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Asymptotic Derivation of Effective Boundary Conditions for Thin Poroelastic Slabs

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Asymptotic Derivation of Effective Boundary Conditions for Thin Poroelastic Slabs

Abstract

This work is devoted to the asymptotic derivation of effective boundary conditions for a thin poroelastic strip in the framework of linear elasticity with voids. We consider a three-dimensional transmission problem coupling a thick elastic body with voids and a thin bonded layer. Using asymptotic expansion techniques, we construct an approximate impedance operator that replaces the thin layer. Rigorous error estimates are established, showing that the error between the solution of the transmission problem and the solution of the approximate impedance problem is of order $\mathcal{O}(\delta^2)$ in the H^1-norm.

Keywords: Elasticity with voids, asymptotic analysis, impedance boundary condition, thin layer, transmission problem.

Résumé

Ce travail est consacré à la dérivation asymptotique de conditions aux limites effectives pour une bande mince poroélastique dans le cadre de l'élasticité linéaire avec vides. Nous considérons un problème de transmission tridimensionnel couplant un corps élastique épais avec vides et une couche mince collée. À l'aide de techniques de développement asymptotique, nous construisons un opérateur d'impédance approché remplaçant la couche mince. Des estimations d'erreur rigoureuses sont tablies, montrant que l'erreur entre la solution du problme de transmission et la solution du problme d'impdance approch est d'ordre $\mathcal{O}(\delta^2)$ dans la norme H^1.

Mots-clés : Élasticité avec vides, analyse asymptotique, condition d'impédance, couche mince, problème de transmission.

Dedication

All praise is due to Allah.

*To my gift from God, the greatest blessing in my life, the source of my strength and inspiration, to those who have given so much and offered what cannot be repaid. " **my mother and my father**, May God grant you both a long life."*

*To my support and my strength in every step, to my sisters" **Makarem, Lina, Roua**" , and to my dear brother" **Mohamed**".*

*To my dear and beloved grandfathers **"Berrabah Aissa ", Layaida Mohamed Eltayeb** and my grandmothers**" Makhfi Bahia and Tiaiba Rebiha"** , my happiness, my strength, and my optimism in life, May Allah protect you and prolong your life.*

*To my grandmother, **Mansour Fatima** may God have mercy on her.*

*To my **uncles** , **aunts** and all my loved ones.*

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E. Berrabah,

Contents

General Introduction

The mathematical modeling of composite materials and structures involving thin layers is a cornerstone of modern engineering and applied mathematics. From coated surfaces in aerospace engineering to biological tissues and geological strata, the presence of a thin, distinct material layer bonded to a thicker substrate profoundly influences the overall mechanical response. Direct numerical simulation of such multi-scale configurations is notoriously challenging, as resolving the fine geometry of the thin layer often requires prohibitively fine meshes, leading to high computational costs and ill-conditioned systems.

To overcome this difficulty, asymptotic methods provide a powerful framework. The core idea is to replace the thin layer governed by complex partial differential equations with an effective impedance boundary condition imposed on the interface of the remaining domain. This reduced model preserves the essential physics of the layer (its stiffness, its internal structure) while eliminating its geometric thickness from the numerical mesh.

In this dissertation, we focus on a specific and advanced continuum theory: linear elasticity with voids. Unlike classical elasticity, this theory, pioneered by Cowin and Nunziato, introduces an additional scalar field representing the volume fraction of voids within the material. This allows for the modeling of porous solids where the pores are dry and do not contain fluid. The presence of voids adds an extra equilibrium equation and couples the displacement field with the void volume fraction, making the derivation of effective boundary conditions a non-trivial task.

We consider a three-dimensional transmission problem consisting of a thick elastic body with voids (Ω_-) bonded to a thin poroelastic (voided) layer (Ω_+^δ) of small thickness δ. The objective is to analyze the asymptotic behavior of the solution as $\delta \to 0$ and to derive a simplified model on the fixed domain Ω_- that encapsulates the effect of the thin layer through a set of effective boundary conditions.

The dissertation is structured as follows:

Chapter 1: Elements of Functional Analysis for PDEs. This chapter serves as a mathematical foundation. We recall the essential concepts and theorems required for the rigorous analysis of partial differential equations, including Lebesgue and Sobolev spaces (L^p, H^m, $H(\mathrm{div})$), the theory of distributions, trace theorems, and the Lax-Milgram theorem. These tools are indispensable for establishing the well-posedness of the transmission problems studied later.

Chapter 2: Study of a model transmission problem for the linear elasticity system with voids. This chapter formulates the exact three-dimensional transmission problem (P^δ) in the domain $\Omega^\delta = \Omega_- \cup \Sigma \cup \Omega_+^\delta$. We present the equilibrium equations, constitutive laws, and transmission conditions for both the displacement field and the void volume fraction. Using the Lax-Milgram theorem, we rigorously prove the existence and uniqueness of the solution in a suitable Hilbert space. Furthermore, we perform a scaling transformation to map the thin layer to a fixed domain and establish critical stability estimates for the scaled problem, which are independent of the small parameter δ.

Chapter 3: Modeling the Effect of a Planar Thin Layer in the Framework of Linear Elasticity with Voids. This is the core chapter dedicated to asymptotic modeling. We formally derive an approximate impedance operator $T_{*\delta}$ that mimics the mechanical action of the thin layer. Two complementary methods are employed:

(i) **Taylor Expansion Method:** A direct expansion of the solution in the thin layer with respect to the thickness coordinate.

(ii) **Asymptotic Expansion Method (Multi-scale):** A systematic expansion of the scaled transmission problem, leading to a hierarchy of boundary value problems.

Both techniques yield a first-order effective boundary condition of order δ, which replaces the thin layer. We prove the well-posedness of the resulting approximate impedance problem and provide rigorous error estimates showing that the error between the exact solution and the approximate solution is of order $\mathcal{O}(\delta^2)$ in the H^1-norm, demonstrating the optimality of the derived model.

Chapter 1

Elements of Functional Analysis for PDEs

In this chapter, we recall the essential tools from functional analysis required for the rigorous analysis of the transmission problems studied in this dissertation. We present the main properties of Lebesgue and Sobolev spaces, the theory of distributions, trace theorems, Korn's inequality, and the Lax-Milgram theorem. This chapter is based on the classical references [6, 9, 21, 17, 15, 10].

1.1 The spaces $L^p(\Omega)$, $(1 \leq p \leq +\infty)$

Ω is an open set of $\mathbb{R}^n$.

Definition 1.1. [6] The Lebesgue space $L^p(\Omega)$ is the set of measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\|f\|_{L^p(\Omega)} < +\infty \quad \text{with} \quad \|f\|_{L^p(\Omega)} = \begin{cases} (\int_\Omega |f(x)|^p \, dx)^{\frac{1}{p}} & \text{if} \quad 1 \leq p < +\infty, \\ \operatorname{ess\,sup}_{x\in\Omega} |f(x)| & \text{if} \quad p = +\infty. \end{cases}$$

Definition 1.2. [6] Let $v = (v_1, ..., v_N)$ be a vector field defined on an open set Ω of $\mathbb{R}^N$, where each component $v_i : \Omega \longrightarrow \mathbb{R}$ is a measurable function. The L^2 norm of v is defined by:

$$\|v\|_{L^2(\Omega)} = \left(\sum_{i=1}^{N} \|v_i\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Definition 1.3. [6, 9] Let $A = (a_{ij}) \in \mathbb{R}^M \times \mathbb{R}^N$ be a matrix whose coefficients $a_{ij} : \Omega \longrightarrow \mathbb{R}$ are measurable functions defined on an open set Ω of $\mathbb{R}^d$. The L^2 norm of A is defined by:

$$\|A\|_{L^2(\Omega)} = \left(\sum_{i=1}^{M} \sum_{j=1}^{N} \|a_{ij}\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Proposition 1.1. *[6, 9] The mapping $v \longmapsto \|v\|_{L^p(\Omega)} \in L^p(\Omega)$ defines a norm on the space $L^p(\Omega)$ and the spaces $L^p(\Omega)$ are Banach spaces for $1 \leq p \leq +\infty$, separable for $1 \leq p < +\infty$ and reflexive for $1 < p < +\infty$.*

Proposition 1.2. *[9] Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leq p \leq +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $f.g \in L^1(\Omega)$ and*

$$\int_\Omega |f(x)g(x)|dx \leq \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q dx \right)^{\frac{1}{q}}.$$

Proposition 1.3. *[9] Let $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $p \in [1, +\infty[$ and let $r > 0$ such that: $\frac{1}{p} + \frac{1}{p'} = \frac{1}{r}$, we have:*

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^{p'}}$$

1.2 Notions on distributions

Definition 1.4. [6, 9] Let Ω be an open set of $\mathbb{R}^n$ and let f be a function from Ω to $\mathbb{C}$. The support of f is the closed set of $\mathbb{R}^n$ which is the closure of

$$\{x \in \Omega : f(x) \neq 0\} \subset \mathbb{R}^n.$$

It is also the complement of the largest open set on which it is interesting to study f.

Proposition 1.4. *[6, 9] $D(\Omega)$ is dense in $L^2(\Omega)$, i.e:*

$$\forall f \in L^2(\Omega),\ \forall \epsilon > 0,\ \exists \varphi \in D(\Omega) : \|f - \varphi\|_{L^2(\Omega)} < \epsilon$$

Definition 1.5. We denote by $D(\Omega)$ or $C_c^\infty(\Omega)$ the space of all functions f of class C^∞ with compact support in Ω, also called the space of test functions.

Definition 1.6. A sequence $(\varphi_n)_{n\in\mathbb{N}}$ of $D(\Omega)$ is said to converge to φ in $D(\Omega)$, if there exists a compact $K \subset \Omega$ such that: for all $n \in \mathbb{N}$ we have

$$\operatorname{supp}\varphi_n \subset K \quad \text{and} \quad \lim_n \sup_{x\in K} |D^p\varphi_n - D^p\varphi| = 0, \forall p \in \mathbb{N}^n,$$

with

$$D^p\varphi = \frac{\partial^{|p|}\varphi}{\partial x_1^{p_1}\partial x_2^{p_2}\ldots\partial x_n^{p_n}} \quad \text{and} \quad |p| = |(p_1, p_2, \ldots, p_n)| = p_1 + p_2 + \ldots + p_n.$$

Definition 1.7. A linear form $T : D(\Omega) \to \mathbb{R}$ is continuous if for every sequence $(\varphi_n)_{n\in\mathbb{N}}$ converging to φ in $D(\Omega)$, we have

$$\lim_n T(\varphi_n) = T(\varphi).$$

Definition 1.8. A distribution on Ω is a continuous linear form on $D(\Omega)$. We denote by $D'(\Omega)$ the space of all distributions on Ω.

Definition 1.9. The derivative of a distribution $T \in D'(\Omega)$ of order $m \in \mathbb{N}^n$ is the mapping

$$D^mT : D(\Omega) \to \mathbb{R}, \varphi \longmapsto D^mT(\varphi) = (-1)^{|m|}\langle T, D^m\varphi\rangle.$$

1.3 Sobolev spaces

1.3.1 Sobolev space $W^{m,p}(\Omega)$

Definition 1.10. [6, 9] The Sobolev space of order $m \in \mathbb{N}$ is the set

$$W^{m,p}(\Omega) = \left\{ \begin{array}{c} u \in L^p(\Omega) : \forall|\alpha| \leq m, \exists g_\alpha \in L^p(\Omega) \\ \int_\Omega u(x)D^\alpha\varphi(x)dx = (-1)^{|\alpha|}\int_\Omega g_\alpha\varphi(x)dx, \forall\varphi \in C_c^\infty(\Omega) \end{array} \right\}.$$

1) For $1 \leq p \leq +\infty$, the space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)} + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

2) For $m = 1$ and $p = 2$, we set $H^1(\Omega) = W^{1,2}(\Omega)$ is equipped with the scalar product

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + \sum_{i=1}^n (\partial_i u, \partial_i v)_{L^2(\Omega)}$$

is a Hilbert space and the norm associated with this scalar product is

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

Theorem 1.1. *[6, 9] The mapping*

$$v \longmapsto \|v\|_{W^{1,p}(\Omega)} \in W^{1,p}(\Omega)$$

defines a norm on the space $W^{1,p}(\Omega)$.

The spaces $W^{1,p}(\Omega)$ are Banach spaces for $1 \leq p \leq +\infty$. For $p = 2$, $W^{1,2}(\Omega)$ is even a Hilbert space.

Moreover, these spaces are separable and reflexive for $1 < p < +\infty$.

1.3.2 The spaces $H^m(\Omega)$ and $H_0^m(\Omega)$

Definition 1.11. Let Ω be a bounded open set of $\mathbb{R}^n$, for $m \in \mathbb{N}$, we denote by $H^m(\Omega)$ the space defined by:

$$H^m(\Omega) = \left\{u \in L^2(\Omega),\ D^\alpha u \in L^2(\Omega),\ \forall \alpha \in \mathbb{N}^n :\ |\alpha| \leq m\right\}$$

we equip it with the scalar product:

$$\forall u, v \in H^m(\Omega) : (u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u D^\alpha v \, d\Omega$$

and the associated norm is:

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(\Omega)}\right)^{1/2}.$$

Definition 1.12. Let $1 \leq p < +\infty$. $W_0^{m,p}(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$. We denote

$$H_0^m(\Omega) = W_0^{m,2}(\Omega).$$

The space $W_0^{m,p}(\Omega)$ equipped with the norm induced by $W^{m,p}(\Omega)$ is a separable Banach space, it is reflexive if $1 < p < +\infty$. $H_0^1(\Omega)$ is a Hilbert space for the scalar product of $H^1(\Omega)$.

Theorem 1.2. *Let Ω be an open set of $\mathbb{R}^n$*
- *If $\Omega = \mathbb{R}^n$, the space of test functions $D(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$.*
- *If $\Omega \neq \mathbb{R}^n$, the space of test functions $D(\Omega)$ is not dense in $H^1(\Omega)$.*

Theorem 1.3. *[17] Let Ω be an open set of $\mathbb{R}^n$ of class C^1 with boundary $\partial\Omega$. We have*

1. *$C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$.*
2. *The trace mapping γ_0 which is linear continuous defined by*

$$\begin{array}{c} C^\infty(\overline{\Omega}) \longrightarrow L^2(\partial\Omega) \\ u \longmapsto u_{|\partial\Omega} \end{array},$$

extends by continuity to a linear continuous mapping from $H^1(\Omega)$ into $L^2(\partial\Omega)$

$$\gamma_0 : H^1(\Omega) \longrightarrow L^2(\partial\Omega),\ u \longmapsto \gamma_0(u) = u_{|\partial\Omega}$$

There exists a constant $C > 0$ independent of Ω such that, for every function v in $H^1(\Omega)$, we have

$$\|\gamma_0(v)\|_{L^2(\partial\Omega)} \leq C \|v\|_{H^1(\Omega)}.$$

Proposition 1.5. *The most important properties of the trace are the following:*
- *If $u \in W^{1,p}(\Omega)$ then in fact $u_{|\partial\Omega} \in W^{1-\frac{1}{p},p}(\partial\Omega)$ and*

$$\left\| u_{|\partial\Omega} \right\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \leq C \left\| u \right\|_{W^{1,p}(\Omega)} \quad \forall u \in W^{1,p}(\Omega),$$

moreover the trace operator $u \longrightarrow u_{|\partial\Omega}$ is surjective from $W^{1,p}(\Omega)$ onto $W^{1-\frac{1}{p},p}(\partial\Omega)$.
- *The kernel of the trace operator is $W_0^{1,p}(\Omega)$, i.e.*

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega), u_{|\partial\Omega} = 0 \right\}.$$

Lemma 1.6. *Assume that Ω is a bounded open set. Then there exists a constant C (depending on Ω and p) such that*

$$\left\| u \right\|_{L^p(\Omega)} \leq C \left\| \nabla u \right\|_{L^p(\Omega)} \quad \forall u \in W_0^{1,p}(\Omega), 1 \leq p < +\infty.$$

In particular the expression $\left\| \nabla u \right\|_{L^p(\Omega)}$ is a norm on $W_0^{1,p}(\Omega)$ which is equivalent to the norm on $W^{1,p}(\Omega)$; on $H_0^1(\Omega)$ the expression $\int_\Omega \nabla u \cdot \nabla v$ is a scalar product which induces the norm $\left\| \nabla u \right\|_{L^2(\Omega)}$ equivalent to the norm $\left\| u \right\|_{H^1(\Omega)}$.

Proposition 1.7. *(Korn's inequality) [10] Let $\Omega \subset \mathbb{R}^N$ be a regular open set, with $N \geq 2$. Then there exists a constant $C > 0$ such that, for all $u \in \left[H_0^1(\Omega) \right]^N$, we have:*

$$\left\| \nabla u \right\|_{L^2(\Omega)} \leq C \left\| e(u) \right\|_{L^2(\Omega)}.$$

where $e(u) = (e_{ij}(u))_{1 \leq i,j \leq N}$ denotes the linearized strain tensor, defined componentwise by:

$$e_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

Proof. In the first member of the inequality there are all the derivatives, whereas in the second member only particular combinations of the derivatives appear. Let $u \in [D(\Omega)]^N$, we have

$$\begin{aligned} 2 \sum_{i,j=1}^N \int_\Omega \left| e_{ij}(u) \right|^2 d\Omega &= \frac{1}{2} \sum_{i,j=1}^N \int_\Omega \left| \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right|^2 d\Omega \\ &= \frac{1}{2} \int_\Omega \sum_{i,j=1}^N \left| \frac{\partial u_i}{\partial x_j} \right|^2 d\Omega + \frac{1}{2} \int_\Omega \sum_{i,j=1}^N \left| \frac{\partial u_j}{\partial x_i} \right|^2 d\Omega + \sum_{i,j=1}^N \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} d\Omega \end{aligned}$$

But since $u \in [D(\Omega)]^N$, integration by parts gives

$$\int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} d\Omega = -\int_\Omega \left(\frac{\partial}{\partial x_j} \left(\frac{\partial u_j}{\partial x_i} \right) \right) u_i d\Omega = \int_\Omega \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_j} d\Omega,$$

Since we have

$$\sum_{i,j=1}^N \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} d\Omega = \int_\Omega (\nabla \cdot u)^2 d\Omega \geq 0,$$

then

$$2 \left\| e(u) \right\|_{L^2(\Omega)}^2 \geq \left\| \nabla u \right\|_{L^2(\Omega)}^2$$

Now using the density of $[D(\Omega)]^N$ in $\left[H_0^1(\Omega) \right]^N$, we obtain

$$\left\| \nabla u \right\|_{L^2(\Omega)} \leq \sqrt{2} \left\| e(u) \right\|_{L^2(\Omega)}, \quad \forall u \in \left[H_0^1(\Omega) \right]^N.$$

□

1.3.3 Duality

Definition 1.13. [6] We define the space $W^{-1,p'}(\Omega)$ as the dual space of $W_0^{1,p}(\Omega)$. In other words an element f of $W^{-1,p'}(\Omega)$ is a continuous linear form on $W_0^{1,p}(\Omega)$, and we denote by $\langle , \rangle$ the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. Recall that

$$\|f\|_{W^{-1,p'}(\Omega)} = \sup_{v \in W_0^{1,p}(\Omega)} \frac{\langle f, v\rangle}{\|v\|_{W_0^{1,p}(\Omega)}} = \sup_{\|v\|_{W_0^{1,p}(\Omega)}=1} \langle f, v\rangle$$

and

$$\langle f, v\rangle \leq \|f\|_{W^{-1,p'}} \|v\|_{W^{1,p}(\Omega)} \text{ for all } v \in W_0^{1,p}(\Omega).$$

Theorem 1.4. *[6, 9] Let $f \in W^{-1,p'}(\Omega)$. Then there exist*

$$f^0, f^1, \ldots, f^N \quad \text{in } L^p(\Omega),$$

such that for all $v \in W_0^{1,p}(\Omega)$ we have

$$\langle f, v\rangle = \int_\Omega \left(f^0 v + \sum_{i=1}^N f^i v_{x_i} \right) \mathrm{d}x$$

and

$$\|f\|_{W^{-1,p'}(\Omega)} = \left(\int_\Omega \sum_{i=0}^N |f^i|^p \right)^{1/p}.$$

In particular if $f \in L^2(\Omega) \subset H^{-1}(\Omega)$, then

$$\langle f, v\rangle = \int_\Omega f v \, dx \text{ for all } v \in H_0^1(\Omega).$$

Theorem 1.5. *[9] Let Ω be a regular open set. If u and v are functions in $H^1(\Omega)$, they satisfy*

$$\int_\Omega u(x) \frac{\partial v}{\partial x_i} dx = - \int_\Omega v(x) \frac{\partial u}{\partial x_i} dx + \int_{\partial\Omega} u(x) v(x) \eta_i(x) ds,$$

with $\eta = (\eta_i)_{1 \leq i \leq n}$ the unit outward normal to $\partial\Omega$.

Theorem 1.6. *[6] Let Ω be a bounded open set with boundary Γ of class C^1. Then if u is a function in $H^2(\Omega)$, the mapping*

$$\begin{aligned} C_c^\infty(\Omega) &\longrightarrow L^2(\partial\Omega) \times L^2(\partial\Omega) \\ u \longmapsto (\gamma_0(u), \gamma_1(u)) &= \left(u_{|\partial\Omega}, \left(\tfrac{\partial u}{\partial n}\right)_{|\partial\Omega} = (\nabla u \cdot n)_{|\partial\Omega} \right) \end{aligned},$$

extends to a linear continuous mapping from $H^2(\Omega)$ into $L^2(\Gamma) \times L^2(\Gamma)$.

Proposition 1.8. *[9] Let Ω be a bounded open set with boundary Γ of class C^1. For any function u in $H^2(\Omega)$ and v in $H^1(\Omega)$, we have the following formula:*

$$\int_\Omega (\Delta u) v \, d\Omega = - \int_\Omega \nabla u \nabla v \, d\Omega + \int_\Gamma \gamma_1(u) \gamma_0(v) \, d\Gamma.$$

Theorem 1.7. *Let H be a Hilbert space equipped with a scalar product denoted $(.,.)$ and f a continuous linear form on H, then*

$$\exists! y \in H, \quad \forall x \in H, \quad f(x) = (y, x).$$

Theorem 1.8. *[21] Let H be a Hilbert space and $a(.,.)$ a continuous and coercive bilinear form. Then for all $\varphi \in H$ there exists a unique $u \in H$ such that*

$$a(u, v) = (\varphi, v),$$

moreover if a is symmetric then u is characterized by the property $u \in H$ and

$$\frac{1}{2} a(u, u) - (\varphi, u) = \min_{v \in H} \left\{ \frac{1}{2} a(v, v) - (\varphi, v) \right\}.$$

1.4 The Hilbert spaces $H(\mathrm{div}, \Omega)$

Definition 1.14. [15] Let Ω be a regular open set of $\mathbb{R}^n$. We set

$$H(\mathrm{div}, \Omega) = \left\{\omega \in [L^2(\Omega)]^n; \quad \mathrm{div}\, \omega \in L^2(\Omega)\right\}.$$

Equipped with the scalar product

$$(u, v) = (u, v)_{[L^2(\Omega)]^n} + (\mathrm{div}\, u, \mathrm{div}\, v)_{L^2(\Omega)}$$

the space $H(\mathrm{div}, \Omega)$ is a Hilbert space.

A consequence of the Riesz representation theorem is the following characterization of the dual of $H(\mathrm{div}, \Omega)$.

Proposition 1.9. *[15] A linear form $F \in [H(\mathrm{div}, \Omega)]'$ (dual of $H(\mathrm{div}, \Omega)$) if and only if there exists $f \in H(\mathrm{div}, \Omega)$ such that*

$$\langle F, \omega\rangle_{[H(\mathrm{div},\Omega)]' \times H(\mathrm{div},\Omega)} = (f, \omega)_{H(\mathrm{div},\Omega)\times H(\mathrm{div},\Omega)}, \forall \omega \in H(\mathrm{div}, \Omega).$$

Proposition 1.10. *[15] The subspace $[C_c^\infty(\overline{\Omega})]^n$ is dense in $H(\mathrm{div}, \Omega)$.*

Theorem 1.9. *[15] Let Ω be a bounded regular open set of class C^1. We define a mapping γ_1 by*

$$\begin{array}{c} \gamma_1 : H(\mathrm{div}, \Omega) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega) \\ \omega \longmapsto \gamma_1(\omega) = (\omega \cdot n)_{|\partial\Omega} \end{array},$$

The mapping γ_1 (the trace of the normal component of an element of $H(\mathrm{div}, \Omega)$) is linear continuous surjective from $H(\mathrm{div}, \Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$. In particular there exists a constant $C > 0$ such that, for all v in $H(\mathrm{div}, \Omega)$, we have

$$\left\|v \cdot \eta_{|\partial\Omega}\right\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C \left\|v\right\|_{H(\mathrm{div},\Omega)}.$$

Theorem 1.10. *[15] Let Ω be a bounded regular open set. If $u \in H^1(\Omega)$ and $\theta \in H(\mathrm{div}, \Omega)$ is a vector function in $[L^2(\Omega)]^n$ such that $\mathrm{div}\, \theta(x) \in L^2(\Omega)$ then*

$$\int_\Omega u(x)\, \mathrm{div}\, \theta\, dx = -\int_\Omega \nabla u(x) \cdot \theta(x)\, dx + \int_{\partial\Omega} u(x)\theta(x) \cdot \eta(x)\, ds.$$

Chapter 2

Study of a model transmission problem for the linear elasticity system with voids

In this chapter, we formulate a transmission problem (P^δ) for an elastic body with voids bonded to a thin poroelastic layer, following the theory of Cowin and Nunziato [11, 24]. Using the Lax-Milgram theorem, we prove the existence and uniqueness of the solution. We then introduce a scaling transformation, as in [1, 2, 3], and establish stability estimates for the transmission problem (P^δ) that are independent of the small parameter δ.

2.1 Elasticity with voids (Poroelasticity)

Classical linear elasticity describes the mechanical behavior of solid materials under small deformations, establishing a linear relationship between stresses and strains via Hooke's law. However, this theory assumes a continuous medium without internal defects, an assumption that becomes restrictive for porous materials or those containing microscopic voids. These voids, whether natural (as in bones, ceramics or foams) or artificial (in architected composites), significantly influence mechanical properties such as stiffness, strength or response to loads.

Theory of elasticity with voids initially developed by Cowin and Nunziato in the 1980s, the theory of elasticity with voids extends the classical framework by introducing an internal variable: the void volume fraction. This scalar quantity, often denoted ω, represents the local volume of pores relative to the material. Unlike poroelasticity, which incorporates the effects of interstitial fluids, this theory focuses on the mechanical interactions between the solid matrix and static voids.

In addition to the classical equations of continuum mechanics, the theory introduces an additional equilibrium equation to describe the redistribution of voids:

1. **Force equilibrium**
$$\sum_{j=1}^{3} D_j \sigma_{ij}\left(u, \omega\right) = -p_i,\ i = 1, 2, 3,$$
where σ_{ij} is the stress tensor, and p the body forces.

2. **Void equilibrium**
$$\sum_{j=1}^{3} D_j h_j\left(\omega^\delta\right) - g\left(u^\delta, \omega^\delta\right) = q$$
where h_j is the equilibrated stress vector, g is the intrinsic equilibrated body force and q represents external void sources

3. **Constitutive law**[11, 24]:: For an isotropic material, the stress tensor is given by:

$$\sigma_{ij}\left(u^{\delta},\omega^{\delta}\right)=2\mu e_{ij}\left(u^{\delta}\right)+\lambda e_{pp}\left(u^{\delta}\right)\delta_{ij}+\beta\omega^{\delta}\delta_{ij},\ i,j,p=1,2,3,$$

$$h_j\left(\omega^{\delta}\right)=\alpha D_j\omega^{\delta},\ j=1,2,3,$$

$$g\left(u^{\delta},\omega^{\delta}\right)=\beta e_{pp}\left(u^{\delta}\right)+\zeta\omega^{\delta},p=1,2,3,$$

Where δ_{ij} is the Kronecker delta, e_{ij} is the strain tensor defined by

$$e_{ij}\left(u^{\delta}\right)=\frac{1}{2}\left(D_i u_j^{\delta}+D_j u_i^{\delta}\right),\ i,j=1,2,3,$$

and μ, α, ζ, λ and β are material constants satisfying the inequalities:

$$\mu>0,\ \ \alpha>0,\ \ \zeta>0,\ \ \lambda>0,\ \ \beta>0,$$
$$\mu+3\lambda>0,\ \ (\mu+3\lambda)\,\zeta>3\beta^2.$$

2.2 Model transmission problem

In this chapter, we consider a three-dimensional model of linear elasticity with voids in a domain $\Omega^{\delta}=\mathbb{R}^2\times\left]-1,\delta\right[$ composed of two bonded porous elastic bodies, $\Omega_-=\mathbb{R}^2\times\left]-1,0\right[$ and a thin slab $\Omega_+^{\delta}=\mathbb{R}^2\times\left]0,\delta\right[$, we also set $\Gamma_-=\mathbb{R}^2\times\{-1\}$, $\Sigma=\mathbb{R}^2\times\{0\}$ and $\Gamma_+^{\delta}=\mathbb{R}^2\times\{\delta\}$, We assume that Ω_- and Ω_+^{δ} are homogeneous and isotropic. We limit our consideration to the case of elastostatics and to a planar geometry, and we denote by the index $+$ (resp. $-$) the restriction to Ω_+^{δ} (resp. to Ω_-). The transmission problem (P^{δ}) given by the model for the displacement $u_{\pm}^{\delta}$ and the change in the volume fraction $\omega_{\pm}^{\delta}$.

(1) Equilibrium equations in Ω_-

$$\left\{\begin{array}{c}\sum_{j=1}^{3}D_j\sigma_{-ij}\left(u_-^{\delta},\omega_-^{\delta}\right)=-p_{-i},\ i=1,2,3,\\ \sum_{j=1}^{3}D_j h_{-j}\left(\omega_-^{\delta}\right)-g_-\left(u_-^{\delta},\omega_-^{\delta}\right)=-q_-.\end{array}\right.\tag{2.1}$$

(2) Equilibrium equations in Ω_+^{δ}

$$\left\{\begin{array}{c}\sum_{j=1}^{3}D_j\sigma_{+ij}\left(u_+^{\delta},\omega_+^{\delta}\right)=0,\ i=1,2,3,\\ \sum_{j=1}^{3}D_j h_{+j}\left(\omega_+^{\delta}\right)-g_+\left(u_+^{\delta},\omega_+^{\delta}\right)=0.\end{array}\right.\tag{2.2}$$

(3) Dirichlet boundary conditions on Γ_-

$$\left\{\begin{array}{c}u_{-i}^{\delta}=0,\ i=1,2,3,\\ \omega_-^{\delta}=0.\end{array}\right.$$

(4) Neumann boundary conditions on Γ_+^{δ}

$$\left\{\begin{array}{c}\sum_{j=1}^{3}\sigma_{+ij}\left(u_+^{\delta},\omega_+^{\delta}\right)\nu_j=0,\ i=1,2,3,\\ D_3\omega_+^{\delta}=0.\end{array}\right.\tag{2.3a}$$

(5) Transmission conditions at the interface Σ

$$\left\{\begin{array}{c}u_{-i}^{\delta}=u_{+i}^{\delta},\ i=1,2,3,\\ \omega_-^{\delta}=\omega_+^{\delta},\\ \sum_{j=1}^{3}\sigma_{-ij}\left(u_-^{\delta},\omega_-^{\delta}\right)\nu_j=\sum_{j=1}^{3}\sigma_{+ij}\left(u_+^{\delta},\omega_+^{\delta}\right)\nu_j,\ i=1,2,3,\\ \alpha_-D_3\omega_-^{\delta}=\alpha_+D_3\omega_+^{\delta}.\end{array}\right.\tag{2.4}$$

Where $\nu = (\nu_1, \nu_2, \nu_3) = (0, 0, 1)$ is the unit normal vector to Σ, $\sigma_{\pm ij}$ is the stress tensor, and p_- is the body force vector, $g_\pm$ is the intrinsic equilibrated body force, $h_\pm$ is the equilibrated stress vector, q_- is the extrinsic equilibrated body force and $D_j = \frac{\partial}{\partial x_j}$; for simplicity in the following sections, we adopt the following notations:

$$\left(u_\pm^\delta, \omega_\pm^\delta\right) = \left(u_{\pm 1}^\delta, u_{\pm 2}^\delta, u_{\pm 3}^\delta, \omega_\pm^\delta\right),$$
$$\sigma_\pm\left(u_\pm^\delta, \omega_\pm^\delta\right)\nu = \left(\sigma_{\pm 13}\left(u_\pm^\delta, \omega_\pm^\delta\right), \sigma_{\pm 23}\left(u_\pm^\delta, \omega_\pm^\delta\right), \sigma_{\pm 33}\left(u_\pm^\delta, \omega_\pm^\delta\right)\right),$$
$$\left(\sigma_\pm\left(u_\pm^\delta, \omega_\pm^\delta\right)\nu, \alpha_\pm D_3\omega_\pm^\delta\right) = \left(\sigma_{\pm 13}\left(u_\pm^\delta, \omega_\pm^\delta\right), \sigma_{\pm 23}\left(u_\pm^\delta, \omega_\pm^\delta\right), \sigma_{\pm 33}\left(u_\pm^\delta, \omega_\pm^\delta\right), \alpha_\pm D_3\omega_\pm^\delta\right).$$

The constitutive equations for isotropic linear elasticity with voids [11, 24] are defined by: are defined by:

$$\sigma_{\pm ij}\left(u_\pm^\delta, \omega_\pm^\delta\right) = 2\mu_\pm e_{\pm ij}\left(u_\pm^\delta\right) + \lambda_\pm e_{\pm pp}\left(u_\pm^\delta\right)\delta_{ij} + \beta_\pm \omega_\pm^\delta \delta_{ij},\ i, j, p = 1, 2, 3,$$
$$h_{\pm j}\left(\omega_\pm^\delta\right) = \alpha_\pm D_j \omega_\pm^\delta,\ j = 1, 2, 3,$$
$$g_\pm\left(u_\pm^\delta, \omega_\pm^\delta\right) = \beta_\pm e_{\pm pp}\left(u_\pm^\delta\right) + \zeta_\pm \omega_\pm^\delta, p = 1, 2, 3,$$

Where δ_{ij} is the Kronecker delta, $e_{\pm ij}$ is the strain tensor defined by:

$$e_{\pm ij}\left(u_\pm^\delta\right) = \frac{1}{2}\left(D_i u_{\pm j}^\delta + D_j u_{\pm i}^\delta\right),\ i, j = 1, 2, 3,$$

and $\mu_\pm$, $\alpha_\pm$, $\zeta_\pm$, $\lambda_\pm$ and $\beta_\pm$ are material constants satisfying the following inequalities (see [11]):

$$\mu_\pm > 0,\ \alpha_\pm > 0,\ \zeta_\pm > 0,\ \lambda_\pm > 0,\ \beta_\pm > 0,$$
$$\mu_\pm + 3\lambda_\pm > 0,\ \ (\mu_\pm + 3\lambda_\pm)\zeta_\pm > 3\beta_\pm^2.$$

2.3 Well-posedness of the transmission problem

In this section, we will prove the existence and uniqueness of the solution to the transmission problem $\left(P^\delta\right)$. We consider the space $W\left(\Omega^\delta\right)$ defined by:

$$W\left(\Omega^\delta\right) = \left\{ \begin{array}{c} (v, \varphi) \in \left[L^2\left(\Omega^\delta\right)\right]^4 : \\ (v, \varphi)_{|\Omega_-} = (v_-, \varphi_-) \in \left[H^1\left(\Omega_-\right)\right]^4,\ (v, \varphi)_{|\Omega_+^\delta} = (v_+, \varphi_+) \in \left[H^1\left(\Omega_+^\delta\right)\right]^4, \\ v_- = v_+ \text{ on } \Sigma,\ \varphi_- = \varphi_+ \text{ on } \Sigma, \\ v_- = 0_{1\times 3} \text{ on } \Gamma_-,\ \varphi_- = 0 \text{ on } \Gamma_-. \end{array} \right\},$$

equipped with the norm

$$\|(v, \varphi)\|_{W(\Omega^\delta)} = \left[\|(v_-, \varphi_-)\|_{[H^1(\Omega_-)]^4}^2 + \|(v_+, \varphi_+)\|_{\left[H^1\left(\Omega_+^\delta\right)\right]^4}^2\right]^{1/2},$$

which is equivalent to the norm

$$|(v, \varphi)|_{W(\Omega^\delta)} = \left[|(v_-, \varphi_-)|_{[H_0^1(\Omega_-)]^4}^2 + |(v_+, \varphi_+)|_{\left[H_0^1\left(\Omega_+^\delta\right)\right]^4}^2\right]^{1/2},$$

with

$$\left\|(v_\pm, \varphi_\pm)\right\|_{\left[H^1\left(\Omega_\pm^\delta\right)\right]^4}^2 = \|\varphi_\pm\|_{H^1\left(\Omega_\pm^\delta\right)}^2 + \sum_{i=1}^3 \|v_{\pm i}\|_{H^1\left(\Omega_\pm^\delta\right)}^2,$$

and

$$\left|(v_\pm, \varphi_\pm)\right|_{\left[H_0^1\left(\Omega_\pm^\delta\right)\right]^4}^2 = |\varphi_\pm|_{H_0^1\left(\Omega_\pm^\delta\right)}^2 + \sum_{i=1}^3 |v_{\pm i}|_{H_0^1\left(\Omega_\pm^\delta\right)}^2.$$

For all $\left[\left(u^\delta, \omega^\delta\right), (v, \varphi)\right] \in \left[W\left(\Omega^\delta\right)\right]^2$, we set

$$a^-\left[\left(u_-^\delta, \omega_-^\delta\right), (v_-, \varphi_-)\right] = \int_{\Omega_-} \left[\begin{array}{c} \mathrm{tr}_3\left(\sigma_-\left(u_-^\delta, \omega_-^\delta\right) e_-(v_-)\right) + \alpha_- \nabla\omega_-^\delta \cdot \nabla\varphi_- \\ +\zeta_- \omega_-^\delta \varphi_- + \beta_-\left(D_1 u_{-1}^\delta + D_2 u_{-2}^\delta + D_3 u_{-3}^\delta\right)\varphi_- \end{array}\right] d\Omega_-,$$

$$a^+\left[\left(u_+^\delta, \omega_+^\delta\right), (v_+, \varphi_+)\right] = \int_{\Omega_+^\delta} \left[\begin{array}{c} \mathrm{tr}_3\left(\sigma_+\left(u_+^\delta, \omega_+^\delta\right) e_+(v_+)\right) + \alpha_+ \nabla\omega_+^\delta \cdot \nabla\varphi_+ \\ +\zeta_+ \omega_+^\delta \varphi_+ + \beta_+\left(D_1 u_{+1}^\delta + D_2 u_{+2}^\delta + D_3 u_{+3}^\delta\right)\varphi_+ \end{array}\right] d\Omega_+^\delta,$$

where tr_3 is the trace of a 3×3 matrix.

Theorem 2.1. *For given (p_-, q_-) in $\left[L^2(\Omega_-)\right]^4$, there exists a unique solution $\left(u^\delta, \omega^\delta\right)$ in $W\left(\Omega^\delta\right)$ to the transmission problem (P^δ). Its weak formulation is given by*

$$a\left[\left(u^\delta, \omega^\delta\right), (v, \varphi)\right] = L(v, \varphi),\ \forall (v, \varphi) \in W\left(\Omega^\delta\right),$$

with

$$a\left[\left(u^\delta, \omega^\delta\right), (v, \varphi)\right] = a^-\left[\left(u_-^\delta, \omega_-^\delta\right), (v_-, \varphi_-)\right] + a^+\left[\left(u_+^\delta, \omega_+^\delta\right), (v_+, \varphi_+)\right],$$

and

$$L(v, \varphi) = \int_{\Omega_-} (p_- \cdot v_- + q_- \varphi_-)\, d\Omega_-.$$

Proof. To prove the existence and uniqueness of the solution to the transmission problem $\left(P^\delta\right)$ in $W\left(\Omega^\delta\right)$, it suffices to verify that the linear form L and the bilinear form a satisfy the hypotheses of the Lax-Milgram theorem. It is obvious that L is continuous on $W\left(\Omega^\delta\right)$. For the continuity of a, we have

$$\begin{aligned}
&\left|a\left[\left(u^\delta, \omega^\delta\right), (v, \varphi)\right]\right| \\
&\leq \left|a^-\left[\left(u_-^\delta, \omega_-^\delta\right), (v_-, \varphi_-)\right]\right| + \left|a^+\left[\left(u_+^\delta, \omega_+^\delta\right), (v_+, \varphi_+)\right]\right| \\
&\leq \int_{\Omega_-} \left[\begin{array}{c} |\lambda_-|\left(\sum_{k=1}^3 \left|D_k u_{-k}^\delta\right|\right)\left(\sum_{k=1}^3 |D_k v_{-k}|\right) + 2|\mu_-|\sum_{i,j=1}^3 \left|e_{ij}\left(u_-^\delta\right)\right| |e_{ij}(v_-)| \\ + |\beta_-|\left|\left(D_1 u_{-1}^\delta + D_2 u_{-2}^\delta + D_3 u_{-3}^\delta\right)\right| |\varphi_-| + |\zeta_-|\left|\omega_-^\delta\right| |\varphi_-| + |\alpha_-|\left|\nabla\omega_-^\delta\right| |\nabla\varphi_-| \end{array}\right] d\Omega_- \\
&+ \int_{\Omega_+^\delta} \left[\begin{array}{c} |\lambda_+|\left(\sum_{k=1}^3 \left|D_k u_{+k}^\delta\right|\right)\left(\sum_{k=1}^3 |D_k v_{+k}|\right) + 2|\mu_+|\sum_{i,j=1}^3 \left|e_{ij}\left(u_+^\delta\right)\right| |e_{ij}(v_+)| \\ + |\beta_+|\left|\left(D_1 u_{+1}^\delta + D_2 u_{+2}^\delta + D_3 u_{+3}^\delta\right)\right| |\varphi_+| + |\zeta_+|\left|\omega_+^\delta\right| |\varphi_+| + |\alpha_+|\left|\nabla\omega_+^\delta\right| |\nabla\varphi_+| \end{array}\right] d\Omega_+^\delta.
\end{aligned}$$

Using the Cauchy-Schwarz inequality and from the definition of the space $W\left(\Omega^\delta\right)$, we obtain

$$\begin{aligned}
\sum_{i,j=1}^3 \left(\int_{\Omega_-} e_{ij}\left(u_-^\delta\right) e_{ij}(v_-)\, d\Omega_-\right) &\leq \left(\sum_{i,j=1}^3 \left\|e_{ij}\left(u_-^\delta\right)\right\|_{L^2(\Omega_-)}\right)\left(\sum_{i,j=1}^3 \|e_{ij}(v_-)\|_{L^2(\Omega_-)}\right) \\
&\leq c\left\|u_-^\delta\right\|_{[H^1(\Omega_-)]^3} \|v_-\|_{[H^1(\Omega_-)]^3} \\
&\leq c\left\|\left(u^\delta, \omega^\delta\right)\right\|_{W(\Omega^\delta)} \|(v, \varphi)\|_{W(\Omega^\delta)},
\end{aligned}$$

where c is a positive constant independent of δ, the remaining terms of a^- can also be bounded by $\left\|\left(u^\delta, \omega^\delta\right)\right\|_{W(\Omega^\delta)} \|(v, \varphi)\|_{W(\Omega^\delta)}$, which gives

$$\left|a^-\left[\left(u_-^\delta, \omega_-^\delta\right), (v_-, \varphi_-)\right]\right| \leq C_1\left\|\left(u^\delta, \omega^\delta\right)\right\|_{W(\Omega^\delta)} \|(v, \varphi)\|_{W(\Omega^\delta)},$$

where C_1 is a positive constant independent of δ. In the same way, we prove that

$$\left|a_+\left[\left(u_+^\delta, \omega_+^\delta\right), (v_+, \varphi_+)\right]\right| \leq C_2\left\|\left(u^\delta, \omega^\delta\right)\right\|_{W(\Omega^\delta)} \|(v, \varphi)\|_{W(\Omega^\delta)},$$

where C_2 is a positive constant independent of δ. Consequently, for all $\left(u^\delta, \omega^\delta\right)$, (v, φ) in $W\left(\Omega^\delta\right)$, there exists a positive constant C independent of δ such that

$$\left|a\left[\left(u^\delta, \omega^\delta\right), (v, \varphi)\right]\right| \leq C\left\|\left(u^\delta, \omega^\delta\right)\right\|_{W(\Omega^\delta)} \|(v, \varphi)\|_{W(\Omega^\delta)},$$

which proves the continuity of a on $\left[W\left(\Omega^\delta\right)\right]^2$. Let us now show that a is coercive, we have

$$
\begin{aligned}
& a\left[\left(u^\delta,\omega^\delta\right),\left(u^\delta,\omega^\delta\right)\right] \\
& = a^-\left[\left(u_-^\delta,\omega_-^\delta\right),\left(u_-^\delta,\omega_-^\delta\right)\right] + a^+\left[\left(u_+^\delta,\omega_+^\delta\right),\left(u_+^\delta,\omega_+^\delta\right)\right] \\
& = \int_{\Omega_-}\left[\begin{array}{c}
2\mu_-\left[e_{-11}\left(u_-^\delta\right)\right]^2 + 2\mu_-\left[e_{-22}\left(u_-^\delta\right)\right]^2 + 2\mu_-\left[e_{-33}\left(u_-^\delta\right)\right]^2 \\
+4\mu_-\left[e_{-21}\left(u_-^\delta\right)\right]^2 + 4\mu_-\left[e_{-13}\left(u_-^\delta\right)\right]^2 + 4\mu_-\left[e_{-23}\left(u_-^\delta\right)\right]^2 \\
+\lambda_-\left[e_{-22}\left(u_-^\delta\right) + e_{-11}\left(u_-^\delta\right) + e_{-33}\left(u_-^\delta\right)\right]^2 \\
+2\beta_-\omega_-^\delta\left[e_{-11}\left(u_-^\delta\right) + e_{-22}\left(u_-^\delta\right) + e_{-33}\left(u_-^\delta\right)\right] + \zeta_-\left(\omega_-^\delta\right)^2 + \alpha_-\left|\nabla\omega_-^\delta\right|^2
\end{array}\right] d\Omega_- \\
& + \int_{\Omega_+^\delta}\left[\begin{array}{c}
2\mu_+\left[e_{+11}\left(u_+^\delta\right)\right]^2 + 2\mu_+\left[e_{+22}\left(u_-^\delta\right)\right]^2 + 2\mu_+\left[e_{+33}\left(u_+^\delta\right)\right]^2 \\
+4\mu_+\left[e_{+21}\left(u_+^\delta\right)\right]^2 + 4\mu_+\left[e_{+13}\left(u_+^\delta\right)\right]^2 + 4\mu_+\left[e_{+23}\left(u_+^\delta\right)\right]^2 \\
+\lambda_+\left[e_{+22}\left(u_+^\delta\right) + e_{+11}\left(u_+^\delta\right) + e_{+33}\left(u_+^\delta\right)\right]^2 \\
+2\beta_+\omega_+^\delta\left[e_{+11}\left(u_+^\delta\right) + e_{+22}\left(u_+^\delta\right) + e_{+33}\left(u_+^\delta\right)\right] + \zeta_+\left(\omega_+^\delta\right)^2 + \alpha_+\left|\nabla\omega_+^\delta\right|^2
\end{array}\right] d\Omega_+^\delta,
\end{aligned}
$$

for all $\left(u^\delta,\omega^\delta\right)$, and let us try to bound $W\left(\Omega^\delta\right)$ from below by $a\left[\left(u^\delta,\omega^\delta\right),\left(u^\delta,\omega^\delta\right)\right]$. We start with $a^-\left[\left(u_-^\delta,\omega_-^\delta\right),\left(u_-^\delta,\omega_-^\delta\right)\right]$. Using the inequality $\left(3a^2+3b^2+3c^2 \geq (a+b+c)^2,\ \forall (a,b,c)\in\mathbb{R}^3\right)$, we obtain

$$
\begin{aligned}
& 2\mu_-\left[e_{-11}\left(u_-^\delta\right)\right]^2 + 2\mu_-\left[e_{-22}\left(u_-^\delta\right)\right]^2 + 2\mu_-\left[e_{-33}\left(u_-^\delta\right)\right]^2 \\
& \geq \mu_-\left[e_{-11}\left(u_-^\delta\right)\right]^2 + \mu_-\left[e_{-22}\left(u_-^\delta\right)\right]^2 + \mu_-\left[e_{-33}\left(u_-^\delta\right)\right]^2 \\
& + \frac{\mu_-}{3}\left[e_{-11}\left(u_-^\delta\right) + e_{-22}\left(u_-^\delta\right) + e_{-33}\left(u_-^\delta\right)\right]^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& a^-\left[\left(u_-^\delta,\omega_-^\delta\right),\left(u_-^\delta,\omega_-^\delta\right)\right] \\
& \geq \int_{\Omega_-}\left[\begin{array}{c}
\mu_-\left[e_{-11}\left(u_-^\delta\right)\right]^2 + \mu_-\left[e_{-22}\left(u_-^\delta\right)\right]^2 + \mu_-\left[e_{-33}\left(u_-^\delta\right)\right]^2 \\
+4\mu_-\left[e_{-21}\left(u_-^\delta\right)\right]^2 + 4\mu_-\left[e_{-13}\left(u_-^\delta\right)\right]^2 + 4\mu_-\left[e_{-23}\left(u_-^\delta\right)\right]^2 \\
+\left(\lambda_+ + \frac{\mu_-}{3}\right)\left[e_{-11}\left(u_-^\delta\right) + e_{-22}\left(u_-^\delta\right) + e_{-33}\left(u_-^\delta\right)\right]^2 \\
+2\beta_-\omega_-^\delta\left[e_{-11}\left(u_-^\delta\right) + e_{-22}\left(u_-^\delta\right) + e_{-33}\left(u_-^\delta\right)\right] + \zeta_-\left(\omega_-^\delta\right)^2 + \alpha_-\left|\nabla\omega_-^\delta\right|^2
\end{array}\right] d\Omega_-,
\end{aligned}
$$

for all $\left(u_-^\delta,\omega_-^\delta\right)$ in $\left[H^1\left(\Omega_-\right)\right]^4$. Using the inequality $\left(a^2+2ab\geq -b^2,\ \forall (a,b)\in\mathbb{R}^2\right)$ with

$$
a = \sqrt{\left(\lambda_- + \frac{\mu_-}{3}\right)}\left[e_{-11}\left(u_-^\delta\right) + e_{-22}\left(u_-^\delta\right) + e_{-33}\left(u_-^\delta\right)\right] \text{ and } b = \frac{\beta_-\omega_-^\delta}{\sqrt{\left(\lambda_- + \frac{\mu_-}{3}\right)}},
$$

we obtain

$$
\begin{aligned}
& \left(\lambda_- + \frac{\mu_-}{3}\right)\left[e_{-11}\left(u_-^\delta\right) + e_{-22}\left(u_-^\delta\right) + e_{-33}\left(u_-^\delta\right)\right]^2 \\
& + 2\beta_-\omega_-^\delta\left[e_{-11}\left(u_-^\delta\right) + e_{-22}\left(u_-^\delta\right) + e_{-33}\left(u_-^\delta\right)\right]^2 + \zeta_-\left(\omega_-^\delta\right)^2 \\
& \geq -\frac{\beta_-^2\left(\omega_-^\delta\right)^2}{\left(\lambda_- + \frac{\mu_-}{3}\right)} + \zeta_-\left(\omega_-^\delta\right)^2 = \frac{\zeta_-\left(\mu_- + 3\lambda_-\right) - 3\beta_-^2}{\left(\mu_- + 3\lambda_+\right)}\left(\omega_-^\delta\right)^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& a^-\left[\left(u_-^\delta,\omega_-^\delta\right),\left(u_-^\delta,\omega_-^\delta\right)\right] \\
& \geq \int_{\Omega_-}\left[\begin{array}{c}
\mu_-\left[e_{-11}\left(u_-^\delta\right)\right]^2 + \mu_-\left[e_{-22}\left(u_-^\delta\right)\right]^2 + \mu_-\left[e_{-33}\left(u_-^\delta\right)\right]^2 \\
+4\mu_-\left[e_{-21}\left(u_-^\delta\right)\right]^2 + 4\mu_-\left[e_{-13}\left(u_-^\delta\right)\right]^2 + 4\mu_-\left[e_{-23}\left(u_-^\delta\right)\right]^2 \\
+\frac{\zeta_-\left(\mu_-+3\lambda_-\right)-3\beta_-^2}{\left(\mu_-+3\lambda_-\right)}\left(\omega_-^\delta\right)^2 + \alpha_-\left|\nabla\omega_-^\delta\right|^2
\end{array}\right] d\Omega_-,
\end{aligned}
$$

for all $\left(u_-^\delta,\omega_-^\delta\right)$ in $\left[H^1\left(\Omega_-\right)\right]^4$. Since we have

$$
\mu_- > 0 \text{ and } \frac{\zeta_-\left(\mu_- + 3\lambda_-\right) - 3\beta_-^2}{\left(\mu_- + 3\lambda_-\right)} > 0 \text{ because } \zeta_-\left(\mu_- + 3\lambda_-\right) \geq 3\beta_-^2 \text{ and } \left(\mu_- + 3\lambda_-\right) > 0,
$$

Then, there exists a positive constant $C_1 = \min\left(\mu_-, \alpha_-, \frac{\zeta_-(\mu_-+3\lambda_-)-3\beta_-^2}{(\mu_-+3\lambda_-)}\right)$ independent of δ such that

$$a^-\left[\left(u_-^\delta, \omega_-^\delta\right), \left(u_-^\delta, \omega_-^\delta\right)\right] \geq C_1 \int_{\Omega_-} \left[\begin{array}{c} \left[e_{-11}\left(u_-^\delta\right)\right]^2 + \left[e_{-22}\left(u_-^\delta\right)\right]^2 + \left[e_{-33}\left(u_-^\delta\right)\right]^2 \\ + \left[e_{-21}\left(u_-^\delta\right)\right]^2 + \left[e_{-13}\left(u_-^\delta\right)\right]^2 + \left[e_{-23}\left(u_-^\delta\right)\right]^2 \\ + \left(\omega_-^\delta\right)^2 + \left|\nabla\omega_-^\delta\right|^2 \end{array}\right] d\Omega_-.$$

In the same way, we prove that for all $\left(u_+^\delta, \omega_+^\delta\right)$ in $\left[H^1\left(\Omega_+^\delta\right)\right]^4$, we have

$$a^+\left[\left(u_+^\delta, \omega_+^\delta\right), \left(u_+^\delta, \omega_+^\delta\right)\right] \geq C_2 \int_{\Omega_+^\delta} \left[\begin{array}{c} \left[e_{+11}\left(u_+^\delta\right)\right]^2 + \left[e_{+22}\left(u_+^\delta\right)\right]^2 + \left[e_{+33}\left(u_+^\delta\right)\right]^2 \\ + \left[e_{+21}\left(u_+^\delta\right)\right]^2 + \left[e_{+13}\left(u_+^\delta\right)\right]^2 + \left[e_{+23}\left(u_+^\delta\right)\right]^2 \\ + \left(\omega_+^\delta\right)^2 + \left|\nabla\omega_+^\delta\right|^2 \end{array}\right] d\Omega_+^\delta,$$

Where $C_2 = \min\left(\mu_+, \alpha_+, \frac{\zeta_+(\mu_++3\lambda_+)-3\beta_+^2}{(\mu_++3\lambda_+)}\right)$ is a positive constant independent of δ. Thus

$$\begin{aligned} a\left[\left(u^\delta, \omega^\delta\right), \left(u^\delta, \omega^\delta\right)\right] \geq\ & C_1 \int_{\Omega_-} \left[\begin{array}{c} \left[e_{-11}\left(u_-^\delta\right)\right]^2 + \left[e_{-22}\left(u_-^\delta\right)\right]^2 + \left[e_{-33}\left(u_-^\delta\right)\right]^2 \\ + \left[e_{-21}\left(u_-^\delta\right)\right]^2 + \left[e_{-13}\left(u_-^\delta\right)\right]^2 + \left[e_{-23}\left(u_-^\delta\right)\right]^2 \\ + \left(\omega_-^\delta\right)^2 + \alpha_-\left|\nabla\omega_-^\delta\right|^2 \end{array}\right] d\Omega_- \\ & + C_2 \int_{\Omega_+^\delta} \left[\begin{array}{c} \left[e_{+11}\left(u_+^\delta\right)\right]^2 + \left[e_{+22}\left(u_+^\delta\right)\right]^2 + \left[e_{+33}\left(u_+^\delta\right)\right]^2 \\ + \left[e_{+21}\left(u_+^\delta\right)\right]^2 + \left[e_{+13}\left(u_+^\delta\right)\right]^2 + \left[e_{+23}\left(u_+^\delta\right)\right]^2 \\ + \left(\omega_+^\delta\right)^2 + \alpha_+\left|\nabla\omega_+^\delta\right|^2 \end{array}\right] d\Omega_+^\delta. \end{aligned}$$

Thanks to Korn's inequality, we obtain

$$a\left[\left(u^\delta, \omega^\delta\right), \left(u^\delta, \omega^\delta\right)\right] \geq C\left\|\left(u^\delta, \omega^\delta\right)\right\|_{W(\Omega^\delta)}^2, \quad \text{for all } \left(u^\delta, \omega^\delta\right) \in W\left(\Omega^\delta\right),$$

Where C is a positive constant independent of δ, which proves the coercivity of a. Consequently, by the Lax-Milgram theorem, the transmission problem (P^δ) admits a unique solution in $W\left(\Omega^\delta\right)$. □

2.4 Stability of the transmission problem

2.4.1 Scaling and the scaled transmission problem

When the parameter δ varies, the domain Ω_+^δ also varies. The solution $\left(u_\pm^\delta, \omega_\pm^\delta\right)$ of the problem $\left(P^\delta\right)$ depends on δ and we cannot compare the solutions corresponding to different values of the parameter δ. We therefore perform a scaling to bring the transmission problem defined in Ω^δ to a transmission problem defined on a fixed domain, which we will denote by Ω, we perform a dilation in the normal direction of Ω_+^δ with ratio δ^{-1}. We obtain $\Omega = \mathbb{R}^2 \times]0,1[$, $\Gamma_+ = \mathbb{R}^2 \times \{1\}$, and for each point $(x_1, x_2, x_3) \in \Omega$, we associate the point $\chi^\delta(x_1, x_2, x_3) \in \Omega^\delta$ as follows:

$$\begin{array}{llll} \chi^\delta: & \Omega & \to & \Omega^\delta \\ & (x_1, x_2, x_3) & \mapsto & \chi^\delta(x_1, x_2, x_3) = \left\{\begin{array}{ll} (x_1, x_2, x_3) & \text{if } x_3 \leq 0 \\ (x_1, x_2, \delta x_3) & \text{if } x_3 > 0 \end{array}\right. . \end{array}$$

and we define the function $\left(\widetilde{u}_\pm^\delta, \widetilde{\omega}_\pm^\delta\right)$ by:

$$\begin{aligned} \widetilde{u}_-^\delta(x_1, x_2, x_3) &= u_-^\delta(x_1, x_2, x_3), \text{ for all } (x_1, x_2, x_3) \in \Omega_-, \\ \widetilde{\omega}_-^\delta(x_1, x_2, x_3) &= \omega_-^\delta(x_1, x_2, x_3), \text{ for all } (x_1, x_2, x_3) \in \Omega_-, \end{aligned}$$

and

$$\begin{array}{llll} \widetilde{u}_+^\delta: & \Omega_+ = \mathbb{R}^2 \times]0,1[& \to & \mathbb{R}^3 \\ & (x_1, x_2, x_3) & \mapsto & \left(u_{+1}^\delta(x_1, x_2, \delta x_3), u_{+2}^\delta(x_1, x_2, \delta x_3), \delta u_{+3}^\delta(x_1, x_2, \delta x_3)\right), \\ \widetilde{\omega}_+^\delta: & \Omega_+ = \mathbb{R}^2 \times]0,1[& \to & \mathbb{R} \\ & (x_1, x_2, x_3) & \mapsto & \omega_+^\delta(x_1, x_2, \delta x_3), \end{array}$$

We then obtain the following scaled problem:
(1) The equations in Ω_+^δ are rewritten in Ω_+ as:

$$(E_{+1}):\ 0 = D_1\left[(2\mu_+ + \lambda_+)\, D_1\widetilde{u}_{+1}^\delta + \lambda_+\left(D_2\widetilde{u}_{+2}^\delta + \frac{1}{\delta^2}D_3\widetilde{u}_{+3}^\delta\right) + \beta_+\widetilde{\omega}_+^\delta\right]$$
$$+\mu_+ D_2\left[D_1\widetilde{u}_{+2}^\delta + D_2\widetilde{u}_{+1}^\delta\right] + \frac{\mu_+}{\delta^2}D_3\left[D_1\widetilde{u}_{+3}^\delta + D_3\widetilde{u}_{+1}^\delta\right],$$

$$(E_{+2}):\ 0 = D_2\left[(2\mu_+ + \lambda_+)\, D_2\widetilde{u}_{+2}^\delta + \lambda_+\left(D_1\widetilde{u}_{+1}^\delta + \frac{1}{\delta^2}D_3\widetilde{u}_{+3}^\delta\right) + \beta_+\widetilde{\omega}_+^\delta\right]$$
$$+\frac{\mu_+}{\delta^2}D_3\left[D_2\widetilde{u}_{+3}^\delta + D_3\widetilde{u}_{+2}^\delta\right] + \mu_+ D_1\left[D_1\widetilde{u}_{+2}^\delta + D_2\widetilde{u}_{+1}^\delta\right],$$

$$(E_{+3}):\ 0 = \frac{1}{\delta}D_1\left[\mu_+\left(D_1\widetilde{u}_{+3}^\delta + D_3\widetilde{u}_{+1}^\delta\right)\right] + \frac{1}{\delta}D_2\left[\mu_+\left(D_2\widetilde{u}_{+3}^\delta + D_3\widetilde{u}_{+2}^\delta\right)\right]$$
$$+\frac{1}{\delta}D_3\left[\frac{(2\mu_+ + \lambda_+)}{\delta^2}D_3\widetilde{u}_{+3}^\delta + \lambda_+\left(D_1\widetilde{u}_{+1}^\delta + D_2\widetilde{u}_{+2}^\delta\right) + \beta_+\widetilde{\omega}_+^\delta\right],$$

$$(E_{+4}):\ 0 = \alpha_+\left(D_1^2\widetilde{\omega}_+^\delta + D_2^2\widetilde{\omega}_+^\delta + \frac{1}{\delta^2}D_3^2\widetilde{\omega}_+^\delta\right) - \beta_+\left(D_1\widetilde{u}_{+1}^\delta + D_2\widetilde{u}_{+2}^\delta + \frac{1}{\delta^2}D_3\widetilde{u}_{+3}^\delta\right) - \zeta_+\widetilde{\omega}_+^\delta.$$

(2) The boundary conditions on Γ_+^δ are rewritten in Γ_+ as:

$$\begin{aligned}
(BC\Gamma_{+1}):&\ 0 = \mu_+\frac{1}{\delta}D_3\widetilde{u}_{+1}^\delta + \mu_+\frac{1}{\delta}D_1\widetilde{u}_{+3}^\delta,\\
(BC\Gamma_{+2}):&\ 0 = \mu_+\frac{1}{\delta}D_3\widetilde{u}_{+2}^\delta + \mu_+\frac{1}{\delta}D_2\widetilde{u}_{+3}^\delta,\\
(BC\Gamma_{+3}):&\ 0 = \frac{(2\mu_+ + \lambda_+)}{\delta^2}D_3\widetilde{u}_{+3}^\delta + \lambda_+\left(D_1\widetilde{u}_{+1}^\delta + D_2\widetilde{u}_{+2}^\delta\right) + \beta_+\widetilde{\omega}_+^\delta,\\
(BC\Gamma_{+4}):&\ 0 = \frac{1}{\delta}D_3\widetilde{\omega}_+^\delta.
\end{aligned}$$

(3) The transmission conditions on Σ are rewritten as follows:

$$\begin{aligned}
(CT\Sigma_1):&\ \widetilde{u}_{+1}^\delta = \widetilde{u}_{-1}^\delta,\\
(CT\Sigma_2):&\ \widetilde{u}_{+2}^\delta = \widetilde{u}_{-2}^\delta,\\
(CT\Sigma_3):&\ \frac{1}{\delta}\widetilde{u}_{+3}^\delta = \widetilde{u}_{-3}^\delta,\\
(CT\Sigma_4):&\ \widetilde{\omega}_+^\delta = \widetilde{\omega}_-^\delta,
\end{aligned}$$

$$\begin{aligned}
(CT\Sigma_5):&\ \sigma_{-13}\left(\widetilde{u}_-^\delta, \widetilde{\omega}_-^\delta\right) = \frac{\mu_+}{\delta}\left(D_3\widetilde{u}_{+1}^\delta + D_1\widetilde{u}_{+3}^\delta\right),\\
(CT\Sigma_6):&\ \sigma_{-23}\left(\widetilde{u}_-^\delta, \widetilde{\omega}_-^\delta\right) = \frac{\mu_+}{\delta}\left(D_3\widetilde{u}_{+2}^\delta + D_2\widetilde{u}_{+3}^\delta\right),\\
(CT\Sigma_7):&\ \sigma_{-33}\left(\widetilde{u}_-^\delta, \widetilde{\omega}_-^\delta\right) = \frac{(2\mu_+ + \lambda_+)}{\delta^2}D_3\widetilde{u}_{+3}^\delta + \lambda_+\left(D_1\widetilde{u}_{+1}^\delta + D_2\widetilde{u}_{+2}^\delta\right) + \beta_+\widetilde{\omega}_+^\delta,
\end{aligned}$$

$$(CT\Sigma_8):\ \alpha_- D_3\widetilde{\omega}_-^\delta = \frac{\alpha_+}{\delta}D_3\widetilde{\omega}_+^\delta.$$

2.4.2 Stability for the scaled transmission problem

After scaling, the space for the study of the transmission problem $\left(P^{\delta}\right)$ in the fixed domain $\Omega_{-}\cup\Sigma\cup\Omega_{+}$ becomes:

$$W_{\delta}\left(\Omega\right)=\left\{\begin{array}{c}(v,\varphi)\in\left[L^{2}\left(\Omega\right)\right]^{4}:\\ (v,\varphi)_{|\Omega_{-}}=(v_{-},\varphi_{-})\in\left[H^{1}\left(\Omega_{-}\right)\right]^{4},\ \ (v,\varphi)_{|\Omega_{+}}=(v_{+},\varphi_{+})\in\left[H^{1}\left(\Omega_{+}\right)\right]^{4},\\ v_{-}=0_{1\times3}\text{ on }\Gamma_{-},\ \varphi_{-}=0\text{ on }\Gamma_{-},\\ (v_{-1},v_{-2},\delta v_{-3})=(v_{+1},v_{+2},v_{+3})\ \text{ on }\Sigma,\ \text{ and }\varphi_{-}=\varphi_{+}\text{ on }\Sigma,\end{array}\right\}$$

and the variational formulation of the scaled transmission problem is written:

$$\begin{array}{l}\text{Find }\left(\widetilde{u}^{\delta},\widetilde{\omega}^{\delta}\right)\in W_{\delta}\left(\Omega\right),\text{ such that }\forall\left(v,\varphi\right)\in W_{\delta}\left(\Omega\right):\\ \quad L_{\delta}\left(v,\varphi\right)=a^{-}\left[\left(\widetilde{u}_{-}^{\delta},\widetilde{\omega}_{-}^{\delta}\right),\left(v_{-},\varphi_{-}\right)\right]+a^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),\left(v_{+},\varphi_{+}\right)\right],\end{array}$$

with

$$a^{-}\left[\left(\widetilde{u}_{-}^{\delta},\widetilde{\omega}_{-}^{\delta}\right),\left(v_{-},\varphi_{-}\right)\right]=\int_{\Omega_{-}}\left[\begin{array}{c}\operatorname{tr}_{3}\left(\sigma_{-}\left(\widetilde{u}_{-}^{\delta},\widetilde{\omega}_{-}^{\delta}\right)e_{-}\left(v_{-}\right)\right)+\alpha_{-}\nabla\widetilde{\omega}_{-}^{\delta}\cdot\nabla\varphi_{-}\\ +\zeta_{-}\widetilde{\omega}_{-}^{\delta}\varphi_{-}+\beta_{-}\left(D_{1}\widetilde{u}_{-1}^{\delta}+D_{2}\widetilde{u}_{-2}^{\delta}+D_{3}\widetilde{u}_{-3}^{\delta}\right)\varphi_{-}\end{array}\right]d\Omega_{-},$$

and

$$\begin{aligned}a^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),\left(v_{+},\varphi_{+}\right)\right]&=\delta a_{+1}^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),\left(v_{+},\varphi_{+}\right)\right]+\frac{1}{\delta}a_{-1}^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),\left(v_{+},\varphi_{+}\right)\right]\\&\quad+\frac{1}{\delta^{3}}a_{-3}^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),\left(v_{+},\varphi_{+}\right)\right],\end{aligned}$$

where

$$a_{+1}^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),\left(v_{+},\varphi_{+}\right)\right]=\int_{\Omega_{+}}\left[\begin{array}{c}\left(2\mu_{+}+\lambda_{+}\right)D_{1}\widetilde{u}_{+1}^{\delta}D_{1}v_{+1}+\lambda_{+}\left(D_{2}\widetilde{u}_{+2}^{\delta}D_{1}v_{+1}+D_{1}\widetilde{u}_{+1}^{\delta}D_{2}v_{+2}\right)\\ +\beta_{+}\left(\widetilde{\omega}_{+}^{\delta}D_{1}v_{+1}+\widetilde{\omega}_{+}^{\delta}D_{2}v_{+2}\right)\\ +\mu_{+}\left[\left(D_{1}\widetilde{u}_{+2}^{\delta}+D_{2}\widetilde{u}_{+1}^{\delta}\right)\right]\left[D_{1}v_{+2}+D_{2}v_{+1}\right]+\\ +\left(2\mu_{+}+\lambda_{+}\right)D_{2}\widetilde{u}_{+2}^{\delta}D_{2}v_{+2}\\ +\alpha_{+}D_{1}\widetilde{\omega}_{+}^{\delta}D_{1}\varphi_{+}+\alpha_{+}D_{2}\widetilde{\omega}_{+}^{\delta}D_{2}\varphi_{+}+\zeta_{+}\widetilde{\omega}_{+}^{\delta}\varphi_{+}\\ +\beta_{+}\left(D_{1}\widetilde{u}_{+1}^{\delta}+D_{2}\widetilde{u}_{+2}^{\delta}\right)\varphi_{+}\end{array}\right]d\Omega,$$

$$a_{-1}^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),\left(v_{+},\varphi_{+}\right)\right]=\int_{\Omega_{+}}\left[\begin{array}{c}\lambda_{+}\left[D_{3}\widetilde{u}_{+3}^{\delta}\left(D_{1}v_{+1}+D_{2}v_{+2}\right)+\left(D_{1}\widetilde{u}_{+1}^{\delta}+D_{2}\widetilde{u}_{+2}^{\delta}\right)D_{3}v_{+3}\right]\\ +\beta_{+}\left[\widetilde{\omega}_{+}^{\delta}D_{3}v_{+3}+D_{3}\widetilde{u}_{+3}^{\delta}\varphi_{+}\right]\\ +\mu_{+}\left(D_{1}\widetilde{u}_{+3}^{\delta}+D_{3}\widetilde{u}_{+1}^{\delta}\right)\left(D_{1}v_{+3}+D_{3}v_{+1}\right)\\ +\mu_{+}\left(D_{2}\widetilde{u}_{+3}^{\delta}+D_{3}\widetilde{u}_{+2}^{\delta}\right)\left(D_{2}v_{+3}+D_{3}v_{+2}\right)\\ +\alpha_{+}D_{3}\widetilde{\omega}_{+}^{\delta}D_{3}\varphi_{+}\end{array}\right]d\Omega_{+},$$

$$a_{-3}^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),\left(v_{+},\varphi_{+}\right)\right]=\int_{\Omega_{+}}\left(2\mu_{+}+\lambda_{+}\right)D_{3}\widetilde{u}_{+3}^{\delta}D_{2}v_{+3}d\Omega_{+},$$

and

$$L_{\delta}\left(v,\varphi\right)=\int_{\Omega_{-}}\left(p_{-}\cdot v_{-}d\Omega_{-}+q_{-}\varphi_{-}\right)d\Omega_{-}.$$

Now, in order to estimate the error between the solution of the transmission problem in Ω_{-} and its first-order approximation, we first set

$$\begin{aligned}&\mathbb{A}\left(v_{-},\varphi_{-}\right)=\left\|v_{-}\right\|_{\left[H^{1}\left(\Omega_{-}\right)\right]^{3}}+\left\|\varphi_{-}\right\|_{H^{1}\left(\Omega_{-}\right)},\\&\mathbb{B}\left(v_{+},\varphi_{+}\right)=\left[\begin{array}{c}\left\|D_{1}v_{+1}\right\|_{L^{2}(\Omega_{+})}+\left\|D_{2}v_{+2}\right\|_{L^{2}(\Omega_{+})}+\left\|D_{1}v_{+2}+D_{2}v_{+1}\right\|_{L^{2}(\Omega_{+})}\\ +\left\|D_{1}\varphi_{+}\right\|_{L^{2}(\Omega_{+})}+\left\|D_{2}\varphi_{+}\right\|_{L^{2}(\Omega_{+})}+\left\|\varphi_{+}\right\|_{L^{2}(\Omega_{+})}\end{array}\right],\\&\mathbb{C}\left(v_{+},\varphi_{+}\right)=\left\|D_{1}v_{+3}+D_{3}v_{+1}\right\|_{L^{2}(\Omega_{+})}+\left\|D_{2}v_{+3}+D_{3}v_{+2}\right\|_{L^{2}(\Omega_{+})}+\left\|D_{3}\varphi_{+}\right\|_{L^{2}(\Omega_{+})},\\&\mathbb{D}\left(v_{+},\varphi_{+}\right)=\left\|D_{3}v_{+3}\right\|_{L^{2}(\Omega_{+})},\end{aligned}$$

For all $(v,\varphi)\in W_{\delta}\left(\Omega\right)$. Then, we state and prove the following stability result:

Theorem 2.2. *Let L_δ be a continuous linear form on $W_\delta(\Omega)$ such that:*

$$|L_\delta(v,\varphi)| \leq l_\delta \left[\mathbb{A}(v_-,\varphi_-) + \sqrt{\delta}\,\mathbb{B}(v_+,\varphi_+) + \frac{1}{\sqrt{\delta}}\,\mathbb{C}(v_+,\varphi_+) + \frac{1}{\delta\sqrt{\delta}}\,\mathbb{D}(v_+,\varphi_+)\right],$$

where l_δ is any function of $\delta > 0$. Then, there exists a constant $C > 0$ (independent of δ) such that the solution $\left(\widetilde{u}^\delta_\pm, \widetilde{\omega}^\delta_\pm\right)$ of the problem

$$\begin{aligned} &\textit{Find } \left(\widetilde{u}^\delta, \widetilde{\omega}^\delta\right) \in W_\delta(\Omega), \textit{ such that } \forall (v,\varphi) \in W_\delta(\Omega): \\ &\quad L_\delta(v,\varphi) = a^-\left[\left(\widetilde{u}^\delta_-, \widetilde{\omega}^\delta_-\right), (v_-,\varphi_-)\right] + a^+\left[\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right), (v_+,\varphi_+)\right], \end{aligned} \tag{2.5}$$

satisfies the following estimates:

$$\mathbb{A}\left(\widetilde{u}^\delta_-, \widetilde{\omega}^\delta_-\right) \leq C l_\delta, \tag{2.6}$$
$$\mathbb{B}\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right) \leq C\delta^{-\frac{1}{2}} l_\delta, \tag{2.7}$$
$$\mathbb{C}\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right) \leq C\delta^{\frac{1}{2}} l_\delta, \tag{2.8}$$
$$\mathbb{D}\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right) \leq C\delta^{\frac{3}{2}} l_\delta. \tag{2.9}$$

Proof. Thanks to Korn's inequality, the expression $\mathbb{A}^2\left(\widetilde{u}^\delta_-, \widetilde{\omega}^\delta_-\right)$ is equivalent to $a^-\left[\left(\widetilde{u}^\delta_-, \widetilde{\omega}^\delta_-\right), \left(\widetilde{u}^\delta_-, \widetilde{\omega}^\delta_-\right)\right]$, and we have:

$$a^+_{-1}\left[\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right), \left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right)\right] = \int_{\Omega_+} \left[\begin{array}{c} 2\lambda_+ D_3\widetilde{u}^\delta_{+3}\left(D_1\widetilde{u}^\delta_{+1} + D_2\widetilde{u}^\delta_{+2}\right) + 2\beta_+\widetilde{\omega}^\delta_+ D_3\widetilde{u}^\delta_{+3} \\ +\mu_+\left[D_1\widetilde{u}^\delta_{+3} + D_3\widetilde{u}^\delta_{+1}\right]^2 + \mu_+\left[D_2\widetilde{u}^\delta_{+3} + D_3\widetilde{u}^\delta_{+2}\right]^2 \\ +\alpha_+\left(D_3\widetilde{\omega}^\delta_+\right)^2 \end{array}\right] d\Omega_+,$$

And from the expressions of $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$, we obtain:

$$a^+_{-1}\left[\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right), \left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right)\right] \leq (2\lambda_+ + 2\beta_+)\,\mathbb{B}\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right)\mathbb{D}\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right) + (2\mu_+ + \alpha_+)\,\mathbb{C}^2\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right).$$

Since we have:

$$(2\lambda_+ + 2\beta_+)\frac{1}{\delta}(\mathbb{B}.\mathbb{D})\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right) \leq (\lambda_+ + \beta_+)\left(\delta\mathbb{B}^2\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right) + \frac{1}{\delta^3}\mathbb{D}^2\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right)\right),$$

Then, taking $\left(\widetilde{v}^\delta, \widetilde{\varphi}^\delta\right) = \left(\widetilde{u}^\delta, \widetilde{\omega}^\delta\right)$ in the variational formulation (2.5), we obtain the following estimate:

$$\left(\begin{array}{c} \mathbb{A}^2\left(\widetilde{u}^\delta_-, \widetilde{\omega}^\delta_-\right) + \delta\,\mathbb{B}^2\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right) \\ +\frac{1}{\delta}\mathbb{C}^2\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right) + \frac{1}{\delta^3}\mathbb{D}^2\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right) \end{array}\right) \leq C l_\delta \left(\begin{array}{c} \mathbb{A}\left(\widetilde{u}^\delta_-, \widetilde{\omega}^\delta_-\right) + \sqrt{\delta}\,\mathbb{B}\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right) \\ +\frac{1}{\sqrt{\delta}}\,\mathbb{C}\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right) + \frac{1}{\delta\sqrt{\delta}}\,\mathbb{D}\left(\widetilde{u}^\delta_+, \widetilde{\omega}^\delta_+\right) \end{array}\right),$$

where $C > 0$ is a positive constant independent of δ. This leads to the estimates (2.6)-(2.9). □

Chapter 3

Modeling the Effect of a Planar Thin Layer in the Framework of Linear Elasticity with Voids

In Chapter 2, we formulated a model transmission problem (P^δ) for an elastic body with voids bonded to a thin poroelastic layer of thickness δ. This chapter derives an approximate impedance condition that replaces the thin layer, reducing (P^δ) to a simpler problem on Ω_- (approximate impedance problem). The exact effect of the thin layer is captured by an impedance operator T_δ. We construct a first-order approximation of T_δ using Taylor expansion (as in [1, 2, 3]) and asymptotic expansion with scaling (see Chapter 2, Section 2.4), and we prove an error estimate between the two solutions.

3.1 Definition of the Exact Impedance Operator

Our objective in this section is to derive an approximate impedance boundary condition on the interface Σ that approximately incorporates the effect of the thin layer Ω_+^δ on Ω_- in order to reduce the transmission problem (P^δ) to a boundary value problem defined in the fixed domain Ω_-, i.e., the equilibrium equations in Ω_+^δ, the transmission conditions on Σ and the Neumann conditions on Γ_+^δ are integrated in the form of an impedance boundary condition on Σ depending on δ.

Impedance concept in modeling. Our objective is to reduce the transmission problem defined in $\Omega^\delta = \Omega_- \cup \Sigma \cup \Omega_+^\delta$ to a boundary value problem defined solely on the fixed domain Ω_-. The exact effect of the thin layer Ω_+^δ on the domain Ω_- is given by the impedance operator T_δ defined by:

$$T_\delta\left(v^\delta, \psi^\delta\right) := \left(\sigma_+\left(u_+^\delta, \omega_+^\delta\right)\nu_{|\Sigma},\ \alpha_+ D_3 \omega_{+|\Sigma}^\delta\right),$$

Where $\left(u_+^\delta, \omega_+^\delta\right)$ is the solution of the following boundary value problem:

$$(P_+^\delta):\begin{cases}\text{Equations (2.2) on } \Omega_+^\delta,\\ \text{Boundary conditions on } \Gamma_+^\delta,\\ u_+^\delta = v^\delta \text{ on } \Sigma,\ \omega_+^\delta = \psi^\delta \text{ on } \Sigma,\end{cases}$$

from the transmission conditions (2.4), it follows that:

$$\left(\sigma_-\left(u_-^\delta, \omega_-^\delta\right)\nu_{|\Sigma},\ \alpha_- D_3 \omega_{-|\Sigma}^\delta\right) = T_\delta\left(u_{-|\Sigma}^\delta, \omega_{-|\Sigma}^\delta\right),$$

and the transmission problem (P^δ) then becomes equivalent to the following impedance problem defined in Ω_-:

$$(P_-^\delta):\begin{cases}\text{Equations (2.1) on } \Omega_-,\\ u_-^\delta = 0_{1\times 3} \text{ on } \Gamma_-,\\ \omega_-^\delta = 0 \text{ on } \Gamma_-,\\ \left(\sigma_-\left(u_-^\delta, \omega_-^\delta\right)\nu_{|\Sigma},\ \alpha_- D_3\omega_{-|\Sigma}^\delta\right) = T_\delta\left(u_{-|\Sigma}^\delta, \omega_{-|\Sigma}^\delta\right) \text{ on } \Sigma.\end{cases}$$

Given that an explicit expression of the exact impedance operator T_δ is not accessible in the general case, we will simply derive an effective approximation $T_{*\delta}$ of T_δ with:

$$T_{*\delta} = \delta T_* \text{ and } T_* \left(v^\delta, \psi^\delta\right) = \left(C_1, C_2, C_3, C_4\right) \left(v^\delta, \psi^\delta\right),$$

where

$$\begin{aligned}
C_1 \left(v^\delta, \psi^\delta\right) &= \frac{4\mu_+ \left(\mu_+ + \lambda_+\right)}{\left(\lambda_+ + 2\mu_+\right)} D_1^2 v_1^\delta + \mu_+ D_2^2 v_1^\delta + \frac{2\mu_+\lambda_+}{2\mu_+ + \lambda_+} D_1 D_2 v_2^\delta + \mu_+ D_1 D_2 v_2^\delta \\
&+ \frac{2\mu_+\beta_+}{2\mu_+ + \lambda_+} D_1 \psi^\delta, \\
C_2 \left(v^\delta, \psi^\delta\right) &= \frac{4\mu_+ \left(\mu_+ + \lambda_+\right)}{\left(\lambda_+ + 2\mu_+\right)} D_2^2 v_2^\delta + \mu_+ D_1^2 v_2^\delta + \frac{2\mu_+\lambda_+}{2\mu_+ + \lambda_+} D_1 D_2 v_1^\delta + \mu_+ D_1 D_2 v_1^\delta \\
&+ \frac{2\mu_+\beta_+}{2\mu_+ + \lambda_+} D_2 \psi^\delta, \\
C_3 \left(v^\delta, \psi^\delta\right) &= 0,
\end{aligned}$$

and

$$C_4 \left(v^\delta, \psi^\delta\right) = \alpha_+ \left(D_1^2 \psi^\delta + D_2^2 \psi^\delta\right) - \frac{2\mu_+\beta_+}{2\mu_+ + \lambda_+} \left(D_1 v_1^\delta + D_2 v_2^\delta\right) - \frac{\zeta_+ \left(2\mu_+ + \lambda_+\right) - \beta_+^2}{2\mu_+ + \lambda_+} \psi^\delta.$$

The solution $(u_-^\delta, \omega_-^\delta)$ of the transmission problem (P^δ) in Ω_- is then approximated by the solution $(u_{-*}^\delta, \omega_{-*}^\delta)$ of the following approximate impedance problem:

$$\left(P_{-*}^\delta\right) : \begin{cases}
\sum_{j=1}^3 D_j \sigma_{-ij} \left(u_{-*}^\delta, \omega_{-*}^\delta\right) = -p_{-i}, \; i = 1,2,3 \text{ in } \Omega_-, \\
\sum_{j=1}^3 D_j h_{-j} \left(\omega_{-*}^\delta\right) - g_- \left(u_{-*}^\delta, \omega_{-*}^\delta\right) = -q_- \text{ on } \Omega_-, \\
u_{-*}^\delta = 0_{1\times 3} \text{ on } \Gamma_-, \; \omega_{-*}^\delta = 0 \text{ on } \Gamma_-, \\
\left(\sigma_- \left(u_{-*}^\delta, \omega_{-*}^\delta\right) \nu_{|\Sigma}, \; \alpha_- D_3 \omega_{-*|\Sigma}^\delta\right) = T_{*\delta} \left(u_{-*|\Sigma}^\delta, \omega_{-*|\Sigma}^\delta\right) \text{ on } \Sigma,
\end{cases}$$

Theorem 3.1. *For given (p_-, q_-) in $\left[L^2 (\Omega_-)\right]^4$, the boundary value problem $\left(P_{-*}^\delta\right)$ has a unique solution in the space*

$$W_* (\Omega_-) = \left\{ \begin{array}{c}
(v_-, \varphi_-) \text{ on } \left[H^1 (\Omega_-)\right]^4 : \\
(D_1 v_{-1}, D_1 v_{-2}, D_1 \varphi_-) \text{ on } \left[L^2 (\Sigma)\right]^3, \\
(D_2 v_{-1}, D_2 v_{-2}, D_2 \varphi_-) \text{ on } \left[L^2 (\Sigma)\right]^3, \\
v_- = 0_{1\times 3} \text{ in } \Gamma_-, \; \varphi_- = 0 \text{ in } \Gamma_-,
\end{array} \right\}$$

and the following error estimate holds

$$\left\| u_-^\delta - u_{-*}^\delta \right\|_{[H^1(\Omega_-)]^3} + \left\| \omega_-^\delta - \omega_{-*}^\delta \right\|_{H^1(\Omega_-)} \leq C\delta^2,$$

Where the constant C depends only on p_-, q_- and the elasticity coefficients.

The proof of Theorem 3.1 is organized in four steps. First, in Section 3.2, we derive the approximate impedance operator $T_{*\delta} = \delta T_*$ using Taylor expansion. Second, in Section 3.3, we validate this approximation via asymptotic expansion of the scaled transmission problem, computing the first terms explicitly and showing that the thin layer's effect appears at order δ. Third, in Section 3.4, we prove that the approximate impedance problem (P_{-*}^δ) is well-posed: existence and uniqueness follow from the Lax-Milgram theorem, and stability estimates are established. Fourth, in Section 3.5, we combine these results to obtain the final error estimate $\|u_-^\delta - u_{-*}^\delta\|_{[H^1]^3} + \|\omega_-^\delta - \omega_{-*}^\delta\|_{H^1} \leq C\delta^2$, which completes the proof of the theorem.

3.2 Derivation of the approximate impedance operator by Taylor expansion

As in [1, 4], we construct a first-order approximation of the impedance using Taylor expansion. The first step to obtain an approximate impedance condition is to explicitly express the normal derivatives of u_+^δ, ω_+^δ, σ_{+13}, σ_{+23}, σ_{+33}, and $D_3\omega_+^\delta$ on Σ in terms of traces on Σ of σ_{+13}, σ_{+23}, σ_{+33}, u_+^δ and ω_+^δ. Since we have

$$\sigma_{+13} = \mu_+\left(D_3u_{+1}^\delta + D_1u_{+3}^\delta\right), \quad \sigma_{+23} = \mu_+\left(D_3u_{+2}^\delta + D_2u_{+3}^\delta\right),$$

and

$$\sigma_{+33} = \left(2\mu_+ + \lambda_+\right)D_3u_{+3}^\delta + \lambda_+\left(D_1u_{+1}^\delta + D_2u_{+2}^\delta\right) + \beta_+\omega_+^\delta,$$

it follows that

$$D_3u_{+1}^\delta = \frac{1}{\mu_+}\left[\sigma_{+13} - \mu_+D_1u_{+3}^\delta\right], \quad D_3u_{+2}^\delta = \frac{1}{\mu_+}\left[\sigma_{+23} - \mu_+D_2u_{+3}^\delta\right], \tag{3.1}$$

and

$$D_3u_{+3}^\delta = \frac{1}{2\mu_+ + \lambda_+}\left[\sigma_{+33} - \lambda_+\left(D_1u_{+1}^\delta + D_2u_{+2}^\delta\right) - \beta_+\omega_+^\delta\right], \tag{3.2}$$

and then, using (3.1) and (3.2), we obtain

$$\begin{aligned}\sigma_{+11} &= \left(2\mu_+ + \lambda_+\right)D_1u_{+1}^\delta + \lambda_+D_2u_{+2}^\delta + \lambda_+D_3u_{+3}^\delta + \beta_+\omega_+^\delta \\ &= \left[\begin{array}{c}\frac{4\mu_+(\mu_++\lambda_+)}{(\lambda_++2\mu_+)}D_1u_{+1}^\delta + \frac{2\mu_+\lambda_+}{2\mu_++\lambda_+}D_2u_{+2}^\delta + \frac{2\mu_+\beta_+}{2\mu_++\lambda_+}\omega_+^\delta \\ +\frac{\lambda_+}{2\mu_++\lambda_+}\sigma_{+33}\end{array}\right],\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\sigma_{+22} &= \left(2\mu_+ + \lambda_+\right)D_2u_{+2}^\delta + \lambda_+D_1u_{+1}^\delta + \lambda_+D_3u_{+3}^\delta + \beta_+\omega_+^\delta \\ &= \left[\begin{array}{c}\frac{4\mu_+(\mu_++\lambda_+)}{(\lambda_++2\mu_+)}D_2u_{+2}^\delta + \frac{2\mu_+\lambda_+}{2\mu_++\lambda_+}D_1u_{+1}^\delta + \frac{2\mu_+\beta_+}{2\mu_++\lambda_+}\omega_+^\delta \\ +\frac{\lambda_+}{2\mu_++\lambda_+}\sigma_{+33}\end{array}\right].\end{aligned} \tag{3.4}$$

Using equations (2.2) and formulas (3.3) and (3.4) of σ_{+11} and σ_{+22} as well as the fact that $\sigma_{+13} = \sigma_{+31}$, $\sigma_{+23} = \sigma_{+32}$, we obtain

$$\left\{\begin{array}{l} D_3\sigma_{+13} = -\frac{4\mu_+(\mu_++\lambda_+)}{(\lambda_++2\mu_+)}D_1^2u_{+1}^\delta - \mu_+D_2^2u_{+1}^\delta - \frac{2\mu_+\lambda_+}{2\mu_++\lambda_+}D_1D_2u_{+2}^\delta - \mu_+D_1D_2u_{+2}^\delta \\ \qquad -\frac{2\mu_+\beta_+}{2\mu_++\lambda_+}D_1\omega_+^\delta - \frac{\lambda_+}{2\mu_++\lambda_+}D_1\sigma_{+33}, \\ D_3\sigma_{+23} = -\frac{4\mu_+(\mu_++\lambda_+)}{(\lambda_++2\mu_+)}D_2^2u_{+2}^\delta - \mu_+D_1^2u_{+2}^\delta - \frac{2\mu_+\lambda_+}{2\mu_++\lambda_+}D_1D_2u_{+1}^\delta - \mu_+D_1D_2u_{+1}^\delta \\ \qquad -\frac{2\mu_+\beta_+}{2\mu_++\lambda_+}D_2\omega_+^\delta - \frac{\lambda_+}{2\mu_++\lambda_+}D_2\sigma_{+33}, \\ D_3\sigma_{+33} = -D_1\sigma_{+13} - D_2\sigma_{+23}, \end{array}\right. \tag{3.5}$$

and

$$\begin{aligned}\alpha_+D_3^2\omega_+^\delta &= -\alpha_+\left(D_1^2\omega_+^\delta + D_2^2\omega_+^\delta\right) + \frac{2\mu_+\beta_+}{2\mu_+ + \lambda_+}\left(D_1u_{+1}^\delta + D_2u_{+2}^\delta\right) \\ &+ \frac{\zeta_+\left(2\mu_+ + \lambda_+\right) - \beta_+^2}{2\mu_+ + \lambda_+}\omega_+^\delta + \frac{\beta_+}{2\mu_+ + \lambda_+}\sigma_{+33}.\end{aligned} \tag{3.6}$$

The second step to obtain an approximate impedance condition is to use the Taylor expansion of σ_{+13}, σ_{+23}, σ_{+33}, and $D_3\omega_+^\delta$ in Ω_+^δ with respect to δ, then we write the boundary conditions on Γ_+^δ as follows:

$$\left\{\begin{array}{l}\sigma_{+13}\left(\delta\right) = \sigma_{+13}\left(0\right) + \delta D_3\sigma_{+13}\left(0\right) + ..., \\ \sigma_{+23}\left(\delta\right) = \sigma_{+23}\left(0\right) + \delta D_3\sigma_{+23}\left(0\right) + ..., \\ \sigma_{+33}\left(\delta\right) = \sigma_{+33}\left(0\right) + \delta D_3\sigma_{+33}\left(0\right) + ...,\end{array}\right. \tag{3.7}$$

and

$$D_3\omega_+^\delta(\delta) = D_3\omega_+^\delta(0) + \delta D_3^2\omega_+^\delta(0) + ... \tag{3.8}$$

Using (3.7), (3.8), (3.5), (3.6) and the boundary conditions (2.3a) on Γ_+^δ, we obtain at first order the following approximate impedance $T_{*\delta}$ defined by its expression:

$$T_{*\delta}\left(v^\delta,\psi^\delta\right) = \delta\left(C_1\left(v^\delta,\psi^\delta\right), C_2\left(v^\delta,\psi^\delta\right), C_3\left(v^\delta,\psi^\delta\right), C_4\left(v^\delta,\psi^\delta\right)\right),$$

and thus we obtain the problem $\left(P_{-*}^\delta\right)$ in Ω_-.

Remark 3.1. In the case of the two-dimensional model of linear elasticity with voids $\left(v^\delta,\psi^\delta\right) = \left(v_1^\delta, v_2^\delta, \psi^\delta\right)$, $\nu = (0,1)$, and in the same way, we can show that the expression of the impedance will be in the form:

$$\begin{aligned} T_{*\delta}\left(v^\delta,\psi^\delta\right) &= \left(\sigma_-\left(v^\delta,\psi^\delta\right)\nu,\ \alpha_- D_2\psi^\delta\right) \\ &= \left(\sigma_{-12}\left(v^\delta,\psi^\delta\right), \sigma_{-22}\left(v^\delta,\psi^\delta\right),\ \alpha_- D_2\psi^\delta\right) \\ &= \delta\left(C_1\left(v^\delta,\psi^\delta\right), C_2\left(v^\delta,\psi^\delta\right), C_3\left(v^\delta,\psi^\delta\right)\right), \end{aligned}$$

where

$$\begin{aligned} C_1\left(v^\delta,\psi^\delta\right) &= \frac{4\mu_+\left(\mu_+ + \lambda_+\right)}{\left(\lambda_+ + 2\mu_+\right)} D_1^2 v_1^\delta + \frac{2\mu_+\beta_+}{2\mu_+ + \lambda_+} D_1\psi^\delta, \\ C_2\left(v^\delta,\psi^\delta\right) &= 0, \end{aligned}$$

and

$$C_3\left(v^\delta,\psi^\delta\right) = \alpha_+ D_1^2\psi^\delta - \frac{\zeta_+\left(2\mu_+ + \lambda_+\right) - \beta_+^2}{2\mu_+ + \lambda_+}\psi^\delta - \frac{2\beta_+\mu_+}{2\mu_+ + \lambda_+} D_1 v_1^\delta.$$

3.3 Construction of the asymptotic expansion and the approximate impedance operator

In this section, we construct a first-order approximation of the impedance operator using asymptotic expansion techniques applied to the scaled transmission problem. The scaling transformation was introduced in Chapter 2, Section 2.4, where we mapped the thin layer Ω_+^δ to a fixed domain Ω_+ by dilating the normal coordinate x_3 with factor δ^{-1}. This scaling allows us to work on a domain independent of δ and to expand the solution in powers of δ.

3.3.1 Asymptotics of the scaled transmission problem

Recall the scaled transmission problem established in Chapter 2, Section 2.4.1, which consists of the equations $(E_{+1}) - (E_{+4})$ in Ω_+, the boundary conditions $(BC\Gamma_{+1}) - (BC\Gamma_{+4})$ on Γ_+, and the transmission conditions $(CT\Sigma_1) - (CT\Sigma_8)$ on Σ. We expand the solution $\left(\widetilde{u}_\pm^\delta, \widetilde{\omega}_\pm^\delta\right)$ of this scaled transmission problem in powers of δ as follows:

$$\left(\widetilde{u}_\pm^\delta, \widetilde{\omega}_\pm^\delta\right) = \left(\widetilde{u}_\pm^0, \widetilde{\omega}_\pm^0\right) + \delta\left(\widetilde{u}_\pm^1, \widetilde{\omega}_\pm^1\right) + \delta^2\left(\widetilde{u}_\pm^2, \widetilde{\omega}_\pm^2\right) + ...,$$

with $\left(\widetilde{u}_\pm^k, \widetilde{\omega}_\pm^k\right)$ for all $k \in \mathbb{N}$ independent of δ. Inserting these asymptotic expansions into the scaled transmission problem and identifying the terms with the same power of δ, we obtain the following hierarchy of boundary value problems:

$$\begin{cases} (EE_{+1})_{k-2}: \ 0 = D_1 \left[\begin{array}{c} (2\mu_+ + \lambda_+) D_1 \widetilde{u}_{+1}^{k-2} + \lambda_+ \left(D_2 \widetilde{u}_{+2}^{k-2} + D_3 \widetilde{u}_{+3}^{k} \right) \\ +\beta_+ \widetilde{\omega}_+^{k-2} \end{array} \right] \\ \quad +\mu_+ D_2 \left(D_1 \widetilde{u}_{+2}^{k-2} + D_2 \widetilde{u}_{+1}^{k-2} \right) + \mu_+ D_3 \left(D_1 \widetilde{u}_{+3}^{k} + D_3 \widetilde{u}_{+1}^{k} \right) \ \text{ in } \Omega_+, \\ (EBC\Gamma_{+1})_{k-1}: \ \mu_+ D_3 \widetilde{u}_{+1}^{k} + \mu_+ D_1 \widetilde{u}_{+3}^{k} = 0 \ \text{ on } \Gamma_+, \\ (ECT\Sigma_5)_{k-1}: \ \sigma_{-13}\left(\widetilde{u}_-^{k-1}, \widetilde{\omega}_-^{k-1} \right) = \mu_+ \left(D_3 \widetilde{u}_{+1}^{k} + D_1 \widetilde{u}_{+3}^{k} \right) \ \text{ on } \Sigma, \\ (ECT\Sigma_1)_k: \ \widetilde{u}_{+1}^{k} = \widetilde{u}_{-1}^{k} \ \text{ on } \Sigma, \end{cases} \tag{3.9}$$

$$\begin{cases} (EE_{+2})_{k-2}: \ 0 = D_2 \left[\begin{array}{c} (2\mu_+ + \lambda_+) D_2 \widetilde{u}_{+2}^{k-2} + \lambda_+ \left(D_1 \widetilde{u}_{+1}^{k-2} + D_3 \widetilde{u}_{+3}^{k} \right) \\ +\beta_+ \widetilde{\omega}_+^{k-2} \end{array} \right] \\ \quad +\mu_+ D_1 \left[\left(D_1 \widetilde{u}_{+2}^{k-2} + D_2 \widetilde{u}_{+1}^{k-2} \right) \right] + \mu_+ D_3 \left[\left(D_2 \widetilde{u}_{+3}^{k} + D_3 \widetilde{u}_{+2}^{k} \right) \right] \ \text{ in } \Omega_+, \\ (EBC\Gamma_{+2})_{k-1}: \ \mu_+ D_3 \widetilde{u}_{+2}^{k} + \mu_+ D_2 \widetilde{u}_{+3}^{k} = 0 \ \text{ on } \Gamma_+, \\ (ECT\Sigma_6)_{k-1}: \ \sigma_{-23}\left(\widetilde{u}_-^{k-1}, \widetilde{\omega}_-^{k-1} \right) = \mu_+ \left(D_3 \widetilde{u}_{+2}^{k} + D_2 \widetilde{u}_{+3}^{k} \right) \ \text{ on } \Sigma, \\ (ECT\Sigma_2)_{k-1}: \widetilde{u}_{+2}^{k} = \widetilde{u}_{-2}^{k} \ \text{ on } \Sigma, \end{cases} \tag{3.10}$$

$$\begin{cases} (EE_{+3})_{k-3}: \ 0 = D_3 \left[\begin{array}{c} (2\mu_+ + \lambda_+) D_3 \widetilde{u}_{+3}^{k} + \lambda_+ \left(D_1 \widetilde{u}_{+1}^{k-2} + D_2 \widetilde{u}_{+2}^{k-2} \right) \\ +\beta_+ \widetilde{\omega}_+^{k-2} \end{array} \right] \\ \quad +\mu_+ D_1 \left[\left(D_1 \widetilde{u}_{+3}^{k-2} + D_3 \widetilde{u}_{+1}^{k-2} \right) \right] + \mu_+ D_2 \left[\left(D_2 \widetilde{u}_{+3}^{k-2} + D_3 \widetilde{u}_{+2}^{k-2} \right) \right] \ \text{ in } \Omega_+, \\ (EBC\Gamma_{+3})_{k-2}: \left[\begin{array}{c} (2\mu_+ + \lambda_+) D_3 \widetilde{u}_{+3}^{k} \\ +\lambda_+ \left(D_1 \widetilde{u}_{+1}^{k-2} + D_2 \widetilde{u}_{+2}^{k-2} \right) + \beta_+ \widetilde{\omega}_+^{k-2} \end{array} \right] = 0 \ \text{ on } \Gamma_+, \\ (ECT\Sigma_7)_{k-2}: \ \sigma_{-33}\left(\widetilde{u}_-^{k-2}, \widetilde{\omega}_-^{k-2} \right) \\ \quad = \left[(2\mu_+ + \lambda_+) D_3 \widetilde{u}_{+3}^{k} + \lambda_+ \left(D_1 \widetilde{u}_{+1}^{k-2} + D_2 \widetilde{u}_{+2}^{k-2} \right) + \beta_+ \widetilde{\omega}_+^{k-2} \right] \ \text{ on } \Sigma, \\ (ECT\Sigma_3)_{k-1}: \ \widetilde{u}_{+3}^{k} = \widetilde{u}_{-3}^{k-1} \ \text{ on } \Sigma, \end{cases} \tag{3.11}$$

and

$$\begin{cases} (EE_{+4})_{k-2}: \ 0 = \left[\begin{array}{c} \alpha_+ \left(D_1^2 \widetilde{\omega}_+^{k-2} + D_2^2 \widetilde{\omega}_+^{k-2} + D_3^2 \widetilde{\omega}_+^{k} \right) \\ -\beta_+ \left(D_1 \widetilde{u}_{+1}^{k-2} + D_2 \widetilde{u}_{+2}^{k-2} + D_3 \widetilde{u}_{+3}^{k} \right) - \zeta_+ \widetilde{\omega}_+^{k-2} \end{array} \right] \ \text{ in } \Omega_+, \\ (EBC\Gamma_{+4})_{k-1}: \ D_3 \widetilde{\omega}_+^{k} = 0 \ \text{ on } \Gamma_+, \\ (ECT\Sigma_8)_{k-1}: \ \alpha_- D_3 \widetilde{\omega}_-^{k-1} = \alpha_+ D_3 \widetilde{\omega}_+^{k} \ \text{ on } \Sigma, \\ (ECT\Sigma_4)_k: \ \widetilde{\omega}_+^{k} = \widetilde{\omega}_-^{k} \ \text{ on } \Sigma, \end{cases} \tag{3.12}$$

where we set

$$\begin{gathered} \widetilde{u}_+^{-2} = \widetilde{u}_+^{-1} = 0_{1\times 2} \text{ in } \Omega_+, \ \widetilde{\omega}_+^{-2} = \widetilde{\omega}_+^{-1} = 0 \ \text{ in } \Omega_+, \\ \widetilde{u}_{+1}^{-2} = \widetilde{u}_{+1}^{-1} = \widetilde{u}_{+2}^{-2} = \widetilde{u}_{+2}^{-1} = \widetilde{\omega}_+^{-2} = \widetilde{\omega}_+^{-1} = 0 \ \text{ on } \Gamma_+, \\ \widetilde{u}_{+1}^{-2} = \widetilde{u}_{+1}^{-1} = \widetilde{u}_{-2}^{-1} = \widetilde{\omega}_+^{-2} = \widetilde{\omega}_+^{-1} = \widetilde{u}_{-3}^{-1} = 0 \ \text{ on } \Sigma, \\ \sigma_{-13}\left(\widetilde{u}_-^{-1}, \widetilde{\omega}_-^{-1} \right) = \sigma_{-23}\left(\widetilde{u}_-^{-1}, \widetilde{\omega}_-^{-1} \right) = \sigma_{-33}\left(\widetilde{u}_-^{-2}, \widetilde{\omega}_-^{-2} \right) = D_3 \widetilde{\omega}_-^{-1} = 0 \ \text{ on } \Sigma. \end{gathered}$$

Remark 3.2. 1) Thanks to the scaling technique in the thin layer Ω_+^δ, the terms of the asymptotic expansion of $\widetilde{u}_+^\delta$ and $\widetilde{\omega}_+$ can be computed explicitly by recurrence as functions of the terms of the asymptotic expansion of $\widetilde{u}_-^\delta$ and $\widetilde{\omega}_-^\delta$ on Σ.

2) The equations in problems (3.9)−(3.12) are linear second-order differential equations with respect to the variable x_3.

3) By integrating the equations of problems (3.9)−(3.12) and using the transmission conditions on Σ as well as the boundary conditions on Γ_+, we can compute $\widetilde{u}_+^\delta$ and $\widetilde{\omega}_+^\delta$ as functions of u_-^k and ω_-^k on Σ.

3.3.2 Calculation of order 0 terms in Ω_+.

For $k = 0$, integration by parts in x_3 in problems (3.9)−(3.12) gives the following results:

$$\begin{aligned}
\widetilde{u}_{+3}^0 &= 0 \ \text{ in } \Omega_+,\\
\widetilde{u}_{+1}^0 &= u_{-1|\Sigma}^0 \ \text{ in } \ \Omega_+,\\
\widetilde{u}_{+2}^0 &= u_{-2|\Sigma}^0 \ \text{ in } \ \Omega_+,\\
\widetilde{\omega}_+^0 &= \omega_{-|\Sigma}^0 \ \text{ in } \Omega_+.
\end{aligned}$$

3.3.3 Calculation of order 1 terms in Ω_+.

For $k = 1$, integration by parts in x_3 in problems (3.9)−(3.12) gives the following results:

$$\begin{aligned}
\widetilde{u}_{+3}^1 &= u_{-3|\Sigma}^0 \ \text{ in } \Omega_+,\\
\widetilde{u}_{+1}^1 &= u_{-1|\Sigma}^1 - x_3 D_1 u_{-3|\Sigma}^0 \ \text{ in } \Omega_+,\\
\widetilde{u}_{+2}^1 &= u_{-2|\Sigma}^1 - x_3 D_2 u_{-3|\Sigma}^0 \ \text{ in } \Omega_+,\\
\widetilde{\omega}_+^1 &= \omega_{-|\Sigma}^1 \ \text{ in } \Omega_+.
\end{aligned}$$

3.3.4 Calculation of order 2 terms in Ω_+.

For $k = 2$, in the same way, we obtain

$$\widetilde{u}_{+3}^2 = u_{-3|\Sigma}^1 - \frac{x_3}{(2\mu_+ + \lambda_+)}\left[\lambda_+\left(D_1 u_{-1|\Sigma}^0 + D_2 u_{-2|\Sigma}^0\right) + \beta_+ \omega_{-|\Sigma}^0\right] \ \text{ in } \Omega_+,$$

$$\begin{aligned}
\widetilde{u}_{+1}^2 = u_{-1|\Sigma}^2 + x_3 &\left[\begin{array}{c} \frac{4(\mu_+ + \lambda_+)}{(2\mu_+ + \lambda_+)} D_1^2 u_{-1|\Sigma}^0 + \frac{2\mu_+ + 3\lambda_+}{(2\mu_+ + \lambda_+)} D_1 D_2 u_{-2|\Sigma}^0 \\ + \frac{2\beta_+}{(2\mu_+ + \lambda_+)} D_1 \omega_{-|\Sigma}^0 + D_2^2 u_{-1|\Sigma}^0 - D_1 u_{-3|\Sigma}^1 \end{array}\right] \\
- \frac{x_3^2}{2} &\left[\frac{(4\mu_+ + 2\lambda_+)}{(2\mu_+ + \lambda_+)} D_1^2 u_{-1|\Sigma}^0 + D_1 D_2 u_{-2|\Sigma}^0 + D_2^2 u_{-1|\Sigma}^0\right] \ \text{ in } \Omega_+,
\end{aligned}$$

$$\begin{aligned}
\widetilde{u}_{+2}^2 = u_{-2|\Sigma}^2 + x_3 &\left[\begin{array}{c} \frac{4(\mu_+ + \lambda_+)}{(2\mu_+ + \lambda_+)} D_2^2 u_{-2|\Sigma}^0 + \frac{2\mu_+ + 3\lambda_+}{(2\mu_+ + \lambda_+)} D_1 D_2 u_{-1|\Sigma}^0 \\ + \frac{2\beta_+}{(2\mu_+ + \lambda_+)} D_2 \omega_{-|\Sigma}^0 + D_1^2 u_{-2|\Sigma}^0 - D_2 u_{-3|\Sigma}^1 \end{array}\right] \\
- \frac{x_3^2}{2} &\left[\begin{array}{c} \frac{(4\mu_+ + 3\lambda_+)}{(2\mu_+ + \lambda_+)} D_2^2 u_{-2|\Sigma}^0 + \frac{2\mu_+ + 2\lambda_+}{(2\mu_+ + \lambda_+)} D_1 D_2 u_{-1|\Sigma}^0 \\ + \frac{\beta_+}{(2\mu_+ + \lambda_+)} D_2 \omega_{-|\Sigma}^0 + D_1^2 u_{-2|\Sigma}^0 \end{array}\right] \ \text{ in } \Omega_+,
\end{aligned}$$

$$\widetilde{\omega}_+^2 = \omega^2_{-|\Sigma} + x_3 \left[\begin{array}{c} D_1^2\omega^0_{-|\Sigma} + D_2^2\omega^0_{-|\Sigma} - \frac{2\mu_+\beta_+}{\alpha_+(2\mu_++\lambda_+)}\left(D_1\widetilde{u}^0_{-1|\Sigma} + D_2 u^0_{-2|\Sigma}\right) \\ -\frac{\zeta_+(2\mu_++\lambda_+)-\beta_+^3}{\alpha_+(2\mu_++\lambda_+)}\omega^0_{-|\Sigma} \end{array} \right]$$
$$+ \frac{x_3^2}{2} \left[\begin{array}{c} \frac{2\mu_+\beta_+}{\alpha_+(2\mu_++\lambda_+)}\left(D_1\widetilde{u}^0_{-1|\Sigma} + D_2 u^0_{-2|\Sigma}\right) + \frac{\zeta_+(2\mu_++\lambda_+)-\beta_+^3}{\alpha_+(2\mu_++\lambda_+)}\omega^0_{-|\Sigma} \\ -D_1^2\omega^0_{-|\Sigma} - D_2^2\omega^0_{-|\Sigma} \end{array} \right] \quad \text{in } \Omega_+,$$

3.3.5 Calculation of order 3 terms in Ω_+.

For $k = 3$, problem (3.11) gives the following relation:

$$\sigma_{-33}\left(u^1_{-|\Sigma}, \omega^1_{-|\Sigma}\right) = (2\mu_+ + \lambda_+) D_3\widetilde{u}^3_{+3} + \lambda_+\left(D_1\widetilde{u}^1_{+1} + D_2\widetilde{u}^1_{+2}\right) + \beta_+\widetilde{\omega}^1_+ = 0 \quad \text{on } \Sigma.$$

Then, the asymptotic expansion of the equations in Ω_-, the boundary conditions on Γ_- and the transmission conditions $(CT\Sigma_5) - (CT\Sigma_8)$ on Σ, allow us to obtain the following results:

3.3.6 Order 0 terms in Ω_- :

At order 0, the terms $\left(u^0_-, \omega^0_-\right)$ satisfy the following boundary value problem in Ω_-:

$$(P_-)_0 : \left\{ \begin{array}{l} \sum_{j=1}^3 D_j\sigma_{-ij}\left(u^0_-, \omega^0_-\right) = -p_{-i}, \; i = 1,2,3 \;\text{ in } \Omega_-, \\ \sum_{j=1}^3 D_j h_{-j}\left(\omega^0_-\right) - g_-\left(u^0_-, \omega^0_-\right) = -q_- \;\text{ in } \Omega_-, \\ u^0_- = \omega^0_- = 0 \;\text{ on } \Gamma_-, \\ \left(\sigma_-\left(u^0_-, \omega^0_-\right)_{|\Sigma}\nu, \; \alpha_- D_3\omega^0_{-|\Sigma}\right) = 0 \;\text{ on } \Sigma. \end{array} \right.$$

This means that at order 0, the thin layer Ω_+^δ has no effect on Ω_-.

3.3.7 Order 1 terms in Ω_- :

At order 1, the terms $\left(u^1_-, \omega^1_-\right)$ satisfy the following boundary value problem in Ω_-:

$$(P_-)_1 : \left\{ \begin{array}{l} \sum_{j=1}^3 D_j\sigma_{-ij}\left(u^1_-, \omega^1_-\right) = 0, \; i = 1,2,3 \;\text{ in } \Omega_-, \\ \sum_{j=1}^3 D_j h_{-j}\left(\omega^1_-\right) - g_-\left(u^1_-, \omega^1_-\right) = 0 \;\text{ in } \Omega_-, \\ u^1_- = \omega^1_- = 0 \;\text{ on } \Gamma_-, \\ \left(\sigma_-\left(u^1_-, \omega^1_-\right)_{|\Sigma}\nu, \; \alpha_- D_3\omega^1_{-|\Sigma}\right) = T_*\left(u^0_{-|\Sigma}, \omega^0_{-|\Sigma}\right) \;\text{ on } \Sigma. \end{array} \right.$$

This means that at order 1, the effect of the thin layer Ω_+^δ on Ω_- is represented by forces and equilibrated forces exerted on Σ.

3.3.8 First-order approximation of the impedance.

Due to the linearity of the constitutive equations, we find that the function $\left(u^\delta_{-*}, \omega^\delta_{-*}\right)$ defined by: $\left(u^\delta_{-*}, \omega^\delta_{-*}\right) = \left(u^0_- + \delta u^1_-, \omega^0_- + \delta\omega^1_-\right)$ satisfies:

$$\left(\sigma_-\left(u^\delta_{-*}, \omega^\delta_{-*}\right)\nu_{|\Sigma}, \; \alpha_- D_3\omega^\delta_{-*|\Sigma}\right) = \left(\sigma_-\left(u^0_-, \omega^0_-\right)\nu_{|\Sigma}, \; \alpha_- D_3\omega^0_{-|\Sigma}\right) + \delta\left(\sigma_-\left(u^1_-, \omega^1_-\right)\nu_{|\Sigma}, \; \alpha_- D_3\omega^1_{-|\Sigma}\right). \tag{3.13}$$

Using the boundary conditions on Σ of problems $(P_-)_0$ and $(P_-)_1$, we obtain from (3.13) the following relation:

$$\left(\sigma_-\left(u^\delta_{-*}, \omega^\delta_{-*}\right)\nu_{|\Sigma}, \; \alpha_- D_3\omega^\delta_{-*|\Sigma}\right) = \delta\left(\sigma_-\left(u^1_-, \omega^1_-\right)\nu_{|\Sigma}, \; \alpha_- D_3\omega^1_{-|\Sigma}\right) = \delta T_*\left(u^0_{-|\Sigma}, \omega^0_{-|\Sigma}\right), \tag{3.14}$$

And since we have $\left(u^0_{-|\Sigma}, \omega^0_{-|\Sigma}\right) = \left(u^\delta_{-*|\Sigma} - \delta u^1_{-|\Sigma}, \omega^\delta_{-*|\Sigma} - \delta\omega^1_{-|\Sigma}\right)$, then relation (3.14) can be written as follows:

$$\left(\sigma_-\left(u^\delta_{-*}, \omega^\delta_{-*}\right)\nu_{|\Sigma},\ \alpha_- D_3\omega^\delta_{-*|\Sigma}\right) = \delta T_*\left(u^\delta_{-*|\Sigma}, \omega^\delta_{-*|\Sigma}\right) - \delta^2 T_*\left(u^1_{-|\Sigma}, \omega^1_{-|\Sigma}\right),$$

Thus, if we neglect the order 2 terms with respect to δ, we find the same approximate impedance as that obtained by the Taylor formula in section 3.2.

3.4 Well-posedness of the approximate impedance problem.

3.4.1 Existence and uniqueness of the solution to the impedance problem

We consider the space:

$$W_*\left(\Omega_-\right) = \left\{\begin{array}{c} \left(v_-, \varphi_-\right) \in \left[H^1\left(\Omega_-\right)\right]^4 : \\ \left(D_1 v_{-1}, D_1 v_{-2}, D_1\varphi_-\right) \in \left[L^2\left(\Sigma\right)\right]^3, \\ \left(D_2 v_{-1}, D_2 v_{-2}, D_2\varphi_-\right) \in \left[L^2\left(\Sigma\right)\right]^3, \\ v_- = 0_{1\times 3} \text{ on } \Gamma_-,\ \varphi_- = 0 \text{ on } \Gamma_-. \end{array}\right\},$$

equipped with the following norm:

$$\left\|\left(v_-, \varphi_-\right)\right\|_{W_*(\Omega_-)} = \left[\left\|\left(v_-, \varphi_-\right)\right\|^2_{[H^1(\Omega_-)]^4} + \left\|\left(v_{-1|\Sigma}, v_{-2|\Sigma}, \varphi_{-|\Sigma}\right)\right\|^2_{[H^1(\Sigma)]^3}\right]^{1/2},$$

with

$$\left\|\left(v_-, \varphi_-\right)\right\|^2_{[H^1(\Omega_-)]^4} = \left\|\varphi_-\right\|^2_{H^1(\Omega_-)} + \sum_{i=1}^{3}\left\|v_{-i}\right\|^2_{H^1(\Omega_-)},$$

and

$$\left\|\left(v_{-1|\Sigma}, v_{-2|\Sigma}, \varphi_{-|\Sigma}\right)\right\|^2_{[H^1(\Sigma)]^3} = \left\|\varphi_{-|\Sigma}\right\|^2_{H^1(\Sigma)} + \sum_{i=1}^{2}\left\|v_{-i|\Sigma}\right\|^2_{H^1(\Sigma)}.$$

For all $\left(u^\delta_{-*}, \omega^\delta_{-*}\right)$ and $\left(v_-, \varphi_-\right)$ in $W_*\left(\Omega_-\right)$, we set:

$$a^-\left[\left(u^\delta_{-*}, \omega^\delta_{-*}\right), \left(v_-, \varphi_-\right)\right] = \int_{\Omega_-}\left[\begin{array}{c} \mathrm{tr}_3\left(\sigma_-\left(u^\delta_{-*}, \omega^\delta_{-*}\right)e_-\left(v_-\right)\right) + \alpha_-\nabla\omega^\delta_{-*}\cdot\nabla\varphi_- \\ +\zeta_-\omega^\delta_{-*}\varphi_- + \beta_-\left(D_1 u^\delta_{-*1} + D_2 u^\delta_{-*2} + D_3 u^\delta_{-*3}\right)\varphi_- \end{array}\right] d\Omega_-,$$

$$a_\Sigma\left[\left(u^\delta_{-*}, \omega^\delta_{-*}\right), \left(v_-, \varphi_-\right)\right]$$
$$= \int_\Sigma\left[\begin{array}{c} \frac{4\mu_+(\mu_+ + \lambda_+)}{(\lambda_+ + 2\mu_+)}D_1 u^\delta_{-*1}D_1 v_{-1} + \mu_+ D_2 u^\delta_{-*1}D_2 v_{-1} + \frac{2\mu_+\lambda_+}{2\mu_+ + \lambda_+}D_2 u^\delta_{-*2}D_1 v_{-1} \\ +\mu_+ D_1 u^\delta_{-*1}D_2 v_{-2} + \frac{2\mu_+\beta_+}{2\mu_+ + \lambda_+}\omega^\delta_{-*}D_1 v_{-1} + \frac{2\mu_+\beta_+}{2\mu_+ + \lambda_+}\omega^\delta_{-*}D_2 v_{-2} \\ +\frac{4\mu_+(\mu_+ + \lambda_+)}{(\lambda_+ + 2\mu_+)}D_2 u^\delta_{-*2}D_2 v_{-2} + \mu_+ D_1 u^\delta_{-*2}D_1 v_{-2} + \frac{2\mu_+\lambda_+}{2\mu_+ + \lambda_+}D_1 u^\delta_{-*1}D_2 v_{-2} \\ +\mu_+ D_1 u^\delta_{-*1}D_2 v_{-2} + \alpha_+ D_1\omega^\delta_{-*}D_1\varphi_- + \alpha_+ D_2\omega^\delta_{-*}D_2\varphi_- \\ -\frac{2\mu_+\beta_+}{2\mu_+ + \lambda_+}\left(D_1 u^\delta_{-*1} + D_2 u^\delta_{-*2}\right)\varphi_- + \frac{\zeta_+(2\mu_+ + \lambda_+) - \beta_+^2}{2\mu_+ + \lambda_+}\omega^\delta_{-*}\varphi_- \end{array}\right] d\Sigma,$$

and we state and prove the following theorem:

Theorem 3.2. *For a pair* $\left(p_-, q_-\right)$ *in* $\left[L^2\left(\Omega_-\right)\right]^4$, *there exists a unique solution* $\left(u^\delta_{-*}, \omega^\delta_{-*}\right)$ *in* $W_*\left(\Omega_-\right)$ *to the impedance problem* $\left(P^\delta_{-*}\right)$. *Its weak formulation is given by:*

$$a^-_\delta\left[\left(u^\delta_{-*}, \omega^\delta_{-*}\right), \left(v_-, \varphi_-\right)\right] = L\left(v_-, \varphi_-\right),\ \forall\left(v_-, \varphi_-\right) \in W_*\left(\Omega_-\right)$$

with

$$a^-_\delta\left[\left(u^\delta_{-*}, \omega^\delta_{-*}\right), \left(v_-, \varphi_-\right)\right] = a^-\left[\left(u^\delta_{-*}, \omega^\delta_{-*}\right), \left(v_-, \varphi_-\right)\right] + \delta a_\Sigma\left[\left(u^\delta_{-*}, \omega^\delta_{-*}\right), \left(v_-, \varphi_-\right)\right],$$

and

$$L\left(v_-, \varphi_-\right) = \int_{\Omega_-}\left(p_-\cdot v_- + q_-\varphi_-\right) d\Omega_-.$$

Proof. It is clear that L is continuous on $W_*(\Omega_-)$. For the continuity of a_δ^-, as in the proof of Theorem 10, we have:

$$\left|a^-\left[\left(u_{-*}^\delta,\omega_{-*}^\delta\right),(v_-,\varphi_-)\right]\right| \leq C_1\left\|\left(u_{-*}^\delta,\omega_{-*}^\delta\right)\right\|_{W_*(\Omega_-)}\|(v_-,\varphi_-)\|_{W_*(\Omega_-)},$$

where C_1 is a positive constant independent of δ, and using the Cauchy-Schwarz inequality as well as the definition of the space $W_*(\Omega_-)$, we obtain:

$$\left|a_\Sigma\left[\left(u_{-*}^\delta,\omega_{-*}^\delta\right),(v_-,\varphi_-)\right]\right| \leq C_2\left\|\left(u_{-*}^\delta,\omega_{-*}^\delta\right)\right\|_{W_*(\Omega_-)}\|(v_-,\varphi_-)\|_{W_*(\Omega_-)},$$

where C_2 is a positive constant independent of δ, and then we obtain:

$$\begin{aligned}\left|a_\delta^-\left[\left(u_{-*}^\delta,\omega_{-*}^\delta\right),(v_-,\varphi_-)\right]\right| &\leq \left|a^-\left[\left(u_{-*}^\delta,\omega_{-*}^\delta\right),(v_-,\varphi_-)\right]\right| + \delta\left|a_\Sigma\left[\left(u_{-*}^\delta,\omega_{-*}^\delta\right),(v_-,\varphi_-)\right]\right| \\ &\leq C_1\left\|\left(u_{-*}^\delta,\omega_{-*}^\delta\right)\right\|_{W_*(\Omega_-)}\|(v_-,\varphi_-)\|_{W_*(\Omega_-)} \\ &\quad + C_2\delta\left\|\left(u_{-*}^\delta,\omega_{-*}^\delta\right)\right\|_{W_*(\Omega_-)}\|(v_-,\varphi_-)\|_{W_*(\Omega_-)},\end{aligned}$$

Since $\delta << 1$, then for all $\left(u_{-*}^\delta,\omega_{-*}^\delta\right),(v_-,\varphi_-)$ in $W_*(\Omega_-)$, we obtain:

$$\left|a_\delta^-\left[\left(u_{-*}^\delta,\omega_{-*}^\delta\right),(v_-,\varphi_-)\right]\right| \leq C\left\|\left(u_{-*}^\delta,\omega_{-*}^\delta\right)\right\|_{W_*(\Omega_-)}\|(v_-,\varphi_-)\|_{W_*(\Omega_-)}.$$

with $C = C_1 + C_2$, which proves the continuity of a_δ^- on $[W_*(\Omega_-)]^2$. For the coercivity of a_δ^-, as in the proof of Theorem 11, for all $\left(u_{-*}^\delta,\omega_{-*}^\delta\right)$ in $W_*(\Omega_-)$, we have:

$$a^-\left[\left(u_{-*}^\delta,\omega_{-*}^\delta\right),\left(u_{-*}^\delta,\omega_{-*}^\delta\right)\right] \geq C_1\left\|\left(u_{-*}^\delta,\omega_{-*}^\delta\right)\right\|_{[H^1(\Omega_-)]^4}^2,$$

with C_1 is a positive constant independent of δ, and since we have:

$$2D_2u_{-*2}^\delta D_1u_{-*1}^\delta \geq -\left(D_2u_{-*2}^\delta\right)^2 - \left(D_1u_{-*1}^\delta\right)^2,$$

then for all $\left(u_{-*}^\delta,\omega_{-*}^\delta\right)$ in $W_*(\Omega_-)$, we obtain:

$$\begin{aligned}a_\Sigma\left[\left(u_{-*}^\delta,\omega_{-*}^\delta\right),\left(u_{-*}^\delta,\omega_{-*}^\delta\right)\right] &= \int_\Sigma\left[\begin{array}{c}\frac{4\mu_-(\mu_-+\lambda_-)}{(\lambda_-+2\mu_-)}\left(D_1u_{-*1}^\delta\right)^2 + \frac{4\mu_-(\mu_-+\lambda_-)}{(\lambda_-+2\mu_-)}\left(D_2u_{-*2}^\delta\right)^2 \\ +\frac{2\mu_-(2\mu_-+3\lambda_-)}{2\mu_-+\lambda_-}D_2u_{-*2}^\delta D_1u_{-*1}^\delta + \mu_-\left(D_1u_{-*2}^\delta\right)^2 + \mu_-\left(D_2u_{-*1}^\delta\right)^2 \\ +\alpha_-\left(D_1\omega_{-*}^\delta\right)^2 + \alpha_-\left(D_2\omega_{-*}^\delta\right)^2 + \frac{\zeta_-(2\mu_-+\lambda_-)-\beta_-^2}{2\mu_-+\lambda_-}\left(\omega_{-*}^\delta\right)^2\end{array}\right] \\ &\geq \int_\Sigma\left[\begin{array}{c}\mu_-\left(D_1u_{-*1}^\delta\right)^2 + \mu_-\left(D_2u_{-*2}^\delta\right)^2 + \mu_-\left(D_1u_{-*2}^\delta\right)^2 + \mu_-\left(D_2u_{-*1}^\delta\right)^2 \\ +\alpha_-\left(D_1\omega_{-*}^\delta\right)^2 + \alpha_-\left(D_2\omega_{-*}^\delta\right)^2 + \frac{\zeta_-(2\mu_-+\lambda_-)-\beta_-^2}{2\mu_-+\lambda_-}\left(\omega_{-*}^\delta\right)^2\end{array}\right]d\Sigma \\ &\geq C_2\left\|\left(u_{-*1|\Sigma}^\delta,u_{-*2|\Sigma}^\delta,\omega_{-*|\Sigma}^\delta\right)\right\|_{[H^1(\Sigma)]^3}^2,\end{aligned}$$

with $C_2 = \min\left(\mu_-,\alpha_-,\frac{\zeta_-(2\mu_-+\lambda_-)-\beta_-^2}{2\mu_-+\lambda_-}\right)$ is a positive constant independent of δ, and then for all $\left(u_{-*}^\delta,\omega_{-*}^\delta\right)$ in $W_*(\Omega_-)$, we obtain:

$$\begin{aligned}a_\delta^-\left[\left(u_{-*}^\delta,\omega_{-*}^\delta\right),\left(u_{-*}^\delta,\omega_{-*}^\delta\right)\right] &= a^-\left[\left(u_{-*}^\delta,\omega_{-*}^\delta\right),\left(u_{-*}^\delta,\omega_{-*}^\delta\right)\right] + \delta a_\Sigma\left[\left(u_{-*}^\delta,\omega_{-*}^\delta\right),\left(u_{-*}^\delta,\omega_{-*}^\delta\right)\right] \\ &\geq C_1\left\|\left(u_{-*}^\delta,\omega_{-*}^\delta\right)\right\|_{[H^1(\Omega_-)]^4}^2 + \delta C_2\left\|\left(u_{-*1|\Sigma}^\delta,u_{-*2|\Sigma}^\delta,\omega_{-*|\Sigma}^\delta\right)\right\|_{[H^1(\Sigma)]^3}^2 \\ &\geq \min(C_1,C_2\delta)\left[\left\|\left(u_{-*}^\delta,\omega_{-*}^\delta\right)\right\|_{[H^1(\Omega_-)]^4}^2 + \left\|\left(u_{-*1|\Sigma}^\delta,u_{-*2|\Sigma}^\delta,\omega_{-*|\Sigma}^\delta\right)\right\|_{[H^1(\Sigma)]^3}^2\right] \\ &\geq C\delta\left\|\left(u_{-*}^\delta,\omega_{-*}^\delta\right)\right\|_{W_*(\Omega_-)}^2,\end{aligned}$$

where C is a positive constant independent of δ, which proves the coercivity of a_δ^-. Consequently, by the Lax-Milgram theorem, the impedance problem (P_{-*}^δ) admits a unique solution in $W_*(\Omega_-)$. $\square$

3.4.2 Stability for the impedance problem

In order to estimate the error between the solution of the approximate impedance and its first-order approximation, we first define

$$\begin{aligned}
A(v_-,\varphi_-) &= \|(v_-,\varphi_-)\|_{[H^1(\Omega_-)]^4},\\
B(v_-,\varphi_-) &= \left[\begin{array}{c} \|D_1 v_{-1}\|_{L^2(\Sigma)} + \|D_2 v_{-1}\|_{L^2(\Sigma)} + \|D_1 v_{-2}\|_{L^2(\Sigma)} + \\ \|D_2 v_{-2}\|_{L^2(\Sigma)} + \|D_1 \varphi_-\|_{L^2(\Sigma)} + \|D_2 \varphi_-\|_{L^2(\Sigma)} + \|\varphi_-\|_{L^2(\Sigma)} \end{array}\right].
\end{aligned}$$

For all $(v_-,\varphi_-) \in W_*(\Omega_-)$. Then, we state and prove the following stability result.

Theorem 3.3. *Let L_δ be a given linear form on $W_*(\Omega_-)$ satisfying the following bound as a function of δ:*

$$|L_\delta(v_-,\varphi_-)| \le m_\delta\left(A(v_-,\varphi_-) + \sqrt{\delta}\ B(v_-,\varphi_-)\right),\quad \text{for all } (v_-,\varphi_-) \in W_*(\Omega_-),$$

where m_δ is any function of $\delta > 0$. Then, there exists a constant $C > 0$ (independent of δ) such that the solution $\left(u^\delta_{-*},\omega^\delta_{-*}\right)$ of the problem

$$\left\{\begin{array}{l} \text{find } \left(u^\delta_{-*},\omega^\delta_{-*}\right) \in W_*(\Omega_-), \text{ such that for all } (v_-,\varphi_-) \in W_*(\Omega_-): \\ a^-\left[\left(u^\delta_{-*},\omega^\delta_{-*}\right),(v_-,\varphi_-)\right] + \delta a_\Sigma\left[\left(u^\delta_{-*},\omega^\delta_{-*}\right),(v_-,\varphi_-)\right] = L_\delta(v_-,\varphi_-), \end{array}\right. \tag{3.15}$$

satisfies the following estimates

$$A\left(u^\delta_{-*},\omega^\delta_{-*}\right) \le C m_\delta, \tag{3.16}$$

$$B\left(u^\delta_{-*},\omega^\delta_{-*}\right) \le C\ \delta^{-1/2} m_\delta. \tag{3.17}$$

Proof. Thanks to Korn's inequality, the expression $A^2\left(u^\delta_{-*},\omega^\delta_{-*}\right)$ is equivalent to $a^-\left[\left(u^\delta_{-*},\omega^\delta_{-*}\right),\left(u^\delta_{-*},\omega^\delta_{-*}\right)\right]$, and we have

$$a_\Sigma\left(\left(u^\delta_{-*},\omega^\delta_{-*}\right),\left(u^\delta_{-*},\omega^\delta_{-*}\right)\right) \ge C\ B^2\left(u^\delta_{-*},\omega^\delta_{-*}\right),$$

where $C > 0$ is a positive constant independent of δ. Then taking $(v_-,\varphi_-) = \left(u^\delta_{-*},\omega^\delta_{-*}\right)$ in the variational formulation (3.15), we obtain the following estimate:

$$A^2\left(u^\delta_{-*},\omega^\delta_{-*}\right) + \delta\ B^2\left(u^\delta_{-*},\omega^\delta_{-*}\right) \le K m_\delta\left(A\left(u^\delta_{-*},\omega^\delta_{-*}\right) + \sqrt{\delta}\ B\left(u^\delta_{-*},\omega^\delta_{-*}\right)\right),$$

where $K > 0$ is a positive constant independent of δ. This directly gives the estimates (3.16) and (3.17). ☐

3.4.3 Asymptotic Expansion for the Approximate Impedance Problem

Setting

$$\begin{aligned}
u^\delta_{-*} &= u^0_{-*} + \delta u^1_{-*} + \delta^2 u^2_{-*} + \ldots\\
\omega^\delta_{-*} &= \omega^0_{-*} + \delta \omega^1_{-*} + \delta^2 \omega^2_{-*} + \ldots
\end{aligned}$$

And inserting these expansions into the approximate impedance problem $\left(P^\delta_{-*}\right)$, we obtain a hierarchy of equations and boundary conditions. At order 0, we obtain:

$$(P_{-*})_0 : \left\{\begin{array}{l} \sum_{j=1}^3 D_j \sigma_{-ij}\left(u^0_{-*},\omega^0_{-*}\right) = -p_{-i},\ i = 1,2,3 \ \text{ in } \Omega_-, \\ \sum_{j=1}^3 D_j h_{-j}\left(\omega^0_{-*}\right) - g_-\left(u^0_{-*},\omega^0_{-*}\right) = -q_- \text{ in } \Omega_-, \\ u^0_{-*} = 0_{1\times 3} \text{ on } \Gamma_-,\ \omega^0_{-*} = 0 \text{ on } \Gamma_-, \\ \left(\sigma_-\left(u^0_{-*},\omega^0_{-*}\right)_{|\Sigma}\nu,\ \alpha_- D_3 \omega^0_{-*|\Sigma}\right) = 0_{1\times 4} \ \text{ on } \Sigma. \end{array}\right.$$

At order 1, we obtain:

$$(P_{-*})_1 : \begin{cases} \sum_{j=1}^{3} D_j \sigma_{-ij}\left(u_{-*}^1, \omega_{-*}^1\right) = 0,\ i = 1,2,3 \text{ in } \Omega_-, \\ \sum_{j=1}^{3} D_j h_{-j}\left(\omega_{-*}^1\right) - g_-\left(u_{-*}^1, \omega_{-*}^1\right) = 0 \text{ in } \Omega_-, \\ u_{-*}^1 = 0_{1\times 3} \text{ on } \Gamma_-,\ \omega_{-*}^1 = 0 \text{ on } \Gamma_-, \\ \left(\sigma_-\left(u_{-*}^1, \omega_{-*}^1\right)_{|\Sigma} \nu,\ \alpha_- D_3 \omega_{-*|\Sigma}^1\right) = T_*\left(u_{-*|\Sigma}^0, \omega_{-*|\Sigma}^0\right) \text{ on } \Sigma, \end{cases}$$

Remark 3.3. The terms $\left(u_{-*}^0, \omega_{-*}^0\right)$ and $\left(u_-^0, \omega_-^0\right)$ (respectively $\left(u_{-*}^1, \omega_{-*}^1\right)$ and $\left(u_-^1, \omega_-^1\right)$) of the expansion of $\left(u_{-*}^\delta, \omega_{-*}^\delta\right)$ and $\left(u_-^\delta, \omega_-^\delta\right)$ are solutions of the same boundary value problem at order 0 (respectively at order 1). Thus, by uniqueness, we have:

$$\left(u_{-*}^0, \omega_{-*}^0\right) = \left(u_-^0, \omega_-^0\right) \quad \text{and} \quad \left(u_{-*}^1, \omega_{-*}^1\right) = \left(u_-^1, \omega_-^1\right).$$

3.5 Error estimates and optimality

3.5.1 Error estimate for the transmission problem

Setting

$$u_-^{(\delta,1)} = u_-^0 + \delta u_-^1,\quad u_+^{(\delta,1)} = \widetilde{u}_+^0 + \delta \widetilde{u}_+^1 + \delta^2\left(0,0,\widetilde{u}_{+3}^2\right),$$
$$\omega_-^{(\delta,1)} = \omega_-^0 + \delta \omega_-^1,\quad \omega_+^{(\delta,1)} = \widetilde{\omega}_+^0 + \delta \widetilde{\omega}_+^1,$$

and using the problems $(P_-)_0$ and $(P_-)_1$ satisfied by $\left(u_-^0, \omega_-^0\right)$ and $\left(u_-^1, \omega_-^1\right)$, as well as problems (3.9)−(3.11) for $k \in \{0,1,2,3\}$, we obtain:

$$\begin{aligned} L_\delta(v,\varphi) &= a_{-3}^+\left[\left(\widetilde{u}_+^3, \widetilde{\omega}_+^3\right), (v_+, \varphi_+)\right] + \delta \int_{\Omega_+} \mu_+ D_3 \widetilde{u}_{+1}^2 D_3 v_{+1} d\Omega_+ \\ &+ \delta \int_{\Omega_+} \mu_+ D_3 \widetilde{u}_{+2}^2 D_3 v_{+2} d\Omega_+ + \delta \int_{\Omega_+} \alpha_+ D_3 \widetilde{\omega}_+^2 D_3 \varphi_+ d\Omega_+ \\ &- \delta \int_{\Omega_+} \left[\mu_+ D_1 \widetilde{u}_{+3}^2 D_1 v_{+3} + \mu_+ D_2 \widetilde{u}_{+3}^2 D_2 v_{+3}\right] d\Omega_+ - \delta^2 a_{+1}^+\left[\left(\widetilde{u}_+^1, \widetilde{\omega}_+^1\right), (v_+, \varphi_+)\right] \end{aligned}$$

where

$$\begin{aligned} L_\delta(v,\varphi) = a^-&\left[\left(\widetilde{u}_-^\delta - u_-^{(\delta,1)}, \widetilde{\omega}_-^\delta - \omega_-^{(\delta,1)}\right), (v_-, \varphi_-)\right] \\ &+ a^+\left[\left(\widetilde{u}_+^\delta - u_+^{(\delta,1)}, \widetilde{\omega}_+^\delta - \omega_+^{(\delta,1)}\right), (v_+, \varphi_+)\right], \end{aligned}$$

and thus, we obtain:

$$|L_\delta(v,\varphi)| \leq C\delta^{3/2}\left[\mathbb{A}(v_-,\varphi_-) + \sqrt{\delta}\,\mathbb{B}(v_+,\varphi_+) + \frac{1}{\sqrt{\delta}}\,\mathbb{C}(v_+,\varphi_+) + \frac{1}{\delta\sqrt{\delta}}\,\mathbb{D}(v_+,\varphi_+)\right],$$

which implies, by the stability result (see Theorem 2.2), the following error estimate:

$$\left\|u_-^\delta - u_-^{(\delta,1)}\right\|_{[H^1(\Omega_-)]^3} + \left\|\omega_-^\delta - \omega_-^{(\delta,1)}\right\|_{H^1(\Omega_-)} \leq C\delta^{3/2}, \tag{3.18}$$

where C is a positive constant independent of δ.

Remark 3.4. We took

$$u_{+3}^{(\delta,1)} = \widetilde{u}_{+3}^0 + \delta \widetilde{u}_{+3}^1 + \delta^2 \widetilde{u}_{+3}^2,$$

in order to satisfy the transmission condition.

$$u_{+3}^{(\delta,1)} = \delta u_{-3}^{(\delta,1)} \quad \text{on } \Sigma,$$

and, consequently, we have $\left(u_\pm^{(\delta,1)}, \omega_\pm^{(\delta,1)}\right)$ in the space $W_\delta(\Omega)$.

Remark 3.5. If p_- and q_- are regular, we can obtain all the terms of the asymptotic expansion $\left(u_-^k, \omega_-^k\right)$ for $k \geq 2$ and prove an analogous error estimate. For example, we can prove:

$$\left\| u_-^\delta - u_-^{(\delta,2)} \right\|_{[H^1(\Omega_-)]^3} + \left\| \omega_-^\delta - \omega_-^{(\delta,2)} \right\|_{H^1(\Omega_-)} \leq C\delta^{5/2}, \tag{3.19}$$

where

$$u_-^{(\delta,2)} = u_-^0 + \delta u_-^1 + \delta^2 u_-^2 \quad \text{and} \quad \omega_-^{(\delta,2)} = \omega_-^0 + \delta \omega_-^1 + \delta^2 \omega_-^2.$$

3.5.2 Error estimate for the impedance problem

Setting

$$u_{-*}^{(\delta,1)} = u_{-*}^0 + \delta u_{-*}^1,$$
$$\omega_{-*}^{(\delta,1)} = \omega_{-*}^0 + \delta \omega_{-*}^1,$$

and

$$\begin{aligned} & a^- \left[\left(u_{-*}^\delta - u_{-*}^{(\delta,1)}, \omega_{-*} - \omega_{-*}^{(\delta,1)} \right), (v_-, \varphi_-) \right] \\ & + \delta a_\Sigma \left[\left(u_{-*}^\delta - u_{-*}^{(\delta,1)}, \omega_{-*} - \omega_{-*}^{(\delta,1)} \right), (v_-, \varphi_-) \right] = L_\delta (v_-, \varphi_-), \end{aligned}$$

and using the problems $(P_{-*})_0$ and $(P_{-*})_1$ satisfied by $(u_{-*}^0, \omega_{-*}^0)$ and $(u_{-*}^1, \omega_{-*}^1)$, we obtain:

$$L_\delta (v_-, \varphi_-) = -\delta^2 a_\Sigma \left[\left(u_{-*}^1, \omega_{-*}^1 \right), (v_-, \varphi_-) \right]$$

and thus, we obtain:

$$|L_\delta (v_-, \varphi_-)| \leq C\delta^{3/2} \left[A (v_-, \varphi_-) + \sqrt{\delta}\; B (v_-, \varphi_-) \right],$$

which implies, by the stability result (see Theorem 3.3), the following error estimate:

$$\left\| u_{-*}^\delta - u_{-*}^{(\delta,1)} \right\|_{[H^1(\Omega_-)]^3} + \left\| \omega_{-*}^\delta - \omega_{-*}^{(\delta,1)} \right\|_{H^1(\Omega_-)} \leq C\delta^{3/2}, \tag{3.20}$$

where C is a positive constant independent of δ.

Remark 3.6. If p_- and q_- are regular, we can obtain all the terms of the asymptotic expansion $\left(u_{-*}^k, \omega_{-*}^k\right)$ for $k \geq 2$ and prove an analogous error estimate. For example, we can prove:

$$\left\| u_{-*}^\delta - u_{-*}^{(\delta,2)} \right\|_{[H^1(\Omega_-)]^3} + \left\| \omega_{-*}^\delta - \omega_{-*}^{(\delta,2)} \right\|_{H^1(\Omega_-)} \leq C\delta^{5/2}, \tag{3.21}$$

where

$$u_{-*}^{(\delta,2)} = u_{-*}^0 + \delta u_{-*}^1 + \delta^2 u_{-*}^2 \text{ and } \omega_{-*}^{(\delta,2)} = \omega_{-*}^0 + \delta \omega_{-*}^1 + \delta^2 \omega_{-*}^2.$$

3.5.3 Final error estimate and optimality

This subsection is devoted to the error estimate between the solution of the transmission problem in Ω_- and the solution of the approximate impedance problem. Since we have $\left(u_{-*}^0, \omega_{-*}^0\right) = \left(u_-^0, \omega_-^0\right)$ and $\left(u_{-*}^1, \omega_{-*}^1\right) = \left(u_-^1, \omega_-^1\right)$ (see Remark 3.3), then, applying the triangle inequality, we can write:

$$\begin{aligned} & \left\| \widetilde{u}_-^\delta - u_{-*}^\delta \right\|_{[H^1(\Omega_-)]^3} + \left\| \widetilde{\omega}_-^\delta - \omega_{-*}^\delta \right\|_{H^1(\Omega_-)} \\ & \leq \left\| \widetilde{u}_-^\delta - u_-^0 - \delta u_-^1 \right\|_{[H^1(\Omega_-)]^3} + \left\| u_{-*}^\delta - u_{-*}^0 - \delta u_{-*}^1 \right\|_{[H^1(\Omega_-)]^3} \\ & + \left\| \widetilde{\omega}_-^\delta - \omega_-^0 - \delta \omega_-^1 \right\|_{H^1(\Omega_-)} + \left\| \omega_{-*}^\delta - \omega_{-*}^0 - \delta \omega_{-*}^1 \right\|_{H^1(\Omega_-)}, \end{aligned}$$

and by virtue of (3.20) and (3.18), we obtain:

$$\left\|\widetilde{u}_{-}^{\delta}-u_{-*}^{\delta}\right\|_{[H^1(\Omega_-)]^3}+\left\|\widetilde{\omega}_{-}^{\delta}-\omega_{-*}^{\delta}\right\|_{H^1(\Omega_-)}\leq C\delta^{3/2},$$

where the constant C depends only on p_-, q_- and the elasticity coefficients. Indeed, if the data p_- and q_- are sufficiently regular so that we can determine $\left(u_{-}^{2},\omega_{-}^{2}\right)$ and $\left(u_{-*}^{2},\omega_{-*}^{2}\right)$, then, using (3.19), the last error estimate, which is not optimal, can be improved as follows:

$$\begin{aligned}\left\|\widetilde{u}_{-}^{\delta}-u_{-}^{0}-\delta u_{-}^{1}\right\|_{[H^1(\Omega_-)]^3} &\leq \left\|\widetilde{u}_{-}^{\delta}-u_{-}^{0}-\delta u_{-}^{1}-\delta^2 u_{-}^{2}\right\|_{[H^1(\Omega_-)]^3}+\delta^2\left\|u_{-}^{2}\right\|_{[H^1(\Omega_-)]^3}\\ &\leq C_1\delta^{5/2}+C_2\delta^2\leq C\delta^2,\end{aligned}$$

and

$$\begin{aligned}\left\|\widetilde{\omega}_{-}^{\delta}-\omega_{-}^{0}-\delta\omega_{-}^{1}\right\|_{H^1(\Omega_-)} &\leq \left\|\widetilde{\omega}_{-}^{\delta}-\omega_{-}^{0}-\delta\omega_{-}^{1}-\delta^2\omega_{-}^{2}\right\|_{H^1(\Omega_-)}+\delta^2\left\|\omega_{-}^{2}\right\|_{H^1(\Omega_-)}\\ &\leq C_1\delta^{5/2}+C_2\delta^2\leq C\delta^2.\end{aligned}$$

Using (3.21), similar estimates for $\left(u_{-*}^{\delta}-u_{-*}^{0}-\delta u_{-*}^{1},\omega_{-*}^{\delta}-\omega_{-*}^{0}-\delta\omega_{-*}^{1}\right)$ can be proved in the same manner as described above.

Conclusion

In this work, we successfully derived effective impedance boundary conditions for a thin poroelastic slab bonded to a thin elastic body with voids. Using asymptotic expansion techniques and Taylor series, we constructed an approximate impedance operator of order 1 that replaces the thin layer. Rigorous error estimates are established, showing that the error between the solution of the transmission problem and the solution of the approximate impedance problem is of order $\mathcal{O}(\delta^2)$ in the H^1-norm. This approach provides a powerful tool for reducing computational costs in multi-scale simulations of porous materials and opens perspectives for extensions to nonlinear or dynamic problems.

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