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***Degenerate elliptic equations with lower-order terms
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“Gratitude is the sign of noble souls.”

— Aesop

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May Allah bless you all.

Dedication

In the Name of Allah, the Most Gracious, the Most Merciful

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who instilled in me the love of knowledge and was the first to awaken my
passion for mathematics.*

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Notation

We introduce the necessary notations and definitions used in the sequel.

$\Omega \subset \mathbb{R}^N$	Open set in the Euclidean space $\mathbb{R}^N$.
X'	Topological dual of the space X .
$(\cdot, \cdot)$	Scalar product.
$\mathbb{R}^N$	Euclidean space of dimension N , where N is a nonzero natural number.
x	Vector in $\mathbb{R}^N$, $x = (x_1, x_2, \dots, x_N)$, $x_i \in \mathbb{R}$ for $1 \leq i \leq N$.
dx	Lebesgue measure in N -dimensional space.
$\partial\Omega = \Gamma$	Boundary of Ω .
$B(x_0, r)$	Open ball of radius r centered at x_0 .
B_E	Unit ball in E : $B_E = \{x \in E : \ x\ \leq 1\}$.
$ \cdot $	Hilbert norm.
$\langle \cdot, \cdot \rangle$	Duality bracket between X and X' .
$\int_{\Omega} f(x) dx$	Integral of f over Ω with respect to the Lebesgue measure.
$\text{supp } u$	Support of the function u .
$\frac{\partial u}{\partial n}$	Outward normal derivative.
$\text{div } u$	Divergence of the vector field u , $\text{div } u = \sum_{i=1}^N \frac{\partial u}{\partial x_i}$.
Δu	Laplacian of u : $\Delta u = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$.
$L^p(\Omega)$	measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that $(\int_{\Omega} u(x) ^p dx)^{1/p} < +\infty$, with $1 \leq p < \infty$.
$L^\infty(\Omega)$	Space of essentially bounded functions on Ω .
$\ u\ _{L^\infty}$	Essential supremum: $\inf\{C > 0 : u(x) \leq C \text{ a.e. in } \Omega\}$.
q	Hölder conjugate of p : $q = \frac{p}{p-1}$ if $p > 1$, and $q = \infty$ if $p = 1$.
$L^q(\Omega) \subseteq L^p(\Omega)$	whenever Ω has finite measure and $1 \leq p < q < \infty$.
$\ u\ _{C_c(\Omega)}$	Supremum norm: $\sup_{x \in \Omega} u(x) $.
$C_c(\Omega)$	Space of continuous functions with compact support in Ω .
$\mathcal{D}(\Omega)$	Space of infinitely differentiable functions with compact support in Ω .
$W^{1,p}(\Omega)$	Sobolev space: $\{u \in L^p(\Omega) : \nabla u \in (L^p(\Omega))^N\}$.

$W_0^{1,p}(\Omega)$ Closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$; functions vanishing on $\partial\Omega$.

a.e. Almost everywhere.

Introduction

The work presented in this dissertation concerns a particular case of elliptic-type partial differential equations involving the divergence operator:

$$Au = -\operatorname{div} \left(M(x) \frac{\nabla u}{1 + |u|} \right).$$

We are interested in studying the following problem

$$(P) \quad \begin{cases} Au + g(x)\nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with $N > 2$, and $f \in L^1(\Omega)$.

We first illustrate the main difficulties that may arise when studying problem (P):

- The non-linearity of the differential operator Au .
- The irregularity caused by $f \in L^1(\Omega)$, which implies that the right-hand side does not belong to the dual space $H_0^{-1}(\Omega)$.
- The operator A is not coercive on $W_0^{1,p}(\Omega)$, despite being well-defined between $H_0^1(\Omega)$ and its dual $H_0^{-1}(\Omega)$. The coercivity degenerates as $|u|$ becomes large, since $\frac{1}{1+|u|} \rightarrow 0$.

To resolve these issues, we approximate the operator by using truncations in $\frac{1}{1+|u|}$ to obtain a coercive differential operator. Then, we prove a priori estimates on the sequence of approximate solutions, and we finally pass to the limit in the approximate problems to establish the existence of a weak solution to problem (P).

Technique

In [1], David Arcoya et al . studied the existence of solutions to problem (P) where $f \in L^{p^*}(\Omega)$ and the operator is defined as:

$$A(v) = -\operatorname{div}(M(x)\nabla u).$$

The problem (P) was also considered in [1] for the case $f \in L^1(\Omega)$. **In this work, we aim to apply the ideas and results presented in [1] to a case of elliptic equations with degenerate coercivity. The method used to prove the existence of solutions $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is the same as that used in [1].**

We approximate problem (P) by a sequence of approximate problems (P_n) with data in $L^\infty(\Omega)$, for which the existence of solutions is guaranteed (see [1]). We then prove uniform estimates on the sequence of solutions and their partial derivatives. Using the linearity of the operator with respect to the gradient and the boundedness and continuity of the map $u \mapsto \nabla u$, we can pass to the limit and obtain a solution.

Thesis Plan

This thesis is divided into three chapters:

- **Chapter 1:** We present some basic definitions and results from functional analysis essential to the study of problem (P). This includes a review of functional spaces (Lebesgue and Sobolev) and brief reminders on weak and weak-* convergence.
- **Chapter 2:** We review properties of operators (bounded, hemicontinuous, monotone, and coercive). We present the method of monotone (pseudo-monotone) operators in a general framework to prove the existence of a solution to the equation

$$Au = -\operatorname{div}\left(M(x)\frac{\nabla u}{1 + |T_k(u)|}\right) + g(x)u = f,$$

with $M(x)$ is a bounded elliptic matrix and $f \in L^\infty(\Omega)$. We use Theorem 2.1 to establish existence of the solution u in $H_0^1(\Omega)$.

- **Chapter 3:** We study the existence of solutions to the equation:

$$(P) \quad \begin{cases} -\operatorname{div} \left(M(x) \frac{\nabla u}{1+|u|} \right) + g(x)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $M(x)$ is a bounded elliptic matrix and $0 \leq g(x) \in L^1$. Even if $f(x)$ only belongs to $L^1(\Omega)$, the assumption

$$\text{there exists } k > 0 \text{ such that } |f(x)| \leq k.g(x). \quad (1)$$

implies the existence of a weak solution u belonging to $H_0^1(\Omega)$ and to $L^\infty(\Omega)$.

Preliminaries

Presenting some analytical tools from the theory of Sobolev spaces that will be utilized throughout this thesis is the aim of this first chapter. Additionally, we will use this chance to present the primary notations.

1.1 Functional spaces

Definition 1.1 (Dual Space). . Let E be a vector space over $\mathbb{R}$ we recall that a functional is a function defined on E or on some subspace of E , with values in $\mathbb{R}$. We denote by $E' = \mathbb{L}(E, \mathbb{R})$ the dual space of E (the norm in E') is defined by:

$$\|f\|_{E'} = \sup_{x \in E, \|x\|=1} |f(x)| = \sup_{x \in E, \|x\| \leq 1} |f(x)| = \sup_{x \in E, \|x\| \neq 0} \frac{|f(x)|}{\|x\|}.$$

Definition 1.2 (Reflexive Spaces). . Let E be a Banach space and let i be the canonical injection $i : E \rightarrow E''$ is surjective , We say that E is reflexive if $i(E) = E''$.

1.1.1 Some properties

- The Mapping from to $L^p(\Omega)$ to $\mathbb{R}^+[6]$

$$\|u\|_p = \begin{cases} \left(\int_{\Omega} |u(x)|^p dx \right)^{\frac{1}{p}}, & 1 \leq p < +\infty, \\ \text{ess sup}_{x \in \Omega} |u(x)|, & p = \infty. \end{cases}$$

defines a norm on $L^p(\Omega)$, under which $L^p(\Omega)$ is a Banach space.

- **Dual Space:**[6] For any real $p \in (1, \infty)$, the dual space of $L^p(\Omega)$ is algebraically and topologically isomorphic to $L^q(\Omega)$, where:

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The duality pairing is given by:

$$\langle f, g \rangle = \int_{\Omega} f(x)g(x) dx, \quad \forall f \in L^p(\Omega), \quad \forall g \in L^q(\Omega).$$

This implies that $L^p(\Omega)$ is reflexive.

1.1.2 Some Inequalities

Theorem 1.1. [3] [Hölder's inequality] Let $1 \leq p \leq \infty$ and p' the conjugate exponent of p , Assume that $f \in L^p(\Omega)$, and $g \in L^{p'}(\Omega)$, then $f \cdot g \in L^1(\Omega)$ and:

$$\|f \cdot g\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}.$$

The case $p=p'=2$ is known as the Schwarz inequality

$$\int_{\Omega} uv dx = \left(\int_{\Omega} |u(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v(x)|^2 dx \right)^{\frac{1}{2}}$$

Theorem 1.2. [3] [Young's inequality] Let $1 < p < \infty$, then we have:

$$ab \leq \frac{1}{p}a^p + \frac{1}{p'}b^{p'}, \quad \forall a \geq 0, \quad \forall b \geq 0 \quad (1.1)$$

Theorem 1.3. [3] [Green's formulas] Let Ω be a domain in $\mathbb{R}^N$, And Let u and v be two regular functions. Then,

$$\int_{\Omega} (\Delta u)v \, dx = - \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, d\sigma$$

1.2 Sobolev Spaces of Order m

Sobolev spaces of order greater than one are necessary for the analysis of several partial differential equations, especially the biharmonic equation. Reference books like those written by R. Adams or J. Necas might be consulted for further information [6]. When considering derivatives in the sense of distributions, if f belongs to $L^p(\Omega)$, we denote its weak derivative as:

$$\frac{\partial u}{\partial x_i}$$

Definition 1.3. [6]

Let p be a real number such that $1 \leq p \leq \infty$, Ω an open subset of $\mathbb{R}^n$, and m an integer with $m \geq 2$. The Sobolev space of order m , denoted $W^{m,p}(\Omega)$ (or $H^m(\Omega)$ if $p = 2$), is defined as:

$$W^{m,p}(\Omega) = \left\{ u \in W^{m-1,p}(\Omega) \mid \forall i, 1 \leq i \leq n, \frac{\partial u}{\partial x_i} \in W^{m-1,p}(\Omega) \right\}.$$

- If $1 \leq p < \infty$, the norm is defined as:

$$\|u\|_{W^{m,p}} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p}^p \right)^{\frac{1}{p}}.$$

- If $p = \infty$, the norm is given by:

$$\|u\|_{W^{m,\infty}} = \max_{|\alpha| \leq m} \|D^\alpha u\|_\infty.$$

Definition 1.4. If p is a real number such that $1 \leq p < \infty$ and $m \geq 2$, the Sobolev space with zero support is defined as the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$, denoted $W_0^{m,p}(\Omega)$ (or $H_0^m(\Omega)$ if $p = 2$).

1.3 First-Order Sobolev Spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 1.5. The Sobolev space $W^{1,p}(\Omega)$, is defined by:

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega), \mid \nabla u \in (L^p(\Omega))^N \right\}$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_p + \|\nabla u\|_p \quad (1.2)$$

and for $p = \infty$

$$\|u\|_{W^{1,\infty}} = \max(\|u\|_{L^\infty}, \|\nabla u\|_{L^\infty}).$$

Definition 1.6. the space $D(\Omega)$ denotes the set of functions in $C^\infty(\Omega)$ with compact support in Ω , we define $1 \leq p < \infty$:

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega), \quad \text{with } u = 0 \quad \text{on } \partial\Omega \right\}.$$

Remark 1.1. Since, for $1 \leq p < \infty$, the space $D(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ by definition, we can identify the dual of $W_0^{1,p}(\Omega)$ with a subspace of the space of distributions $D'(\Omega)$. we denote this as: $W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))'$

In the particular case where $p = 2$, $H^1(\Omega)$, $H_0^1(\Omega)$, $H_0^{-1}(\Omega)$, are respectively denoted $H^1(\Omega)$, $H_0^1(\Omega)$, $H^{-1}(\Omega)$

The space H^1 is a Hilbert space equipped with the scalar product.

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + (u', v')_{L^2(\Omega)}$$

The associated norm

$$\|u\|_{H^1(\Omega)} = (\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2)^{1/2}$$

is a norm equivalent to that of $H^1(\Omega)$

The following proposition represents the fundamental properties of the spaces $W^{1,p}$, $W_0^{1,p}$. For the proof we refer the reader to reference [6].

Proposition 1.1. • The space $W^{1,p}$ is a Banach space for $1 \leq p \leq \infty$.

- The space $W^{1,p}$ is a separable space for $1 \leq p < \infty$.
- The space $W^{1,p}$ is a reflexive space for $1 < p < \infty$.
- The space $W_0^{1,p}$ is a separable space . Also reflexive for $1 < p < \infty$.
- The space H_0^1 is a separable Hilbert space.

1.4 Sobolev injections

Sobolev injections are widely used. they provide inequalities between the norms of sobolev spaces and L^p norms. This compactness result is a powerful tool in the study of PDEs, allowing us to transition from a sobolev space to a space of Lebesgue.

Theorem 1.4. [*Continuous Injection*[6]] Assume that Ω is bounded open subset of $\mathbb{R}^N$ with $N \geq 1$ and $1 \leq p < \infty$.

- $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q \in [1, p^*[$ if $p < N$. Where $p^* = \frac{Np}{N-p}$
- $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q \in [1, \infty[$ if $p = N$.
- $W_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. ,if $p > N$.

Theorem 1.5 (Compact Injection (Rellich-Kondrachon)[6]). Assume that Ω is bounded domain of class C^1 we have :

- $W_0^{1,p}(\Omega) \hookrightarrow_c L^q(\Omega)$, $\forall q \in [1, p^*[$ if $p < N$, where $p^* = \frac{Np}{N-p}$
- $W_0^{1,p}(\Omega) \hookrightarrow_c L^q(\Omega)$, $\forall q \in [1, \infty[$ if $p = N$.
- $W_0^{1,p}(\Omega) \hookrightarrow_c C(\overline{\Omega})$,if $p > N$.

Remark 1.2.

1. If Ω is not bounded, the injection $W^{1,p}(\Omega) \subset L^p(\Omega)$ is not compact in general.
2. The injection $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ is never compact even if Ω is bounded and regular

Lemma 1.1 (Poincare's inequality). Suppose that $1 \leq p < \infty$, and Ω is a bounded open set. Then there exists a constant C (depending on Ω and p) such that:

$$\| u \|_{L^p(\Omega)} \leq C \| \nabla u \|_{L^p(\Omega)} . \quad (1.3)$$

1.4.1 Weak convergence and (weak-*) convergence

Weak convergence

Definition 1.7. Let E be a Banach space, E' its dual and $\langle \cdot, \cdot \rangle$ the duality bracket EE' .

We say that the sequence (x_n) in E weakly converges to $x \in E$ if and only if:

$$\langle f, x_n \rangle \longrightarrow^{n \rightarrow +\infty} \langle f, x \rangle, \quad \forall f \in E'.$$

we denote this as

$$x_n \rightharpoonup x \quad \text{in } E.$$

Proposition 1.2. • if $x_n \longrightarrow x$ strongly then $x_n \rightharpoonup x$ in E .

• if $x_n \rightharpoonup x$ in E then $\|x_n\|_E$ is bounded.

•

$$\begin{cases} \text{If } x_n \rightharpoonup x, \text{ weakly in } E \\ \text{And } f_n \longrightarrow f \text{ strongly in } E' \end{cases}$$

then $\langle f_n, x_n \rangle \longrightarrow \langle f, x \rangle$ as $n \longrightarrow +\infty$.

Theorem 1.6. Let E be a reflexive Banach space. Then, for any bounded sequence $(x_n)_n \subset E$, there exists a sub-sequence on $(x_{n_k})_{n_k}$ that converges weakly in E .

(weak*) convergence

Definition 1.8. We say that the sequence (f_n) in E' weakly* converges to $f \in E'$ if and only if:

$$\langle f_n, g \rangle \longrightarrow^{n \rightarrow +\infty} \langle f, g \rangle, \quad \forall g \in E$$

we denote this as

$$f_n \rightharpoonup^* f \quad \text{in } E'.$$

Proposition 1.3.

$$f_n \longrightarrow f \quad \text{strongly} \implies f_n \rightharpoonup^* f \quad \text{in } E'$$

For more details, you can refer to the book[6].

1.4.2 Convergence theorems

Theorem 1.7 (Lebesgue's Dominated Convergence Theorem). *Let $p \in [1, \infty)$, and (u_n) be a sequence of functions in $L^p(\Omega)$ such that*

- $u_n \rightarrow u$ a.e in Ω .
- there is a function $g \in L^p(\Omega)$ satisfying:

$$|u_n(x)| \leq g(x) \quad \text{a.e in } \Omega \quad \forall n \in \mathbb{N}.$$

Then $u_n \rightarrow u$ in $L^p(\Omega)$, that is :

$$u \in L^p(\Omega), \quad \text{and} \quad \|u_n - u\|_{L^p(\Omega)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Theorem 1.8 (Lebesgue's Dominated Convergence Inverse Theorem). *Let (f_n) be a sequence in $L^p(\Omega)$ and let $f \in L^p(\Omega)$ be such that $\|f_n - f\|_{L^p(\Omega)} \rightarrow 0$*

Then ,there exist a subsequence (f_{n_k}) and a function $h \in L^p(\Omega)$ such that

- $f_{n_k}(x) \rightarrow f(x)$ a.e in Ω .
- $|f_{n_k}(x)| \leq h(x) \quad \forall K$ a.e in Ω .

Theorem 1.9. [2] *Let $1 < p < +\infty$, and let $\{u_n\}$ be a bounded sequence in $W_0^{1,p}(\Omega)$. Then, there exists a subsequence u_{n_k} and a function $u \in W_0^{1,p}(\Omega)$ such that:*

$$\nabla u_{n_k} \rightharpoonup \nabla u \quad \text{weakly in } L^p(\Omega).$$

Theorem 1.10 (Fatou's Lemma). *Let (u_n) be a sequence of function on of $L^1(\Omega)$ such that:*

- (1) for evry n , $u_n(x) \geq 0$
- (2) $\sup_n \int_{\Omega} u_n(x) \leq \infty$

for every $x \in \Omega$ we put $\liminf_{n \rightarrow \infty} u_n(x) dx$

$$\int_{\Omega} \liminf_{n \rightarrow \infty} u_n(x) dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} u_n(x) dx.$$

Elliptic Equations with Regular Data

In this chapter, we present some basic tools and results from functional analysis that are necessary to study the existence and properties of solutions to elliptic equations. We focus particularly on the theory of monotone and pseudo-monotone operators, which we will use later in the approximation framework in Chapter 3. For that purpose, we introduce fundamental notions such as monotonicity, hemicontinuity, coercivity, and pseudo-monotonicity, with simple illustrative examples for each. These concepts will enable us to prove a general existence theorem, which will serve as a basis for the approximation process later on.

2.1 Some Operators

2.1.1 Monotone Operators

Definition 2.1. *Let X be a reflexive and separable Banach space, and let A be a mapping from X to X' (possibly nonlinear). We say that:*

- A is **monotone** if:

$$\forall u, v \in X, \quad \langle Au - Av, u - v \rangle \geq 0.$$

- A is **strictly monotone** if additionally:

$$\forall u, v \in X, \quad \langle Au - Av, u - v \rangle = 0 \Rightarrow u = v.$$

Example 2.1. *The p -Laplacian is a typical example of a monotone operator and plays a fundamental role in the theory of nonlinear elliptic equations. It is defined as:*

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u), \quad \text{for } p > 1.$$

This operator acts on the Sobolev space $W_0^{1,p}(\Omega)$, and is monotone because the mapping $\xi \mapsto |\xi|^{p-2}\xi$ is monotone in $\mathbb{R}^N$.

More precisely, for all $u, v \in W_0^{1,p}(\Omega)$, we have:

$$\langle A(u) - A(v), u - v \rangle = \int_{\Omega} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot (\nabla u - \nabla v) dx \geq 0.$$

In the particular case where $p = 2$, the operator reduces to the classical Laplacian:

$$\Delta_2 u = \operatorname{div}(\nabla u) = \Delta u.$$

Example 2.2. Consider the linear operator $Au = -\Delta u$. The operator maps $H_0^1(\Omega)$ into its dual $H^{-1}(\Omega)$. It is monotone because, for all $u, v \in H_0^1(\Omega)$:

$$\langle A(u) - A(v), u - v \rangle = \int_{\Omega} \nabla(u - v) \cdot \nabla(u - v) dx = \|u - v\|_{H_0^1(\Omega)}^2 \geq 0.$$

2.1.2 Hemicontinuous Operators

Definition 2.2. Let X be a reflexive and separable Banach space, and let A be a mapping from X to X' . We say that A is **hemicontinuous** if for all $u, v \in X$, the function:

$$t \mapsto \langle A(u + tv), v \rangle$$

is continuous from $\mathbb{R}$ to $\mathbb{R}$.

Example 2.3. The operator $A : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ defined by:

$$Au = -\Delta u$$

is hemicontinuous. Indeed, for any $u, v \in H_0^1(\Omega)$ and $t \in \mathbb{R}$, we have:

$$\langle A(u + tv), v \rangle = \int_{\Omega} \nabla(u + tv) \cdot \nabla v dx.$$

Expanding the integral, we get:

$$\int_{\Omega} \nabla u \cdot \nabla v dx + t \int_{\Omega} |\nabla v|^2 dx.$$

This shows that $t \mapsto \langle A(u + tv), v \rangle$ is a linear function of t and hence continuous.

2.1.3 Coercive Operators

Definition 2.3. An operator $A : V \rightarrow V'$ is said to be **coercive** if:

$$\lim_{\|v\|_V \rightarrow +\infty} \frac{\langle Av, v \rangle}{\|v\|_V} = +\infty.$$

Example 2.4. The operator $A : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ defined by:

$$Au = -\operatorname{div}(\nabla u) + a(x)u = -\Delta u + a(x)u, \quad \text{where } a(x) \geq 0, a(x) \in L^\infty(\Omega)$$

is coercive. Indeed, we have:

$$\frac{\langle Av, v \rangle}{\|v\|_{H_0^1}} = \frac{\int_\Omega |\nabla v|^2 dx + \int_\Omega a(x)v^2 dx}{\|v\|_{H_0^1}}.$$

Since $a(x) \geq 0$, we obtain:

$$\frac{\langle Av, v \rangle}{\|v\|_{H_0^1}} \geq \alpha \|v\|_{H_0^1} \rightarrow +\infty \quad \text{as } \|v\|_{H_0^1} \rightarrow +\infty.$$

2.1.4 Bounded Operators

Definition 2.4. Let V and W be two Banach spaces, and let $A : V \rightarrow W$ be an operator.

We say that A is **bounded** if it maps every bounded set in V to a bounded set in W , i.e.,

$$\forall \rho > 0, \exists C_\rho > 0 : \quad A(B_V(0, \rho)) \subset B_W(0, C_\rho).$$

where $B_V(0, \rho)$ denotes the open ball in V centered at 0 with radius $\rho > 0$, and $B_W(0, C_\rho)$

denotes the open ball in W centered at 0 with radius $C_\rho > 0$.

Example 2.5. The operator $Au = -\Delta_p$ is bounded from $W_0^{1,p}(\Omega)$ in $W^{-1,p'}(\Omega)$

From the expression of the norm in a dual space, let $\rho > 0$, for $u \in B_V(0, \rho)$, we can write:

$$\|Au\|_{V'} = \sup_{\substack{\varphi \in V \\ \|\varphi\| \leq 1}} |\langle Au, \varphi \rangle| = \sup_{\substack{\varphi \in V \\ \|\varphi\| \leq 1}} \left| \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \right|.$$

So,

$$\begin{aligned}
 \left| \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \right| &\leq \int_{\Omega} |\nabla u|^{p-1} |\nabla \varphi| \\
 &\leq \left(\int_{\Omega} |\nabla u|^p \right)^{\frac{1}{p}} \left(\int_{\Omega} |\nabla \varphi|^p \right)^{\frac{1}{p}} \\
 &= \|u\|_V^{p-1} \|\varphi\|_V \\
 &\leq \rho^{p-1}
 \end{aligned}$$

Hence $\|Au\|_{V'} \leq \rho^{p-1}$. this shows that $A(B_V(0, \rho)) \subset B_{V'}(0, \rho^{p-1})$

2.1.5 Pseudo-monotone Operators

Definition 2.5. Let V be a separable and reflexive Banach space. An operator $A : V \rightarrow V'$ is pseudo-monotone if:

- (i) A is bounded, that is, the image of a bounded subset of V is a bounded subset of V'
- (ii) whenever $u_j \rightharpoonup u$ in V weak and if $\limsup_{j \rightarrow +\infty} \langle A(u_j), u_j - v \rangle \leq 0$ then

$$\liminf_{j \rightarrow +\infty} \langle A(u_j), u_j - v \rangle \geq \langle A(u), u - v \rangle, \quad \forall v \in V. \quad (2.1)$$

where $\langle . \rangle$ refers to the duality product between V' and V .

Proposition 2.1. If $A : V \rightarrow V'$ is bounded, hemicontinuous and monotonic, then A is pseudo-monotone.

Proof. a) Let $\{u_j\}$ a sequence weakly converging to u in V . Suppose that

$$\limsup_{j \rightarrow \infty} \langle Au_j, u_j - u \rangle \leq 0$$

If A is monotone, we have

$$\lim_{j \rightarrow \infty} \langle Au_j, u_j - u \rangle \rightarrow 0 \quad (2.2)$$

Indeed, the monotonicity of A and the weak convergence of $\{u_j\}$ towards u implies that

$$\langle Au_j, u_j - u \rangle \geq \langle Au, u_j - u \rangle \rightarrow 0 \quad \text{pour } j \rightarrow \infty$$

And so

$$0 \geq \limsup_{j \rightarrow \infty} \langle Au_j, u_j - u \rangle \geq \liminf_{j \rightarrow \infty} \langle Au_j, u_j - u \rangle \geq \limsup_{j \rightarrow \infty} \langle Au, u_j - u \rangle = 0$$

Hence (2.2)

b) for $v \in V$ and $t \in]0, 1[$, let $w = (1 - t)u + tv$. We have $\langle Au_j - Aw, u_j - w \rangle$ so that:

$$t \langle Au_j, u - v \rangle \geq -\langle Au_j, u_j - u \rangle + \langle Aw, u_j - u \rangle - t \langle Aw, v - u \rangle.$$

From which, thanks to (2.2):

$$t \liminf_j \langle Au_j, u - v \rangle \geq -t \langle Aw, v - u \rangle,$$

from which, dividing by t and taking into account (2.2):

$$\liminf_j \langle Au_j, u_j - v \rangle \geq \langle Aw, u - v \rangle \quad (2.3)$$

$$w = (1 - t)u + tv \quad \forall t \in]0, 1[$$

By making t tend towards 0 in (2.3), and using hemicontinuity, we deduce

$$\liminf_{j \rightarrow \infty} \langle Au_j, u_j - v \rangle \geq \langle Au, u - v \rangle, \quad \forall v \in V$$

Which means that A is pseudo-monotonic. □

2.2 Monotonic operator theorems

2.2.1 General result

Let V be a reflexive and separable Banach space, and $A : V \rightarrow V'$ be an operator.

Theorem 2.1 (Existence theorem). *Suppose that $A : V \rightarrow V'$ is an operator :*

- *bounded*
- *hemicontinuous*
- *coercive*

- *monotone*

Let $f \in V'$. Then, there exists a function $u \in V$ and only one such that:

$$Au = f$$

For the proof this theorem, we refer the reader to the book [7]

Corollary 2.1. *If A is a coercive pseudomonotone operator, then*

$$\forall f \in V', \quad \exists u \in V \quad \text{such that} \quad Au = f.$$

2.3 Application

We consider the following problem:

$$\begin{cases} -\operatorname{div} \left(\frac{M(x)\nabla u}{1+|T_n(u)|} \right) + g(x)u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.4)$$

Where Ω be a bounded open set in $\mathbb{R}^n$, and $M(x)$ be a bounded elliptic matrix satisfying, for some positive constants α, β , the following condition for every $\xi \in \mathbb{R}^N$:

$$M(x)\xi \cdot \xi \geq \alpha|\xi|^2, \quad |M(x)| \leq \beta. \quad (2.5)$$

Additionally, we assume that

$$0 \leq g(x) \in L^\infty(\Omega), \quad f \in L^\infty(\Omega).$$

We define the following operator:

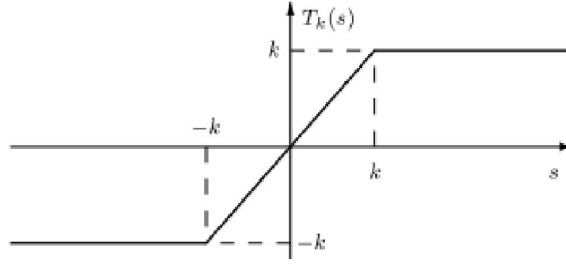
$$A_n(u) = -\operatorname{div} \left(\frac{M(x)\nabla u}{1+|T_n(u)|} \right) + g(x)u,$$

where:

- T_n is th function of $\mathbb{R}$ from $\mathbb{R}$ given by

$$T_k(s) = \begin{cases} k, & \text{if } s \geq k, \\ s, & \text{if } |s| < k, \\ -k, & \text{if } s \leq -k. \end{cases}$$

It can be verified that the function T_k is a globally Lipschitz function, The graph of T_k is:



We can verify that the function T_k is a globally Lipschitz function satisfying $|T_k(s)| \leq k$ and $|T_k(s)| \leq |s|$.

Lemma 2.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$. The operator A_n is pseudo-monotone from $H_0^1(\Omega)$ into $H_0^{-1}(\Omega)$. Moreover, A_n is coercive in the following sense:*

$$\frac{\langle A_n v, v \rangle}{\|v\|_{H_0^1(\Omega)}} \rightarrow \infty \quad \text{as} \quad \|v\|_{H_0^1(\Omega)} \rightarrow \infty.$$

Proof. The proof is based on the assertion that the operator A_n is pseudomonotone and coercive.

(1) The operator A_n is coercive. From the ellipticity condition (2.5), we obtain:

$$\langle A_n v, v \rangle = \int_{\Omega} \frac{M(x) \nabla v \cdot \nabla v}{1 + |T_n(v)|} dx + \int_{\Omega} g(x) v^2 dx.$$

Using the given bound:

$$1 + |T_n(v)| \leq 1 + n \quad \Rightarrow \quad \frac{1}{1 + |T_n(v)|} \geq \frac{1}{1 + n}.$$

and dropping the nonnegative term, we find

$$\langle A_n v, v \rangle = \int_{\Omega} \frac{M(x) \nabla v \cdot \nabla v}{1 + |T_n(v)|} dx \geq \frac{\alpha}{1 + n} \int_{\Omega} |\nabla v|^2 dx = \frac{\alpha}{1 + n} \|v\|_{H_0^1(\Omega)}^2.$$

which implies

$$\lim_{\|v\|_{H_0^1(\Omega)} \rightarrow +\infty} \frac{\langle A_n v, v \rangle}{\|v\|_{H_0^1(\Omega)}} \longrightarrow +\infty.$$

as $\|v\|_{H_0^1(\Omega)} \longrightarrow +\infty$ since $\frac{\alpha}{1+n} > 0$.

(2) The operator A_n is pseudomonotone.

(i) A_n is bounded. Indeed, let u be a bounded function in $H_0^1(\Omega)$ that is : let $\rho > 0$, for $u \in B_{H_0^1(\Omega)}(0, \rho)$, and for all $v \in H_0^1(\Omega)$, we have

$$\|A_n u\|_{H_0^{-1}(\Omega)} = \sup_{\substack{v \in H_0^1(\Omega) \\ \|v\| \leq 1}} |\langle A_n u, v \rangle|.$$

then, we have

$$\langle A_n u, v \rangle = \int_{\Omega} \frac{M(x) \nabla u}{1 + |T_n(u)|} \nabla v dx + \int_{\Omega} g(x) u \cdot v dx$$

because that $1 + |T_n(u)| \geq 1$, we get

$$|\langle A_n u, v \rangle| \leq \int_{\Omega} |M(x)| |\nabla u| \cdot |\nabla v| dx + \int_{\Omega} g(x) |u| \cdot |v| dx$$

Since (2.5) and the fact that $g \in L^\infty(\Omega)$, we have

$$|\langle A_n u, v \rangle| \leq \beta \int_{\Omega} |\nabla u| \cdot |\nabla v| dx + \sup_{x \in \Omega} g(x) \int_{\Omega} |u| \cdot |v| dx$$

Using the cauchy Schwarz inequality, we obtain

$$|\langle A_n u, v \rangle| \leq \beta \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} + M \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}, \quad M = \sup_{x \in \Omega} g(x).$$

Since $\|u\|_{L^2(\Omega)} \leq \|u\|_{H_0^1(\Omega)}$ and $\|v\|_{H_0^1(\Omega)} \leq 1$, we have

$$|\langle A_n u, v \rangle| \leq (\beta + M) \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \leq (\beta + M) \rho.$$

Hence $\|A_n u\|_{H_0^{-1}(\Omega)} \leq (\beta + M) \rho$. this shows that

$$A_n(B_{H_0^1(\Omega)}(0, \rho)) \subset B_{H_0^{-1}(\Omega)}(0, (\beta + M) \rho).$$

(ii) If $u_m \rightharpoonup u$ weakly in $H_0^1(\Omega)$, as $m \rightarrow +\infty$, and for any $v \in H_0^1(\Omega)$

$$\begin{aligned} 0 &\geq \limsup_{m \rightarrow +\infty} \langle A_n u_m, u_m - v \rangle \\ &= \limsup_{m \rightarrow +\infty} \left(\int_{\Omega} \frac{M(x) \nabla u_m}{1 + |T_n(u_m)|} \nabla (u_m - v) dx + \int_{\Omega} g(x) u \cdot (u_m - v) dx \right). \end{aligned}$$

The compact embedding (see Theorem 1.4), yields that $u_m \rightarrow u$ in $L^2(\Omega)$ for a subsequence still denoted as (u_m) . Moreover, we assume that $u_m \rightarrow u$ a.e. in Ω and using the fact that $\nabla u_m \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$, and by the definition of weak convergence, we get

$$\frac{M(x)\nabla u_m}{1 + |T_n(u_m)|} \rightarrow \frac{M(x)\nabla u}{1 + |T_n(u)|} \quad \text{strongly in } L^2(\Omega), \quad (2.6)$$

and

$$g(x)u \cdot (u_m - u)dx \rightarrow 0 \quad \text{as } m \rightarrow +\infty$$

and we have that

$$\int_{\Omega} \left(\frac{M(x)\nabla u_m}{1 + |T_n(u_m)|} - \frac{M(x)\nabla u}{1 + |T_n(u)|} \right) \nabla(u_m - u)dx \rightarrow 0 \quad \text{as } m \rightarrow +\infty \quad (2.7)$$

Now, let us prove that

$$\liminf_{m \rightarrow +\infty} \langle A_n u_m, u_m - v \rangle \geq \langle A_n u, u - v \rangle, \quad \forall v \in H_0^1(\Omega).$$

Because that

$$\frac{M(x)\nabla u_m}{1 + |T_n(u_m)|} \rightharpoonup \frac{M(x)\nabla u}{1 + |T_n(u)|} \quad \text{weakly in } L^2(\Omega),$$

we have

$$\lim_{m \rightarrow +\infty} \int_{\Omega} \frac{M(x)\nabla u_m \nabla v}{1 + |T_n(u_m)|} dx = \int_{\Omega} \frac{M(x)\nabla u \nabla v}{1 + |T_n(u)|} dx, \quad \forall v \in H_0^1(\Omega). \quad (2.8)$$

By virtue of Fatou's lemma, we get

$$\liminf_{m \rightarrow +\infty} \int_{\Omega} \frac{M(x)\nabla u_m \nabla u_m}{1 + |T_n(u_m)|} dx \geq \int_{\Omega} \frac{M(x)\nabla u \nabla u}{1 + |T_n(u)|} dx \quad (2.9)$$

Finally, combining (3.15) and (2.9), we obtain

$$\begin{aligned} \liminf_{m \rightarrow +\infty} \langle A_n u_m, u_m - v \rangle &= \liminf_{m \rightarrow +\infty} \int_{\Omega} \frac{M(x)\nabla u_m \nabla(u_m - v)}{1 + |T_n(u_m)|} dx \\ &\geq \int_{\Omega} \frac{M(x)\nabla u \nabla(u - v)}{1 + |T_n(u)|} dx = \langle A_n u, u - v \rangle. \end{aligned}$$

Therefore A_n is pseudomonotone. Then, according to the corollary 2.1, there exists at least one solution u in $H_0^1(\Omega)$ to Problem 2.4.

□

Degenerate elliptic equations with lower-order terms and L^1 data

In this chapter, we aim to prove the existence of weak solutions for a class of **degenerate elliptic equations** involving lower-order terms. Specifically, we consider the problem:

$$(P) \quad \begin{cases} -\operatorname{div} \left(\frac{1}{1+|u|} M(x) \nabla u \right) + g(x)u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, M is a bounded elliptic matrix which satisfies, for some positive constants α, β , for every $\xi \in \mathbb{R}$

$$\alpha|\xi|^2 \leq M(x)\xi \cdot \xi, \tag{3.1}$$

$$|M(x)| \leq \beta. \tag{3.2}$$

and $0 \leq g(x) \in L^1$. Even if $f(x)$ only belongs to $L^1(\Omega)$, the assumption

$$f(x) \in L^1(\Omega) \tag{3.3}$$

$$\text{there exists } k > 0 \text{ such that } |f(x)| \leq k.g(x). \tag{3.4}$$

implies the existence of a weak solution u belonging to $H_0^1(\Omega)$ and to $L^\infty(\Omega)$.

3.1 Weak Solutions

3.1.1 Definition

Definition 3.1. A function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is called a weak solution to problem (P) if it satisfies the integral equation:

$$\int_{\Omega} \frac{M(x) \nabla u \cdot \nabla \phi}{1 + |u|} dx + \int_{\Omega} g(x) u \phi dx = \int_{\Omega} f \phi dx, \quad \forall \phi \in H_0^1(\Omega) \cap L^\infty(\Omega). \quad (3.5)$$

3.1.2 Existence Theorem

Theorem 3.1. Let $f \in L^1(\Omega)$. Then, the problem (P) has at least one weak solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ in the sense of Definition 3.1.

Remark 3.1. The differential operator:

$$u \mapsto -\operatorname{div} \left(\frac{M(x) \nabla u}{1 + |u|} \right) \quad (3.6)$$

is not coercive on $H_0^1(\Omega)$, despite being well-defined between $H_0^1(\Omega)$ and its dual $H_0^{-1}(\Omega)$. Degenerate coercivity implies that as $|u|$ becomes large, $\frac{1}{1+|u|}$ tends to zero. This prevents the application of classical methods for elliptic operators. To address this, we approximate the operator by using truncations in $\frac{1}{1+|u|}$ to obtain a coercive differential operator. We then establish a priori estimates on the sequence of approximate solutions and finally pass to the limit in the approximate problems to prove the existence of a weak solution for problem (P).

3.1.3 the truncation functions

Lemma 3.1 (Stampacchia). Let $T : \mathbb{R} \rightarrow \mathbb{R}$ be a globally lipschitz function, i.e.

$$\exists C > 0 \quad \text{such that} \quad |T(s) - T(t)| \leq C|s - t|, \quad \forall s, t \in \mathbb{R},$$

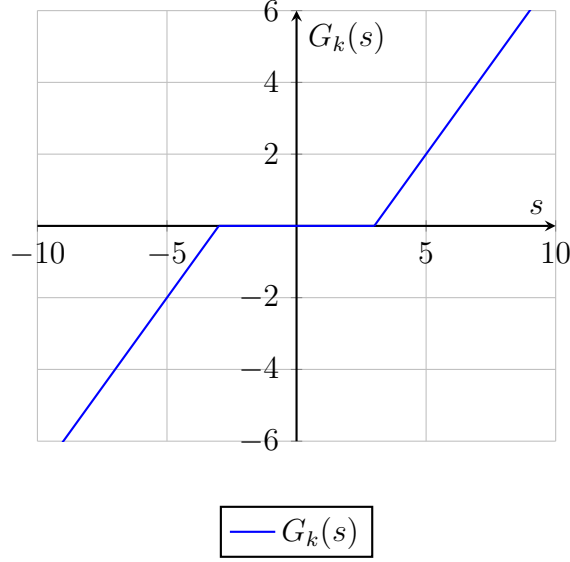
such that $T(0) = 0$.

Then, $\forall v \in W_0^{1,p}(\Omega)$ with $1 \leq p \leq \infty$ we have :

$$T(v) \in W_0^{1,p}(\Omega) \quad \text{and} \quad \nabla T(v) = T'(v) \nabla v \quad \text{in} \quad \mathcal{D}'(\Omega) \quad \text{and} \quad \text{almost everywhere in } \Omega.$$

Example 3.1. Let $k > 0$. We will use the following function defined for $r \in \mathbb{R}$ by

$$G_k(s) = s - T_k(s) = \begin{cases} s - k, & \text{if } s > k, \\ 0, & \text{if } |s| \leq k, \\ s + k, & \text{if } s < -k. \end{cases}$$



so,

$$G_k(u_n) = u_n - T_k(u_n) = \begin{cases} u_n - k, & \text{if } u_n > k, \\ 0, & \text{if } |u_n| \leq k, \\ u_n + k, & \text{if } u_n < -k. \end{cases}$$

and

$$\nabla G_k(u_n) = \begin{cases} \nabla u_n, & \text{if } |u_n| > k, \\ 0, & \text{if } |u_n| \leq k. \end{cases}$$

3.2 Approximation of Problem (P)

We consider the following approximated problem (P_n) :

$$(P_n) \quad \begin{cases} -\operatorname{div} \left(\frac{M(x) \nabla u_n}{1 + |T_n(u_n)|} \right) + g_n(x) u_n = f_n(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

i.e., satisfying

$$\int_{\Omega} M(x) \frac{\nabla u_n}{1 + |T_n(u_n)|} \nabla \varphi dx + \int_{\Omega} g_n u_n \varphi dx = \int_{\Omega} f_n \varphi dx, \quad (3.7)$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, where

$$T_n(s) = \max(-n, \min(n, s)),$$

is the truncation at level n ,

$$f_n(x) = \frac{f(x)}{1 + \frac{1}{n}|f(x)|} \in L^\infty(\Omega),$$

and

$$g_n(x) = \frac{g(x)}{1 + \frac{1}{n}|g(x)|} \in L^\infty(\Omega).$$

Justification of the Estimates for f_n and g_n

1. Boundedness of the approximating functions. We recall the definition of the approximating functions:

$$f_n(x) = \frac{f(x)}{1 + \frac{1}{n}|f(x)|}, \quad g_n(x) = \frac{g(x)}{1 + \frac{1}{n}|g(x)|}.$$

Taking the absolute value of $f_n(x)$, we obtain

$$|f_n(x)| = \left| \frac{f(x)}{1 + \frac{1}{n}|f(x)|} \right| = \frac{|f(x)|}{1 + \frac{1}{n}|f(x)|}.$$

Let us define the function $\varphi(s) = \frac{s}{1 + \frac{1}{n}s}$ for $s \geq 0$. We observe that φ is increasing and satisfies $\varphi(s) \leq s$ for all $s \geq 0$. Therefore, applying this to $|f(x)|$, we deduce:

$$|f_n(x)| = \varphi(|f(x)|) \leq |f(x)|, \quad \forall x \in \Omega.$$

In the same way, we obtain

$$|g_n(x)| \leq |g(x)|, \quad \forall x \in \Omega.$$

2. Uniform boundedness by n . We aim to prove that $|f_n(x)| \leq n$ for all $x \in \Omega$. Recall

$$|f_n(x)| = \frac{|f(x)|}{1 + \frac{1}{n}|f(x)|}.$$

Let us define again the function $\varphi(s) = \frac{s}{1+\frac{1}{n}s}$ for $s \geq 0$. We observe:

$$\varphi(s) = \frac{s}{1 + \frac{1}{n}s} = \frac{ns}{n + s}.$$

Since $\frac{ns}{n+s} \leq n$ for all $s \geq 0$, it follows that:

$$|f_n(x)| = \varphi(|f(x)|) \leq n, \quad \forall x \in \Omega.$$

Similarly, we obtain

$$|g_n(x)| \leq n, \quad \forall x \in \Omega.$$

Hence, both f_n and g_n are uniformly bounded in $L^\infty(\Omega)$ by n .

L^1 norm estimate. We show that $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$. Indeed, for all $x \in \Omega$,

$$|f_n(x)| = \frac{|f(x)|}{1 + \frac{1}{n}|f(x)|} \leq |f(x)|.$$

Therefore, we have the pointwise inequality

$$|f_n(x)| \leq |f(x)| \quad \text{a.e. in } \Omega.$$

Using this and applying the Lebesgue monotone convergence theorem (or directly integrating the inequality), we obtain

$$\|f_n\|_{L^1(\Omega)} = \int_{\Omega} |f_n(x)| dx \leq \int_{\Omega} |f(x)| dx = \|f\|_{L^1(\Omega)}.$$

Similarly, we have

$$\|g_n\|_{L^1(\Omega)} \leq \|g\|_{L^1(\Omega)}.$$

3. Dominance condition. Assume that there exists a constant $k > 0$ such that:

$$|f(x)| \leq kg(x) \quad \text{a.e. in } \Omega.$$

We want to prove that this implies

$$|f_n(x)| \leq kg_n(x), \quad \forall x \in \Omega.$$

Using the definitions of f_n and g_n , we write

$$|f_n(x)| = \frac{|f(x)|}{1 + \frac{1}{n}|f(x)|}, \quad g_n(x) = \frac{g(x)}{1 + \frac{k}{n}|g(x)|}.$$

Using the definition of ψ , we estimate

$$|f_n(x)| = \frac{|f(x)|}{1 + \frac{1}{n}|f(x)|} \leq \frac{kg(x)}{1 + \frac{1}{n}kg(x)} = kg_n(x).$$

Hence, we deduce

$$|f_n(x)| \leq kg_n(x), \quad \forall x \in \Omega. \quad (3.8)$$

This proves the dominance condition for the approximated data.

Conclusion. As a result, the sequences (f_n) and (g_n) satisfy the following properties

- $f_n, g_n \in L^\infty(\Omega)$ and are uniformly bounded by n ,
- $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}, \|g_n\|_{L^1(\Omega)} \leq \|g\|_{L^1(\Omega)},$
- If $|f(x)| \leq kg(x)$ in Ω , then $|f_n(x)| \leq kg_n(x).$

These approximated functions allow us to define a family of regularized problems (P_n) with bounded data, which are suitable for the application of existence results.

The operator associated with (P_n) is defined by

$$A_n(u) := -\operatorname{div} \left(\frac{M(x)\nabla u}{1 + |T_n(u)|} \right) + g_n(x)u.$$

From Chapter 2, it follows that $A_n : H_0^1(\Omega) \rightarrow H_0^{-1}(\Omega)$ is bounded, coercive, and pseudo-monotone. Hence, by the general existence theorem (Theorem 2.1), there exists a weak solution $u_n \in H_0^1(\Omega)$ to problem (P_n) .

3.3 Uniform Estimates of Approximate solutions

Lemma 3.2. *Let $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be the weak solution to the approximate problem (P_n) . Then, there exists a constant $C > 0$, independent of n , such that*

$$\|u_n\|_{L^\infty(\Omega)} \leq C, \quad \|u_n\|_{H_0^1(\Omega)} \leq C.$$

Proof. In order to establish uniform estimates for the approximate solutions u_n , we follow the method of truncation. Let $k > 0$ and consider the truncation function defined by

$$G_k(u_n) = u_n - T_k(u_n) = \begin{cases} u_n - k & \text{if } u_n > k, \\ 0 & \text{if } |u_n| \leq k, \\ u_n + k & \text{if } u_n < -k, \end{cases}$$

which belongs to $H_0^1(\Omega)$ by Stampacchia's lemma.

We choose $G_k(u_n)$ as a test function in the weak formulation of problem (P_n) , that is:

$$\int_{\Omega} \frac{1}{1 + |T_n(u_n)|} M_n(x) \nabla u_n \cdot \nabla G_k(u_n) + \int_{\Omega} g_n(x) u_n G_k(u_n) = \int_{\Omega} f_n(x) G_k(u_n).$$

Using the ellipticity condition (3.1), and the fact that $\nabla G_k(u_n) = \nabla u_n$ on the set $\{|u_n| > k\}$, we obtain

$$\frac{\alpha}{1+n} \int_{\Omega} |\nabla G_k(u_n)|^2 + \int_{\Omega} g_n(x) u_n G_k(u_n) \leq \int_{\Omega} |f_n(x)| |G_k(u_n)|.$$

Dropping the non negative term and using that $u_n G_k(u_n) \geq 0$, we have that by (3.8)

$$\int_{\Omega} g_n(x) (|u_n| - k) |G_k(u_n)| \leq 0,$$

which implies that

$$|u_n| \leq k.$$

Hence, we have the uniform estimate

$$\boxed{\|u_n\|_{L^\infty(\Omega)} \leq k.} \tag{3.9}$$

To estimate the norm in $H_0^1(\Omega)$, we take u_n itself as a test function in the weak formulation of (P_n) , yielding:

$$\int_{\Omega} \frac{1}{1 + |T_n(u_n)|} M_n(x) |\nabla u_n|^2 dx + \int_{\Omega} g_n(x) |u_n|^2 dx \leq \int_{\Omega} |f_n(x)| |u_n| dx.$$

Using again the condition (3.1), (3.9) and dropping the non negative term, we obtain

$$\frac{\alpha}{1+k} \int_{\Omega} |\nabla u_n|^2 dx \leq k \int_{\Omega} |f_n(x)| dx,$$

which implies that

$$\boxed{\|u_n\|_{H_0^1(\Omega)} \leq C.}$$

where $C = \frac{k(1+k)\|f\|_{L^1(\Omega)}}{\alpha}$. This completes the proof of the a priori estimates, independently of n . $\square$

3.4 Passage to the limit

Proposition 3.1.

$$u_n \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \quad (3.10)$$

$$u_n \longrightarrow u \quad \text{strongly in } L^2(\Omega). \quad (3.11)$$

Therefore

$$u_n \longrightarrow u \quad \text{a.e in } \Omega, \quad (3.12)$$

For $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (See (3.7)), we have

$$\int_{\Omega} M(x) \frac{\nabla u_n}{1 + |T_n(u)|} \nabla \varphi dx + \int_{\Omega} g_n(x) u_n \varphi dx = \int_{\Omega} f_n \varphi dx,$$

Proof. 1) **Weak convergence in $H_0^1(\Omega)$:**

From Lemma 3.2, we know that the sequence (u_n) is uniformly bounded in $H_0^1(\Omega)$, i.e.,

$$\|u_n\|_{H_0^1(\Omega)} \leq C, \quad \forall n \in \mathbb{N},$$

for some constant $C > 0$ independent of n . Since $H_0^1(\Omega)$ is a reflexive Banach space, by (theorem 1.6), there exists a function $u \in H_0^1(\Omega)$ and a subsequence (still denoted u_n) such that:

$$u_n \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega).$$

2) Strong convergence in $L^2(\Omega)$:

The compact embedding of Sobolev spaces (Rellich–Kondrachov theorem) implies that:

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega),$$

i.e., the injection is compact. Thus, the weak convergence in $H_0^1(\Omega)$ and the uniform boundedness of (u_n) yield:

$$u_n \rightarrow u \quad \text{strongly in } L^2(\Omega).$$

3) Almost everywhere convergence (via Inverse Dominated Convergence Theorem):

From Lemma 3.2, we know that :

$$\|u_n\|_{L^\infty(\Omega)} \leq C.$$

Moreover, from step 2 we have that:

$$u_n \rightarrow u \quad \text{strongly in } L^2(\Omega).$$

Then, by the inverse of the Dominated Convergence Theorem (see Theorem 1.8), there exists a subsequence (u_{n_k}) and a function $h \in L^2(\Omega)$ such that:

- $u_{n_k}(x) \rightarrow u(x)$ almost everywhere in Ω ,
- $|u_{n_k}(x)| \leq h(x)$ a.e. in Ω .

But since $|u_n(x)| \leq C$ uniformly, we may take $h(x) \equiv C \in L^2(\Omega)$.

Therefore, we conclude:

$$u_n(x) \rightarrow u(x) \quad \text{almost everywhere in } \Omega.$$

□

1) Passage to the limit in $\int_{\Omega} f_n \varphi dx$

Using that $f_n \rightarrow f$ strongly in $L^1(\Omega)$, we have $\forall \varphi \in L^\infty(\Omega)$

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f_n \varphi dx = \int_{\Omega} f \varphi dx.$$

2) Passage to the limit in $\int_{\Omega} g_n(x) u_n \varphi dx$.

Thanks to (3.9), we have

$$\int_{\Omega} g_n(x) u_n dx \leq \int_{\Omega} g_n(x) |u_n| dx \leq k \int_{\Omega} g_n(x) dx. \quad (3.13)$$

Since $g_n(x) \in L^\infty(\Omega)$, the sequence $\{g_n(x) u_n\}_n$ is bounded in $L^1(\Omega)$ and thanks to (3.12), we have that

$$g_n(x) u_n \rightarrow g(x) u \quad \text{strongly in } L^1(\Omega). \quad (3.14)$$

So, we have

$$\begin{aligned} \left| \int_{\Omega} g_n(x) u_n \varphi - \int_{\Omega} g(x) u \varphi \right| &= \left| \int_{\Omega} (g_n(x) u_n - g(x) u) \varphi \right| \\ &\leq C \|g_n(x) u_n - g(x) u\|_{L^1(\Omega)} \rightarrow 0, \end{aligned}$$

as $n \rightarrow +\infty$

ensure that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} g_n(x) u_n \varphi dx = \int_{\Omega} g(x) u \varphi dx.$$

3) Passage to the limit in $\int_{\Omega} M(x) \frac{\nabla u_n}{1+|T_n(u_n)|} \nabla \varphi dx$.

Using the fact that $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$ and (3.11), we have

$$\frac{M(x) \nabla u_n}{1+|T_n(u_n)|} \rightharpoonup \frac{M(x) \nabla u}{1+|T(u)|} \quad \text{strongly in } L^2(\Omega),$$

So,

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \frac{M(x) \nabla u_n \nabla \varphi}{1 + |T_n(u_n)|} dx = \int_{\Omega} \frac{M(x) \nabla u \nabla \varphi}{1 + |T(u)|} dx, \quad \forall \varphi \in H_0^1(\Omega). \quad (3.15)$$

Finally, we pass to the limit in the formulation (3.7), we obtain

$$\int_{\Omega} M(x) \frac{\nabla u}{1 + |u|} \nabla \varphi dx + \int_{\Omega} g(x) u \varphi dx = \int_{\Omega} f \varphi dx,$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Therefore, we have to prove that u is a solution to problem (P). This finishes the proof of theorem 3.1.

Conclusion

In this work, we have studied the existence of weak solutions to a class of degenerate elliptic problems of the form

$$\begin{cases} -\operatorname{div} \left(\frac{M(x)\nabla u}{1+|u|} \right) + g(x)u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $M(x)$ is a bounded elliptic matrix, $g \in L^1(\Omega)$ is a nonnegative function, and $f \in L^1(\Omega)$ satisfies the domination condition $|f(x)| \leq kg(x)$ for some constant $k > 0$.

The main difficulty in this problem lies in the degeneracy of the operator, which causes the lack of coercivity, and the low regularity of the data. To address these challenges, we introduced an approximation scheme using truncation functions and constructed a sequence of regularized problems of the form

$$A_n(u) = -\operatorname{div} \left(\frac{M(x)\nabla u}{1+|T_n(u)|} \right) + g_n(x)u,$$

where T_n is a standard truncation function and g_n is a bounded approximation of g .

By employing pseudo-monotonicity arguments and the theory of monotone operators, we proved the existence of bounded weak solutions $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ for the approximate problems. We then established uniform a priori estimates and used compactness tools to pass to the limit and obtain a weak solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ for the original degenerate problem.

This result highlights the effectiveness of regularization methods and the pseudo-monotonicity framework in addressing elliptic equations with degenerate coercivity and minimal assumptions on the data.

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Abstract

This thesis investigates the existence of weak solutions for a class of degenerate elliptic equations with lower-order terms and right-hand side data in $L^1(\Omega)$. The problem under consideration is of the form:

$$\begin{cases} -\operatorname{div} \left(\frac{M(x)\nabla u}{1+|u|} \right) + g(x)u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open domain, $M(x)$ is a bounded and elliptic matrix, $g(x) \in L^1(\Omega)$ is a nonnegative lower-order coefficient, and $f(x) \in L^1(\Omega)$ satisfies a domination condition of the type $|f(x)| \leq kg(x)$.

Due to the lack of coercivity and low regularity of the data, we introduce a sequence of approximate problems using truncation functions to regularize the nonlinear operator. We then establish uniform a priori estimates for the approximate solutions in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Finally, we pass to the limit in the nonlinear terms using compactness and weak convergence techniques to prove the existence of a bounded weak solution to the original problem.

Keywords: degenerate elliptic equations, lower-order terms, pseudo-monotone operator, L^1 -data, bounded weak solution.

Résumé

Ce mémoire est consacré à l'étude de l'existence de solutions faibles pour une classe d'équations elliptiques dégénérées comportant des termes d'ordre inférieur et des données du second membre dans l'espace $L^1(\Omega)$. Le problème considéré s'écrit :

$$\begin{cases} -\operatorname{div} \left(\frac{M(x)\nabla u}{1+|u|} \right) + g(x)u = f(x) & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

où $\Omega \subset \mathbb{R}^N$ est un domaine ouvert borné, $M(x)$ est une matrice elliptique bornée, $g(x) \in L^1(\Omega)$ est une fonction positive ou nulle, et $f(x) \in L^1(\Omega)$ satisfait la condition de domination $|f(x)| \leq kg(x)$.

En raison du caractère dégénéré de l'opérateur et de la faible régularité des données, nous introduisons une suite de problèmes approchés en utilisant des fonctions de troncature pour régulariser l'opérateur non linéaire. Nous établissons ensuite des estimations a priori uniformes des solutions approchées dans $H_0^1(\Omega) \cap L^\infty(\Omega)$. Enfin, nous passons à la limite à l'aide de techniques de compacité et de convergence faible afin de démontrer l'existence d'une solution faible bornée du problème initial.

Mots-clés : équations elliptiques dégénérées, termes d'ordre inférieur, opérateur pseudo-monotone, données L^1 , solution faible bornée.

الملخص

تتناول هذه المذكرة دراسة وجود حلول ضعيفة لفئة من المعادلات الإهليلجية المنحنية التي تتضمن حدوداً دنياً وبيانات تنتمي إلى فضاء $L^1(\Omega)$. المسألة المدروسة تتمثل في النظام التالي:

$$\begin{cases} -\operatorname{div} \left(\frac{M(x)\nabla u}{1+|u|} \right) + g(x)u = f(x), \\ u = 0, \end{cases}$$

حيث أن المعادلة معرفة في Ω ، والحل منعدم على $\partial\Omega$.

حيث تمثل $\Omega \subset \mathbb{R}^N$ مجموعة مفتوحة ومحدودة، و $M(x)$ مصفوفة معاملات محدودة وقابلة للإهليلجية، و $g(x) \in L^1(\Omega)$ دالة غير سالبة، و $f(x) \in L^1(\Omega)$ تحقق شرط الهيمنة من النوع $|f(x)| \leq kg(x)$ لـ $k > 0$.

نظراً لانعدام التماسك وضعف انتظام البيانات، تم تقديم مسائل تقريبية باستخدام دوال القطع لتقريب المؤثر غير الخطي. بعد ذلك، تم إثبات تقديرات أولية منتظمة للحلول التقريبية في فضاء $H_0^1(\Omega) \cap L^\infty(\Omega)$. وأخيراً، باستخدام أدوات التراص والتقارب الضعيف، تم تمرير النهاية في المصطلحات غير الخطية لإثبات وجود حل ضعيف محدود للمسألة الأصلية.

الكلمات المفتاحية: معادلات إهليلجية منحنية، حدود دنياً، مؤثر شبه رتبي، بيانات في L^1 ، حلول ضعيفة محدودة.