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ORIGINAL PAPER

Lipschitz *p*-lattice summing operators

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Abstract

In this paper, we introduce and study the notion of Lipschitz p-lattice summing operators in the category of Lipschitz operators which generalizes the class of p-lattice summing operators in the linear case. Some interesting properties are given. Also, some connections with other classes of operators are presented.

Keywords Lipschitz p-summing operators · p-Lattice summing operators · Concave and convex operators · Order bounded operators

Mathematics Subject Classification 46B28 · 46T99 · 47H99 · 47L20

1 Introduction

The notion of Lipschitz p-summing operators was introduced and studied by Farmer and Johnson [14]. They showed that it is really a good generalization of the concept of linear p-summing operators [14, Theorem 2]. This class of Lipschitz operators marked the beginning of the theory of nonlinear summability. Motivated by the importance of this theory, several authors, have developed and studied many concepts relating to summability. Chen and Zheng introduced in [10] (strongly) Lipschitz p-integral and p-nuclear operators. In [6] Chávez-Domínguez introduced

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the notion of Lipschitz (r, p, q)-summing operators and Lipschitz (q, p)-mixing in [7]. Some papers in this domain are due to Yahi, Achour and Rueda [31] and Saadi [27]. Independently, they introduced and studied the class of Lipschitz strongly p-summing operators. The first authors introduced also summing Lipschitz conjugates and (p, σ)-summability with an appropriate factorization. They characterized also those Lipschitz operators whose Lipschitz conjugates are absolutely p-summing. Other Lipschitz versions of different types of summability of linear operators were investigated also by many authors such as, [9, 22].

In [32], Yannovskii introduced and studied the notion of 1-lattice summing linear operators. Nielson and Szulga in [24] and [29], extended this notion to p $(1 \leq p \leq \infty)$. It was extended later in [21], to the theory of operator spaces (or the non-commutative case) and to sublinear operators in [3]. In the present work, we generalize this class of linear operators to Lipschitz operators. We give some characterizations and properties. We end our paper by studying some relations between other summabilities.

The paper is organized as follows. In Sect. 2, we recall some basic definitions and properties concerning the Lipschitz dual and many concepts relating to summability. In Sect. 3, we study the notion of Lipschitz p-lattice summing operators between a pointed metric space X and a Banach lattice E, which extends the class of p-lattice summing operators. This generalization is a natural nonlinear analogous of the class of p-lattice summing linear operators. We give some properties concerning this notion and we prove certain characterizations of this type of operators. Finally in Sect. 4, we try to give some properties in connection with the category of Lipschitz p-summing operators.

2 Notations, definitions and properties

Unless otherwise stated X, Y, Z will always denote pointed metric spaces, i.e., with a distinguished element at all time denoted by 0, whereas E, F, G will denote real Banach spaces. As customary, $\mathcal{B}_E$ will denote the closed unit ball of E, E^* its topological linear dual and $\mathcal{B}(E, F)$ is the linear space of bounded linear maps from E to F. The space $\mathrm{Lip}_0(X, E)$ is the Banach space of Lipschitz functions $T : X \to E$ such that $T(0) = 0$ with pointwise addition and the Lipschitz norm $\mathrm{Lip}(.)$ given by

$$\mathrm{Lip}(T) = {}_{x \neq y}\left\{\frac{\|T(x) - T(y)\|}{d(x, y)}\right\}.$$

We denote the Lipschitz dual of X by $X^{\#} = \mathrm{Lip}_0(X) = \mathrm{Lip}_0(X, \mathbb{R})$. The closed unit ball $\mathcal{B}_{X^{\#}}$ of $X^{\#}$ is a compact Hausdorff space for the topology of pointwise convergence on X. A molecule on X is a real valued function m on X with finite support and satisfying $\sum_{x \in \mathrm{supp(m)}} m(x) = 0$. Denote by $\mathcal{M}(X)$ the real linear space of molecules on X. The condition $\sum_{i=1}^{n} m(x_i) = 0$ ensures that m can be represented by $m = \sum_{i=1}^{n} \alpha_i m_{x_i y_i}$, where $m_{x_i y_i} = \mathbf{1}_{\{x_i\}} - \mathbf{1}_{\{y_i\}}$ ($\mathbf{1}_{\{x\}}$ is the indicator function of the subset $\{x\}$ of X). Put $\|m\|_{\mathcal{M}(X)} = \inf\left\{\sum_{i=1}^{n} |\alpha_i| d(x_i, y_i)\right\}$, where the infimum is taken over all representations of $m = \sum_{i=1}^{n} \alpha_i \left(\mathbf{1}_{\{x_i\}} - \mathbf{1}_{\{y_i\}}\right)$. It follows that $\|.\|_{\mathcal{M}(X)}$ is a norm on the vector space $\mathcal{M}(X)$. Denote by Æ(X) the completion of the normed space $(\mathcal{M}(X), \|.\|_{\mathcal{M}(X)})$. This space was first introduced by Arens and Eells in 1956 [1]. Originally, the basic idea goes back to Kantorovich [18]. The terminology Arens-Eells space Æ(X) is due to Weaver [30]. The application $i_X : X \longrightarrow$ Æ (X) defined by

$$i_X(x) = m_{x0} = \mathbf{1}_{\{x\}} - \mathbf{1}_{\{0\}} \tag{1}$$

is an isometric embedding of X into Æ(X). Also the family $\left(\mathbf{1}_{\{x\}} - \mathbf{1}_{\{0\}}\right)_{x \in X}$ is a Hamel basis in $\mathcal{M}(X)$. For more details on the basic theory of the spaces of Lipschitz operators and their preduals, one can consult for example the papers of Godefroy and Kalton [15] or Weaver [30].

The following theorem due to [30, Theorem 2.2] is known as the linearization of Lipschitz operators.

Theorem 2.1 *Let $T \in Lip_0(X, E)$. Then there is a unique bounded linear operator $T_{\mathrm{L}} \in \mathcal{B}(Æ(X), E)$ such that $T = T_{\mathrm{L}} \circ i_X$ and $\|T_{\mathrm{L}}\| = \mathrm{Lip}(T)$ ($i_X : X \longrightarrow Æ(X)$ is defined as in (1)).*

The linear operator T_{L} is called the linearization of T. We know from the above that every molecule m is uniquely expressible in the form

$$m = \sum_{i=1}^{n} \alpha_i m_{x_i 0}$$

where the points x_i are all distinct and none equals to 0. Then, T_{L} is defined by

$$T_{\mathrm{L}}(m) = \sum_{i=1}^{n} \alpha_i T(x_i). \tag{2}$$

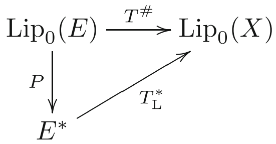
The dual E^* of E is naturally isometric to a subspace of $\mathrm{Lip}_0(E)$. By [4, p. 37], there is a norm one projection $P : \mathrm{Lip}_0(E) \longrightarrow E^*$. Sawashima in [28] defined the Lipschitz adjoint (or dual) $T^{\#} : \mathrm{Lip}_0(E) \longrightarrow \mathrm{Lip}_0(X)$ of a Lipschitz map $T \in \mathrm{Lip}_0(X, E)$ by the formula

$$T^{\#}(f) = f \circ T, \qquad f \in \mathrm{Lip}_0(E).$$

He showed that $T^{\#}$ is a continuous linear operator and

$$\left\|T^{\#}\right\| = \mathrm{Lip}(T) = \left\|T^{\#}\mid_{E^*}\right\|.$$

We notice that $T^t = T^{\#}\mid_{E^*}$ corresponds in a canonical way to the usual adjoint of the linear operator attached to T by Theorem 2.1, i.e., $T^{\#}\mid_{E^*} = T_{\mathrm{L}}^*$.



We recall now some standard notations and some properties concerning Banach lattices. We denote by $\|.\|_p$ the norm on l_p of a sequence of real numbers. For a sequence of vectors $(a_i)_i$ in a Banach space E, its strong p-norm is the l_p-norm of the sequence $(\|a_i\|)_i$ and we denote its weak p-norm (cf. [11]) by

$$\omega_p\big((a_i)_i\big) = \sup_{a^* \in \mathcal{B}_{E^*}} \big\|(a^*(a_i))_i\big\|_p.$$

We denote respectively these spaces by $l_p(E)$ and $l_p^{\omega}(E)$ ($l_p^n(E)$ and $l_p^{n\omega}(E)$ if we take finite sequences $(x_i)_{1 \le i \le n} \subset E$). Analogously for a sequence $(\lambda_i)_i$ of real numbers and $(x_i)_i$, $(y_i)_i$ of points in a metric space X, we denote their "*weak Lipschitz p-norm*" by

$$\omega_p^{\mathrm{L}}((\lambda_i),(x_i),(y_i)) = \sup_{f \in \mathcal{B}_{X^\#}} \big\|(\lambda_i(f(x_i) - f(y_i)))_i\big\|_p.$$

Let n be a natural number. For a Banach lattice E and $1 \le p \le \infty$, we let $E\left(l_p^n\right)$ the space of sequences $a = (a_1, \ldots, a_n) \in E^n$ for which

$$\|a\|_{E\left(l_p^n\right)} = \left\| \left(\sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}} \right\|, \quad \text{if } 1 \le p \le \infty$$

and

$$\|a\|_{E\left(l_\infty^n\right)} = \left\| \max_{1 \le i \le n} |a_i| \right\|, \quad \text{if } p = \infty.$$

The space $E\left(l_p^n\right)$ equipped with the natural order

$$a \le b \Longleftrightarrow a_i \le b_i, \quad \text{for all } 1 \le i \le n$$

is a Banach lattice.

Let E be a Banach lattice then E^* is a complete Banach lattice w.r.t the ordering $\le$ defined by

$$a^* \ge 0 \Longleftrightarrow \langle a^*, a \rangle \ge 0 \quad \text{for all } a \in E_+^*. \tag{3}$$

Definition 2.2 Let E, F be two Banach lattices. A linear operator $T : E \longrightarrow F$ is called positive ($T \ge 0$) if

$$|T(a)| \leq T(|a|) \quad \text{for all } a \in E.$$

We denote by $\mathcal{B}(E,F)_+$ the set of all bounded positive operators.

Inspired by the useful concept of absolutely summing operators, Farmer and Johnson introduced in [14] the following definition: a Lipschitz map $T : X \to E$ is called Lipschitz p-summing ($1 \leq p < \infty$) if there exists a positive constant C such that for all $n \in \mathbb{N}$ and all $x_1, ..., x_n, y_1, ..., y_n \in X$ and all non negative reals $\lambda_1, ..., \lambda_n$ we have

$$\left\| \left(\lambda_i^{\frac{1}{p}} (T(x_i) - T(y_i)) \right) \right\|_p \leq C \omega_p^{\mathrm{L}} \left(\left(\lambda_i^{\frac{1}{p}} \right)_i, (x_i)_i, (y_i)_i \right). \tag{4}$$

The infimum of such constants verifying (4) is denoted by $\pi_p^L(T)$. This is a true generalization of the concept of linear p-summing operators, since it is shown in [14, Theorem 2] that the Lipschitz p-summing norm of a linear operator is the same as its p-summing norm. In the sequel, it will be useful to note that the above definition is the same if we restrict to $\lambda_i = 1$ (see [14] for an implicit proof).

We shall give another equivalent definition in terms of Lipschitz tensor product. This notion of tensor product between metric and Banach spaces; which is a nonlinear generalization of linear tensor product, was introduced by Cabrera-Padilla et al. in [5]. The Lipschitz tensor product $X \boxtimes E$ is defined as the vector subspace of $\mathrm{Lip}_0(X, E^*)'$ spanned by the set

$$\left\{ \delta_{(x,y)} \boxtimes a : (x,y) \in X^2,\ a \in E \right\}$$

where $\left(\delta_{(x,y)} \boxtimes a\right)(g) = \langle g(x) - g(y), a \rangle$, for any $g \in \mathrm{Lip}\ _0(X, E^*)$ (whither $\delta_{(x,y)}(g) = g(x) - g(y)$). The Lipschitz injective norm on $X \boxtimes E$, is defined for $u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes a_i$ by

$$\epsilon(u) = \sup \left\{ \left| \sum_{i=1}^n (f(x_i) - f(y_i)) \langle \xi, a_i \rangle \right| : f \in \mathcal{B}_{X^\#},\ \xi \in E^* \right\}.$$

Remark 2.3 A Lipschitz map $T : X \to E$ is Lipschitz p-summing ($1 \leq p < \infty$) if there exists a positive constant C such that for all $n \in \mathbb{N}$ and for all $x_1, ..., x_n,\ y_1, ..., y_n \in X$, we have for all linear operator $v : l_{p^*}^n \longrightarrow \text{Æ}(X)$, $v(e_i) = \mathbf{1}_{\{x_i\}} - \mathbf{1}_{\{y_i\}}$ (namely $v = \sum_{i=1}^n \left(\mathbf{1}_{\{x_i\}} - \mathbf{1}_{\{y_i\}} \right) \boxtimes a_i \in X \boxtimes l_p^n$)

$$\left(\sum_{i=1}^n \| T_{\mathrm{L}}(v(e_i)) \|^p \right)^{\frac{1}{p}} \leq C \|v\|$$

where e_i denotes the unit vector basis of $l_{p^*}^n$ and p^* denotes the exponent conjugate to p (i.e., the one that satisfies $1/p + 1/p^* = 1$).

We have

$$\begin{aligned}\epsilon(v) = &\sup\left\{\left|\sum_{i=1}^{n}(f(x_i)-f(y_i))\langle\xi, a_i\rangle\right| : f\in\mathcal{B}_{X^{\#}},\ \xi\in l_{p^*}^{n}\right\}\\ =&\sup\left\{\left|\sum_{i=1}^{n}\langle\xi, (f(x_i)-f(y_i))a_i\rangle\right| : f\in\mathcal{B}_{X^{\#}},\ \xi\in l_{p^*}^{n}\right\}\\ =&\sup\left\{\left|\left\langle\xi, \sum_{i=1}^{n}(f(x_i)-f(y_i))a_i\right\rangle\right| : f\in\mathcal{B}_{X^{\#}},\ \xi\in l_{p^*}^{n}\right\}\\ =&\sup\left\{\left(\sum_{i=1}^{n}|(f(x_i)-f(y_i))|^p\right)^{\frac{1}{p}} : f\in\mathcal{B}_{X^{\#}}.\right\}\end{aligned} \tag{5}$$

$$\|v\| = \epsilon(v) = \sup_{\|f\|_{X^{\#}}=1}\left(\sum_{i=1}^{n}|f(x_i)-f(y_i)|^p\right)^{\frac{1}{p}}.$$

Remark 2.3 is equivalent to

$$\left(\sum_{i=1}^{n}\|T(x_i)-T(y_i)\|^p\right)^{\frac{1}{p}} \le C\|w\|$$

where $w : X^{\#} \longrightarrow l_p^n$ be the linear operator defined by

$$w(f) = (f(x_i)-f(y_i))_{1\le i\le n}.$$

The following definition is due to [8].

Definition 2.4 Let X be a arbitrary pointed metric space and let E be a Banach lattice. Consider $1\le p\le\infty$.

(i) A Lipschitz map $T : X \longrightarrow E$ is called Lipschitz p-convex if there exists a positive constant C such that for every $n\in\mathbb{N}$, for all $x_1, ..., x_n;\ y_1, ..., y_n \in X$ and all non negative reals $\lambda_1, ..., \lambda_n$, we have

$$\left\|\left(\lambda_i^{\frac{1}{p}}(T(x_i)-T(y_i))\right)\right\|_{E\left(l_p^n\right)} \le C\left(\sum_{i=1}^{n}\lambda_i d(x_i, y_i)^p\right)^{\frac{1}{p}}$$

for $p<\infty$ and we take the maximum for $p=\infty$.

(ii) A Lipschitz map $T : E \longrightarrow X$ is called Lipschitz p-concave if there exists a positive constant C such that for every $n\in\mathbb{N}$, for all $a_1, ..., a_n;\ a'_1, ..., a'_n \in E$ and all non negative reals $\lambda_1, ..., \lambda_n$, we have

$$\left(\sum_{i=1}^{n}\lambda_i d\left(T(a_i), T\left(a'_i\right)\right)^p\right)^{\frac{1}{p}} \le C\left\|\left(\lambda_i^{\frac{1}{p}}\left(a_i-a'_i\right)\right)\right\|_{E\left(l_p^n\right)}$$

for $p<\infty$. For the case $p=\infty$, the sum should be replaced by max.

The smallest such constant C is called the Lipschitz p-convexity constant of T (resp. Lipschitz p-concavity constant of T) and is denoted by $C_p^{\mathrm{L}}(T)$ (resp. $M_p^{\mathrm{L}}(T)$). We shall also denote by $C_p^{\mathrm{L}}(X,E)$ and $M_p^{\mathrm{L}}(E,X)$ the sets of Lipschitz p-convex and Lipschitz p-concave operators respectively.

Note that this is a generalization of the linear case: a linear map $T : X \to E$ from a Banach space X to a Banach lattice E is p-convex if, and only if, it is Lipschitz p-convex with the same constant. The same for Lipschitz p-concave. Because the unit ball $\mathcal{B}_{X^\#}$ is not involved. We know also in [8, Theorem 3.3] that a Lipschitz map $T : X \to E$ is Lipschitz p-convex if, and only if, $T_L : Æ(X) \to E$ is p-convex. Moreover, in this case the p-convexity constants are the same. If X is a Banach space, then $C_p(X) = C_p(\mathrm{Id}_X)$ and $M_p(X) = M_p(\mathrm{Id}_X)$.

3 Lipschitz p-lattice summing operators

The following definition was introduced and studied in linear case by Yanovskii in [32] for $p = 1$ and generalized by Nielsen and Szulga in [24] and [29] for $p > 1$. In this paper we extend this notion and some results to the theory of Lipschitz operators. Recall that a linear operator T from a Banach space X to a Banach lattice E is called p-lattice summing (for $1 \le p \le \infty$) if there is a positive constant C so that for all finite sets $\{x_1, x_2, ..., x_n\} \subset X$ we have

$$\left\| \left(\sum_{i=1}^{n} |T(x_i)|^p \right)^{\frac{1}{p}} \right\| \le C \sup_{x^* \in \mathcal{B}_{X^*}} \left(\sum_{i=1}^{n} |\langle x^*, x_i \rangle|^p \right)^{\frac{1}{p}}$$

if p is finite and

$$\left\| \sup_{1 \le i \le n} |T(x_i)| \right\| \le C \sup_{1 \le i \le n} \|x_i\|$$

if p is infinite. If T is p-lattice summing we denote by $\lambda_p(T)$ the smallest constant C, which is a norm on the space $\Lambda_p(X,E)$ of all p-lattice summing operators from X to E turning it into a Banach space.

Definition 3.1 An operator $T : X \longrightarrow E$ is Lipschitz p-lattice (or order) summing if, there is a positive constant C such that for every n in $\mathbb{N}$, for all $x_1, ..., x_n;\ y_1, ..., y_n$ in X and all non negative reals $\lambda_1, ..., \lambda_n$, we have

$$\left\| \left(\lambda_i^{\frac{1}{p}} (T(x_i) - T(y_i)) \right)_i \right\|_{E(l_p^n)} \le C \omega_p^{\mathrm{L}} \left(\left(\lambda_i^{\frac{1}{p}} \right)_i, (x_i)_i, (y_i)_i \right) \tag{6}$$

if p is finite and

$$\left\| \lambda_i |T(x_i) - T(y_i)| \right\|_{1 \le i \le n} \le C \lambda_i d(x_i, y_i)_{1 \le i \le n}. \tag{7}$$

if p is infinite.

We denote by $\Lambda_p^L(X, E)$ the vector space of all Lipschitz p-lattice summing operators T from X into E, which is a Banach space if we consider the norm $\lambda_p^{\mathrm{L}}(T)$, the infimum of all C verifying (6) and (7).

Remark 3.2 If T is in $\Lambda_p^{\mathrm{L}}(X, E)$, then T is Lipschitz p-convex and $C_p^{\mathrm{L}}(T) \le \lambda_p^{\mathrm{L}}(T)$ for p finite and $C_\infty^{\mathrm{L}}(T) = \lambda_\infty^{\mathrm{L}}(T)$.

Remark 3.3 If X is a Banach space and T is a bounded linear operator from X into E, then if T is p-lattice summing then T is Lipschitz p-lattice summing and $\lambda_p^{\mathrm{L}}(T) \le \lambda_p(T)$ for all $1 \le p < \infty$. If p is infinite, we have $\lambda_\infty^{\mathrm{L}}(T) = \lambda_\infty(T) = C_\infty(T)$. We do not know if the converse is true, because the unit ball $\mathcal{B}_{X^\#}$ is involved and we do not have a factorization theorem.

Proposition 3.4 *Consider T in $Lip_0(X, E), R$ in$Lip_0(Z, X)$ and φ in $\mathcal{B}(E, F)_+$. If T is Lipschitz p-lattice summing, then $\varphi \circ T \circ R$ is Lipschitz p-lattice summing ($1 \le p \le \infty$) and $\lambda_p^{\mathrm{L}}(\varphi \circ T \circ R) \le \|\varphi\| \lambda_p^{\mathrm{L}}(T) \mathrm{Lip}(R)$.*

Proof Let $n \in \mathbb{N}$, $(z_i)_{1 \le i \le n}$; $(z_i')_{1 \le i \le n} \subset Z$ and $(\lambda_i)_{1 \le i \le n} \subset \mathbb{R}_+$. Thus

$$\begin{aligned}
&\left\| \left(\lambda_i^{\frac{1}{p}} \varphi \circ T \circ R(z_i) - \varphi \circ T \circ R(z_i') \right) \right\|_{F(l_p^n)} \\
&= \left\| \left(\lambda_i^{\frac{1}{p}} \varphi\left(T \circ R(z_i) - T \circ R(z_i')\right) \right) \right\|_{F(l_p^n)} \\
&\quad \text{(by [20 p. 55])} \\
&\le \|\varphi\| \left\| \left(\lambda_i^{\frac{1}{p}} \left(T \circ R(z_i) - T \circ R(z_i')\right) \right) \right\|_{E(l_p^n)} \\
&\quad \text{(by Krivine's theorem)} \\
&\le \|\varphi\| \lambda_p^{\mathrm{L}}(T) \sup_{f \in \mathcal{B}_{X^\#}} \left(\sum_{i=1}^n \lambda_i \left| \left(f(R(z_i)) - f\left(R(z_i')\right) \right) \right|^p \right)^{\frac{1}{p}}. \\
&\le \|\varphi\| \lambda_p^{\mathrm{L}}(T) \mathrm{Lip}(R) \sup_{f \in \mathcal{B}_{X^\#}} \left(\sum_{i=1}^n \lambda_i \left| \frac{f \circ R(z_i)}{\mathrm{Lip}(R)} - \frac{f \circ R(z_i')}{\mathrm{Lip}(R)} \right|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Since $\dfrac{f \circ R}{\mathrm{Lip}(R)} \in \mathcal{B}_{Z^\#}$ we obtain

$$\left\|\left(\sum_{i=1}^{n}\lambda_i\left|\varphi\circ T\circ R(z_i)-\varphi\circ T\circ R\left(z_i'\right)\right|^p\right)^{\frac{1}{p}}\right\|$$
$$\leq\|\varphi\|\lambda_p^L(T)\mathrm{Lip}(R)\sup_{\|g\|_{Z^\#}\leq 1}\left(\sum_{i=1}^{n}\lambda_i\left|g(z_i)-g\left(z_i'\right)\right|^p\right)^{\frac{1}{p}}.$$

This implies that $\varphi\circ T\circ R$ is Lipschitz p-lattice summing with $\lambda_p^{\mathrm{L}}(\varphi\circ T\circ R)\leq\|\varphi\|\lambda_p^{\mathrm{L}}(T)\mathrm{Lip}(R)$ and this ends the proof. □

Remark 3.5 Clearly the notion of Lipschitz p-lattice summing operators is not an ideal in Pietsch's sense but it is an ideal on left.

By Proposition 3.4, we have the following proposition. For the converse, we can see [27, Remark 3.3].

Proposition 3.6 *Consider* $1\leq p<\infty$. *Let* $T:X\longrightarrow E$ *be a Lipschitz map and* T_{L} *its linearization (as in Theorem 2.1). If* T_{L} *is p-lattice summing, then T is Lipschitz p-lattice summing. The converse is false.*

Proposition 3.7 *Let T be in Lip* $_0(X,E)$, *then T is in* $\Lambda_\infty^{\mathrm{L}}(X,E)$ *if, and only if,* T_{L} *is in* $\Lambda_\infty(Æ(X),E)$ *and* $\lambda_\infty(T_{\mathrm{L}})=\lambda_\infty^{\mathrm{L}}(T)$.

Proof Suppose that T_{L} is in $\Lambda_\infty(Æ(X),E)$. Then, for every $n\in\mathbb{N}$, for all $x_1,\ldots,x_n,y_1,\ldots,y_n\in X$ and all non negative reals $\lambda_1,\ldots,\lambda_n$, we have

$$\begin{aligned}\left\|\bigvee_{1\leq i\leq n}\lambda_i\left|T(x_i)-T(y_i)\right|\right\|&=\left\|\bigvee_{1\leq i\leq n}\lambda_i\left|T_{\mathrm{L}}(m_{x_i0})-T_{\mathrm{L}}\left(m_{y_i0}\right)\right|\right\|\\&=\left\|\bigvee_{1\leq i\leq n}\lambda_i\left|T_{\mathrm{L}}\left(m_{x_iy_i}\right)\right|\right\|\\&=\left\|\bigvee_{1\leq i\leq n}\left|T_{\mathrm{L}}\left(\lambda_i m_{x_iy_i}\right)\right|\right\|\\&\leq\lambda_\infty(T_{\mathrm{L}})\sup_{1\leq i\leq n}\lambda_i d(x_i,y_i).\end{aligned}$$

This gives that T is in $\Lambda_\infty^{\mathrm{L}}(X,E)$ and $\lambda_\infty^{\mathrm{L}}(T)\leq\lambda_\infty(T_{\mathrm{L}})$.

For the converse, suppose that T is in $\Lambda_\infty^{\mathrm{L}}(X,E)$. This yields by Remark 3.2 that, T is in $C_\infty^{\mathrm{L}}(X,E)$ and $C_\infty^{\mathrm{L}}(T)=\lambda_\infty^{\mathrm{L}}(T)$. By [8, Theorem 3.3], T_{L} is in $C_\infty(Æ(X),E)$ and hence is in $\Lambda_\infty(Æ(X),E)$. □

Proposition 3.8 *Let* T *be inLip* $_0(X,E)$. *If* $(T^t)^*\in\Lambda_p\left(\left(X^\#\right)^*,E^{**}\right)$ *then,* $T\in\Lambda_p^{\mathrm{L}}(X,E)$.

Proof Suppose that $(T^t)^*$ is in $\Lambda_p\left(\left(X^\#\right)^*,E^{**}\right)$. Then , $(T_{\mathrm{L}})^{**}$ is in $\Lambda_p\left(\left(X^\#\right)^*,E^{**}\right)$ and by [26] the operator T_{L} is in $\Lambda_p(Æ(X),E)$. So we conclude by Proposition 3.6, that $T\in\Lambda_p^{\mathrm{L}}(X,E)$. □

Remark 3.9 By Proposition 3.7, the converse is true for $p=\infty$. But it is not valid for $1\leq p<\infty$ by Proposition 3.6.

Theorem 3.10 *Let $T:X\longrightarrow E$ be a Lipschitz operator such that $T(0)=0$ and $1\leq p<\infty$. Then, we have the following properties for a positive constant C.*

(i) If the operator $T\in\Lambda_p^{\mathrm{L}}(X,E)$ with $\lambda_p^{\mathrm{L}}(T)\leq C$, $u\in\mathcal{B}(l_{p^*}^n,Æ(X))$, then for all $n\in\mathbb{N}$ and $(x_i)_{1\leq i\leq n},(y_i)_{1\leq i\leq n}\subset X$ the operator $T_{\mathrm{L}}u$ is in $\Lambda_\infty(l_{p^*}^n,E)$ and $\lambda_\infty(T_{\mathrm{L}}u)\leq C\|u\|$, where $u(e_i)=m_{x_iy_i}$.

(ii) If $\lambda_\infty(T_{\mathrm{L}}u)\leq C\|u\|$ for all $n\in\mathbb{N}$ and $u\in\mathcal{B}(l_{p^*}^n,Æ(X))$ then, the operator T is in $\Lambda_p^{\mathrm{L}}(X,E)$ and $\lambda_p^{\mathrm{L}}(T)\leq C\|u\|$.

Proof (i) Suppose that T is in $\Lambda_p^{\mathrm{L}}(X,E)$. Let n be in $\mathbb{N}$ and $(x_i)_{1\leq i\leq n},(y_i)_{1\leq i\leq n}\subset X$. Consider $u(e_i)=m_{x_iy_i}$, where $(e_i)_{1\leq i\leq n}$ denote the unit vector basis of $l_{p^*}^n$. By (5), we have $\|u\|=\sup\limits_{\|f\|_{X\#}=1}\left(\sum_{i=1}^n|f(x_i)-f(y_i)|^p\right)^{\frac{1}{p}}$ and for $\alpha\in l_{p^*}^n$

$$\begin{aligned}|T_{\mathrm{L}}u(\alpha)| &=\left|T_{\mathrm{L}}u\left(\sum_{i=1}^n\alpha_ie_i\right)\right|=\left|\sum_{i=1}^n\alpha_iT_{\mathrm{L}}\left(m_{x_i0}-m_{y_i0}\right)\right|\\ &=\left|\sum_{i=1}^n\alpha_i(T(x_i)-T(y_i))\right|([20\text{ Part II, page 42}])\\ &\leq\left(\sum_{i=1}^n|T(x_i)-T(y_i)|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^n|\alpha_i|^{p^*}\right)^{\frac{1}{p^*}}\\ &\leq\left(\sum_{i=1}^n|T(x_i)-T(y_i)|^p\right)^{\frac{1}{p}}\|\alpha\|_{l_{p^*}^n}.\end{aligned}$$

By [24, Proposition 1.2 and Theorem 1.3], we have

$$\begin{aligned}\lambda_\infty(T_{\mathrm{L}}u)&\leq\left\|\left(\sum_{i=1}^n|T(x_i)-T(y_i)|^p\right)^{\frac{1}{p}}\right\|\\ &\leq\lambda_p^{\mathrm{L}}(T)\|u\|\leq C\|u\|.\end{aligned}$$

This implies that $T_{\mathrm{L}}u$ is in $\Lambda_\infty(l_{p^*}^n,E)$ and $\lambda_\infty(T_{\mathrm{L}}u)\leq C\|u\|$.

(ii) Suppose that $\lambda_\infty(T_{\mathrm{L}}u) \leq C\|u\|$ for all $n \in \mathbb{N}$ and all $u \in \mathcal{B}(l^n_{p^*}, Æ(X))$. This implies by [24, Theorem 1.3] that $\lambda_p(T_{\mathrm{L}}) \leq C\|u\|$ and by Proposition 3.4, we have $\lambda^{\mathrm{L}}_p(T) \leq C\|u\|$. □

Theorem 3.11 *Let X be a pointed metric space and E be a Banach lattice. Consider T in* $\mathrm{Lip}_0(X,E)$. *If the operator T is in* $\Lambda^{\mathrm{L}}_\infty(X,E)$, *then T is in* $\Lambda^{\mathrm{L}}_p(X,E)$ *for* $1 \leq p \leq \infty$.

Proof Suppose that T is in $\Lambda^L_\infty(X,E)$. This implies by Proposition 3.7 that T_{L} is in $\Lambda_\infty(Æ(X), E)$. We observe by [24, Theorem 1.5] that $\Lambda_\infty(Æ(X), E)$ is a subspace of $\Lambda_p(Æ(X), E)$. Therefore $T_{\mathrm{L}} \in \Lambda_p(Æ(X), E)$ and consequently by Proposition 3.6, we conclude that $T \in \Lambda^{\mathrm{L}}_p(X,E)$. □

We give now the following definition to have an equivalent relation between the operator T and its linearization T_{L}. In a way, it is the reciprocal of Proposition 3.6.

Definition 3.12 An operator $T : X \longrightarrow E$ is Lipschitz factorable p-lattice summing if, there is a positive constant C such that for all $n, m \in \mathbb{N}$, $(x_{ij})_{i,j}; (y_{ij})_{i,j} \in X$ and all reals $(\lambda_{ij})_{i,j}$, $(1 \leq i \leq n, 1 \leq j \leq m)$, we have

$$\left\| \left(\sum_{i=1}^{n} \left| \sum_{j=1}^{m} \lambda_{ij}(T(x_{ij}) - T(y_{ij})) \right|^p \right)^{\frac{1}{p}} \right\| \leq C \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^{n} \left| \sum_{j=1}^{m} \lambda_{ij}(f(x_{ij}) - f(y_{ij})) \right|^p \right)^{\frac{1}{p}}. \tag{8}$$

The class of all Lipschitz factorable p-lattice summing operators is denoted by $\Lambda^{\mathrm{L,f}}_p(X,E)$. In this case, we define $\lambda^{\mathrm{L,f}}_p(T)$ as the infimum of all constants C verifying (8).

Remark 3.13 Every Lipschitz factorable p-lattice summing operator is Lipschitz p-lattice summing operator, i.e., $\Lambda^{\mathrm{L,f}}_p(X,E) \subset \Lambda^{\mathrm{L}}_p(X,E)$, and $\lambda^{\mathrm{L}}_p(T) \leq \lambda^{\mathrm{L,f}}_p(T)$ (we take $m = 1$).

Theorem 3.14 *Let X be a pointed metric space and E be a Banach lattice. Consider T in* $\mathrm{Lip}_0(X,E)$. *Then T is Lipschitz factorable p-lattice summing if, and only if,* T_{L} *is p-lattice summing, and we have* $\lambda^{\mathrm{L,f}}_p(T) = \lambda_p(T_{\mathrm{L}})$.

Proof Let n, m be in $\mathbb{N}$, for $(x_{ij})_{i,j}; (y_{ij})_{i,j}$ in X and $(\lambda_{ij})_{i,j}$ in $\mathbb{R}$, $(1 \leq i \leq n, 1 \leq j \leq m)$, we have

$$\begin{aligned}
&\left\| \left(\sum_{i=1}^{n} \left| \sum_{j=1}^{m} \lambda_{ij} (T(x_{ij}) - T(y_{ij})) \right|^{p} \right)^{\frac{1}{p}} \right\| \\
&= \left\| \left(\sum_{i=1}^{n} \left| \sum_{j=1}^{m} \lambda_{ij} T_{\mathrm{L}}(m_{x_{ij}y_{ij}}) \right|^{p} \right)^{\frac{1}{p}} \right\| \\
&= \left\| \left(\sum_{i=1}^{n} T_{\mathrm{L}} \left(\sum_{j=1}^{m} \lambda_{ij} m_{x_{ij}y_{ij}} \right)^{p} \right)^{\frac{1}{p}} \right\| \\
&= \left\| \left(\sum_{i=1}^{n} |T_{\mathrm{L}}(m_i)|^{p} \right)^{\frac{1}{p}} \right\| \\
&\leq \lambda_p(T_{\mathrm{L}}) \sup_{\xi \in \mathcal{B}_{\text{Æ}(X)^*}} \left(\sum_{i=1}^{n} |\xi(m_i)|^{p} \right)^{\frac{1}{p}} \\
&\leq \lambda_p(T_{\mathrm{L}}) \sup_{\xi \in \mathcal{B}_{\text{Æ}(X)^*}} \left(\sum_{i=1}^{n} \left| \xi \left(\sum_{j=1}^{m} \lambda_{ij} m_{x_{ij}y_{ij}} \right) \right|^{p} \right)^{\frac{1}{p}} \\
&\leq \lambda_p(T_{\mathrm{L}}) \sup_{\xi \in \mathcal{B}_{\text{Æ}(X)^*}} \left(\sum_{i=1}^{n} \left| \sum_{j=1}^{m} \lambda_{ij} \xi(m_{x_{ij}y_{ij}}) \right|^{p} \right)^{\frac{1}{p}} \\
&\leq \lambda_p(T_{\mathrm{L}}) \sup_{f \in \mathcal{B}_{X^\#}} \left(\sum_{i=1}^{n} \left| \sum_{j=1}^{m} \lambda_{ij} (f(x_{ij}) - f(y_{ij})) \right|^{p} \right)^{\frac{1}{p}} .
\end{aligned}$$

This gives that T is Lipschitz factorable p-lattice summing and $\lambda_p^{\mathrm{L,f}}(T) \leq \lambda_p(T_{\mathrm{L}})$. Following the inverse schema, we will have the reciprocal. □

We give here a characterization of Lipschitz factorable p-lattice summing operators.

Proposition 3.15 *Let $1 \leq p \leq \infty$. Then $T \in \Lambda_p^{\mathrm{L,f}}(X,E)$ if, and only if, $T_{\mathrm{L}} u \in \Lambda_\infty(l_{p^*},E)$ for all $n \in \mathbb{N}$ and $u \in \mathcal{B}(l_{p^*}^n$,Æ(X)). In this case we have*

$$\lambda_p^{\mathrm{L},f}(T) = \sup \left\{ \lambda_\infty(T_{\mathrm{L}} u) : n \in \mathbb{N} \text{ and } u \in \mathcal{B}(l_{p^*}^n, \text{Æ}(X)) \right\}.$$

Proof Consider $T \in \Lambda_p^{\mathrm{L,f}}(X,E)$. This means that $T_{\mathrm{L}} \in \Lambda_p$(Æ($X$), E). As T_{L} is linear operator and according Nielsen-Szulga theorem [24, Theorem 1.3], we have $T_{\mathrm{L}} u \in \Lambda_\infty(l_{p^*}^n, E)$ for all $u \in \mathcal{B}(l_{p^*}$,Æ($X$)) and

$$\lambda_p(T_{\mathrm{L}}) = \sup\left\{\lambda_\infty(T_{\mathrm{L}}u) : n \in \mathbb{N} \text{ and } u \in \mathcal{B}(l_{p^*}^n, \mathbb{AE}(X))\right\}.$$

We conclude by Theorem 3.14. □

Let now $T : E \longrightarrow F$ be a Lipschitz operators between Banach lattices E, F. We say that T is *Lipschitz p-regular*, $(1 \leq p \leq \infty)$ and we write $T \in \rho_p^{\mathrm{L}}(E, F)$, if there is a positive constant C such that for every n in $\mathbb{N}$, for all $x_1, ..., x_n; y_1, ..., y_n$ in X and all non negative reals $\lambda_1, ..., \lambda_n$, we have

$$\left\|\left(\lambda_i^{\frac{1}{p}}(T(x_i) - T(y_i))\right)\right\|_{F\left(l_p^n\right)} \leq C\left\|\left(\lambda_i^{\frac{1}{p}}(x_i - y_i)\right)\right\|_{E\left(l_p^n\right)}$$

for p finite and we take the sup if p is infinite. The best possible constant will be denoted by $\rho_p^{\mathrm{L}}(T)$.

If T is linear then the notions of regular and Lipschitz regular coincide with $\rho_p(T) = \rho_p^L(T)$. It was proved by Krivine in [19] (see also [20]) that every bounded linear operator is 2-regular and every positive linear operator is p-regular for, $1 \leq p \leq \infty$. In general, the Lipschitz operators are not 2-regular, we can see [2].

Remark 3.16 Let $T : X \longrightarrow E$ be a Lipschitz p-lattice summing operator. If $R : E \longrightarrow F$ is p-regular, then $R \circ T$ is *Lipschitz* p-lattice summing and $\lambda_p^L(R \circ T) \leq \rho_p^{\mathrm{L}}(R)\lambda_p^{\mathrm{L}}(T)$.

Proposition 3.17 *Let E, F, E_1, F_1 be Banach lattices and $1 \leq p \leq \infty$. Consider a Lipschitz p-regular operator $T : E \longrightarrow F$, a positive linear bounded operator $u : F \longrightarrow F_1$ and a Lipschitz operator $R : E_1 \longrightarrow E$. Then, $u \circ T \circ R$ is Lipschitz p-regular and $\rho_p^{\mathrm{L}}(u \circ T \circ R) \leq \|u\|\rho_p^{\mathrm{L}}(T)\mathrm{Lip}(R)$.*

Proof Since u is positive, we have

$$\begin{aligned}
&\left(\sum_{i=1}^n \lambda_i |(u \circ T \circ R(a_i) - u \circ T \circ R(b_i))|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^n \left|u\left(\lambda_i^{\frac{1}{p}}(T \circ R(a_i) - T \circ R(b_i))\right)\right|^p\right)^{\frac{1}{p}} \\
&= \sup_{(\alpha_i) \in \mathcal{B}_{l_{p^*}^n}} u\left(\sum_{i=1}^n \lambda_i^{\frac{1}{p}}(T \circ R(a_i) - T \circ R(b_i))\alpha_i\right) \\
&\leq u\left(\sup_{(\alpha_i) \in \mathcal{B}_{l_{p^*}^n}} \sum_{i=1}^n \lambda_i^{\frac{1}{p}}(T \circ R(a_i) - T \circ R(b_i))\alpha_i\right) \\
&\leq u\left(\left(\sum_{i=1}^n \lambda_i |T \circ R(a_i) - T \circ R(b_i)|^p\right)^{\frac{1}{p}}\right).
\end{aligned}$$

This yields

$$\begin{aligned}
&\left\| \left(\lambda_i^{\frac{1}{p}} u \circ T \circ R(a_i) - u \circ T \circ R(b_i) \right) \right\|_{F_1\left(l_p^n\right)} \\
&= \left\| u \left(\lambda_i^{\frac{1}{p}} (T \circ R(a_i) - T \circ R(b_i)) \right) \right\|_{F_1\left(l_p^n\right)} \\
&\quad \text{(by [20 p. 55])} \\
&\leq \|u\| \left\| \left(\lambda_i^{\frac{1}{p}} (T \circ R(a_i) - T \circ R(b_i)) \right) \right\|_{F\left(l_p^n\right)} \\
&\leq \|u\| \rho_p^{\mathrm{L}}(T) \left\| \left(\lambda_i^{\frac{1}{p}} (R(a_i) - R(b_i)) \right) \right\|_{E\left(l_p^n\right)} \\
&\leq \|u\| \rho_p^{\mathrm{L}}(T) \mathrm{Lip}(R) \left\| \left(\lambda_i^{\frac{1}{p}} (a_i - b_i) \right) \right\|_{E_1\left(l_p^n\right)}
\end{aligned}$$

and thus the announced result is obtained. □

Remark 3.18 Let E, F, F_1 be Banach lattices and $1 \leq p \leq \infty$.

(1) Consider a Lipschitz p-regular operator $T : E \longrightarrow F$ and a Lipschitz p-concave operator $R : F \longrightarrow F_1$. Then, $R \circ T$ is Lipschitz p-concave and $M_{\mathrm{Lip}}^p(R \circ T) \leq M_{\mathrm{Lip}}^p(R) \rho_p^{\mathrm{L}}(T)$. Indeed,

$$\begin{aligned}
\left\| \left(\lambda_i^{\frac{1}{p}} (R \circ T(a_i) - R \circ T(b_i)) \right) \right\|_{l_p^n(F_1)} &\leq M_{\mathrm{Lip}}^p(R) \left\| \left(\lambda_i^{\frac{1}{p}} (T(a_i) - T(b_i)) \right) \right\|_{F\left(l_p^n\right)} \\
&\leq M_{\mathrm{Lip}}^p(R) \rho_p^{\mathrm{L}}(T) \left\| \left(\lambda_i^{\frac{1}{p}} (a_i - b_i) \right) \right\|_{E\left(l_p^n\right)}.
\end{aligned}$$

(2) Consider a Lipschitz p-regular operator $T : E \longrightarrow F$ and a Lipschitz p-convex operator $S : E_1 \longrightarrow E$. Then, $T \circ S$ is Lipschitz p-convex and $C_{\mathrm{Lip}}^p(T \circ S) \leq C_{\mathrm{Lip}}^p(S) \rho_p^{\mathrm{L}}(T)$.

(3) If $p = 2$ and R is a bounded linear operator, we have $\rho_2^{\mathrm{L}}(T \circ R) \leq K_G \rho_2^{\mathrm{L}}(T) \|R\|$; because every bounded linear operator R is 2-regular and $\rho_2(R) \leq K_G \|R\|$ (where K_G is the universal Grothendieck constant).

Now, we define the Lipschitz order bounded operator. This definition is inspired from the linear case (see [24]) which is: if X is a Banach space and E is a Banach lattice then a bounded linear operator $T : X \longrightarrow E$ is called *order bounded* if there exists an element $a \in E_+$ such that

$$|T(x)| \leq a\|x\|, \quad \text{for all } x \in X. \tag{9}$$

In this case, we define the order bounded norm $\|T\|_m$ by

$$\|T\|_m = \inf\{\|a\| : a \text{ satisfies (3.4)}\}$$

which is by [23] a complete norm on $\mathfrak{B}(X, E)$, the space of all order bounded operators from X into E. By Schaefer [13], the m-tensor product $X\widehat{\otimes}_m E$ is defined by the closure of $X \otimes E$ in $\mathfrak{B}(X^*, E)$.

An operator T in $\mathrm{Lip}_0(X, E)$ is called *Lipschitz order bounded* if there exists an element a in E_+ such that

$$|T(x) - T(y)| \leq ad(x, y), \quad \text{for all } x, y \in X. \tag{10}$$

The space of such operators is denoted by $\mathfrak{Lip}_0(X, E)$. We let

$$\mathfrak{Lip}(T) = \inf\{\|a\| : a \text{ satisfies } (3.5)\}.$$

or

$$\mathfrak{Lip}(T) = \left\| \sup_{x \neq y} \frac{|T(x) - T(y)|}{d(x, y)} \right\| \tag{11}$$

which exists by (3); because T is order bounded as a map into E^{**} (also we can take E a complete Banach lattice). We have also

$$\mathrm{Lip}(T) \leq \mathfrak{Lip}(T). \tag{12}$$

Indeed, we have for all $x, y \in X$

$$\frac{|T(x) - T(y)|}{d(x, y)} \leq \sup_{x \neq y} \frac{|T(x) - T(y)|}{d(x, y)}$$

and consequently

$$\frac{\|T(x) - T(y)\|}{d(x, y)} \leq \left\| \sup_{x \neq y} \frac{|T(x) - T(y)|}{d(x, y)} \right\|$$

for all $x, y \in X$. By taking the sup, we obtain by (11) that $\mathrm{Lip}(T) \leq \mathfrak{Lip}(T)$. It is clear that $\mathfrak{Lip}_0(X, E)$ is a vector space and that $\mathfrak{Lip}(.)$ is a norm on $\mathfrak{Lip}_0(X, E)$. Let us prove that the space $\mathfrak{Lip}_0(X, E)$ equipped with the norm $\mathfrak{Lip}(.)$ is a Banach space. By (12) and [12, Lemma 3.100], it is enough to prove that if $(T_n)_{n \geq 1} \subset \mathfrak{Lip}_0(X, E)$ such that $\sum_{n=1}^{+\infty} \mathfrak{Lip}(T_n) < +\infty$, then the operator $T = \sum_{i=1}^{n} T_n$ is in $\mathfrak{Lip}_0(X, E)$ and $\mathfrak{Lip}(T) \leq \sum_{n=1}^{+\infty} \mathfrak{Lip}(T_n)$. In this case

$$\mathfrak{Lip}(T) = \inf\left\{ \sum_{n=1}^{+\infty} \mathfrak{Lip}(T_n) \right\}$$

where the infimum is taken over all series in $\mathfrak{Lip}_0(X, E)$ summing up to T. Consider $\epsilon > 0$ and choose a sequence $(a_n) \subset E_+$ such that

$$\|a_n\| \leq \mathfrak{Lip}(T_n) + \frac{\epsilon}{2^n}, \quad for n \geq 1.$$

Then $\sum_{n=1}^{+\infty} \|a_n\| \leq \sum_{n=1}^{+\infty} \mathfrak{Lip}(T_n) + \epsilon$ and hence from [12, Lemma 3.100] again , we can find $a \in E_+$, such that

$$\|a\| \leq \sum_{n=1}^{+\infty} \mathfrak{Lip}(T_n) + \epsilon.$$

We have $\mathfrak{Lip}(T) \leq \sum_{n=1}^{+\infty} \mathfrak{Lip}(T_n) + \epsilon$. As ϵ is arbitrary, this proves the statement.

Theorem 3.19 *Let Xbe a pointed metric space and E be a Banach lattice. Consider T in* $\mathrm{Lip}_0(X, E)$. *Then T is Lipschitz order bounded if, and only if,* T_{L} *is order bounded and* $\|T_{\mathrm{L}}\|_m = \mathfrak{Lip}(T)$.

Proof Suppose that T_{L} is order bounded. Then there is a in E such that

$$|T_{\mathrm{L}}(m)| \leq a\|m\| \text{ for all } m \in \mathcal{M}(X).$$

This implies for all $x, y \in X$

$$\left|T_{\mathrm{L}}\left(m_{xy}\right)\right| \leq a\left\|m_{xy}\right\|$$

and hence

$$|T(x) - T(y)| \leq ad(x, y)$$

by [30, Lemma 3.5] and (2).

For the converse, suppose that T is Lipschitz order bounded. So there is a in E such that for all x, y in X, we have $|T(x) - T(y)| \leq ad(x, y)$. Let m be in $\mathcal{M}(X)$. For $\epsilon > 0$, consider a representation of $m = \sum_{i=1}^{n} \alpha_i m_{x_i y_i}$ such that $\sum_{i=1}^{n} |\alpha_i| d(x_i, y_i) < \|m\| + \epsilon$. We have

$$\begin{aligned} |T_{\mathrm{L}}(m)| &= \left|T_{\mathrm{L}}\left(\sum_{i=1}^{n} \alpha_i m_{x_i y_i}\right)\right| = \left|\sum_{i=1}^{n} \alpha_i (T(x_i) - T(y_i))\right| \\ &\leq a \sum_{i=1}^{n} |\alpha_i| d(x_i, y_i) \leq a(\|m\| + \epsilon) \end{aligned}$$

and this for all $\epsilon > 0$. Thus $|T_{\mathrm{L}}(m)| \leq a\|m\|$ for all m in $\mathcal{M}(X)$ and by continuity $|T_{\mathrm{L}}(m)| \leq a\|m\|$ for all m in Æ(X). □

The following corollary is immediate.

Corollary 3.20 *Consider T in* $\mathrm{Lip}_0(X, E)$. *Then T is Lipschitz order bounded if, and only if,* $(T^t)^*$ *is order bounded and* $\|(T^t)^*\|_m = \mathfrak{Lip}(T)$.

Remark 3.21 If $w : E \longrightarrow F$ is a positive linear operator, then $w \circ T$ is order bounded. Indeed, consider x, y in X and a in E_+ such that $|T(x) - T(y)| \leq ad(x, y)$. We have

$$\begin{aligned} |w \circ T(x) - w \circ T(y)| &= |w(T(x) - T(y))| \\ &\leq w(|T(x) - T(y)|) \\ &\leq w(a) d(x, y). \end{aligned}$$

As w is positive, then $w(a) \in F_+$ and this implies that $w \circ T$ is order bounded.

We recall now the following, as announced in [17]. Let X be a metric space and E be a Banach space. The Lipschitz image of a mapping $T : X \to E$ is the set

$$\left\{ \frac{T(x) - T(y)}{d(x,y)} : x, y \in X, \ x \neq y \right\}.$$

An immediate consequence, $T : X \to E$ is a Lipschitz mapping if its Lipschitz image is a bounded subset of E.

Definition 3.22 [17] A Lipschitz operator T in $\mathrm{Lip}_0(X, E)$ has Lipschitz finite dimensional rank if the linear hull of its Lipschitz image is a finite dimensional subspace of E. In that case we define the Lipschitz rank, Lrank(T) of T to be the dimension of this subspace.

We regard now the norm $\mathfrak{Lip}$ as Lipschitz tensor product.

Definition 3.23 Let X be a pointed metric space and E be a Banach lattice. The space $X^{\#} \circledast E$ is defined as the linear subspace of $(X \boxtimes E^*)'$ spanned by the set

$$\left\{ f \boxtimes a : f \in X^{\#}, a \in E \right\}.$$

This space is called the associated Lipschitz tensor product of X, E^*. For this concept, we can also consult [16].

The Lipschitz operators of finite rank from X into E are majorizing; that is $X^{\#} \circledast E \subset \mathfrak{Lip}_0(X, E)$. Indeed, consider $T = \sum_{i=1}^{n} f_i \boxtimes a_i$ in $X^{\#} \circledast E$. We have

$$\begin{aligned} |T(x) - T(y)| &= \left| \sum_{i=1}^{n} (f_i(x) - f_i(y)) a_i \right| \\ &\leq \sum_{i=1}^{n} |f_i(x) - f_i(y)| |a_i| \\ &\leq \left(\sum_{i=1}^{n} \mathrm{Lip}(f_i) |a_i| \right) d(x, y). \end{aligned}$$

This implies that $X^{\#} \circledast E$ is in $\mathfrak{Lip}_0(X, E)$ and suggests us the following definition.

Definition 3.24 The $\mathfrak{Lip}$-tensor product $X^{\#} \widehat{\circledast}_{\mathfrak{Lip}} E$ is the closure of $X^{\#} \circledast E$ in $\mathfrak{Lip}_0(X, E)$.

The following proposition generalizes, Proposition 1.2 of [24].

Proposition 3.25 *Consider T in $\mathrm{Lip}_0(X, E)$. The operator T is in $\Lambda_{\infty}^{\mathrm{L}}(X, E)$ if, and only if, T is in $\mathfrak{Lip}_0(X, E)$. In this case, $\lambda_{\infty}^{\mathrm{L}}(T) = \mathfrak{Lip}(T) = \lambda_{\infty}(T_{\mathrm{L}}) = \|T_{\mathrm{L}}\|_m$.*

Proof Suppose that T is in $\mathfrak{Lip}_0(X, E)$. For $\epsilon > 0$, there is a in E_+ such that for all x, y in X, we have

$$|T(x) - T(y)| \leq ad(x, y)$$

with $\mathfrak{Lip}(T) \leq \|e\| \leq (1+\epsilon)\mathfrak{Lip}(T)$. This implies that, for every n in $\mathbb{N}$, all $x_1, ..., x_n;\ y_1, ..., y_n$ in X and all non negative reals $\lambda_1, ..., \lambda_n$, we have

$$\sup_{1 \leq i \leq n} \lambda_i |T(x_i) - T(y_i)| \leq a \sup_{1 \leq i \leq n} \lambda_i d(x_i, y_i)$$

and hence

$$\begin{aligned} \left\| \lambda_{1 \leq i \leq n_i} |T(x_i) - T(y_i)| \right\| &\leq \|a\| \lambda_{1 \leq i \leq n_i} d(x_i, y_i) \\ &\leq (1+\epsilon)\mathfrak{Lip}(T) \lambda_{1 \leq i \leq n_i} d(x_i, y_i). \end{aligned}$$

As ϵ is arbitrary, then $\lambda_\infty^{\mathrm{L}}(T) \leq \mathfrak{Lip}(T)$.

Consider now T in $\Lambda_\infty^{\mathrm{L}}(X, E)$. By Proposition 3.7, this gives that T_{L} *is* in $\Lambda_\infty(Æ(X), E)$ and consequently for all $x_1, ..., x_n;\ y_1, ..., y_n$ in X, we have

$$\left\| _{1 \leq i \leq n} \left| T_{\mathrm{L}}\left(m_{x_i y_i}\right) \right| \right\| \leq \lambda_\infty(T)_{1 \leq i \leq n} \|m_{x_i y_i}\|.$$

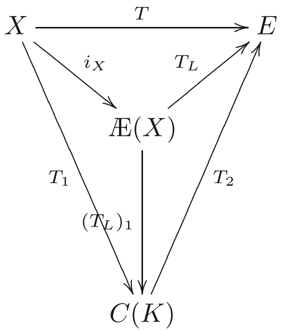
By [13, Proposition 3.4], there exists a in E_+, $\|a\| \leq \lambda_\infty(T)$ such that for each x, y in X, one has $\left| T_{\mathrm{L}}\left(\frac{m_{xy}}{\|m_{xy}\|}\right) \right| \leq a$. We conclude that for all $x, y \in X$, $|T(x) - T(y)| \leq ad(x, y)$ and thus by (10), T is Lipschitz order bounded and $\mathfrak{Lip}(T) \leq \lambda_\infty^{\mathrm{L}}(T)$. □

Corollary 3.26 *The $\mathfrak{Lip}$-tensor product $X^{\#}\widehat{\circledast}_{\mathfrak{Lip}}E$ is a subspace of $\Lambda_p^{\mathrm{L}}(X, E)$* for all $1 \leq p \leq \infty$.

Proof We have by definition $X^{\#}\widehat{\circledast}_{\mathfrak{Lip}}E$ and by Proposition 3.25 $\mathfrak{Lip}_0(X, E) = \Lambda_\infty^{\mathrm{L}}(X, E)$ with $\lambda_\infty^{\mathrm{L}}(T) = \mathfrak{Lip}(T)$. If T is in $\Lambda_\infty^{\mathrm{L}}(X, E)$ then T_{L} is in $\Lambda_\infty(Æ(X), E)$ which is included in $\Lambda_p(Æ(X), E)$ by [24, Theorem 1.5(i)] and consequently $T \in \Lambda_p^{\mathrm{L}}(X, E)$. □

Lemma 3.27 *Let X be a pointed metric space and E be a Banach lattice. Consider T in* $\mathrm{Lip}_0(X, E)$. *Then, T is Lipschitz order bounded operator if, and only if, there are a compact Hausdorff space K and operators $T_1 \in$* Lip $_0(X, C(K))$ *and $T_2 \in \mathcal{B}_+(C(K), E)$ so that $Lip(T_1) \leq 1$, $\|T_2\|_m = \mathfrak{Lip}(T)$ and $T = T_2 \circ T_1$.*

Proof Let $T \in \mathfrak{Lip}_0(X, E)$. This means by Theorem 3.19 that T_L is order bounded operator. Then, by a classical result we can find a compact Hausdorff space K and operators $(T_L)_1 \in \mathcal{B}(Æ(X), C(K))$ and $T_2 \in \mathcal{B}_+(C(K), E)$ so that $\|(T_L)_1\| \leq 1$, $\|T_2\|_m = \|T_{\mathrm{L}}\|_m$, $T_L = T_2 \circ (T_{\mathrm{L}})_1$ and the following diagram is commutative



Note that $T_1 = (T_L)_1 \circ i_X \in \mathrm{Lip}_0(X, C(K))$ and $\mathrm{Lip}(T_1) \leq 1$. Indeed,

$$\mathrm{Lip}(T_1) = \mathrm{Lip}((T_L)_1 \circ i_X) \leq \|(T_L)_1\| \leq 1.$$

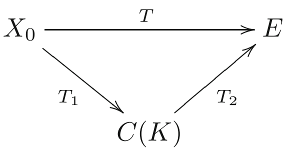
Conversely, suppose that $T_1 \in \mathrm{Lip}_0(X, C(K))$ and $T_2 \in \mathcal{B}_+(C(K), E)$ such that $T = T_2 \circ T_1$. Let $x, y \in X$ and $a \in E_+$. We have

$$\begin{aligned} |T(x) - T(y)| &= |T_2 \circ T_1(x) - T_2 \circ T_1(y)| \\ &= |T_2(T_1(x) - T_1(y))| \\ &\leq a\|T_1(x) - T_1(y)\| \\ &\leq a\mathrm{Lip}(T_1)d(x, y). \end{aligned}$$

This implies that $T \in \mathfrak{Lip}_0(X, E)$. □

Theorem 3.28 *Let X_0 be a subspace of a pointed metric space X which contained the distinguished element and E be a Banach lattice space. Then, each operator T in $\mathfrak{Lip}_0(X_0, E)$ admits a Lipschitz order bounded extension $\widetilde{T} : X \longrightarrow E$, with $\mathfrak{Lip}(\widetilde{T}) = \mathfrak{Lip}(T)$.*

Proof Consider T in $\mathfrak{Lip}_0(X_0, E)$. By Lemma 3.27, T has the following factorization



where $T_1 \in \mathrm{Lip}_0(X_0, C(K))$ and $T_2 \in \mathcal{B}_+(C(K), E)$ such that $\mathrm{Lip}(T_1) \leq 1$, $\|T_2\|_m = \mathfrak{Lip}(T)$. By the non linear Hahn-Banach theorem [4, Proposition 1.2], T_1 admits an extension $\widetilde{T}_1 \in \mathrm{Lip}_0(X, C(K))$ with $\mathrm{Lip}(\widetilde{T}_1) = \mathrm{Lip}(T_1)$. Hence, T admits a extension $\widetilde{T}$ such that $\widetilde{T} = T_2 \circ \widetilde{T}_1$. We deduce from Lemma 3.27, that $\widetilde{T}$ is Lipschitz order bounded operator with

$$\mathfrak{Lip}(\widetilde{T}) \leq \|T_2\|_m \mathrm{Lip}(\widetilde{T}_1) = \mathfrak{Lip}(T)\mathrm{Lip}(T_1) \leq \mathfrak{Lip}(T).$$

For the reverse inequality, note that $T = \widetilde{T} \circ i$ ($i : X_0 \longrightarrow X$ is the natural isometric embedding) and hence $\mathfrak{Lip}(T) = \mathfrak{Lip}(\widetilde{T} \circ i) \leq \mathfrak{Lip}(\widetilde{T})$. □

4 Connection with Lipschitz *p*-summing operators

In this section, we try to make the connection between Lipschitz p-lattice summing and Lipschitz p-summing operators.

Theorem 4.1 *Consider $1 \leq p < \infty$. Let $T : X \longrightarrow E$ be a Lipschitz map. If T is in $\Lambda_p^{\mathrm{L}}(X, E)$, then for all Lipschitz p-concave operator $S : E \longrightarrow F$, $S \circ T$ is Lipschitz p-summing and $\pi_p^{\mathrm{L}}(ST) \leq M_p^{\mathrm{L}}(S)\lambda_p^{\mathrm{L}}(T)$.*

Proof Let T be in $\Lambda_p^{\mathrm{L}}(X, E)$ and consider $S : E \longrightarrow F$ a Lipschitz p-concave operator. We have for every n in $\mathbb{N}$, all (x_i), $(y_i) \subset X$ and all non negative reals $\lambda_1, \ldots, \lambda_n$

$$\begin{aligned}
&\left\| \left(\lambda_i^{\frac{1}{p}} (ST(x_i) - ST(y_i)) \right)_{1 \leq i \leq n} \right\|_{l_p^n(F)} \\
&\leq M_p^{\mathrm{L}}(S) \left\| \left(\lambda_i^{\frac{1}{p}} (T(x_i) - T(y_i)) \right)_{1 \leq i \leq n} \right\|_{E\left(l_p^n\right)} \\
&\leq M_p^{\mathrm{L}}(S)\lambda_p^{\mathrm{L}}(T)\omega_p^L\left(\left(\lambda_i^{\frac{1}{p}} \right), (x_i), (y_i) \right).
\end{aligned}$$

This implies that $S \circ T$ is Lipschitz p-summing and $\pi_p^{\mathrm{L}}(ST) \leq M_p^{\mathrm{L}}(S)\lambda_p^{\mathrm{L}}(T)$. □

Proposition 4.2 *Consider $1 \leq p < \infty$. Let $T : X \longrightarrow E$ be a Lipschitz map. If T is in $\Lambda_p^{\mathrm{L}}(X, E)$, then for all p-concave space F and all positive bounded linear operator $\varphi : E \longrightarrow F$, $\varphi \circ T$ is in $\Pi_p^{\mathrm{L}}(X, F)$ and $\pi_p^{\mathrm{L}}(\varphi T) \leq \|\varphi\| M_p(F)\lambda_p^{\mathrm{L}}(T)$.*

Proof We have for every n in $\mathbb{N}$, all $(x_i), (y_i) \subset X$ and all non negative reals $\lambda_1, \ldots, \lambda_n$

$$\begin{aligned}
&\left\| \left(\lambda_i^{\frac{1}{p}} (\varphi T(x_i) - \varphi T(y_i)) \right)_{1 \leq i \leq n} \right\|_{l_p^n(F)} \\
&\leq M_p(F) \left\| \left(\sum_{i=1}^{n} \lambda_i^{\frac{1}{p}} |\varphi T(x_i) - \varphi T(y_i)|^p \right)^{\frac{1}{p}} \right\| \\
&\text{(Proposition 3.4)} \leq \|\varphi\| M_p(F) \left\| \left(\sum_{i=1}^{n} \lambda_i^{\frac{1}{p}} |T(x_i) - T(y_i)|^p \right)^{\frac{1}{p}} \right\| \\
&\leq \|\varphi\| M_p(F)\lambda_p^{\mathrm{L}}(T)\omega_p^{\mathrm{L}}\left(\left(\lambda_i^{\frac{1}{p}} \right), (x_i)_i, (y_i)_i \right).
\end{aligned}$$

This implies that $\varphi \circ T$ is Lipschitz p-summing and $\pi_p^{\mathrm{L}}(\varphi T) \leq \|\varphi\| M_p(F) \lambda_p^{\mathrm{L}}(T)$. □

Corollary 4.3 *Let X be a pointed metric space and E be a p-concave Banach lattice. Then, every Lipschitz p-lattice summing operator T from X into E is Lipschitz p-summing and* $\pi_p^{\mathrm{L}}(T) \leq M_p(E) \lambda_p^{\mathrm{L}}(T)$.

Theorem 4.4 *Let X be a pointed metric space and let E be a Banach lattice. Consider* $T : X \longrightarrow E$ *a Lipschitz map. If the operator T is in* $\Pi_p^{\mathrm{L}}(X, E)$,*then for all Lipschitz p-convex* $S : E \longrightarrow F$, *we have* $S \circ T$ *is in* $\Lambda_p^{\mathrm{L}}(X, F)$ *and* $\lambda_p^{\mathrm{L}}(S \circ T) \leq C_p^{\mathrm{L}}(S) \pi_p^{\mathrm{L}}(T)$.

Proof As F is p-convex, we have for every n in $\mathbb{N}$, all $(x_i), (y_i) \subset X$ and all non negative reals $\lambda_1, ..., \lambda_n$

$$\left\| \left(\sum_{i=1}^{n} \lambda_i |ST(x_i) - ST(y_i)|^p \right)^{\frac{1}{p}} \right\|_F \leq C_p^{\mathrm{L}}(S) \left(\sum_{i=1}^{n} \lambda_i \|T(x_i) - T(y_i)\|_F^p \right)^{\frac{1}{p}}$$

$$(\text{Proposition 3.4}) \leq C_p^{\mathrm{L}}(S) \left(\sum_{i=1}^{n} \lambda_i \|T(x_i) - T(y_i)\|_E^p \right)^{\frac{1}{p}}$$

$$\leq C_p^{\mathrm{L}}(S) \pi_p^{L}(T) \omega_p^{\mathrm{L}} \left(\left(\lambda_i^{\frac{1}{p}} \right), (x_i), (y_i) \right).$$

This implies that $S \circ T$ *is* Lipschitz p-lattice summing and $\lambda_p^{\mathrm{L}}(S \circ T) \leq C_p^{\mathrm{L}}(S) \pi_p^{\mathrm{L}}(T)$. □

Corollary 4.5 *Let X be a pointed metric space and E be a p-convex Banach lattice. Then, every Lipschitz p-summing operator T from X into E is Lipschitz p-lattice summing and* $\lambda_p^{\mathrm{L}}(T) \leq C_p(E) \pi_p^{\mathrm{L}}(T)$.

Corollary 4.6 *If* $E = F = L_p(\Omega, \mu)$ *and* $S = Id_{L_p(\Omega,\mu)}$ *then,* $\Pi_p^{\mathrm{L}}(X, L_p(\Omega, \mu)) = \Lambda_p^{\mathrm{L}}(X, L_p(\Omega, \mu))$ *and* $\pi_p^{\mathrm{L}}(T) = \lambda_p^{\mathrm{L}}(T)$.

Proof This comes from the fact that $L_p(\Omega, \mu)$ is p-convex and p-concave with $C_p^{\mathrm{L}}(L_p(\Omega, \mu)) = M_p^{\mathrm{L}}(L_p(\Omega, \mu)) = 1$. □

Remark 4.7 Let $T : X \longrightarrow E, S : Z \longrightarrow X$ and $R : E \longrightarrow F$ *be Lipschitz maps. Then*

(i) *If T is* Lipschitz p- lattice summing operator and S is Lipschitz p-summing, then $T \circ S$ is Lipschitz p-lattice summing and $\lambda_p^{\mathrm{L}}(T \circ S) \leq \lambda_p^{\mathrm{L}}(T) \pi_p^{\mathrm{L}}(S)$.

(ii) If T is p-regular and R is Lipschitz p-summing, then $R \circ T$ is Lipschitz p-summing and $\pi_p^{\mathrm{L}}(R \circ T) \leq \rho_p^{\mathrm{L}}(T) \pi_p^{\mathrm{L}}(R)$.

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References

1. Arens, R.-F., Eels, J.: On embedding uniform and topological spaces. Pac. J. Math. **6**, 397–403 (1956)
2. Achour, D., Mezrag, L.: Little Grothendieck's theorem for sublinear operators. J. Math. Anal. Appl. **296**, 541–552 (2004)
3. Achour, D., Mezrag, L.: Sur les opérateurs sous linéaires p-lattice sommants. An. Univ. Oradea Fasc. Mat. **XIV**, 237–250 (2007)
4. Benyamini, Y., Lindenstrauss, J.: Geometric Nonlinear Functional Analysis, vol. 48. Amer. Math. Soc., Providence (2000)
5. Cabrera-Padilla, M.-G., Chávez-Domínguez, J.-A., Jiménez-Vargas, A., Villegas-Vallecillos, M.: Lipschitz tensor product. Khayyam J. Math. **1**(2), 185–218 (2015)
6. Chávez-Domínguez, J.-A.: Duality for Lipschitz p-summing operators. J. Funct. Anal. **261**(2), 387–407 (2011)
7. Chávez-Domínguez, J.-A.: Lipschitz (q,p)-mixing operators. Proc. Am. Math. Soc. **140**, 3101–3115 (2012)
8. Chávez-Domínguez, J.-A.: Lipschitz p-convex and q-concave maps. Preprint
9. Chen,D., Zheng, B.: Remarks on Lipschitz p-summing operators. Proc. Am. Math. Soc. **139**(8), 2891–2898 (2011)
10. Chen, D., Zheng, B.: Lipschitz p-integral operators and Lipschitz p-nuclear operators. Nonlinear Anal. **75**, 5270–5282 (2012)
11. Defant, A., Floret, K.: Tensor Norms and Operator Ideals, North-Holland Math. Stud., vol. 176. North-Holland Publishing Co., Amsterdam (1993)
12. Fabian, M., Habala, P., Hájek, P., Montesinos, V., Zizler, V.: Banach Space Theory. Springer, Berlin (2011)
13. Schaefer, H.-H.: Banach Lattices and Positive Operators, (Grundlehner der Mathematischen Weissenschaften), vol. 215. Springer, Berlin (1974)
14. Farmer, J.-D., Johnson, W.-B.: Lipschitz p-summing operators. Proc. Am. Math. Soc. **137**(9), 2989–2995 (2009)
15. Kalton, N.-J., Godefroy, G.: Lipschitz-free Banach spaces. Stud. Math. **159**, 121–141 (2003)
16. Johnson, J.-A.: Banach spaces of Lipschitz functions and vector-valued Lipschitz functions. Trans. Am. Math. Soc. **148**, 147–169 (1970)
17. Jiménez-Vargas, A., Sepulcre, J.-M.: Moisés Villegas–Vallecillos, Lipschitz compact operators. J. Math. Anal. Appl. **415**(2), 889–901 (2014)
18. Kantorovich, L.-V.: On the translocation of masses. Dokl. Akad. Nauk SSSR **37**, 227–229 (1942)
19. Krivine, J.-L.: Théorèmes de factorisation dans les espaces réticulés, Séminaire Maurey-Schwartz 1973–1974: Espaces L_p, applications radonifiantes et géométrie des espaces de Banach, Exp. Nos. 22 et 23, Centre de Math., Ecole Polytech., Paris (1974)
20. Lindenstrauss, J., Tzafriri, L.: Classical Banach Spaces I and II. Springer, Berlin (1996)
21. Mezrag, L.: Little G. T. for l_p-lattice summing operators. Serdica. Math. J. **32**, 39–56 (2006)
22. Mezrag, L., Tallab, A.: On Lipschitz $\tau(p)$-summing operators. Colloq. Math. **147**(1), 95–114 (2017)
23. Nielson, N.-J.: On Banach ideals determined by Banach lattices and their applications. Diss. Math. (Rozprawy Mat.) **CIX**, 1–62 (1973)
24. Nielsen, N.-J., Szulga, J.: p-Lattice summing operators. Math. Nachr. **119**, 219–230 (1984)
25. Pietsch, A.: Absolut p-summierende Abbildungen in normierten Räumen. Stud. Math. **28**, 333–353 (1967)
26. Pomares, B.-P.: Biduals of p-lattice summing operators. Extracta Math. **1**(3), 136–138 (1986)
27. Saadi, K.: Some properties for Lipschitz strongly p-summing operators. J. Math. Anal. Appl. **423**, 1410–1426 (2015)
28. Sawashima, I.: Methods of Lipschitz duals. In: Lecture Notes Ec. Math. Sust., vol. 419, pp. 247–259. Springer, Berlin (1975)
29. Szulga, J.: On lattice summing operators. Proc. Am. Math. Soc. **87**(2), 258–262 (1983)
30. Weaver, N.: Lipschitz Algebras, 2nd edn. World Scientific Publishing Co., River Edge (2018)
31. Yahi, R., Achour, D., Rueda, P.: Absolutely summing Lipschitz conjugates. Mediterr. J. Math. **13**, 1949–1961 (2016)
32. Yanovskii, L.-P.: On summing and lattice summing operators and characterizations of AL-spaces. Sibirskii Mat. Zh. **20**(2), 401–408 (1979). (**in Russian**)