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**On the certain realized homogeneous spaces defined from the
Littlewood-Paley decomposition theory**

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Abstract

This thesis in the field of functional analysis, it concerns the study of some characterizations of realized homogeneous spaces satisfying

$$r > 0: \quad \|f(\lambda \cdot)\| \sim \lambda^r \|f\| \quad \forall \lambda > 0, \forall f.$$

In particular, we have obtained some embeddings and inequalities in homogeneous Besov-Triebel-Lizorkin spaces, their realized versions, *BMO* and Morrey spaces.

Also, this work contains further results as quasi-norms in smooth functions spaces.

Keywords: Besov spaces, Besov-type spaces, homogeneous spaces, *BMO* spaces, Triebel-Lizorkin spaces, realizations, Littlewood-Paley decomposition, Riesz operator.

ملخص

هذه الأطروحة في مجال التحليل الدالي، تختص بدراسة الفضاءات المنتجانسة المحققة التي تحقق العلاقة التالية:

$$r > 0: \|f(\lambda)\| \sim \lambda^r \|f\| \forall \lambda > 0, \forall f$$

وبصفه خاصة لقد وجدنا إحتواءات و متراجحات في فضاءات بيزوف و ليزوركين-تريبال المتجانسة، فضاءاتها المحققة، فضاء بيامو و موري. بالإضافة إلى ذلك هذا العمل يحتوي على نتائج اخرى المتمثلة في وجود تنظيم لفضاء الدوال الملساء.

كلمات المفتاحية: فضاءات بيزوف، فضاءات بيزوف المعممة، الفضاءات المتجانسة، فضاء بيامو، فضاءات تريبال-ليزوركين، التحقيقات، مؤثر ريز.

Résumé

Cette thèse dans le domaine de l'analyse fonctionnelle, elle concerne l'étude de certaines caractérisations des espaces homogènes réalisés satisfaisant

$$r > 0: \quad \|f(\lambda \cdot)\| \sim \lambda^r \|f\| \quad \forall \lambda > 0, \forall f.$$

En particulier, nous avons obtenu des inclusions et des inégalités dans les espaces homogènes de Besov-Triebel-Lizorkin, leurs versions réalisées, les espaces de *BMO* et Morrey.

De plus, ce travail contient des résultats supplémentaires en tant que quasi-norme dans les espaces de fonctions lisses.

Mots clés: les espaces de Besov, les espaces de type Besov, les espaces homogènes, l'espace de *BMO*, les espaces de Triebel-Lizorkin, les réalisations, la décomposition de Littlewood-Paley, l'opérateur de Riesz.

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Introduction

Harmonic analysis is one of the fundamental theories in functional analysis. Various functional spaces, including Sobolev, Besov, and Triebel-Lizorkin spaces, have been defined and characterized using it. Many authors have been drawn to describing these spaces since the 1970s.

Besov and Triebel-Lizorkin spaces are utilized in the domain of nonlinear analysis, which involves the study of nonlinear partial differential equations see e.g., [48], variational problems and geometric analysis. They are employed in the study of mappings between manifolds in the regularity theory, as well as in the analysis of singularities and complex features of solutions.

In this thesis we are interested by studying the homogeneous spaces which satisfied the following equivalent

$$r > 0: \quad \|f(\lambda \cdot)\| \sim \lambda^r \|f\| \quad \forall \lambda > 0, \forall f.$$

The advantage of these spaces that is the sample manipulation of their norms with dilatation, in particular homogeneous Besov space $\dot{B}_{p,q}^s$, homogeneous Triebel-Lizorkin space $\dot{F}_{p,q}^s$ and homogeneous Besov type space $\dot{B}_{p,q}^{s,\tau}$, where these spaces are defined by distributions modulo all polynomials, since for any polynomial f on \mathbb{R}^n it holds $\|f\| = 0$. To eliminate this observation we use the realization to resolve some mathematical problems, where the concept of realizations was first introduced by G. Bourdaud in [11] for the homogeneous Besov spaces $\dot{B}_{p,q}^s$ and he has established this theory and demonstrated the existence of realizations in Sobolev spaces and homogeneous Triebel-Lizorkin space $\dot{F}_{p,q}^s$ for all $1 \leq p < \infty$ and $1 \leq q \leq \infty$, see [14, 15], then M. Moussai extended these results to the cases if $0 < p, q < 1$, see [32]. Recently, M. Benallia and M. Moussai in their paper [3] treated the case $p = \infty$. Then, the realization of $\dot{B}_{p,q}^{s,\tau}$ has been introduced by F. Bensaid and M. Moussai see [6]. Realization of homogeneous spaces are used in a lot of areas such as Navier-Stokes equation, see [20], pointwise multiplications, Hardy type estimations, etc see. e.g [4, 9, 10, 34].

The objective of our research is to demonstrate the possibility of extending certain findings from inhomogeneous areas to homogeneous ones. However, this cannot be achieved easily, which is why we employ the idea of realization.

Our thesis is divided into four chapters

- In the first one, we give some classical inequalities, we defined Besov spaces, Triebel-Lizorkin spaces, Besov type spaces and Lizorkin-Triebel type spaces by using the Littlewood-Paley decomposition. After that, we review some fundamental concepts about these spaces. A lot of these properties will be used in the upcoming chapters.
- In the second chapter, we recall the identification between BMO space and the realized space $\dot{F}_{\infty,2}^0$ of Triebel-Lizorkin proved recently in [23]. This result is based on the fact that BMO space cannot coincides with $\dot{F}_{\infty,2}^0$, since for any polynomial f of degree ≥ 1 it holds that $\|f\|_{\dot{F}_{\infty,2}^0} = 0$ while $\|f\|_{BMO} = \infty$, and embedding between BMO space and the realized space $\dot{F}_{p,q}^{n/p}$. Then, we give some applications of our results such as embeddings between the homogeneous Sobolev space, the inhomogeneous Besov spaces, their homogeneous and realized counterparts, and the relation of Riesz operator and Bessel potentials operator with BMO .
- In the third chapter, we extend some functional characterizations of $\dot{A}_{p,q}^s$ space related to BMO (here $\dot{A}_{p,q}^s$ denotes the realized spaces of Besov or Triebel-Lizorkin) given in the recent paper [24], where we generalize these properties to the realized Besov type spaces $\dot{B}_{p,q}^{s,\tau}$, and we prove that the John-Nirenberg Theorem also valid in $\dot{A}_{p,q}^s$ and $\dot{B}_{p,q}^{s,\tau}$.
- In the forth chapter, we give tow independent paragraphs:
 - A characterization by differences operator of smooth function spaces,
 - An Hirschmann property for classical Besov and Triebel-Lizorkin.
- In the end of these chapters, we add an Annexe in which there are some works in the way to finish.

Finlay, the results given in chapters 2 and 3 have been published, that are:

• **B. Gheribi, M. Moussai.** *Some embedding related to homogeneous Triebel-Lizorkin spaces and the BMO functions.* **Probl. Anal. Issues Anal.** **13(31) no. 2 (2024), pp. 24-48.**

• **B. Gheribi, M. Moussai.** *Functional characterizations of realized homogeneous Besov and Triebel-Lizorkin spaces.* **FILOMAT, 38(13) (2024), 4417–4440.**

Notation

Usual notation

- The set of natural numbers denoted by \mathbb{N} and \mathbb{N}_0 denote the natural numbers including 0 i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- The set of all integer numbers denoted by \mathbb{Z} .
- \mathbb{R}^n is the n -dimensional real Euclidean space.
- The integer part of the real number x denoted by $[x]$.
- If $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we set $E(x) := ([x_1], \dots, [x_n]) \in \mathbb{Z}^n$.
- For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we write $|\alpha| = \alpha_1 + \dots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$.
- Let $a \in \mathbb{R}$, we put $a_+ := \max(0, a)$.
- For two quasi Banach spaces E and B , we denote $E \hookrightarrow F$ the continuous embedding, there exists $c > 0$ such that $\|f\|_F \leq c\|f\|_E$ for any $f \in E$.
- $\text{supp } f$ is the support of the function f , i.e., $\text{supp } f = \overline{\{x : f(x) \neq 0\}}$.
- p' is the conjugate exponent of p , i.e., $p' := p/(p-1)$ if $1 < p \leq \infty$ and $p' := \infty$ if $0 < p \leq 1$.
- Let $0 < q \leq \infty$. We denote by $\ell_q(\mathbb{Z})$ the space of sequences $(g_j)_{j \in \mathbb{Z}}$ endowed by the quasi-norm

$$\|(g_j)_{j \in \mathbb{Z}}\|_{\ell_q} := \left(\sum_j |g_j|^q \right)^{1/q} \text{ if } q < \infty,$$

$$\|(g_j)_{j \in \mathbb{Z}}\|_{\ell_\infty} := \sup_j |g_j|.$$

- $L_p(\mathbb{R}^n)$ is the Lebesgue space endowed by the quasi-norm

$$\|f\|_p := \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} \text{ if } p < \infty,$$

$$\|f\|_\infty := \operatorname{esssup}_{x \in \mathbb{R}^n} |f(x)|.$$

- If we take $E \subset \mathbb{R}^n$ instead of \mathbb{R}^n in Lebesgue space i.e., $L_p(E)$ we denote the norm of this space by $\|\cdot\|_{L_p(E)}$
- Let $E \subset \mathbb{R}^n$ be a measurable set. $|E|$ stands for the Lebesgue measure of E .
- L_p^{loc} is the space of functions in $L_p(\Omega)$ for any compact subset Ω in \mathbb{R}^n .
- $\mathcal{D}(\mathbb{R}^n)$ is the set of all C^∞ function with compact support.
- $\mathcal{D}'(\mathbb{R}^n)$ is the topological dual space of $\mathcal{D}(\mathbb{R}^n)$, called distributions space.
- For $m \in \mathbb{N}$, $\mathcal{P}_m(\mathbb{R}^n)$ is the set of all polynomials of degree $< m$; we put $\mathcal{P}_0(\mathbb{R}^n) = \{0\}$, $\mathcal{P}_1(\mathbb{R}^n) = \{c\}$ and $\mathcal{P}_\infty(\mathbb{R}^n)$ the set of all polynomials in \mathbb{R}^n .
- $\mathcal{S}(\mathbb{R}^n)$ is the set of all Schwartz functions on \mathbb{R}^n .
- $\mathcal{S}'(\mathbb{R}^n)$ is the set of all tempered distributions on \mathbb{R}^n , the topological dual space of $\mathcal{S}(\mathbb{R}^n)$.
- For $m \in \mathbb{N} \cup \{\infty\}$, $\mathcal{S}_m(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n) : \langle x^\alpha, f \rangle = \int_{\mathbb{R}^n} x^\alpha f(x) dx = 0, \forall |\alpha| < m\}$.
- $\mathcal{S}'_m(\mathbb{R}^n)$ is the topological dual of $\mathcal{S}_m(\mathbb{R}^n)$ the so-called tempered distributions modulo polynomials of degree less than m .
- For all $f \in \mathcal{S}'(\mathbb{R}^n)$, $[f]_\infty$ denotes the equivalence class of f modulo $\mathcal{P}_\infty(\mathbb{R}^n)$.
- χ_E denotes the characteristic function on the set $E \subset \mathbb{R}^n$.
- For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in \mathbb{R}^n with center x , and radius r .
- The Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx, \quad x \in \mathbb{R}^n.$$

Its inverse is denoted by $\mathcal{F}^{-1}f(x) = \check{f}(x) := (2\pi)^{-n} \hat{f}(-x)$, \mathcal{F} and \mathcal{F}^{-1} can be extended on \mathcal{S}' in the usual way.

- The convolution $f * g$ is defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy, \quad f, g \in L^1(\mathbb{R}^n).$$

- The translation and dilation operators τ_a and h_λ ($a \in \mathbb{R}^n$, $\lambda > 0$) respectively, are defined by $\tau_a f := f(\cdot - a)$, and $h_\lambda f := f(\lambda^{-1} \cdot)$, respectively.
- We denote by $Q_{(x_0, \lambda)}$ the cube with side length λ centered at $x_0 \in \mathbb{R}^n$, i.e.,

$$Q_{(x_0, \lambda)} := x_0 + [-\lambda, \lambda]^n.$$

Special notation

In all this thesis we use:

- c, c_1, \dots strictly positive constants, depending only on the fixed parameters, as n, s, p, \dots and some fixed functions, their values may vary from line to another.
- C^∞, L_p, \dots instead of $C^\infty(\mathbb{R}^n), L_p(\mathbb{R}^n), \dots$ i.e., we omit \mathbb{R}^n in all function spaces of this work, since all spaces are defined on \mathbb{R}^n .
- If a function space E_p is defined for $p \leq \infty$, we limit ourselves to $p < \infty$, then $p = \infty$ will be defined in the usual way.

Chapter 1

Preliminaries

In this chapter, we give several essential tools necessary for this work, such as some classical inequalities, the definition of homogeneous Besov-Triebel-Lizorkin type spaces, their realized counterparts and some properties.

For more details we quote e.g., [37, 46, 47, 52].

1.1 Elementary inequalities

We will give some useful inequalities in our work.

Proposition 1.1. (Hölder inequality) *Let $0 < p, q, r \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, if $f \in L_p$ and $g \in L_q$ then $fg \in L_r$ and*

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

Proposition 1.2. (Young inequality) *Let $1 \leq p, q, r \leq \infty$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then, for all $f \in L_p$ and $g \in L_q$ we have $f * g \in L_r$ and*

$$\|f * g\|_r \leq c \|f\|_p \|g\|_q.$$

Proposition 1.3. (Bernstein inequality) *Let $0 < p \leq q \leq \infty$ and $\alpha \in \mathbb{N}^n$, then there exists a constant $c(\alpha, p, q, n) > 0$ such that for all $f \in L_p \cap \mathcal{S}'$ and all $R > 0$ with $\text{supp } \widehat{f} \subset \{\xi : |\xi| \leq R\}$, we have*

$$\|f^{(\alpha)}\|_q \leq c R^{|\alpha| + n(1/p - 1/q)} \|f\|_p.$$

Proof. See e.g., [46, Remark. 1.3.2/1]. In case $|\alpha| = 0$, the constant c can be given explicitly, $c = p_0^{n(1/p - 1/q)}$ where p_0 is the smallest integer not less than $p/2$, cf. [36, Theorem. 4]. \square

Proposition 1.4. (Minkowski inequality) Let $0 < p, q \leq \infty$. Let E_i (where $i = p$ or q) and $E \in \{\ell, L\}$. Let F_j (where $j = p$ or q) and $F \in \{\ell, L\}$. If $j \geq i$, then

$$\| \|f\|_{E_i} \|_{F_j} \leq \| \|f\|_{F_j} \|_{E_i}.$$

Proposition 1.5. Let $0 < \alpha \leq 1$ and let a_j be positive numbers, then

$$\left(\sum_j a_j \right)^\alpha \leq \sum_j a_j^\alpha.$$

1.2 Littlewood-Paley decompositions

Here we recall the Littlewood-Paley decomposition, where we need to introduce two functions, ρ and γ , which are an infinitely differentiable radial functions satisfying

$$0 \leq \rho(\xi) \leq 1, \quad \rho(\xi) = 1 \text{ if } |\xi| \leq 1 \text{ and } \rho(\xi) = 0 \text{ if } |\xi| \geq 3/2.,$$

γ defined as

$$\gamma(\xi) := \rho(\xi) - \rho(2\xi)$$

then γ is supported by the annulus $1/2 \leq |\xi| \leq 3/2$, and $\gamma(\xi) = 1$ if $1 \leq |\xi| \leq 3/4$.

These functions enable us to obtain the following two partitions of unit, the homogeneous partition of unit:

$$\sum_{j \in \mathbb{Z}} \gamma(2^j \xi) = 1, \quad (\xi \in \mathbb{R}^n \setminus \{0\}). \quad (1.1)$$

The inhomogeneous partition of unit:

$$\rho(2^{-k} \xi) + \sum_{j > k} \gamma(2^{-j} \xi) = 1, \quad (\xi \in \mathbb{R}^n, k \in \mathbb{Z}). \quad (1.2)$$

To obtain (1.1) and (1.2) we refer to e.g. [38, 37, 46]. Using ρ and γ we introduce the operators S_j and Q_j defined by

$$\widehat{S_j f} := \rho(2^{-j} \cdot) \hat{f} \quad \text{and} \quad \widehat{Q_j f} := \gamma(2^{-j} \cdot) \hat{f}, \quad (\forall j \in \mathbb{Z}).$$

Remark 1.6. The operators S_j and Q_j take values in the space of analytical functions of exponential type, see Paley-Wiener theorem [26, Theorem 1.7.7].

We need the following notation: $f \in \mathcal{S}'$ we denote by $[f]_m$ the equivalent class of f in \mathcal{S}'_m ,

where $m \in \mathbb{N} \cup \{\infty\}$, then we use the following convention:

$$\forall f \in \mathcal{S}'_\infty \text{ we define } Q_j f := Q_j f_1, \forall f_1 \in \mathcal{S}' \text{ such that } [f_1]_\infty = f,$$

then Q_j is defined on \mathcal{S}'_∞ since $Q_j f = 0$ ($\forall j \in \mathbb{Z}$) if and only if $f \in \mathcal{P}_\infty$.

Remark 1.7. The operators S_j and Q_j are uniformly bounded in $\mathcal{L}(L_p)$, for any $p \in [1, \infty]$ (the Young inequality).

The Littlewood-Paley decompositions are given by the following assertion:

Proposition 1.8. • For all $f \in \mathcal{S}'_\infty$, it holds that

$$f = \sum_{j \in \mathbb{Z}} Q_j f, \text{ in } \mathcal{S}'_\infty;$$

this decomposition is so-called homogeneous version. We can replace \mathcal{S}'_∞ by \mathcal{S}_∞ .

• For all $f \in \mathcal{S}$ and all $k \in \mathbb{Z}$, it holds that

$$f = S_k f + \sum_{j > k} Q_j f, \text{ in } \mathcal{S};$$

this decomposition is so-called inhomogeneous version. We can replace \mathcal{S} by \mathcal{S}' .

Proof. See [32, Proposition 2.7]. □

1.3 Besov type and Triebel-Lizorkin type spaces

In this section we will give the definitions and some proprieties of Besov-type and Lizorkin-Triebel-type spaces, where we can find their in [52].

1.3.1 Dyadic cubes

We define the dyadic cubes. Let $\eta \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$, where $\eta := (\eta_1, \eta_2, \dots, \eta_n)$, the dyadic cube of side length 2^{-k} and sides parallel to the coordinate axes is subset of \mathbb{R}^n defined by

$$P_{k,\eta} := \{x \in \mathbb{R}^n : 2^{-k}\eta_j \leq x_j < 2^{-k}(\eta_j + 1), \quad j = 1, 2, \dots, n\},$$

that means $P_{k,\eta} := 2^{-k}([0, 1]^n + \eta)$.

Proposition 1.9. For any $k \in \mathbb{Z}$ we have $\mathbb{R}^n = \bigcup_{\eta \in \mathbb{Z}^n} P_{k,\eta}$.

We can cover any cube $Q_{(x_0,r)}$ by union of disjoint dyadic cubes. Let k integer number such that $2^{-k} \leq r < 2^{-k+1}$. If $x \in Q_{(x_0,r)}$, we have

$$\begin{aligned} -r &\leq x_i - x_i^0 \leq r \\ -2^{-k+1} + x_i^0 &\leq x_i \leq 2^{-k+1} + x_i^0 \\ -2 + 2^k x_i^0 &\leq 2^k x_i \leq 2 + 2^k x_i^0 \\ -2 + [2^k x_i^0] &\leq 2^k x_i \leq 3 + [2^k x_i^0] \quad (i = 1, \dots, n). \end{aligned}$$

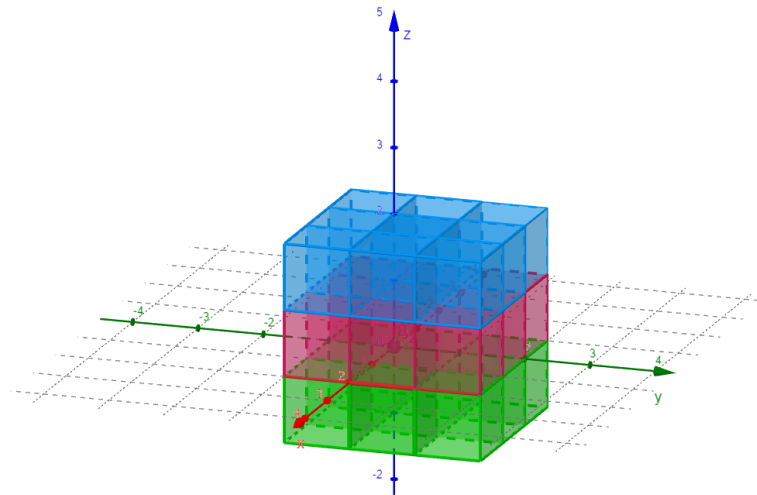
Then we obtain 5^n disjoint dyadic cubes $\{P_{k,E(2^k x_0) - u + w_j}\}_{r=1}^{5^n}$ such that $Q_{(x_0,r)} \subset \bigcup_{j=1}^{5^n} P_{k,E(2^k x_0) - u + w_j}$, where $E(2^k x_0) := ([2^k x_1^0], \dots, [2^k x_n^0]) \in \mathbb{Z}^n$, $u := (2, \dots, 2)$, $w_j := (w_{1,j}, \dots, w_{n,j}) \in \mathbb{Z}^n$ with $w_{i,j} \in \{0, 1, 2, 3, 4\}$ ($i = 1, \dots, n$) and $w_j \neq w_{j'}$ if $j \neq j'$, i.e. $w_1 := (0, \dots, 0)$, $w_2 := (1, 0, \dots, 0)$, ..., $w_{5^n} := (4, \dots, 4)$. See, e.g. [35].

Example 1.10. We have $Q_{(0,1)} \subset \bigcup_{j=1}^{3^n} P_{0,w_j}$ where $\{P_{0,w_j}\}_{r=1}^{3^n}$ are disjoint dyadic cubes, indeed; if $x \in Q_{(0,1)}$ i.e., $-1 \leq 2^0 x_i \leq 1$ ($i = 1, \dots, n$) then,

$$-1 \leq 2^0 x_i < -1 + 1 \mid 0 \leq 2^0 x_i < 0 + 1 \mid 1 \leq 2^0 x_i < 1 + 1 \quad (i = 1, \dots, n)$$

$w_j := (w_{1,j}, \dots, w_{n,j}) \in \mathbb{Z}^n$ with $w_{i,j} \in \{-1, 0, 1\}$ ($i = 1, \dots, n$) and $w_j \neq w_{j'}$ if $j \neq j'$, i.e. $w_1 := (-1, \dots, -1)$, $w_2 := (-1, -1, \dots, -1, 0)$, ..., $w_{3^n} := (1, \dots, 1)$.

Now, if we take $n = 3$ then we obtain 27 disjoint dyadic cubes, where its defined by the following vectors: $w_1 := (-1, -1, -1)$, $w_2 := (-1, -1, 0)$, $w_3 := (-1, 0, 0)$, ..., $w_{27} := (1, \dots, 1)$.



1.3.2 Definitions and some properties

Definition 1.11. Let $\tau \geq 0$ and $p \in]0, \infty]$. The Lebesgue-type space L_p^τ is the set of functions $f \in L_{\text{loc}}^p$ such that

$$\|f\|_{L_p^\tau} = \sup_{k \leq 0} \sup_{\eta \in \mathbb{Z}^n} 2^{kn\tau} \left(\int_{P_{k,\eta}} |f(x)|^p dx \right)^{1/p} < \infty. \quad (1.3)$$

Remark 1.12. • If we take $\tau = 0$, then $L_p^\tau = L_p$.

- If $\tau < 0$, then the set of functions satisfying (1.3) is $\{0\}$.
- $L_p^\tau \hookrightarrow \mathcal{S}'$ if $p \geq 1$.
- If $\tau_1 \leq \tau_2$, then $L_p^{\tau_1} \hookrightarrow L_p^{\tau_2}$.

Definition 1.13. (i) Let $s, \tau \in \mathbb{R}$ and $p, q \in]0, \infty]$. The homogeneous Besov-type spaces $\dot{B}_{p,q}^{s,\tau}$ is the set of all $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{B}_{p,q}^{s,\tau}} := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{kn\tau} \left(\sum_{j>k} 2^{jsq} \|Q_j f\|_{L_p(P_{k,\eta})}^q \right)^{1/q} < \infty.$$

(ii) Let $s, \tau \in \mathbb{R}$, $q \in]0, \infty]$ and $p \in]0, \infty[$. The homogeneous Triebel-Lizorkin-type space $\dot{F}_{p,q}^{s,\tau}$ is the set of all $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{F}_{p,q}^{s,\tau}} := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{kn\tau} \left\| \left(\sum_{j>k} 2^{jsq} |Q_j f|^q \right)^{1/q} \right\|_{L_p(P_{k,\eta})} < \infty.$$

Definition 1.14. (i) Let $s, \tau \in \mathbb{R}$ and $p, q \in]0, \infty]$. The inhomogeneous Besov-type space $B_{p,q}^{s,\tau}$ is the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{B_{p,q}^{s,\tau}} := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{n\tau k} \left(\sum_{j>k_+} 2^{jsq} \|Q_j f\|_{L_p(P_{k,\eta})}^q \right)^{1/q} < \infty, \text{ here } Q_0 := S_0.$$

(ii) Let $s, \tau \in \mathbb{R}$, $q \in]0, \infty]$ and $p \in]0, \infty[$. The inhomogeneous Triebel-Lizorkin-type space $F_{p,q}^{s,\tau}$ is the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{F_{p,q}^{s,\tau}} := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{kn\tau} \left\| \left(\sum_{j>k_+} 2^{jsq} |Q_j f|^q \right)^{1/q} \right\|_{L_p(P_{k,\eta})} < \infty, \text{ here } Q_0 := S_0.$$

Throughout this work, we employ the notation \dot{A} to indicate \dot{B} or \dot{F} , and A to indicate B or F when there is no confusion arises.

Remark 1.15. The spaces \dot{A} and A are quasi-Banach for the above defined quasi-norms, see e.g., [52, Corollary 2.1]. The spaces \dot{A} are endowed with quasi-seminorms in \mathcal{S}'_∞ . Their definitions are independent of the choice ρ .

In other words, if we select a different functions that possesses the same properties as ρ , the definitions of \dot{A} and A remain independent on the choice of ρ . Consequently, we obtain quasi-seminorms and quasi-norms, respectively, that are equivalent to those defined by ρ (see, for example, [50, Corollary 3.1]).

Definition 1.16. (i) Let $s \in \mathbb{R}$ and $p, q \in]0, \infty]$. The homogeneous Besov space $\dot{B}_{p,q}^s$ is the set of $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{B}_{p,q}^s} := \left(\sum_{j \in \mathbb{Z}} (2^{js} \|Q_j f\|_p)^q \right)^{1/q} < \infty.$$

(ii) Let $s \in \mathbb{R}$, $p \in]0, \infty[$ and $q \in]0, \infty]$. The homogeneous Triebel-Lizorkin space denoted by $\dot{F}_{p,q}^s$ is the set of $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{F}_{p,q}^s} := \left\| \left(\sum_{j \in \mathbb{Z}} (2^{js} |Q_j f|)^q \right)^{1/q} \right\|_p < \infty.$$

Definition 1.17. (i) Let $s \in \mathbb{R}$ and $q \in]0, \infty[$. The homogeneous space $\dot{F}_{\infty,q}^s$ is the set of $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{F}_{\infty,q}^s} := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} \left(2^{kn} \int_{P_{k,\eta}} \sum_{j \geq k} 2^{jsq} |Q_j f(x)|^q dx \right)^{1/q} < \infty.$$

(ii) For $q = \infty$, we put $\dot{F}_{\infty,\infty}^s = \dot{B}_{\infty,\infty}^s$.

Definition 1.18. (i) Let $s \in \mathbb{R}$ and $p, q \in]0, \infty]$ The inhomogeneous Besov space denoted by $B_{p,q}^s$ is the set of $f \in \mathcal{S}'$ such that

$$\|f\|_{B_{p,q}^s} := \|S_0 f\|_p + \left(\sum_{j \geq 1} 2^{jsq} \|Q_j f\|_p^q \right)^{1/q} < \infty.$$

(ii) Let $s \in \mathbb{R}$, $p \in]0, \infty[$ and $q \in]0, \infty]$. The inhomogeneous Triebel-Lizorkin space $F_{p,q}^s$ is the set of $f \in \mathcal{S}'$ such that

$$\|f\|_{F_{p,q}^s} := \|S_0 f\|_p + \left\| \left(\sum_{j \geq 1} 2^{jsq} |Q_j f|^q \right)^{1/q} \right\|_p < \infty.$$

Proposition 1.19. Let $s, \tau \in \mathbb{R}$ and $0 < p, q \leq \infty$ ($p < \infty$ in the F -case)

(i) we have $\dot{A}_{p,q}^{s,0} = \dot{A}_{p,q}^s$ and $A_{p,q}^{s,0} = A_{p,q}^s$.

(ii) If $\tau < 0$ then $A_{p,q}^{s,\tau} = \{0\}$ and $\dot{A}_{p,q}^{s,\tau} = \mathcal{P}_\infty$.

Proof. We refer to [52, Lemma 2.1, Proposition 2.1] and [51, Proposition 3.1]. \square

1.3.3 Embeddings

Proposition 1.20. Let $s \in \mathbb{R}$, $\tau \geq 0$ and $0 < p, q \leq \infty$, ($p < \infty$ in the F -space), then one has:

(i) $\mathcal{S}_\infty \hookrightarrow \dot{A}_{p,q}^{s,\tau} \hookrightarrow \mathcal{S}'_\infty$ and $\mathcal{S} \hookrightarrow A_{p,q}^{s,\tau} \hookrightarrow \mathcal{S}'$,

(ii) if $0 < q_1 \leq q_2 \leq \infty$, then $\dot{A}_{p,q_1}^{s,\tau} \hookrightarrow \dot{A}_{p,q_2}^{s,\tau}$.

(iii) $\dot{B}_{p,\min(p,q)}^{s,\tau} \hookrightarrow \dot{F}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{p,\max(p,q)}^{s,\tau}$.

Proof. For (i) we refer to [50, Proposition 3.1 and 3.4] and [52, Proposition 2.3]. For (ii) we apply the fact that $\ell_{q_1} \hookrightarrow \ell_{q_2}$. The assertion (iii) follows from the following inequality:

$$\left(\sum_{j \geq k} \|g_j\|_p^v \right)^{1/v} \leq \left\| \left(\sum_{j \geq k} |g_j|^q \right)^{1/q} \right\|_p \leq \left(\sum_{j \geq k} \|g_j\|_p^u \right)^{1/u},$$

where $u = \min(p, q)$ and $v = \max(p, q)$. \square

Proposition 1.21. Let $s, s_1, s_2 \in \mathbb{R}$, $0 \leq \tau < \infty$ and $0 < q \leq \infty$

(i) Let $s_1 > s_2$ and $0 < p_1 < p_2 \leq \infty$ ($p_2 < \infty$ in the F -space). Then

$$\dot{A}_{p_1,q}^{s_1,\tau} \hookrightarrow \dot{A}_{p_2,q}^{s_2,\tau} \hookrightarrow \dot{B}_{\infty,q}^{s_2-n/p_2,\tau} \quad \text{if} \quad s_1 - n/p_1 = s_2 - n/p_2.$$

(ii) $\dot{A}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{\infty,\infty}^{s+n\tau-n/p}$.

(iii) If ($\tau > 1/p$ and $0 < q < \infty$) or ($\tau = 1/p$ and $q = \infty$) then $\dot{A}_{p,q}^{s,\tau} = \dot{B}_{\infty,\infty}^{s+n\tau-n/p}$.

Proof. For (i) see [50, Proposition 3.3]. For (ii) and (iii) see [51, Theorem 1]. \square

The following statement give us the relation between homogeneous and inhomogeneous spaces:

Proposition 1.22. Let $p, q \in]0, \infty]$, $\tau \geq 0$ and $s > (n/p - n)_+$. Then $A_{p,q}^{s,\tau} := L_p^\tau \cap \dot{A}_{p,q}^{s,\tau}$, where this intersection means that $f \in L_p^\tau$ and $[f]_\infty \in \dot{A}_{p,q}^{s,\tau}$. In addition, the expression $\|f\|_{L_p^\tau} + \|[f]_\infty\|_{\dot{A}_{p,q}^{s,\tau}}$ and $\|f\|_{A_{p,q}^{s,\tau}}$ are equivalent quasi-norms in $A_{p,q}^{s,\tau}$.

Proof. See [4, Theorem 3.1]. \square

Proposition 1.23. *There exist two constants $c_1, c_2 > 0$ such that*

$$c_1 \|f\|_{\dot{A}_{p,q}^{s,\tau}} \leq \lambda^{s+n\tau-n/p} \|h_\lambda f\|_{\dot{A}_{p,q}^{s,\tau}} \leq c_2 \|f\|_{\dot{A}_{p,q}^{s,\tau}}$$

holds, for all $f \in \dot{A}_{p,q}^{s,\tau}$ and all $\lambda > 0$. Moreover, the expression $\lambda^{s+n\tau-n/p} \|h_\lambda f\|_{\dot{A}_{p,q}^{s,\tau}}$, defines an equivalent seminorm in $\dot{A}_{p,q}^{s,\tau}$.

Proof. See [52, Remark 8.1]. □

Proposition 1.24. *Let $m \in \mathbb{N}$. A function f of \mathcal{S}' belongs to $\dot{A}_{p,q}^{s,\tau}$ if and only if its derivatives $f^{(\alpha)}$ belongs to $\dot{A}_{p,q}^{s-m,\tau}$ for all $|\alpha| = m$. Moreover, the expression $\sum_{|\alpha|=m} \|f^{(\alpha)}\|_{\dot{A}_{p,q}^{s-m,\tau}}$ is an equivalent quasi-norm in $\dot{A}_{p,q}^{s,\tau}$.*

Proof. See [49, Theorem 1.5]. □

Proposition 1.25. *Let $s, \tau \in \mathbb{R}$ and $p, q \in]0, +\infty]$. Let a, b be a real numbers such that $0 < a < b$ and $(u_j)_{j \in \mathbb{Z}}$ be a sequence in \mathcal{S}' such that*

- \hat{u}_j is supported by the annulus $a2^j \leq |\xi| \leq b2^j$.
- $A := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{n\tau k} \left(\sum_{j \geq k} 2^{jsq} \|u_j\|_{L_p(P_{k,\eta})}^q \right)^{1/q} < \infty$ in the B -space,
- $A := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{n\tau k} \left\| \left(\sum_{j \geq k} 2^{jsq} |u_j|^q \right)^{1/q} \right\|_{L_p(P_{k,\eta})} < \infty$ in the F -space.

- (i) *Then the series $\sum_{j \in \mathbb{Z}} u_j$ converges in \mathcal{S}' to a limit u satisfying $\|u\|_{\dot{A}_{p,q}^{s,\tau}} \leq cA$, where the constant c depends only on n, s, τ, p, q, a and b .*
- (ii) *If in addition $s > (n/p - n)_+$, the same result holds for $a = 0$.*

Proof. This assertion is a Nikol'skij-type estimate. To show the proof of (i) in the B -space see [6, Theorem 2] and for the F -space see [8, Theorem 2.3.1] and for (ii) see [7, Theorem 3.2]. □

Chapter 2

Riesz operator and realized homogeneous Triebel-Lizorkin spaces

In this chapter, we see the acting operation of the Riesz operator on $\dot{F}_{p,q}^s$ in a certain sens. We need to recall some proprieties of BMO space. However, as the realizations play an important role in this work, we first begin by giving an overview on realizations, definition and some properties.

2.1 Realizations

The realizations have been introduced by G. Bourdaud in [11] by studying the Besov case. Nowadays there is a variant of spaces where we know their realizations counterpart; see. e.g., [6, 14, 15, 30, 32].

2.1.1 Generalities on realizations

Definition 2.1. Let $m \in \{0, 1, 2, \dots\} \cup \{\infty\}$. Let E be a vector subspace of \mathcal{S}'_m endowed with a structure of Banach space such that the natural mapping $E \hookrightarrow \mathcal{S}'_m$ is continuous, then we call E a Banach distribution space in $\mathcal{S}'_m(\mathbb{R}^n)$ abbreviated by (B.D.S).

Definition 2.2. Let $m \in \{0, 1, 2, \dots\} \cup \{\infty\}$ and $k \in \{0, 1, 2, \dots, m\}$. Let E be B.D.S of \mathcal{S}'_m , such that the continuous embedding $E \hookrightarrow \mathcal{S}'_m$ holds. We call a realization σ of E in \mathcal{S}'_k a continuous linear mapping $\sigma : E \rightarrow \mathcal{S}'_k$ such that for all function f in E the equivalent class of $[\sigma(f)]_m$ modulo polynomial equal to f , i.e., $[\sigma(f)]_m = f$. The image set $\sigma(E)$ is so-called the realized space of E with respect to σ .

The following proposition give us a classification of realizations

Proposition 2.3. *Let E be B.D.S. in \mathcal{S}'_m . Let $\sigma_0 : E \rightarrow \mathcal{S}'_k$ be a realization. For any finite family $(\mathcal{L}_\alpha)_{k \leq |\alpha| \leq m}$ of continuous linear functionals on E , the following formula defines a realization of E in \mathcal{S}'_k :*

$$\sigma(f)(x) = \sigma_0(f)(x) + \sum_{k \leq |\alpha| \leq m} \mathcal{L}_\alpha(f)x^\alpha.$$

Conversely, any realization σ of E modulo $\mathcal{P}_k(\mathbb{R}^n)$ is given in such a way.

Proof. See [11, Proposition 1]. □

For every f in E , the element $\sigma(f)$ is the unique representative of f in $\sigma(E)$; consequently, σ is completely characterized by its range. We say that a realization σ of E commutes with translations (resp. dilations) if $\tau_a \circ \sigma = \sigma \circ \tau_a$, $a \in \mathbb{R}^n$, (resp. $h_\lambda \circ \sigma = \sigma \circ h_\lambda$, $\lambda > 0$); this goes if and only if the range of σ is translation (resp. dilation) invariant.

Remark 2.4. A subspace E of \mathcal{S}'_m has generally infinitely many realizations in \mathcal{S}'_k if $k < m$, in the case of $k = m$ the identity is the unique realization. Of course, with additional conditions such as *translations or dilations invariance*, a realization of E in \mathcal{S}'_k for $k < m$ has some chances to be unique.

2.1.2 The realized spaces

We recall the definition of distributions vanish at infinity in the weak sense where we need it later on.

Definition 2.5. *We say that a tempered distribution $f \in \mathcal{S}'$ vanishes at the infinity in the weak sense, if $f(\lambda^{-1}(\cdot))$ tends to 0 in \mathcal{S}' when λ tends to 0, i.e.,*

$$\lim_{\lambda \downarrow 0} \langle f(\lambda^{-1}(\cdot)), \varphi \rangle = 0, \quad \forall \varphi \in \mathcal{S}.$$

The set of all such distributions is denoted by $\tilde{\mathcal{C}}_0$.

Example 2.6. (i) $f \in \tilde{\mathcal{C}}_0$ if $f \in L_p$ ($1 \leq p < \infty$).

(ii) $\partial_j f \in \tilde{\mathcal{C}}_0$ ($j = 1, \dots, n$) if $f \in L_\infty$ or $f \in \tilde{\mathcal{C}}_0$.

(iii) If $f \in \tilde{\mathcal{C}}_0$ is a polynomial, then $f = 0$, i.e., $\tilde{\mathcal{C}}_0 \cap \mathcal{P}_\infty = \{0\}$.

For other examples can be found in [5, p. 26].

As mentioned in the precedent Chapter we have $\dot{B}_{p,q}^{s,\tau} \hookrightarrow \mathcal{S}'_\infty$, however, there is a smaller integer $\mu \in \mathbb{N}_0$ such that $\dot{B}_{p,q}^{s,\tau} \hookrightarrow \mathcal{S}'_\mu$, for this purpose, we fix the natural number μ as,

$$\mu := \begin{cases} ([s+n\tau-n/p]+1)_+ & \text{if } s+n\tau-n/p \notin \mathbb{N}_0 \text{ or } q > 1, \\ s+n\tau-n/p & \text{if } s+n\tau-n/p \in \mathbb{N}_0 \text{ and } q \leq 1; \end{cases} \quad (2.1)$$

For a no confusion if $\tau = 0$ in (2.1), we will use the following notation:

$$\mu_0 := \mu, \quad (2.2)$$

since, we are going to use the classical Besov and Triebel-Lizorkin realized. Concerning the realized spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$ and $\dot{F}_{p,q}^s(\mathbb{R}^n)$ we refer to [13, 14, 32].

The number μ has the following property, that is the convergence of the Littlewood-Paley series of any element of $\dot{B}_{p,q}^{s,\tau}$:

Proposition 2.7. *Let $s, \tau \in \mathbb{R}$ and $p, q \in]0, \infty]$. Let $f \in \dot{B}_{p,q}^{s,\tau}$, then the Littlewood-Paley sum $\sum_{j \in \mathbb{Z}} Q_j f$ converges in \mathcal{S}'_μ to element denoted $\sigma_\mu(f)$. Then the mapping $\sigma_\mu : \dot{B}_{p,q}^{s,\tau} \rightarrow \mathcal{S}'_\mu$ defined in such a way is a translation and dilatation commuting realization of $\dot{B}_{p,q}^{s,\tau}$ into \mathcal{S}'_μ , and σ_μ is the unique representative of f satisfying $\partial^\alpha \sigma_\mu(f) \in \tilde{C}_0$ for all $|\alpha| = \mu$.*

Proof. See [6, Theorem 1]. □

Definition 2.8. *The realized space $\dot{B}_{p,q}^{s,\tau}$ is the set of all $f \in \mathcal{S}'_\mu$ such that $[f]_\infty \in \dot{B}_{p,q}^{s,\tau}$ and $\partial^\alpha f \in \tilde{C}_0$ for all $|\alpha| = \mu$, where μ is defined in (2.1). This space is endowed with the quasi-seminorm $\|f\|_{\dot{B}_{p,q}^{s,\tau}} := \|[f]_\infty\|_{\dot{B}_{p,q}^{s,\tau}}$.*

Proposition 2.7 guarantees the convergence of the Littlewood-Paley series in \mathcal{S}'_μ , however if we want its convergence in \mathcal{S}' , we are forced to add a polynomial of degree depends on μ , that is the following statement:

Proposition 2.9. *Let $f \in \dot{B}_{p,q}^{s,\tau}$. We define $\sigma_i(f)$ ($i = 1, 2, 3$) as the following:*

- if either $s+n\tau < n/p$ or ($s+n\tau = n/p$ and $0 < q \leq 1$),

$$\sigma_1(f) := \sum_{j \in \mathbb{Z}} Q_j f, \quad (2.3)$$

- if either $s+n\tau-n/p \in \mathbb{R}^+ \setminus \mathbb{N}_0$ or ($s+n\tau-n/p \in \mathbb{N}$ and $0 < q \leq 1$),

$$\sigma_2(f) := \sum_{j \in \mathbb{Z}} \left(Q_j f - \sum_{|\alpha| < \mu} (Q_j f)^{(\alpha)}(0) x^\alpha / \alpha! \right), \quad (2.4)$$

- if $s + n\tau - n/p \in \mathbb{N}_0$ and $q > 1$

$$\sigma_3(f) := \sum_{j \geq 1} Q_j f + \sum_{j \leq 0} \left(Q_j f - \sum_{|\alpha| < \mu} (Q_j f)^{(\alpha)}(0) x^\alpha / \alpha! \right). \quad (2.5)$$

Then σ_i ($i = 1, 2, 3$) is a realization of $\dot{B}_{p,q}^{s,\tau}$ into \mathcal{S}' such that $\sigma_i(f)$ ($i = 1, 2, 3$) converges in \mathcal{S}' , with

$$[\sigma_i(f)]_\infty = f \text{ in } \mathcal{S}'_\infty \text{ and } \partial^\alpha \sigma_i(f) \in \tilde{C}_0 \text{ if } |\alpha| = \mu.$$

Proof. In [11, Section 2] the author has given the construction of realizations of the homogeneous Besov spaces $\dot{B}_{p,q}^s$ in \mathcal{S}' , then recently in [6, Remark 5] they extend this construction to $\dot{B}_{p,q}^{s,\tau}$. \square

Remark 2.10. The function γ in Proposition 2.9 can be replaced with any function $v \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ such that $v \geq 0$ and $v(\xi) \geq c > 0$ if $a \leq |\xi| \leq b$. In that case, we replace the operators Q_j by $v_j(D)$ defined as the following:

$$\widehat{v_j(D)f}(\xi) := v(2^{-j}\xi)\check{f}.$$

Also, we called $\sigma_i(\dot{B}_{p,q}^{s,\tau})$ ($i = 1, 2, 3$) the realized space of $\dot{B}_{p,q}^{s,\tau}$, which has the following definition:

Definition 2.11. The realized space $\sigma_i(\dot{B}_{p,q}^{s,\tau})$ ($i = 1, 2, 3$) is defined as the following:

1. if either $s + n\tau < n/p$ or $s + n\tau = n/p$ and $0 < q \leq 1$, here $\mu = 0$ or $\mu = 1$,

$$\sigma_1(\dot{B}_{p,q}^{s,\tau}) := \{f \in \mathcal{S}' : [f]_\infty \in \dot{B}_{p,q}^{s,\tau} \text{ and } f \in \tilde{C}_0\},$$

2. if either $s + n\tau - n/p \in \mathbb{R}^+ \setminus \mathbb{N}_0$ or $s + n\tau - n/p \in \mathbb{N}$ and $0 < q \leq 1$, here $\mu = [s + n\tau - n/p] + 1$ or $\mu = s + n\tau - n/p$, respectively,

$$\sigma_2(\dot{B}_{p,q}^{s,\tau}) := \{f \in \mathcal{S}' : [f]_\infty \in \dot{B}_{p,q}^{s,\tau}, f \in C^{\mu-1}, f^{(\beta)}(0) = 0 \text{ for } |\beta| \leq \mu - 1\},$$

3. if $s + n\tau - n/p \in \mathbb{N}_0$ and $q > 1$, here $\mu = [s + n\tau - n/p] + 1$,

$$\begin{aligned} \sigma_3(\dot{B}_{p,q}^{s,\tau}) := \{f \in \mathcal{S}' : [f]_\infty \in \dot{B}_{p,q}^{s,\tau}, f \in C^{\mu-1}, \\ f^{(\beta)}(0) = \sum_{j \geq 1} (Q_j f)^{(\beta)}(0) \text{ for } |\beta| \leq \mu - 1\}. \end{aligned}$$

This space is endowed with the quasi-seminorm $\|f\|_{\sigma_i(\dot{B}_{p,q}^{s,\tau})} := \|[f]_\infty\|_{\dot{B}_{p,q}^{s,\tau}}$.

In connection with translations and dilations, we add the following assertion:

Proposition 2.12. *Let σ_i ($i = 1, 2, 3$) be realizations given in (i)–(iii) of Proposition 2.9. Then*

- (i) σ_1 commutes with translations and dilations,
- (ii) σ_2 commutes with dilations only,
- (iii) σ_3 does not commute with translations nor dilations.

Proof. See [6, Remark 7]. The case $\tau = 0$ can be found in [15, Section 4] and [32, Section 3].
□

Remark 2.13. We define $\check{F}_{p,q}^{s,\tau}$ in a manner similar to Definition 2.8. In Subsection 2.1.2, we can substitute \check{F} and \check{B} for \dot{B} and \dot{F} , respectively, by replacing the conditions $0 < q \leq 1$ and $1 < q \leq \infty$ in (2.1), Proposition 2.9, and Definition 2.11 with $0 < p \leq 1$ and $1 < p \leq \infty$, respectively. We refer to [8], for further information.

Remark 2.14. The realizations of $\dot{B}_{p,q}^s$ and $\dot{F}_{p,q}^s$ obtained by taken $\tau = 0$ in all this section See.e.g., [11] and [14, 15, 32].

Finally, for general properties of the realizations, we refer to e.g., [3]–[6] and [11, 14, 15, 23, 30, 32].

2.2 Triebel Lizorkin and BMO spaces

This paragraph (Preparations) is also useful for the next chapters.

2.2.1 BMO space

The BMO space was first introduced by F. John and L. Nirenberg in [28]. To define this space we need the following notation: For a defined function f , we put its mean value (or the average value):

$$m_Q f := \frac{1}{|Q|} \int_Q f(x) dx.$$

Definition 2.15. *The space of bounded mean oscillation (abbreviated by BMO) is defined as the set of locally integrable functions i.e., $f \in L_1^{\text{loc}}$ such that*

$$\|f\|_{BMO} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - m_Q f| dx < \infty;$$

where the supremum is taken over all finite cubes Q in \mathbb{R}^n .

Remark 2.16. • The $(BMO, \|\cdot\|_{BMO})$ space is a vector space, endowed with the seminorm vanishing on the constant functions.

- The BMO space is invariant by translation and dilatation, i.e.,

$$\|\tau_a f\|_{BMO} = \|f\|_{BMO} \quad \text{and} \quad \|h_\lambda f\|_{BMO} = \|f\|_{BMO}.$$

Example 2.17. (i) L_∞ is a proper subspace of BMO .

(ii) The function $f(x) := \log|x|$ belongs to BMO , see [42, p. 140-141].

(iii) We have $\log|\mathcal{P}(x)| \in BMO$ for all polynomial \mathcal{P} on \mathbb{R}^n , see.e.g., [42, p. 177].

(iv) The function $h(x) := \operatorname{sgn}(x) \log|x|$ is not in BMO . Indeed, we have the mean value of this function equal zero on $[-\lambda, \lambda]$

$$\begin{aligned} m_{[-\lambda, \lambda]} h(x) &= \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} \operatorname{sgn}(x) \log|x| dx \\ &= \frac{1}{2\lambda} \int_{-\lambda}^0 -\log(-x) dx + \frac{1}{2\lambda} \int_0^{\lambda} \log(x) dx = 0, \end{aligned}$$

while

$$\frac{1}{2\lambda} \int_{-\lambda}^{\lambda} |h(x) - m_{[-\lambda, \lambda]}| dx = \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} |\log|x|| dx = 1 - \log \lambda \quad \forall \lambda \in]0, 1[,$$

then $\sup_Q \frac{1}{|Q|} \int_Q |f(x) - m_Q f| dx$ diverges.

For the example $f(x) = \log|x|$ see Section 2.3 below.

2.2.2 Characterization of BMO space

The following assertion proves that functions in BMO space are "nearly bounded", see [42, 21].

Proposition 2.18. *If f is a BMO function then*

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|^{n+1}} dx < \infty$$

The space BMO is characterized by the following statement:

Proposition 2.19. *A function $f \in L_2^{\text{loc}}$ belongs to BMO if, and only if*

(i) $\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|^{n+1}} dx < \infty,$

- (ii) $\sup_{B_k} |B_k|^{-1} \int_{B_k} \sum_{j \geq k} |Q_j f(x)|^2 dx < \infty$, where the supremum is taken over all finite balls B_k in \mathbb{R}^n of radius 2^{-k} ($k \in \mathbb{Z}$).

Proof. See [12], p. 147]. □

The proof of this characterization holds a significant historical background given in [23]. We recall the following proposition:

Proposition 2.20. *A function $f \in BMO$ if, and only if*

- (i) $\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|^{n+1}} dx < \infty$,
- (ii) $\sup_{y \in \mathbb{R}^n, h > 0} h^{-n} \int_{|x-y| < h} \int_0^h t |\nabla P_t f(x)|^2 dt dx < \infty$, where $P_t f$ is the Poisson integral of f .

Proof. See [21], Theorem 3]. We recall the definition of Poisson integral $P_t f$ that mentioned in last reference

$$P_t f(x) = c_n \int_{\mathbb{R}^n} \frac{t f(y)}{(t^2 + |x-y|^2)^{(n+1)/2}} dy, \quad t > 0.$$

□

The difference between Propositions 2.19 and 2.20 lies in condition (ii). For the below proposition we refer to [25, 42].

Proposition 2.21. *Let f be a function in BMO space. For all ball B and $k \in \mathbb{N}$, it holds*

$$|m_B f - m_{2^k B} f| \leq 2^k \|f\|_{BMO}$$

Proposition 2.22. *For all $f \in BMO$.*

- (i) *It holds that*

$$\int_{\mathbb{R}^n} \frac{|f(x) - m_{Q_{(0,1)}} f|}{1 + |x|^{n+1}} dx \leq c \|f\|_{BMO} \quad (2.6)$$

where $Q_{(0,1)}$ is the cube with side length 1 centered at the origin.

- (ii) *Let $d > 0$. There exists a constant c such that*

$$R^d \int_{\mathbb{R}^n} \frac{|f(x) - m_{Q_{(x_0, R)}} f|}{(R + |x - x_0|)^{n+d}} dx \leq c \|f\|_{BMO} \quad (2.7)$$

holds, for all $f \in BMO$, all $R > 0$ and all $x_0 \in \mathbb{R}^n$, where $Q_{(x_0, R)}$ is the cube with side length R and center x_0 .

Proof. To prove (i), we utilize Proposition 2.21. For (ii), we substitute $f(Rx + x_0)$ instead of $f(x)$ in equation (2.6). Since $\|\cdot\|_{BMO}$ remains unchanged under translation and dilation, as stated in Remark 2.16, we obtain the desired outcome. For further elaboration, refer to sources such as [21, 25, 42]. □

Proposition 2.23. *If $f \in BMO$ then $f \in L_p^{\text{loc}}$ and*

$$\frac{1}{|Q|} \int_Q |f(x) - m_Q f|^p dx < c \|f\|_{BMO}^p,$$

holds for all cubes Q in \mathbb{R}^n

Proof. See e.g., [21, 25, 28, 42]. □

The following statement is called the John-Nirenberg Theorem:

Theorem 2.24. *Let $f \in BMO$. Then there exist two constants c_1 and c_2 , such that*

$$\left| \left\{ x \in Q : |f(x) - m_Q f| > \alpha \right\} \right| \leq c_1 |Q| \exp\left(-\frac{c_2 \alpha}{\|f\|_{BMO}}\right)$$

holds, for all cubes Q in \mathbb{R}^n and all $\alpha > 0$.

Proof. See e.g., [25, 29, 39] and [42, p. 144]. □

As a consequence of the John-Nirenberg Theorem that all BMO functions are exponentially integrable over any cube Q in \mathbb{R}^n , we present it in the following assertion:

Corollary 2.25. *All BMO functions are exponentially integrable over any cube; such that the inequality*

$$\frac{1}{|Q|} \int_Q \exp\left(\frac{b}{\|f\|_{BMO}} |f(x) - m_Q f|\right) dx \leq c, \quad c := c(n, b),$$

holds, for all $f \in BMO$ and all cubes Q in \mathbb{R}^n .

Proof. See e.g., [25]. □

2.2.3 Bessel potentials operator and BMO space

We firstly defined the Bessel potentials operator \mathcal{J}_α by

$$\mathcal{J}_\alpha f := \mathcal{F}^{-1} \left((1 + |\xi|^2)^{-\alpha/2} \widehat{f} \right) \quad \forall \alpha \in \mathbb{R}.$$

Then we recall that this operator maps $A_{p,q}^s$ isomorphically onto $A_{p,q}^{s+\alpha}$ see e.g., [41, p. 149], [37, p. 67] or [46, p. 58].

Proposition 2.26. *Let $1 < p < \infty$. If $f \in L_p$ then*

$$\|\mathcal{J}_{n/p}f\|_{BMO} \leq c \sup_{t>0} t |\{x: |f(x)| > t\}|^{1/p}, \quad (2.8)$$

where the notation $|\{\dots\}|$ denotes the Lebesgue measure of the set $\{\dots\}$.

Proof. We refer to, e.g., [44, Theorem 2(a)]. □

Proposition 2.27. *For all $1 < p < \infty$, we have*

$$H_p^{n/p} \hookrightarrow BMO, \quad (2.9)$$

where $H_p^{n/p}$ is the Bessel-potential space defined as the set of functions f satisfying

$$\|f\|_{H_p^{n/p}} := \|\mathcal{J}_{-n/p}f\|_p < \infty.$$

Proof. This statement follows from the previous one, where the right-hand side of inequality (2.8) is bounded by $c\|f\|_p$. Consequently, we have

$$\|f\|_{BMO} \leq c \|\mathcal{J}_{-n/p}f\|_p,$$

which yields the result. □

Proposition 2.28. *For all $0 < p < \infty$ and $0 < q \leq \infty$, we have*

$$F_{p,q}^{n/p} \hookrightarrow BMO$$

Proof. By applying Proposition 2.27 and as $H_{p_1}^{n/p_1}$ coincides with the inhomogeneous Triebel-Lizorkin space $F_{p_1,2}^{n/p_1}$ with $\max(p, 1) < p_1 < \infty$. Due to the embedding $F_{p,q}^{n/p} \hookrightarrow F_{p_1,2}^{n/p_1}$, we obtain the required result. □

As a consequence of [44, Theorem 2(b)] we have the following statement:

Proposition 2.29.

$$BMO \cap (L_1 + L_\infty) \subset B_{\infty,\infty}^0. \quad (2.10)$$

Proof. In [44, Theorem 2(b)], it was proved that if $\alpha > 0$, then

$$\|\mathcal{J}_\alpha f\|_{B_{\infty,\infty}^\alpha} \leq c \|f\|_{BMO}$$

holds, for all $f \in BMO \cap (L_1 + L_\infty)$. As mentioned in the beginning of this section \mathcal{J}_α maps $B_{\infty,\infty}^\alpha$ isomorphically onto $B_{\infty,\infty}^0$ i.e., $\|\mathcal{J}_\alpha f\|_{B_{\infty,\infty}^\alpha} \equiv \|f\|_{B_{\infty,\infty}^0}$ see e.g., [37, p. 67] or [46, p. 58].

Now, we have

$$\|f\|_{B_{\infty,\infty}^0} \leq c\|f\|_{BMO},$$

and we obtain the desired embedding. □

Remark 2.30. Here we want to eliminate the assumption $L_1 + L_\infty$ in (2.10).

2.2.4 The realized space of $\dot{F}_{\infty,q}^s$

We first recall the definition of realized space of $\dot{F}_{\infty,q}^s$ denoted by $\tilde{F}_{\infty,q}^s$, see [3]. Here we define a natural number μ as

$$\mu := ([s] + 1)_+.$$

Definition 2.31. *The continuous linear mapping $\sigma : \dot{F}_{\infty,q}^s \rightarrow \mathcal{S}'_\mu$ is a realization of $\dot{F}_{\infty,q}^s$ such that for all $[f]_\infty \in \dot{F}_{\infty,q}^s$ we have $[\sigma(f)]_\mu = f$ in \mathcal{S}'_μ . The realized space $\tilde{F}_{\infty,q}^s$ is the set of $f \in \mathcal{S}'_\mu$ such that $[f]_\infty \in \dot{F}_{\infty,q}^s$ and $\partial^\alpha f \in \tilde{C}_0$ if $|\alpha| = \mu$. This space is endowed with the quasi-seminorm $\|f\|_{\tilde{F}_{\infty,q}^s} := \|[f]_\infty\|_{\dot{F}_{\infty,q}^s}$.*

Proposition 2.32. *Let $0 < q \leq \infty$ and $s \in \mathbb{R}^n$. Let $f \in \dot{F}_{\infty,q}^s$ and σ_μ is defined as a sum in \mathcal{S}'_μ . Then $\sigma : \dot{F}_{\infty,q}^s \rightarrow \mathcal{S}'_\mu$ is a realization of $\dot{F}_{\infty,q}^s$ such that σ commutes with translations and dilations. Moreover, $\partial^\alpha \sigma(f) \in \tilde{C}_0$, ($\forall |\alpha| = \mu$).*

Proof. See [3, Theorem 1]. □

2.2.5 Some main properties on realized spaces

Here we formulate certain features of $\tilde{F}_{p,q}^{n/p}$ that used to prove the results in [23], which hold true for Besov.

Proposition 2.33. It holds $\dot{F}_{\infty,q}^0 \hookrightarrow \dot{B}_{\infty,\infty}^0$.

Proof. [3, Lemma 2] □

Proposition 2.34. *Let either $(0 < p < \infty)$ or $(p = \infty \text{ and } q = 2)$. It holds that $\tilde{F}_{p,q}^{n/p} \hookrightarrow L_1^{\text{loc}} \cap \dot{B}_{\infty,\infty}^0$.*

Proof. See [23, Proposition 7]. □

Remark 2.35. We have $\tilde{F}_{\infty,2}^0 \hookrightarrow L_2^{\text{loc}}$ where the proof is similar to that given for the preceding proposition

Proposition 2.36. *There exists a constant $c > 0$ such that the inequality*

$$\frac{1}{|Q|} \int_Q |S_k f(x)| \, dx \leq c \| [f]_\infty \|_{\dot{F}_{p,q}^{n/p}}$$

holds, for all $f \in \dot{F}_{p,q}^{n/p}$, all $k \in \mathbb{Z}$ and all cube Q in \mathbb{R}^n .

Proof. See [23, Proposition 8]. □

Proposition 2.37. *Let either $(0 < p < \infty)$ or $(p = \infty$ and $q = 2)$. If $f \in \dot{F}_{p,q}^{n/p}$, then*

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|^{n+1}} \, dx < \infty. \quad (2.11)$$

Proof. See [23, Proposition 9]. □

Remark 2.38. We note that (2.11) shows that $\dot{F}_{p,q}^{n/p} \subset L_1^{\text{loc}}$ if either $(0 < p < \infty)$ or $(p = \infty$ and $q = 2)$.

Now, we define the Hardy space H^p (see, e.g., [42, Theorem III.1.2/1, p. 91].) to give the relation between the Hardy space H^p and the realized space $\dot{F}_{p,2}^0$

Definition 2.39. *Let $0 < p < \infty$. The Hardy space H^p is the set of all distributions f such that*

$$\| \sup_{t>0} |\Phi_t * f| \|_p < \infty,$$

where $\Phi \in \mathcal{S}$, $\widehat{\Phi}(0) = 1$ and $\Phi_t := t^{-n} \Phi(\cdot/t)$, $(t > 0)$

Then, as $\dot{F}_{p,2}^0$ and H^p cannot be identified since for a polynomial f it holds $\Phi_t * f \in \mathcal{P}_\infty$. we give the relation between the Hardy space H^p and the realized space $\dot{F}_{p,2}^0$

Proposition 2.40. (i) *If $0 < p \leq 1$ then $H^p = \dot{F}_{p,2}^0$, and their quasi-norms are equivalent i.e.,*

$$\|f\|_{H^p} \sim \|f\|_{\dot{F}_{p,2}^0}.$$

(ii) *If $1 < p < \infty$ then $L_p = \dot{F}_{p,2}^0$, and their quasi-norms are equivalent i.e., $\|f\|_{L_p} \sim \|f\|_{\dot{F}_{p,2}^0}$.*

Proof. For (i) we refer to [39, Theorem 3.17] and for (ii) see, [32, Proposition 5.2]. □

2.3 *BMO* space and the realized spaces

The spaces $\dot{F}_{p,q}^s$, which are defined by the Littlewood-Paley decomposition (abbreviated as LPd), specifically $\dot{F}_{\infty,2}^0$ as defined in [22], (5.1), are characterized by distributions modulo all polynomials. On the other hand, the space *BMO* is defined by distributions that are modulo constants, as defined in [21]. It is important to note that $\dot{F}_{\infty,2}^0$ cannot be identified with *BMO*, as evidenced by the fact that for every polynomial f with a degree of at least 1, $\|f\|_{\dot{F}_{\infty,2}^0} = 0$ while $\|f\|_{BMO} = \infty$. In [46, p. 243], the author substituted the space *BMO*-modulo constants with a space modulo polynomials (where this spaces denoting by $BMO^{(*)}$) to obtain this identification.

To further examine the above identification, we use the realized space $\dot{F}_{p,q}^s$ of $\dot{F}_{p,q}^s$. This space is a subset of \mathcal{S}'_{μ_0} , (see (2.2) by replacing the conditions $0 < q \leq 1$ and $1 < q \leq \infty$ with $0 < p \leq 1$ and $1 < p \leq \infty$, respectively). Here, we present some of the results that were previously published in [23].

Theorem 2.41. *The identity $BMO = \dot{F}_{\infty,2}^0$ holds with equivalent seminorms.*

Now, we turn to the example $f(x) = \log|x|$, $x \in \mathbb{R}^n$ in *BMO*. Indeed, we prove that $f \in \dot{F}_{\infty,2}^0$. We have $\widehat{f}(\xi)$ coincides with $c/|\xi|^n$ for $\xi \in \mathbb{R}^n \setminus \{0\}$, then

$$\widehat{Q_j \partial_1 f}(\xi) = c \xi_1 |\xi|^{-n} \gamma(2^{-j} \xi) = 2^{j(1-n)} \psi(2^{-j} \xi)$$

where $\psi(\xi) := c \xi_1 |\xi|^{-n} \gamma(\xi)$ and $\psi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$.

We obtain

$$\begin{aligned} \|[\partial_1 f]_{\infty}\|_{\dot{F}_{\infty,2}^{-1}} &= \sup_{\eta \in \mathbb{Z}^n} \sup_{k \in \mathbb{Z}} \left(2^{kn} \int_{P_{k,\eta}} \sum_{j \geq k} 2^{-2j} (2^j |\mathcal{F}^{-1} \psi(2^j x)|)^2 dx \right)^{1/2} \\ &\leq c_1 \|\mathcal{F}^{-1} \psi\|_2 \sup_{k \in \mathbb{Z}} \left(\sum_{j \geq k} 2^{(k-j)n} \right) = c_2 \|\mathcal{F}^{-1} \psi\|_2 \end{aligned}$$

then by applying Proposition 1.24, we get $[f]_{\infty} \in \dot{F}_{\infty,2}^0$.

To prove that $\partial_1 f \in \widetilde{C}_0$ it suffices to proceed as the following. We write

$$\begin{aligned} f(x) &= \log|x| \rho(x) + \log|x| (1 - \rho(x)) \\ &=: f_1(x) + f_2(x). \end{aligned}$$

We will prove that $\partial_1 f_1$ and $\partial_1 f_2$ in \tilde{C}_0 . For any $\varphi \in \mathcal{S}$, let $\lambda > 0$, we have:

$$\begin{aligned} |\langle \partial_1 f_1(\lambda^{-1} \cdot), \varphi \rangle| &= \lambda^{n+1} |\langle f_1, \partial_1 \varphi(\lambda \cdot) \rangle| \\ &\leq \lambda^{n+1} \|\partial_1 \varphi\|_\infty \int_{|x| \leq 1} |\log |x|| \, dx \\ &\leq c \lambda^{n+1}, \end{aligned}$$

by letting in the last inequality λ to 0, we obtain that $|\langle \partial_1 f_1(\lambda^{-1} \cdot), \varphi \rangle| \rightarrow 0$, it holds the desired result.

To prove that $\partial_1 f_2 \in \tilde{C}_0$, we apply the following lemma proved in [116, Lemma 1]:

Lemma 2.42. *Let f be a continuous function on \mathbb{R}^n such that $f(x) = o(|x|)$ as $|x| \rightarrow +\infty$. Then $\partial_j f \in \tilde{C}_0$, $j = 1, \dots, n$.*

Here we have f_2 is of C^∞ class on \mathbb{R}^n , such that $\lim_{|x| \rightarrow \infty} \frac{f_2(x)}{|x|} = 0$.

Theorem 2.43. *If $0 < p < \infty$, the embedding $\tilde{F}_{p,q}^{n/p} \hookrightarrow BMO$ is proper.*

An application of Theorem 2.41 yields the following assertions for the homogeneous Besov spaces $\dot{B}_{p,q}^s$ and inhomogeneous counterparts, where the first one can be found in [112, p. 154] and in [37, p. 169, lines 2–3 p. 252]:

Corollary 2.44. *If $f \in BMO$, then its equivalence class modulo all polynomials $[f]_\infty$ belongs to $\dot{B}_{\infty,\infty}^0$.*

Proof. It suffices to apply Theorem 2.41 and Proposition 2.33. □

Corollary 2.45. *The embedding $BMO \hookrightarrow B_{\infty,\infty}^0$ is proper.*

Proof. The embedding $BMO \hookrightarrow B_{\infty,\infty}^0$ follows immediately by Proposition 2.34 and Theorem 2.41. In order to establish the strictly of the embedding, it is enough to note that $B_{\infty,\infty}^0 \not\subset L_1^{\text{loc}}$. We recall that $B_{\infty,q}^0 \subset L_1^{\text{loc}}$ iff $q \leq 2$, cf. [40, Theorem 3.3.2]. □

We have the same results for the realized space $\tilde{B}_{p,q}^s$ of $\dot{B}_{p,q}^s$:

Corollary 2.46. *It holds that $\tilde{B}_{\infty,2}^0 \hookrightarrow BMO \hookrightarrow \tilde{B}_{\infty,\infty}^0$.*

Proof. This is an immediate application of the embeddings $\tilde{B}_{\infty,2}^0 \hookrightarrow \tilde{F}_{\infty,2}^0 \hookrightarrow \tilde{B}_{\infty,\infty}^0$, which can be easily obtained by $\tilde{B}_{\infty,2}^0 \hookrightarrow \tilde{F}_{\infty,2}^0$ and Proposition 2.33. □

Hence, to deal with $\tilde{F}_{p,q}^{n/p}$ ($0 < p < \infty$) presents the contribution of Theorem 2.43, since (2.9) becomes a particular case.

2.4 Riesz operator and realized spaces

Here, we give some claims related to our results. To observe the performance of the Riesz potential \mathcal{I}_α on BMO , which is defined as

$$\mathcal{I}_\alpha f := \mathcal{F}^{-1}(|\xi|^{-\alpha} \widehat{f}) \quad \forall \alpha \in \mathbb{R}.$$

Remark 2.47. We can refer to the fact that \mathcal{I}_α maps \mathcal{S}_∞ to itself, which can be easily proven. In order to provide a consistent definition of $\mathcal{I}_\alpha : \mathcal{S}'_\infty \rightarrow \mathcal{S}'_\infty$, it is appropriate to state that if $f \in \mathcal{S}'_\infty$, then for all $\varphi \in \mathcal{S}_\infty$ and all $f_1 \in \mathcal{S}'$ such that $[f_1]_\infty = f$, the following holds:

$$\langle \mathcal{I}_\alpha f, \varphi \rangle := \langle f_1, \mathcal{I}_\alpha \varphi \rangle.$$

Specifically, we have the following proposition:

Proposition 2.48. *We have f in $\dot{F}_{p,q}^s$ if, and only if $\mathcal{I}_\alpha f$ in $\dot{F}_{p,q}^{s+\alpha}$. Moreover, $\|\mathcal{I}_\alpha f\|_{\dot{F}_{p,q}^{s+\alpha}} \sim \|f\|_{\dot{F}_{p,q}^s}$.*

Proof. We refer to [46, Theorem 5.2.3/1]. □

From the last statement, we get

Proposition 2.49. • $\mathcal{I}_\alpha([f]_\infty) \in \dot{F}_{p,q}^{s+\alpha}$ if $f \in \tilde{F}_{p,q}^s$, and there exists a function $g \in \tilde{F}_{p,q}^{s+\alpha}$ such that $\mathcal{I}_\alpha([f]_\infty) = [g]_\infty$ in \mathcal{S}' .

• In particular, $\mathcal{I}_\alpha([f]_\infty) \in \dot{F}_{\infty,2}^\alpha$ if $f \in BMO$.

Proof. We put $g := \sigma_\mu(\mathcal{I}_\alpha([f]_\infty))$. Where σ_μ is defined in (2.3)–(2.5). □

Proposition 2.50. *Let $0 < p < q < \infty$. We introduce $\alpha := n(1/p - 1/q)$.*

(i) *Then it holds $\|\mathcal{I}_\alpha([f]_\infty)\|_{\dot{F}_{q,2}^0} \leq c \|f\|_{\dot{F}_{p,2}^0}$ for all $f \in \tilde{F}_{p,2}^0$.*

(ii) *If $g \in \tilde{F}_{q,2}^0$ such that $\mathcal{I}_{-\alpha}([g]_\infty) \in \tilde{F}_{p,2}^0$, then it holds $\|g\|_{\dot{F}_{q,2}^0} \leq c \|\mathcal{I}_{-\alpha}([g]_\infty)\|_{\dot{F}_{p,2}^0}$ where the constant c is independent of g .*

Proof. *Step 1: Proof of (i).* Let $f \in \tilde{F}_{p,2}^0$, by using Proposition 2.49 then there exists a function $g \in \tilde{F}_{p,2}^\alpha$ such that $\mathcal{I}_\alpha([f]_\infty) = [g]_\infty$ in \mathcal{S}' . i.e.,

$$\|[g]_\infty\|_{\dot{F}_{p,2}^\alpha} = \|\mathcal{I}_\alpha([f]_\infty)\|_{\dot{F}_{p,2}^\alpha} \tag{2.12}$$

With $0 < p < q < \infty$ and $\alpha = n/p - n/q$, we apply the embedding $\dot{F}_{p,2}^\alpha \hookrightarrow \dot{F}_{q,2}^0$ and Proposition 2.48, we get

$$\|[g]_\infty\|_{\dot{F}_{q,2}^0} \leq c_1 \|[g]_\infty\|_{\dot{F}_{p,2}^\alpha} \tag{2.13}$$

and

$$\|\mathcal{I}_\alpha([f]_\infty)\|_{\dot{F}_{p,2}^\alpha} \leq c_2 \| [f]_\infty \|_{\dot{F}_{p,2}^0} \quad (2.14)$$

By (2.12), (2.13) and (2.14) we obtain the desired result.

Step 2: Proof of (ii). On the other hand, by the definition of \mathcal{I}_α and by Proposition 2.49, thus, for $f \in \dot{F}_{p,2}^0$ we have $\mathcal{I}_\alpha([f]_\infty) - g \in \mathcal{P}_\infty$, i.e., $\widehat{\mathcal{I}_\alpha([f]_\infty)} = \widehat{g}$ on $\mathbb{R}^n \setminus \{0\}$. Also, for all φ in \mathcal{S}_∞ and all f_1 in \mathcal{S}' with $[f_1]_\infty = [f]_\infty$, we have

$$\langle \mathcal{I}_\alpha([f]_\infty), \mathcal{I}_{-\alpha}\varphi \rangle \stackrel{\text{def}}{=} \langle f_1, \varphi \rangle = \langle g, \mathcal{I}_{-\alpha}\varphi \rangle \stackrel{\text{def}}{=} \langle \mathcal{I}_{-\alpha}([g]_\infty), \varphi \rangle;$$

then similarly, for $f \in \dot{F}_{p,2}^0$ we get $\mathcal{I}_{-\alpha}([g]_\infty) - f \in \mathcal{P}_\infty$. □

We now present the Hausdorff-Young inequality where it is required below, see e.g., [43, p. 178].

Lemma 2.51. (Hausdorff-Young inequality) *Let $1 \leq p \leq 2$. For any $f \in L_p$ we have $\widehat{f} \in L_{p'}$, and*

$$\|\widehat{f}\|_{p'} \leq \|f\|_p.$$

Corollary 2.52. (i) *If $0 < p \leq 2$, then*

$$\left(\int_{\mathbb{R}^n} |\xi|^{-2(n/p-n/2)} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2} \leq c \|f\|_{\dot{F}_{p,2}^0}, \quad \forall f \in \dot{F}_{p,2}^0. \quad (2.15)$$

(ii) *If $2 \leq q < \infty$ and $g \in \mathcal{S}'$ such that $\widehat{g} \in L_1^{\text{loc}}$, then*

$$\|g\|_q \leq c \left(\int_{\mathbb{R}^n} |\xi|^{2(n/2-n/q)} |\widehat{g}(\xi)|^2 d\xi \right)^{1/2} \quad (2.16)$$

where the positive constant c is independent of g ; we assume that the right-hand side of (2.16) is finite.

Proof. *Step 1: proof of (i).* The case $p = 2$ is easy. Then we assume $0 < p < 2$. By taking $q = 2$ in Proposition 2.50 (i) it suffices to prove

$$\| |\xi|^{-\alpha} \widehat{f} \|_2 \leq c \| \mathcal{I}_\alpha([f]_\infty) \|_2, \quad \forall f \in \dot{F}_{p,2}^0, \quad \alpha = n(1/p - 1/2). \quad (2.17)$$

Indeed we will use standard arguments. We first show that $|\xi|^{-\alpha} \widehat{f}$ is a regular element in \mathcal{S}' , in the sense that

$$\left| \int_{\mathbb{R}^n} |\xi|^{-\alpha} \widehat{f}(\xi) \varphi(\xi) d\xi \right| < \infty, \quad \forall \varphi \in \mathcal{S}.$$

In case $1 < p < 2$ we have $\widehat{f} \in L_{p'}$ (the Hausdorff-Young inequality since $f \in L_p$) then by Hölder inequality, for every $\varphi \in \mathcal{S}$,

$$\int_{|\xi| \leq 1} |\xi|^{-\alpha} |\widehat{f}\varphi(\xi)| d\xi \leq \|\widehat{f}\|_{p'} \|\varphi\|_{\infty} \left(\int_{|\xi| \leq 1} |\xi|^{-\alpha p} d\xi \right)^{1/p} \leq c \|\widehat{f}\|_{p'} \|\varphi\|_{\infty} \quad (2.18)$$

since $\alpha < n/p$; for $N > n/p - \alpha$ we have

$$\int_{|\xi| > 1} |\xi|^{-\alpha} |\widehat{f}\varphi(\xi)| d\xi \leq c_1 \|\widehat{f}\|_{p'} \left(\int_{|\xi| > 1} |\xi|^{-(N+\alpha)p} d\xi \right)^{1/p} \leq c_2 \|\widehat{f}\|_{p'}.$$

In case $0 < p \leq 1$ we have $|\widehat{f}(\xi)| \leq c|\xi|^{n(1/p-1)}\|f\|_{H^p}$ proved in e.g. [42, III.5.4, p. 128], then, for every $\varphi \in \mathcal{S}$,

$$\int_{|\xi| \leq 1} |\xi|^{-\alpha} |\widehat{f}\varphi(\xi)| d\xi \leq c_1 \|f\|_{H^p} \|\varphi\|_{\infty} \int_{|\xi| \leq 1} |\xi|^{-\alpha+n(1/p-1)} d\xi \leq c_2 \|f\|_{H^p} \quad (2.19)$$

since $n - \alpha + n(1/p - 1) = n/2$; for $L > n/2$ we have

$$\int_{|\xi| > 1} |\xi|^{-\alpha} |\widehat{f}\varphi(\xi)| d\xi \leq c_1 \|f\|_{H^p} \left(\int_{|\xi| > 1} |\xi|^{-L-n/2} d\xi \right)^{1/p} \leq c_2 \|f\|_{H^p}.$$

We note that this proof also contains the argument of that $|\xi|^{-\alpha}\widehat{f}$ is a regular distribution, i.e., in L_1^{loc} (in (2.18) and (2.19) take $\xi \in \mathbb{R}^n$ and $\varphi \in \mathcal{D}$). In the second, by duality in L_2 , density of \mathcal{S} in L_2 and density of \mathcal{S}_{∞} in \mathcal{S} endowed with the L_2 norm (this can be done as follows, if $f \in \mathcal{S}$ then $((id - S_{-j})f)$ is a sequence in \mathcal{S}_{∞} , where

$$\|S_{-j}f\|_2 \leq c \|\rho(2^j \cdot)\widehat{f}\|_2 \leq c2^{-jn/2} \|\rho\|_2 \|f\|_1,$$

thus $(id - S_{-j})f \rightarrow f$ in L_2), together with Parseval's identity ($\|\varphi\|_2 = (2\pi)^{-n/2} \|\widehat{\varphi}(-\cdot)\|_2$) we get

$$\begin{aligned} \left\| |\xi|^{-\alpha} \widehat{f} \right\|_2 &\leq c \sup_{\varphi \in \mathcal{S}_{\infty}, \|\varphi\|_2 \leq 1} \left| \int_{\mathbb{R}^n} |\xi|^{-\alpha} \widehat{f}(\xi) \widehat{\varphi}(-\xi) d\xi \right| \\ &= c \sup_{\varphi \in \mathcal{S}_{\infty}, \|\varphi\|_2 \leq 1} \left| \langle \widehat{f}, |\xi|^{-\alpha} \widehat{\varphi}(-\cdot) \rangle \right|, \end{aligned}$$

since the function $|\xi|^{-\alpha}\widehat{\varphi}$ belongs to \mathcal{S} and $\widehat{f} \in \mathcal{S}'$, recall that $f \in \mathcal{S}'$. We continue where the elementary equality $\mathcal{F}^{-1}[|\xi|^{-\alpha}\widehat{\varphi}(-\cdot)](-x) = \mathcal{F}^{-1}[|\xi|^{-\alpha}\widehat{\varphi}](x)$ implies

$$\langle f, \widehat{|\xi|^{-\alpha}\widehat{\varphi}} \rangle = (2\pi)^n \langle f, \mathcal{F}^{-1}[|\xi|^{-\alpha}\widehat{\varphi}(-\cdot)](-\cdot) \rangle = (2\pi)^n \langle f, \mathcal{I}_{\alpha}\varphi \rangle.$$

Then $\| |\xi|^{-\alpha} \widehat{f} \|_2 \leq c_1 \sup_{\varphi \in \mathcal{S}_\infty, \|\varphi\|_2 \leq 1} |\langle \mathcal{S}_\alpha([f]_\infty), \varphi \rangle|$, and by Cauchy-Schwarz inequality we obtain (2.17); then (2.15).

Step 2: proof of (ii). As above the case $q = 2$ is easy, then we assume that $2 < q < \infty$. Also, here by taking $p = 2$ in Proposition 2.50(ii) it suffices to use the Plancherel formula, and obtain (recall the assumption $\widehat{g} \in \mathcal{S}' \cap L_1^{\text{loc}}$)

$$\begin{aligned} \|\mathcal{S}_{-\alpha}([g]_\infty)\|_2 &= \sup_{\varphi \in \mathcal{S}_\infty, \|\varphi\|_2 \leq 1} |\langle g, \mathcal{S}_{-\alpha} \varphi \rangle| = \sup_{\varphi \in \mathcal{S}_\infty, \|\varphi\|_2 \leq 1} |\langle \widehat{g}, |\xi|^\alpha \widehat{\varphi} \rangle| \\ &\leq \sup_{\varphi \in \mathcal{S}_\infty, \|\varphi\|_2 \leq 1} |\langle |\xi|^\alpha \widehat{g}, |\xi|^{-\alpha} (|\xi|^\alpha \widehat{\varphi}) \rangle| \leq c \| |\xi|^\alpha \widehat{g} \|_2, \end{aligned}$$

and thus (2.16) is proved. □

Remark 2.53. In this remark we present other inequalities similar to (2.15).

The inequality outlined below is given in [9, Theorem 1.4.1]:

$$\left(\int_{\mathbb{R}^n} |\xi|^{-n(2-p)} |\widehat{f}(\xi)|^p d\xi \right)^{1/p} \leq c \|f\|_p, \quad \text{for all } 1 \leq p \leq 2 \quad \text{and all } f \in L_p.$$

After that, the author in [27, Theorem 3.1] provided the following inequality:

$$\left(\int_{\mathbb{R}^n} |\xi|^{-n(2-p)} |\widehat{f}(\xi)|^p d\xi \right)^{1/p} \leq c \|f\|_{\dot{F}_{p,q}^0}, \quad \text{for all } 0 < p < 2 \quad \text{and } 0 < q \leq \infty.$$

holds, for all $f \in \dot{F}_{p,q}^0$. Then, we refer to [42, p. 128] for this inequality

$$\left(\int_{\mathbb{R}^n} |\xi|^{-n(2-p)} |\widehat{f}(\xi)|^p d\xi \right)^{1/p} \leq c \|f\|_{H^p}, \quad \text{for all } 0 < p \leq 1.$$

Here, we give the reason for adding the assumption $\widehat{g} \in L_1^{\text{loc}}$ in (2.16), and quote [43, I.4.13, p. 34] for the following proposition:

Proposition 2.54. *If $q > 2$ there exist functions $g \in L_q$ such that $\widehat{g} \notin L_1^{\text{loc}}$, i.e., \widehat{g} is not a function.*

2.5 Embedding between homogeneous Sobolev space and $\dot{F}_{p,q}^s$

We consider the homogeneous Sobolev space \dot{H}^s of tempered distributions f such that $\widehat{f} \in L_1^{\text{loc}}$ and

$$\|f\|_{\dot{H}^s} := \| |\xi|^s \widehat{f} \|_2 < \infty.$$

See e.g., [2, Section 1.3] for other properties of \dot{H}^s .

Proposition 2.55. (i) If $1 < p \leq 2$ then $L_p \hookrightarrow \dot{H}^{-n(1/p-1/2)}$.

(ii) If $0 \leq s < n/2$ then $\dot{H}^s \hookrightarrow L_{2n/(n-2s)}$.

Proof. Let $1 < p \leq 2$. By applying Proposition 2.40 $L_p = \dot{F}_{p,2}^0$ if $1 < p < \infty$, and by applying inequality (2.15) we obtain the first embedding.

In estimation (2.16), we set $s = n/2 - n/q$ that means $q = 2n/(n - 2s)$ and as $H^p = \dot{F}_{p,2}^0$ if $0 < p \leq 1$ (see Proposition 2.40) we get the second embedding. \square

Proposition 2.56. We have

$$\dot{H}^s \hookrightarrow \dot{F}_{2,2}^s \quad \text{and} \quad \dot{H}^{n/2} \hookrightarrow BMO. \quad (2.20)$$

Proof. Theorem 2.41 yields the second embedding in (2.20). We then prove the first one. Let $f \in \dot{H}^s$, then

$$\int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} (2^{js} |Q_j f|)^2 dx \leq (2\pi)^{-n} \sum_{j \in \mathbb{Z}} \int_{\Omega_j} (|\xi|^s |\gamma(2^{-j}\xi) \widehat{f}(\xi)|)^2 d\xi$$

where Ω_j denotes the annulus $(1/2)2^j \leq |\xi| \leq (3/2)2^j$, and here $|\xi| \sim 2^j$. Thus,

$$\int_{\Omega_j} (|\xi|^s |\gamma(2^{-j}\xi) \widehat{f}(\xi)|)^2 d\xi \leq \int_{\mathbb{R}^n} (|\xi|^s |\widehat{f}(\xi)|)^2 \gamma(2^{-j}\xi)^2 d\xi,$$

but $\sum_{j \in \mathbb{Z}} \gamma(2^{-j}\xi)^2 \leq 3\|\gamma\|_\infty^2$ since this sum contains at most three non-vanishing terms, which gives the desired estimate and $[f]_\infty \in \dot{F}_{2,2}^s$.

We now prove that $f^{(\alpha)} \in \widetilde{C}_0$ for $|\alpha| = \mu$; see (2.1) for the definition of μ :

$$\mu = ([s - n/2] + 1)_+. \quad (2.21)$$

Let $\lambda > 0$ and $r \in \mathbb{Z}$ be such that $2^{-r-1} < \lambda \leq 2^{-r}$. The function $\mathcal{F}(h_\lambda(Q_{j-r}f^{(\alpha)}))$ (where $h_\lambda f := f(\lambda^{-1}\cdot)$) is supported by the annulus $2^{j-1} \leq |\xi| \leq 3 \cdot 2^j$. We have

$$\mathcal{F}(Q_k h_\lambda(Q_{j-r}f^{(\alpha)})) = 0 \quad \text{if} \quad k - j \geq 3 \quad \text{or} \quad k - j \leq -2,$$

and

$$h_\lambda f^{(\alpha)} = \sum_{j,k \in \mathbb{Z}} Q_k h_\lambda(Q_{j-r}f^{(\alpha)}) = \sum_{j \in \mathbb{Z}} \sum_{-2 \leq k \leq 3} Q_{j+k} h_\lambda(Q_{j-r}f^{(\alpha)}).$$

We set $\widetilde{Q}_j := \sum_{-2 \leq k \leq 3} Q_{j+k}$ associated to the function $\widetilde{\gamma} := \sum_{-2 \leq k \leq 3} \gamma(2^{-k}\cdot)$. Thus for every

$\varphi \in \mathcal{S}$, and by using the elementary formula $\langle f, g \rangle = (2\pi)^{-n} \langle \widehat{f}, \widehat{g}(-\cdot) \rangle$, we have

$$\begin{aligned} \sum_{-2 \leq k \leq 3} \langle Q_{j+k} h_\lambda(Q_{j-r} f^{(\alpha)}), \varphi \rangle &= \lambda^n \langle Q_{j-r} f^{(\alpha)}, \widetilde{Q}_j \varphi(\lambda \cdot) \rangle \\ &= (2\pi)^{-n} |\alpha| \langle \xi^\alpha \gamma(2^{r-j} \cdot) \widehat{f}, \widetilde{\gamma}(2^{-j} \lambda^{-1} \cdot) \widehat{\varphi}(-\lambda^{-1} \cdot) \rangle, \end{aligned}$$

the function $\xi^\alpha \gamma(2^{r-j} \cdot) \widehat{f} \in L_1^{\text{loc}}$ since $\xi^\alpha \gamma(2^{r-j} \cdot) \in \mathcal{D}$, then we can apply Cauchy-Schwarz inequality in the last term, and obtain

$$|\langle h_\lambda f^{(\alpha)}, \varphi \rangle| \leq c \lambda^{n/2} \sum_{j \in \mathbb{Z}} \|\xi^\alpha \gamma(2^{r-j} \cdot) \widehat{f}\|_2 \|\widetilde{\gamma}(2^{-j} \cdot) \widehat{\varphi}(-\cdot)\|_2. \quad (2.22)$$

Clearly on the one hand we have (again $|\xi|^{\mu-s} \gamma \in \mathcal{D}$)

$$\begin{aligned} \|\xi^\alpha \gamma(2^{r-j} \cdot) \widehat{f}\|_2 &= \| |\xi|^{\mu-s} \gamma(2^{r-j} \cdot) |\xi|^s \widehat{f} \|_2 \leq c_1 2^{(j-r)(\mu-s)} \|f\|_{\dot{H}^s} \\ &\leq c_2 2^{j(\mu-s)} \lambda^{\mu-s} \|f\|_{\dot{H}^s}. \end{aligned} \quad (2.23)$$

On the other hand, we exploit the following assertion, a classical consequence of Taylor formula: $\|\widetilde{Q}_j \varphi\|_2 = O(2^{-jN})$ as $j \rightarrow +\infty$, for all $N \in \mathbb{N}$, then we have

$$\|\gamma(2^{-j} \cdot) \widehat{\varphi}(-\cdot)\|_2 \leq c_1 \|\widetilde{Q}_j \varphi\|_2 \leq c_2 2^{-jN} \quad (\text{recall that } \widetilde{\gamma} \text{ is even}),$$

and

$$\begin{aligned} \sum_{j \in \mathbb{Z}} 2^{j(\mu-s)} \|\widetilde{\gamma}(2^{-j} \cdot) \widehat{\varphi}(-\cdot)\|_2 &\leq c_1 \sum_{j \geq 1} 2^{j(\mu-s)} \|\widetilde{Q}_j \varphi\|_2 + \|\widehat{\varphi}\|_\infty \|\widetilde{\gamma}\|_2 \sum_{j \leq 0} 2^{j(\mu-s+n/2)} \\ &\leq c_1 \sum_{j \geq 1} 2^{j(\mu-s-N)} + c_2 \sum_{j \leq 0} 2^{j(\mu-s+n/2)}, \end{aligned} \quad (2.24)$$

choosing the natural number N such that $N > \mu - s$, and by (2.21) we have $\mu - s + n/2 > 0$, thus we get that the right-hand side of (2.24) is bounded by a positive and finite constant c_3 . Finally, we turn to (2.22), where from (2.23) and (2.24) it holds that

$$|\langle h_\lambda f^{(\alpha)}, \varphi \rangle| \leq c \lambda^{\mu-s+n/2} \|f\|_{\dot{H}^s}.$$

Taking $\lambda \downarrow 0$, then by (2.21) (i.e. $\mu - s + n/2 > 0$) we obtain $f^{(\alpha)} \in \widetilde{C}_0$, and (2.20) is proved. \square

As $\dot{B}_{p,p}^0 \hookrightarrow \dot{F}_{p,2}^0 \hookrightarrow \dot{B}_{p,2}^0$ if $0 < p \leq 2$, and $\dot{F}_{p,2}^0 \hookrightarrow \dot{B}_{p_1,p}^{-s}$ if $p < p_1$ and $s := n/p - n/p_1$, then we can obtain embeddings between H^p and realized of Besov spaces; we omit details.

Chapter 3

Functional characterizations of homogeneous Besov-type spaces

Under certain conditions on s , p and q , several function characteristics of BMO are valid on realized homogeneous Besov $\dot{B}_{p,q}^s$ and Triebel-Lizorkin spaces $\dot{F}_{p,q}^s$ cf. [24]. We propose to extend the results of [24] to the realized homogeneous Besov-type space $\dot{B}_{p,q}^{s,\tau}$.

3.1 Results already obtained on $\dot{A}_{p,q}^s$

Besov and Triebel-Lizorkin spaces are defined with respect to polynomials, as indicated in chapter 2 $\|f\|_{\dot{B}_{p,q}^s} = \|f\|_{\dot{F}_{p,q}^s} = 0$ if and only if f is a polynomial on \mathbb{R}^n .

However, Propositions 2.18, 2.22 and 2.23 are not valid if we replace BMO with $\dot{B}_{p,q}^s$ or $\dot{F}_{p,q}^s$. See the introduction of the paper [24] we give an example in this context.

In [24], we have extended some properties given in Chapter 2 Section 2.22 to the realized spaces $\dot{A}_{p,q}^s$. To recall this, we use the following brevity:

If for a given function $f \in \dot{A}_{p,q}^s$ there exists a polynomial $u_f \in \mathcal{P}_{\mu_0}$, then we set

$$\tilde{f} := f + u_f.$$

Theorem 3.1. *Let $0 < p, q \leq \infty$ and $s > (\frac{n}{p} - n)_+$. Let d be a real number such that $d > (s - \frac{n}{p})_+$. If $f \in \dot{A}_{p,q}^s$, then there exists a polynomial $u_f \in \mathcal{P}_{\mu_0}$ such that the inequality*

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x)|}{1 + |x|^{n+d}} dx \leq c_1 \| [f]_\infty \|_{\dot{A}_{p,q}^s}$$

holds; the constant c_1 is independent of f .

In particular, if either $s < \frac{n}{p}$ ($p < \infty$ in F -spaces) or $s = \frac{n}{p}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces); here, realizations commute with translations, then the inequality

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x - x_0|^{n+d}} dx \leq c_2 \| [f]_\infty \|_{\dot{A}_{p,q}^s}$$

holds for all $x_0 \in \mathbb{R}^n$; the constant c_2 is independent of f and x_0 .

Proof. See [24, Theorem 2.1]. □

Remark 3.2. It is required that $p < \infty$ if $s < \frac{n}{p}$ or $s = \frac{n}{p}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces) and $s > 0$.

Remark 3.3. By the fact that $BMO = \dot{F}_{\infty,2}^0$ and $\mu_0 = 1$ (see Theorem 2.41), we obtain Proposition 2.18 by applying Theorem 3.1.

Theorem 3.4. Let s, p, q and d be identical to those in Theorem 3.1.

(i) If $f \in \dot{A}_{p,q}^s$, then there exists a polynomial $u_f \in \mathcal{P}_{\mu_0}$ such that the inequality

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x) - m_{Q(0,1)} \tilde{f}|}{1 + |x|^{n+d}} dx \leq c \| [f]_\infty \|_{\dot{A}_{p,q}^s}$$

holds; the constant c is independent of f .

(ii) Assume that $p < \infty$ in F -spaces and that

- either $s < \frac{n}{p}$ or $s = \frac{n}{p}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces),
- either $s - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}_0$ or $s - \frac{n}{p} \in \mathbb{N}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces);

here, realizations commute with dilations. If $f \in \dot{A}_{p,q}^s$, then there exists a polynomial $u_f \in \mathcal{P}_{\mu_0}$ such that the inequality

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x) - m_{Q(0,\lambda)} \tilde{f}|}{\lambda^{n+d} + |x|^{n+d}} dx \leq c \lambda^{s-d-n/p} \| [f]_\infty \|_{\dot{A}_{p,q}^s}$$

holds for all $\lambda > 0$; the constant c is independent of f and λ .

(iii) In particular, if either $s < \frac{n}{p}$ ($p < \infty$ in F -spaces) or $s = \frac{n}{p}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces); here, realizations commute with translations, then the inequality

$$\int_{\mathbb{R}^n} \frac{|f(x) - m_{Q(x_0,\lambda)} f|}{\lambda^{n+d} + |x - x_0|^{n+d}} dx \leq c \lambda^{s-d-n/p} \| [f]_\infty \|_{\dot{A}_{p,q}^s} \quad (3.1)$$

holds for all $f \in \dot{A}_{p,q}^s$, all $\lambda > 0$ and all $x_0 \in \mathbb{R}^n$; the constant c is independent of f , λ and x_0 .

Proof. See [24, Theorem 2.4]. □

For modulo constants ($\mu_0 = 1$, that is, $u_f = \text{constant}$), we can extend the claim (iii) of Theorem 3.4 to the cases if either $s - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}_0$ or $s - \frac{n}{p} \in \mathbb{N}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces) or $s - \frac{n}{p} \in \mathbb{N}_0$ and $q > 1$ in B -spaces ($p > 1$ in F -spaces); that is:

Corollary 3.5. *Let $0 < p, q \leq \infty$. Suppose that $\mu_0 = 1$ in either $s - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}_0$ either $s - \frac{n}{p} \in \mathbb{N}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces) or $s - \frac{n}{p} \in \mathbb{N}_0$ and $q > 1$ in B -spaces ($p > 1$ in F -spaces). Then, the inequality (3.1) remains true.*

Proof. It is Corollary 2.5 in [24]. □

Theorem 3.6. *Let $0 < p, q \leq \infty$ and $s > (\frac{n}{p} - n)_+$. Let $0 < p_1 < \infty$. In case $1 < p_1 < \infty$ we assume that*

$$\text{in } F\text{-spaces: } \frac{n}{p_1} \geq \frac{n}{p} - s, \quad (3.2)$$

$$\text{in } B\text{-spaces: either } \frac{n}{p_1} > \frac{n}{p} - s, \text{ or } \frac{n}{p_1} = \frac{n}{p} - s \text{ and } q \leq p_1. \quad (3.3)$$

(i) *If $f \in \dot{A}_{p,q}^s$, then there exists a polynomial $u_f \in \mathcal{P}_{\mu_0}$ such that the inequality*

$$\left(\int_{Q(0,1)} |\tilde{f}(x) - m_{Q(0,1)} \tilde{f}|^{p_1} dx \right)^{1/p_1} \leq c \| [f]_\infty \|_{\dot{A}_{p,q}^s} \quad (3.4)$$

holds; the constant c is independent of f .

(ii) *Assume that $p < \infty$ in F -spaces and*

- *either $s < \frac{n}{p}$ or $s = \frac{n}{p}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces),*
- *either $s - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}_0$ or $s - \frac{n}{p} \in \mathbb{N}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces);*

here, we have realizations commute with dilations. If $f \in \dot{A}_{p,q}^s$, then there exists a polynomial $u_f \in \mathcal{P}_{\mu_0}$ such that the inequality

$$\left(\frac{1}{|Q(0,\lambda)|} \int_{Q(0,\lambda)} |\tilde{f}(x) - m_{Q(0,\lambda)} \tilde{f}|^{p_1} dx \right)^{1/p_1} \leq c \lambda^{s-n/p} \| [f]_\infty \|_{\dot{A}_{p,q}^s} \quad (3.5)$$

holds for all $\lambda > 0$; the constant c is independent of f and λ .

(iii) In particular, if either $s < \frac{n}{p}$ or $s = \frac{n}{p}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces); here, we have realizations commute with translations, then the inequality

$$\left(\frac{1}{|Q_{(x_0, \lambda)}|} \int_{Q_{(x_0, \lambda)}} |f(x) - m_{Q_{(x_0, \lambda)}} f|^{p_1} dx \right)^{1/p_1} \leq c \lambda^{s-n/p} \| [f]_\infty \|_{\dot{A}_{p,q}^s} \quad (3.6)$$

holds for all $f \in \dot{A}_{p,q}^s$, all $\lambda > 0$ and all $x_0 \in \mathbb{R}^n$; the constant c is independent of f , λ and x_0 .

Proof. See [24, Theorem 2.6]. □

As in Corollary 3.5, we extend the assertion Theorem 3.6/(iii) to functions $f \in \dot{A}_{p,q}^s$ such that $\mu_0 = 1$. Where in this formulation we will find interesting examples with cases $s = \frac{n}{p}$ or $1 + \frac{n}{p}$.

Corollary 3.7. Let s, p, p_1 and q be the same as in Theorem 3.6, in particular the conditions (3.2)–(3.3). Let either $(\frac{n}{p} - n)_+ < s < 1 + \frac{n}{p}$ or $s = 1 + \frac{n}{p}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces). Then, the inequality (3.6) remains true.

$\mu_0 = 0$ or 1 in Corollary 3.7. See [24]. The inequality (3.6) is false for functions $f \in \dot{A}_{p,q}^s$ if $\mu_0 \geq 2$.

Example 3.8 (counterexample). The function define by

$$h(x) := x_1^2$$

belongs to $\dot{A}_{p,q}^{3+n/p}$ since $\mu_0 \geq 3$. Through simple computation, one determines

$$\int_{Q_{(0,1)}} |h(x) - m_{Q_{(0,1)}} h| dx = \frac{4}{9\sqrt{3}} 2^n.$$

Thus, if $p_1 \geq 1$ by Hölder inequality we obtain

$$\int_{Q_{(0,1)}} |h(x) - m_{Q_{(0,1)}} h| dx \leq c \left(\int_{Q_{(0,1)}} |h(x) - m_{Q_{(0,1)}} h|^{p_1} dx \right)^{1/p_1},$$

if $0 < p_1 < 1$, since $|h(x) - m_{Q_{(0,1)}} h| \leq |h(x)| + \frac{1}{3} \leq \frac{4}{3}$ for all $x \in Q_{(0,1)}$, we have

$$\begin{aligned} \int_{Q_{(0,1)}} |h(x) - m_{Q_{(0,1)}} h| dx &= \int_{Q_{(0,1)}} |h(x) - m_{Q_{(0,1)}} h|^{p_1} |h(x) - m_{Q_{(0,1)}} h|^{1-p_1} dx \\ &\leq \left(\frac{4}{3}\right)^{1-p_1} \int_{Q_{(0,1)}} |h(x) - m_{Q_{(0,1)}} h|^{p_1} dx, \end{aligned}$$

this the left-hand side of (3.6), while $\| [h]_\infty \|_{\dot{A}_{p,q}^{3+n/p}} = 0$.

Example 3.9. The function $g(x) := x_1$ belongs to $\dot{A}_{p,q}^{1+n/p}$ (with $q > 1$ in B-case and $p > 1$ in F-case, here $\mu_0 = 2$) for with the same calculations given for h .

Remark 3.10. The constant c in the right hand of (3.4)–(3.6) with the form $c_1 p_1$, where $c_1 > 1$ independent of p_1 , as can be shown by closely examining the proof of Theorem 3.6 in the case $s \geq \frac{n}{p}$ and $1 \leq p_1 < \infty$.

The purpose of Corollary 3.7 is to extend to realized spaces the so-called John-Nirenberg inequality, which has been proven on BMO (see, for example, [25, 29, 39] and [42, p. 144]).

Theorem 3.11. Let $0 < p < \infty$ and $0 < q \leq \infty$. Let either $\frac{n}{p} \leq s < 1 + \frac{n}{p}$ or $s = 1 + \frac{n}{p}$ and $q \leq 1$ in B-spaces ($p \leq 1$ in F-spaces). Then, there exist two positive constants c_1, c_2 , such that the inequality

$$\left| \left\{ x \in Q_{(x_0, \lambda)} : |f(x) - m_{Q_{(x_0, \lambda)}} f| > \alpha \right\} \right| \leq c_1 |Q_{(x_0, \lambda)}| \exp \left(- \frac{c_2 \alpha}{|Q_{(x_0, \lambda)}|^{s/n-1/p} \| [f]_\infty \|_{\dot{A}_{p,q}^s}} \right) \quad (3.7)$$

holds for all $f \in \dot{A}_{p,q}^s$, all $\alpha > 0$ and all cubes $Q_{(x_0, \lambda)}$ in \mathbb{R}^n .

Proof. See [24, Theorem 2.12]. □

The previous result has the following consequence

Corollary 3.12. Let p, q and s given as in Theorem 3.11. Then

$$\frac{1}{|Q|} \int_Q \exp \left(\frac{b}{|Q|^{s/n-1/p} \| [f]_\infty \|_{\dot{A}_{p,q}^s}} |f(x) - m_Q f| \right) dx \leq c, \quad c := c(n, b),$$

holds for all $f \in \dot{A}_{p,q}^s$, all $\alpha > 0$ and all cubes Q in \mathbb{R}^n with sides parallel to the axes. where $b < c_2$ (c_2 is the constant given in (3.7)).

Proof. The proof is similar as in [25, Corollary 2.1.7]. □

Remark 3.13. In [24, Section 5.1], we have proved that the condition $d > (s - \frac{n}{p})_+$ given in Theorem 3.1 is optimal.

3.2 Recent results

3.2.1 Statement of assertions in Besov-type spaces

Here, we will extend the results discussed in section 3.1 to the realized homogeneous Besov-type space $\dot{B}_{p,q}^{s,\tau}$. We will focus on the case where $0 < \tau \leq 1/p$. However, if $\tau > 1/p$ and $0 < q < \infty$,

or if $\tau = 1/p$ and $q = \infty$, the homogeneous Besov-type space coincides with the homogeneous Besov space $\dot{B}_{p,q}^{s,\tau} = \dot{B}_{\infty,\infty}^{s+n\tau-n/p}$, as described in [51].

Theorem 3.14. *Let $0 < p < \infty$, $0 < q \leq \infty$, $0 < \tau \leq \frac{1}{p}$ and $s > (\frac{n}{p} - n)_+$. Let d be a real number such that $d > (s + n\tau - \frac{n}{p})_+$. If $f \in \dot{B}_{p,q}^{s,\tau}$, then there exists a polynomial $u_f \in \mathcal{P}_\mu$ such that the inequality*

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x)|}{1 + |x|^{n+d}} dx \leq c_1 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}} \quad (3.8)$$

holds; the constant c_1 is independent of f .

In particular, if either $s + n\tau < \frac{n}{p}$ or $s + n\tau = \frac{n}{p}$ and $q \leq 1$; here we have realizations commute with translations, then the inequality

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x - x_0|^{n+d}} dx \leq c_2 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}} \quad (3.9)$$

holds for all $x_0 \in \mathbb{R}^n$; the constant c_2 is independent of f and x_0 .

Theorem 3.15. *Let $0 < p < \infty$, $0 < q \leq \infty$, $0 < \tau \leq \frac{1}{p}$ and $s > (\frac{n}{p} - n)_+$. Let d be a real number such that $d > (s + n\tau - \frac{n}{p})_+$.*

(i) *If $f \in \dot{B}_{p,q}^{s,\tau}$, then there exists a polynomial $u_f \in \mathcal{P}_\mu$ such that the inequality*

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x) - m_{Q(0,1)} \tilde{f}|}{1 + |x|^{n+d}} dx \leq c \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}} \quad (3.10)$$

holds; the constant c is independent of f and u_f .

(ii) *Assume that either $s + n\tau < \frac{n}{p}$ either $s + n\tau = \frac{n}{p}$ and $q \leq 1$ either $s + n\tau - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}_0$ or $s + n\tau - \frac{n}{p} \in \mathbb{N}$ and $q \leq 1$; here we have realizations commute with dilations. If $f \in \dot{B}_{p,q}^{s,\tau}$, then there exists a polynomial $u_f \in \mathcal{P}_\mu$ such that the inequality*

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x) - m_{Q(0,\lambda)} \tilde{f}|}{\lambda^{n+d} + |x|^{n+d}} dx \leq c \lambda^{s+n\tau-d-n/p} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}} \quad (3.11)$$

holds for all $\lambda > 0$; the constant c is independent of f , u_f and λ .

(iii) *Assume that either $s + n\tau < \frac{n}{p}$ or $s + n\tau = \frac{n}{p}$ and $q \leq 1$; here we have realizations commute with translations. If $f \in \dot{B}_{p,q}^{s,\tau}$, then there exists a polynomial $u_f \in \mathcal{P}_\mu$ such that the inequality*

$$\int_{\mathbb{R}^n} \frac{|f(x) - m_{Q(x_0,\lambda)} f|}{\lambda^{n+d} + |x - x_0|^{n+d}} dx \leq c \lambda^{s+n\tau-d-n/p} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}, \quad (3.12)$$

holds for all $\lambda > 0$ and all $x_0 \in \mathbb{R}^n$; the constant c is independent of f , u_f , λ and x_0 .

Theorem 3.16. *Let $0 < p < \infty$, $0 < q \leq \infty$, $0 < \tau \leq \frac{1}{p}$ and $s > (n/p - n)_+$. Let $0 < p_1 < \infty$. In case $1 < p_1 < \infty$ we assume that either $n/p_1 > n/p - s$ or $n/p_1 = n/p - s$ and $q \leq p_1$.*

(i) *If $f \in \dot{B}_{p,q}^{s,\tau}$, then there exists a polynomial $u_f \in \mathcal{P}_\mu$ such that the inequality*

$$\left(\int_{Q(0,1)} |\tilde{f}(x) - m_{Q(0,1)} \tilde{f}|^{p_1} dx \right)^{1/p_1} \leq c \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}, \quad (3.13)$$

holds; the constant c is independent of f and u_f .

(ii) *Assume that either $s + n\tau < \frac{n}{p}$ either $s + n\tau = \frac{n}{p}$ and $q \leq 1$ or $s + n\tau - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}_0$; here, we have realizations commute with dilations. If $f \in \dot{B}_{p,q}^{s,\tau}$, then there exists a polynomial $u_f \in \mathcal{P}_\mu$ such that the inequality*

$$\left(\frac{1}{|Q(0,\lambda)|} \int_{Q(0,\lambda)} |\tilde{f}(x) - m_{Q(0,\lambda)} \tilde{f}|^{p_1} dx \right)^{1/p_1} \leq c \lambda^{s+n\tau-n/p} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}, \quad (3.14)$$

holds for all $\lambda > 0$; the constant c is independent of f , λ and u_f .

(iii) *Now, assume that either $s + n\tau < \frac{n}{p}$ or $s + n\tau = \frac{n}{p}$ and $q \leq 1$; here we have realizations commute with translations, then the inequality*

$$\left(\frac{1}{|Q(x_0,\lambda)|} \int_{Q(x_0,\lambda)} |\tilde{f}(x) - m_{Q(x_0,\lambda)} \tilde{f}|^{p_1} dx \right)^{1/p_1} \leq c \lambda^{s+n\tau-n/p} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}, \quad (3.15)$$

holds for all $f \in \dot{B}_{p,q}^{s,\tau}$, all $\lambda > 0$ and all x_0 in \mathbb{R}^n ; the constant c is independent of f , λ and u_f .

Remark 3.17. In the left-hand side of the inequalities (3.8), (3.10), (3.11), (3.13), and (3.14), it is possible to replace \int with $\inf_{u \in \mathcal{P}_\infty} \int$ and \tilde{f} with $f + u$, respectively. This substitution is in accordance with the concept presented in [46, 5.2.4(2)].

Remark 3.18. The condition $\frac{n}{p_1} \geq \frac{n}{p} - s$ is required for Theorem 3.16, specifically in equation (3.14), this condition implies that $\frac{n}{p_1} \geq \frac{n}{p} - s - n\tau$. If we suppose that $s + n\tau + \frac{n}{p_1} - \frac{n}{p} < 0$. By dividing each term in equation (3.14) by $\lambda^{s+n\tau-n/p}$, we obtain

$$\lambda^{n/p-n/p_1-s-n\tau} \left(\int_{Q(0,\lambda)} |\tilde{f}(x) - m_{Q(0,\lambda)} \tilde{f}|^{p_1} dx \right)^{1/p_1} \leq c \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}},$$

and letting $\lambda \rightarrow \infty$. Then its contradiction with the assumption. Then, probably the condition $\frac{n}{p_1} \geq \frac{n}{p} - s - n\tau$ is the correct one for Theorem 3.16.

Remark 3.19. It is interested to show the optimality of the above conditions given in Theorems 3.14–3.16 in the sens of Remark 3.13. See again [24, Section 5.1].

3.2.2 Necessary tools

The following assertions are in the sequel.

Proposition 3.20. *Let $g \in \mathcal{P}_\mu$, μ is defined in (3.15) if $d > (s + n\tau - \frac{n}{p})_+$ then*

$$\int_{\mathbb{R}^n} \frac{|g(x)|}{1 + |x|^{n+d}} dx < \infty. \quad (3.16)$$

Proof. We need only that x^α with $|\alpha| = \mu - 1$ satisfies (3.16), it suffices to choose $d > \mu - 1$. Since

$$\left(s + n\tau - \frac{n}{p}\right)_+ > \mu - 1, \quad (3.17)$$

we obtain the desired result. We now demonstrate that estimate (3.17) is true.

If $\mu = [s + n\tau - \frac{n}{p}] + 1$ or $\mu = s + n\tau - \frac{n}{p}$. Thus, $\mu - 1 \leq s + n\tau - \frac{n}{p}$.

If $\mu = 1$ or $\mu = 0$ for $s + n\tau - \frac{n}{p} \leq 0$. We have $\mu - 1 = 0 < d$ or $\mu - 1 = -1 < (s + n\tau - \frac{n}{p})_+$, respectively. \square

Proposition 3.21. *Let $0 < p < \infty$ and $N \in \mathbb{N}_0$. Then there exists a constant $c > 0$, such that*

$$|f(x)| \leq c \left(R^n \int_{\mathbb{R}^n} |f(y)|^p (1 + R|x-y|)^{-Np} dy \right)^{1/p} \quad (3.18)$$

holds, for all $R > 0$, all $f \in \mathcal{S}'$ such that $\text{supp } \widehat{f} \subset \{\xi : |\xi| \leq R\}$ and all $x \in \mathbb{R}^n$.

Proof. To show (3.18) we consider two cases:

Case $p \geq 1$: Let us consider $\theta \in \mathcal{S}$ such that $\widehat{\theta}(\xi) = 1$ if $|\xi| \leq 1$. We put $\widehat{\theta}_R(\xi) = \widehat{\theta}(\xi/R)$. Then it holds

$$f = \theta_R * f.$$

We apply Hölder inequality

$$\begin{aligned} |f(x)|^p &= |\theta_R * f(x)|^p = \left| \int_{\mathbb{R}^n} \theta(y) f(x - R^{-1}y) dy \right|^p \\ &\leq \left[\int_{\mathbb{R}^n} |\theta(y)| (1 + |y|)^N (1 + |y|)^{-N} |f(x - R^{-1}y)| dy \right]^p \\ &\leq \left(\int_{\mathbb{R}^n} |\theta(y)|^{p'} (1 + |y|)^{Np'} dy \right)^{p/p'} \int_{\mathbb{R}^n} (1 + |y|)^{-Np} |f(x - R^{-1}y)|^p dy, \end{aligned}$$

then by changing variables $z := x - R^{-1}y$, we obtain the result.

Case $0 < p < 1$: Let us consider $\theta \in \mathcal{S}$ such that $\text{supp } \widehat{\theta} \subset \{\xi : |\xi| \leq 3/2\}$ and $\widehat{\theta}(\xi) = 1$ if $|\xi| \leq 1$. We put

$$g_x(y) := \theta(y) f(x - R^{-1}y)$$

for a fixed $x \in \mathbb{R}^n$. We have

$$\widehat{g}_x(\xi) = \int_{\mathbb{R}^n} \widehat{\theta}(\xi - \eta) e^{-iR x \cdot \eta} \widehat{f}(-R\eta) d\eta.$$

Form the last equality, we get $\text{supp } \widehat{g}_x \subset \{\xi : |\xi| \leq 5/2\}$, then by Bernstein inequality

$$\|g_x\|_1 \leq \left(\frac{5}{2}\right)^{n/p-n} \|g_x\|_p, \quad (\forall x \in \mathbb{R}^n).$$

An easy calculation gives

$$f(x) = \int_{\mathbb{R}^n} g_x(y) dy, \quad \forall x \in \mathbb{R}^n$$

then using this inequality, we obtain

$$\begin{aligned} |f(x)| &\leq \int_{\mathbb{R}^n} |\theta(y) f(x - R^{-1}y)| dy \\ &\leq c_1 \left(\int_{\mathbb{R}^n} |\theta(y)|^p |f(x - R^{-1}y)|^p dy \right)^{1/p} \\ &\leq c_2 \left(\int_{\mathbb{R}^n} |\theta(y)|^p (1 + |y|)^{Np} (1 + |y|)^{-Np} |f(x - R^{-1}y)|^p dy \right)^{1/p} \\ &\leq c_3 \left(\int_{\mathbb{R}^n} (1 + |y|)^{-Np} |f(x - R^{-1}y)|^p dy \right)^{1/p} \sup_{z \in \mathbb{R}^n} |\theta(z)| (1 + |z|)^N. \end{aligned}$$

Then the desired result. □

Corollary 3.22. *Let $0 < \beta < \infty$ and $N \in \mathbb{N}_0$, then there exists a constant $c > 0$ such that*

$$|Q_j g(x)| \leq c \left(2^{jn} \int_{\mathbb{R}^n} |Q_j g(y)|^\beta (1 + 2^j |x - y|)^{-N\beta} dy \right)^{1/\beta}$$

holds, for all $j \in \mathbb{Z}$, all $x \in \mathbb{R}^n$ and all $f \in \mathcal{S}'$.

Proof. The proof already exists in [45, Chapter V, Theorem 5]. We can prove it by using another way for that it suffices to apply Proposition 3.21 with $R := 2^j$, $f = Q_j g$ and $p = \beta$. □

Remark 3.23. In Proposition 3.21 if we take $N \in \mathbb{R}$ it remains true.

3.2.3 Proof of Theorem 3.14

We will subdivide the proof into several steps.

Step 1. Let $f \in \dot{B}_{p,q}^{s,\tau}$. As we have $\sigma([f]_\infty) - f \in \mathcal{P}_\mu$, we will show (3.8) with $\sigma([f]_\infty)$ instead of \tilde{f} where $\sigma = \sigma_i$ ($i = 1, 2, 3$) with σ_i are defined in Proposition 2.9, see (2.3)–(2.5). We are reduced to prove

$$\int_{\mathbb{R}^n} \frac{|\sigma_i([f]_\infty)(x)|}{1 + |x|^{n+d}} dx \leq c \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}} \quad (i = 1, 2, 3), \quad (3.19)$$

under the restriction on parameters n, s, p, q and τ given in Proposition [3.9](#) and $s > (n/p - n)_+$. For this and brevity, we set

$$\begin{aligned} g_1 &:= \sum_{j \geq 1} Q_j f, & g_2 &:= \sum_{j \geq 1} \left\{ Q_j f - \sum_{|\alpha| < \mu} (Q_j f)^{(\alpha)}(0) \frac{x^\alpha}{\alpha!} \right\}, \\ g_3 &:= \sum_{j \leq 0} Q_j f, & g_4 &:= \sum_{j \leq 0} \left\{ Q_j f - \sum_{|\alpha| < \mu} (Q_j f)^{(\alpha)}(0) \frac{x^\alpha}{\alpha!} \right\}, \end{aligned}$$

where

$$\begin{aligned} \sigma_1([f]_\infty) &:= g_1 + g_3, \\ \sigma_2([f]_\infty) &:= g_2 + g_4, \\ \sigma_3([f]_\infty) &:= g_1 + g_4. \end{aligned}$$

We put

$$U_i := \int_{\mathbb{R}^n} \frac{|g_i(x)|}{1 + |x|^{n+d}} dx \quad (i = 1, 2, 3, 4).$$

Also, for reason of clarity we will subdivide the proof of this step into several substeps. We will essentially prove that the above series converges normally on every compact subset of \mathbb{R}^n .

Substep 1.1: estimate of $\sigma_1([f]_\infty)$ with either $(n/p - n)_+ + n\tau < s + n\tau < n/p$ or $s + n\tau = n/p$ and $0 < q \leq 1$; recall that here $\mu = 0$.

1.1.1 The case: $s + n\tau = n/p$ and $0 < q \leq 1$.

We consider $p \geq 1$ and $p < 1$, separately.

- If $p \geq 1$, by Corollary [3.22](#) with $\beta := p$ and $g := f$ we have

$$|Q_k f(x)| \leq c 2^{kn/p} \left(\sum_{\eta \in \mathbb{Z}^n} \int_{P_{l,\eta}} |Q_k f(y)|^p (1 + 2^k |x - y|)^{-(n+1)p} dy \right)^{1/p}, \quad (3.20)$$

If $x \in P_{l,w}$ and $y \in P_{l,\eta}$ then

$$1 + |w - \eta| \leq 2\sqrt{2n}(1 + 2^k |x - y|) \quad (\forall k \geq l). \quad (3.21)$$

By inserting the last estimation in [\(3.20\)](#), and using the fact that $(\sum_{\eta \in \mathbb{Z}^n} \dots)^{1/p} \leq \sum_{\eta \in \mathbb{Z}^n} (\dots)^{1/p}$, since $1/p \leq 1$, see Proposition [1.5](#), it holds

$$|Q_k f(x)| \leq c 2^{kn/p} \sum_{\eta \in \mathbb{Z}^n} (1 + |w - \eta|)^{-(n+1)} \left(\int_{P_{l,\eta}} |Q_k f(y)|^p dy \right)^{1/p} \quad \forall x \in P_{l,w} \quad k \geq l. \quad (3.22)$$

By using characteristic of the support of $\widehat{Q_k f}$ such that

$$\text{supp } \mathcal{F}(Q_k f) \cap \text{supp } \mathcal{F}(Q_j f) = \emptyset \quad \text{if } |k - j| \geq \log_2 3,$$

the serie $\sum_{k \in \mathbb{Z}} \widehat{Q_k f}(\xi)$, for all $\xi \in \mathbb{R}^n \setminus \{0\}$, contains at most 3 non-vanishing terms corresponding to the compact annulus $2^{k-r-1} \leq |\xi| \leq 3 \cdot 2^{k+r}$ ($r = -1, 0, 1$). We have $\sum_{k \in \mathbb{Z}} \widehat{Q_k f}(\xi) = \sum_{k \in \mathbb{Z}, k \in \Lambda} \widehat{Q_k f}(\xi)$, where the set Λ is constituted by *consecutive elements*, i.e., $\Lambda := \{J, J+1, J+2\}$.

Now, we choose $l := J$ in (3.22), then we get

$$\begin{aligned} \sum_{k \geq J} |Q_k f(x)| &= \sum_{k=J}^{J+2} |Q_k f(x)| \\ &\leq \sum_{\eta \in \mathbb{Z}^n} (1 + |w - \eta|)^{-(n+1)} \sum_{k=J}^{J+2} 2^{kn/p} \|Q_k f\|_{L_p(P_{J,\eta})}, \quad \forall x \in P_{l,w}, \end{aligned} \quad (3.23)$$

but, we have (recall $\frac{n}{p} = n\tau + s$)

$$\begin{aligned} 2^{kn/p} \|Q_k f(y)\|_{L_p(P_{J,\eta})} &\leq 2^{2n\tau} \left(2^{qJn\tau} \sum_{k \geq J} 2^{jsq} \|Q_j f\|_{L_p(P_{J,\eta})}^q \right)^{1/q} \\ &\leq 2^{2n\tau} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}. \end{aligned}$$

then inserting the last estimation into (3.23), we obtain

$$\sum_{k \geq J} |Q_k f(x)| \leq c \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}} \sum_{\eta \in \mathbb{Z}^n} (1 + |w - \eta|)^{-(n+1)}, \quad \forall x \in P_{J,w}. \quad (3.24)$$

we turn to σ_1 , we have

$$|\sigma_1([f]_\infty)(x)| \leq \sum_{k \geq J} |Q_k f(x)| \leq c \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}} \quad \forall x \in P_{J,w}, \quad (3.25)$$

Recall that we used the fact that $\sum_{\eta \in \mathbb{Z}^n} (1 + |w - \eta|)^{-(n+1)} := c < \infty$.

Now, to obtain (3.19) it suffices to observe that $d > 0$ and

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|\sigma_1([f]_\infty)(x)|}{1 + |x|^{n+d}} dx &= \sum_{w \in \mathbb{Z}^n} \int_{P_{J,w}} \frac{|\sigma_1([f]_\infty)(x)|}{1 + |x|^{n+d}} dx \\ &\leq c \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}} \int_{\mathbb{R}^n} \frac{dx}{1 + |x|^{n+d}}. \end{aligned}$$

- Now, if $0 < p < 1$, we introduce a parameter a satisfying $0 < a < p < 1$ which will be

chosen later, and use Corollary 3.22 with $\beta := a$, $g := f$ and $N := n + \varepsilon$ (ε , also will be chosen later) then we apply Hölder inequality (with $v := p/a$ and $v' := p/(p-a)$), we obtain for all $x \in P_{l,w}$,

$$\begin{aligned} |Q_k f(x)| &\leq c 2^{kn/a} \left(\sum_{\eta \in \mathbb{Z}^n} \int_{P_{l,\eta}} |Q_k f(y)|^a (1 + 2^k |x-y|)^{-(n+\varepsilon)a} dy \right)^{1/a} \\ &\leq 2^{kn/a} \left[\sum_{\eta \in \mathbb{Z}^n} \|Q_k f\|_{L_p(P_{l,\eta})}^a \left(\int_{P_{l,\eta}} (1 + 2^k |x-y|)^{-(n+\varepsilon)v'a} dy \right)^{1/v'} \right]^{1/a}. \end{aligned}$$

Then, choosing as above $l := J$ using the equality $\sigma_1([f]_\infty)(x) = \sum_{k=J}^{J+2} Q_k f$, we get

$$|\sigma_1([f]_\infty)(x)| \leq \sum_{k \geq J} \left[2^{kn} \sum_{\eta \in \mathbb{Z}^n} \|Q_k f\|_{L_p(P_{J,\eta})}^a \left(\int_{P_{J,\eta}} (1 + 2^k |x-y|)^{-(n+\varepsilon)v'a} dy \right)^{1/v'} \right]^{1/a},$$

for all $x \in P_{J,w}$.

By the Minkowski inequality, see Proposition 1.4, we get, ($\forall x \in P_{J,w}$),

$$|\sigma_1([f]_\infty)(x)| \leq c \left\{ \sum_{\eta \in \mathbb{Z}^n} \left[\sum_{k \geq J} 2^{kn/a} \|Q_k f\|_{L_p(P_{J,\eta})} \left(\int_{P_{J,\eta}} (1 + 2^k |x-y|)^{-(n+\varepsilon)av'} dy \right)^{1/(av')} \right]^a \right\}^{1/a}.$$

By applying (3.21), we obtain

$$\begin{aligned} |\sigma_1([f]_\infty)(x)| &\leq c \left(\sum_{\eta \in \mathbb{Z}^n} (1 + |w - \eta|)^{-(n+\varepsilon)a} \left[\sum_{k \geq J} 2^{(kn/a)(1-1/v')} \|Q_k f\|_{L_p(P_{J,\eta})} \right]^a \right)^{1/a} \\ &\leq c \left(\sum_{\eta \in \mathbb{Z}^n} (1 + |w - \eta|)^{-(n+\varepsilon)a} \left[\sum_{k \geq J} 2^{kn/p} \|Q_k f\|_{L_p(P_{J,\eta})} \right]^a \right)^{1/a}. \end{aligned}$$

We now proceed as in the preceding case, since $\frac{n}{p} = s + n\tau$, we obtain

$$|\sigma_1([f]_\infty)(x)| \leq c \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}} \left(\sum_{\eta \in \mathbb{Z}^n} (1 + |w - \eta|)^{-(n+\varepsilon)a} \right)^{1/a}, \quad \forall x \in P_{J,w}. \quad (3.26)$$

We have

$$\sum_{\eta \in \mathbb{Z}^n} (1 + |w - \eta|)^{n-1-(n+\varepsilon)a} \sim \sum_{j \geq 1} j^{n-1-(n+\varepsilon)a}$$

then it suffices to choose ε and a such that

$$n + \varepsilon > \frac{n}{p} \quad \text{and} \quad \frac{n}{n + \varepsilon} < a < p.$$

We conclude that (3.19) holds, since $d > 0$.

1.1.2 The case: $n\tau < s + n\tau < n/p$ and $1 < p \leq \infty$ (recall here $\mu = 0$), we have

$$|g_3(x)| \leq \| [f]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}} \sum_{j \leq 0} 2^{-j(s+n\tau-n/p)} \leq c \| [f]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}}, \quad (3.27)$$

then we apply $\dot{B}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{\infty,\infty}^{s+n\tau-n/p}$ which yields $U_3 \leq c \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}$ since $d > 0$.

By Hölder inequality we also have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|g_1(x)|}{1+|x|^{n+d}} dx &= \sum_{\eta \in \mathbb{Z}^n} \int_{P_{1,\eta}} \frac{|g_1(x)|}{1+|x|^{n+d}} dx \\ &\leq \sum_{\eta \in \mathbb{Z}^n} \|g_1\|_{L_p(P_{1,\eta})} \left(\int_{P_{1,\eta}} (1+|x|^{n+d})^{-p'} dx \right)^{1/p'} \\ &\leq \sum_{\eta \in \mathbb{Z}^n} 2^{-n\tau} 2^{n\tau} \sum_{k \geq 1} 2^{ks} 2^{-ks} \|Q_k f\|_{L_p(P_{1,\eta})} \left(\int_{P_{1,\eta}} (1+|x|^{n+d})^{-p'} dx \right)^{1/p'} \\ &\leq c \| [f]_\infty \|_{\dot{B}_{p,\infty}^{s,\tau}} 2^{-n\tau} \sum_{k \geq 1} 2^{-ks} \sum_{\eta \in \mathbb{Z}^n} \left(\int_{P_{1,\eta}} (1+|x|^{n+d})^{-p'} dx \right)^{1/p'}. \end{aligned}$$

Thus, by the fact that $1 + |\eta|^{n+d} \leq c(1 + |x|^{n+d})$ and $\dot{B}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{p,\infty}^{s,\tau}$, we get

$$U_1 \leq c_1 \| [f]_\infty \|_{\dot{B}_{p,\infty}^{s,\tau}} \sum_{k \geq 1} 2^{-ks} \sum_{\eta \in \mathbb{Z}^n} (1 + |\eta|^{n+d})^{-1} \leq c_2 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}.$$

Recall that

$$\sum_{\eta \in \mathbb{Z}^n} (1 + |\eta|^{n+d})^{-1} \sim \sum_{j \geq 1} j^{-(d+1)} < \infty,$$

since $d > 0$. It holds the desired result.

1.1.3 The case: $n/p - n + n\tau < s + n\tau < n/p$ and $0 < p \leq 1$. We introduce a parameter p_1 such that we apply Proposition 2.1 and obtain

$$\dot{B}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{p_1,q}^{s-n/p+n/p_1,\tau}$$

then we first choose $p < p_1$. By assumption we have

$$s - \frac{n}{p} + \frac{n}{p_1} + n\tau < \frac{n}{p_1},$$

then, if we put μ_1 and μ_2 the associated numbers to $\dot{B}_{p,q}^{s,\tau}$ and $\dot{B}_{p_1,q}^{s-n/p+n/p_1,\tau}$, respectively, we get

$\mu_1 = \mu_2 = 0$ and

$$\dot{B}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{p_1,q}^{s-n/p+n/p_1,\tau}$$

Now, we choose p_1 such that

$$n/p - s < n/p_1$$

and apply 1.1.2 for $\dot{B}_{p_1,q}^{s-n/p_1+n/p,\tau}$ instead of $\dot{B}_{p,q}^{s,\tau}$. Then the desired result holds since $d > 0$.

Substep 1.2: estimate of $\sigma_2([f]_\infty)$ with either $s + n\tau - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}_0$ or $s + n\tau - \frac{n}{p} \in \mathbb{N}$ and $0 < q \leq 1$.

1.2.1 *The case: $s + n\tau - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}_0$.* For all integer N , we can control it, we set

$$f_N(x) := f(2^{-N}x).$$

We have

$$\begin{aligned} \widehat{Q_j f}(\xi) &= 2^{-Nn} \gamma(2^{-j+N} 2^{-N} \xi) \widehat{f_N}(2^{-N} \xi) \\ &= 2^{-Nn} \widehat{Q_{j-N} f_N}(2^{-N} \xi) = \mathcal{F}[Q_{j-N} f_N(2^N \cdot)](\xi), \end{aligned}$$

then

$$Q_j f(x) = Q_{j-N} f_N(2^N x).$$

By inserting this equality in $\sigma_2([f]_\infty)$, we get

$$\begin{aligned} \sigma_2([f]_\infty)(x) &= g_2(x) + g_4(x) = \sum_{j \in \mathbb{Z}} \left(Q_{j-N} f_N(2^N x) - \sum_{|\alpha| < \mu} (Q_{j-N} f_N)^{(\alpha)}(0) \frac{(2^N x)^\alpha}{\alpha!} \right) \\ &= \sum_{k \in \mathbb{Z}} \left(Q_k f_N(2^N x) - \sum_{|\alpha| < \mu} (Q_k f_N)^{(\alpha)}(0) \frac{(2^N x)^\alpha}{\alpha!} \right) \end{aligned}$$

We divide $\sigma_2([f]_\infty)$ as $g_5 + g_6$ with $g_5 := \sum_{k \geq 1} \dots$ and $g_6 := \sum_{k \leq 0} \dots$. We have

$$|g_5(x)| \leq \sum_{k \geq 1} \left(\|Q_k f_N\|_\infty + c \sum_{|\alpha| < \mu} |2^N x|^{|\alpha|} \|(Q_k f_N)^{(\alpha)}\|_\infty \right).$$

By Bernstein inequality (see Proposition [1.3](#)) we also have $\|(Q_k f_N)^{(\alpha)}\|_\infty \leq c 2^{k|\alpha|} \|Q_k f_N\|_\infty$.

Thus,

$$|g_5(x)| \leq c \|[f_N]_\infty\|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}} \sum_{k \geq 1} \left(2^{-k(s+n\tau-n/p)} + 2^{-k(s-n/p+n\tau-\mu+1)} \sum_{|\alpha| < \mu} |2^N x|^{|\alpha|} \right).$$

By using

$$\| [f_N]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}} \equiv 2^{-(s+n\tau-n/p)N} \| [f]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}},$$

and as $s + n\tau - \frac{n}{p} - \mu + 1 > 0$, it follows

$$|g_5(x)| \leq c 2^{-(s+n\tau-n/p)N} \| [f]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}} \left(1 + \sum_{|\alpha| < \mu} |2^N x|^{|\alpha|} \right), \quad (3.28)$$

the constant c is independent of f , N and x .

To estimate g_6 we introduce a parameter b such that $0 < b < 1$, we will fixed this parameter later, and use Taylor's formula, then we obtain

$$\begin{aligned} |g_6(x)| &\leq \sum_{k \leq 0} \left(\| Q_k f_N \|_\infty + \sum_{|\alpha| < \mu} \| (Q_k f_N)^{(\alpha)} \|_\infty |2^N x|^{|\alpha|} (\alpha!)^{-1} \right)^{1-b} \\ &\quad \times \left(\mu \sum_{|\alpha| = \mu} (\alpha!)^{-1} |2^N x|^{|\alpha|} \int_0^1 (1-t)^{\mu-1} | (Q_k f_N)^{(\alpha)} (2^N t x) | dt \right)^b. \end{aligned} \quad (3.29)$$

Using the well-known inequality

$$x \geq 0, y \geq 0, a \geq 0: \quad (x+y)^a \leq \max(1, 2^{a-1}) (x^a + y^a), \quad (3.30)$$

we get

$$\begin{aligned} |g_6(x)| &\leq c_1 \| [f_N]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}} \sum_{k \leq 0} 2^{-k(s+n\tau-n/p)(1-b)} \\ &\quad \times \left(1 + \sum_{|\alpha| < \mu} 2^{k|\alpha|} |2^N x|^{|\alpha|} \right)^{1-b} \left(2^{-k(s+n\tau-n/p-\mu)} |2^N x|^\mu \right)^b \\ &\leq c_2 \| [f_N]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}} |2^N x|^{\mu b} \left(1 + \sum_{|\alpha| < \mu} |2^N x|^{|\alpha|(1-b)} \right) \sum_{k \leq 0} 2^{-k(s+n\tau-n/p-\mu b)}. \end{aligned} \quad (3.31)$$

Then we need the condition $\mu b + \frac{n}{p} - s - n\tau > 0$. Hence, the number b must satisfy

$$\frac{1}{\mu} \left(s + n\tau - \frac{n}{p} \right) < b < 1. \quad (3.32)$$

Thus, as in (3.28), we obtain

$$|g_6(x)| \leq c 2^{-(s+n\tau-n/p)N} \| [f]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}} |2^N x|^{\mu b} \left(1 + \sum_{|\alpha| < \mu} |2^N x|^{|\alpha|(1-b)} \right), \quad (3.33)$$

where the constant c is independent of f , N and x .

- Notice that (B.33) holds true when we assume that $s + n\tau = \frac{n}{p}$ and that $1 < q \leq \infty$, meaning that $\mu = 1$; for more information, go to Substep 1.3 below.

Now, to estimate $\sigma_2([f]_\infty)$. We choose $N := N(x) \in \mathbb{Z}$ such that $2^{-N} \leq |x| < 2^{-N+1}$ (i.e., $|x| \sim 2^{-N}$), then from (B.28) and (B.33), it holds

$$|\sigma_2([f]_\infty)(x)| \leq c|x|^{s+n\tau-n/p} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}},$$

where the constant c is independent of f and x . Hence, (B.19) is satisfied since $d > s + n\tau - \frac{n}{p}$.

1.2.2 The case: $s + n\tau - \frac{n}{p} \in \mathbb{N}$ and $0 < q \leq 1$. We use the Taylor's formula, then

$$\sigma_2([f]_\infty)(x) = \mu \sum_{k \in \mathbb{Z}} \sum_{|\alpha|=\mu} (\alpha!)^{-1} |x|^{|\alpha|} \int_0^1 (1-t)^{\mu-1} (Q_k f)^{(\alpha)}(tx) dt.$$

We have $\sum_{k \in \mathbb{Z}^n} \cdots = \sum_{k=J}^{J+2} \cdots$, where J is defined in Case 1.1.1, its definition is given in the same way by Fourier transform, i.e.

$$\widehat{Q_k f^{(\alpha)}}(t \cdot)(\xi) = t^{-n} \widehat{Q_k f^{(\alpha)}}(t^{-1} \xi),$$

and the fact that

$$2^{k-r-1} \leq t^{-1} |\xi| \leq 3 \cdot 2^{k+r} \quad \text{with } r = -1, 0, 1,$$

hence $J \in \Lambda \subset \mathbb{Z}$ with $\text{Card} \Lambda = 3$ (recall $\Lambda := \{J, J+1, J+2\}$).

Now

$$\begin{aligned} |\sigma_2([f]_\infty)(x)| &\leq c_1 \sum_{|\alpha|=\mu} \sum_{k=J}^{J+2} |x|^\mu \|Q_k f^{(\alpha)}\|_\infty \int_0^1 (1-t)^{\mu-1} dt \\ &\leq c_2 |x|^\mu \sum_{|\alpha|=\mu} \| [f^{(\alpha)}]_\infty \|_{\dot{B}_{\infty,\infty}^0} \\ &\leq c_3 |x|^\mu \| [f]_\infty \|_{\dot{B}_{\infty,\infty}^\mu} \leq c_4 |x|^\mu \| [f] \|_{\dot{B}_{p,q}^{s,\tau}}. \end{aligned} \tag{3.34}$$

We obtain the desired estimate since $d > \mu$; recall that $\mu = s + n\tau - \frac{n}{p}$.

Substep 1.3: estimate of $\sigma_3([f]_\infty)$ with $s + n\tau - \frac{n}{p} \in \mathbb{N}_0$ and $1 < q \leq \infty$. Recall that here $\mu = s + n\tau - \frac{n}{p} + 1 \geq 1$.

1.3.1 The case: $s + n\tau = \frac{n}{p}$. We introduce a parameter p_1 satisfying $\max(p, 1) < p_1 < \infty$, we

will use the same way as in Case 1.1.1 by applying (3.23) with p_1

$$|g_1(x)| \leq \sum_{\eta \in \mathbb{Z}^n} (1 + |w - \eta|)^{-(n+1)} \sum_{k=J}^{J+2} 2^{kn/p_1} \|Q_k f(y)\|_{L_{p_1}(P_{J,\eta})}, \quad \forall x \in P_{l,w}$$

and applying (3.24) with replacing s , p and q by $n/p_1 - n\tau$, p_1 and ∞ respectively, we obtain

$$|g_1(x)| \leq \sum_{\eta \in \mathbb{Z}^n} (1 + |w - \eta|)^{-(n+1)} \sum_{k \geq J} 2^{kn/p_1} \|Q_k f\|_{L_{p_1}(P_{J,w})} \leq c \| [f]_\infty \|_{\dot{B}_{p_1, \infty}^{n/p_1 - n\tau, \tau}}. \quad (3.35)$$

For estimate U_1 , we combine the last inequality, the embedding $\dot{B}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{p_1, \infty}^{n/p_1 - n\tau, \tau}$ (see Proposition 1.21) and the fact that $d > 0$, it follows

$$U_1 \leq c_1 \| [f]_\infty \|_{\dot{B}_{p_1, \infty}^{n/p_1 - n\tau, \tau}} \int_{\mathbb{R}^n} \frac{1}{1 + |x|^{n+d}} dx \leq c_2 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}. \quad (3.36)$$

To estimate U_4 , we proceed as in Substep 1.2 for g_6 , thus by (3.32) with Choosing $0 < b < \min(1, d)$ and (3.33), it holds

$$|g_4(x)| \leq c |x|^b \| [f]_\infty \|_{\dot{B}_{\infty, \infty}^0}, \quad (3.37)$$

then by using the embedding $\dot{B}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{\infty, \infty}^0$, we get

$$U_4 \leq c_1 \| [f]_\infty \|_{\dot{B}_{\infty, \infty}^0} \int_{\mathbb{R}^n} \frac{|x|^b}{1 + |x|^{n+d}} dx \leq c_2 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}. \quad (3.38)$$

Hence, (3.36) and (3.38) yield the desired estimate (3.19).

1.3.2 The case: $s + n\tau - n/p \in \mathbb{N}$. We observe that the estimate

$$\sum_{k>0} \|Q_k f\|_\infty = \sum_{k>0} 2^{-k(s+n\tau-n/p)} 2^{k(s+n\tau-n/p)} \|Q_k f\|_\infty \leq c \| [f]_\infty \|_{\dot{B}_{\infty, \infty}^{s+n\tau-n/p}}$$

gives

$$U_1 \leq c_1 \| [f]_\infty \|_{\dot{B}_{\infty, \infty}^{s+n\tau-n/p}} \leq c_2 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}. \quad (3.39)$$

The estimate for U_4 , which is the most complicated one, relies on the Case 1.2.1. We take into account the following two cases:

- If $|x| < 1$, we introduce an integer $N \geq 1$ such that $|x| \sim 2^{-N}$. As a result, using (3.32) with

$$\frac{s + n\tau - n/p}{s + n\tau - n/p + 1} < b < 1$$

and by (3.33), we arrive at the following equation

$$|g_4(x)| \leq c|x|^{s+n\tau-n/p} \| [f]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}} \leq c \| [f]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}}. \quad (3.40)$$

• If $|x| \geq 1$. Defining an integer $N \geq 0$ such that $|x| \sim 2^N$. We set $f_N(x) := f(2^N x)$, and we apply the equality (for $j \leq 0$)

$$Q_j f(x) = Q_{j+N} f_N(2^{-N} x) = Q_k f_N(2^{-N} x) \quad \text{with } k := j + N \leq N,$$

then, we can write $g_4 = g_7 + g_8$ where

$$g_7(x) := \sum_{k < 0} \left(Q_k f_N(2^{-N} x) - \sum_{|\alpha| < \mu} (Q_k f_N)^{(\alpha)}(0) \frac{(2^{-N} x)^\alpha}{\alpha!} \right),$$

$$g_8(x) := \sum_{0 \leq k \leq N} \left(Q_k f_N(2^{-N} x) - \sum_{|\alpha| < \mu} (Q_k f_N)^{(\alpha)}(0) \frac{(2^{-N} x)^\alpha}{\alpha!} \right).$$

For g_7 , we have as in g_6 (see (3.32) with $\frac{s+n\tau-n/p}{s+n\tau-n/p+1} < b < 1$ and (3.33)),

$$|g_7(x)| \leq c|x|^{s+n\tau-n/p} \| [f]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}}. \quad (3.41)$$

To estimate g_8 , we use an inequality similar to (3.29) with $0 < b_1 < 1$ instead of b , that is

$$|g_8(x)| \leq \sum_{0 \leq j \leq N} \left(\| Q_j f_N \|_\infty + \sum_{|\alpha| < \mu} \| (Q_j f_N)^{(\alpha)} \|_\infty |2^{-N} x|^{|\alpha|} (\alpha!)^{-1} \right)^{1-b_1}$$

$$\times \left(\mu \sum_{|\alpha| = \mu} (\alpha!)^{-1} |2^{-N} x|^{|\alpha|} \int_0^1 (1-t)^{\mu-1} |(Q_j f_N)^{(\alpha)}(2^{-N} t x)| dt \right)^{b_1}.$$

We get

$$|g_8(x)| \leq c_1 \| [f_N]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}} \sum_{0 \leq j \leq N} 2^{j(\mu b_1 - s + n\tau - n/p)} \left(1 + \sum_{|\alpha| < \mu} 2^{j|\alpha|(1-b_1)} \right)$$

$$\leq c_2 \| [f_N]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}} \left(\sum_{0 \leq j \leq N} 2^{j(\mu b_1 - s + n\tau - n/p)} + \sum_{0 \leq j \leq N} 2^{j b_1} \right). \quad (3.42)$$

Now, we choose b_1 such that $\mu b_1 - s + n\tau - n/p < 0$

$$\text{(i.e., } \sum_{0 \leq j \leq N} 2^{j(\mu b_1 - s + n\tau - n/p)} \leq \sum_{j \geq 0} 2^{j(\mu b_1 - s + n\tau - n/p)} = c < \infty) \quad (3.43)$$

where the constant c is independent of N . Using the following elementary inequality

$$\sum_{0 \leq j \leq N} 2^{jb_1} \leq c_1 2^{Nb_1} \leq c_2 |x|^{b_1}$$

(the constants c_1, c_2 depend only on b_1 and n), and

$$\| [f_N]_\infty \|_{\dot{B}_{\infty, \infty}^{s+n\tau-n/p}} \leq c |x|^{s+n\tau-n/p} \| [f]_\infty \|_{\dot{B}_{\infty, \infty}^{s+n\tau-n/p}},$$

then (3.42) becomes

$$|g_8(x)| \leq c |x|^{s+n\tau-n/p} (1 + |x|^{b_1}) \| [f]_\infty \|_{\dot{B}_{\infty, \infty}^{s+n\tau-n/p}}. \quad (3.44)$$

Summarizing, (3.39)–(3.41) and (3.44) we arrive at

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|\sigma_3([f]_\infty)(x)|}{1 + |x|^{n+d}} dx &\leq c \| [f]_\infty \|_{\dot{B}_{\infty, \infty}^{s+n\tau-n/p}} \left(\int_{\mathbb{R}^n} \frac{1}{1 + |x|^{n+d}} dx \right. \\ &\quad \left. + \int_{|x| \geq 1} \frac{|x|^{s+n\tau-n/p}}{1 + |x|^{n+d}} dx + \int_{|x| \geq 1} \frac{|x|^{s+n\tau-n/p+b_1}}{1 + |x|^{n+d}} dx \right). \end{aligned}$$

Thus, we need the condition:

$$s + n\tau - n/p + b_1 < d.$$

Then, from this condition and (3.43), we choose b_1 satisfying

$$0 < b_1 < \min \left(\frac{s + n\tau - n/p}{s + n\tau - n/p + 1}, d - s - n\tau + n/p \right),$$

and by the fact that $\dot{B}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{\infty, \infty}^{s+n\tau-n/p}$ we obtain the desired estimate (3.19) for σ_3 .

Step 2: proof of (3.9). Replacing f by $\tau_{-x_0} f$ in (3.19), applying Proposition 2.12 and using the fact that $\sigma_1([\tau_{-x_0} f]_\infty)(x) = \sigma_1([f]_\infty)(x + x_0)$, then

$$\int_{\mathbb{R}^n} \frac{|\sigma_1([f]_\infty)(x + x_0)|}{1 + |x|^{n+d}} dx \leq c \| [\tau_{-x_0} f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}},$$

by changing variable $y = x + x_0$ and as $\| \cdot \|_{\dot{B}_{p,q}^{s,\tau}}$ is translation invariant, it follows

$$\int_{\mathbb{R}^n} \frac{|\sigma_1([f]_\infty)(y)|}{1 + |y - x_0|^{n+d}} dy \leq c \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}},$$

According with the requirements on the parameters n, s, τ, p , and q given in Proposition 2.9

(i). The fact that $\sigma_1([f]_\infty) - f \in \mathcal{P}_0 = \{0\}$ leads us to the required estimate. \square

3.2.4 Proof of Theorem 3.15

The proof is divided into three steps. Let $f \in \dot{B}_{p,q}^{s,\tau}$.

Step 1: proof of (i). Similar to the proof of Theorem 3.14 and by the fact that $\sigma_i([f]_\infty) - f \in \mathcal{P}_\mu$ ($i = 1, 2, 3$) it is enough to deal with realizations σ_i ($i = 1, 2, 3$) defined in Proposition 2.9. By (3.19), it is evident that

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|\sigma_i([f]_\infty)(x) - m_{Q(0,1)} \sigma_i([f]_\infty)|}{1 + |x|^{n+d}} dx &\leq c_1 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}} + |m_{Q(0,1)} \sigma_i([f]_\infty)| \int_{\mathbb{R}^n} \frac{1}{1 + |x|^{n+d}} dx \\ &\leq c_2 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}} \\ &\quad + \frac{1}{|Q(0,1)|} \int_{Q(0,1)} (1 + |x|^{n+d}) \frac{|\sigma_i([f]_\infty)(x)|}{1 + |x|^{n+d}} dx \int_{\mathbb{R}^n} \frac{1}{1 + |y|^{n+d}} dy \\ &\leq c_2 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}} \left(1 + \int_{\mathbb{R}^n} \frac{1}{1 + |x|^{n+d}} dx \right) \\ &\leq c_2 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}, \end{aligned} \quad (3.45)$$

and then the desired estimate (3.10) holds.

Step 2: proof of (ii). The same as in the previous step, we apply $\sigma_i([f]_\infty) - f \in \mathcal{P}_\mu$ and Proposition 2.12. We only proceed with realizations commute with dilations σ_i ($i = 1, 2$) defined in Proposition 2.9. Changing f by $f(\lambda \cdot)$ ($\lambda > 0$) in (3.45) and using the equality $\sigma_i([h_{1/\lambda} f]_\infty)(x) = \sigma_i([f]_\infty)(\lambda x)$, we then obtain

$$\int_{\mathbb{R}^n} \frac{|\sigma_i([f]_\infty)(x) - m_{Q(0,\lambda)} \sigma_i([f]_\infty)|}{\lambda^{n+d} + |x|^{n+d}} dx \leq c \lambda^{s+n\tau-d-n/p} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}} \quad (i = 1, 2). \quad (3.46)$$

The desired inequality (3.11).

Step 3: proof of (iii). We use (3.46) with only σ_1 , by replacing f with $\tau_{-x_0} f$ ($x_0 \in \mathbb{R}^n$) and by considering the fact that σ_1 is translation invariant i.e., $\sigma_1([\tau_{-x_0} f]_\infty) = \sigma_1([f]_\infty)(x + x_0)$ and $m_{Q(0,\lambda)} \sigma_1([\tau_{-x_0} f]_\infty)(x) = m_{Q(x_0,\lambda)} \sigma_1([f]_\infty)$

$$\int_{\mathbb{R}^n} \frac{|\sigma_1([f]_\infty)(x + x_0) - m_{Q(x_0,\lambda)} \sigma_1([f]_\infty)|}{\lambda^{n+d} + |x|^{n+d}} dx \leq c \lambda^{s+n\tau-d-n/p} \| [\tau_{-x_0} f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}. \quad (3.47)$$

by changing variable $y = x + x_0$ and the fact that $\| \cdot \|_{\dot{B}_{p,q}^{s,\tau}}$ commute with translation we obtain the desired result (3.12). \square

3.2.5 Proof of Theorem 3.16

Let $f \in \dot{B}_{p,q}^{s,\tau}$. Here also we apply $\sigma_i([f]_\infty) - f \in \mathcal{P}_\mu$ ($i = 1, 2, 3$), we use the realizations σ_i ($i = 1, 2, 3$) given in Proposition 2.9.

Step 1: proof of (i). We will prove (3.13) in several steps, where we will present explicitly the constant which appears in its right side.

Substep 1.1: If $0 < p_1 \leq 1$. Initially, we observe the case $p_1 = 1$. Selecting $d > (s + n\tau - \frac{n}{p})_+$ as previously, it holds according to (3.10)

$$\int_{Q_{(0,1)}} |\tilde{f}(x) - m_{Q_{(0,1)}} \tilde{f}| dx \leq c_1 \int_{Q_{(0,1)}} \frac{|\tilde{f}(x) - m_{Q_{(0,1)}} \tilde{f}|}{1 + |x|^{n+d}} dx \leq c_2 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}. \quad (3.48)$$

Let us suppose $0 < p_1 < 1$ in the second case. We put $\nu := \frac{1}{p_1}$. Using exponents ν and ν' and Hölder inequality, we then obtain

$$\left(\int_{Q_{(0,1)}} |\tilde{f}(x) - m_{Q_{(0,1)}} \tilde{f}|^{p_1} dx \right)^{1/p_1} \leq c_2^{n(1-p_1)/p_1} \int_{Q_{(0,1)}} |\tilde{f}(x) - m_{Q_{(0,1)}} \tilde{f}| dx,$$

Now, to reach the required estimate, we apply the first case $p_1 = 1$.

Substep 1.2: If $1 < p_1 < \infty$. Using σ_i ($i = 1, 2, 3$), we divide the input, and we will apply (3.30), as in the Proof of Theorem 3.14, we put

$$g_1 := \sum_{j \geq 1} Q_j f, \quad g_2 := \sum_{j \geq 1} \left\{ Q_j f - \sum_{|\alpha| < \mu} (Q_j f)^{(\alpha)}(0) \frac{x^\alpha}{\alpha!} \right\},$$

$$g_3 := \sum_{j \leq 0} Q_j f, \quad g_4 := \sum_{j \leq 0} \left\{ Q_j f - \sum_{|\alpha| < \mu} (Q_j f)^{(\alpha)}(0) \frac{x^\alpha}{\alpha!} \right\},$$

1.2.1: Estimate of $\sigma_1([f]_\infty)$. Recall that $\sigma_1([f]_\infty) = g_1 + g_3$.

- If $s + n\tau = \frac{n}{p}$ and $0 < q \leq 1$, we apply (3.25) and (3.26), then we have

$$\begin{aligned} |\sigma_1([f]_\infty)(x) - m_{Q_{(0,1)}}(\sigma_1([f]_\infty))|^{p_1} &\leq (2c_1)^{p_1-1} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}^{p_1-1} |\sigma_1([f]_\infty)(x) - m_{Q_{(0,1)}}(\sigma_1([f]_\infty))| \\ &\leq (2c_1)^{p_1-1} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}^{p_1-1} |\tilde{f}(x) - m_{Q_{(0,1)}} \tilde{f}|, \end{aligned}$$

consequently, applying (3.48) is sufficient to get the desired outcome. with the constant $(c_1 2)^{1-1/p_1} c_2 \leq c_3$.

• If $s + n\tau < \frac{n}{p}$, and $p_1 \leq p < \infty$, by Hölder inequality (with exponents $v := \frac{p}{p_1}$ and v'), since $s > 0$, $2^{n/p_1 - n/p} < 2^n$ and as $Q(0, 1) \subset \cup_{r=1}^{3^n} P_{0, w_r}$ (see Example □□), it holds

$$\begin{aligned} \|g_1\|_{L_{p_1}(Q(0,1))} &\leq 2^{n/p_1 - n/p} \left(\int_{Q(0,1)} |g_1(x)|^p dx \right)^{1/p} \\ &\leq 2^n \sum_{r=1}^{3^n} \sum_{j>0} \|Q_j f\|_{L_p(P_{0, w_r})} \\ &\leq c \| [f]_\infty \|_{\dot{B}_{p, \infty}^{s, \tau}} \sum_{j>0} 2^{-js} \end{aligned} \quad (3.49)$$

• If $s + n\tau < \frac{n}{p}$, $p < p_1 < \infty$ and $\frac{n}{p} - s < \frac{n}{p_1} < \frac{n}{p}$, as $Q(0, 1) \subset \cup_{r=1}^{3^n} P_{0, w_r}$ we have

$$\begin{aligned} \|g_1\|_{L_{p_1}(Q(0,1))} &\leq \sum_{r=1}^{3^n} \sum_{j>0} \|Q_j f\|_{L_{p_1}(P_{0, w_r})} \\ &\leq c \| [f]_\infty \|_{\dot{B}_{p_1, \infty}^{s - n/p + n/p_1, \tau}} \sum_{j>0} 2^{-j(s - n/p + n/p_1)}. \end{aligned}$$

Using the embedding $\dot{B}_{p, q}^{s, \tau} \hookrightarrow \dot{B}_{p_1, \infty}^{s - n/p + n/p_1, \tau}$ and

$$\sum_{j \geq 1} 2^{-j\beta} \leq \frac{1}{\beta \log 2} \quad (\forall \beta > 0)$$

it holds

$$\|g_1\|_{L_{p_1}(Q(0,1))} \leq c \left(\frac{n}{p_1} - \frac{n}{p} + s \right)^{-1} \| [f]_\infty \|_{\dot{B}_{p, q}^{s, \tau}}.$$

• If $s + n\tau < \frac{n}{p}$, $p < p_1 < \infty$ and $\frac{n}{p} - s = \frac{n}{p_1} < \frac{n}{p}$, we have

$$\begin{aligned} \|Q_j f\|_{L_{p_1}(Q(0,1))} &\leq \|Q_j f\|_{L_{p_1}(Q(0,1))}^{p/p_1} \|Q_j f\|_\infty^{1 - p/p_1} \\ &\leq \sum_{r=1}^{3^n} \|Q_j f\|_{L_{p_1}(P_{0, w_r})}^{p/p_1} \|Q_j f\|_\infty^{1 - p/p_1}, \end{aligned}$$

the second term in the last inequality is bounded by $2^{-jsp/p_1} \| [f]_\infty \|_{\dot{B}_{p, q}^{s, \tau}}^{p/p_1}$ and the third one is embedded by $2^{-j(s + n\tau - n/p)(1 - p/p_1)} \| [f]_\infty \|_{\dot{B}_{\infty, \infty}^{s + n\tau - n/p}}^{1 - p/p_1}$ then by applying the embedding $\dot{B}_{p, q}^{s, \tau} \hookrightarrow \dot{B}_{\infty, \infty}^{s - n/p + n\tau}$, we get that $\|g_1\|_{L_{p_1}(Q(0,1))}$ is bounded by

$$\|g_1\|_{L_{p_1}(Q(0,1))} \leq \sum_{j \geq 1} \|Q_j f\|_{L_{p_1}(Q(0,1))} \leq c \| [f]_\infty \|_{\dot{B}_{p, q}^{s, \tau}} \sum_{j \geq 1} 2^{jn\tau(p/p_1 - 1)}.$$

by the assumption $p < p_1$ we obtain the desired result.

- We now see g_3 if $s + n\tau < \frac{n}{p}$. By (3.27) we obtain

$$\|g_3\|_{L_{p_1}(Q_{(0,1)})} \leq c_3 \| [f]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}} \left(\int_{Q_{(0,1)}} dx \right)^{1/p_1} = 2^{n/p_1} c_4 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}.$$

On the other hand, since

$$\int_{Q_{(0,1)}} |\sigma_1([f]_\infty)(x)| dx \leq c \int_{Q_{(0,1)}} \frac{|\sigma_1([f]_\infty)(x)|}{1 + |x|^{n+d}} dx,$$

then by (3.19) and the fact that $2^{n/p_1} \leq 2^n$ we have

$$\left(\int_{Q_{(0,1)}} |m_{Q_{(0,1)}}(\sigma_1([f]_\infty))|^{p_1} dx \right)^{1/p_1} = 2^{n/p_1} |m_{Q_{(0,1)}}(\sigma_1([f]_\infty))| \leq 2^n c \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}. \quad (3.50)$$

Hence (3.13) is proved for σ_1 .

1.2.2: Estimate of $\sigma_2([f]_\infty)$. As in the proof of Theorem 3.14, we have $\sigma_2([f]_\infty) = g_2 + g_4$.

Assume first $s + n\tau - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}_0$. To estimate $\|g_2\|_{L_{p_1}(Q_{(0,1)})}$, thus $\|\sum_{j \geq 1} Q_j f\|_{L_{p_1}(Q_{(0,1)})}$ can be treated as in (3.49) if $p_1 \leq p$, however, if $p_1 > p$, we have

$$\begin{aligned} \left\| \sum_{j \geq 1} Q_j f \right\|_{L_{p_1}(Q_{(0,1)})} &\leq 2^{n/p_1} \sum_{j \geq 1} \|Q_j f\|_\infty \\ &\leq c 2^{n/p_1} \| [f]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}} \sum_{j \geq 1} 2^{-j(s+n\tau-n/p)} \\ &\leq c 2^{n/p_1} \left(s + n\tau - \frac{n}{p} \right)^{-1} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}. \end{aligned}$$

We apply the embedding $\dot{B}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{\infty,\infty}^{s+n\tau-n/p}$ and Bernstein inequality (Proposition 1.3), in order to proceed with the second term in g_2 .

$$\begin{aligned} \left(\int_{Q_{(0,1)}} \left| \sum_{j \geq 1} \sum_{|\alpha| < \mu} (Q_j f)^{(\alpha)}(0) \frac{x^\alpha}{\alpha!} \right|^{p_1} dx \right)^{1/p_1} &\leq c_1 \sum_{|\alpha| < \mu} \left(\int_{Q_{(0,1)}} |x|^{p_1|\alpha|} dx \right)^{1/p_1} \sum_{j \geq 1} \|Q_j f^{(\alpha)}\|_\infty \\ &\leq c_1 \sum_{|\alpha| < \mu} \left(\int_{Q_{(0,1)}} dx \right)^{1/p_1} \sum_{j \geq 1} 2^{j|\alpha|} \|Q_j f\|_\infty \\ &\leq 2^{n/p_1} c_2 \| [f]_\infty \|_{\dot{B}_{p,q}^{s+n\tau-n/p}} \sum_{j \geq 1} 2^{j(n/p-s-n\tau+\mu-1)} \\ &\leq 2^n c_3 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}, \end{aligned}$$

since $\frac{n}{p} - s - n\tau + \mu - 1 < 0$.

We now estimate $\|g_4\|_{L_{p_1}(\mathcal{Q}_{(0,1)})}$; by (3.31) with $N = 0$, we obtain

$$\begin{aligned} \|g_4\|_{L_{p_1}(\mathcal{Q}_{(0,1)})} &\leq c_1 \| [f]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}} \left(\int_{\mathcal{Q}_{(0,1)}} |x|^{\mu b p_1} \left(1 + \sum_{|\alpha| < \mu} |x|^{|\alpha|(1-b)} \right)^{p_1} dx \right)^{1/p_1} \\ &\leq 2^{n/p_1} c_2 \| [f]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}} \leq 2^n c_2 \| [f]_\infty \|_{\dot{B}_{\infty,\infty}^{s+n\tau-n/p}}, \end{aligned}$$

where b satisfies the condition (3.32), and we conclude with the embedding $\dot{B}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{\infty,\infty}^{s+n\tau-n/p}$.

Second, if $s + n\tau - \frac{n}{p} \in \mathbb{N}$ and $0 < q \leq 1$, we apply (3.34) and obtain

$$\| \sigma_2([f]_\infty) \|_{L_{p_1}(\mathcal{Q}_{(0,1)})} \leq 2^{n/p_1} c \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}.$$

Finally, the estimate of $\left(\int_{\mathcal{Q}_{(0,1)}} |m_{\mathcal{Q}_{(0,1)}}(\sigma_2([f]_\infty))|^{p_1} dx \right)^{1/p_1}$ can be done similar to (3.50). The desired estimate is obtained with a constant $c > 1$ independent of p_1 .

1.2.3: Estimate of $\sigma_3([f]_\infty)$. Here $\sigma_3([f]_\infty) = g_1 + g_4$ where $g_1 = \sum_{j \geq 1} \mathcal{Q}_j f$ and $g_4 = \sum_{j \leq 1} (\mathcal{Q}_j f - \sum_{|\alpha| < \mu} \frac{1}{\alpha!} (\mathcal{Q}_j f)^{(\alpha)}(0) x^\alpha)$. Assume that $s + n\tau - \frac{n}{p} \in \mathbb{N}$ and $q > 1$; this case can be done similar to the preceding substep when $s + n\tau - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}_0$ and will be omitted. We now see the case $s + n\tau = \frac{n}{p}$. Recall that as $s > (n/p - n)_+$ implies that $p < \infty$. By (3.37) we get

$$\|g_4\|_{L_{p_1}(\mathcal{Q}_{(0,1)})} \leq c_1 \left(\int_{\mathcal{Q}_{(0,1)}} |x|^{b p_1} dx \right)^{1/p_1} \| [f]_\infty \|_{\dot{B}_{\infty,\infty}^0} \leq 2^{n/p_1} c_2 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}$$

where $0 < b < 1$.

Now, if $p_1 > p$, by applying both (3.35) and the embedding $\dot{B}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{p_1,\infty}^{n/p_1-n\tau,\tau}$, we obtain

$$\begin{aligned} \|g_1\|_{L_{p_1}(\mathcal{Q}_{(0,1)})} &\leq c 2^{n/p_1} \| [f]_\infty \|_{\dot{B}_{p_1,\infty}^{n/p_1-n\tau,\tau}} \\ &\leq c 2^n \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}} \end{aligned}$$

if $p_1 \leq p$, by the same way as in (3.49) i.e., Using Hölder inequality (with exponents $v := \frac{p}{p_1}$ and v') and the fact that $\mathcal{Q}_{(0,1)} \subset \cup_{r=1}^{3^n} P_{0,w_r}$, we have

$$\begin{aligned} \|g_1\|_{L_{p_1}(\mathcal{Q}_{(0,1)})} &\leq 2^{n/p_1} c_1 \|g_1\|_{L_p(\mathcal{Q}_{(0,1)})} \\ &\leq 2^{n/p_1} c_1 \sum_{r=0}^{3^n} \sum_{j \geq 1} 2^{-js} (2^{js} \| \mathcal{Q}_j f \|_{P_{0,w_r}}) \\ &\leq 2^{n/p_1} c_2 \| [f]_\infty \|_{\dot{B}_{p,\infty}^{s,\tau}} \leq 2^n c_3 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}. \end{aligned}$$

Also, the estimate of $(\int_{Q_{(0,1)}} |m_{Q_{(0,1)}}(\sigma_3([f]_\infty))|^{p_1} dx)^{1/p_1}$ can be done similar to (3.50). By these estimates we obtain the desired result for $\sigma_3([f]_\infty)$.

Step 2: proof of (ii). The realizations σ_i ($i = 1, 2$) commute with dilations, we then proceed as in the proof of Theorem 3.15/Step 2.

Step 3: proof of (iii). Here as the proof of Theorem 3.14/Step 2 since the realization σ_1 commutes with translations. Let $f \in \dot{B}_{p,q}^{s,\tau}$. Here also we apply $\sigma_i([f]_\infty) - f \in \mathcal{P}_\mu$ ($i = 1, 2, 3$), we use the realizations σ_i ($i = 1, 2, 3$) given in Proposition 2.9. \square

3.2.6 Application

We recall the definition of Morrey and refer to [52, sect. 1.3.2].

Definition 3.24. The Morrey space \mathcal{M}_r^u ($0 < r \leq u \leq \infty$) is the set of all functions $f \in L_r^{\text{loc}}$ such that

$$\|f\|_{\mathcal{M}_r^u} := \sup_Q |Q|^{1/u-1/r} \left(\int_Q |f(x)|^r dx \right)^{1/r} < \infty,$$

with the supremum is taken over all cubes Q in \mathbb{R}^n .

Proposition 3.25. We have $\mathcal{M}_r^r = L_r$, $\mathcal{M}_r^\infty = L_\infty$

As mentioned before, the presence of the polynomials causes that $\dot{B}_{p,q}^{s,\tau}$ can not embedded in \mathcal{M}_r^u . However, in realized spaces one can obtain the following result:

Theorem 3.26. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s > (\frac{n}{p} - n)_+$. Suppose that either $s + n\tau < \frac{n}{p}$ or $s + n\tau = \frac{n}{p}$ and $q \leq 1$. Let $0 < p_1 < \infty$ be a real number such that either $\frac{n}{p_1} > \frac{n}{p} - s$, or $\frac{n}{p_1} = \frac{n}{p} - s$ and $q \leq p_1$ if $1 < p_1 < \infty$. We put $\frac{1}{u} := \frac{1}{p} - \frac{s}{n} - \tau$. Then it holds $\dot{B}_{p,q}^{s,\tau} \hookrightarrow \mathcal{M}_{p_1}^u$.

Proof. Let $f \in \dot{B}_{p,q}^{s,\tau}$. By applying Theorem 3.14, we can establish that $\dot{B}_{p,q}^{s,\tau} \hookrightarrow L_{p_1}^{\text{loc}}$ for every $0 < p_1 < \infty$, this implies that for all $N > 0$

$$\left(\int_{|x| \leq N} |\sigma_1([f]_\infty)(x)|^{p_1} dx \right)^{1/p_1} \leq c_N \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}},$$

By substituting f with $f(\lambda \cdot)$ in the previous inequality and using Proposition 2.23, we get

$$\left(\lambda^{-n} \int_{|x| \leq \lambda N} |\sigma_1([f]_\infty)(x)|^{p_1} dx \right)^{1/p_1} \leq c_N \lambda^{s+n\tau-n/p} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}.$$

Taking $Q_{(x_0, \lambda)}$ rather than $B(x_0, \lambda)$ in the last estimation and $N = 1$. Since $\sigma_1([f]_\infty) - f \in \mathcal{P}_0$, we obtain

$$|Q_{(x_0, \lambda)}|^{(1/p-s/n-\tau)-1/p_1} \left(\int_{Q_{(x_0, \lambda)}} |f(x)|^{p_1} dx \right)^{1/p_1} \leq c \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}, \quad (3.51)$$

Since $s + n\tau < n/p$, $p_1 \in]0, \infty]$, and since $n/p - n\tau - s \leq n/p - s \leq n/p_1$ we obtain $1/p - \tau - s/n \leq 1/p_1$, then we can derive the required conclusion from the inequality (3.51). \square

The following result give us the John-Nirenberg Theorem for $\dot{B}_{p,q}^{s,\tau}$:

Theorem 3.27. *Let $0 < p < \infty$ and $0 < q \leq \infty$. Let either $\frac{n}{p} \leq s + n\tau < 1 + \frac{n}{p}$ or $s + n\tau = 1 + \frac{n}{p}$ and $q \leq 1$. Then, there exist two positive constants c_1, c_2 , such that the inequality*

$$\left| \left\{ x \in Q_{(x_0, \lambda)} : |f(x) - m_{Q_{(x_0, \lambda)}} f| > \alpha \right\} \right| \leq c_1 |Q_{(x_0, \lambda)}| \exp \left(- \frac{c_2 \alpha}{|Q_{(x_0, \lambda)}|^{s/n+\tau-1/p} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}} \right) \quad (3.52)$$

holds for all $f \in \dot{B}_{p,q}^{s,\tau}$, all $\alpha > 0$ and all cubes $Q_{(x_0, \lambda)}$ in \mathbb{R}^n .

Proof. The proof is similar to that given in [42, p. 145] using Theorem 3.16. \square

3.3 To Triebel-Lizorkin-type spaces

As the realized homogeneous Triebel-Lizorkin-type spaces are defined in the recent thesis [8, Section 4.3], we propose that we can extend the results given in the last section to $\dot{F}_{p,q}^{s,\tau}$, and this will be an eventually future paper to publishing in collaboration with the advisor of the thesis.

Chapter 4

Further proprieties via differences and Fourier transform

This chapter contains tow independent sections, where the first one is a characterization of smooth functions by differences operator, and the second concerns the Hirschmann property in Besov and Triebel-Lizorkin spaces.

4.1 Quasi-norms in \mathcal{D}

4.1.1 Preparations

We first give the definition of this operators :

Definition 4.1. *The difference operates are define by $\Delta_h f := f(\cdot + h) - f$. We put $\Delta_h^1 f := \Delta_h f$ and $\Delta_h^m f := \Delta_h^1(\Delta_h^{m-1} f)$ for $m = 2, 3, \dots$*

The explicit formula can be demonstrated via mathematical induction

$$\Delta_h^m f(x) = \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} f(x + jh). \quad (4.1)$$

We are inspired the main result of this paragraph form the following assertion that proved in [III, Lemma 1.4]:

Proposition 4.2. *Let $0 < r < 1$. Let f be a function of class C^1 , with compact support in \mathbb{R}^n . Then the function*

$$F(x) := \sup_{t>0} t^{-r-n} \int_{|h|\leq t} |\Delta_h f(x)| dh$$

is in L_p for all $p \in [1, \infty]$.

Notation: To formulate the result, we give the following notation which will be used throughout this paragraph:

$$\mathcal{N}_{p,q}^{s,m}(f) := \left\| \left(\int_0^\infty t^{-sq} \left(t^{-n} \int_{|h|\leq t} |\Delta_h^m f(\cdot)| dh \right)^q \frac{dt}{t} \right)^{1/q} \right\|_p.$$

4.1.2 A quasi-norm in \mathcal{D}

We have the following result:

Theorem 4.3. *Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $0 < s < m$, then $\mathcal{N}_{p,q}^{s,m}(f)$ defines a quasi-norm in \mathcal{D} .*

Proof. Let $f \in \mathcal{D}$; for simplicity we subdivide the proof into two steps.

Step 1: Convergence of $\mathcal{N}_{p,q}^{s,m}(f)$. We set

$$F(x) := \left(\int_0^\infty t^{-sq} \left(t^{-n} \int_{|h|\leq t} |\Delta_h^m f(x)| dh \right)^q \frac{dt}{t} \right)^{1/q}.$$

We will prove that F is in L_p . As f in \mathcal{D} we have $\text{supp } f \subset B(0, M)$, where $B(0, M)$ is a ball centered at 0 with radius M .

For technical reasons we write

$$\|F\|_p = (A + B)^{1/p},$$

where

$$A := \int_{|x|\leq(m+2)M} |F(x)|^p dx$$

and

$$B := \int_{|x|>(m+2)M} |F(x)|^p dx.$$

Substep 1.1: Estimate of A . If $|x| \leq (m+2)M$, by using the explicit form of $\Delta_h^m f$ see (4.1), we have

$$|\Delta_h^m f(x)| \leq \|f\|_\infty \sum_{j=0}^m \binom{m}{j} \leq c_1 \|f\|_\infty, \quad (4.2)$$

and

$$|\Delta_h^m f(x)| \leq c_2 |h|^m. \quad (4.3)$$

Applying (4.2) and (4.3), we obtain

$$|\Delta_h^m f(x)| \leq \min(c_1 \|f\|_\infty, c_2 |h|^m) \leq c_3 \min(1, |h|^m).$$

To show (4.3) we apply Taylor's formula to $f(x + jh)$ for all $(j \in \{0, \dots, m\})$ we obtain

$$f(x + jh) = f(x) + \sum_{i=1}^n jh_i \int_0^1 (1-u) \partial_i f(x + juh) du \quad \forall j = 0, \dots, m.$$

By taking summation in the last equality on j we get

$$\begin{aligned} \Delta_h^m f(x) &= \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} \sum_{i=1}^n jh_i \int_0^1 (1-u) \partial_i f(x + juh) du \\ &= m \sum_{i=1}^n h_i \int_0^1 (1-u) \sum_{j=1}^m (-1)^{m-j} \binom{m-1}{j-1} \partial_i f(x + juh) du. \end{aligned}$$

Then it holds (4.3).

We will show two cases.

Case 1.1.1: if $t \leq 1$, we have

$$t^{-n-s} \int_{|h| \leq t} |\Delta_h^m f(x)| \leq c_1 t^{-n-s} \int_{|h| \leq t} |h|^m dh = c_2 t^{m-s}.$$

Case 1.1.2: if $t > 1$, we have

$$t^{-n-s} \int_{|h| \leq t} |\Delta_h^m f(x)| dh \leq c_1 t^{-n-s} \left(\int_{|h| \leq 1} |h|^m dh + \int_{1 < |h| \leq t} dh \right) \leq c_2 t^{-s}$$

then, as $0 < s < m$ it holds that F in $|x| \leq (m+2)M$ is bounded as well as A . Indeed; by inserting the two last cases we obtain

$$\begin{aligned} A &= \int_{|x| \leq (m+2)M} |F(x)|^p dx \\ &\leq c_1 \int_{|x| \leq (m+2)M} \left(\int_0^1 t^{(m-s)q} \frac{dt}{t} + \int_1^\infty t^{-sq} \frac{dt}{t} \right)^{p/q} dx \\ &\leq c_2 (m+2)^n M^n. \end{aligned}$$

Substep 1.2: Estimate of B . If $|x| > (m+2)M$. As $\text{supp } f \subset B(0, M)$, here we have $f(x) = 0$.

Case 1.2.1: if $t \leq M$, since $|h| \leq t$ we have

$$|x + jh| \geq |x| - |jh| \geq (m+2-j)M \quad \forall j = 0, \dots, m$$

then

$$|x + jh| \geq 2M \quad \forall j = 0, \dots, m,$$

it holds

$$\Delta_h^m f(x) = 0.$$

Case 1.2.2: if $t > M$ and $t \geq |x|/(m+2)$, we have

$$t^{-n-s} \int_{|h| \leq t} |\Delta_h^m f(x)| dh = t^{-n-s} \sum_{j=1}^m \binom{m}{j} \int_{|h| \leq t} |f(x+jh)| dh \leq c \|f\|_{\infty} t^{-s}.$$

Case 1.2.3: if $t > M$ and $t \leq |x|/(m+2)$, we have

$$|x+jh| < \frac{m+2+j}{m+2}|x| < \frac{2m+2}{m+2}|x|$$

and

$$|x+jh| > |x| - |jh| > \frac{2}{m+2}|x|.$$

Let δ be a real number which will be choosing later, then

$$\begin{aligned} t^{-n-s} \int_{|h| \leq t} |\Delta_h^m f(x)| dh &= t^{-n-s} \sum_{j=1}^m \binom{m}{j} \int_{|h| \leq t} |x+jh|^{\delta} |x+jh|^{-\delta} |f(x+jh)| dh \\ &\leq t^{-n-s} \left(\frac{2}{m+2}|x|\right)^{-\delta} \sum_{j=1}^m \binom{m}{j} \int_{|y-x| \leq jt} |y|^{\delta} |f(y)| dy \\ &\leq ct^{-n-s} |x|^{-\delta} \int_{\mathbb{R}^n} |y|^{\delta} |f(y)| dy \\ &\leq c|x|^{-\delta} t^{-n-s}. \end{aligned}$$

Now, we have by inserting the precedent cases

$$\begin{aligned} F(x)^q &\leq c \left(|x|^{-\delta q} \int_M^{|x|/(m+2)} t^{(-n-s)q} \frac{dt}{t} + \int_{|x|/(m+2)}^{\infty} t^{-sq} \frac{dt}{t} \right) \\ &\leq c|x|^{-\delta q}, \end{aligned}$$

then, if we take $\delta > n/p$ we obtain

$$B \leq c \int_{|x| > 3M} |x|^{-\delta p} dx \leq c \int_{3M}^{\infty} r^{n-1-\delta p} dr$$

if we take $\delta > n/p$ we obtain that B is bounded.

Step 2: $\mathcal{N}_{p,q}^{s,m}(f)$ defines a quasi-norm:

$$f = 0 \iff \mathcal{N}_{p,q}^{s,m}(f) = 0.$$

As $\Delta_h^m(f)$ is linear operator, then

$$\mathcal{N}_{p,q}^{s,m}(\lambda f) = |\lambda| \mathcal{N}_{p,q}^{s,m}(f), \quad \forall \lambda \in \mathbb{C}$$

and

$$\exists k \geq 1 : \quad \mathcal{N}_{p,q}^{s,m}(f + g) \leq k(\mathcal{N}_{p,q}^{s,m}(f) + \mathcal{N}_{p,q}^{s,m}(g)).$$

□

4.2 Hirschmann property for $B_{p,q}^s$ and $F_{p,q}^s$ spaces

We propose to generalize Hirschmann theorem to Besov spaces $B_{p,q}^s$ and Triebel-Lizorkin spaces $F_{p,q}^s$ by changing H^s with H_p^s , in previous works, see e.g. [12].

4.2.1 The Fourier multipliers

The Fourier multipliers play a fundamental role in this chapter. We begin this paragraph, then we need to define the pointwise multipliers.

4.2.1.1 Pointwise multipliers

Definition 4.4. *Let E be a distribution Banach space (D.B.S). contain D as a dense subspace, we say that the distribution A is pointwise multiplier, if there exists a constant $c > 0$, such that for all $\psi \in C^\infty \cap E$ we have $A\psi \in E$ and*

$$\|A\psi\|_E \leq c\|\psi\|_E.$$

The space of linear multiplier noted by $M(E)$ with the norm

$$\|A\|_{M(E)} = \{\|A\psi\|_E; \|\psi\|_E = 1, \text{ for all } \psi \in C^\infty \cap E.\}$$

Proposition 4.5. • $(M(E), \|\cdot\|_{M(E)})$ is a Banach space.

- If $D \subset M(E)$ then $M(E)$ is algebra, i.e., $A_1, A_2 \in M(E) \Rightarrow A_1 A_2 \in M(E)$.
- $M(L_p) = L_\infty$, for all $1 \leq p \leq \infty$.

4.2.1.2 The Fourier multipliers for L_p

Let $f \in \mathcal{S}$ and $f \in \mathcal{S}'$ then we have $\mathcal{F}^{-1}m * f \in \mathcal{S}'$. As we have \mathcal{S} dense in L_p ($1 \leq p < \infty$), then $T : f \mapsto \mathcal{F}^{-1}m * f$ may be extended to continuous operator in L_p to itself. We can define it on \mathcal{S}' . Let $m \in \mathcal{S}'$, m is a Fourier multiplier for L_p if there exists a constant $c > 0$ such that

$$\|\mathcal{F}^{-1}m * f\|_{L_p} \leq c\|f\|_{L_p} \quad \text{for all } f \in \mathcal{S},$$

M_p denotes the collection of all Fourier multipliers for L_p , with the norm

$$\|m\|_{M_p} = \sup_{\|f\|_p=1} \|\mathcal{F}^{-1}m * f\|_p.$$

Remark 4.6. We can define the space M_p by: $M_p = M(\mathcal{F}L_p)$, where

$$\mathcal{F}L_p := \{f \in \mathcal{S}' : \|f\|_{\mathcal{F}L_p} = \|\mathcal{F}^{-1}f\|_p\}.$$

Proposition 4.7. • $M_1 = M_\infty = \mathcal{F}B$, with B is the bounded measure space.

- $M_2 = L_\infty$.
- $M_p = M_{p'}$ if $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.
- $M_{p_0} \cap M_{p_1} \subset M_q$ where $\frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $0 < \theta < 1$ and $\|m\|_{M_q} \leq \|m\|_{M_{p_0}}^{1-\theta} \|m\|_{M_{p_1}}^\theta$.
- $M_1 \subset M_p \subset M_q \subset M_2$ if $1 \leq p \leq q \leq 2$.

Proof. See [9] or [12]. □

4.2.1.3 Fourier multipliers for $B_{p,q}^s$

Definition 4.8. Let $s \in \mathbb{R}$, $0 \leq p, q \leq \infty$. Let $m \in \mathcal{S}'$. m is a Fourier multiplier for $B_{p,q}^s$ if there exists a constant $c > 0$ such that

$$\|\mathcal{F}^{-1}m * f\|_{B_{p,q}^s} \leq c\|f\|_{B_{p,q}^s} \quad \text{for all } f \in \mathcal{S}.$$

The set of all Fourier multipliers for $B_{p,q}^s$ denoted by \tilde{M}_p is independent of s and q , see [46].

4.2.1.4 Fourier multipliers for $F_{p,q}^s$

Definition 4.9. Let $s \in \mathbb{R}$, $0 \leq p < \infty$ and $0 \leq q \leq \infty$. Let $m \in \mathcal{S}'$. m is a Fourier multiplier for $F_{p,q}^s$, if there exists a constant $c > 0$ such that

$$\|\mathcal{F}^{-1}m * f\|_{F_{p,q}^s} \leq c\|f\|_{F_{p,q}^s} \quad \text{for all } f \in \mathcal{S}.$$

The set of all Fourier multipliers for $F_{p,q}^s$ denoted by $\tilde{M}_{p,q}$ is independent of s , see [46].

Remark 4.10. We can defined the space \tilde{M}_p by: $\tilde{M}_p = M(\mathcal{F}B_{p,q}^s)$, where

$$\mathcal{F}B_{p,q}^s := \{f \in \mathcal{S}' : \|f\|_{\mathcal{F}B_{p,q}^s} = \|\mathcal{F}^{-1}f\|_p\},$$

with the quasi-norm

$$\|m\|_{\tilde{M}_p} = \sup_{\|f\|_{B_{p,q}^s} = 1} \|\mathcal{F}^{-1}m * f\|_{B_{p,q}^s}.$$

Proposition 4.11. • $\tilde{M}_p = \tilde{M}_{p'}$ if $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

- $\tilde{M}_{p_0} \cap \tilde{M}_{p_1} \subset \tilde{M}_q$ where $\frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $0 < \theta < 1$ and $\|m\|_{\tilde{M}_q} \leq \|m\|_{\tilde{M}_{p_0}}^{1-\theta} \|m\|_{\tilde{M}_{p_1}}^\theta$.
- $\tilde{M}_p \subset \tilde{M}_q \subset \tilde{M}_2 = M_2 = L_\infty$ if $0 < p \leq q \leq 2$.
- $\tilde{M}_1 \subset \tilde{M}_p$, $p \geq 1$.

Proof. See [46]. □

Remark 4.12. We can defined the space $\tilde{M}_{p,q}$ by: $\tilde{M}_{p,q} = M(\mathcal{F}F_{p,q}^s)$, where

$$\mathcal{F}L_p := \{f \in \mathcal{S}' : \|f\|_{\mathcal{F}F_{p,q}^s} = \|\mathcal{F}^{-1}f\|_p\},$$

with the quasi-norm

$$\|m\|_{\tilde{M}_{p,q}} = \sup_{\|f\|_{F_{p,q}^s} = 1} \|\mathcal{F}^{-1}m * f\|_{F_{p,q}^s}.$$

Proposition 4.13. • $\tilde{M}_{p,p} = \tilde{M}_p$ if $0 < p < \infty$.

- $\tilde{M}_{p,2} = \tilde{M}_p$ if $1 < p < \infty$.
- $\tilde{M}_{p,q} = \tilde{M}_{p',q'}$ if $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.
- $\tilde{M}_{p,q} \subset \tilde{M}_p$ if $0 < p < \infty$ and $0 < q \leq \infty$.

- $\tilde{M}_{p,q_0} \subset \tilde{M}_{p,q_1}$ if $0 < q_0 \leq q_1 \leq p$ or $p \leq q_1 \leq q_0 \leq \infty$ and $p \neq \infty$.
- $\tilde{M}_{p_0,q_0} \subset \tilde{M}_{p_1,q_1}$ if $\frac{1}{p_1} = \frac{1-\theta}{p_0} + \frac{\theta}{2}$, $\frac{1}{q_1} = \frac{1-\theta}{q_0} + \frac{\theta}{2}$ with $0 < \theta < 1$.

Proof. See [46]. □

4.3 Hirschmann's theorem

4.3.1 Bessel-potential space

To demonstrate that the space of Fourier multipliers \tilde{M}_p contains $H_p^s \cap B_{\infty,\infty}^r$ under certain parameter conditions, we recall the definition of Bessel-potential space H_p^s .

Definition 4.14. Let $s \in \mathbb{R}$ and $1 < p < \infty$ the Bessel-potential space H_p^s is the set all $f \in \mathcal{S}'$, such that

$$\|f\|_{H_p^s} := \|\mathcal{F}^{-1}[(1+|\xi|^2)^{s/2} \mathcal{F}f(\xi)](\cdot)\|_p < \infty.$$

4.3.2 Hirschmann's theorem in M_p

Proposition 4.15. Let $s \leq \frac{n}{2}$ and $r \in \mathbb{R}^+ \setminus \mathbb{N}$, if $f \in B_{\infty,\infty}^s \cap H^s$ then $f \in M_p$ for all

$$\frac{2r+2(n/2-s)}{2r+n/2-s} < p < \frac{2r}{n/2-s}.$$

The preceding theorem is valid also in \tilde{M}_p and in $\tilde{M}_{p,q}$:

Proposition 4.16. Let $s \leq \frac{n}{2}$ and $r \in \mathbb{R}^+ \setminus \mathbb{N}$, if $f \in B_{\infty,\infty}^s \cap H^s$ then $f \in \tilde{M}_p$ for all

$$\frac{2r+2(n/2-s)}{2r+n/2-s} < p < \frac{2r}{n/2-s}.$$

Proposition 4.17. Let $s \leq \frac{n}{2}$ and $r \in \mathbb{R}^+ \setminus \mathbb{N}$, if $f \in B_{\infty,\infty}^s \cap H^s$ then $f \in \tilde{M}_{p,q}$ for all

$$\frac{2r+2(n/2-s)}{2r+n/2-s} < p < \frac{2r}{n/2-s} \quad \text{and} \quad p = q \quad \text{or} \quad q = 2.$$

4.3.3 Statement of the results

Theorem 4.18. Let $s \leq \frac{n}{p}$ and $r \in \mathbb{R}^+ \setminus \mathbb{N}$, if $f \in B_{\infty,\infty}^r \cap H_p^s$ then $f \in M_u$ for all

$$\frac{2r+2(\frac{n}{p}-s)}{2r+\frac{n}{p}-s} < u < \frac{2r}{\frac{n}{p}-s} + 2.$$

Theorem 4.19. *Let $s \leq \frac{n}{p}$ and $r \in \mathbb{R}^+ \setminus \mathbb{N}$, if $f \in B_{\infty, \infty}^r \cap H_p^s$ then $f \in \widetilde{M}_u$ for all*

$$\frac{2r + 2(\frac{n}{p} - s)}{2r + \frac{n}{p} - s} < u < \frac{2r}{\frac{n}{p} - s} + 2.$$

Corollary 4.20. *Let $s \leq \frac{n}{p}$ and $r \in \mathbb{R}^+ \setminus \mathbb{N}$, if $f \in B_{\infty, \infty}^r \cap H_p^s$ then $f \in \widetilde{M}_{u, q}$ for all*

$$\frac{2r + 2(\frac{n}{p} - s)}{2r + \frac{n}{p} - s} < u < \frac{2r}{\frac{n}{p} - s} + 2 \text{ and } u = q \text{ or } q = 2.$$

4.3.4 Proof of main results

Proof of Theorem 4.18. It's clearly $\frac{2r + 2(\frac{n}{p} - s)}{2r + \frac{n}{p} - s}$ is the conjugate of $\frac{2r}{\frac{n}{p} - s} + 2$, by Proposition 4.7, it sufficient to prove the theorem only on

$$\frac{2r + 2(\frac{n}{p} - s)}{2r + \frac{n}{p} - s} < u \leq 2.$$

Let $f \in B_{\infty, \infty}^r \cap H_p^s$, we recall that $f = \sum_{j \geq 0} Q_j f$ with $\widehat{Q_j f} = \gamma(2^{-j} \xi) \widehat{f}(\xi)$ and $\widehat{Q_0 f} = \rho(\xi) \widehat{f}(\xi)$. Thanks to the interpolation, we have

$$\|Q_j f\|_{M_u} \leq c \|Q_j f\|_{M_1}^{1-\theta} \|Q_j f\|_{M_2}^{\theta}, \quad \text{for all } 0 < \theta < 1. \quad (4.4)$$

We will estimate the right hand side of (4.4). To begin, we estimate $\|Q_j f\|_{M_2}$, using Proposition 4.7, we find

$$\|Q_j f\|_{M_2} = \|Q_j f\|_{\infty} \leq c 2^{-jr} \|f\|_{B_{\infty, \infty}^r}. \quad (4.5)$$

Then, using $\|Q_j f\|_{M_1}$ by using Hölder inequality and the fact that $\|g\|_{p'} = (2\pi)^{\frac{n}{p'}} \|\mathcal{F}^{-1} g\|_p$ with $1 \leq p \leq 2$, we have

$$\begin{aligned} \|Q_j f\|_{M_1} &= \|\widehat{Q_j f}\|_1 \\ &\leq \|\widehat{f}(\xi) (1 + |\xi|^2)^{s/2}\|_{p'} \|\gamma(2^{-j} \cdot) (1 + |\xi|^2)^{-s/2}\|_p \\ &\leq c (2\pi)^{\frac{n}{p'}} 2^{j(\frac{n}{p} - s)} \|\gamma\|_{\infty} \|f\|_{H_p^s}, \end{aligned} \quad (4.6)$$

by inserting (4.6) and (4.5) into (4.4), we get

$$\|Q_j f\|_{M_u} \leq c 2^{j(\frac{n}{p}-s)(1-\theta)-jr\theta},$$

where $\frac{1}{u} = (1-\theta) + \frac{\theta}{2}$ to obtain that $f \in M_u$, we have to suppose that $(\frac{n}{p}-s)(1-\theta) - r\theta < 0$ by simple calculate, we get $\frac{2(\frac{n}{p}-s) + 2r}{\frac{n}{p}-s+2r} < u$.

We obtain the desired result with $1 < p \leq 2$ and use the duality in H_p^s to get the rest when $p > 2$. \square

Proof of Theorem 4.19. As what we say in proof of Theorem 4.18, and by Proposition 4.11, it sufficient to suppose

$$\frac{2r + 2(\frac{n}{p}-s)}{2r + \frac{n}{p} - s} < u \leq 2.$$

Let $f \in B_{\infty,\infty}^r \cap H_p^s$. To prove that $f \in \tilde{M}_u$ we will use the same method as in Proof of Theorem 4.18, by replacing M_u with \tilde{M}_u in (4.4) which comes from Proposition 4.11.

We use the fact that $\|Q_j f\|_{\tilde{M}_2} = \|Q_j f\|_{M_2}$ and applying (4.5) to estimate $\|Q_j f\|_{\tilde{M}_2}$

$$\|Q_j f\|_{\tilde{M}_2} \leq c 2^{-jr} \|f\|_{B_{\infty,\infty}^r}. \quad (4.7)$$

We will now estimate $\|Q_j f\|_{\tilde{M}_1}$, using the young equality

$$\|\mathcal{F}^{-1} Q_j f * g\|_{B_{1,\infty}^0} \leq \sup_{k \geq 0} \|\mathcal{F}^{-1} Q_j f\|_1 \|2^{kn} \mathcal{F}^{-1}[\gamma(2^k \cdot)] * g\|_1 = \|\mathcal{F}^{-1} Q_j f\|_1 \|g\|_{B_{1,\infty}^0},$$

then we obtain that $\|Q_j f\|_{\tilde{M}_1} = \|\mathcal{F}^{-1} Q_j f\|_1$ in the other hand we have $\|\mathcal{F}^{-1} Q_j f\|_1 = \|Q_j f\|_{M_1}$ by applying (4.6) we get

$$\|Q_j f\|_{\tilde{M}_1} \leq c 2^{j(\frac{n}{p}-s)} \|f\|_{H_p^s} \quad \text{for all } 1 < p \leq 2 \quad (4.8)$$

then inserting (4.7) and (4.8) in $\|Q_j f\|_{\tilde{M}_u} \leq c \|Q_j f\|_{\tilde{M}_1}^{1-\theta} \|Q_j f\|_{\tilde{M}_2}^\theta$, with $0 < \theta < 1$ we get

$$\|Q_j f\| \leq c 2^{j(\frac{n}{p}-s)(1-\theta)} 2^{-jr\theta}$$

where $\frac{1}{u} = (1-\theta) + \frac{\theta}{2}$, we suppose that $(\frac{n}{p}-s)(1-\theta) - r\theta < 0$ to obtain that $f \in \tilde{M}_u$ then by simple calculation we get $u > \frac{2r + 2(\frac{n}{p}-s)}{2r + \frac{n}{p} - s}$.

When $p > 2$ we use the duality of H_p^s . \square

Proof of Corollary 4.20 . If $q = u$, we have $\tilde{M}_{u,u} = \tilde{M}_u$, then we apply Theorem 4.19.

If $q = 2$, we have $\tilde{M}_{u,2} = \tilde{M}_u$, then we use Theorem 4.18. □

Chapter 5

Annexe

5.1 Further obtained results

The following assertion can be found in [24]:

Theorem 5.1. *Let $0 < a < b$. Let $(u_j)_{j \in \mathbb{Z}}$ be a sequence in \mathcal{S}' such that*

- \hat{u}_j is supported by the compact annulus $a2^j \leq |\xi| \leq b2^j$,
- $A := \left(\sum_{j \in \mathbb{Z}} 2^{jsq} \|u_j\|_p^q \right)^{1/q} < \infty$ in B -space,
- $A := \left\| \left(\sum_{j \in \mathbb{Z}} 2^{jsq} |u_j|^q \right)^{1/q} \right\|_p < \infty$ in F -space if $p < \infty$,
- $A := \sup_{k \in \mathbb{Z}, \mu \in \mathbb{Z}^n} \left(2^{kn} \int_{P_{k,\mu}} \sum_{j \geq k} 2^{jsq} |u_j(x)|^q dx \right)^{1/q}$ in F -space if $p = \infty$.

- (i) Then the series $\sum_{j \in \mathbb{Z}} u_j$ converges in \mathcal{S}' to a limit u satisfying $\|u\|_{\dot{A}_{p,q}^s} \leq cA$, where the constant c depends only on n, s, p, q, a and b .
- (ii) If in addition $s > (\frac{n}{p} - n)_+$ in B -space, $s > (\frac{n}{\min(p,q)} - n)_+$ in F -space, the same conclusion holds for $a = 0$.

Proof. See [24, Proposition 3.7]. □

Remark 5.2. Only the case $p = \infty$ in Triebel-Lizorkin spaces is the aim result here, where the proof is obtained using the decomposition on dyadic cubes, since other cases can be found in, e.g., [17, Proposition 4], [31, Proposition 3.4] and [32, Propositions 2.15, 2.17].

5.2 The Besov-Triebel-Lizorkin type spaces

5.2.1 The algebra spaces

If we consider the space

$$\dot{\mathcal{A}}_{p,q}^s = \dot{A}_{p,q}^s \cap L_\infty,$$

we obtain an algebra space for the product. This space plays an important role in some fields, see e.g. [18], [19] and [31].

Here, we will give some functionals characterizations of the algebra spaces $\dot{\mathcal{A}}_{p,q}^{s,\tau} = \dot{A}_{p,q}^{s,\tau} \cap L_\infty$ in collaboration with F. Bensaïd, we note that these spaces have recently studied in [7].

5.2.2 Characterization via differences

We are working in the proof of the following results,

where we need first the notation:

For $1 \leq u \leq \infty$, $s > 0$ and $m \in \mathbb{N}$, we put

$$\mathcal{M}_{p,q}^{s,\tau,m}(f) := \sup_{k \in \mathbb{Z}, \mu \in \mathbb{Z}^n} 2^{n\tau k} \left(\int_0^\infty t^{-sq} \left(\int_{P_{k,\mu}} t^{-n} \int_{t/2 \leq |h| \leq t} |\Delta_h^m f(x)|^p dh dx \right)^{q/p} \frac{dt}{t} \right)^{1/q},$$

$$\widetilde{\mathcal{M}}_{p,q}^{s,\tau,m}(f) := \sup_{k \in \mathbb{Z}, \mu \in \mathbb{Z}^n} 2^{n\tau k} \left(\int_0^\infty t^{-sq} \left(\int_{P_{k,\mu}} \left(t^{-n} \int_{t/2 \leq |h| \leq t} |\Delta_h^m f(x)|^u dh \right)^{p/u} dx \right)^{q/p} \frac{dt}{t} \right)^{1/q},$$

$$\mathcal{N}_{p,q}^{s,\tau,m}(f) := \sup_{k \in \mathbb{Z}, \mu \in \mathbb{Z}^n} 2^{n\tau k} \left(\int_{P_{k,\mu}} \left(\int_0^\infty t^{-sq} t^{-n} \int_{t/2 \leq |h| \leq t} |\Delta_h^m f(x)|^q dh \frac{dt}{t} \right)^{p/q} dx \right)^{1/p},$$

$$\widetilde{\mathcal{N}}_{p,q}^{s,\tau,m}(f) := \sup_{k \in \mathbb{Z}, \mu \in \mathbb{Z}^n} 2^{n\tau k} \left(\int_{P_{k,\mu}} \left(\int_0^\infty t^{-sq} \left(t^{-n} \int_{t/2 \leq |h| \leq t} |\Delta_h^m f(x)|^u dh \right)^{q/u} \frac{dt}{t} \right)^{p/q} dx \right)^{1/p}.$$

For the simplicity, we put $a_k := 2 \min(1, 2^{-k})$ and

$$\|f\|_{B_{p,q}^{s,\tau}}^* := \sup_{k \in \mathbb{Z}, \mu \in \mathbb{Z}^n} 2^{n\tau k} \left(\int_0^{a_k} t^{-sq} \left(\int_{P_{k,\mu}} t^{-n} \int_{t/2 \leq |h| \leq t} |\Delta_h^m f(x)|^p dh dx \right)^{q/p} \frac{dt}{t} \right)^{1/q},$$

$$\|f\|_{B_{p,q}^{s,\tau}}^{(*,u)} := \sup_{k \in \mathbb{Z}, \mu \in \mathbb{Z}^n} 2^{n\tau k} \left(\int_0^{a_k} t^{-sq} \left(\int_{P_{k,\mu}} \left(t^{-n} \int_{t/2 \leq |h| \leq t} |\Delta_h^m f(x)|^u dh \right)^{p/u} dx \right)^{q/p} \frac{dt}{t} \right)^{1/q},$$

$$\|f\|_{F_{p,q}^{s,\tau}}^* := \sup_{k \in \mathbb{Z}, \mu \in \mathbb{Z}^n} 2^{n\tau k} \left(\int_{P_{k,\mu}} \left(\int_0^{a_k} t^{-sq} t^{-n} \int_{t/2 \leq |h| \leq t} |\Delta_h^m f(x)|^q dh \frac{dt}{t} \right)^{p/q} dx \right)^{1/p},$$

$$\|f\|_{F_{p,q}^{s,\tau}}^{(*,u)} := \sup_{k \in \mathbb{Z}, \mu \in \mathbb{Z}^n} 2^{n\tau k} \left(\int_{P_{k,\mu}} \left(\int_0^{a_k} t^{-sq} \left(t^{-n} \int_{t/2 \leq |h| \leq t} |\Delta_h^m f(x)|^u dh \right)^{q/u} \frac{dt}{t} \right)^{p/q} dx \right)^{1/p}.$$

Then we will characterize the realized homogeneous Besov and Triebel-Lizorkin type spaces as in [B3]. We are also characterize the algebra versions with the help of these functional formulas.

Conclusion and perspectives

In this work, we prove that BMO coincides with the realized homogeneous space $\dot{F}_{\infty,2}^0$. For $p < \infty$, we show that the realized space of $\dot{F}_{p,q}^{n/p}$ is strictly embedded in BMO . Then we derive additional results that extend to homogeneous and inhomogeneous Besov spaces. We present an embeddings between BMO and Besov space $B_{\infty,\infty}^0$, as well as the realized spaces $\dot{B}_{\infty,2}^0$ and $\dot{B}_{\infty,\infty}^0$. We apply the Riesz operator \mathcal{I}_α to BMO space to produce embeddings of realized versions.

We extend some functional characterization given in [24] to the realized homogeneous Besov-type spaces $\dot{B}_{p,q}^{s,\tau}$, we prove embedding between Morrey space and $\dot{B}_{p,q}^{s,\tau}$ under certain conditions on parameters and we also show the John-Nirenberg inequality on $\dot{B}_{p,q}^{s,\tau}$.

We obtain a quasi-norm in smooth functions spaces. We are now going to discuss some planned future research in a number of different fields.

- We propose that we can extend the results given in Chapter 3 to $\dot{F}_{p,q}^{s,\tau}$.
- It is interesting to see if $\dot{F}_{\infty,2}^0$ can be endowed with seminorms defined by the Poisson semi-group, we refer to [23].
- As $\mathcal{I}_\alpha : \dot{B}_{p,q}^s \rightarrow \dot{B}_{p,q}^{s+\alpha}$, we have $\mathcal{I}_\alpha : \dot{B}_{p,q}^s \rightarrow \dot{B}_{p,q}^{s+\alpha}$ is an open question, similarly in the \dot{F} -spaces, cf. [34, Remark 2.11].

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