1.9] we have $u \in \Pi_{p,\sigma}(\ell_{p^*}, \ell_{\frac{p}{1-\sigma}})$. It is easy to see that S is well-defined. Since $\frac{1-\sigma}{p} = \frac{1-\sigma}{mp} + ... + \frac{1-\sigma}{mp}$, by Hölder's inequality we have

$$\begin{aligned}
\|S(x^1, ..., x^m)\| &= \left\|(x_i^1 ... x_i^m)_{i=1}^{\infty}\right\|_{\frac{p}{1-\sigma}} \\
&\leq \left\|(x_i^1)_{i=1}^{\infty}\right\|_{\frac{mp}{1-\sigma}} ... \left\|(x_i^m)_{i=1}^{\infty}\right\|_{\frac{mp}{1-\sigma}} \\
&\leq \left\|(x_i^1)_{i=1}^{\infty}\right\|_{\frac{p}{1-\sigma}} ... \left\|(x_i^m)_{i=1}^{\infty}\right\|_{\frac{p}{1-\sigma}} \quad (\text{because } \tfrac{mp}{1-\sigma} \geq \tfrac{p}{1-\sigma}) \\
&= \|x^1\| ... \|x^m\|.
\end{aligned}$$

It follows that S is continuous and $\|S\| \leq 1$. Then by Theorem 2.2.10 we have

$$T \in \mathcal{L}^{\sigma}_{d,(p,...,p)}\left(\ell_{p^*}, ..., \ell_{p^*}; \ell_{\frac{p}{1-\sigma}}\right).$$

On the other hand,

$$\|(e_i)_{i=1}^{\infty}\|_{p,\omega} = \sup_{\phi \in (\ell_{p^*})^*, \|\phi\| \leq 1} \left(\sum_{i=1}^{\infty} |\phi(e_i)|^p\right)^{\frac{1}{p}} = \sup_{(\phi_i)_{i=1}^{\infty} \in \ell_p, \left\|(\phi_i)_{i=1}^{\infty}\right\|_p \leq 1} \left(\sum_{i=1}^{\infty} |\phi_i|^p\right)^{\frac{1}{p}} = 1.$$

But

$$\|(T(e_i, ..., e_i))_{i=1}^{\infty}\|_{\frac{p}{m}} = \left\|\left(\frac{1}{i^{\frac{m}{p}}}\right)_{i=1}^{\infty}\right\|_{\frac{p}{m}} = \left(\sum_{i=1}^{\infty} \frac{1}{i}\right)^{\frac{m}{p}} = \infty,$$

i.e. $(e_i)_{i=1}^{\infty} \in \ell_p^{\omega}(\ell_{p^*})$ but $(T(e_i, ..., e_i))_{i=1}^{\infty} \notin \ell_{\frac{p}{m}}(\ell_{\frac{p}{1-\sigma}})$.
By Definition 1.4.16, this implies that,

$$T \notin \mathcal{L}_{d,(p,...,p)}\left(\ell_{p^*}, ..., \ell_{p^*}; \ell_{\frac{p}{1-\sigma}}\right).$$

Example 2.2.14. A bit more elaborated example of this kind can be given in the setting of the Hilbert spaces. Let L^2 be the Hilbert space $L^2[0,1]$, and L^1 the corresponding L^1-space. For the case, $p = 2$ and $0 \leq \sigma < 1$ it is known that the ideal of $(2,\sigma)$-absolutely continuous operators between a couple of Hilbert spaces coincides with the one of $\frac{2}{1-\sigma}$-approximable operators (see 15.5 in [61] for the definition, and Proposition 5.1 in [46] for the result). It is also known that for all $1 \leq r < \infty$, r-summing operators and 2-approximable operators (Hilbert-Schmidt operators) coincide on Hilbert spaces (see Theorem 22.1.8 in [61]). This allows to construct an easy example of a bilinear mapping $T : L^2 \times L^2 \to L^1$ which is dominated $(2,2;\sigma)$-continuous but is not $(\frac{2}{1-\sigma}, \frac{2}{1-\sigma})$-dominated. Let $0 < \sigma < 1$, and define such a T as $T(f^1, f^2) := u(f^1) \cdot u(f^2)$, where $u : L^2 \to L^2$ is a $(2,\sigma)$-absolutely continuous that is not a $\frac{2}{1-\sigma}$-summing operator and "$\cdot$" is the pointwise product of elements in L^2. Take n couples $(f_1^1, f_1^2)...(f_n^1, f_n^2)$ of elements of L^2. Then, using Hölder's inequality and taking into account that u is $(2,\sigma)$-absolutely continuous, we obtain

$$\begin{aligned}
&\left(\sum_{i=1}^{n} \|T(f_i^1, f_i^2)\|_{L^1}^{\frac{1}{1-\sigma}}\right)^{1-\sigma} \\
= &\left(\sum_{i=1}^{n} \|u(f_i^1).u(f_i^2)\|_{L^1}^{\frac{1}{1-\sigma}}\right)^{1-\sigma} \\
\leq &\left(\sum_{i=1}^{n} (\|u(f_i^1)\|_{L^2} . \|u(f_i^2)\|_{L^2})^{\frac{1}{1-\sigma}}\right)^{1-\sigma} \\
\leq &\left(\sum_{i=1}^{n} (\|u(f_i^1)\|_{L^2})^{\frac{2}{1-\sigma}}\right)^{\frac{1-\sigma}{2}} . \left(\sum_{i=1}^{n} (\|u(f_i^2)\|_{L^2})^{\frac{2}{1-\sigma}}\right)^{\frac{1-\sigma}{2}} \\
\leq &\; C.\delta_{2\sigma}((f_i^1)_{i=1}^n).\delta_{2\sigma}((f_i^2)_{i=1}^n),
\end{aligned}$$

and so T is dominated $(2,2;\sigma)$-continuous. However, since u is not $\frac{2}{1-\sigma}$-summing, there is a sequence $(g_i)_{i=1}^{\infty}$ of functions in L^2 such that

$$(g_i)_{i=1}^{\infty} \in \ell^{\omega}_{\frac{2}{1-\sigma}}(L^2) \text{ but } (u(g_i))_{i=1}^{\infty} \notin \ell_{\frac{2}{1-\sigma}}(L^2).$$

Since

$$\|(T(g_i, g_i))_{i=1}^{\infty}\|_{\frac{1}{1-\sigma}} = \left(\sum_{i=1}^{\infty} \|u(g_i).u(g_i)\|_{L^1}^{\frac{1}{1-\sigma}}\right)^{1-\sigma} = \left(\sum_{i=1}^{\infty} (\|u(g_i)\|_{L^2})^{\frac{2}{1-\sigma}}\right)^{1-\sigma} = \infty,$$

we have

$$(T(g_i, g_i))_{i=1}^{\infty} \notin \ell_{\frac{1}{1-\sigma}}(L^1).$$

Therefore, the bilinear mapping T is not $(\frac{2}{1-\sigma}, \frac{2}{1-\sigma})$-dominated (see Definition 1.4.16)
.

2.2.3 Tensorial representation

In this section we introduce a tensor norm, $\beta_{p,\sigma}$, of order $m+1$ represent our multi-ideal $\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}$ i.e. the topological dual of the normed space $(X_1 \otimes_{\beta_{p,\sigma}} ... \otimes_{\beta_{p,\sigma}} X_m \otimes_{\beta_{p,\sigma}} Y)$ is isometric to

$$(\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}(X_1, ..., X_m; Y^*), \|.\|_{\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}}).$$

Let $u \in X_1 \otimes ... \otimes X_m \otimes Y$, for $1 \leq p, p_1, ..., p_m, r < \infty$, $0 \leq \sigma < 1$ with $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ and $\frac{1}{r} + \frac{1-\sigma}{p} = 1$, we consider

$$\beta_{p,\sigma}(u) = \inf \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \, \|(y_i)_{i=1}^n\|_r \,,$$

where the infimum is taken over all representations of u of the form

$$u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \otimes y_i$$

with $x_i^j \in X_j, y_i \in Y, i = 1, ..., n, j = 1, ..., m$ and $n, m \in \mathbb{N}$.

Proposition 2.2.15. *$\beta_{p,\sigma}$ is a reasonable crossnorm.*

Proof. First, we show that $\beta_{p,\sigma}(\lambda u) = |\lambda| \, \beta_{p,\sigma}(u)$ for all $u \in X_1 \otimes ... \otimes X_m \otimes Y$ and $\lambda \in \mathbb{K}$. This is obvious when $\lambda = 0$, so suppose that $\lambda \neq 0$. If $u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \otimes y_i$ is a representation of u then $\lambda u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \otimes (\lambda y_i)$, and so we have

$$\beta_{p,\sigma}(\lambda u) \leq \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \, \|(\lambda y_i)_{i=1}^n\|_r = |\lambda| \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \, \|(y_i)_{i=1}^n\|_r \,.$$

Since this holds for every representation of u, it follows that $\beta_{p,\sigma}(\lambda u) \leq |\lambda| \, \beta_{p,\sigma}(u)$. In the same way, we have

$$\beta_{p,\sigma}(u) = \beta_{p,\sigma}(\frac{1}{\lambda} \lambda u) \leq \frac{1}{|\lambda|} \beta_{p,\sigma}(\lambda u),$$

giving $|\lambda| \, \beta_{p,\sigma}(u) \leq \beta_{p,\sigma}(u)$. Therefore $\beta_{p,\sigma}(\lambda u) = |\lambda| \, \beta_{p,\sigma}(u)$.
Now, to prove that $\beta_{p,\sigma}$ satisfies the triangle inequality, let $u', u'' \in X_1 \otimes ... \otimes X_m \otimes Y$, and let $\varepsilon > 0$. Choose representations of u' and u'' of the form

$$u' = \sum_{i=1}^{n'} x_i'^1 \otimes ... \otimes x_i'^m \otimes y_i', \quad u'' = \sum_{i=1}^{n''} x_i''^1 \otimes ... \otimes x_i''^m \otimes y_i''$$

such that

$$\beta_{p,\sigma}(u')+\varepsilon \geq \prod_{j=1}^{m}\delta_{p_j\sigma}((x_i'^j)_{i=1}^{n'}).\left\|(y_i')_{i=1}^{n'}\right\|_r \text{ and } \beta_{p,\sigma}(u'')+\varepsilon \geq \prod_{j=1}^{m}\delta_{p_j\sigma}((x_i''^j)_{i=1}^{n''}).\left\|(y_i'')_{i=1}^{n''}\right\|_r.$$

We can write u', u'' in the following way

$$u' = \sum_{i=1}^{n'} z_i'^1 \otimes ... \otimes z_i'^m \otimes t_i', \quad u'' = \sum_{i=1}^{n''} z_i''^1 \otimes ... \otimes z_i''^m \otimes t_i''$$

with

$$\begin{aligned}
z_i'^j &= \frac{(\beta_{p,\sigma}(u')+\varepsilon)^{\frac{1-\sigma}{p_j}}}{\delta_{p_j\sigma}\left((x_i'^j)_{i=1}^{n'}\right)} x_i'^j, \quad j=1,...,m, i=1,...,n', \\
t_i' &= \frac{\prod_{j=1}^{m}\delta_{p_j\sigma}\left((x_i'^j)_{i=1}^{n'}\right)}{(\beta_{p,\sigma}(u')+\varepsilon)^{\frac{1-\sigma}{p}}} y_i', \quad i=1,...,n', \\
z_i''^j &= \frac{(\beta_{p,\sigma}(u'')+\varepsilon)^{\frac{1-\sigma}{p_j}}}{\delta_{p_j\sigma}\left((x_i''^j)_{i=1}^{n''}\right)} x_i''^j, \quad j=1,...,m, i=1,...,n'', \\
t_i'' &= \frac{\prod_{j=1}^{m}\delta_{p_j\sigma}\left((x_i''^j)_{i=1}^{n''}\right)}{(\beta_{p,\sigma}(u'')+\varepsilon)^{\frac{1-\sigma}{p}}} y_i'', \quad i=1,...,n''.
\end{aligned}$$

It follows that

$$\begin{aligned}
\delta_{p_j\sigma}((z_i'^j)_{i=1}^{n'}) = (\beta_{p,\sigma}(u')+\varepsilon)^{\frac{1-\sigma}{p_j}} \quad &\text{and} \quad \left\|(t_i')_{i=1}^{n'}\right\|_r \leq (\beta_{p,\sigma}(u')+\varepsilon)^{\frac{1}{r}}, \\
\delta_{p_j\sigma}((z_i''^j)_{i=1}^{n''}) = (\beta_{p,\sigma}(u'')+\varepsilon)^{\frac{1-\sigma}{p_j}} \quad &\text{and} \quad \left\|(t_i'')_{i=1}^{n''}\right\|_r \leq (\beta_{p,\sigma}(u'')+\varepsilon)^{\frac{1}{r}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\prod_{j=1}^{m}\delta_{p_j\sigma}((z_i'^j)_{i=1}^{n'}).\left\|(t_i')_{i=1}^{n'}\right\|_r &\leq \beta_{p,\sigma}(u')+\varepsilon \leq \beta_{p,\sigma}(u')+\beta_{p,\sigma}(u'')+\varepsilon, \\
\prod_{j=1}^{m}\delta_{p_j\sigma}((z_i''^j)_{i=1}^{n''}).\left\|(t_i'')_{i=1}^{n''}\right\|_r &\leq \beta_{p,\sigma}(u'')+\varepsilon \leq \beta_{p,\sigma}(u')+\beta_{p,\sigma}(u'')+\varepsilon.
\end{aligned}$$

The last two inequalities imply that

$$\beta_{p,\sigma}(u'+u'') \leq \beta_{p,\sigma}(u')+\beta_{p,\sigma}(u'')+\varepsilon.$$

Since this holds for every $\varepsilon > 0$, we have $\beta_{p,\sigma}(u'+u'') \leq \beta_{p,\sigma}(u')+\beta_{p,\sigma}(u'')$.

Let ϵ denote the injective tensor norm on $X_1 \otimes ... \otimes X_m \otimes Y$. Let $u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \otimes y_i \in X_1 \otimes ... \otimes X_m \otimes Y$. By Hölder's inequality and (2.1) we get

$$\begin{aligned}
\epsilon(u) &= \sup\left\{\left|\sum_{i=1}^{n}\phi_1(x_i^1)...\phi_m(x_i^m)\psi(y_i)\right|; \phi_j \in B_{X_j^*}, \psi \in B_{Y^*}\right\} \\
&\leq \sup_{\phi_j \in B_{X_j^*}} \left\|(\phi_1(x_i^1)...\phi_m(x_i^m))_{i=1}^{n}\right\|_{\frac{p}{1-\sigma}} \|(y_i)_{i=1}^{n}\|_r \\
&\leq \prod_{j=1}^{m}\left\|(x_i^j)_{i=1}^{n}\right\|_{\frac{p_j}{1-\sigma},\omega} \|(y_i)_{i=1}^{n}\|_r \\
&\leq \prod_{j=1}^{m}\delta_{p_j\sigma}((x_i^j)_{i=1}^{n}) \|(y_i)_{i=1}^{n}\|_r.
\end{aligned}$$

Since this holds for every representation of u, we obtain $\epsilon(u) \le \beta_{p,\sigma}(u)$. Thus $\beta_{p,\sigma}(u) = 0$ imply $u = 0$. Hence $\beta_{p,\sigma}$ is a norm on $X_1 \otimes ... \otimes X_m \otimes Y$.
It is clear that

$$\beta_{p,\sigma}(x^1 \otimes ... \otimes x^m \otimes y) \le \left\|x^1\right\| ... \|x^m\| \|y\|$$

for every $x^j \in X_j, j = 1, ..., m$ and $y \in Y$.
Let $\phi_j \in X_j^*$ with $\phi_j \neq 0, j = 1, ..., m$, let $\psi \in Y^*$ and let $u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \otimes y_i$.
Then an application of Hölder's inequality yields

$$\begin{aligned}
|\phi_1 \otimes ... \otimes \phi_m \otimes \psi(u)| &= \left|\phi_1 \otimes ... \otimes \phi_m \otimes \psi(\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \otimes y_i)\right| \\
&= \left|\sum_{i=1}^{n} \phi_1(x_i^1)...\phi_m(x_i^m)\psi(y_i)\right| \\
&\le \sum_{i=1}^{n} |\phi_1(x_i^1)...\phi_m(x_i^m)\psi(y_i)| \\
&\le \prod_{j=1}^{m} \left(\sum_{i=1}^{n} \left|\phi_j(x_i^j)\right|^{\frac{p_j}{1-\sigma}}\right)^{\frac{1-\sigma}{p_j}} \|(\psi(y_i))_{i=1}^n\|_r \\
&\le \|\phi_1\| ... \|\phi_m\| \|\psi\| \prod_{j=1}^{m} \left(\sum_{i=1}^{n} \left|\frac{\phi_j}{\|\phi_j\|}(x_i^j)\right|^{\frac{p_j}{1-\sigma}}\right)^{\frac{1-\sigma}{p_j}} \|(y_i)_{i=1}^n\|_r \\
&\le \|\phi_1\| ... \|\phi_m\| \|\psi\| \prod_{j=1}^{m} \left\|(x_i^j)_{i=1}^n\right\|_{\frac{p_j}{1-\sigma},\omega} \|(y_i)_{i=1}^n\|_r \\
&\le \|\phi_1\| ... \|\phi_m\| \|\psi\| \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \|(y_i)_{i=1}^n\|_r ,
\end{aligned}$$

from which it follows that

$$|\phi_1 \otimes ... \otimes \phi_m \otimes \psi(u)| \le \|\phi_1\| ... \|\phi_m\| \|\psi\| \beta_{p,\sigma}(u).$$

Therefore $\phi_1 \otimes ... \otimes \phi_m \otimes \psi$ is bounded and satisfies

$$\|\phi_1 \otimes ... \otimes \phi_m \otimes \psi\| \le \|\phi_1\| ... \|\phi_m\| \|\psi\|$$

and we have shown that $\beta_{p,\sigma}$ is a reasonable crossnorm. □

In particular, note that when $m = 1$, the norm $\beta_{p,\sigma}$ coincides with the norm $d_{p,\sigma}$ on $X_1 \otimes Y$, that was introduced by J. A. López Molina and E. A. Sánchez Pérez in [41].
In what follows we consider the tensor product of linear operators in connection with the reasonable crossnorm $\beta_{p,\sigma}$. We show that the reasonable crossnorm $\beta_{p,\sigma}$ is actually a tensor norm (see Definition 1.3.16).

Proposition 2.2.16. *Let X_j, Y_j, X, Y be Banach spaces, and $T \in \mathcal{L}(X, Y)$, $T_j \in \mathcal{L}(X_j, Y_j)$, $(j = 1, ..., m)$. Then there is a unique continuous linear operator*

$$T_1 \otimes_{\beta_{p,\sigma}} ... \otimes_{\beta_{p,\sigma}} T_m \otimes_{\beta_{p,\sigma}} T : \left(X_1 \widehat{\otimes} ... \widehat{\otimes} X_m \widehat{\otimes} X, \beta_{p,\sigma}\right) \to \left(Y_1 \widehat{\otimes} ... \widehat{\otimes} Y_m \widehat{\otimes} Y, \beta_{p,\sigma}\right)$$

such that

$$T_1 \otimes_{\beta_{p,\sigma}} ... \otimes_{\beta_{p,\sigma}} T_m \otimes_{\beta_{p,\sigma}} T(x^1 \otimes ... \otimes x^m \otimes x) = (T_1 x^1) \otimes ... \otimes (T_m x^m) \otimes (Tx)$$

for every $x^j \in X_j, (j = 1, ..., m)$ *and* $x \in X$. *Moreover*

$$\left\|T_1 \otimes_{\beta_{p,\sigma}} ... \otimes_{\beta_{p,\sigma}} T_m \otimes_{\beta_{p,\sigma}} T\right\| = \|T_1 \otimes ... \otimes T_m \otimes T\| = \|T\| \prod_{j=1}^{m} \|T_j\|.$$

Proof. By [68, Page 7] there is a unique linear operator

$$T_1 \otimes ... \otimes T_m \otimes T : (X_1 \otimes ... \otimes X_m \otimes X) \to (Y_1 \otimes ... \otimes Y_m \otimes Y)$$

such that

$$T_1 \otimes ... \otimes T_m \otimes T(x^1 \otimes ... \otimes x^m \otimes x) = (T_1 x^1) \otimes ... \otimes (T_m x^m) \otimes (Tx)$$

for every $x^j \in X_j, j = 1, ..., m$ *and* $x \in X$. We may suppose $T_j \neq 0$, $j = 1, ..., m$ and $T \neq 0$. Let $u \in X_1 \otimes ... \otimes X_m \otimes X$ such that

$$u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \otimes x_i,$$

hence the sum

$$\sum_{i=1}^{n} \left(T_1 x_i^1\right) \otimes ... \otimes (T_m x_i^m) \otimes (T x_i)$$

is a representation of $T_1 \otimes ... \otimes T_m \otimes T(u)$ in $Y_1 \otimes ... \otimes Y_m \otimes Y$. Then, for every $1 \leq p, p_1, ..., p_m, r < \infty, 0 \leq \sigma < 1$ with $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ and $\frac{1}{r} + \frac{1-\sigma}{p} = 1$, we have

$$\begin{aligned} \beta_{p,\sigma}\left(T_1 \otimes ... \otimes T_m \otimes T(u)\right) &\leq \prod_{j=1}^{m} \delta_{p_j\sigma}((T_j x_i^j)_{i=1}^n) \left\|(T x_i)_{i=1}^n\right\|_r \\ &\leq \|T\| \prod_{j=1}^{m} \|T_j\| \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \left\|(x_i)_{i=1}^n\right\|_r. \end{aligned}$$

Since this holds for every representation of u, we obtain

$$\beta_{p,\sigma}\left(T_1 \otimes ... \otimes T_m \otimes T(u)\right) \leq \|T\| \prod_{j=1}^{m} \|T_j\| \, \beta_{p,\sigma}(u).$$

So that the linear operator

$$T_1 \otimes ... \otimes T_m \otimes T : (X_1 \otimes ... \otimes X_m \otimes X, \beta_{p,\sigma}) \to (Y_1 \otimes ... \otimes Y_m \otimes Y, \beta_{p,\sigma})$$

is continuous and we have $\|T_1 \otimes ... \otimes T_m \otimes T\| \leq \|T\| \prod_{j=1}^{m} \|T_j\|$.

On the other hand, as $\beta_{p,\sigma}$ is an reasonable crossnorm we get that

$$\begin{aligned}
\|T\| \prod_{j=1}^{m} \|T_j\| &= \sup_{x \in B_X} \|T(x)\| . \prod_{j=1}^{m} \sup_{x^j \in B_{X_j}} \left\| T_j(x^j) \right\| \\
&= \sup_{x \in B_X, x^j \in B_{X_j}} \left(\|T(x)\| . \prod_{j=1}^{m} \left\| T_j(x^j) \right\| \right) \\
&= \sup_{x \in B_X, x^j \in B_{X_j}} \beta_{p,\sigma} \left((T_1 x^1) \otimes ... \otimes (T_m x^m) \otimes (Tx) \right) \\
&= \sup_{x \in B_X, x^j \in B_{X_j}} \beta_{p,\sigma} \left((T_1 \otimes ... \otimes T_m \otimes T)(x^1 \otimes ... \otimes x^m \otimes x) \right) \\
&\leq \sup_{\beta_{p,\sigma}(u) \leq 1} \beta_{p,\sigma} \left((T_1 \otimes ... \otimes T_m \otimes T)(u) \right) \\
&= \|T_1 \otimes ... \otimes T_m \otimes T\| .
\end{aligned}$$

Thus

$$\|T_1 \otimes ... \otimes T_m \otimes T\| \geq \|T\| \prod_{j=1}^{m} \|T_j\| ,$$

and therefore $\|T_1 \otimes ... \otimes T_m \otimes T\| = \|T\| \prod_{j=1}^{m} \|T_j\|$.
Now taking the unique continuous extension of the linear operator $T_1 \otimes ... \otimes T_m \otimes T$ to the completions of $X_1 \otimes ... \otimes X_m \otimes X$ and $Y_1 \otimes ... \otimes Y_m \otimes Y$ which we denote by

$$T_1 \otimes_{\beta_{p,\sigma}} ... \otimes_{\beta_{p,\sigma}} T_m \otimes_{\beta_{p,\sigma}} T$$

we obtain a unique linear operator from $\left(X_1 \widehat{\otimes} ... \widehat{\otimes} X_m \widehat{\otimes} X, \beta_{p,\sigma} \right)$ into $\left(Y_1 \widehat{\otimes} ... \widehat{\otimes} Y_m \widehat{\otimes} Y, \beta_{p,\sigma} \right)$ with the norm

$$\left\| T_1 \otimes_{\beta_{p,\sigma}} ... \otimes_{\beta_{p,\sigma}} T_m \otimes_{\beta_{p,\sigma}} T \right\| = \|T\| \prod_{j=1}^{m} \|T_j\| .$$

□

Following the idea of [43, Theorem 3.7] we present the representation of the multi-ideal $\mathcal{L}_{d,(p_1,...,p_m)}$ by the tensor norm $\beta_{p,\sigma}$.

Theorem 2.2.17. *The space* $(\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)} (X_1, ..., X_m; Y^*) , \|.\|_{\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}})$ *is isometrically isomorphic to* $(X_1 \otimes ... \otimes X_m \otimes Y, \beta_{p,\sigma})^*$ *through the mapping* Ψ *defined by*

$$\Psi(T)(x^1 \otimes ... \otimes x^m \otimes y) = T(x^1, ..., x^m)(y),$$

for every $T \in \mathcal{L}^{\sigma}_{d,(p_1,...,p_m)} (X_1, ..., X_m; Y^*) , x^j \in X_j, j = 1, ..., m$ *and* $y \in Y$.

Proof. It is easy to see that the correspondence Ψ defined as above is linear. It remains to show the surjectivity and that

$$\|\Psi(T)\|_{(X_1 \otimes ... \otimes X_m \otimes Y, \beta_{p,\sigma})^*} = \|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}}$$

for all T in $(\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)} (X_1, ..., X_m; Y^*))$.
Take now $T \in \mathcal{L}^{\sigma}_{d,(p_1,...,p_m)} (X_1, ..., X_m; Y^*)$, and let

$$u=\sum_{i=1}^{n}x_i^1\otimes ...\otimes x_i^m\otimes y_i\in X_1\otimes ...\otimes X_m\otimes Y,$$

where $m\in\mathbb{N},(x_i^j)_{i=1}^n\subset X_j,(y_i)_{i=1}^n\subset Y,j=1,...,m$. Hence, by Hölder's inequality it follows that

$$\begin{aligned}
|\Psi(T)(u)| &= \left|\sum_{i=1}^{n}T(x_i^1,...,x_i^m)(y_i)\right| \\
&\leq \left\|(T(x_i^1,...,x_i^m))_{i=1}^n\right\|_{\frac{p}{1-\sigma}}\left\|(y_i)_{i=1}^n\right\|_r \\
&\leq \|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}}\prod_{j=1}^{m}\delta_{p_j\sigma}((x_i^j)_{i=1}^n)\left\|(y_i)_{i=1}^n\right\|_r .
\end{aligned}$$

So

$$|\Psi(T)(u)|\leq\|T\|_{\mathcal{L}^{\sigma}_{d(p_1,...,p_m)}}\cdot\beta_{p,\sigma}(u).$$

Since u is arbitrary it follows that

$$\|\Psi(T)\|_{(X_1\otimes...\otimes X_m\otimes Y,\beta_{p,\sigma})^*}\leq\|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}} .$$

In order to establish the reverse inequality, let $\phi\in(X_1\otimes ...\otimes X_m\otimes Y,\beta_{p,\sigma})^*$ and define the m-linear mapping $T\in\mathcal{L}\left(X_1,...,X_m;Y^*\right)$ by

$$T(x^1,...,x^m)(y)=\phi(x^1\otimes ...\otimes x^m\otimes y).$$

Let $(x_i^1,...,x_i^m)_{i=1}^n\subset X_1\times ...\times X_m$. Since

$$\left\|T(x_i^1,...,x_i^m)\right\|=\sup_{y_i\in Y,\|y_i\|=1}\left|T(x_i^1,...,x_i^m)(y_i)\right|,$$

for each $\varepsilon>0$, choose $(y_i)_{i=1}^n\subset Y,\|y_i\|=1,j=1,...,m$ such that

$$\left\|T(x_i^1,...,x_i^m)\right\|^{\frac{p}{1-\sigma}}-\frac{\varepsilon}{n}\leq\left|T(x_i^1,...,x_i^m)(y_i)\right|^{\frac{p}{1-\sigma}}$$

then

$$\sum_{i=1}^{n}\left\|T(x_i^1,...,x_i^m)\right\|^{\frac{p}{1-\sigma}}\leq\varepsilon+\sum_{i=1}^{n}\left|T(x_i^1,...,x_i^m)(y_i)\right|^{\frac{p}{1-\sigma}}=(*).$$

With a convenient choice of $\lambda_i\in\mathbb{K}$, $|\lambda_i|=1$, $i=1,...,n$ we can write

$$
\begin{aligned}
(*) &= \varepsilon + \sum_{i=1}^{n} \left| |\phi(x_i^1 \otimes ... \otimes x_i^m \otimes y_i)|^{\frac{p}{1-\sigma}-1} \phi(x_i^1 \otimes ... \otimes x_i^m \otimes y_i) \right| \\
&= \varepsilon + \phi \left(\sum_{i=1}^{n} \lambda_i \, |\phi(x_i^1 \otimes ... \otimes x_i^m \otimes y_i)|^{\frac{p}{1-\sigma}-1} x_i^1 \otimes ... \otimes x_i^m \otimes y_i \right) \\
&\leq \varepsilon + \|\phi\| \, \beta_{p,\sigma} \left(\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \otimes \left(\lambda_i \, |\phi(x_i^1 \otimes ... \otimes x_i^m \otimes y_i)|^{\frac{p}{1-\sigma}-1} y_i \right) \right) \\
&\leq \varepsilon + \|\phi\| \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \left\| \left(\lambda_i \, |\phi(x_i^1 \otimes ... \otimes x_i^m \otimes y_i)|^{\frac{p}{1-\sigma}-1} y_i \right)_{i=1}^n \right\|_r \\
&= \varepsilon + \|\phi\| \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \left(\sum_{i=1}^{n} |\phi(x_i^1 \otimes ... \otimes x_i^m \otimes y_i)|^{\left(\frac{p}{1-\sigma}-1\right)r} \right)^{\frac{1}{r}} \\
&= \varepsilon + \|\phi\| \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \left(\sum_{i=1}^{n} |T(x_i^1, ..., x_i^m)(y_i)|^{\left(\frac{p}{1-\sigma}-1\right)r} \right)^{\frac{1}{r}} \\
&\leq \varepsilon + \|\phi\| \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \left(\sum_{i=1}^{n} \|T(x_i^1, ..., x_i^m)\|^{\left(\frac{p}{1-\sigma}-1\right)r} \right)^{\frac{1}{r}}.
\end{aligned}
$$

Since $(\frac{p}{1-\sigma} - 1)r = \frac{p}{1-\sigma}$, these inequalities imply

$$
\left(\sum_{i=1}^{n} \left\| T(x_i^1, ..., x_i^m) \right\|^{\frac{p}{1-\sigma}} \right)^{1-\frac{1}{r}} \leq \varepsilon + \|\phi\| \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n).
$$

Since this holds for every $\varepsilon > 0$, we have

$$
\left(\sum_{i=1}^{n} \left\| T(x_i^1, ..., x_i^m) \right\|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \leq \|\phi\| \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n)
$$

showing that

$$
T \in \mathcal{L}_{d,(p_1,...,p_m)}^{\sigma} (X_1, ..., X_m; Y^*)
$$

and

$$
\|T\|_{\mathcal{L}_{d,(p_1,...,p_m)}^{\sigma}} \leq \|\phi\| = \|\Psi(T)\|_{(X_1 \otimes ... \otimes X_m \otimes Y, \beta_{p,\sigma})^*}.
$$

□

Now we are ready to introduce a new formula of the tensor norm $\beta_{p,\sigma}$ in such a way that we characterize the space of dominated $(p_1, ..., p_m; \sigma)$-continuous multilinear forms.

Let $u \in X_1 \otimes ... \otimes X_m \otimes Y$. For $1 \leq p, p_1, ..., p_m, r < \infty, 0 \leq \sigma < 1$ with $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ and $\frac{1}{r} + \frac{1-\sigma}{p} = 1$, we consider

$$
\widetilde{\nu}_{p,\sigma}(u) = \inf \|(\lambda_i)_{i=1}^n\|_r \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \, \|(y_i)_{i=1}^n\|_\infty
$$

taking the infimum over all representations of u of the form

$$u = \sum_{i=1}^{n} \lambda_i x_i^1 \otimes ... \otimes x_i^m \otimes y_i$$

with $(x_i^j)_{i=1}^n \subset X_j, (y_i)_{i=1}^n \subset Y, (\lambda_i)_{i=1}^n \subset \mathbb{K}, j = 1, ..., m$ and $n, m \in \mathbb{N}$.

Proposition 2.2.18. *We have* $\widetilde{\nu}_{p,\sigma}(u) = \beta_{p,\sigma}(u)$ *for all* $u \in X_1 \otimes ... \otimes X_m \otimes Y$.

Proof. We note first that every representation of u of the form $\sum_{i=1}^{n} \lambda_i x_i^1 \otimes ... \otimes x_i^m \otimes y_i$ can be written as $\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \otimes (\lambda_i y_i)$ and hence

$$\begin{aligned} \beta_{p,\sigma}(u) &\leq \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \, \|(\lambda_i y_i)_{i=1}^n\|_r \\ &\leq \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \, \|(\lambda_i)_{i=1}^n\|_r \, \|(y_i)_{i=1}^n\|_\infty \end{aligned}$$

from which it follows that $\beta_{p,\sigma}(u) \leq \widetilde{\nu}_{p,\sigma}(u)$.
On the other hand, let $\sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \otimes y_i$ be a representation of u. We can write u as $\sum_{i=1}^{n} \lambda_i x_i^1 \otimes ... \otimes x_i^m \otimes z_i$, where $\lambda_i = \|y_i\|$ and $\|z_i\| \leq 1$ for every $i = 1, ..., n$. Then

$$\widetilde{\nu}_{p,\sigma}(u) \leq \|(y_i)_{i=1}^n\|_r \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \, \|(z_i)_{i=1}^n\|_\infty \leq \|(y_i)_{i=1}^n\|_r \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n)$$

and hence $\widetilde{\nu}_{p,\sigma}(u) \leq \beta_{p,\sigma}(u)$.

□

Remark 2.2.19. Making $F = \mathbb{K}$, in Theorem 2.2.17 we obtain that for every family of Banach spaces $X_1, ..., X_m$, the space of dominated $(p_1, ..., p_m; \sigma)$-continuous multilinear forms

$$(\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}(X_1, ..., X_m), \|.\|_{\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}})$$

is isometric to $(X_1 \otimes ... \otimes X_m \otimes \mathbb{K}, \widetilde{\nu}_{p,\sigma})^*$.

We recall that by the universal property of tensor products (Theorem 1.3.6), there is an algebraic isomorphism between the m-linear mapping from $X_1 \times ... \times X_m$ into Y and the linear mapping from $X_1 \otimes ... \otimes X_m$ into Y. To each m-linear mapping T corresponds the unique linear mapping T_L such that

$$T_L(x^1 \otimes ... \otimes x^m) = T(x^1, ..., x^m).$$

In Proposition 2.2.15 if we take $Y = \mathbb{K}$, then we identify $X_1 \otimes ... \otimes X_m \otimes \mathbb{K}$ with $X_1 \otimes ... \otimes X_m$, and in this case the corresponding tensor norm will be denoted by $\nu_{p,\sigma}$ and can be described as follows:

$$\nu_{p,\sigma}(u) = \inf \|(\lambda_i)_{i=1}^n\|_r \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n)$$

where the infimum is taken over all representations of $u \in X_1 \otimes ... \otimes X_m$ of the form $u = \sum_{i=1}^n \lambda_i x_i^1 \otimes ... \otimes x_i^m$ with $(\lambda_i)_{i=1}^n \subset \mathbb{K}, (x_i^j)_{i=1}^n \subset X_j, j = 1, ..., m$.
The next theorem and its proof are similar to Theorem 2.2.17.

Theorem 2.2.20. *$(\mathcal{L}_{d,(p_1,...,p_m)}^\sigma (X_1, ..., X_m), \|.\|_{\mathcal{L}_{d,(p_1,...,p_m)}^\sigma})$ is isometrically isomorphic to $(X_1 \otimes ... \otimes X_m, \nu_{p,\sigma})^*$ through the mapping $T \mapsto T_L$.*

Proof. We consider $\phi \in (X_1 \otimes ... \otimes X_m, \nu_{p,\sigma})^*$ and T the m-linear scalar function on $X_1 \times ... \times X_m$ given by $T(x^1, ..., x^m) = \phi(x^1 \otimes ... \otimes x^m)$. This means that $T_L = \phi$. We can choose scalars λ_i, $|\lambda_i| = 1$, $i = 1, ..., n$ such that

$$\begin{aligned}
& \sum_{i=1}^n |T(x_i^1, ..., x_i^m)|^{\frac{p}{1-\sigma}} \\
= & \sum_{i=1}^n |\phi(x_i^1 \otimes ... \otimes x_i^m)|^{\frac{p}{1-\sigma}} \\
= & \left| \sum_{i=1}^n |\phi(x_i^1 \otimes ... \otimes x_i^m)|^{\frac{p}{1-\sigma}-1} \lambda_i \phi(x_i^1 \otimes ... \otimes x_i^m) \right| \\
= & \left| \phi\left(\sum_{i=1}^n \lambda_i |\phi(x_i^1 \otimes ... \otimes x_i^m)|^{\frac{p}{1-\sigma}-1} x_i^1 \otimes ... \otimes x_i^m \right) \right| \\
\leq & \|\phi\| \, \nu_{p,\sigma} \left(\sum_{i=1}^n \lambda_i |\phi(x_i^1 \otimes ... \otimes x_i^m)|^{\frac{p}{1-\sigma}-1} x_i^1 \otimes ... \otimes x_i^m \right) \\
\leq & \|\phi\| \left\| \left(\lambda_i |\phi(x_i^1 \otimes ... \otimes x_i^m)|^{\frac{p}{1-\sigma}-1} \right)_{i=1}^n \right\|_r \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \\
= & \|\phi\| \left(\sum_{i=1}^n |\phi(x_i^1 \otimes ... \otimes x_i^m)|^{\left(\frac{p}{1-\sigma}-1\right)r} \right)^{\frac{1}{r}} \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \\
= & \|\phi\| \left(\sum_{i=1}^n |T(x_i^1, ..., x_i^m)|^{\left(\frac{p}{1-\sigma}-1\right)r} \right)^{\frac{1}{r}} \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \\
= & \|\phi\| \left(\sum_{i=1}^n |T(x_i^1, ..., x_i^m)|^{\frac{p}{1-\sigma}} \right)^{\frac{1}{r}} \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n).
\end{aligned}$$

These inequalities imply

$$\left(\sum_{i=1}^n \left| T(x_i^1, ..., x_i^m) \right|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \leq \|\phi\| \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n).$$

Thus T is dominated $(p_1, ..., p_m; \sigma)$-continuous and

$$\|T\|_{\mathcal{L}_{d,(p;p_1,...,p_m)}^\sigma} \leq \|\phi\| = \|T_L\|_{(X_1 \otimes ... \otimes X_m, \nu_{p,\sigma})^*}.$$

Conversely if T is dominated $(p_1, ..., p_m; \sigma)$-continuous we have

$$T_L(u) = \sum_{i=1}^{n} \lambda_i T(x_i^1, ..., x_i^m)$$

for $u = \sum\limits_{i=1}^{n} \lambda_i x_i^1 \otimes ... \otimes x_i^m \in X_1 \otimes ... \otimes X_m$.
Then, by Hölder's inequality, we have

$$\begin{aligned} |T_L(u)| &\leq \sum_{i=1}^{n} |\lambda_i| \, |T(x_i^1, ..., x_i^m)| \\ &\leq \|(\lambda_i)_{i=1}^n\|_r \cdot \left\|(T(x_i^1, ..., x_i^m))_{i=1}^n\right\|_{\frac{p}{1-\sigma}} \\ &\leq \|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}} \|(\lambda_i)_{i=1}^n\|_r \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n). \end{aligned}$$

It follows that $|T_L(u)| \leq \|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}} \nu_{p,\sigma}(u)$. Consequently

$$\|T_L\|_{(X_1 \otimes ... \otimes X_m, \nu_{p,\sigma})^*} \leq \|T\|_{\mathcal{L}^{\sigma}_{d,(p_1,...,p_m)}}.$$

□

Chapter 3

Strongly (p,σ)-continuous multilinear operators

The results in this chapter were published in collaboration with D. Achour, E. A. Sánchez Pérez and P. Rueda in the journal of "Linear and multilinear algebra" [4] and are presented as follows. Section 3.1 is devoted to study the notion of *strongly (p,σ)-continuous linear operator.* We present a characterization given by a summability property and an integral domination. In Section 3.2 we construct a new multi-ideal by the composition method starting from the ideal of strongly (p,σ)-continuous linear operators. We give an analogue of the Pietsch domination theorem and we characterize the adjoint operators of strongly (p,σ)-continuous m-linear operators. In section 3.3 we find the trace duality representation of the strongly (p,σ)-continuous m-linear operators by presenting a tensor norm $g_{p,\sigma}$ on $X_1\otimes\cdots\otimes X_m\otimes X$ that satisfies that the topological dual of $g_{p,\sigma}$ on $X_1\otimes\cdots\otimes X_m\otimes Y^*$ is isometric to the space of strongly (p,σ)-continuous m-linear operators from $X_1\times\cdots\times X_m$ into Y. Finally, we present the factorization theorem for the strongly (p,σ)-continuous m-linear operators (Theorem 3.2.13). Two particular cases of this result are relevant and new: the linear case, that is given in Theorem 3.2.15, and Theorem 3.2.16 which provides the factorization theorem for the Cohen strongly p-summing operators.

3.1 Strongly (p,σ)-continuous linear operators.

3.1.1 Preliminary.

The ideal $\Pi_{p,\sigma}$ is a particular case of the family $\mathcal{D}_{q,\nu,p,\sigma}$ of operator ideals introduced in [41], which generalizes the classical ideal $\mathcal{D}_{q,p}$ of (q,p) -dominated operators [61]. They are defined as follows. Let $1\leq r,p,q<\infty$ and $0\leq\sigma,\nu<1$ such that

$$\frac{1}{r}+\frac{1-\sigma}{p^*}+\frac{1-\nu}{q}=1.$$

Definition 3.1.1. *An operator $T\in\mathcal{L}(X,Y)$ is said to be (q,ν,p,σ)-dominated if there exist Banach spaces G,H, linear operators $R\in\Pi_q(X,G), S\in\Pi_{p^*}(Y^*,H)$ and* *a constant $C > 0$ such that*

$$|\langle Tx, y^*\rangle| \le C\,\|x\|^{\nu}\,\|Rx\|^{1-\nu}\,\|y^*\|^{\sigma}\,\|Sy^*\|^{1-\sigma} \qquad x \in X, \quad y^* \in Y^*. \tag{3.1}$$

In such case, we put

$$d_{q,p}^{\nu,\sigma}(T) = \inf\left\{C\pi_q(R)^{1-\nu}\pi_{p^*}(S)^{1-\sigma}\right\},$$

taking the infimum over all $C > 0$, $R \in \Pi_q(X, G)$ and $S \in \Pi_{p^*}(Y^*, H)$ such that (3.1) holds.

We denote by $\left(\mathcal{D}_{q,\nu,p,\sigma}, d_{q,p}^{\nu,\sigma}\right)$ the Banach ideal of (q,ν,p,σ)-dominated linear operators. The following result is a characterization of this ideal (see [41, Theorem 2.4]).

Theorem 3.1.2. *Let $1 \le r, p, q \le \infty$ and $0 \le \sigma, \nu \le 1$ such that $\frac{1}{r} + \frac{1-\sigma}{p^*} + \frac{1-\nu}{q} = 1$. The following assertions are equivalent.*
(i) $T \in \mathcal{D}_{q,\nu,p,\sigma}(X, Y)$
(ii) There exist a constant $C > 0$ and regular Borel probability measures μ and τ on B_{X^*} and $B_{Y^{**}}$, respectively, such that for every $x \in X$ and $y^* \in Y^*$, the following inequality holds

$$|\langle Tx, y^*\rangle| \le C\left(\int_{B_{X^*}} (|\langle x, x^*\rangle|^{1-\nu}\,\|x\|^{\nu})^{\frac{q}{1-\nu}} d\mu\right)^{\frac{1-\nu}{q}}$$
$$\times\left(\int_{B_{Y^{**}}} (|\langle y^*, y^{**}\rangle|^{1-\sigma}\,\|y^*\|^{\sigma})^{\frac{p^*}{1-\sigma}} d\tau\right)^{\frac{1-\sigma}{p^*}}. \tag{3.2}$$

(iii) There exist a constant $C > 0$ such that for every $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$ the following inequality holds

$$\|(\langle Tx_i, y_i^*\rangle)_{i=1}^n\|_{r^*} \le C\delta_{q\nu}((x_i)_{i=1}^n)\delta_{p^*\sigma}((y_i^*)_{i=1}^n). \tag{3.3}$$

Moreover, $d_{q,p}^{\nu,\sigma}(T) = \inf C$, where the infimum is taken over all constants C either in (3.2) or in (3.3).

Consider a (q,σ,p,σ)-dominated operators $T : X \longrightarrow Y$. The domination (3.2) is in fact the same thing that characterizes that B_T (the bilinear form associated to T), is dominated $(q, p^*; \sigma)$-continuous. This provides the domination

$$|B_T(x, y^*)| := |\langle T(x), y^*\rangle| \le C(\int_{B_{X^*}} (|\langle x, x^*\rangle|^{1-\sigma}\,\|x\|^{\sigma})^{\frac{q}{1-\sigma}} d\eta_1)^{\frac{1-\sigma}{q}}$$
$$\times(\int_{B_{Y^{**}}} (|\langle y^*, y^{**}\rangle^{1-\sigma}\,\|y^*\|^{\sigma}\, d\eta_2)^{\frac{p^*}{1-\sigma}})^{\frac{1-\sigma}{p^*}},$$

where η_1 and η_2 are regular Borel probability measures on the corresponding unit balls. The last inequality implies that

$$\|T(x)\| \le C(\int_{B_{X^*}} (|\langle x, x^*\rangle|^{1-\sigma}\,\|x\|^{\sigma})^{\frac{q}{1-\sigma}} d\eta_1)^{\frac{1-\sigma}{q}},$$

and
$$\|T^*(y^*)\| \leq C(\int_{B_{Y^{**}}} (|\langle y^*, y^{**}\rangle^{1-\sigma} \|y^*\|^{\sigma} d\eta_2)^{\frac{p^*}{1-\sigma}})^{\frac{1-\sigma}{p^*}},$$
for all $x \in X$ and $y^* \in Y^*$. After Theorem 2.1.20, we can find the following factorization scheme for the (q, σ, p, σ)-dominated operators (we use the same notation as that in Theorem 2.1.20). Then there are regular Borel probability measures η_1 and η_2 on B_{X^*} and $B_{Y^{**}}$, respectively, such that T factorizes as

$$\begin{array}{ccc} X & \stackrel{T}{\longrightarrow} & Y \\ i \downarrow & & \uparrow \widetilde{T} \\ i_X(X) & \stackrel{J_{q,\sigma}}{\longrightarrow} & L_{q,\sigma}(\eta_1) \end{array}$$

and T^* factorizes as

$$\begin{array}{ccc} Y^* & \stackrel{T^*}{\longrightarrow} & X^* \\ i \downarrow & & \uparrow \widetilde{T^*} \\ i_{Y^*}(Y^*) & \stackrel{J_{p^*,\sigma}}{\longrightarrow} & L_{p^*,\sigma}(\eta_2) \end{array}$$

In fact, our multilinear factorization result Theorem 2.2.10 gives that the bilinear form B_T associated to T factorizes as
$$X \times Y^* \to L_{q,\sigma}(\eta_1) \times L_{p^*,\sigma}(\eta_2) \to \mathbb{R}.$$

3.1.2 Ideal of strongly (p, σ)-continuous linear operators.

This subsection is devoted to analyze the linear ideal of strongly (p, σ)-continuous operators. This ideal can be obtained as a particular class of (q, ν, p, σ)-dominated operators, for $\nu = 1$ (see Definition 3.1.1).

Definition 3.1.3. *Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$, such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. A mapping $T \in \mathcal{L}(X, Y)$ is strongly (p, σ)-continuous if there are Banach spaces H, a linear operator $S \in \Pi_{p^*}(Y^*, H)$ and a constant $C > 0$ such that for all $x \in X$ and $y^* \in Y^*$ we have*

$$|\langle T(x), y^*\rangle| \leq C \|x\| \|y^*\|^{\sigma} \|S(y^*)\|^{1-\sigma}. \tag{3.4}$$

The class of all strongly (p, σ)-continuous linear operators from X into Y is denoted by $\mathcal{D}_p^{\sigma}(X, Y)$ and by $d_p^{\sigma}(T)$ the strongly (p, σ)-continuous norm which is defined by
$$d_p^{\sigma}(T) = \inf \left\{ C \pi_{p^*}(S)^{1-\sigma} \right\}.$$
where the infimum is taken over all $C > 0$ and $S \in \Pi_{p^*}(Y^*, H)$ such that the inequality (3.4) holds.
We can use Theorem 3.1.2 in order to obtain the subsequent characterization of strongly (p, σ)-continuous linear operators in terms of a summability property and an integral domination. This is a particular case of the general characterization of (q, ν, p, σ)-dominated operators; the equivalence with *(iii)* is new. We write the proof for the aim of completeness.

Theorem 3.1.4. *Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$, such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. For $T \in \mathcal{L}(X,Y)$, the following statements are equivalent.*
(i) $T \in \mathcal{D}_p^\sigma(X,Y)$.
(ii) There exist a constant $C > 0$ and a regular Borel probability measure μ on $B_{Y^{**}}$, such that for every $x \in X$ and $y^* \in Y^*$ the following inequality holds

$$|\langle T(x), y^* \rangle| \leq C \|x\| \left(\int_{B_{Y^{**}}} (|\langle y^*, \varphi \rangle|^{1-\sigma} \|y^*\|^\sigma)^{\frac{p^*}{1-\sigma}} d\mu \right)^{\frac{1-\sigma}{p^*}}.$$

(iii) There exist a constant $C > 0$ such that for every $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$ the following inequality holds

$$\|(\langle T(x_i), y_i^* \rangle)_{i=1}^n\|_1 \leq C \left\|(x_i)_{i=1}^n\right\|_r \delta_{p^*\sigma}((y_i^*)_{i=1}^n).$$

(iv) There exist a constant $C > 0$ such that for every $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$ the following inequality holds

$$\|(\langle T(x_i), y_i^* \rangle)_{i=1}^n\|_{\frac{p^*}{1-\sigma}} \leq C \|(x_i)_{i=1}^n\|_\infty \delta_{p^*\sigma}((y_i^*)_{i=1}^n).$$

Moreover, $d_p^\sigma(T) = \inf C$, where the infimum is taken over all constants C either in (ii) or (iii) or in (iv).

Proof. The equivalence (i)$\Longleftrightarrow$(ii) and the implication (iv)$\Longrightarrow$(i) is given by Theorem 3.1.2 and taking into account the equality (2.2).
(ii)$\Longrightarrow$(iii) Let $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$. An application of Hölder's inequality reveals that

$$\begin{aligned}
& \|(\langle T(x_i), y_i^* \rangle)_{i=1}^n\|_1 \\
\leq \; & C \sum_{i=1}^n \|x_i\| \left(\int_{B_{Y^{**}}} (|\langle y_i^*, \varphi \rangle|^{1-\sigma} \|y_i^*\|^\sigma)^{\frac{p^*}{1-\sigma}} d\mu \right)^{\frac{1-\sigma}{p^*}} \\
\leq \; & C \|(x_i)_{i=1}^n\|_r \left(\sum_{i=1}^n \int_{B_{Y^{**}}} (|\langle y_i^*, \varphi \rangle|^{1-\sigma} \|y_i^*\|^\sigma)^{\frac{p^*}{1-\sigma}} d\mu \right)^{\frac{1-\sigma}{p^*}} \\
= \; & C \|(x_i)_{i=1}^n\|_r \left(\int_{B_{Y^{**}}} \sum_{i=1}^n (|\langle y_i^*, \varphi \rangle|^{1-\sigma} \|y_i^*\|^\sigma)^{\frac{p^*}{1-\sigma}} d\mu \right)^{\frac{1-\sigma}{p^*}} \\
\leq \; & C \|(x_i)_{i=1}^n\|_r \delta_{p^*\sigma}((y_i^*)_{i=1}^n).
\end{aligned}$$

(iii)$\Longrightarrow$(iv) For all $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$ we have

$$\begin{aligned}
\|(\langle T(x_i), y_i^* \rangle)_{i=1}^n\|_{\frac{p^*}{1-\sigma}} &= \left(\sum_{i=1}^n |\langle x_i, T^*(y_i^*) \rangle|^{r^*} \right)^{\frac{1}{r^*}} \\
&\leq \left(\sum_{i=1}^n \|x_i\|^{r^*} \|T^*(y_i^*)\|^{r^*} \right)^{\frac{1}{r^*}} \\
&\leq \|(x_i)_{i=1}^n\|_\infty \|(T^*(y_i^*))_{i=1}^n\|_{r^*}.
\end{aligned}$$

On the other hand, by the equality (1.3) we have

$$\|(T^*(y_i^*))_{i=1}^n\|_{r^*} = \sup\left\{\left|\sum_{i=1}^n \langle T^*(y_i^*), z_i\rangle\right| : (z_i)_{i=1}^n \subset X, \left\|(z_i)_{i=1}^n\right\|_r \leq 1\right\}.$$

Thus, for all $(z_i)_{i=1}^n \subset X$ we obtain

$$\begin{aligned}\left|\sum_{i=1}^n \langle T^*(y_i^*), z_i\rangle\right| &\leq \sum_{i=1}^n |\langle y_i^*, T(z_i)\rangle| \\ &\leq C\left\|(z_i)_{i=1}^n\right\|_r \delta_{p^*\sigma}((y_i^*)_{i=1}^n).\end{aligned}$$

By taking the supremum over the unit ball in $\ell_r^n(X)$ we obtain

$$\|(T^*(y_i^*))_{i=1}^n\|_{r^*} \leq C\delta_{p^*\sigma}((y_i^*)_{i=1}^n).$$

Therefore

$$\|(\langle T(x_i), y_i^*\rangle)_{i=1}^n\|_{\frac{p^*}{1-\sigma}} \leq C\,\|(x_i)_{i=1}^n\|_\infty\,\delta_{p^*\sigma}((y_i^*)_{i=1}^n).$$

□

Note that if we take $\sigma = 0$, we obtain $\mathcal{D}_p^0 = \mathcal{D}_p$.

Remark 3.1.5. According to [70, Definition 2.2.2] and Definition 3.1.1 we obtain

$$\mathcal{D}_p^\sigma = (\Pi_{p^*,\sigma})^{dual} = \{T \in \mathcal{L}(X, Y) : T^* \in \Pi_{p^*,\sigma}(Y^*, X^*)\}. \quad (3.5)$$

Thus $(\mathcal{D}_p^\sigma, d_p^\sigma)$ is a Banach operator ideal (see Proposition 1.2.5).

Corollary 3.1.6. *Consider* $1 < p, q < \infty$ *such that* $p \leq q$. *Then,*

$$\mathcal{D}_q^\sigma(X, Y) \subset \mathcal{D}_p^\sigma(X, Y).$$

Proof. If $u \in \mathcal{D}_q^\sigma(X, Y) = (\Pi_{q^*,\sigma})^{dual}(X, Y)$, then $u^* \in \Pi_{q^*,\sigma}(Y^*, X^*)$. Now, by the inclusion theorem for the class of (p,σ)-absolutely continuous linear mappings (see Proposition 2.1.7) and the relation (3.5), we have

$$u \in \mathcal{D}_p^\sigma(X, Y).$$

□

Remark 3.1.7. *If Y is a reflexive Banach space, then every strongly (p,σ)-continuous linear operator $T : X \to Y$ is compact.* Certainly, since $T^* : Y^* \to X^*$ is (p^*,σ)-absolutely continuous, we can conclude from Corollary 2.1.22 that T^* is compact. Consequently, T is compact.

We can establish the following comparison between the classes of strongly (p,σ)-continuous linear operators and strongly p-summing linear operators.

Proposition 3.1.8. *Let $p > 1$ and $0 < \sigma < 1$. Then,*

$$\mathcal{D}_p(X, Y) \subset \mathcal{D}_p^\sigma(X, Y).$$

Moreover we have

$$d_p^\sigma(T) \leq d_p(T) \quad \textit{for all} \quad T \in \mathcal{D}_p(X, Y).$$

Proof. Let $T \in \mathcal{D}_p(X, Y)$. Then its adjoint $T^* : Y^* \to X^*$ is (p^*, σ)-absolutely continuous and $\pi_{p^*,\sigma}(T^*) \leq d_p(T)$ (see Theorem 1.2.10 and Proposition 2.1.9) and the result is obtained by (3.5). □

In what follows we prove more general results, also with the aim of proving that strongly p-summing and strongly (p, σ)-continuous linear operators are in fact different classes (Example 3.1.10). We will show first that in fact the strongly (p, σ)-continuous linear operators are the adjoints of (p, σ)-absolutely continuous linear operators. For the proof of this result we will use Proposition 2.1.16.

Corollary 3.1.9. *Let $1 < p < \infty$ and $0 \leq \sigma < 1$. Let $T \in \mathcal{L}(X, Y)$ and $T^* \in \mathcal{L}(Y^*, X^*)$ its adjoint. Then T is (p, σ)-absolutely continuous if and only if T^* is strongly (p^*, σ)-continuous.*

Proof. By the relation (3.5), the linear mapping $T^* : Y^* \longrightarrow X^*$ is strongly (p^*, σ)-continuous if and only if its adjoint $T^{**} : X^{**} \longrightarrow Y^{**}$ is (p, σ)-absolutely continuous and this equivalent to $T \in \Pi_{p,\sigma}(X, Y)$. □

We show in the next example that in general $\mathcal{D}_p \neq \mathcal{D}_p^\sigma$.

Example 3.1.10. *Let $p > 1$* and *$0 < \sigma < 1$ such that $p^* < \frac{p}{1-\sigma}$. Let $T \in \mathcal{L}(\ell_{p^*}, \ell_{\frac{p}{1-\sigma}})$ defined by $T(e_i) = (\frac{1}{i})^{\frac{1}{p}} e_i$, where $(e_i)_{i=1}^\infty$ is the unit vector canonical basis of ℓ_{p^*}. The adjoint operator of T is strongly (p^*, σ)-continuous but it is not strongly p^*-summing.* In order to see this, note that by [41, Example 1.9] we have $T \in \Pi_{p,\sigma}(\ell_{p^*}, \ell_{\frac{p}{1-\sigma}})$ and $T \notin \Pi_p\left(\ell_{p^*}, \ell_{\frac{p}{1-\sigma}}\right)$. Then by Theorem 1.2.10 and Corollary 3.1.9 we get the result.

3.2 The multi-ideal of strongly (p, σ)-continuous multilinear operators

3.2.1 Properties of strongly (p, σ)-continuous multilinear mappings

In this subsection we extend to multilinear mappings the concept of strongly (p, σ)-continuous linear operators, for which the resulting vector space $\mathcal{D}_p^{m,\sigma}$ of the strongly (p, σ)-continuous multilinear operators is a normed (Banach) multi-ideal. We also show the Pietsch's domination theorem for such operators. We prove that $\mathcal{D}_p^{m,\sigma}$ is generated by the composition method from the operator ideal $\mathcal{D}_p^\sigma$ and we study the connection between a mapping belonging to this multi-ideal and its adjoint.

Definition 3.2.1. *Let $1 \leq p, r < \infty$ and $0 \leq \sigma < 1$ such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. An m-linear mapping $T : X_1 \times \cdots \times X_m \to Y$ is strongly (p, σ)-continuous if there is a constant $C > 0$ such that for any $x_1^j, \ldots, x_n^j \in X_j$, $(1 \leq j \leq m)$ and any $y_1^*, \ldots, y_n^* \in Y^*$, we have*

$$\left\| \left(\left\langle T\left(x_i^1, \ldots, x_i^m\right), y_i^* \right\rangle \right)_{i=1}^n \right\|_1 \leq C \left(\sum_{i=1}^n \prod_{j=1}^m \left\| x_i^j \right\|^r \right)^{\frac{1}{r}} \delta_{p^*\sigma}((y_i^*)_{i=1}^n) \tag{3.6}$$

for all choices of $n \in \mathbb{N}$.

The collection of all strongly (p, σ)-continuous m-linear mappings $X_1 \times \cdots \times X_m \to Y$ will be denoted $\mathcal{D}_p^{m,\sigma}(X_1, \ldots, X_m; Y)$.
The least C for which (3.6) holds will be written $\|.\|_{\mathcal{D}_p^{m,\sigma}}$. It is easy to check that if $T \in \mathcal{D}_p^{m,\sigma}(X_1, \ldots, X_m; Y)$, then

$$\|T\| \leq \|T\|_{\mathcal{D}_p^{m,\sigma}}.$$

For $\sigma = 0$, we have $\mathcal{D}_p^{m,0}(X_1, \ldots, X_m; Y) = \mathcal{D}_p^m(X_1, \ldots, X_m; Y)$, the space of Cohen strongly p-summing m-linear operators.

The next result provides a characterization of this class by means of an inequality.

Proposition 3.2.2. *Let $1 \leq p, r < \infty$ $0 \leq \sigma < 1$ such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. The mapping $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is strongly (p, σ)-continuous if and only if*

$$\left\| \left(\left\langle T\left(x_i^1, \ldots, x_i^m\right), y_i^* \right\rangle \right)_{i=1}^n \right\|_1 \leq C \prod_{j=1}^m \left\| (x_i^j)_{i=1}^n \right\|_{rm} \delta_{p^*\sigma}((y_i^*)_{i=1}^n) \tag{3.7}$$

whenever $x_1^j, \ldots, x_n^j \in X_j$, $(1 \leq j \leq m)$ and $y_1^*, \ldots, y_n^* \in Y^*$.

Proof. Indeed, starting from (3.7) we obtain (3.6) by replacing x_i^j by $\dfrac{\left(\prod_{k=1}^m \|x_i^k\|\right)^{\frac{1}{m}}}{\|x_i^j\|} x_i^j$.
The reverse is immediate by Hölder's inequality and taking into account the equality $\frac{1}{r} = \frac{1}{rm} + \overset{(m)}{\cdots} + \frac{1}{rm}$. □

This class satisfies a Pietsch's domination theorem. For the proof we will use the full general Pietsch's domination theorem (see [56, Theorem 4.6]).

Theorem 3.2.3. *An m-linear operator $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is strongly (p, σ)-continuous if and only if there is a constant $C > 0$ and a regular Borel probability measure μ on $B_{Y^{**}}$,(with the weak star topology) so that for all $(x^1, \ldots, x^m) \in X_1 \times \cdots \times X_m$ and for all $y^* \in Y^*$, the inequality*

$$\left| \left\langle T\left(x^1, \ldots, x^m\right), y^* \right\rangle \right| \leq C \prod_{j=1}^m \left\| x^j \right\| \left(\int_{B_{Y^{**}}} (|\varphi(y^*)|^{1-\sigma} \|y^*\|^\sigma)^{\frac{p^*}{1-\sigma}} d\mu \right)^{\frac{1-\sigma}{p^*}} \tag{3.8}$$

holds.

Proof. A strongly (p,σ)-continuous m-linear operator T is R_1, R_2-S-abstract $(r, \frac{p^*}{1-\sigma})$-summing (see [56, Definition 4.4]) for the parameters

$$\begin{cases} t = 2 \text{ and } k = m-1 \\ G_1 = X_m \text{ and } G_2 = Y^* \\ E_j = X_j \text{ and } j = 1, \ldots, m-1 \\ K_1 = B_{X_1^* \times \cdots \times X_m^*} \text{ and } K_2 = B_{Y^{**}} \\ \mathcal{H} = \mathcal{L}(X_1, \ldots, X_m; Y) \\ q = 1, q_1 = r \text{ and } q_2 = \frac{p^*}{1-\sigma} \\ S(T, x^1, \ldots, x^m, y^*) = |\langle T(x^1, \ldots, x^m), y^* \rangle| \\ R_1(\varphi, x^1, \ldots, x^m) = \|x^1\| \cdots \|x^m\| \\ R_2(\varphi, x^1, \ldots, x^{m-1}, y^*) = |\varphi(y^*)|^{1-\sigma} \|y^*\|^\sigma, \end{cases}$$

[56, Theorem 4.6] gives the result. $\square$

An immediate consequence of Theorem 3.2.3 is the following corollary.

Corollary 3.2.4. *Consider* $1 < p, q < \infty$ *and* $0 \leq \sigma < 1$ *such that* $p \leq q$. *Then*

$$\mathcal{D}_q^{m,\sigma}(X_1, \ldots, X_m; Y) \subset \mathcal{D}_p^{m,\sigma}(X_1, \ldots, X_m; Y).$$

Moreover we have $\|T\|_{\mathcal{D}_p^{m,\sigma}} \leq \|T\|_{\mathcal{D}_q^{m,\sigma}}$ *for all* $T \in \mathcal{D}_q^{m,\sigma}(X_1, \ldots, X_m; Y)$.

Proof. Let $T \in \mathcal{D}_q^{m,\sigma}(X_1, \ldots, X_m; Y)$. There is a regular Borel probability measure μ on $B_{Y^{**}}$ so that for all $(x^1, \ldots, x^m) \in X_1 \times \cdots \times X_m$ and for all $y^* \in Y^*$ we have

$$\begin{aligned} \left|\left\langle T\left(x^1, \ldots, x^m\right), y^*\right\rangle\right| &\leq \|T\|_{\mathcal{D}_q^{m,\sigma}} \prod_{j=1}^m \left\|x^j\right\| \left(\int_{B_{Y^{**}}} \left(|\varphi(y^*)|^{1-\sigma} \|y^*\|^\sigma\right)^{\frac{q^*}{1-\sigma}} d\mu\right)^{\frac{1-\sigma}{q^*}} \\ &\leq \|T\|_{\mathcal{D}_q^{m,\sigma}} \prod_{j=1}^m \left\|x^j\right\| \left(\int_{B_{Y^{**}}} \left(|\varphi(y^*)|^{1-\sigma} \|y^*\|^\sigma\right)^{\frac{p^*}{1-\sigma}} d\mu\right)^{\frac{1-\sigma}{p^*}} \end{aligned}$$

then $T \in \mathcal{D}_p^{m,\sigma}(X_1, \ldots, X_m; Y)$ and $\|T\|_{\mathcal{D}_p^{m,\sigma}} \leq \|T\|_{\mathcal{D}_q^{m,\sigma}}$. $\square$

We show in what follows that the multi-ideal generated by the *composition method* from the operator ideal $\mathcal{D}_p^\sigma$ coincide with the space of strongly (p,σ)-continuous multilinear operators.

Proposition 3.2.5. *For* $1 < p, r \leq \infty$ *and* $0 \leq \sigma < 1$ *such that* $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. *We have* T *is strongly* (p,σ)*-continuous* m*-linear operator if and only if its linearization* T_L *is strongly* (p,σ)*-continuous linear operator. In this case* $\|T\|_{\mathcal{D}_p^{m,\sigma}} = d_p^\sigma(T_L)$.

Proof. Suppose that T_L is strongly (p,σ)-continuous. Let $x^j \in X_j (j = 1, ..., m)$ and $y^* \in Y^*$. By Theorem 3.1.4 there exists a regular Borel probability measure μ on $B_{Y^{**}}$, (with the weak star topology) such that

$$\begin{aligned} \left|\left\langle T(x^1, ..., x^m), y^*\right\rangle\right| &= \left|\left\langle T_L(x^1 \otimes ... \otimes x^m), y^*\right\rangle\right| \\ &\leq d_p^\sigma(T_L) \pi(x^1 \otimes ... \otimes x^m) \left(\int_{B_{Y^{**}}} \left(|\varphi(y^*)|^{1-\sigma} \|y^*\|^\sigma\right)^{\frac{p^*}{1-\sigma}} d\mu\right)^{\frac{1-\sigma}{p^*}} \\ &= d_p^\sigma(T_L) \prod_{j=1}^m \left\|x^j\right\| \left(\int_{B_{Y^{**}}} \left(|\varphi(y^*)|^{1-\sigma} \|y^*\|^\sigma\right)^{\frac{p^*}{1-\sigma}} d\mu\right)^{\frac{1-\sigma}{p^*}}. \end{aligned}$$

From Theorem 3.2.3 we get $T \in \mathcal{D}_p^{m,\sigma}(X_1, ..., X_m; Y)$ and $\|T\|_{\mathcal{D}_p^{m,\sigma}} \leq d_p^\sigma(T_L)$.
Conversely, suppose that T is an strongly (p,σ)-continuous m-linear operator. Let $v = \sum_{i=1}^{n} x_i^1 \otimes \cdots \otimes x_i^m \in X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m$ such that $v \neq 0$ and $y^* \in Y^*$. Then there is a regular Borel probability measure μ on $B_{Y^{**}}$, such that

$$\begin{aligned}
|\langle T_L(v), y^*\rangle| &= \left|\sum_{i=1}^{n} \langle T_L(x_i^1 \otimes ... \otimes x_i^m), y^*\rangle\right| \\
&\leq \sum_{i=1}^{n} \left|\langle T(x_i^1, \ldots, x_i^m), y^*\rangle\right| \\
&\leq \sum_{i=1}^{n} \|T\|_{\mathcal{D}_p^{m,\sigma}} \prod_{j=1}^{m} \|x_i^j\| \left(\int_{B_{Y^{**}}} (|\varphi(y^*)|^{1-\sigma} \|y^*\|^\sigma)^{\frac{p^*}{1-\sigma}} d\mu\right)^{\frac{1-\sigma}{p^*}} \\
&= \|T\|_{\mathcal{D}_p^{m,\sigma}} (\sum_{i=1}^{n} \prod_{j=1}^{m} \|x_i^j\|) \left(\int_{B_{Y^{**}}} (|\varphi(y^*)|^{1-\sigma} \|y^*\|^\sigma)^{\frac{p^*}{1-\sigma}} d\mu\right)^{\frac{1-\sigma}{p^*}}.
\end{aligned}$$

Taking the infimum over all representations of v we get

$$|\langle T_L(v), y^*\rangle| \leq \|T\|_{\mathcal{D}_p^{m,\sigma}} \pi(v) \left(\int_{B_{Y^{**}}} (|\varphi(y^*)|^{1-\sigma} \|y^*\|^\sigma)^{\frac{p^*}{1-\sigma}} d\mu\right)^{\frac{1-\sigma}{p^*}}.$$

Therefore, by Theorem 3.1.4, the linear mapping T_L is strongly (p,σ)-continuous and

$$\|T\|_{\mathcal{D}_p^{m,\sigma}} = d_p^\sigma(T_L).$$

□

As a consequence, we obtain the following corollary which is a straightforward consequence of the preceding proposition, Theorem 1.4.5 and Proposition 1.4.6.

Corollary 3.2.6. *The class $\mathcal{D}_p^{m,\sigma}$ is the Banach multi-ideal generated by the composition method from the Banach operator ideal $\mathcal{D}_p^\sigma$, i.e.,*

$$\mathcal{D}_p^{m,\sigma}(X_1, \ldots, X_m; Y) = \mathcal{D}_p^\sigma \circ \mathcal{L}(X_1, \ldots, X_m; Y)$$

for all Banach spaces $X_1, \ldots, X_m$ and Y.

The preceding corollary has more straightforward consequences.

Remark 3.2.7. *Every strongly (p,σ)-continuous m-linear operator with a reflexive range is compact.* Indeed, if $T \in \mathcal{D}_p^{m,\sigma}(X_1, \ldots, X_m; Y)$, then there is a Banach space G, an operator $u \in \mathcal{D}_p^\sigma(G; Y)$ and $R \in \mathcal{L}(X_1, \ldots, X_m; G)$ such that $T = u \circ R$. Since every strongly (p,σ)-continuous linear operator with a reflexive range is compact (see Remark 3.1.7), the mappings T is compact.

A natural question is to study the connection between multilinear operators and their adjoints for different classes of summability. In the next result, we characterize the class of strongly (p,σ)-continuous m-linear operators by using the adjoint operators in a similar way that is done in the linear case.

Theorem 3.2.8. *Let $1 < p \leq \infty$, $0 \leq \sigma < 1$, $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ and T^* its adjoint. Then T is strongly (p, σ)-continuous if and only if T^* is (p^*, σ)-absolutely continuous. In this case,*

$$\|T\|_{\mathcal{D}_p^{m,\sigma}} = \pi_{p^*,\sigma}(T^*).$$

Proof. By the Proposition 3.2.5 we have that T belongs to $D_p^{m,\sigma}(X_1, \ldots, X_m; Y)$ if and only if its linearization T_L is strongly (p, σ)-continuous and by (3.5) this is equivalent to $(T_L)^* : Y^* \to \left(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m\right)^*$ is (p^*, σ)-absolutely continuous. In the other hand, we have $(T_L)^* = \Psi \circ T^*$ (then $T^* = \Psi^{-1} \circ (T_L)^*$) where

$$\Psi : \mathcal{L}\left(X_1, \ldots, X_m\right) \longrightarrow (X_1 \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi X_m)^*$$

is the isomorphism isometry given by $\Psi(\phi) = \phi_L$ (see Theorem 1.3.10). By the ideal property concerning the operator ideal $\Pi_{p^*,\sigma}$, we have that $(T_L)^*$ is (p^*, σ)-absolutely continuous if and only if $T^* \in \Pi_{p^*,\sigma}(Y^*, \mathcal{L}\left(X_1, \ldots, X_m\right))$ and

$$\pi_{p^*,\sigma}(T^*) = \pi_{p^*,\sigma}((T_L)^*) = d_p^\sigma(T_L) = \|T\|_{\mathcal{D}_p^{m,\sigma}}$$

□

Corollary 3.2.9. *For a Banach space Y, the following assertions are equivalent.*

(i) $id_Y \in \mathcal{D}_p^\sigma(Y; Y)$.

(ii) $\mathcal{D}_p^{m,\sigma}(X_1, \ldots, X_m; Y) = \mathcal{L}(X_1, \ldots, X_m; Y)$ for all Banach spaces $X_1, \ldots, X_m$ and Y.

(iii) Y is finite dimensional.

Proof. By Proposition 1.4.8 we have, id_Y is strongly (p, σ)-continuous if and only if

$$\mathcal{D}_p^\sigma \circ \mathcal{L}(X_1, \ldots, X_m; Y) = \mathcal{L}(X_1, \ldots, X_m; Y).$$

Then the equivalence between (i) and (ii) is obtained by Corollary 3.2.6. For the implication (ii) $\Rightarrow$ (iii) we can define the multilinear mapping $T_0 : Y \times \mathbb{R} \times \cdots \times \mathbb{R} \to Y$ given by $T_0(y, r_1, \ldots, r_{m-1}) = y r_1 \cdots r_{m-1}$. It is clear that T_0 is continuous hence it belongs to $\mathcal{D}_p^{m,\sigma}(Y, \mathbb{R}, \ldots, \mathbb{R}; Y)$. Using the domination theorem for strongly (p, σ)-continuous multilinear operators (Theorem 3.2.3), we obtain for all $y^* \in Y^*$ and $y \in Y$

$$\begin{aligned} & |\langle T_0(y, r_1, \ldots, r_{m-1}), y^* \rangle| \\ = \; & |r_1, \ldots r_{m-1}| \, |\langle y, y^* \rangle| \\ \leq \; & C \, \|y\| \, |r_1| \ldots |r_{m-1}| \left(\int_{B_{Y^{**}}} \left(|\varphi(y^*)|^{1-\sigma} \, \|y^*\|^\sigma \right)^{\frac{p^*}{1-\sigma}} d\mu(\varphi) \right)^{\frac{1-\sigma}{p^*}} . \end{aligned}$$

Then

$$|\langle y, y^* \rangle| \leq C \, \|y\| \, \|y^*\|^\sigma \left(\int_{B_{Y^{**}}} |\varphi(y^*)|^{p^*} \, d\mu(\varphi) \right)^{\frac{1-\sigma}{p^*}} .$$

Taking the supremum over all $y \in B_Y$ we get

$$\|y^*\| = \|Id_{Y^*}(y^*)\| \leq C^{\frac{1}{1-\sigma}} \left(\int_{B_{Y^{**}}} |\varphi(y^*)|^{p^*} \, d\mu(\varphi) \right)^{\frac{1}{p^*}},$$

and so Id_{Y^*} is p^*-summing. Therefore, by the Dvoretsky-Rogers Theorem (see Theorem 1.2.7) Y^* is finite dimensional. To finish the proof let us show (iii) $\Rightarrow$ (i). If Y is finite dimensional, then Id_Y is a finite rank operator, and so $Id_Y \in \Pi_{p^*,\sigma}$. Therefore, by (3.5) we have $Id_Y \in \mathcal{D}_p^{\sigma}(Y;Y)$. □

3.2.2 Representation of the multi-Ideal $\mathcal{D}_p^{m,\sigma}$ by tensor norms

After the results of the previous subsections, we are ready to introduce a tensor norm which represents the multi-ideal $\mathcal{D}_p^{m,\sigma}$ i.e. to show that the strongly (p,σ)-continuous m-linear operators are a dual space of an $(m+1)$-fold tensor product.
Let $X_1, \ldots, X_m, X$ be Banach spaces. Let $u \in X_1 \otimes \cdots \otimes X_m \otimes X$, for $1 < p, r < \infty, 0 \leq \sigma < 1$ such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. We define

$$g_{p,\sigma}(u) = \inf \left(\sum_{i=1}^{n} \prod_{j=1}^{m} \|x_i^j\|^r \right)^{\frac{1}{r}} \delta_{p^*\sigma}((x_i)_{i=1}^n), \tag{3.9}$$

where the infimum is taken among all the representations of u as

$$u = \sum_{i=1}^{n} x_i^1 \otimes \cdots \otimes x_i^m \otimes x_i,$$

with $(x_i^j)_{i=1}^n \subset X_j, (x_i)_{i=1}^n \subset X, j = 1, \ldots, m$ and $n, m \in \mathbb{N}$.
Using the representations given in the proof of Proposition 3.2.2 we obtain a new formula for the norm $g_{p,\sigma}$,

$$g_{p,\sigma}(u) = \inf \prod_{j=1}^{m} \left\|(x_i^j)_{i=1}^n\right\|_{rm} \delta_{p^*\sigma}((x_i)_{i=1}^n). \tag{3.10}$$

Proposition 3.2.10. *$g_{p,\sigma}$ is a tensor norm of order $m+1$.*

Proof. Let $u', u'' \in X_1 \otimes ... \otimes X_m \otimes Y$, and let $\varepsilon > 0$. Choose representations of u' and u'' of the form

$$u' = \sum_{i=1}^{n'} x_i'^1 \otimes ... \otimes x_i'^m \otimes x_i', \quad u'' = \sum_{i=1}^{n''} x_i''^1 \otimes ... \otimes x_i''^m \otimes x_i'',$$

such that

$$g_{p,\sigma}(u') + \varepsilon \geq \left(\sum_{i=1}^{n'} \prod_{j=1}^{m} \|x_i'^j\|^r \right)^{\frac{1}{r}} \delta_{p^*\sigma}\left((x_i')_{i=1}^{n'}\right),$$

and

$$g_{p,\sigma}(u'')+\varepsilon \geq \left(\sum_{i=1}^{n''}\prod_{j=1}^{m}\left\|x_i''^j\right\|^r\right)^{\frac{1}{r}}\delta_{p^*\sigma}\left((x_i'')_{i=1}^{n''}\right).$$

We can write u', u'' in the following way

$$u'=\sum_{i=1}^{n'} z_i'^1\otimes ...\otimes z_i'^m\otimes t_i', \quad u''=\sum_{i=1}^{n''} z_i''^1\otimes ...\otimes z_i''^m\otimes t_i'',$$

with

$$\begin{aligned}
z_i'^1 &= \frac{\delta_{p^*\sigma}\left((x_i')_{i=1}^{n'}\right)}{(g_{p,\sigma}(u')+\varepsilon)^{\frac{1-\sigma}{p^*}}}x_i'^1, \quad \text{and} \quad z_i'^j = x_i'^j, j=2,...,m, i=1,...,n'\\
t_i' &= \frac{(g_{p,\sigma}(u')+\varepsilon)^{\frac{1-\sigma}{p^*}}}{\delta_{p^*\sigma}\left((x_i')_{i=1}^{n'}\right)}x_i', \quad i=1,...,n'\\
z_i''^1 &= \frac{\delta_{p^*\sigma}\left((x_i'')_{i=1}^{n''}\right)}{(g_{p,\sigma}(u'')+\varepsilon)^{\frac{1-\sigma}{p^*}}}x_i''^1, \quad \text{and} \quad z_i''^j = x_i''^j, j=2,...,m, i=1,...,n''\\
t_i'' &= \frac{(g_{p,\sigma}(u'')+\varepsilon)^{\frac{1-\sigma}{p^*}}}{\delta_{p^*\sigma}\left((x_i'')_{i=1}^{n''}\right)}x_i'', \quad i=1,...,n''.
\end{aligned}$$

It follows that

$$\delta_{p^*\sigma}\left((t_i')_{i=1}^{n'}\right)=(g_{p,\sigma}(u')+\varepsilon)^{\frac{1-\sigma}{p^*}} \quad \text{and} \quad \left(\sum_{i=1}^{n'}\prod_{j=1}^{m}\left\|z_i'^j\right\|^r\right)^{\frac{1}{r}}\leq (g_{p,\sigma}(u')+\varepsilon)^{\frac{1}{r}}$$

$$\delta_{p^*\sigma}\left((t_i'')_{i=1}^{n''}\right)=(g_{p,\sigma}(u'')+\varepsilon)^{\frac{1-\sigma}{p^*}} \quad \text{and} \quad \left(\sum_{i=1}^{n''}\prod_{j=1}^{m}\left\|z_i''^j\right\|^r\right)^{\frac{1}{r}}\leq (g_{p,\sigma}(u'')+\varepsilon)^{\frac{1}{r}}.$$

Thus

$$\delta_{p^*\sigma}\left((t_i')_{i=1}^{n'}\right)\cdot\left(\sum_{i=1}^{n'}\prod_{j=1}^{m}\left\|z_i'^j\right\|^r\right)^{\frac{1}{r}}\leq g_{p,\sigma}(u')+\varepsilon \leq g_{p,\sigma}(u')+g_{p,\sigma}(u'')+\varepsilon,$$

$$\delta_{p^*\sigma}\left((t_i'')_{i=1}^{n''}\right)\cdot\left(\sum_{i=1}^{n''}\prod_{j=1}^{m}\left\|z_i''^j\right\|^r\right)^{\frac{1}{r}}\leq g_{p,\sigma}(u'')+\varepsilon \leq g_{p,\sigma}(u')+g_{p,\sigma}(u'')+\varepsilon.$$

The two last inequalities imply that

$$g_{p,\sigma}(u'+u'')\leq g_{p,\sigma}(u')+g_{p,\sigma}(u'')+\varepsilon,$$

for all $\varepsilon>0$, hence the triangular inequality is proved for $g_{p,\sigma}$.
We show that $g_{p,\sigma}(\lambda u)=|\lambda|\, g_{p,\sigma}(u)$ for all $u\in X_1\otimes ...\otimes X_m\otimes X$ and $\lambda\in\mathbb{K}$. This is obvious when $\lambda=0$, so suppose that $\lambda\neq 0$. If $u=\sum_{i=1}^{n} x_i^1\otimes ...\otimes x_i^m\otimes x_i$ is a

representation of u then $\lambda u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \otimes (\lambda x_i)$, and so we have

$$\begin{aligned} g_{p,\sigma}(\lambda u) &\leq \left(\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_i^j\right\|^r\right)^{\frac{1}{r}} \delta_{p^*\sigma}\left((\lambda x_i)_{i=1}^n\right) \\ &= |\lambda| \left(\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_i^j\right\|^r\right)^{\frac{1}{r}} \delta_{p^*\sigma}\left((x_i)_{i=1}^n\right). \end{aligned}$$

It follows that $g_{p,\sigma}(\lambda u) \leq |\lambda|\, g_{p,\sigma}(u)$. In the same way, we have

$$g_{p,\sigma}(u) = g_{p,\sigma}(\frac{1}{\lambda}\lambda u) \leq \frac{1}{|\lambda|} g_{p,\sigma}(\lambda u),$$

giving $|\lambda|\, g_{p,\sigma}(u) \leq g_{p,\sigma}(u)$. Therefore $g_{p,\sigma}(\lambda u) = |\lambda|\, g_{p,\sigma}(u)$.
Let $\phi_j \in B_{X_j^*}, \psi \in B_{X^*} (j = 1, \ldots, m)$ and let

$$u = \sum_{i=1}^{n} x_i^1 \otimes \cdots \otimes x_i^m \otimes x_i \in X_1 \otimes \cdots \otimes X_m \otimes X.$$

It follows directly from Hölder's inequality and (2.1) that

$$\begin{aligned} \left|\sum_{i=1}^{n} \phi_1(x_i^1)\cdots\phi_m(x_i^m)\psi(x_i)\right| &\leq \left(\sum_{i=1}^{n}\prod_{j=1}^{m}\left|\phi_j(x_i^j)\right|^r\right)^{\frac{1}{r}} \left(\sum_{i=1}^{n}|\psi(x_i)|^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}} \\ &\leq \left(\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_i^j\right\|^r\right)^{\frac{1}{r}} \|(x_i)_{i=1}^n\|_{\frac{p^*}{1-\sigma},\omega} \\ &\leq \left(\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_i^j\right\|^r\right)^{\frac{1}{r}} \delta_{p^*\sigma}((x_i)_{i=1}^n). \end{aligned}$$

Then, if ϵ is the injective norm, we have

$$\epsilon(u) = \sup_{\phi_j \in B_{X_j^*}, \psi \in B_{X^*}} \left|\sum_{i=1}^{n} \phi_1(x_i^1)\cdots\phi_m(x_i^m)\psi(x_i)\right| \leq \left(\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_i^j\right\|^r\right)^{\frac{1}{r}} \delta_{p^*\sigma}((x_i)_{i=1}^n).$$

Since it holds for every representation of u, consequently $\epsilon(u) \leq g_{p,\sigma}(u)$. Thus $g_{p,\sigma}(u) = 0$ imply $u = 0$. Hence $g_{p,\sigma}$ is a norm on $X_1 \otimes ... \otimes X_m \otimes X$ and $\epsilon \leq g_{p,\sigma}$.
On the other hand we have

$$\begin{aligned} g_{p,\sigma}(u) &\leq \left(\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_i^j\right\|^r\right)^{\frac{1}{r}} \delta_{p^*\sigma}((x_i)_{i=1}^n) \\ &\leq \left(\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_i^j\right\|^r\right)^{\frac{1}{r}} \left(\sum_{i=1}^{n}\|x_i\|^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}. \end{aligned}$$

By replacing in the representation of u the x_i^j by

$$\frac{\left(\prod_{k=1}^{m}\left\|x_i^k\right\|\left\|x_i\right\|\right)^{\frac{1}{rm}}}{\left\|x_i^j\right\|}x_i^j,$$

and x_i by

$$\frac{\left(\prod_{k=1}^{m}\left\|x_i^k\right\|\left\|x_i\right\|\right)^{\frac{1-\sigma}{p^*}}}{\left\|x_i\right\|}x_i,$$

a simple calculation gives

$$g_{p,\sigma}(u)\leq\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_i^j\right\|\left\|x_i\right\|.$$

Taking the infimum over all representations of u, we find $g_{p,\sigma}(u)\leq\pi(u)$. And we have shown that $g_{p,\sigma}$ is a reasonable crossnorm.
It only remains to show that $g_{p,\sigma}$ satisfies the metric mapping property. Let X_j, Y_j, X, Y be normed vector spaces and let $s\in\mathcal{L}(X,Y)$, $s_j\in\mathcal{L}(X_j,Y_j)$, $(j=1,...,m)$. We may suppose that $s_j\neq 0$, $j=1,...,m$ and $s\neq 0$. Let

$$u=\sum_{i=1}^{n}x_i^1\otimes...\otimes x_i^m\otimes x_i\in X_1\otimes...\otimes X_m\otimes X,$$

hence the sum

$$\sum_{i=1}^{n}s_1(x_i^1)\otimes...\otimes s_m(x_i^m)\otimes s\left(x_i\right)$$

is a representation of $s_1\otimes...\otimes s_m\otimes s(u)$ in $Y_1\otimes...\otimes Y_m\otimes Y$. Then,

$$\begin{aligned}
& g_{p,\sigma}\left(s_1\otimes...\otimes s_m\otimes s(u)\right)\\
\leq\ & \left(\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|s_j(x_i^j)\right\|^r\right)^{\frac{1}{r}}\delta_{p^*\sigma}\left(\left(s\left(x_i\right)\right)_{i=1}^n\right)\\
\leq\ & \|s\|\prod_{j=1}^{m}\|s_j\|\left(\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_i^j\right\|^r\right)^{\frac{1}{r}}\delta_{p^*\sigma}\left(\left(x_i\right)_{i=1}^n\right).
\end{aligned}$$

Since it holds for every representation of u, consequently

$$g_{p,\sigma}\left(s_1\otimes...\otimes s_m\otimes s(u)\right)\leq\|s\|\prod_{j=1}^{m}\|s_j\|\,g_{p,\sigma}(u).$$

So that the linear operator

$$s_1\otimes...\otimes s_m\otimes s:\left(X_1\otimes_{g_{p,\sigma}}...\otimes_{g_{p,\sigma}}X_m\otimes_{g_{p,\sigma}}X\right)\to\left(Y_1\otimes_{g_{p,\sigma}}...\otimes_{g_{p,\sigma}}Y_m\otimes_{g_{p,\sigma}}Y\right)$$

is continuous and we have

$$\|s_1 \otimes ... \otimes s_m \otimes s\| \leq \|s\| \prod_{j=1}^{m} \|s_j\| .$$

□

The main result of this subsection is the following

Theorem 3.2.11. *Let $X_1, \ldots, X_m, Y$ be Banach spaces and let $1 < p, r < \infty, 0 \leq \sigma < 1$ such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. The space $(\mathcal{D}_p^{m,\sigma}(X_1, \ldots, X_m; Y), \|.\|_{\mathcal{D}_p^{m,\sigma}})$ is isometrically isomorphic to $(X_1 \otimes \cdots \otimes X_m \otimes Y^*, g_{p,\sigma})^*$.*

Proof. It is easy to see that the correspondence

$$\Psi : (\mathcal{D}_p^{m,\sigma}(X_1, \ldots, X_m; Y), \|.\|_{\mathcal{D}_p^{m,\sigma}}) \to (X_1 \otimes \cdots \otimes X_m \otimes Y^*, g_{p,\sigma})^*$$

defined by

$$\Psi(T)(x^1 \otimes \cdots \otimes x^m \otimes y^*) = \left\langle T(x^1, \ldots, x^m), y^* \right\rangle,$$

for every $T \in (\mathcal{D}_p^{m,\sigma}(X_1, \ldots, X_m; Y)$, $x^j \in X_j$ $(j = 1, \ldots, m)$ and $y^* \in Y^*$, is linear. It remains to show the surjectivity and that

$$\|\Psi(T)\|_{(X_1 \otimes \cdots \otimes X_m \otimes Y^*, g_{p,\sigma})^*} = \|T\|_{\mathcal{D}_p^{m,\sigma}} .$$

Take $T \in (\mathcal{D}_p^{m,\sigma}(X_1, \ldots, X_m; Y)$ and let $u \in X_1 \otimes \cdots \otimes X_m \otimes Y^*$ with the representation

$$u = \sum_{i=1}^{n} x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i^*.$$

We have

$$\begin{aligned} |\Psi(T)(u)| &= \left| \sum_{i=1}^{n} \langle T(x_i^1, \ldots, x_i^m), y_i^* \rangle \right| \\ &\leq \|T\|_{\mathcal{D}_p^{m,\sigma}} \left(\sum_{i=1}^{n} \prod_{j=1}^{m} \left\|x_i^j\right\|^r \right)^{\frac{1}{r}} \delta_{p^*\sigma}((y_i^*)_{i=1}^n). \end{aligned}$$

Since it holds for every representation of u we get

$$|\Psi(T)(u)| \leq \|T\|_{\mathcal{D}_p^{m,\sigma}} g_{p,\sigma}(u).$$

It follows that

$$\|\Psi(T)\|_{(X_1 \otimes \cdots \otimes X_m \otimes Y^*, g_{p,\sigma})^*} \leq \|T\|_{\mathcal{D}_p^{m,\sigma}} .$$

In order to establish the reverse inequality, let $\phi \in (X_1 \otimes \cdots \otimes X_m \otimes Y^*, g_{p,\sigma})^*$, define the m-linear mapping $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ by

$$\langle T(x^1,\ldots,x^m),y^*\rangle = \phi(x^1\otimes\cdots\otimes x^m\otimes y^*).$$

Let $(x_i^j)_{i=1}^n \subset X_j (j=1,\ldots,m)$ and $(y_i^*)_{i=1}^n \subset Y^*$, for $\lambda_1,\ldots,\lambda_n \geq 0$ we can write

$$\begin{aligned}
&\left|\sum_{i=1}^n \lambda_i \langle T(x_i^1,\ldots,x_i^m),y_i^*\rangle\right| \\
= &\left|\sum_{i=1_i}^n \lambda_i \phi\left(x_i^1\otimes\cdots\otimes x_i^m\otimes y_i^*\right)\right| \\
\leq &\|\phi\|\, g_{p,\sigma}\left(\sum_{i=1}^n \lambda_i x_i^1\otimes\cdots\otimes x_i^m\otimes y_i^*\right) \\
\leq &\|\phi\|\,\|(\lambda_i)_{i=1}^n\|_\infty \left(\sum_{i=1}^n\prod_{j=1}^m \left\|x_i^j\right\|^r\right)^{\frac{1}{r}} \delta_{p^*\sigma}\left((y_i^*)_{i=1}^n\right).
\end{aligned}$$

By taking the supremum over all $\lambda_1,\ldots,\lambda_n \geq 0$ with $\|(\lambda_i)_{i=1}^n\|_\infty \leq 1$ and using the equality (1.4) we get that

$$\left\|\left(\langle T(x_i^1,\ldots,x_i^m),y_i^*\rangle\right)_{i=1}^n\right\|_1 \leq \|\phi\| \left(\sum_{i=1}^n\prod_{j=1}^m \left\|x_i^j\right\|^r\right)^{\frac{1}{r}} \delta_{p^*\sigma}\left((y_i^*)_{i=1}^n\right);$$

this becomes $T \in \mathcal{D}_p^{m,\sigma}(X_1,\ldots,X_m;Y)$ and

$$\|T\|_{\mathcal{D}_p^{m,\sigma}} \leq \|\phi\| = \|\Psi(T)\|_{(X_1\otimes\cdots\otimes X_m\otimes Y^*, g_{p,\sigma})^*}.$$

□

3.2.3 The factorization theorem

Let $X_1,\ldots,X_m,Y$ be Banach spaces, $1 \leq p,r < \infty$, $0 \leq \sigma < 1$ such that $\frac{1}{r}+\frac{1-\sigma}{p^*}=1$ and a regular Borel probability measure η on $B_{Y^{**}}$, (with the weak star topology). We denote by e the isometric embedding $Y^* \to C(B_{Y^{**}})$ given by $e(y^*) = \langle y^*,\cdot\rangle$. For $f \in e(Y^*)$ consider the seminorm

$$\|f\|_{p,\sigma} = \inf \sum_{k=1}^n \|f_k\|_{e(Y^*)}^\sigma \left(\int_{B_{Y^{**}}} |f_k|^p\, d\eta\right)^{\frac{1-\sigma}{p}},$$

the infimum computed over all decompositions of f as $f = \sum_{k=1}^n f_k$ in $e(Y^*)$. Following (see subsection 2.1.3), let $L_{p,\sigma}(\eta)$ be the completion of the quotient normed space

$$e(B_{Y^*})/\|\cdot\|_{p,\sigma}^{-1}(0)$$

of all classes of functions as $\langle y^*,\cdot\rangle \in e(B_{Y^*}) \subset C(B_{Y^{**}})$, $y^* \in Y^*$, with the quotient norm $\|\cdot\|_{p,\sigma}$. Let us call $J_{p,\sigma} : e(Y^*) \to L_{p,\sigma}(\eta)$ the projection on the quotient.

In what follows, let the m-linear mapping $K : X_1 \times \cdots \times X_m \to \mathcal{L}(X_1, \ldots, X_m)^*$ defined by

$$K(x^1, \ldots, x^m)(\phi) := \phi(x^1, \ldots, x^m) \text{ for all } \phi \in \mathcal{L}(X_1, \ldots, X_m). \tag{3.11}$$

For the proof of the Factorization Theorem concerning the class $\mathcal{D}_p^{m,\sigma}$ we need the following proposition.

Proposition 3.2.12. *The multilinear mapping K, us defined by (3.11), is continuous and $\|K\| = 1$. On the other hand, if $T \in \mathcal{L}(X_1, ..., X_m; Y)$ and $k_Y : Y \hookrightarrow Y^{**}$ is the natural embedding. Then the following diagram commutes*

$$\begin{array}{ccc} X_1 \times \cdots \times X_m & \xrightarrow{\quad K \quad} & \mathcal{L}(X_1, \ldots, X_m)^* \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle T^{**}} \\ Y & \xrightarrow{\quad k_Y \quad} & Y^{**} \end{array}$$

i.e., $k_Y \circ T = T^{**} \circ K$.

Proof. By the definition of the m-linear mapping K, we obtain directly

$$\begin{aligned} \|K\| &= \sup_{\|x^j\| \le 1, j=1,\ldots,m} \left\| K(x^1, ..., x^m) \right\| \\ &= \sup_{\|x^j\| \le 1, j=1,\ldots,m} \left\{ \sup_{\|\phi\| \le 1} \left\| K(x^1, ..., x^m)(\phi) \right\| \right\} \\ &= \sup_{\|\phi\| \le 1} . \sup_{\|x^j\| \le 1, j=1,\ldots,m} \left| \phi(x^1, ..., x^m) \right| \\ &= \sup_{\|\phi\| \le 1} \|\phi\| \\ &= 1. \end{aligned}$$

For all $x^j \in X_j (j = 1, ..., m)$ and $y^* \in Y^*$ we have

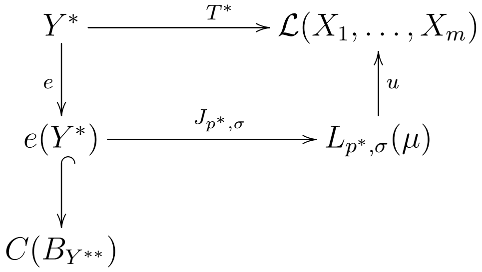
$$\begin{aligned} (k_Y \circ T)(x^1, ..., x^m)(y^*) &= y^*(T(x^1, ..., x^m)) \\ &= T^*(y^*)(x^1, ..., x^m) \\ &= K(x^1, ..., x^m)(T^* y^*) \\ &= T^{**} \circ K(x^1, ..., x^m)(y^*) \end{aligned}$$

this gives $k_Y \circ T = T^{**} \circ K$. □

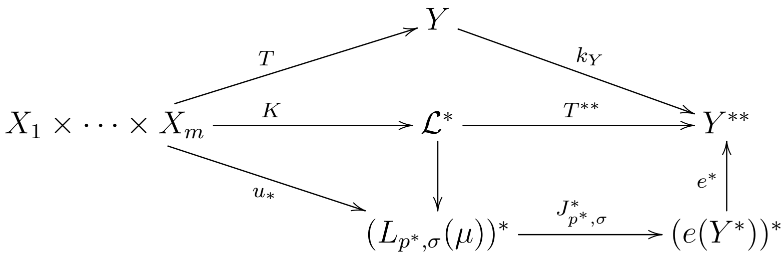
Theorem 3.2.13. *(Factorization Theorem)*
For every multilinear operator $T : X_1 \times \cdots \times X_m \to Y$, the following statements are equivalent.
(i) T is strongly (p, σ)-continuous.
(ii) There exist a regular Borel probability measure μ on $B_{Y^{**}}$ and a continuous m-linear mapping $u_* : X_1 \times \cdots \times X_m \to (L_{p^*,\sigma}(\mu))^*$ such that

$$k_Y \circ T = e^* \circ J^*_{p^*,\sigma} \circ u_*.$$

Proof. To simplify the notation, let us write $\mathcal{L}^*$ instead of $\mathcal{L}(X_1,\ldots,X_m)^*$.
$(i) \Rightarrow (ii)$ Assume that T is strongly (p,σ)-continuous. By Theorem 3.2.8 we have that $T^* : Y^* \to \mathcal{L}(X_1,\ldots,X_m)$ is (p^*,σ)-absolutely continuous with $\|T\|_{\mathcal{D}_p^{m,\sigma}} = \pi_{p^*,\sigma}(T^*)$. By Theorem 2.1.20, there exist a regular Borel probability measure μ on $B_{Y^{**}}$ and a bounded linear operator u such that the following diagram commutes,



since $L_{p^*,\sigma}(\mu)$ is the closure of $(J_{p^*,\sigma}\circ e)(Y^*)$. By transposing the diagram above and the preceding proposition, we obtain the following diagram, which commutes



where u_* is the m-linear mapping defined by

$$\begin{aligned}\langle u_*(x^1,\ldots,x^m), J_{p^*,\sigma}\circ e(y^*)\rangle \quad &:= \langle u^*\circ K(x^1,\ldots,x^m), J_{p^*,\sigma}\circ e(y^*)\rangle \\ &= K(x^1,\ldots,x^m)\left(u\left(J_{p^*,\sigma}\circ e(y^*)\right)\right) \\ &= u\left(J_{p^*,\sigma}\circ e(y^*)\right)(x^1,\ldots,x^m)\end{aligned}$$

for all $y^*\in Y^*, x^j\in X_j (j=1,\ldots,m)$. It is clear that u_* is well-defined because

$$u\left(J_{p^*,\sigma}\circ e(y^*)\right)\in\mathcal{L}(X_1,...,X_m).$$

On the other hand we have

$$\begin{aligned}\left|\left\langle u_*(x^1,\ldots,x^m), J_{p^*,\sigma}\circ e(y^*)\right\rangle\right| \quad &= \quad \left|u(J_{p^*,\sigma}\circ e(y^*))(x^1,\ldots,x^m)\right| \\ &\leq \quad \|u(J_{p^*,\sigma}\circ e(y^*))\| \prod_{j=1}^m \left\|x^j\right\| \\ &\leq \quad \|u\|\,\|J_{p^*,\sigma}\circ e(y^*)\|_{L_{p^*,\sigma}(\mu)} \prod_{j=1}^m \left\|x^j\right\|.\end{aligned}$$

We take the supremum over all $J_{p^*,\sigma}\circ e(y^*)$ with $\|J_{p^*,\sigma}\circ e(y^*)\|_{L_{p^*,\sigma}(\mu)} \leq 1$ in order to obtain

$$\left\|u_*(x^1,\ldots,x^m)\right\| \leq \|u\| \prod_{j=1}^m \left\|x^j\right\|.$$

Therefore, u_* is continuous with norm $\leq \|u\|$.
$(ii) \Rightarrow (i)$ Assume that $k_Y \circ T = e^* \circ J^*_{p^*,\sigma} \circ u_*$. The natural inclusion/quotient mapping $J_{p^*,\sigma}$ is (p^*, σ)-absolutely continuous (see Lemma 2.1.19). Then $J^*_{p^*,\sigma}$ is strongly (p, σ)-continuous by Corollary 3.1.9. Consequently, the m-linear mapping is $J^*_{p,\sigma} \circ u_*$ is strongly (p, σ)-continuous by Corollary 3.2.6 and so

$$e^* \circ J^*_{p^*,\sigma} \circ u_* = k_Y \circ T \in \mathcal{D}^{m,\sigma}_p(X_1, \ldots, X_m; Y^{**})$$

by the ideal property.
It remains to show that the mapping T is in $\mathcal{D}^{m,\sigma}_p(X_1, \ldots, X_m; Y)$. By Theorem 3.2.8 we have

$$(k_Y \circ T)^* = T^* \circ k^*_Y \in \Pi_{p^*,\sigma}(Y^{***}, \mathcal{L}(X_1, \ldots, X_m)).$$

The fact that $k^*_Y \circ k_{Y^*} = id_{Y^*}$ implies

$$T^* = (T^* \circ k^*_Y) \circ k_{Y^*} \in \Pi_{p^*,\sigma}(Y^*, \mathcal{L}(X_1, \ldots, X_m)).$$

Thus $T \in \mathcal{D}^{m,\sigma}_p(X_1, \ldots, X_m; Y)$. □

A direct consequence of the previous theorem is the following

Corollary 3.2.14. *A multilinear operator $T : X_1 \times \cdots \times X_m \to Y$ belongs to $\mathcal{D}^{m,\sigma}_p(X_1, \ldots, X_m; Y)$ if and only if $T^{**} \in \mathcal{D}^{\sigma}_p\big(\mathcal{L}(X_1, \ldots, X_m)^*, Y^{**}\big)$.*

For the case $m = 1$ we obtain the factorization theorem for the linear case, which as we said in the introduction of this chapter is also new. Let us write it separately.

Theorem 3.2.15. *For every linear operator $T : X \to Y$, the following statements are equivalent.*
(i) T is strongly (p, σ)-continuous.
(ii) There exist a regular Borel probability measure μ on $B_{Y^{**}}$ and a continuous linear mapping $u_* : X \to (L_{p^*,\sigma}(\mu))^*$ such that

$$k_Y \circ T = e^* \circ J^*_{p^*,\sigma} \circ u_*.$$

Let us finish this chapter by writing the factorization theorem for Cohen strongly p-summing multilinear operators. The linear case (i.e. for $m = 1$) is also new. Putting $\sigma = 0$ in Theorem 3.2.13, we obtain the following

Theorem 3.2.16. *For every multilinear operator $T : X_1 \times \cdots \times X_m \to Y$, the following assertions are equivalent.*
(i) T is Cohen strongly p-summing.
(ii) There exist a regular Borel probability measure μ on $B_{Y^{**}}$, a subspace S of the Lebesgue space $L_{p^*}(\mu)$ and a continuous m-linear mapping $u_* : X_1 \times \cdots \times X_m \to S^*$ such that

$$k_Y \circ T = e^* \circ J^*_{p^*} \circ u_*.$$

Chapter 4

Absolutely continuous polynomials

The results obtained in this chapter have been published in the Journal of Mathematical Analysis and Applications [3]. In this chapter we introduce and study the polynomial version of the $(p; p_1, \ldots, p_m; \sigma)$-absolutely continuous multilinear operators, that will be called $(p; q; \sigma)$-absolutely continuous polynomials. Special attention is paid to the particular case of dominated $(p; \sigma)$-continuous polynomials. Inspired by the factorization theorem for dominated polynomials [19], we prove a more general factorization scheme for dominated $(p; \sigma)$-continuous polynomials. The factorization in [19] is based in finding a prototype of a p-dominated polynomial with values in a linear subspace of an L_p space, endowed with a suitable norm, through which any p-dominated polynomial factors. Our approximation to this problem is slightly different.
This chapter is organized as follows. In Section 1, we recall some notation and basic facts on polynomials on Banach spaces. In Section 2, we study and characterize the ideal of $(p; q; \sigma)$-absolutely continuous polynomials. In Section 3 we present a particular case: the dominated $(p; \sigma)$-continuous polynomials, where a factorization should apply. Far from being trivial, the expectations are met. First we establish a domination theorem for such operators similar to th one that holds in the m-linear case, comparing also dominated $(p; \sigma)$-continuous and p-dominated polynomials and, in Section 4, we show our main result: the factorization theorem for dominated $(p; \sigma)$-continuous polynomials.

4.1 Definitions and general results

A mapping $P : X \to Y$ is an *m-homogeneous polynomial* if there exists a unique symmetric m-linear mapping $\check{P} : X \times \overset{(m)}{\cdots} \times X \to Y$ such that $P(x) = \check{P}\left(x, \overset{(m)}{\ldots}, x\right)$ for every $x \in X$. Both are related by the polarization formula [50, Theorem 1.10]

$$\check{P}\left(x^1, \ldots, x^m\right) = \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_m P(\varepsilon_1 x^1 + \cdots + \varepsilon_m x^m), \quad (x^j)_{j=1}^m \subset X.$$

We denote by $\mathcal{P}(^m X;Y)$ the Banach space of all continuous m-homogeneous polynomials from X into Y endowed with the norm

$$\|P\| = \sup\{\|P(x)\| : \|x\| \leq 1\} = \inf\{C : \|P(x)\| \leq C\|x\|^m, x \in X\}.$$

The (finite) linear combinations of the m-homogeneous polynomials $x \to \phi(x)^m y$, where $\phi \in X^*$ and $y \in Y$, are called *polynomials of finite type*. For the general theory of homogeneous polynomials we refer to [28].

R. Ryan, in his thesis [67], introduced the projective tensor norm on the symmetric tensor product of Banach spaces to study homogeneous polynomials.
Let X be a Banach space and denote by $\otimes^{m,s}X = X \otimes \overset{(m)}{\cdots} \otimes X$ the m-fold symmetric tensor product of X. *The projective symmetric tensor norm*, π_s is the norm,

$$\pi_s(z) = \inf\left\{\sum_{i=1}^{n} \|x_i\|^m : m \in \mathbb{N}, z = \sum_{i=1}^{n} x_i \otimes \overset{(m)}{\cdots} \otimes x_i\right\},$$

and $\widehat{\otimes}_{\pi_s}^{m,s} X$ stands for the completion of $X \otimes_{\pi_s} \overset{(m)}{\cdots} \otimes_{\pi_s} X$. We use P^L to denote the *linearization* of the polynomial $P \in \mathcal{P}(^m X, Y)$, that is, P^L is a linear operator from $\widehat{\otimes}_{\pi_s}^{m,s} X$ into Y such that $P(x) = P^L(x \otimes \cdots \otimes x)$ for every $x \in X$. The correspondence between a polynomial and its linearization $P \leftrightarrow P_L$ establishes an isometric isomorphism between $\mathcal{P}(^m X, Y)$ and $\mathcal{L}(\widehat{\otimes}_{\pi_s}^{m,s} X, Y)$. For definitions and basic properties of symmetric tensor products, the s-tensor norm and the interplay with homogeneous polynomials we refer to [33].

In this chapter we follow the standard definition of ideal of polynomials which can be found for example in [15]. An ideal of homogeneous polynomials (or polynomial ideal) is a subclass $\mathcal{Q}$ of the class of every continuous homogeneous polynomials between Banach spaces such that, for all $m \in \mathbb{N}$ and any Banach spaces X and Y the components $\mathcal{Q}(^m X, Y) = \mathcal{P}(^m X, Y) \cap \mathcal{Q}$ satisfy:
(i) $\mathcal{Q}(^m X, Y)$ contains the m-homogeneous polynomials of finite type.
(ii) $\mathcal{Q}$ has the ideal property: if $u \in \mathcal{L}(E, X), P \in \mathcal{Q}(^m X, Y)$ and $v \in \mathcal{L}(Y, F)$, then $v \circ P \circ u$ is in $\mathcal{Q}(^m E, F)$.

$(\mathcal{Q}; \|.\|_{\mathcal{Q}})$ is a normed (Banach) polynomial ideal if
$(i\prime)$ $(\mathcal{Q}(^m X, Y), \|.\|_{\mathcal{Q}})$ is a normed (Banach) space for all X, Y,
$(ii\prime)$ $\|id_{\mathbb{K}^m} : \mathbb{K} \to \mathbb{K} : id_{\mathbb{K}^m}(x) = x^m\|_{\mathcal{Q}} = 1$ for all m,
$(iii\prime)$ If $u \in \mathcal{L}(E, X), P \in \mathcal{Q}(^m X, Y)$ and $v \in \mathcal{L}(Y, F)$, then $\|v \circ P \circ u\|_{\mathcal{Q}} \leq \|v\| \|P\|_{\mathcal{Q}} \|u\|^m$.

In the Definition 2.2.3, if we take $p_1 = \ldots = p_m = q$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m} = \frac{m}{q}$ we say that T is *dominated* (q, σ)*-continuous* and we denote the corresponding vector space and norm by $\mathcal{L}_{d,q}^{\sigma}(X_1, ..., X_m; Y)$ and $\|.\|_{\mathcal{L}_{d,q}^{\sigma}}$ respectively. In this case, the inequality (2.10) can be written as

$$\left\|\left(T\left(x_i^1, \ldots, x_i^m\right)\right)_{i=1}^n\right\|_{\frac{q}{m(1-\sigma)}} \leq C \prod_{j=1}^{m} \delta_{q\sigma}\left((x_i^j)_{i=1}^n\right). \tag{4.1}$$

4.2 $(p;q;\sigma)$-Absolutely continuous m-homogeneous polynomials

In this section we define and characterize the notion of $(p;q;\sigma)$-absolutely continuous m-homogeneous polynomials, according to the definition of $(p;p_1,\ldots,p_m;\sigma)$-absolutely continuous multilinear mappings.

Definition 4.2.1. *Let $m\in\mathbb{N}$, $1\leq p,q<+\infty$ such that $mp\geq q$ and $0\leq\sigma<1$. A polynomial $P\in\mathcal{P}(^mX;Y)$ is called $(p;q;\sigma)$-absolutely continuous if there exists a constant $C>0$ such that for every $(x_i)_{i=1}^n\subset X$,*

$$\|(P(x_i))_{i=1}^n\|_{\frac{p}{1-\sigma}}\leq C.\left(\delta_{q\sigma}((x_i)_{i=1}^n)\right)^m. \tag{4.2}$$

The space of all such polynomials is denoted by $\mathcal{P}^\sigma_{as(p,q)}(^mX,Y)$. It is equipped with the complete norm $\|.\|_{\mathcal{P}^\sigma_{as(p,q)}}$, which is computed as the infimum of all constants C such that the inequality (4.2) holds.

For $\sigma=0$ we have $\mathcal{P}^0_{as(p,q)}(^mX;Y)=\mathcal{P}_{p,q}(^mX;Y)$, the space of absolutely $(p;q)$-summing polynomials (see [44]).

Remark 4.2.2. Let us show some basic ways of constructing polynomials belonging to our new class. Let X, Y and Z be Banach spaces and let $m\in\mathbb{N}$, $1\leq p,q<+\infty$ *such that $mp\geq q$ and $0\leq\sigma<1$.*
(a) *Every m-homogeneous polynomial of finite type from X into Y is $(p;q;\sigma)$-absolutely continuous.* A simple calculation shows this result.
(b) *Let us show a particular example of the case mentioned above. Let $P=id_{\mathbb{K}^m}$ be the polynomial $id_{\mathbb{K}^m}:\mathbb{K}\to\mathbb{K}$ given by $id_{\mathbb{K}^m}(x)=x^m$. The following calculations show that it is $(p;q;\sigma)$-absolutely continuous and that $\|id_{\mathbb{K}^m}\|_{\mathcal{P}^\sigma_{as(p,q)}}=1$.* Let $(x_i)_{i=1}^n\subset\mathbb{K}$. By the inequality (2.1) we can write

$$\begin{aligned}\|(id_{\mathbb{K}^m}(x_i))_{i=1}^n\|_{\frac{p}{1-\sigma}} &= \|(x_i)_{i=1}^n\|^m_{\frac{mp}{1-\sigma}}\\ &\leq \|(x_i)_{i=1}^n\|^m_{\frac{q}{1-\sigma},\omega}\\ &\leq \left(\delta_{q\sigma}\left((x_i)_{i=1}^n\right)\right)^m.\end{aligned}$$

It follows that $id_{\mathbb{K}^m}\in\mathcal{P}^\sigma_{as(p,q)}(^m\mathbb{K};\mathbb{K})$ and $\|id_{\mathbb{K}^m}\|_{\mathcal{P}^\sigma_{as(p,q)}}\leq 1$. In fact, it can be easily shown that $\|id_{\mathbb{K}^m}\|_{\mathcal{P}^\sigma_{as(p,q)}}\geq\|id_{\mathbb{K}^m}\|=1$.
(c) *Let $Q\in\mathcal{P}(^mX;Y)$ and let $u:Z\to X$ be a $(p;\sigma)$-absolutely continuous linear operator. Then the polynomial $P=Q\circ u$ is $(\frac{p}{m};p;\sigma)$-absolutely continuous and*

$$\|P\|_{\mathcal{P}^\sigma_{as(\frac{p}{m},p)}}\leq\|Q\|\left(\pi_{p,\sigma}(u)\right)^m.$$

In order to see this, note that if $(z_i)_{i=1}^n\subset Z$, then

$$\|(P(z_i))_{i=1}^n\|_{\frac{p}{m(1-\sigma)}}\leq\|Q\|.\|u(z_i)_{i=1}^n\|^m_{\frac{p}{1-\sigma}}\leq\|Q\|.\left(\pi_{p,\sigma}(u)\right)^m.\left(\delta_{p\sigma}((z_i)_{i=1}^n)\right)^m.$$

As a consequence of parts (a) and (b) of the remark above and the next result — which proof is straightforward using calculation as in Remark 4.2.2(c)— we obtain that $\left(\mathcal{P}^{\sigma}_{as(p,q)}, \|.\|_{\mathcal{P}^{\sigma}_{as(p,q)}}\right)$ is a normed polynomial ideal.

Proposition 4.2.3. *(Ideal property). Let $u \in \mathcal{L}(X,G)$ and $v \in \mathcal{L}(F,Y)$. If $P \in \mathcal{P}(^mG,F)$ is $(p;q;\sigma)$-absolutely continuous, then $v \circ P \circ u$ is $(p;q;\sigma)$-absolutely continuous and*

$$\|v \circ P \circ u\|_{\mathcal{P}^{\sigma}_{as(p,q)}} \leq \|v\| \, \|P\|_{\mathcal{P}^{\sigma}_{as(p,q)}} \, \|u\|^m .$$

Although $(p;q;\sigma)$-absolutely continuous polynomials have been introduced independently of $(p;p_1,\dots,p_m)$-absolutely continuous multilinear mappings, in order to relate both classes we characterize first these classes of non linear operators by means of their summability properties.
Adapting the proof of Proposition 2.2.6, we easily get the characterization of $(p;q;\sigma)$-absolutely continuous polynomials by means of transformations of vector valued sequence spaces.

Proposition 4.2.4. *Let $m \in \mathbb{N}$, $1 \leq p,q < +\infty$ such that $mp \geq q$ and $0 \leq \sigma < 1$. Let $P \in \mathcal{P}(^mX;Y)$. Then the polynomial P is $(p;q;\sigma)$-absolutely continuous if and only if $(P(x_i))_{i=1}^{\infty} \in \ell_{\frac{p}{1-\sigma}}(Y)$ for every $(x_i)_{i=1}^{\infty} \in \ell^{q\sigma}(X)$.*

The above characterization allows to relate the properties of the $(p;q;\sigma)$-absolutely continuous polynomials with the ones of their corresponding symmetric multilinear mappings.

Corollary 4.2.5. *Let $P \in \mathcal{P}(^mX;Y)$. Then $P \in \mathcal{P}^{\sigma}_{as(p,q)}(^mX;Y)$ if and only if $\check{P} \in \mathcal{L}^{\sigma}_{as(p;q,\dots,q)}(^mX;Y)$.*

Proof. It follows by means of a standard argument from the polarization formula and Proposition 2.2.6. □

The interest of inclusion theorems yielded to establish a very general setting where an abstract inclusion theorem holds [56]. Let us see that the class of $(p;q;\sigma)$-absolutely continuous polynomials satisfies an inclusion theorem by using this abstract result, although a standard calculation gives also the inclusion.

Proposition 4.2.6. *(Inclusion theorem). Let $1 \leq p \leq q < \infty$ and $1 \leq p_1 \leq q_1 < \infty$ be such that $\frac{m}{p_1} - \frac{1}{p} \leq \frac{m}{q_1} - \frac{1}{q}$. Then $\mathcal{P}^{\sigma}_{as(p,p_1)}(^mX;Y) \subset \mathcal{P}^{\sigma}_{as(q,q_1)}(^mX;Y)$. Moreover, we have $\|.\|_{\mathcal{P}^{\sigma}_{as(q,q_1)}} \leq \|.\|_{\mathcal{P}^{\sigma}_{as(p,p_1)}}$.*

Proof. Any polynomial in $\mathcal{P}^{\sigma}_{as(r,s)}(^mX;Y)$ is (r',s')-abstract (R,S)-summing (see [56] for the definition), for $r' := \frac{mr}{1-\sigma}$, $s' := \frac{s}{1-\sigma}$, $R : X \times \mathbb{R} \times B_{X^*} \to [0,\infty)$ and $S : \mathcal{P}(^mX;Y) \times X \times \mathbb{R} \times B_{X^*} \to [0,\infty)$ given by

$$S(Q,x,b,\phi) := \|Q(x)\|^{1/m} \quad \text{and} \quad R(x,b,\phi) := |\phi(x)|^{1-\sigma}\|x\|^{\sigma}.$$

The mappings R and S satisfy $S(Q,tx,b,\phi) = |t|S(Q,x,b,\phi)$ and $R(tx,b,\phi) = |t|R(x,b,\phi)$ for all $x \in X$, $b \in \mathbb{R}$, $Q \in \mathcal{P}(^mX;Y)$ and $\phi \in B_{X^*}$. Then [56, Theorem 3.15] gives the result. □

4.3 Dominated (p,σ)-continuous polynomials

A relevant special case of $(p;q;\sigma)$-absolutely continuous polynomial is when we have $mp = q$. In this situation —and following the standard notations in similar cases— we will call the mappings dominated $(p;\sigma)$-continuous polynomials, and we will denote the corresponding vector space and norm by $\mathcal{P}^{\sigma}_{d,p}({}^mX;Y)$ and $\|.\|_{\mathcal{P}^{\sigma}_{d,p}}$, respectively, for $p \geq m$. Actually, we have $\mathcal{P}^{\sigma}_{d,p}({}^mX;Y) = \mathcal{P}^{\sigma}_{as(\frac{p}{m},p)}({}^mX;Y)$, i.e. a polynomial $P \in \mathcal{P}({}^mX;Y)$ is *dominated* $(p;\sigma)$*-continuous* if there is a constant $C > 0$ such that for every $(x_i)_{i=1}^n \subset X$ we have

$$\|(P(x_i))_{i=1}^n\|_{\frac{p}{m(1-\sigma)}} \leq C.\left[\delta_{p\sigma}\left((x_i)_{i=1}^n\right)\right]^m. \tag{4.3}$$

Notice that for $m = 1$ we recover also the ideal of (p,σ)-absolutely continuous linear operators. When $\sigma = 0$, $\mathcal{P}^{0}_{d,p}({}^mX;Y)$ is the space of all p-dominated m-homogeneous polynomials, which is denoted simply by $\mathcal{P}_{d,p}({}^mX;Y)$. The definition and some fundamental results on p-dominated homogeneous polynomials between Banach spaces can be found in [15], [44] or [49].

Remark 4.3.1. From Corollary 4.2.5, Theorem 2.2.10 and [15, Proposition 9] it follows that the decomposition in part (c) of Remark 4.2.2 actually characterizes dominated $(p;\sigma)$-continuous polynomials. Indeed, $P \in \mathcal{P}^{\sigma}_{d,p}({}^mX;Y)$ if and only if there is a Banach space Z, a $(p;\sigma)$-absolutely continuous linear operator $u : X \to Z$ and a polynomial $Q \in \mathcal{P}({}^mZ;Y)$ such that $P = Q \circ u$. This factorization will be used several times and is essential for our purposes of getting a Pietsch type factorization theorem for dominated $(p;\sigma)$-continuous polynomials.

It is well known that $(\mathcal{P}_{d,p}, \|.\|_{d,p})$ is a Banach ideal of polynomials if $p \geq m$. Although a domination theorem for dominated polynomials follows easily as in the linear case, to get the corresponding factorization theorem requires new techniques that use mainly symmetric tensor products and an adequate representation of these spaces. This has been done in [20] and [21], where it is shown that any p-dominated polynomial factors through a canonical prototype of a p-dominated polynomial in the spirit of Pietsch's classical result. Our aim is to obtain a domination/factorization result for the larger class of dominated $(p;\sigma)$-continuous polynomials. We will see that new constructions of renormed subspaces of L_p spaces different from the ones used in [20] and [21] are required.

Theorem 4.3.2. *(Domination theorem) Let $m \in \mathbb{N}$ and $1 \leq p < \infty$. An m-homogeneous polynomial $P \in \mathcal{P}({}^mX;Y)$ is dominated $(p;\sigma)$-continuous if and only if there is a regular Borel probability measure μ on B_{X^*} (with the weak* topology) and a constant $C > 0$ such that for all $x \in X$*

$$\|P(x)\| \leq C\,\|x\|^{m\sigma}\left(\int_{B_{X^*}} |\phi(x)|^p \,d\mu(\phi)\right)^{\frac{m(1-\sigma)}{p}}. \tag{4.4}$$

Moreover, in this case $\|P\|_{\mathcal{P}^{\sigma}_{d,p}} = \inf\{C > 0 : C$ *satisfies (4.4)*$\}$.

Proof. A dominated $(p;\sigma)$-continuous m-homogeneous polynomial P is RS-abstract q-summing (see [18, 55] for the definition), for $q := \frac{p}{(1-\sigma)m}$, $R : B_{X^*} \times X \times \mathbb{R} \to [0,\infty)$ and $S : \mathcal{P}(^m X; Y) \times X \times \mathbb{R} \to [0,\infty)$ given by

$$R(\phi, x, b) := |\phi(x)|^{(1-\sigma)m} \|x\|^{\sigma m} \quad \text{and} \quad S(Q, x, b) := \|Q(x)\|,$$

$x \in X, \phi \in B_{X^*}, b \in \mathbb{R}, Q \in \mathcal{P}(^m X; Y)$. Theorem 2.2 in [18] or Theorem 3.1 in [55] gives the result. $\square$

Definition 4.3.3. *Any regular Borel probability measure μ on B_{X^*}, with the weak* topology that satisfies (4.4) is called a Pietsch measure for P.*

An inclusion between the classes of the dominated $(p;\sigma)$-continuous polynomials and the p-dominated polynomials follows easily from the definitions.

Proposition 4.3.4. *Let $1 \leq p < \infty$ and $0 \leq \sigma < 1$. Then $\mathcal{P}_{d,\frac{p}{(1-\sigma)}}(^m X; Y) \subset \mathcal{P}^\sigma_{d,p}(^m X; Y)$. Consequently, $\mathcal{P}_{d,p}(^m X; Y) \subset \mathcal{P}^\sigma_{d,p}(^m X; Y)$.*

Proof. Let $P \in \mathcal{P}_{d,\frac{p}{(1-\sigma)}}(^m X; Y)$. Let $(x_i)_{i=1}^n$ be a sequence in X. Using inequality (2.1) we have

$$\left(\sum_{i=1}^n \|P(x_i)\|^{\frac{p}{m(1-\sigma)}}\right)^{\frac{m(1-\sigma)}{p}} \leq \|P\|_{d,\frac{p}{(1-\sigma)}} \left[\|(x_i)_{i=1}^n\|_{\frac{p}{1-\sigma},\omega}\right]^m \leq \|P\|_{d,\frac{p}{(1-\sigma)}} \left[\delta_{p\sigma}((x_i)_{i=1}^n)\right]^m.$$

Then $P \in \mathcal{P}^\sigma_{d,p}(^m X; Y)$ and $\|P\|_{\mathcal{P}^\sigma_{d,p}} \leq \|P\|_{d,\frac{p}{(1-\sigma)}}$. Hence $\mathcal{P}_{d,\frac{p}{(1-\sigma)}}(^m X; Y) \subset \mathcal{P}^\sigma_{d,p}(^m X; Y)$. Since $p \leq \frac{p}{1-\sigma}$ it follows that $\mathcal{P}_{d,p}(^m X; Y) \subset \mathcal{P}_{d,\frac{p}{(1-\sigma)}}(^m X; Y)$ (see [49]). Hence the inclusion $\mathcal{P}_{d,p}(^m X; Y) \subset \mathcal{P}^\sigma_{d,p}(^m X; Y)$ is proved. $\square$

By [14, Example 1] there is a m-dominated polynomial $P \in \mathcal{P}(^m X; Y)$, $m \geq 2$, which is not weakly compact. Then Proposition 4.3.4 gives the existence of a dominated (m,σ)-continuous polynomial which is not weakly compact.

Remark 4.3.5. In general, $\mathcal{P}^\sigma_{d,p} \neq \mathcal{P}_{d,\frac{p}{(1-\sigma)}}$. Let us show an example of a polynomial belonging to $\mathcal{P}^\sigma_{d,p}$ that is not in $\mathcal{P}_{d,\frac{p}{(1-\sigma)}}$. Let $L^1 := L^1[0,1]$ and L^2 the corresponding Hilbert space. We know by Example 2.2.14 that there is a *symmetric* bilinear operator $T : L^2 \times L^2 \to L^1$ such that

$$T \in \mathcal{L}^\sigma_{as(1;2,2)}(^2 L^2; L^1) \quad \text{but} \quad T \notin \mathcal{L}_{as(\frac{1}{(1-\sigma)};\frac{2}{(1-\sigma)},\frac{2}{(1-\sigma)})}(^2 L^2; L^1).$$

Then, by Corollary 4.2.5 and [49, Theorem 6], the polynomial $\hat{T} \in \mathcal{P}(^2 L^2; L^1)$ associated to T satisfies that $\hat{T} \in \mathcal{P}^\sigma_{d,2}(^2 L^2; L^1)$, but $\hat{T} \notin \mathcal{P}_{d,\frac{2}{1-\sigma}}(^2 L^2; L^1)$.

4.4 The factorization theorem

As in the case of the p-dominated multilinear operators, we will show that there is a factorization theorem that characterizes when a polynomial is dominated $(p;\sigma)$-continuous. In fact, this theorem presents the prototype of dominated $(p;\sigma)$-continuous polynomial, i.e. the polynomial belonging to this class through which each polynomial of the class factors. The ideas for proving the factorization follow the lines of the one that are used in [21] but there are some meaningful differences based on the fact that the domination for the dominated $(p;\sigma)$-continuous polynomials is not based in a norm but in some interpolated expression between the norm of X and the one of $L^p(\mu)$. To deal with, we use techniques inspired in the convexification of Banach lattices.

If X is a Banach space and $m \in \mathbb{N}$, we define the m-homogeneous polynomial

$$\Delta\colon X \longrightarrow C(B_{X^*});\ \Delta(x)(\varphi) = \varphi(x)^m, x \in X, \varphi \in C(B_{X^*}).$$

We consider the restriction δ of its linearization to the m-fold symmetric tensor product $\otimes_{\pi_s}^{m,s} X$. So defined, δ is a linear operator

$$\delta\colon \otimes_{\pi_s}^{m,s} X \longrightarrow C(B_{X^*});\ \delta(x \otimes \cdots \otimes x)(\varphi) = \varphi(x)^m.$$

By Lemma 4.1. in [20], this mapping is injective. To simplify the notation, sometimes we shall write $\otimes_m x := x \otimes \cdots \otimes x$. Let δ_m stand for the canonical m-homogeneous polynomial

$$\delta_m : X \longrightarrow \otimes_{\pi_s}^{m,s} X;\ \delta_m(x) = x \otimes \cdots \otimes x.$$

Let $i_X : X \to C(B_{X^*})$ be the canonical isometric inclusion given by the evaluation. Given μ a Borel probability measure on B_{X^*}, $j_p : C(B_{X^*}) \to L_p(\mu)$ denotes the canonical mapping and $j^m_{\frac{p}{m}}$ the continuous m-homogeneous polynomial,

$$j^m_{\frac{p}{m}} : i_X(X) \subset C(B_{X^*}) \longrightarrow L_{\frac{p}{m}}(\mu);\ j^m_{\frac{p}{m}}(f) = j_{\frac{p}{m}}(f^m).$$

On $j_p \circ i_X(X)$ consider the seminorm

$$\|j_p \circ i_X(x)\|_{p,\sigma} := \inf \left\{ \sum_{j=1}^n \|x_j\|^\sigma \|j_p \circ i_X(x_j)\|^{1-\sigma}_{L_p(\mu)} : x = \sum_{j=1}^n x_j, x_j \in X,\ n \in \mathbb{N}. \right\}$$

Consider the relation

$$j_p \circ i_X(x) \equiv j_p \circ i_X(y) \text{ if and only if } \|i_X(x-y)\| = 0,$$

and denote by $L_{p,\sigma}(\mu)$ the quotient space and by $j_{p,\sigma} : i_X(X) \to L_{p,\sigma}(\mu)$ the quotient mapping. Then $\|\cdot\|_{p,\sigma}$ becomes a norm on $L_{p,\sigma}(\mu)$.

Following a general construction that is well-known for the case of Banach function spaces (see for example [53, Chapter 2]), we can define what we call the m-th power $L_{p,\sigma}(\mu)_{[m]}$ of $L_{p,\sigma}(\mu)$. However, notice that in this case this new space is not a

Banach function space. It is a vector space that is defined as the linear span of all polynomials of the form $(j_{\frac{p}{m}} \circ i_X(x))^m$ for x being an element of X, i.e.

$$\begin{aligned} L_{p,\sigma}(\mu)_{[m]} &= \Big\{ h \in L_{\frac{p}{m}}(\mu) : h = \textstyle\sum_{j=1}^n \lambda_j j_{\frac{p}{m}} \circ \Delta(x_j), x_j \in X, \lambda_j \in \mathbb{K}, n \in \mathbb{N} \Big\} \\ &= \Big\{ h \in L_{\frac{p}{m}}(\mu) : h = \textstyle\sum_{j=1}^n \lambda_j j_{\frac{p}{m}} \left(i_X(x_j)\right)^m, x_j \in X, \lambda_j \in \mathbb{K}, n \in \mathbb{N} \Big\}. \end{aligned}$$

Consider the m-homogeneous polynomial Q given by

$$Q : j_{p,\sigma} \circ i_X(X) \to L_{p,\sigma}(\mu)_{[m]};\ Q(j_{p,\sigma} \circ i_X(x)) := j_{\frac{p}{m}} \circ \Delta(x),$$

and let Q^L be its linearization. Given

$$h = \sum_{j=1}^n \lambda_j j_{\frac{p}{m}} \circ \Delta(x_j) \in L_{p,\sigma}(\mu)_{[m]},$$

we will denote by θ the tensor $\theta := \sum_{j=1}^n \lambda_j \otimes_m x_j$. By T we denote the symmetric m-fold tensor product of the linear operator $j_{p,\sigma} \circ i_X$. So

$$T\colon \otimes^{m,s} X \longrightarrow \otimes^{m,s} j_{p,\sigma} \circ i_X(X);\ T(\otimes_m x) = \otimes_m j_{p,\sigma}\left(i_X(x)\right) \text{ for every } x \in X.$$

For each $h \in L_{p,\sigma}(\mu)_{[m]}$ define

$$\pi_{p;\sigma,m}(h) := \inf \left\{ \sum_{j=1}^n |\lambda_j| \|j_{p,\sigma} \circ i_X(x_j)\|_{p,\sigma}^m : h = \sum_{j=1}^n \lambda_j j_{\frac{p}{m}} \circ \Delta(x_j) \right\},$$

and this is equal to

$$\inf \left\{ \sum_{j=1}^n |\lambda_j| \cdot \left(\sum_{k=1}^{n_j} \|x_k^j\|^\sigma \|j_p \circ i_X(x_k^j)\|_{L_p}^{1-\sigma} \right)^m : h = \sum_{j=1}^n \lambda_j j_{\frac{p}{m}} \circ \Delta(\sum_{k=1}^{n_j} x_k^j) \right\}.$$

Note that

$$Q^L \circ T(\theta) = j_{\frac{p}{m}} \circ \delta(\theta) = j_{\frac{p}{m}} \circ \delta \left(\sum_{j=1}^n \lambda_j \otimes_m (\sum_{k=1}^{n_j} x_k^j) \right) = \sum_{j=1}^n \lambda_j j_{\frac{p}{m}} \circ \Delta(\sum_{k=1}^{n_j} x_k^j) = h$$

whenever $\theta = \sum_{j=1}^n \lambda_j \otimes_m (\sum_{k=1}^{n_j} x_k^j)$.

Proposition 4.4.1. *If $p \geq m$ then $\pi_{p;\sigma,m}$ is a norm on $L_{p,\sigma}(\mu)_{[m]}$.*

Proof. Following [20, Proposition 4.2], on the space $X^{\frac{p}{m}} := j_{\frac{p}{m}}^m \circ i_X(X) \subseteq L_{\frac{p}{m}}(\mu)$ a norm is defined by

$$\pi_{\frac{p}{m}}(h) := \inf \left\{ \sum_{i=1}^n |\lambda_i| \cdot \|(j_{\frac{p}{m}} \circ \delta) \otimes_m x_i)\|_{L_{\frac{p}{m}}} \right\},$$

where the infimum is taken over all representations $\sum_{i=1}^n \lambda_i \otimes_m x_i$ of all $\theta \in \otimes_{\pi_s}^{m,s} X$ such that $(j_{p/m} \circ \delta)(\theta) = h$. We have that

$$\begin{aligned}
\pi_{\frac{p}{m}}(h) = \inf &\left\{ \sum_{i=1}^n |\lambda_i| \cdot \|(j_{\frac{p}{m}} \circ \delta)(\otimes_m x_i)\|_{L_{\frac{p}{m}}} : \theta = \sum_{i=1}^n \lambda_i \otimes_m x_i,\ (j_{\frac{p}{m}} \circ \delta)(\theta) = h \right\} \\
= \ \inf &\left\{ \sum_{i=1}^n |\lambda_i| \cdot \|(j_p \circ i_X(x_i)\|_{L_p}^m : \theta = \sum_{i=1}^n \lambda_i \otimes_m x_i,\ (j_{\frac{p}{m}} \circ \delta)(\theta) = h \right\} \\
\leq \ \inf &\left\{ \sum_{i=1}^n |\lambda_i| \cdot \left(\sum_{k=1}^{n_i} \|j_p \circ i_X(x_k^i)\|_{L_p} \right)^m : \theta = \sum_{i=1}^n \lambda_i \otimes_m (\sum_{k=1}^{n_i} x_k^i),\ (j_{\frac{p}{m}} \circ \delta)(\theta) = h \right\} \\
\leq \ \inf &\left\{ \sum_{i=1}^n |\lambda_i| \cdot \left(\sum_{k=1}^{n_i} \|x_k^i\|^\sigma \|j_p \circ i_X(x_k^i)\|_{L_p}^{1-\sigma} \right)^m : \theta = \sum_{i=1}^n \lambda_i \otimes_m (\sum_{k=1}^{n_i} x_k^i),\ (j_{\frac{p}{m}} \circ \delta)(\theta) = h \right\} \\
= \ &\pi_{p;\sigma,m}(h).
\end{aligned}$$

Therefore, if $\pi_{p;\sigma,m}(h) = 0$ then $\pi_{\frac{p}{m}}(h) = 0$, hence $h = 0$.
Let $h', h'' \in L_{p,\sigma}(\mu)_{[m]}$ and $\varepsilon > 0$. Choose representations of h' and h'' of the form

$$h' = \sum_{j=1}^{n'} \lambda'_j j_{\frac{p}{m}} \circ \Delta(x'_j), h'' = \sum_{j=1}^{n''} \lambda''_j j_{\frac{p}{m}} \circ \Delta(x''_j)$$

such that

$$\sum_{j=1}^{n'} |\lambda'_j| \|j_{p,\sigma} \circ i_X(x'_j)\|_{p,\sigma}^m \leq \pi_{p;\sigma,m}(h') + \frac{\varepsilon}{2},$$

and

$$\sum_{j=1}^{n''} |\lambda''_j| \|j_{p,\sigma} \circ i_X(x''_j)\|_{p,\sigma}^m \leq \pi_{p;\sigma,m}(h'') + \frac{\varepsilon}{2}.$$

Then $\sum_{j=1}^{n'} \lambda'_j j_{\frac{p}{m}} \circ \Delta(x'_j) + \sum_{j=1}^{n''} \lambda''_j j_{\frac{p}{m}} \circ \Delta(x''_j)$ is a representation of $h' + h''$ and so

$$\begin{aligned}
\pi_{p;\sigma,m}(h' + h'') &\leq \sum_{j=1}^{n'} |\lambda'_j| \|j_{p,\sigma} \circ i_X(x'_j)\|_{p,\sigma}^m + \sum_{j=1}^{n''} |\lambda''_j| \|j_{p,\sigma} \circ i_X(x''_j)\|_{p,\sigma}^m \\
&\leq \pi_{p;\sigma,m}(h') + \pi_{p;\sigma,m}(h'') + \varepsilon.
\end{aligned}$$

Since this holds for every $\varepsilon > 0$, we obtain the triangular inequality. Now given $\alpha \in \mathbb{K}, \alpha \neq 0$ and $h \in L_{p,\sigma}(\mu)_{[m]}$ with the representation $\sum_{j=1}^n \lambda_j j_{\frac{p}{m}} \circ \Delta(x_j)$. We have

$$\alpha h \ = \ \sum_{j=1}^n \lambda_j j_{\frac{p}{m}} (i_X(\alpha^{\frac{1}{m}} x_j))^m = \sum_{j=1}^n \lambda_j j_{\frac{p}{m}} \circ \Delta(\alpha^{\frac{1}{m}} x_j).$$

Using the homogeneity of $\|.\|_{p,\sigma}$ we obtain

$$\begin{aligned}
\pi_{p;\sigma,m}(\alpha h) &\leq \sum_{j=1}^n |\lambda_j| \|j_{p,\sigma} \circ i_X(\alpha^{\frac{1}{m}} x_j)\|_{p,\sigma}^m \\
&= |\alpha| \sum_{j=1}^n |\lambda_j| \|j_{p,\sigma} \circ i_X(x_j)\|_{p,\sigma}^m.
\end{aligned}$$

Since this holds for every representation of h, it follows that

$$\pi_{p;\sigma,m}(\alpha h) \leq |\alpha| \, \pi_{p;\sigma,m}(h).$$

In the same way, we have

$$\pi_{p;\sigma,m}(h) = \pi_{p;\sigma,m}(\frac{1}{\alpha}\alpha h) \leq \frac{1}{|\alpha|}\pi_{p;\sigma,m}(\alpha h),$$

giving $|\alpha| \, \pi_{p;\sigma,m}(h) \leq \pi_{p;\sigma,m}(\alpha h)$. Therefore $|\alpha| \, \pi_{p;\sigma,m}(h) = \pi_{p;\sigma,m}(\alpha h)$. The case $\alpha = 0$ is obvious. □

The next proposition shows that Q^L is an isometric isomorphism between the spaces $\hat{\otimes}_{\pi_s}^{m,s} L_{p,\sigma}(\mu)$ and $(L_{p,\sigma}(\mu)_{[m]}, \pi_{p;\sigma,m})$ whenever we assume its injectivity.

Proposition 4.4.2. *Assume that Q^L is injective. The completion of the space $(L_{p,\sigma}(\mu)_{[m]}, \pi_{p;\sigma,m})$ is isometrically isomorphic to $\hat{\otimes}_{\pi_s}^{m,s} L_{p,\sigma}(\mu)$.*

Proof. Consider Q^L restricted to $\otimes_{\pi_s}^{m,s} j_{p,\sigma} \circ i_X(X)$, that is

$$Q^L : \otimes_{\pi_s}^{m,s} j_{p,\sigma} \circ i_X(X) \to (L_{p,\sigma}(\mu)_{[m]}, \pi_{p;\sigma,m}).$$

Let us see first that Q^L is onto. Given $h = \sum_{j=1}^{n} \lambda_j j_{\frac{p}{m}} \circ \Delta(x_j) \in L_{p,\sigma}(\mu)_{[m]}$, let $\theta = \sum_{j=1}^{n} \lambda_j x_j \otimes \cdots \otimes x_j$. Then

$$\begin{aligned} Q^L(T(\theta)) &= \sum_{j=1}^{n} \lambda_j Q^L(\otimes_m j_{p,\sigma} \circ i_X(x_j)) = \sum_{j=1}^{n} \lambda_j Q(j_{p,\sigma} \circ i_X(x_j)) \\ &= \sum_{j=1}^{n} \lambda_j j_{\frac{p}{m}} \circ \Delta(x_j) = h. \end{aligned}$$

From the definitions of the norms and the injectivity of Q^L it follows that $\pi_{p;\sigma,m}(h) = \pi_s(T(\theta))$ and so Q^L is an isometry. Therefore, the extension of Q^L to the completions is the required isometric isomorphism. □

To simplify the notation we will use $(L_{p,\sigma}(\mu)_{[m]}, \pi_{p;\sigma,m})$ for its completion too. Let us define a polynomial which shall play the role of the canonical prototype of dominated $(p;\sigma)$-continuous m-homogeneous polynomial through which any other polynomial of the class must factor. Define the polynomial

$$j_{p;\sigma,m} := Q \circ j_{p,\sigma} : i_X(X) \longrightarrow L_{p,\sigma}(\mu)_{[m]}$$

For each $x \in X$,

$$j_{p;\sigma,m}(i_X(x)) = Q \circ j_{p,\sigma}(i_X(x)) = j_{\frac{p}{m}} \circ \Delta(x) = (j_{\frac{p}{m}} \circ i_X(x))^m$$

and so $j_{p;\sigma,m}$ can be identified with the restriction of the m-homogeneous polynomial $j_{\frac{p}{m}}^m$ to the space $i_X(X)$ and with values in $L_{p,\sigma}(\mu)_{[m]}$.

Proposition 4.4.3. *The m-homogeneous polynomial $j_{p;\sigma,m}$ is dominated $(p;\sigma)$-continuous.*

Proof. Given $x_1, \ldots, x_n \in X$. Since the canonical linear mapping $j_{p,\sigma} : i_X(X) \to L_{p,\sigma}(\eta)$ is (p,σ)-absolutely continuous and $\pi_{p,\sigma}(j_{p,\sigma}) \leq 1$ (see Lemma 2.1.19) we have

$$\begin{aligned} & \left(\sum_{i=1}^{n} \pi_{p;\sigma,m}(j_{p;\sigma,m}(i_X(x_i)))^{\frac{p}{m(1-\sigma)}} \right)^{\frac{m(1-\sigma)}{p}} \\ = & \left(\sum_{i=1}^{n} \pi_{p;\sigma,m}((j_{\frac{p}{m}} \circ i_X(x_i))^m)^{\frac{p}{m(1-\sigma)}} \right)^{\frac{m(1-\sigma)}{p}} \\ \leq & \left(\sum_{i=1}^{n} \|(j_{p,\sigma} \circ i_X(x_i)\|_{p,\sigma}^{\frac{p}{1-\sigma}} \right)^{\frac{m(1-\sigma)}{p}} \\ \leq & \ [\delta_{p\sigma}((i_X(x_i)_{i=1}^n)]^m, \end{aligned}$$

proving that $j_{p;\sigma,m}$ is dominated (p,σ)-continuous and $\|j_{p;\sigma,m}\|_{\mathcal{P}^{\sigma}_{d,p}} \leq 1$. □

To get the factorization of dominated $(p;\sigma)$-continuous polynomials through $j_{p;\sigma,m}$ we need some preliminary results.
Although the following proposition intends to generalize [21, Proposition 3.4], it only applies for some specific Pietsch measures of a dominated $(p;\sigma)$-continuous m-homogeneous polynomial P (see Definition 4.3.3) and a different proof is required. From Remark 4.3.1, P can be written as $P = Q \circ u$, where u is a $(p;\sigma)$-absolutely continuous linear operator from X into some Banach space Z and $Q : Z \to Y$ is a continuous m-homogeneous polynomial. An easy calculation shows that any Pietsch measure μ for u is a Pietsch measure for P. In the following result we are considering μ such a measure.

Proposition 4.4.4. *Let $P \in \mathcal{P}^{\sigma}_{d,p}(^mX;Y)$. If $x, y \in X$ are such that $j_{p,\sigma} \circ i_X(x) = j_{p,\sigma} \circ i_X(y)$ then $P(x) = P(y)$.*

Proof. By Remark 4.3.1, there exist a Banach space Z, a $(p;\sigma)$-absolutely continuous linear operator $u : X \to Z$ and a polynomial $Q \in \mathcal{P}(^mZ;Y)$ such that P can be written as $P = Q \circ u$. Let μ be a Pietsch measure for u with constant C. Therefore, by the polarization formula we obtain $\check{P} = \check{Q} \circ (u, \ldots, u)$. Take $\varepsilon > 0$. The equality $j_{p,\sigma} \circ i_X(x) = j_{p,\sigma} \circ i_X(y)$ says that $\|j_{p,\sigma} \circ i_X(x-y)\|_{p,\sigma} = 0$. Then, there exist $x_1, \ldots, x_n \in X$ such that $x - y = \sum_{k=1}^n x_k$ and

$$\sum_{k=1}^{n} \|x_k\|^{\sigma} \|j_p \circ i_X(x_k)\|_{L_p(\mu)}^{1-\sigma} < \varepsilon. \tag{4.5}$$

It follows that

$$\begin{aligned}
\|P(x)-P(y)\| &= \|\check{P}(x,\overset{(m)}{\dots},x)-\check{P}(y,\overset{(m)}{\dots},y)\| \\
&= \|\sum_{j=1}^{m}\check{P}(y,\overset{(j-1)}{\dots},y,x-y,x,\overset{(m-j)}{\dots},x)\| \\
&\le \sum_{j=1}^{m}\sum_{k=1}^{n}\|\check{P}(y,\overset{(j-1)}{\dots},y,x_k,x,\overset{(m-j)}{\dots},x)\| \\
&= \sum_{j=1}^{m}\sum_{k=1}^{n}\|\check{Q}\left(u(y),\overset{(j-1)}{\dots},u(y),u(x_k),u(x),\overset{(m-j)}{\dots},u(x)\right)\| \\
&\le \sum_{j=1}^{m}\sum_{k=1}^{n}\|\check{Q}\|\|u(y)\|^{j-1}\|u(x_k)\|\|u(x)\|^{m-j} \\
&\le \|\check{Q}\|\sum_{j=1}^{m}\|u(y)\|^{j-1}\|u(x)\|^{m-j}\sum_{k=1}^{n}\pi_{p,\sigma}(u)\|x_k\|^{\sigma}\|j_p\circ i_X(x_k)\|_{L_p(\mu)}^{1-\sigma} \\
&< \pi_{p,\sigma}(u)\|\check{Q}\|\sum_{j=1}^{m}\|u(y)\|^{j-1}\|u(x)\|^{m-j}\varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary we conclude that $P(x) = P(y)$. □

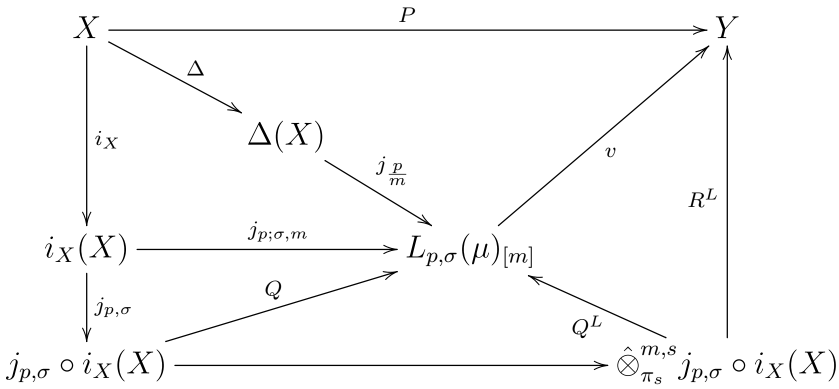
We present now the factorization theorem for dominated $(p;\sigma)$-continuous polynomials.

Theorem 4.4.5. *Let $m \in \mathbb{N}$ and $p \ge m$. Assume that (B_{X^*}, w^*) is separable. A polynomial $P \in \mathcal{P}(^mX;Y)$ is dominated $(p;\sigma)$-continuous if and only if there exist a regular Borel probability measure μ on B_{X^*}, with the weak* topology, and a continuous linear operator $v : (L_{p,\sigma}(\mu)_{[m]}, \pi_{p;\sigma,m}) \to Y$ such that the following diagram commutes*

$$\begin{array}{ccc}
X & \xrightarrow{\quad P \quad} & Y \\
\downarrow{\scriptstyle i_X} & & \uparrow{\scriptstyle v} \\
i_X(X) & \xrightarrow{j_{p;\sigma,m}} & L_{p,\sigma}(\mu)_{[m]} \\
\downarrow & & \downarrow \\
C(B_{X^*}) & & L_{\frac{p}{m}}(\mu)
\end{array} \quad . \tag{4.6}$$

Proof. Assume that $P \in \mathcal{P}_{d,p}^{\sigma}(^mX;Y)$. By Remark 4.3.1, there exist a Banach space Z, a $(p;\sigma)$-absolutely continuous linear operator $u : X \to Z$ and a polynomial $Q \in \mathcal{P}(^mZ;Y)$ such that P can be written as $P = Q \circ u$. Let μ be a Pietsch measure for u. By [21, Proposition 2.5] we can assume that $j_{\frac{p}{m}} : \Delta(X) \to L_{\frac{p}{m}}(\mu)$ is injective. For each $x \in X$ define $R(j_{p,\sigma} \circ i_X(x)) := P(x)$. By Proposition 4.4.4 the mapping $R : j_{p,\sigma} \circ i_X(X) \to Y$ is a well-defined m-homogeneous polynomial. Let $Q^L : \otimes_{\pi_s}^{m,s} j_{p,\sigma} \circ i_X(X) \to L_{p,\sigma}(\mu)_{[m]}$ be the surjective isometric isomorphism given by

Proposition 4.4.2. Indeed Q^L is injective by $Q^L \circ T = j_{\frac{p}{m}} \circ \delta$. Taking into account the commutative diagrams



let us define $v := R^L \circ (Q^L)^{-1}$. It is clear that, so defined, v is continuous and closes the diagram (4.6).
The converse follows from Proposition 4.4.3 and the ideal property. □

Conditions that assure the separability of (B_{X^*}, w^*) are easy to find (see for example [32, Page 127]).

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