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Some characterizations of fuzzy ideals in ordered semirings

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Contents

List of Symbols

Notation	Name
$\cdot$	A binary operation.
$*$	A binary operation.
$+$	Addition operation.
$\circ$	The composition operation.
$\leq$	A binary ordered relation.
$\geq$	A binary dual ordered relation.
ϕ	The empty set.
d	Derivation.
f	A fuzzy set.
$\cup$	The union operation.
$\cap$	The intersection operation.
$\subseteq$	The subset operation.
χ	Characteristic function.
f^{-1}	The preimage.

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I dedicate this thesis to my parents. Thanks for your constant love and support. I owe you.

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BEN ZIANE Ahmed

Introduction

The idea of ideals created by Dedekind for the theory of algebraic numbers, was generalized by Emmy Noether for associative rings. The one and two-sided ideals introduced by her, are as yet central concepts in ring theory. Since then many papers on ideals for rings and semigroups appeared showing the importance of the concept [A.H. Clifford , L.M. Gluskin, M. P. Schützenberger, S. Lajos, K. Iséki and many others]. Ideals are just a generalization of the lattice-theoretical methods which was given by G. Birkhoff , O. Steinfeld, and N. Kehayopulu.

After the concept of the fuzzy set was introduced by professor Zadeh in 1965 ([20]) in his article "Fuzzy sets" as an extension of the classical notion of set, this theory has been developed by many authors.

Using the notion of fuzzy set theory the concepts of fuzzy one- and two-sided ideals in groupoids have been introduced by A. Rosenfeld in [19], the concepts of fuzzy bi-ideals in semigroups have been introduced by N. Kuroki in [12] and [13], respectively.

Semirings [4] which is just a generalization of rings and distributive lattices applied many diverse areas of mathematics as combinatorics, functional analysis, graph theory, automata theory, mathematical modelling and parallel computation systems etc.(for example, see [4] , [5]). Semirings have also been proved to be an important algebraic tool in theoretical computer science, see [5], for some details and examples. Ideals of semirings play an important role in the structure theory of ordered semirings and useful for much purposes.

This work is structured as follows. In Chapter 1, we provide basic notions to ordered semiring, many type of ideals in ordered semirings and derivation in ordered semirings. Also, we recall some definitions, basic concepts and their fundamental properties, and we give some examples. In Chapter 2, we recall some characterizations on fuzzy interior ideals in ordered semirings. Further some of their characterizations are obtained through regularity criterion. At the end of this Chapter, some proprieties of fuzzy bi-ideals in ordered semirings are given. In chapter 3, We focus to study the properties of derivations of fuzzy ideals in an ordered semiring, in this way some results are presented.

Chapter 1

Preliminaries

The purpose of this first chapter to provide a basic introduction to crisp ideals in a ring, ordered semirings, ideals in ordered semirings and derivation in ordered semirings. Many of the definitions and properties of these concepts will be used in next chapters.

1.1 Definitions

In this section, we will give a definition for each one of the following vocabularies: semigroups, groups, subgroups, semirings, subsemirings and ideals in semiring.

1.1.1 Semigroups

Definition 1.1 *A semigroup is a set S equipped with a binary operation "$*$" that satisfies associativity, i.e., for all $a, b, c \in S$, $(a * b) * c = a * (b * c)$.*

Example 1.2 *Consider the set of positive integers with multiplication. This forms a semigroup as multiplication is associative.*

1.1.2 Groups

In the following we give definitions and examples for groups, abelian groups and subgroups.

Definition 1.3 *A group $(G, \cdot)$ is a non empty set with a binary operation "$\cdot$" satisfying the following axioms*

(i) G is closed under the operation "$\cdot$", i.e., $x \cdot y \in G$, for all $x, y \in G$;

(ii) *the operation* " $\cdot$ " *is associative, i.e.,* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, *for all* $x, y, z \in G$;

(iii) *there is an identity element* $e \in G$ *such that* $x \cdot e = e \cdot x = x$, *for all* $x \in G$;

(iv) *each element* $x \in G$ *has an inverse element* $x^{-1} \in G$ *such that* $x \cdot x^{-1} = x^{-1} \cdot x = e$.

Definition 1.4 *Let* G *be a group.* G *is called an abelian group, or a commutative group, if* $x \cdot y = y \cdot x$, *for all* $x, y \in G$.

Example 1.5 $(\mathbb{Z}, +), (\mathbb{R}^*, \times)$ *are abelian groups.*

Definition 1.6 *A non empty-sub set* H *of a group* G *is called a subgroup of* G *if* H *is itself a group with respect to the operation of* G.

In the following theorem, we present an equivalent definition of a subgroup.

Theorem 1.7 *A subset* K *of a group* G *is a subgroup of* G *if and only if the following conditions are hold*

(i) K *is a non empty;*

(ii) $x \in K$ *and* $y \in K$ *imply* $x \cdot y \in K$, *for all x,y in* K;

(iii) $x \in K$ *imply* $x^{-1} \in K$, *for all* $x \in K$.

Example 1.8 $(\mathbb{Z}, +)$ *is a subgroup of* $(\mathbb{R}, +)$

1.1.3 Rings

Now, we give definitions and examples of rings, abelian rings and subrings.

Definition 1.9 *A ring* $(R, +, \cdot)$ *is a non-empty set* R *with two binary operations* "+" *and* " $\cdot$ " *satisfying the following axioms*

(i) $(R, +)$ *is an abelian group;*

(ii) *associativity of multiplication, i.e.,* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, *for all* $x, y, z \in R$;

(iii) *multiplication is distributive with respect to addition, meaning that*

1. *$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, for all $a, b, c \in R$ (left distributivity).*

2. *$(b + c) \cdot a = (b \cdot a) + (c \cdot a)$, for all $a, b, c \in R$ (right distributivity).*

If there is an element 1 in R such that $a \cdot 1 = a$ and $1 \cdot a = a$, for all $a \in R$ then R is a unit ring (that is, 1 is the multiplicative identity).

Definition 1.10 *Let $(R, +, \cdot)$ be a ring. Then $(R, +, \cdot)$ is called abelian ring or commutative rings, if $x \cdot y = y \cdot x$, for all $x, y \in R$.*

Example 1.11 *The mathematical systems $(\mathbb{Z}, +, \times)$, $(\mathbb{Q}, +, \times)$, $(\mathbb{R}, +, \times)$ and $(\mathbb{C}, +, \times)$ are commutative ring with identity.*

Definition 1.12 *A ring R is called an integral domain if it is a commutative ring with non zero identity in which $a \cdot b = 0$ implies $a = 0$ or $b = 0$.*

Definition 1.13 *Let $(R, +, \cdot)$ be a ring. A non empty subset S of R is called a subring of R if S itself is a ring with respect to the operations of R.*

Next, we provide an equivalent definition of a subring.

Theorem 1.14 *A subset S of the ring R is a subring of R if and only if all three conditions are satisfied*

(i) S is non empty,

(ii) $(S, +)$ is a subgroup of R,

(iii) $x \cdot y \in S$, for all $x, y \in S$.

Example 1.15 *$(\mathbb{Z}, +, \times)$ is a subring of $(\mathbb{R}, +, \times)$.*

1.1.4 Ideals in a ring

Now, we give the definition and an example of ideals in a rings.

Definition 1.16 *A non empty set H of a ring R is called an ideal in R if it holds that*

(i) $(H, +)$ *is a subgroup of the group* $(R, +)$*,*

(ii) for all $a \in H$*, for all* $x \in R$*,* $a \cdot x \in H$ *and* $x \cdot a \in H$*.*

Example 1.17 *In the ring* $(\mathbb{Z}, +, \times)$*. For* $n \in \mathbb{N}$ *the set* $I = \{qn \mid q \in \mathbb{Z}\}$ *is an ideal of* $\mathbb{Z}$*.*

1.1.5 Semirings

Definition 1.18 *A semiring is a non-empty set S with two associative binary operations called addition denote* $(+)$ *and multiplication denote* $(\cdot)$*, such that*

(i) addition is a commutative operation,

(ii) multiplication distributes over addition both from the left and from the right,

(iii) there exists $0 \in S$ *such that* $x + 0 = x$ *and* $x \cdot 0 = 0 \cdot x = 0$*, for all* $x \in S$*.*

We denote $A.B = AB = \{ab : a \in A, b \in B\}$ and $A + B = \{a + b : a \in A, b \in B\}$.

Definition 1.19 *An element x of a semiring S is called a multiplicatively idempotent (an additively idempotent) element if* $xx = x$ $(x + x = x)$*.*

Example 1.20 *Let* $\mathbb{N}$ *be the set of natural numbers. Then*

1. $(\mathbb{N}, max, min)$ *is a semiring.*

2. $(\mathbb{N}, +, \cdot)$ *is a semiring.*

1.1.6 Subsemirings

Definition 1.21 *A subsemiring of a semiring H is a subset R of H such that R is itself a semiring with the same operations of S.*

Example 1.22 *Consider the set of integers* $\{\ldots, -4, -2, 0, 2, 4, \ldots\}$ *as a subsemiring of the integers.*

1.1.7 Ideals of Semiring

Definition 1.23 *An ideal of a semiring S is a subset $H \subseteq S$ satisfying*

1. *H is an additive semigroup of $(S, +)$.*
2. *For all $a \in H$ and $b \in S$, $a * b \in H$ and $b * a \in H$.*

Example 1.24 *Let $\mathbb{N}$ be the set of natural numbers. Then*

1. *$2\mathbb{N}$ is an ideal from $(\mathbb{N}, max, min)$.*
2. *$2\mathbb{N}$ is an ideal from $(\mathbb{N}, +, \cdot)$.*

Definition 1.25 *[16]*

A non-empty subset A of a semiring S is called

(i) *a sub-semiring of S if $(A, +)$ is a sub-semigroup of $(S, +)$ and $AA \subseteq A$.*

(ii) *a quasi-ideal of S if A is a sub-semiring of S and $AS \cap SA \subseteq A$.*

(iii) *a bi-ideal of S if A is a sub-semiring of S and $ASA \subseteq A$.*

(iv) *an interior ideal of S if A is a sub-semiring of S and $SAS \subseteq A$.*

(v) *a left (right) ideal of S if A is a sub-semiring of S and $SA \subseteq A$ $(AS \subseteq A)$.*

(vi) *an ideal if A is a subsemiring of $S, AS \subseteq A$ and $SA \subseteq A$.*

(vii) *a bi-interior ideal of S if A is a sub-semi ring of S and $SAS \cap ASA \subseteq A$.*

(viii) *a left bi-quasi ideal (right bi-quasi ideal) of S if A is a sub-semiring of S and $SA \cap ASA \subseteq A$ $(AS \cap ASA \subseteq A)$.*

(ix) *a left (right) tri-ideal of S if A is a subsemiring of S and $ASAA \subseteq A$ $(AASA \subseteq A)$.*

(x) *a tri-ideal of S if A is a subsemiring of S and $ASAA \subseteq A$ and $AASA \subseteq A$.*

(xi) *a left (right) weak-interior ideal of* S *if* A *is a sub-semiring of* S *and* $SAA \subseteq A$ $(AAS \subseteq A)$. *A weak-interior ideal of* S *if* A *is a subsemiring of* S *and* A *is a left weak-interior ideal and a right weak-interior ideal of* S.

(xii) *a* k*-ideal of* S *if* A *is an ideal of* S *and* $x \in S$, $x + y \in A, y \in A$ *then* $x \in A$.

(xiii) *a* $m - k$*-ideal of* S *if* A *is an ideal of* S *and* $x \in A, xy \in A, 1 \neq y \in M$ *then* $y \in A$.

1.2 Ordered semirings

Definition 1.26 *Let* P *be a non-empty set, and let* $\leq$ *be a relation on* P. $\leq$ *is a partial order in* P *if it holds*

1. *(Reflexive) For all* $x \in P$, $x \leq x$.
2. *(Antisymmetric) For all* $x, y \in P$, *if* $x \leq y$ *and* $y \leq x$, *then* $x = y$.
3. *(Transitive) For all* $x, y, z \in P$, *if* $x \leq y$ *and* $y \leq z$, *then* $x \leq z$.

Definition 1.27 *[16] A semiring* S *is called an ordered semiring if it admits a compatible relation* $\leqslant$, *i.e.,* $\leqslant$ *is a partial ordering on* S *satisfies the following conditions. If* $a \leqslant b$ *and* $c \leqslant d$, *for all* $a, b, c, d \in S$

(i) $a + c \leqslant b + d, c + a \leqslant d + b$,

(ii) $ac \leqslant bd$,

(iii) $ca \leqslant db$.

Example 1.28 *(1) Let* $S = [0, 1]$. *A binary operation* $+$ *is defined as* $a + b = \max\{a, b\}$, *for all* $a, b \in S$ *and* $x \cdot y = \min\{x, y\}$, *for all* $x, y \in S$. *Then* S *is an ordered semiring* S *with usual ordering. All ideals of* S *are closed intervals,* $[0, a]$ *for some* $a \in S$.

(2) $(\mathbb{N}, +, \cdot)$ *is an ordered semiring with the usual ordering.*

Definition 1.29 *[16] An ordered semiring* S *is said to have zero element if there exists an element* $0 \in S$ *such that* $0 + x = x = x + 0$ *and* $0 \cdot x = x \cdot 0 = 0$, *for all* $x \in S$.

Definition 1.30 *[16] An ordered semiring S is said to be commutative semiring if $xy = yx$, for all $x, y \in S$.*

Definition 1.31 *[16] A non-zero element a in an ordered semiring S is said to be a zero divisor if there exists non zero element $b \in S$, such that $ab = ba = 0$.*

Definition 1.32 *[16] An ordered semiring S with unity 1 and zero element 0 is called an integral ordered semiring if it has no zero divisors.*

Definition 1.33 *[16] A non-empty subset A of an ordered semiring S is called a sub-semiring S if $(A, +)$ is a subsemigroup of $(S, +)$ and $ab \in A$, for all $a, b \in A$.*

Definition 1.34 *In an ordered semiring S*

1. *$(M, +)$ is a positively ordered if $a + b \geq a, b$, for all $a, b \in S$.*
2. *$(M, +)$ is a negatively ordered if $a + b \leq a, b$, for all $a, b \in S$.*
3. *$(M, .)$ is a positively ordered if $ab \geq a, b$, for all $a, b \in S$.*
4. *$(M, .)$ is a negatively ordered if $ab \leq a, b$, for all $a, b \in S$..*

Definition 1.35 *An ordered semiring S is said to be totally ordered semiring if any two elements of S are comparable.*

1.2.1 Ideals of ordered semirings

Definition 1.36 *[15] A left (resp. right) ideal H of the semiring S is called a left (resp. right) ordered ideal, if for any $a \in S$, $b \in H$, $a \leq b$ implies $a \in H$. $H$is called an ordered ideal of S if it is both a left and a right ordered ideal of S.*

Example 1.37 *Let $S = ([0, 1], \vee, \cdot, 0)$ where $[0, 1]$ is the unit interval $a \vee b = \max\{a, b\}$ and $a \cdot b = (a + b - 1) \vee 0$ for $a, b \in [0, 1]$. Then it is easy to verify that S equipped with the usual ordering $\leq$ is an ordered semiring and $H = [0, \frac{1}{2}]$ is an ordered ideal of S.*

1.2.2 Derivations in ordered semirings

In this section, we introduce the notion of a derivation of an ordered semiring and study some of their properties. The must of this definitions and proprieties can be fond in [17].

Definition 1.38 *[17] Let S be an ordered semiring. A mapping $d : S \longrightarrow S$ is said to be derivation if*

(i) $d(a+b) = d(a) + d(b)$;

(ii) $d(ab) = d(a)b + ad(b)$;

(iii) If $a \leq b$ *then* $d(a) \leq d(b)$, *for all* $a, b \in S$.

Example 1.39 *Let $\mathbb{N}$ be a the set of all natural numbers.*

(i) Let $S = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} | a, b, c \in \mathbb{N} \cup \{0\} \right\}$. *$S$ is a semiring with respect to usual addition and multiplication of matrices. Define $[a_{ij}] \leq [b_{ij}]$ if and only if $a_{ij} \leq b_{ij}$, for all i, j, where $a_{ij}, b_{ij} \in S$. Then S is an ordered semiring.*
Define $d : S \longrightarrow S$ *by* $d \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$. *Then d is a derivation of S.*

(ii) Let $S = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} | a, b, c \in \mathbb{N} \cup \{0\} \right\}$. *$S$ is a semiring with respect to usual addition and multiplication of matrices. Define $[a_{ij}] \leq [b_{ij}]$ if and only if $a_{ij} \leq b_{ij}$, for all i, j, where $a_{ij}, b_{ij} \in S$. Then S is an ordered semiring.*
Define $d : S \longrightarrow S$ *by* $d \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & 2b \\ 0 & 0 \end{pmatrix}$.
Then d is not a derivation of M.

(iii) For $S = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a, b, c \in \mathbb{N} \cup \{0\} \right\}$. *$S$ is a semiring with respect to usual addition and multiplication of matrices.*
Define $[a_{ij}] \leq [b_{ij}]$ if and only if $a_{ij} \leq b_{ij}$, for all i, j, where $a_{ij}, b_{ij} \in S$. Then S is an ordered semiring.
Define $d : S \longrightarrow S$ *by* $d \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$.
Then d is not a derivation of S.

Theorem 1.40 *[17] Let S be an ordered semiring in which $(S,+)$ is a positively ordered with unity element and d be a derivation of S.*

(i) If $x \in S$ then $xd(1) \leq d(x)$;

(ii) If $d(1) = 1$ then $x \leq d(x)$.

Proof. Let S be an ordered semiring in which $(S,+)$ is positively ordered with unity element and d be a derivation of S.

(i) Let $x \in S$. Then $x1 = x$.

$$\begin{aligned} d(x) &= d(x1) \\ &= d(x)1 + xd(1) \end{aligned}$$

since $xd(1) \leq d(x1)$ then $xd(1) \leq d(x)$

(ii) Suppose $d(1) = 1$. By (i), we have $xd(1) \leq d(x)$.

$$\begin{aligned} d(1) = 1 &\Longrightarrow x1 \leq d(x) \quad (since \quad x1 = xd(1)) \\ &\Longrightarrow x \leq d(x). \end{aligned}$$

□

Theorem 1.41 *[17] Let d be a derivation of an ordered semiring S with a zero element. Then $d(0) = 0$.*

Theorem 1.42 *[17] Let S be an idempotent ordered semiring with $a + ba = a$, for all $a, b \in S$, and d be a derivation of S.*

1. *if $d(d(x)) = d(x)$ then $d(xd(x)) = d(x)$;*

2. *if $d(d(x)) = x$ then $d(d(x)d(y)) = d(xy)$, for all $x \in S$.*

Theorem 1.43 *[17] Let S be an idempotent ordered semiring in which $(M,+)$ is a positively ordered satisfying $a + ab = a$ and $a + ba = a$, for all $a, b \in S$. Then $d(x) \leq x$, for all $x \in S$.*

Theorem 1.44 *[17] Let S be an idempotent ordered semiring in which $(M,+)$ is a positively ordered with $a+ab=a$, $a+ba=a$, for all $a,b \in S$, and d be a derivation of S. Then*

1. $d(xy) = d(x)d(y)$,
2. $d(xy) \leq d(x+y)$, *for all $x,y \in S$.*

Chapter 2

Fuzzy interior ideals and fuzzy bi-ideals in ordered semirings

In this chapter, we recall definitions of fuzzy bi-ideals and fuzzy interior ideals in ordered semi-rings, also we study some of their related properties. At the end of this chapter, fuzzy interior ideals and some of their characterizations are obtained through regularity criterion. The majority of these results can be found in the paper of D. Mandal see[15] .

2.1 Definitions

Definition 2.1 *[20] Let S be a non-empty set. A mapping $f : S \to [0,1]$ is called a fuzzy subset of S.*

Example 2.2 *Let A be non-empty subset of a semiring S. The characteristic function of A is a fuzzy subset of S and it is defined by*

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.3 *[20] Let f be a fuzzy subset of a set S and $t \in [0,1]$. The set*

$$f_t = \{x \in S : f(x) \geq t\}$$

is called the t-level subset of f. Clearly, $f_t \subseteq f_s$, whenever $t \geq s$.

Definition 2.4 *[20]*

Let f and g be two fuzzy subsets of a set S.

If $f(x) \leq g(x)$, for all $x \in S$, then we say that f is a subset of g and denote by $f \subseteq g$.

Definition 2.5 *[20]*

The union and intersection of two fuzzy subsets f and g of S, denoted by $f \cup g$ and $f \cap g$ respectively, are defined by

$$(f \cup g)(x) = \max\{f(x), g(x)\} \text{ for all } x \in S.$$
$$(f \cap g)(x) = \min\{f(x), g(x)\} \text{ for all } x \in S.$$

Definition 2.6 *Let S and T be non-empty sets and $\varphi : S \to T$ be a any function. A fuzzy subset f of S is called a φ-invariant if $\varphi(x) = \varphi(y) \Rightarrow f(x) = f(y)$.*

Definition 2.7 *[15]*

Let f be a non empty fuzzy subset of an ordered semiring S (i.e., $f(x) \neq 0$ for some $x \in S$). Then, f is called a fuzzy left ideal (resp. fuzzy right ideal) of S if

(i) $f(x + y) \geq \min\{f(x), f(y)\}$;

(ii) $f(xy) \geq f(y)$ *(resp.* $f(xy) \geq f(x)$);

(iii) $x \leq y$ *implies* $f(x) \geq f(y)$;

for all $x, y \in S$.

A fuzzy ideal of an ordered semiring S is a non empty fuzzy subset of S which is a fuzzy left ideal as well as a fuzzy right ideal of S.

Example 2.8 *Let $S = \{0, x, y, z\}$ with the ordered relation $0 \prec z \prec y \prec x$. Define operations on S by following*

$\oplus$	0	x	y	z
0	0	x	y	z
x	x	x	y	z
y	y	y	y	z
z	z	z	z	z

and

$\otimes$	0	x	y	z
0	0	0	0	0
x	0	x	x	x
y	0	x	x	x
z	0	x	x	x

Then $(S, \oplus, \otimes)$ forms an ordered semiring. Now if we define a fuzzy subset f of S by $f(0) = 1$, $f(c) = 0.3$, $f(b) = 0.2$, and $f(a) = 0.1$, then f will be a fuzzy ideal of S.

Theorem 2.9 *[15] A fuzzy subset f of S is a fuzzy ordered ideal if and only if its level subset $f_t,\ t \in [0,1]$ is an ordered ideal of S.*

Proof. We proof the theorem only for left ordered ideal. For right ordered ideal, it is follows similarly. To proof this, it is sufficient to consider the case only for ordered semiring and left ordered ideal. Let f be a fuzzy left ordered ideal of S.

1. Let $x, y \in f_t$, then $f(x) \geq t$ and $f(y) \geq t$ and f since fuzzy left ordered ideal $f(x+y) \geq \min\{f(x), f(y)\} \geq t$, so $x + y \in f_t$.

2. For $x \in S, y \in f_t$, then $f(x) \geq t$ and $f(y) \geq t$ and f since fuzzy left ordered ideal $f(xy) \geq f(y) \geq t$, so $xy \in f_t$.

3. Suppose that $x \in S$ and $y \in f_t$ with $x \leq y$, since f is a fuzzy ordered left ideal of S then $f(x) \geq f(y) \geq t$, so that $x \in f_t$, i.e., f_t is a ordered left ideal of S.

Conversely, if f_t is a ordered left ideal of S.

1. For $x, y \in S$, setting $\min\{f(x), f(y)\} = t$, hence $f(x) \geq t$ and $f(y) \geq t$, so $x, y \in f_t$, since f_t is a ordered left ideal of S then $x + y \in f_t$, it follows that $f(x+y) \geq \min\{f(x), f(y)\} = t$.

2. For any $x, y \in S$, setting $f(y) = t$, hence $f(y) \geq t$, so $y \in f_t$, since f_t is a ordered left ideal of S then $xy \in f_t$, it follows that $f(xy) \geq f(y) = t$.

3. Let $x, y \in S$ and $x \leq y$, we have to show that $f(x) \geq f(y)$. By absurd suppose that $f(x) < f(y)$ then ther exists $t_1 \in [0,1]$ such that $f(x) < t_1 < f(y)$. Then $y \in f_{t_1}$ but $x \notin f_{t_1}$ which is contradiction to fact that f_{t_1} is a left ordered ideal of S. Hence the proof. □

Corollary 2.10 *[15] Let H be a subset of an ordered semiring S. Then H is a left (resp. right) ideal of S if and only if its charactiristic function χ_H is a fuzzy ordered left (resp. right) ideal of S.*

Definition 2.11 *[15] Let f be a fuzzy set of ordered semiring S and $a \in S$. We denote H_a the subset of S defined as follows*

$$H_a = \{x \in S : f(x) \geq f(a)\}.$$

Proposition 2.12 *[15] Let S be an ordered semiring and f be a fuzzy ordered right (resp. left) ideal of S. Then H_a is a right (resp. left) ideal of S for every $a \in S$.*

Proof. Let f be a fuzzy ordered right ideal of S and $a \in S$. $H_a \neq \phi$ because $a \in H_a$ for every $a \in S$. Let $x, y \in H_a$, and $z \in S$,

1. we have $f(x) \geq f(a)$ and $f(y) \geq f(a)$, it follows $f(x+y) \geq \min\{f(x), f(y)\} \geq f(a)$, ($f$ is a fuzzy right ideal) which implies $x + y \in H_a$.

2. $f(xz) \geq f(x) \geq f(a)$, it follows that $xz \in H_a$.

3. if $z \leq x$ then $f(z) \geq f(x) \geq f(a)$, i.e., $z \in H_a$.

Thus H_a is a right ideal of S.

Similarly, we can proof the result for left ideal. □

Remark 2.13 *[15] The converse of this proposition is not true which can seen by the next example.*

Example 2.14 *Let $S = \{0, a, b, c\}$ with the ordered relation $0 < c < b < a$ define the operations on S by the following Cayley tables*

$\boxplus$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	a	a
c	c	a	a	a

and

$\boxtimes$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

Then $(S, \boxplus, \boxtimes)$ forms an ordered semiring.

Now, suppose that f is a fuzzy subset of S defined by $f(0) = 1$, $f(c) = 0.3$, $f(b) = 0.2$ and $f(a) = 0.1$. Then $H_0 = \{0\}$, $H_c = \{0, c\}$, $H_b = \{0, c, b\}$ and $H_a = S$ all are right ideals of S. But f is not fuzzy ordered right ideal of S, since $f(b \boxplus c) = f(a) \geq 0.1 \ngeq 0.2 = \min\{0.2, 0.3\} = \min\{f(b), f(c)\}$.

Proposition 2.15 *[15] Intersection of a non-empty collection of fuzzy right (resp. left) ideals of an ordered semiring S is also a fuzzy right (resp. left) ideal of S.*

Proof. Let $\{f_i : i \in I\}$ be a family of fuzzy ideals of S and $x, y \in S$, so

$$\begin{aligned}(\cap_{i\in I} f_i)(x+y) &= \inf_{i\in I}(f_i(x+y)) \\ &\geq \inf_{i\in I}(\min(f_i(x), f_i(y))) \\ &= \min\left(\inf_{i\in I}(f_i(x)), \inf_{i\in I}(f_i(y))\right) \\ &= \min((\cap_{i\in I} f_i)(x), (\cap_{i\in I} f_i)(y))\end{aligned}$$

$$\begin{aligned}(\cap_{i\in I} f_i)(xy) &= \inf_{i\in I}(f_i(xy)) \\ &\geq \inf_{i\in I}(f_i(x)) \\ &= \cap_{i\in I} f_i(x)\end{aligned}$$

Suppose that $x \leq y$, then $f_i(x) \geq f_i(y)$, so $\inf_{i\in I}(f_i(x)) \geq \inf_{i\in I}(f_i(y))$, it follows that $(\cap_{i\in I} f_i)(x) \geq (\cap_{i\in I} f_i)(y)$

Thus $\cap_{i\in I} f_i$ is a fuzzy right ideal of S.

Similarly, we can prove that the result for fuzzy left ideal is true.

□

Proposition 2.16 *[15] Let $\{f_i : i \in I\}$ be a family of fuzzy right (resp. left) ideals of S such that $f_i \subseteq f_j$ or $f_j \subseteq f_i$ for $i, j \in I$. Then $\cup_{i\in I} f_i$ is a fuzzy right (resp. left) ideals of an ordered semiring S.*

Proof. Let $\{f_i : i \in I\}$ be a family of fuzzy right ideals of S and $x, y \in S$, so

$$\begin{aligned}(\cup_{i\in I} f_i)(x+y) &= \sup_{i\in I}(f_i(x+y)) \\ &= \sup_{i\in I}(\min(f_i(x), f_i(y))) \\ &\geq \min\left(\sup_{i\in I}(f_i(x)), \sup_{i\in I}(f_i(y))\right) \\ &= \min((\cup_{i\in I} f_i)(x), (\cup_{i\in I} f_i)(y))\end{aligned}$$

$$\begin{aligned}(\cup_{i\in I} f_i)(xy) &= \sup_{i\in I}(f_i(xy)) \\ &\geq \sup_{i\in I}(f_i(x)) \\ &= \cup_{i\in I} f_i(x)\end{aligned}$$

Suppose that $x \leq y$, then $f_i(x) \geq f_i(y)$, so $\sup_{i\in I}(f_i(x)) \geq \sup_{i\in I}(f_i(y))$, it follows that $(\cup_{i\in I} f_i)(x) \geq (\cup_{i\in I} f_i)(y)$

Thus $\cup_{i\in I} f_i$ is a fuzzy right ideal of S.

Similarly, we can prove that the result for fuzzy left ideal is true.

□

2.2 Fuzzy interior ideals

Definition 2.17 *[15] Let f be a non empty fuzzy subset of an ordered semiring S (i.e., $f(x) \neq 0$ for some $x \in S$). Then, f is called a fuzzy interior ideal of S if*

(i1) $f(x+y) \geq \min\{f(x), f(y)\}$;

(i2) $f(xy) \geq \min\{f(x), f(y)\}$;

(i3) $f(xyz) \geq f(y)$;

(i4) $x \leq y$ *implies* $f(x) \geq f(y)$.

for all $x, y, z \in S$.

Example 2.18 *The fuzzy subset defined in Example 2.8 is also an example of interior fuzzy ideal.*

Example 2.19 *Let $S = \{0, a, b, c\}$ with the ordered relation $0 < a < b < c$ define the operations on S by the following Cayley tables*

$\boxplus$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

and

$\boxtimes$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	b	c

Then $(S, \boxplus, \boxtimes)$ forms an ordered semiring with the ordering relation defined by $x \leq y$ if and only if $x \boxplus y = y$ and $x \boxtimes y = x$. Let f be the fuzzy subset of S defined by $f(0) = 1$, $f(a) = 0.8$, $f(b) = 0.6$ and $f(c) = 0.3$. Then, f is a fuzzy interior ideals of S.

Theorem 2.20 *[15] A fuzzy subset f of S is a fuzzy interior ideal if and only if its level subset f_t, $t \in [0, 1]$ is an interior ideal of S.*

Proof. The proof is similar to Theorem 2.9. □

Corollary 2.21 *[15] Let H be a subset of an ordered semiring S. Then H is an interior ideal of S if and only if its charactiristic function χ_H is a fuzzy interior ideal of S.*

2.3 Fuzzy interior ideals and regularity criterion

Definition 2.22 *[15] An ordered semiring S is called regular (resp intra-regular) if for each $a \in S$, there exists $x, y \in S$ such that $a \leq axa$ (resp $a \leq xa^2y$).*

Proposition 2.23 *[15] Every fuzzy interior ideal of a regular ordered semiring S is a fuzzy ideal.*

Proof. Assume that f is a fuzzy interior ideals of a regular ordered semiring S and $a, b \in S$. It sufficient to prove that $f(ab) \geq f(a)$ and $f(ab) \geq f(b)$.

Since S is regular then there exists $x \in S$ such that $a \leq axa$, so we have

$$
\begin{aligned}
f(ab) &\geq f((axa)b) \text{ (since } ab \leq (axa)b \text{ and } f \text{ is an ordered ideal)} \\
&\geq f((ax)ab) \\
&\geq f(a) \text{ (since } f \text{ is an fuzzy interior ideal)}
\end{aligned}
$$

Similarly, since S is regular then there exists $y \in S$ such that $b \leq byb$, so we have

$$
\begin{aligned}
f(ab) &\geq f(a(byb)) \text{ (since } ab \leq a(byb) \text{ and } f \text{ is an ordered ideal)} \\
&\geq f(ab(yb)) \\
&\geq f(b) \text{ (since } f \text{ is an fuzzy interior ideal)}
\end{aligned}
$$

Therefore f is an fuzzy interior ideal. □

Proposition 2.24 *[15] Every fuzzy interior ideal of an intra-regular ordered semiring S is also a fuzzy ideal.*

Proof. Suppose that f is a fuzzy interior ideals of an intra-regular ordered semiring S and $a, b \in S$. It is sufficient to prove that $f(ab) \geq f(a)$ and $f(ab) \geq f(b)$.

Since S is intra-regular then there exists $x, y \in S$ such that $a \leq xa^2y$, so we have

$$f(ab) \geq f\left(xa^2yb\right) \left(\text{since } ab \leq xa^2yb \text{ and } f \text{ is an ordered ideal}\right)$$

$$\geq f(xa)a(yb))$$

$$\geq f(a)\,(\text{since } f \text{ is an fuzzy interior ideal})$$

Similarly, since S is regular then there exists $x, y \in S$ such that $b \leq xb^2y$ so we have

$$f(ab) \geq f\left(axb^2y\right) \left(\text{since } ab \leq axb^2y) \text{ and } f \text{ is an ordered ideal}\right)$$

$$\geq f((ax)b(by))$$

$$\geq f(b)\,(\text{since } f \text{ is an fuzzy interior ideal})$$

Therefore f is an fuzzy interior ideal. □

2.4 Bi-ideals in ordered semiring

Definition 2.25 *[14] Let f be a non empty fuzzy subset of an ordered semiring S. Then, f is called a fuzzy interior idea of S if it satisfies*

(bi1) $f(x+y) \geq \min\{f(x), f(y)\}$;

(bi2) $f(xy) \geq \min\{f(x), f(y)\}$;

(bi3) $f(xyz) \geq \min\{f(x), f(z)\}$;

(bi4) $x \leq y$ *implies* $f(x) \geq f(y)$.

for all $x, y, z \in S$.

Example 2.26 *Let $S = \{0, a, b, c\}$ with the ordered relation $0 < c < b < a$ define the operations on S by the following Cayley tables*

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

and

$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	b	c

Then $(S, \oplus, \odot)$ is an ordered semiring. Let f be the fuzzy subset of S defined by $f(0) = 1$, $f(a) = 0.3$, $f(b) = 0.2$ and $f(c) = 0.1$. Then, f is a fuzzy bi-ideal of S.

Proposition 2.27 *[14] Intersection of a non-empty collection of fuzzy bi-ideals of an ordered semiring S is also a fuzzy bi-ideal of S.*

Proof. Let $\{f_i : i \in I\}$ be a family of fuzzy bi-ideals of S and $x, y \in S$, so

$$\begin{aligned}(\cap_{i\in I} f_i)(x+y) &= \inf_{i\in I}(f_i(x+y)) \\ &\geq \inf_{i\in I}(\min(f_i(x), f_i(y))) \\ &= \min\left(\inf_{i\in I}(f_i(x)), \inf_{i\in I}(f_i(y))\right) \\ &= \min((\cap_{i\in I} f_i)(x), (\cap_{i\in I} f_i)(y))\end{aligned}$$

and

$$\begin{aligned}(\cap_{i\in I} f_i)(xy) &= \inf_{i\in I}(f_i(xy)) \\ &\geq \inf_{i\in I}(\min(f_i(x), f_i(y))) \\ &= \min\left(\inf_{i\in I}(f_i(x)), \inf_{i\in I}(f_i(y))\right) \\ &= \min((\cap_{i\in I} f_i)(x), (\cap_{i\in I} f_i)(y)) \\ (\cap_{i\in I} f_i)(xyz) &= \inf_{i\in I}(f_i(xyz)) \\ &\geq \inf_{i\in I}(\min(f_i(x), f_i(z))) \\ &= \min\left(\inf_{i\in I}(f_i(x)), \inf_{i\in I}(f_i(z))\right) \\ &= \min((\cap_{i\in I} f_i)(x), (\cap_{i\in I} f_i)(z))\end{aligned}$$

Suppose that $x \leq y$, then $f_i(x) \geq f_i(y)$, so $\inf_{i\in I}(f_i(x)) \geq \inf_{i\in I}(f_i(y))$, it follows that $(\cap_{i\in I} f_i)(x) \geq (\cap_{i\in I} f_i)(y)$

Thus $\cap_{i\in I} f_i$ is a fuzzy bi-ideal of S.

□

Chapter 3

Derivations of fuzzy ideals in ordered semiring

In this chapter, we discuss some properties of derivations of fuzzy ideals in an ordered semiring based on the paper of "M. K. M. Rao and R. K. Kona" titeled "Derivations of fuzzy ideals in ordered semirings, for more details see [17].

Definition 3.1 *[17] Let f be a non-empty fuzzy subset of an ordered semiring S. Then f is said to be a fuzzy prime ideal of S if it satisfies*

(i) $f(x+y) \geq min\{f(x), f(y)\}$;

(ii) $f(xy) = max\{f(x), f(y)\}$;

(iii) $x \leq y \Longrightarrow f(x) \geq f(y)$, *for all* $x, y \in S$.

Definition 3.2 *[17] An ideal H of an ordered semiring S is said to be characteristic if $d(H) = H$, for all derivations d of S.*

Definition 3.3 *[17] A fuzzy ideal f of an ordered semiring S is called a fuzzy ideal characteristic if $f(d(x)) = f(x)$, for all derivations d of S and $x \in S$.*

Theorem 3.4 *[17]*

Let f be a fuzzy ideal and d be a derivation of an idempotent ordered semiring S with $a + ab = a$ and $a + ba = a$, for all $a, b \in S$. Then the mapping $f^d : S \longrightarrow [0, 1]$, defined by $f^d(x) = f(d(x))$, for all $x \in M$, is a fuzzy ideal of S.

Proof. Let f be a fuzzy ideal, d be a derivation of an idempotent ordered semiring S with $a+ab=a$ and $a+ba=a$, for all $a,b\in S$. Suppose $x,y\in S$. Then

$$\begin{aligned} f^d(x+y) &= f(d(x+y)) \\ &= f(d(x)+d(y)) \\ &\geq min\{f(d(x)), f(d(y))\} \\ &= min\left\{f^d(x), f^d(y)\right\} \end{aligned}$$
$$\begin{aligned} f^d(xy) &= f(d(xy)) \\ &= f(d(x)d(y)) \text{ by Theorem 1.44} \\ &\geq \max\{f(d(x)), f(d(y))\} \\ &= \max\left\{f^d(x), f^d(y)\right\}. \end{aligned}$$

Let $x,y\in S$ and $x\leq y$. Then

$$\begin{aligned} x\leq y &\Longrightarrow d(x)\leq d(y) \\ &\Longrightarrow f(d(x))\geq f(d(y)) \\ &\Longrightarrow f^d(x)\geq f^d(y). \end{aligned}$$

Hence f^d is a fuzzy ideal of S. □

Theorem 3.5 *[17] Let f be a fuzzy ideal and d be an onto derivation of an idempotent ordered semiring S with identity $a+ab=a$, for all $a,b\in S$. Then f is a fuzzy ideal characteristic if and only if each level ideal f of S is a characteristic ideal.*

Proof. Suppose f is a fuzzy ideal characteristic and $x\in f_s$ where $s\in[0,1]$ and d is a derivation of S. Then

$$\begin{aligned} f(x)\geq s &\Longrightarrow f((d(x))\geq s \\ &\Longrightarrow d(x)\in f_s \\ &\Longrightarrow d(f_s)\subseteq f. \end{aligned}$$

Let $x\in f_s$. Then there exists $y\in S$ such that $d(y)=x$.

$d(y)=x \Longrightarrow (f(d(y))=f(x)\geq s \text{ and } f(y)\geq s) \Longrightarrow y\in f_s \Longrightarrow (d(y)\in f_s \text{ and } x=d(y)\in d(f_s))$. Therefore $f_s\subseteq d(f_s)$.

Hence $f_s = d(f_s)$.

Conversely, suppose that each level ideal f_s of S is a characteristic, $x \in S$, d is an onto derivation of M and $f(x) = s$.

Then $x \in f_s$ and $x \notin f_t$ for $s < t \Longrightarrow d(x) \in d(f_s) = f_s$.

By Theorem 3.4, $f^d : M \longrightarrow [0,1]$ defined by $f^d(x) = f(d(x))$, is an fuzzy ideal.

Let $w = f^d(x)$ and $w > s$. Then $d(x) \in f_w = d(f_w) \Longrightarrow x \in f_w$.

Which is a contradiction, since $w > s$.

Therefore $f^d(x) = f(d(x)) = s = f(x)$.

Hence f is a fuzzy ideal characteristic. $\square$

Definition 3.6 *[17] Let d be a derivation of an ordered semiring S and f be a fuzzy subset of S. We define a fuzzy subset $d(f)$ of S by*

$$d(f)(x) = \begin{cases} \sup\limits_{y \in d^{-1}(x)} f(y), & \text{if } d^{-1}(x) \neq \phi \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.7 *[17] Let d be an onto derivation of an ordered semiring S. If f is a fuzzy subset of S then the preimage of f under d is the fuzzy subset of S defined by $d^{-1}(f)(x) = f(d(x))$, for all $x \in S$ and it is denoted by $d^{-1}(f)$.*

Theorem 3.8 *[17] Let d be a derivation of an ordered semiring S and f be a d-invariant fuzzy subset of S. If $x = d(a)$ then $d(f)(x) = f(a)$, for all $a \in S$.*

Proof. Let d be a derivation of an ordered semiring S and f be a d-invariant fuzzy subset of S. Suppose $x = d(a)$. Then $d^{-1}(x) = a$. Let $t \in d^{-1}(x)$. Then $x = d(t) \Longrightarrow d(a) = x = d(t)$. Since f is a d-invariant fuzzy subset of S so $f(a) = f(t)$. Therefore $d(f)(x) = sup_{t \in d^{-1}(x)} f(t) = f(a)$. $\square$

Theorem 3.9 *[17] Let d be a derivation of an ordered idempotent semiring S and f be a fuzzy ideal of S. If $f \circ d = \eta$ then η is a fuzzy ideal of an ordered semiring S.*

Proof. Let $x, y \in S$. Then

$$\begin{aligned}\eta(x+y) &= f(d(x+y)) \\ &= f(d(x)+d(y)) \\ &\geq min\{f(d(x)), f(d(y))\} \\ &= min\{\eta(x), \eta(y)\}\end{aligned}$$

and

$$\begin{aligned}\eta(xy) &= f(d(xy)) \\ &= f[d(x)d(y)] \\ &\geq max\{f(d(x)), f(d(y))\} \\ &= max\{\eta(x), \eta(y)\}.\end{aligned}$$

Suppose $x, y \in S$ and $x \leq y$. Since d is a derivation of S, we have

$$\begin{aligned}x \leq y &\Longrightarrow d(x) \leq d(y) \\ &\Longrightarrow f(d(x)) \geq f(d(y)) \\ &\Longrightarrow \eta(x) \geq \eta(y).\end{aligned}$$

Hence η is a fuzzy ideal of S. □

Theorem 3.10 *[17] Let S be an idempotent ordered semiring with identity $a + ab = a$, for all $a, b \in S$, and $d : S \longrightarrow S$ be an onto derivation of S. If f is a d-invariant fuzzy prime ideal of S then $d(f)$ is a fuzzy prime ideal of S.*

Proof. Let S be an idempotent ordered semiring with $a + ab = a$, for all $a, b \in S$.

Suppose $d(a) = x$. By Theorem 3.8, we have $d(f)(x) = f(a)$. Let $x, y \in S$. Then there exist $a, b \in S$ such that $d(a) = x, d(b) = y$.

Then $d(a + b) = d(a) + d(b) = x + y$ which implies $d(f)(x + y) = f(a + b) \geq min\{f(a), f(b)\} = min\{d(f)(x), d(f)(y)\}$. By Theorem 1.44 $d(ab) = d(a)d(b) = xy$. Then $d(f)(xy) = sup_{t \in d^{-1}(x)} f(t) = f(ab) = max\{f(a), f(b)\} = max\{d(f(x)), d(f(y))\}$.

Let $x, y \in S$ and $x \leq y$. Then there exist $a, b \in S$ such that $d(a) = x$ and $d(b) = y$.

$$
\begin{aligned}
x \leq y &\Longrightarrow d(a) \leq d(b) \\
&\Longrightarrow f(d(a)) \geq f(d(b)) \\
&\Longrightarrow d(f)(x) \geq d(f)(y).
\end{aligned}
$$

Hence $d(f)$ is a fuzzy ideal of S. □

Theorem 3.11 *[17] If f is a fuzzy prime ideal and d is a derivation of an idempotent ordered semiring S with $a + ab = a$, for all $a, b \in S$, then $d^{-1}(f)$ is a fuzzy prime ideal of S.*

Proof. Suppose d is a derivation of an idempotent ordered semiring S, f is a fuzzy ideal of S and x_1, $x_2 \in S$.

$$
\begin{aligned}
d^{-1}(f)(x_1 + x_2) &= f(d(x_1 + x_2)) \\
&= f\left[d(x_1) + d(x_2)\right] \\
&\geq min\left\{f(d(x_1) + fd(x_2)\right\} \\
&= min\left\{d^{-1}(f(x_1)), d^{-1}(f(x_2))\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
d^{-1}(f)(x_1 x_2) &= f(d(x_1 x_2)) \\
&= f(d(x_1)d(x_2)) \\
&= max\left\{f(d(x_1)), f(d(x_2))\right\} \\
&= max\left\{d^{-1}(f(x_1)), d^{-1}(f(x_2))\right\}
\end{aligned}
$$

Let $x_1, x_2 \in S$ and $x_1 \leq x_2$. Then

$$
\begin{aligned}
x_1 \leq x_2 &\Longrightarrow d(x_1) \leq d(x_2) \\
&\Longrightarrow f(d(x_1)) \geq f(d(x_2)) \\
&\Longrightarrow d^{-1}(f(x_1)) \geq d^{-1}(f(x_2)).
\end{aligned}
$$

Therefore $d^{-1}(f)$ is a fuzzy prime ideal of S.

□

Conclusion

In conclusion, this work has provided a comprehensive exploration of several fundamental concepts in the study of ordered semirings. The first chapter laid the groundwork by presenting key definitions and properties that are essential for grasping the more complex ideas discussed later.

Second chapter specifically focusing on fuzzy bi-ideals and fuzzy interior ideals within ordered semirings. By recalling and expanding upon the definitions of these concepts, we were able to explore their intrinsic properties. Notably, this chapter culminated in the characterization of fuzzy interior ideals through a regularity criterion, thereby contributing to a deeper understanding of their structural features and significance.

Finally, we disscus the notion of derivation of fuzzy ideals. This section built upon the prior discussions by integrating the concept of derivation, which is crucial for practical applications in this area. By exploring the coniction between fuzzy ideals and derivation, we have added a new dimension to the study of ordered semirings.

Overall, this work have systematically addressed the key aspects of fuzzy ideals in ordered semirings, providing valuable insights and characterizations that can serve as a foundation for further research. Future research could further investigate additional properties and applications of fuzzy ideals and their derivations, continuing to expand the knowledge and impact of this study.

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