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coercivity and L^1 data**

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Dedication

I dedicate this modest work :

-To my mom and dad,

-To my brother Fares and my sisters Chaima and Somia,

-To the little buds Sajida, Jori, Qusay,

-To all my family,

-To all friends and all my department family,

-To all my adorable ones that i have known during all my life ...

Mohammed Bouguerra

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Notation

We introduce the necessary notations and definition which are used in the sequel.

$\Omega \subset \mathbb{R}^N$	open set in \mathbb{R}^N
X'	topological dual of X .
(\cdot, \cdot)	scalar product.
\mathbb{R}^N	Euclidean space of dimension N , where N is a nonzero natural number.
x	vector in \mathbb{R}^N , $x = (x_1, x_2, \dots, x_N)$, $x_i \in \mathbb{R}$, $1 \leq i \leq N$.
dx	Lebesgue measure in N -dimensional space.
$\partial\Omega = \Gamma$	boundary of Ω .
$B(x_0, r)$	open ball of radius r centered at x_0 .
B_E	$= \{x \in E; \ x\ \leq 1\}$.
$ \cdot $	Hilbert norm.
Ω	open subset of \mathbb{R}^N .
$\langle \cdot, \cdot \rangle$	duality bracket between X and its dual space.
$\int_{\Omega} f(x) dx$	integral of f in Ω with respect to the Lebesgue measure.
supp u	Support of the function u .
$\frac{\partial u}{\partial n}$	outward normal derivative.
div u	divergence of the vector u , $div u = \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + \dots + \frac{\partial u}{\partial x_N}$.
$\Delta u = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$	Laplacian of u .
$L^p(\Omega)$	$= \{u : \Omega \rightarrow \mathbb{R} \text{ measurable and } \left(\int_{\Omega} u(x) ^p dx\right)^{1/p} < +\infty \text{ such that } 1 \leq p < \infty\}$.
$L^p(\Omega, w)$	$= \{u : \Omega \rightarrow \mathbb{R} \text{ measurable and } w(x) > 0 \left(\int_{\Omega} u(x) ^p w(x) dx\right)^{1/p} < +\infty \text{ such that } 1 \leq p < \infty\}$
\mathcal{H}	Hilbert space.
$ u _p$	$= \left[\int_{\Omega} u(x) ^p dx\right]^{1/p} = u _{L^p}$.
$L^{\infty}(\Omega)$	$= \{u : \Omega \rightarrow \mathbb{R} \text{ measurable, } \exists M > 0 \mid u(x) \leq M \text{ a.e.}\}$.
$\ u\ _{L^{\infty}}$	$= \inf\{C : u(x) \leq C \text{ a.e on } \Omega\}$.
q	Hölder conjugate of p : $q = \frac{p}{p-1}$ if $p > 1$ and $q = \infty$ if $p = 1$.
$L^q(\Omega) \subset L^p(\Omega)$,	$\forall 1 \leq p \leq q < \infty$.

$\|u\|_{C_c(\Omega)} = \sup_{x \in \Omega} |u(x)| = \max_{x \in \Omega} |u(x)|.$

$L^p(\Omega)$ Lebesgue space.

$L^p(\Omega, w)$ Weighted Lebesgue space.

$C_c(\Omega)$ space of continuous functions with compact.

$\mathcal{D}(\Omega)$ space of infinitely differentiable functions on Ω with compact support in Ω .

$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \mid \nabla u \in (L^p(\Omega))^N \right\}.$

$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega), \text{ with } u = 0 \text{ on } \partial\Omega \right\}.$

$W^{1,p}(\Omega, w) = \left\{ u \in L^p(\Omega, w) \mid \nabla u \in (L^p(\Omega, w))^N \right\}.$

$(a + b)^p \leq 2^{p-1}(a^p + b^p)$ for $1 \leq p < \infty$ and $a, b \geq 0$.

a.e. almost everywhere.

General Introduction

Partial differential equations, which will be abbreviated as "PDE" in the following, constitute an important branch of applied mathematics and this field is becoming increasingly important in modern times. PDEs have great utility in modeling of many phenomena of different natures such as physics, chemistry, biology and other sciences. In other words, PDEs intervene in many other fields: in chemistry to model reactions, in economics to study market behavior, in finance to study derivative products and in image processing to restore damage. These problems boil down to mathematical models usually written in the form :

$$L(u) = f, \quad (1)$$

where L is a definite operator from a function space E to a function space F , u is the unknown function and f a given function. PDEs probably first appeared during the 17th century. In the field of PDEs has been enriched as the sciences have developed and especially physics. However, the systematic study of PDEs is much more recent and it is only during of the 20th century that mathematicians began to develop and indeed this theory has recently experienced a great theoretical and practical advancement. The mathematical analysis of these partial differential equations requires an appropriate choice of functional spaces and a clear definition of the notion of solution (the existence and sometimes uniqueness). One of the things to keep in mind about PDEs is to ask the question: can we get solutions explicitly?. So what mathematics can do on the other hand, is to say if one or more solutions exist, and to describe sometimes very precisely- ment some properties of these solutions. So usually an exact solution of problem (1) is not easy to find and sometimes we cannot

even find the explicit solution. This leads to the introduction of the notion of the weak solution which appeared in 1934 in the work of Jean Leray. therefore, in most cases it is very difficult, if not impossible, to exhibit the solutions of a PDE. In some cases we manage to try to show that the problem admits a unique solution (it is said to be well-posed). But sometimes we can calculate numerical approximations of the solutions.

Weighted elliptic equations have been widely studied in the literature, starting from the paper by Trudinger in the linear case $p = 2$ (see [9]), looking for solutions in weighted Sobolev spaces. For a comprehensive study of weighted Sobolev spaces, and its applications to the existence of solutions for weighted elliptic equations, see for example the book [7] A more direct approach, which does not use weighted Sobolev spaces, is the one of [5] (with weights in $L^r(\Omega)$, $r > 1$, which can be zero on a subset of Ω) and [4], where strictly positive weights in $L^1(\Omega)$ have been considered.

Moreover, the work presented in this dissertation concerns a particular case of the equations to partial derivatives of the elliptic type involving the divergent operator:

$$Au = -\operatorname{div} \left(a(x) \frac{|\nabla u|^{p-2} \nabla u}{1 + |u|} \right), \quad 1 < p < \infty.$$

We are interested in this memory to study the following problem:

$$(P) \quad \begin{cases} Au + e(x)|\nabla u|^{p-2}\nabla u = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Ω is a bounded open set of \mathbb{R}^N with $N > 2$, we assume that $f \in L^1(\Omega)$.

We first illustrate the main difficulties that may arise when studying of the problem:

(P)

1) The non-linearity of the differential operator Au .

2) The irregularity in the case where $f \in L^1(\Omega)$ i.e. the second member of (P) does not belong to space $W^{-1,p'}(\Omega)$.

3) the differential operator Au is not coercive on $W_0^{1,p}(\Omega)$ despite being well-defined between $W_0^{1,p}(\Omega)$ and its dual $W^{-1,p'}(\Omega)$. Degenerate coercivity implies that as $|u|$ becomes large, $\frac{1}{1+|u|}$ tends to zero.

To solve this issue, we are going to approximate the operator by employing truncations in $\frac{1}{1+|u|}$ to obtain a coercive differential operator. Next, we will prove some a priori estimates on the sequence of approximate solutions, and we shall finally pass to the limit in the approximate problems to establish the existence of a weak solution for the problem (P).

Technique

In [1], L. Boccardo et al. have studied the existence of the solutions to problem (P) where the data f in $L^{p^*}(\Omega)$ with the operator:

$$A : v \longrightarrow -\operatorname{div} \left(a(x) |\nabla u|^{p-2} \nabla u \right),$$

The problem (P) was also considered in [1], when f in $L^1(\Omega)$. **In this work, I would like to apply the ideas and results mentioned in [1] to the case of nonlinear weighted elliptic equations with degenerate coercivity. The method used to prove the existence of solutions $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ for the problem (P) is the same as used in [1].**

We are going to prove the existence of the weak solution of the problem (P). To do this, we approximate the problem (P) by a sequence of approximate problems (P_n) given in L^∞ whose existence of the solution approximate is guaranteed (See [8]). Then we will prove some estimates uniform on the sequence of solutions of these problems (P_n) and their partial derivatives. Once this is done, the linearity of the operator with respect to the gradient as well as the boundedness and the continuity of $u \mapsto |\nabla u|^{p-2} \nabla u$ will make it possible to pass to the limit, thus finding the solution.

Thesis plan

This thesis is divided into three chapters as follows: The first chapter is dedicated to giving some basic definitions and results with functional analysis tools essential to the achievement

of the objectives for the study of the (P) problem. For example we recall functional spaces (Lebesgue, Sobolev) and we give brief reminders of weak and weak star convergence.

In the second chapter we recall some definitions on the operators (Bounded, hime-continuous, Monotone and coercive). We also present the method of monotonic (pseudo-monotonic) operators in the general framework to prove the existence of a solution for the equation:

$$Au = -\operatorname{div}(w(x)|\nabla u|^{p-2}\nabla u) = f.$$

According to hypotheses on the functions $a = |\nabla u|^{p-2}\nabla u$ and f ($f \in W^{-1,p'}(\Omega)$) and for to prove the existence and the regularity of the solution we use the Theorem 2.1.

In the third chapter, we will study existence of solutions for the equation:

$$(P) \quad \begin{cases} -\operatorname{div}(a(x)\frac{|\nabla u|^{p-2}\nabla u}{1+|u|}) + e(x)|u|^{p-2}u = f & \Omega; \\ u = 0 & \partial\Omega, \end{cases}$$

with $a(x)$ such that (3.1) holds, and the functions $e(x)$ in $L^1(\Omega)$ and $f(x)$ in $L^1(\Omega)$ are such that there exists $k_0 > 0$ such that:

$$|f(x)| \leq k_0 \cdot e(x)$$

For this equation we will prove existence of a solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Preliminaries

The purpose of this first chapter is to present a number of analytical tools from the theory of Sobolev spaces that will be used throughout this thesis. We will also take the opportunity to introduce the main notations.

1.1 Reminders and some definitions

Throughout this chapter Ω will denote a bounded domain in \mathbb{R}^N , $N \geq 1$, i.e. a connected and bounded open subset of \mathbb{R}^N . Its boundary will be denoted by Γ or $\partial\Omega$ and its closure by $\bar{\Omega}$.

Let $u = u(x_1, \dots, x_N)$ be a function defined in $\Omega \subset \mathbb{R}^N$. Assuming that exists, the gradient of u at point x is defined as the vector:

$$\nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), \dots, \frac{\partial u}{\partial x_N}(x) \right).$$

The Euclidean norm of ∇u is denoted by $|\nabla u|$:

$$|\nabla u| = \left[\left| \frac{\partial u}{\partial x_1} \right|^2 + \dots + \left| \frac{\partial u}{\partial x_N} \right|^2 \right]^{1/2}.$$

Definition 1.1. (Space of test functions): The space $\mathcal{D}(\Omega)$ or $C_c^\infty(\Omega)$ is the set of infinitely differentiable functions $\varphi : \Omega \rightarrow \mathbb{R}$ (of class $C^\infty(\Omega)$) with compact support in Ω .

$\mathcal{D}(\Omega)$ is a vector space, and each element of this space is called a test function.

- $\text{supp } \varphi = \overline{\{x \in \Omega, \varphi(x) \neq 0\}}$,
- $\mathcal{D}(\Omega) = C^\infty(\Omega) \cap C_c(\Omega)$.

Definition 1.2 (Reflexive Spaces). . The space E is said to be reflexive if the canonical injection $i : E \rightarrow E''$ is surjective , i.e. $i(E) = E''$.

Definition 1.3 (Separable Spaces). . A Banach space E is said to be separable if there exists a countable set that is dense in E .

Definition 1.4 (Dual Space). . Let E be a vector space over \mathbb{R} we recall that a functional is a function defined on E or on some subspace of E , with values in \mathbb{R} . We denote by $E' = \mathbb{L}(E, \mathbb{R})$ the dual space of E (the norm in E') is defined by:

$$\|f\|_{E'} = \sup_{x \in E, \|x\|=1} |f(x)| = \sup_{x \in E, \|x\| \leq 1} |f(x)| = \sup_{x \in E, \|x\| \neq 0} \frac{|f(x)|}{\|x\|}.$$

1.2 Functional spaces

1.2.1 L^p Spaces

Let Ω be an open subset of \mathbb{R}^N and let $p \in \mathbb{R}$ with $1 \leq p < \infty$, we set

$L^p(\Omega)$ is defined as:

$$L^p(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ is measurable and } \int_{\Omega} |f(x)|^p < \infty \right\}.$$

with

$$\|f\|_{L^p(\Omega)} = \|f\|_p = \left(\int_{\Omega} |f(x)|^p \right)^{\frac{1}{p}}, \quad p \geq 1.$$

and we set

$$L^\infty(\Omega) = \{f : \Omega \rightarrow \mathbb{R} \text{ measurable, } \exists C > 0, |f(x)| \leq C \text{ a.e. on } \Omega\}.$$

with c is a constant

$$\|f\|_{\infty, \Omega} = \inf \{C > 0 : |f| \leq C \text{ a.e. in } \Omega\}.$$

The space $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{2, \Omega} = \int_{\Omega} uv dx.$$

Theorem 1.1. [3] $L^p(\Omega)$ is a Banach space for any $1 \leq p \leq \infty$ and separable for $1 \leq p < \infty$.

Let $1 \leq p \leq \infty$.be denoted by p' the conjugate exponent of p :

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

1.2.2 Some Inequalities

Let $1 \leq p \leq q \leq \infty$ be two real numbers; and let p' be the conjugate exponent of p .

Lemma 1.1 (Hölder's inequality). Assume that $f \in L^p(\Omega)$, and $g \in L^{p'}(\Omega)$ with $1 \leq p \leq \infty$, then $f \cdot g \in L^1(\Omega)$ and:

$$\| f \cdot g \|_{L^1(\Omega)} \leq \| f \|_{L^p(\Omega)} \| g \|_{L^{p'}(\Omega)} .$$

If more $|\Omega| < \infty$ and $f \in L^q(\Omega)$, so $f \in L^p(\Omega)$ and

$$\|f\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(\Omega)} .$$

In particular,

$$L^q(\Omega) \subset L^p(\Omega), \quad \forall 1 \leq p \leq q < \infty .$$

Lemma 1.2. Let $p_i \in [1, +\infty]$ be exponents with $1 \leq i \leq k$ such that:

$1/p = 1/p_1 + \dots + 1/p_k \leq 1$. Then, for all functions $f_i \in L^{p_i}(\Omega)$, We have $f = f_1 \dots f_k \in L^p(\Omega)$ and generalized Hölder inequality:

$$\|f\|_p \leq \| f \|_{p_1} \dots \| f \|_{p_k} .$$

Lemma 1.3 (Young's inequality). For all non-negative real numbers a, b and every $1 < p < \infty$, then we have:

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad p' = \frac{p}{p-1}. \tag{1.1}$$

Corollary 1.1. *Generalized Young's inequality: for $1 < p < \infty$, $p' = \frac{p}{p-1}$ and any positive ϵ we have:*

$$ab \leq \epsilon^p \frac{a^p}{p} + \frac{1}{\epsilon^{p'}} \frac{b^{p'}}{p'}.$$

Lemma 1.4 (Divergence and Green's formulas). *Let Ω be a domain in \mathbb{R}^N , and $n(x)$ its exterior normal. Let u and v be two regular functions, and w be a vector field defined on Ω . Then,*

$$\int_{\Omega} \operatorname{div} w \, dx = \int_{\partial\Omega} w \cdot n \, d\sigma \quad (\text{divergence formula}).$$

$$\int_{\Omega} (\Delta u)v \, dx = - \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, d\sigma \quad (\text{green formula}).$$

1.3 Sobolev Spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 1.5. *The Sobolev space $W^{1,p}(\Omega)$, is defined by:*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega), \mid \nabla u \in (L^p(\Omega))^N \right\}.$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\| u \|_{W^{1,p}(\Omega)} = \| u \|_p + \| \nabla u \|_p. \tag{1.2}$$

and for $p = \infty$

$$\| u \|_{W^{1,\infty}} = \max(\| u \|_{L^\infty}, \| \nabla u \|_{L^\infty}).$$

The space H^1 is a Hilbert¹ space equipped with the scalar product.

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + (u', v')_{L^2(\Omega)}.$$

The associated norm:

$$\| u \|_{H^1(\Omega)} = (\| u \|_{L^2(\Omega)}^2 + \| \nabla u \|_{L^2(\Omega)}^2)^{1/2}.$$

is a norm equivalent to that of $W^{1,2}(\Omega)$

¹David Hilbert (1862-1943) is a German mathematician. He is often considered one of the greatest mathematicians of the 20th century.

Proposition 1.1. $W^{1,p}(\Omega)$ is a Banach space for $1 \leq p \leq \infty$ reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$.

In particular $H^1(\Omega)$ is reflexive, separable and Hilbert space.

Definition 1.6. The space $D(\Omega)$ denotes the set of functions in $C^\infty(\Omega)$ with compact support in Ω , we define $1 \leq p < \infty$

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega), \text{ with } u = 0 \text{ on } \partial\Omega \right\}.$$

Remark 1.1. Since, for $1 \leq p < \infty$, the space $D(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ by definition, we can identify the dual of $W_0^{1,p}(\Omega)$ with a subspace of the space of distributions $D'(\Omega)$. we denote this as: $W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))'$.

1.3.1 Sobolev Injections and Inequalities

Sobolev injections are widely used. they provide inequalities between the norms of sobolev spaces and L^p norms. This compactness result is a powerful tool in the study of PDEs, allowing us to transition from a sobolev space to a space of Lebesgue.

Theorem 1.2 (continuous Injection). Let Ω be a bounded open subset of \mathbb{R}^N with $N \geq 1$ and $1 \leq p < \infty$.

- $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q \in [1, p^*[$ if $p < N$ and $p^* = \frac{Np}{N-p}$.
- $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q \in [1, \infty[$ if $p = N$.
- $W_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, if $p > N$.

Theorem 1.3 (compact Injection (Rellich-Kondrachon)). Let Ω be a bounded domain of class C^1

- $W_0^{1,p}(\Omega) \hookrightarrow_c L^q(\Omega)$, $\forall q \in [1, p^*[$ if $p < N$ and $p^* = \frac{Np}{N-p}$.
- $W_0^{1,p}(\Omega) \hookrightarrow_c L^q(\Omega)$, $\forall q \in [1, \infty[$ if $p = N$

- $W_0^{1,p}(\Omega) \subset C(\overline{\Omega})$, if $p > N$.

Lemma 1.5 (Poincaré's inequality). *Suppose that $1 \leq p < \infty$, and Ω is a bounded open set. Then there exists a constant C (depending on Ω and p) such that:*

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}. \quad (1.3)$$

Moreover, by the density of $\mathcal{D}(\Omega)$ in $W_0^{1,p}(\Omega)$, the inequality (1.3) stay true for any function $u \in W_0^{1,p}(\Omega)$.

1.3.2 Convergence theorems

In this section, we present some definitions and results on the convergence sequences of measurable functions

Lemma 1.6 ([2]). *Let (u_n) the sequence in $L^p(\Omega)$ with $1 < p < \infty$.*

Suppose that

- (u_n) is bounded in $L^p(\Omega)$;
- $u_n \rightarrow u$ a.e. on Ω .

Then, $u_n \rightarrow u$ in $L^q(\Omega)$, for all $1 \leq q < p$ and weakly in $L^p(\Omega)$, i.e.,

$$\lim_{n \rightarrow +\infty} \int_{\Omega} u_n v dx = \int_{\Omega} u v dx, \text{ for all } v \in L^{p'}(\Omega).$$

Theorem 1.4 (Fatou's Lemma [3]). *Let (u_n) be a sequence of non-negative measurable function on Ω . Then*

$$\int_{\Omega} \liminf_{n \rightarrow \infty} u_n(x) dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} u_n(x) dx.$$

Theorem 1.5 (Lebesgue's Dominated Convergence Theorem[3]). *Let $p \in [1, \infty)$, and (u_n) be a sequence of functions in $L^p(\Omega)$ such that*

- $u_n \rightarrow u$ a.e in Ω .

- there is a function $v \in L^p(\Omega)$ satisfying:

$$|u_n(x)| \leq v(x) \quad \text{a.e in } \Omega \quad \forall n \in \mathbb{N}.$$

Then $u_n \rightarrow u$ in $L^p(\Omega)$, that is :

$$u \in L^p(\Omega), \quad \text{and} \quad \|u_n - u\|_{L^p(\Omega)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Theorem 1.6 (Lebesgue's Dominated Convergence Inverse Theorem). *Let (f_n) be a sequence in $L^p(\Omega)$ and let $f \in L^p(\Omega)$ be such that $\|f_n - f\|_{L^p(\Omega)} \rightarrow 0$*

Then ,there exist a subsequence (f_{n_k}) and a function $h \in L^p(\Omega)$ such that

- $f_{n_k}(x) \rightarrow f(x)$ a.e in Ω .
- $|f_{n_k}(x)| \leq h(x) \quad \forall k$ a.e in Ω .

1.4 Some notation of convergence

1.4.1 strong convergence

Definition 1.7. *Convergence of a sequence $(x_n)_{n \in \mathbb{N}}$ in a normed vector space $(E, \|\cdot\|_E)$ to an element $x \in E$, defined in the following way :*

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_0 \implies \|x_n - x\|_E \leq \epsilon.$$

or

$$\lim_{n \rightarrow +\infty} \|x_n - x\|_E = 0.$$

We denote by $x_n \rightarrow x$ as $n \rightarrow +\infty$

1.4.2 Weak convergence

Definition 1.8. *Let E be a Banach space, E' its dual and $\langle \cdot, \cdot \rangle$ the duality bracket EE' .*

We say that the sequence (x_n) in E weakly converges to $x \in E$ if and only if:

$$\langle g, x_n \rangle \xrightarrow{n \rightarrow +\infty} \langle g, x \rangle, \quad \forall g \in E'.$$

we denote this as

$$x_n \rightharpoonup x \quad \text{in } E.$$

Proposition 1.2. • if $x_n \rightarrow x$ strongly then $x_n \rightharpoonup x$ in E .

• if $x_n \rightharpoonup x$ in E then $\|x_n\|_E$ is bounded.

•

$$\begin{cases} \text{If } x_n \rightharpoonup x, \text{ weakly in } E \\ \text{And } g_n \rightarrow g \text{ strongly in } E' \end{cases}$$

then $\langle g_n, x_n \rangle \rightarrow \langle g, x \rangle$ as $n \rightarrow +\infty$.

Theorem 1.7. Let E be a reflexive Banach space. Then, for any bounded sequence $(x_n)_n \subset E$, there exists a sub-sequence on $(x_{n_k})_{n_k}$ that converges weakly in E .

1.4.3 (weak*) convergence

Definition 1.9. We say that the sequence (f_n) in E' weakly* converges to $f \in E'$ if and only if:

$$\langle f_n, g \rangle \xrightarrow{n \rightarrow +\infty} \langle f, g \rangle, \quad \forall g \in E.$$

we denote this as

$$f_n \rightharpoonup^* f \quad \text{in } E'.$$

Proposition 1.3.

$$f_n \rightarrow f \quad \text{strongly} \quad \implies \quad f_n \rightharpoonup^* f \quad \text{in } E'.$$

For more details, you can refer to the book[3].

Weighted space

In this chapter, we study weighted Lebesgue and Sobolev spaces defined in the domains of n -dimensional Euclidean space \mathbb{R}^N , $N \geq 2$. By a weight, we shall mean a locally integrable function w on \mathbb{R}^N such that $w(x) > 0$ a.e.. Every weight w gives rise to a measure on the measurable subsets of \mathbb{R}^n through integration.

2.1 Weighted Sobolev spaces

2.1.1 The Weighted Lebesgue space

Definition 2.1. Let Ω be a measurable subset of \mathbb{R}^N , $N \geq 2$, and $w : \mathbb{R}^N \rightarrow [0, +\infty)$ be a locally integrable nonnegative function, i.e., a weight. The weighted Lebesgue space $L^p(\Omega, w)$, $1 \leq p < \infty$, is defined as a Banach space of locally integrable functions $f : \Omega \rightarrow \mathbb{R}$ endowed with the following norm:

$$\|f\|_{L^p(\Omega, w)} = \left(\int_{\Omega} |f(x)|^p w(x) dx \right)^{\frac{1}{p}}.$$

If $w = 1$, we write $L^p(\Omega, w) = L^p(\Omega)$.

2.1.2 The Weighted Sobolev spaces

Definition 2.2. Let Ω be an open subset of \mathbb{R}^N , $N \geq 2$. We define the weighted Sobolev space $W^{1,p}(\Omega, w)$, $1 \leq p < \infty$, as a normed space of locally integrable weakly differentiable functions $f : \Omega \rightarrow \mathbb{R}$ endowed with the following norm:

$$\|f\|_{W^{1,p}(\Omega; w)} = \|f\|_{L^p(\Omega; w)} + \|\nabla f\|_{L^p(\Omega; w)}.$$

We also define $W_0^{1,p}(\Omega, w)$ as the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega, w)$ and The dual space of $W_0^{1,p}(\Omega, w)$ is $W^{-1,p'}(\Omega, w)$. The spaces $W_0^{1,p}(\Omega, w)$ and $W^{1,p}(\Omega, w)$ are Banach spaces.

When $w = 1$, these spaces will be denoted $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, respectively.

Remark 2.1. *It is evident that the weight function w which satisfies*

$$0 < c_1 \leq w(x) \leq c_2, \quad \text{for } x \in \Omega, (c_1 \text{ and } c_2 \text{ positive constants}),$$

gives nothing new (the space $W_0^{1,p}(\Omega, w)$ is then identical with the classical Sobolev space $W_0^{1,p}(\Omega)$).

We now determine conditions on the weight w that guarantee that functions in $L_w^p(\Omega)$ are locally integrable on Ω . Let $1 \leq p < \infty$ and let w be a weight such that $w^{-1/(p-1)}$ is locally integrable, when $p > 1$, and such that w locally is bounded from below away from 0, when $p = 1$, *i.e.*, such that

$$\operatorname{ess\,sup}_{x \in B} \frac{1}{w(x)} < \infty.$$

for every ball B . Suppose that $f \in L_w^p(\Omega)$, and let $B \subset \Omega$ be a ball. If $1 < p < \infty$, then, by Hölder's inequality,

$$\int_B |f| dx \leq \left(\int_B w^{-1/(p-1)} dx \right)^{1/p'} \left(\int_B |f|^p w dx \right)^{1/p}. \quad (2.1)$$

and if $p = 1$,

$$\int_B |f| dx \leq \operatorname{ess\,sup}_{x \in B} \frac{1}{w(x)} \int_B |f| w dx. \quad (2.2)$$

It follows that $L^p(\Omega, w) \subset L_{loc}^1(\Omega)$ and that convergence in $L^p(\Omega, w)$ implies local convergence in $L^1(\Omega)$. If Ω is bounded, one obtains in the same way that $L^p(\Omega, w)$ is continuously embedded in $L^1(\Omega)$.

2.1.3 Monotone Operators

Definition 2.3. *An operator $A : V \rightarrow V'$ is said to be:*

- *Monotone if* $\langle Au - Av, u - v \rangle \geq 0, \forall u, v \in V.$
- *strictly monotone if* $\langle Au - Av, u - v \rangle > 0, \forall u, v \in V, u \neq v.$

Example 2.1. The operator $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ defined by $Au = -\Delta u$ is monotone, where $H_0^1(\Omega)$ is equipped with the norm of the gradient. Indeed, for $u, v \in H_0^1(\Omega)$, and we Use lemma 1.4

we have:

$$\begin{aligned} \langle Au - Av, u - v \rangle &= \int_{\Omega} \nabla(u - v) \cdot \nabla(u - v) \\ &= \|u - v\|_{H_0^1(\Omega)}^2 \\ &\geq 0. \end{aligned}$$

2.1.4 Bounded Operator

Definition 2.4. Let V and W be two Banach spaces and let $A : V \rightarrow W$ be an operator. We say that A is bounded if it maps every bounded set in V to a bounded set in W ; i.e

$$\forall \rho > 0, \quad \exists C_{\rho} > 0 : \quad A(B_V(0, \rho)) \subset B_W(0, C_{\rho}).$$

where $B_V(0, \rho)$ designates the open ball in V with center 0 and radius $\rho > 0$ and $B_W(0, C_{\rho})$ designates the open ball in W with center 0 radius $C_{\rho} > 0$.

2.1.5 Hemicontnuous Operators

Definition 2.5. An operator $A : V \rightarrow W$ is said to be hemicontinuous at the point u_{∞} in V if, for any sequence $\{u_n\}$ converging to u_{∞} , the sequence $\{Au_n\}$ converges weakly to Au_{∞} in W , in other words :

$$\forall v \in V, \quad \forall \{\lambda_n\} \subset \mathbb{R}, \quad \lambda_n \rightarrow 0, \quad A(u_{\infty} + \lambda_n v) \rightharpoonup Au_{\infty}.$$

If A is hemicontinuous at every point in V , we say that it is hemicontinuous on V . In reflexive spaces and when $W = V'$, and passing from sequential to continuous, we can define hemicontinuity on V by requiring that:

$$\forall u, v, w \in V \quad \text{the map} \quad \mathbb{R} \ni \lambda \mapsto \langle A(u + \lambda v), w \rangle \in \mathbb{R}$$

is continuous .

Example 2.2. The operator $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ defined by

$Au = -\Delta u = -\operatorname{div}(\nabla u)$ is hemicontinuous. Indeed, let $u, v, w \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$, we use lemma 1.4.

we have :

$$\begin{aligned} \langle A(u + \lambda v), w \rangle &= \int_{\Omega} \nabla(u + \lambda v) \cdot \nabla w \\ &= \int_{\Omega} \nabla u \cdot \nabla w + \lambda \int_{\Omega} \nabla v \cdot \nabla w \\ &= a + b\lambda. \end{aligned}$$

This shows that $\lambda \longrightarrow \langle A(u + \lambda v), w \rangle$ is continuous.

2.1.6 Coercive Operator

Definition 2.6. An operator $A : V \rightarrow V'$ is said to be coercive if :

$$\lim_{\|v\|_V \rightarrow +\infty} \frac{\langle Av, v \rangle}{\|v\|_V} = +\infty.$$

2.2 Pseudo-monotonic Operators

Definition 2.7. An operator $A : V \rightarrow V'$

i) We say that A is pseudo-monotonic (in sense 1) if it is

- for all $u_n \rightharpoonup u$ in V with $\lim_{n \rightarrow \infty} \sup \langle Au_n, u_n - u \rangle \leq 0$

we have

$$\liminf_{n \rightarrow \infty} \langle Au_n, u_n - v \rangle \geq \langle Au_n, u - v \rangle, \quad \forall v \in V$$

ii) We say that A is pseudo-monotonic (in sense 2) if it is

- for all $u_n \rightharpoonup u$ in V with $A(u_n) \rightharpoonup \xi$ in V' and

$$\limsup_{n \rightarrow \infty} \langle Au_n, u_n \rangle \leq \langle \xi, u \rangle$$

We have

$$\xi = A(u) \quad \text{and} \quad \langle Au_n, u_n \rangle \rightarrow \langle Au, u \rangle.$$

Proposition 2.1. *If A is bounded, then A is pseudo-monotonic in sense 1 if and only if it is pseudo-monotonic in sense 2.*

Proof. Let us first assume that A is pseudo-monotonic in sense 1 (but not necessarily bounded).

Let u_n be a sequence such that $u_n \rightharpoonup u$, $A(u_n) \rightharpoonup \xi$ and

$$\limsup \langle A(u_n), u_n \rangle \leq \langle \xi, u \rangle$$

We deduce that

$$\limsup \langle A(u_n), u_n - u \rangle = \limsup (\langle A(u_n), u_n \rangle - \langle A(u_n), u \rangle) \leq 0.$$

So, by pseudo-monotony in sense 1, it follows that for all $v \in V$,

$$\liminf \langle A(u_n), u_n - v \rangle \geq \langle A(u), u - v \rangle,$$

hence by developing the left hand side,

$$\liminf \langle A(u_n), u_n \rangle - \langle \xi, v \rangle \geq \langle A(u), u - v \rangle.$$

Taking $v = u$, we deduce that $\liminf \langle A(u_n), u_n \rangle \geq \langle \xi, u \rangle$, from where

$$\langle A(u_n), u_n \rangle \rightarrow \langle \xi, u \rangle$$

and by reporting in the inequality above

$$\langle \xi, u - v \rangle \geq \langle A(u), u - v \rangle.$$

Taking $v = u + w$, we deduce that $\xi = A(u)$ and that A is pseudo-monotonic in sense 2.

Now suppose that A is pseudo-monotonic in sense 2 and bounded. Let u_n be a sequence such that $u_n \rightharpoonup u$ and $\limsup \langle A(u_n), u_n - u \rangle \leq 0$. Let $v \in V$. As A is bounded, we can extract a sequence such that

$$A(u_{n'}) \rightharpoonup \xi \quad \text{and} \quad \langle A(u_{n'}), u_{n'} - v \rangle \rightarrow \liminf \langle A(u_n), u_n - v \rangle.$$

As $\langle A(u_{n'}), u \rangle \rightarrow \langle \xi, u \rangle$, we first have $\limsup \langle A(u_{n'}), u' \rangle \leq \langle \xi, u \rangle$. So, by pseudo-monotonicity in sense 2, it follows that $\xi = A(u)$ and $\langle A(u_{n'}), u' \rangle \rightarrow \langle A(u), u \rangle$. We therefore see that

$$\langle A(u_{n'}), u' - v \rangle \rightarrow \langle A(u), u - v \rangle$$

So A is pseudo-monotonic in sense 1. □

Proposition 2.2. *If $A : V \rightarrow V'$ is bounded, hemicontinuous and monotonic, then A is pseudo-monotonic (in sense 1).*

Proof. Let $\{u_j\}$ be a sequence that converges weakly to u in V . Suppose that :

$$\limsup_{j \rightarrow \infty} \langle Au_j, u_j - u \rangle \leq 0.$$

If A is monotonic, we have

$$\lim_{j \rightarrow \infty} \langle Au_j, u_j - u \rangle \rightarrow 0 \tag{2.3}$$

Indeed, the monotonicity of A and the weak convergence of $\{u_j\}$ to u implies that

$$\langle Au_j, u_j - u \rangle \geq \langle Au, u_j - u \rangle \rightarrow 0 \quad \text{for } j \rightarrow \infty.$$

And so

$$0 \geq \limsup_{j \rightarrow \infty} \langle Au_j, u_j - u \rangle \geq \liminf_{j \rightarrow \infty} \langle Au_j, u_j - u \rangle \geq \limsup_{j \rightarrow \infty} \langle Au, u_j - u \rangle = 0.$$

Hence (2.3)

b) for $v \in V$ and $t \in]0, 1[$, let $w = (1 - t)u + tv$. we have $\langle Au_j - Aw, u_j - w \rangle$ of fate that:

$$t \langle Au_j, u - v \rangle \geq - \langle Au_j, u_j - u \rangle + \langle Aw, u_j - u \rangle - t \langle Aw, v - u \rangle.$$

Hence , thanks to (2.3):

$$t \liminf_j \langle Au_j, u - v \rangle \geq -t \langle Aw, v - u \rangle,$$

hence , dividing by t and taking into account (2.3):

$$\liminf_j \langle Au_j, u_j - v \rangle \geq \langle Aw, u - v \rangle \tag{2.4}$$

$$w = (1 - t)u + tv \quad \forall t \in]0, 1[.$$

By finishing tend t to 0 in (2.4), and using hemicontinuity, we deduce

$$\lim_{j \rightarrow \infty} \inf \langle Au_j, u_j - v \rangle \geq \langle Au, u - v \rangle, \quad \forall v \in V.$$

Which means that A is pseudo-monotonic in sense 1 . □

2.3 The weighted p -Laplacian operator

Let $w : \mathbb{R}^N \rightarrow [0, +\infty)$ be a locally integrable nonnegative function, i.e., a weight. The operator :

$$\Delta_{p,w}(u) = \operatorname{div}(w(x) |\nabla u|^{p-2} \nabla u), \quad 1 < p < \infty.$$

is known as weighted p -Laplacian (see, for example,[6, 7]). The weighted p -Laplacian is a non-linear operator when $p \neq 2$ and is linear when $p = 2$.

2.4 Properties of the weighted p -Laplacian operator

Let Ω be a bounded open subset of \mathbb{R}^N , and let the operator

$$A : V = W_0^{1,p}(\Omega) \rightarrow V' = W^{-1,p'}(\Omega)$$

defined by

$$Au = -\operatorname{div}(w(x) |\nabla u|^{p-2} \nabla u), \quad 1 < p < \infty.$$

where $w(x)$ be a function in $W^{1,p}(\Omega)$ such that

$$0 < \alpha \leq w(x) \leq \beta. \quad (2.5)$$

According to the green's formula, we have

$$\forall \varphi \in W_0^{1,p}(\Omega) : \langle Au, \varphi \rangle = \int_{\Omega} w(x) |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi.$$

Proposition 2.3. *The operator $Au = -\operatorname{div}(w(x) |\nabla u|^{p-2} \nabla u)$, $p \neq 2$. is bounded from $W_0^{1,p}(\Omega)$ in $W^{-1,p'}(\Omega)$*

Proof. From the expression of the norm in a dual space, Let ρ strictly positive and let $u \in B_{W_0^{1,p}(\Omega)}(0, \rho)$ then $\|u\|_{W_0^{1,p}(\Omega)} \leq \rho$, one can get

$$\|Au\|_{V'} = \sup_{\substack{\varphi \in V \\ \|\varphi\| \leq 1}} |\langle Au, \varphi \rangle| = \sup_{\substack{\varphi \in V \\ \|\varphi\| \leq 1}} \left| \int_{\Omega} w(x) |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \right|.$$

Using the Hölder's inequality and (2.5), we obtain

$$\begin{aligned} \|Au\|_{V'} &\leq \sup_{\substack{\varphi \in V \\ \|\varphi\| \leq 1}} \int_{\Omega} |w(x)| |\nabla u|^{p-1} \cdot |\nabla \varphi| \\ &\leq \beta \sup_{\substack{\varphi \in V \\ \|\varphi\| \leq 1}} \left(\int_{\Omega} |\nabla u|^p \right)^{\frac{1}{p}} \left(\int_{\Omega} |\nabla \varphi|^p \right)^{\frac{1}{p}} \\ &= \beta \sup_{\substack{\varphi \in V \\ \|\varphi\| \leq 1}} \|u\|_{W_0^{1,p}(\Omega)}^{p-1} \|\varphi\|_{W_0^{1,p}(\Omega)} \\ &\leq \beta \rho^{p-1}. \end{aligned}$$

Which implies the boundedness of A . □

Proposition 2.4. *the operator $Au = -\operatorname{div}(w(x) |\nabla u|^{p-2} \nabla u)$, $p \neq 2$. is hemicontinuous from V to V' .*

Proof. Let $v \in V$, and $\{\lambda_n\}$ be a real sequence converging to 0.

$$\forall g \in V, \quad \langle A(u_{\infty} + \lambda_n v), g \rangle \rightarrow \langle Au_{\infty}, g \rangle.$$

we use lemma 1.4, we have

$$\langle Au_{\infty}, g \rangle = \int_{\Omega} w(x) |\nabla u_{\infty}|^{p-2} \nabla u_{\infty} \cdot \nabla g$$

then

$$\begin{aligned}\langle A(u_\infty + \lambda_n v), g \rangle &= \int_{\Omega} w(x) |\nabla(u_\infty + \lambda_n v)|^{p-2} \nabla(u_\infty + \lambda_n v) \nabla g \\ &= \int_{\Omega} w(x) |(\nabla u_\infty + \lambda_n \nabla v)|^{p-2} (\nabla u_\infty + \lambda_n \nabla v) \nabla g.\end{aligned}$$

$$1. w(x) |\nabla(u_\infty + \lambda_n \nabla v)|^{p-2} (\nabla u_\infty + \lambda_n \nabla v) \nabla g \rightarrow^{p \cdot p} w(x) |\nabla u_\infty|^{p-2} (\nabla u_\infty) \nabla g.$$

we can assume $|\lambda_n| \leq 1$, and $w(x) \leq \beta$, $\beta > 0$ then

$$\begin{aligned}\left| w(x) |\nabla(u_\infty + \lambda_n \nabla v)|^{p-2} (\nabla u_\infty + \lambda_n \nabla v) \nabla g \right| &= w(x) |\nabla(u_\infty + \lambda_n \nabla v)|^{p-1} |\nabla g| \\ &\leq \beta \left[|\nabla u_\infty| + |\lambda_n| |\nabla v| \right]^{p-1} |\nabla g| \\ &\leq \beta \left[|\nabla u_\infty| + |\nabla v| \right]^{p-1} |\nabla g|.\end{aligned}$$

we recall that

$$\left(\sum_{i=1}^N a_i \right)^\alpha \leq \max\{1, N^{\alpha-1}\} \sum_{i=1}^N a_i^\alpha, \quad \forall a_i \geq 0, \alpha > 0. \quad (2.6)$$

Using Young's inequality (1.3) and (2.6), we can write

$$\begin{aligned}\left| w(x) |\nabla(u_\infty + \lambda_n \nabla v)|^{p-2} (\nabla u_\infty + \lambda_n \nabla v) \nabla g \right| &\leq \frac{\beta |\nabla g|^p}{p} + \frac{\beta \left[|\nabla u_\infty| + |\nabla v| \right]^p}{p'} \\ &\leq \frac{\beta}{p} |\nabla g|^p + \frac{\max(1, 2^{p-1})}{p'} \left(|\nabla u_\infty|^p + |\nabla v|^p \right) \beta.\end{aligned}$$

the fact that $g, v, u_\infty \in W_0^{1,p}(\Omega)$, and $\beta > 0$ implies that

$$\frac{\beta}{p} |\nabla g|^p + \frac{2^{p-1}}{p'} \left(|\nabla u_\infty|^p + |\nabla v|^p \right) \beta \in L^1(\Omega).$$

According to Lebesgue's dominated convergence theorem

$$\lim_{n \rightarrow +\infty} \langle A(u_\infty + \lambda_n v), g \rangle = \langle A(u_\infty), g \rangle.$$

Which shows the hemicontinuity of A . □

Proposition 2.5. *the operator $Au = -\operatorname{div}(w(x) |\nabla u|^{p-2} \nabla u)$, $p \neq 2$. is monotone from V to V' .*

Proof. we recall that

$$\forall x, y \in \mathbb{R}^N : (|x|^{p-2}x - |y|^{p-2}y)(x - y) \geq \mu|x - y|^p, \mu > 0.$$

So, $\forall u, v \in V$ and we use lemma 1.4

$$\begin{aligned} \langle Au - Av, u - v \rangle &= \int_{\Omega} w(x) \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \nabla(u - v) \\ &\geq \alpha \int_{\Omega} \mu |\nabla u - \nabla v|^p \\ &\geq 0. \end{aligned}$$

□

Proposition 2.6. *the operator $Au = -\operatorname{div}(w(x) |\nabla u|^{p-2} \nabla u)$, $p > 1$. is coercive from V to V' .*

Proof. We use lemma 1.4 we have :

$$\begin{aligned} \lim_{\|v\|_V \rightarrow +\infty} \frac{\langle Av, v \rangle}{\|v\|_V} &= \lim_{\|v\|_V \rightarrow +\infty} \frac{\int_{\Omega} w(x) |\nabla v|^p}{\|v\|_V} \\ &\geq \lim_{\|v\|_V \rightarrow +\infty} \frac{\alpha \|v\|_V^p}{\|v\|_V} \\ &= \lim_{\|v\|_V \rightarrow +\infty} \alpha \|v\|_V^{p-1} = +\infty \quad \text{car } \alpha > 0 \quad \text{and } 1 < p < +\infty. \end{aligned}$$

□

2.5 Monotonic Operator Theorems

2.5.1 General result

Let V be a reflexive and separable Banach space, and $A : V \rightarrow V'$ be an operator.

Theorem 2.1 (Existence theorem). *Suppose that $A : V \rightarrow V'$ is an operator :*

- *bounded.*

- *hemicontinuous.*
- *coercive.*
- *monotone.*

Let $f \in V'$. Then, there exists a function $u \in V$ and only one such that:

$$Au = f.$$

For the proof this theorem, we refer the reader to the book [8].

Remark 2.2. Let $p > 1$ and let $w(x)$ be a function in $W^{1,p}(\Omega)$ such that

$$0 < \alpha \leq w(x) \leq \beta.$$

Then, the operator A defined by

$$Au = -\operatorname{div}(w(x) |\nabla u|^{p-2} \nabla u), \quad 1 < p < \infty.$$

to verify all the hypotheses of the theorem 2.1, so

$$\forall f \in W^{-1,p'}(\Omega), \quad \exists u \in W_0^{1,p}(\Omega) \quad \text{such that} \quad Au = f.$$

Now, we have the following result.

Lemma 2.1. Let $p > 1$, g belong to $L^\infty(\Omega)$ and let $w(x)$ be a function in $W^{1,p}(\Omega)$ such that:

$$0 < \alpha \leq w(x) \leq \beta. \tag{2.7}$$

Let v in $W_0^{1,p}(\Omega)$ be the weak solution of the equation

$$\begin{cases} -\operatorname{div}(w(x) |\nabla v|^{p-2} \nabla v) = g(x) & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

Then v is such that

$$\int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi = \int_{\Omega} G(x) \varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

where $G(x)$ is the $L^1(\Omega)$ function

$$G(x) = \left[|\nabla v|^{p-2} \nabla v \cdot \nabla w + g(x) \right] \frac{1}{w(x)}. \quad (2.9)$$

Remark 2.3. Note that the solution v of (2.8) exists since the differential weighted operator

$$A(v) = -\operatorname{div}(w(x) |\nabla v|^{p-2} \nabla v)$$

is both coercive and pseudomonotone on $W_0^{1,p}(\Omega)$, so that it is surjective on the dual of $W_0^{1,p}(\Omega)$.

Proof. Let φ belong to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Since $w(x)$ belongs to $W^{1,p}(\Omega)$, and $\frac{1}{w(x)}$ is a bounded function, we have that $\psi = \frac{\varphi}{w(x)}$ belongs to $W_0^{1,p}(\Omega)$, so that it can be chosen as test function in the weak formulation of (2.8). We obtain

$$\int_{\Omega} w(x) |\nabla v|^{p-2} \nabla v \cdot \nabla \left(\frac{\varphi}{w(x)} \right) dx = \int_{\Omega} g(x) \frac{\varphi}{w(x)} dx.$$

Using that

$$\nabla \left(\frac{\varphi}{w(x)} \right) = \frac{w(x) \nabla \varphi - \varphi \nabla w(x)}{w^2(x)},$$

we have

$$\int_{\Omega} \frac{w(x)}{w(x)} |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi - \int_{\Omega} \frac{w(x)}{w^2(x)} |\nabla v|^{p-2} \nabla v \cdot \nabla w \varphi = \int_{\Omega} g(x) \frac{\varphi}{w(x)}.$$

which can be rewritten as

$$\int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi = \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla w \frac{\varphi}{w(x)} + \int_{\Omega} g(x) \frac{\varphi}{w(x)} = \int_{\Omega} G(x) \varphi.$$

where in the last passage we have used the definition (2.9) of $G(x)$. Because that $w(x) \in W^{1,p}(\Omega)$ and the fact that $v \in W_0^{1,p}(\Omega)$, we have that $|\nabla v|^{p-2} \nabla v \cdot \nabla w$ is bounded in $L^1(\Omega)$, while the function g is bounded ($g \in L^\infty$). Furthermore, thanks to assumption (2.7), the term $\frac{1}{w(x)}$ is bounded from above by $\frac{1}{\alpha}$, so that $G(x)$ is bounded in $L^1(\Omega)$, as desired. \square

Nonlinear weighted elliptic equations with degenerate coercivity and L^1 data

In this chapter, we aim to prove the existence of weak solutions for nonlinear elliptic equation with degenerate coercivity. The right-hand side of the equation given in $L^1(\Omega)$

3.1 The problem (P)

The problem considered in this work is the nonlinear weighted elliptic problem having a degenerate coercivity:

$$(P). \quad \begin{cases} -\operatorname{div} \left(a(x) \frac{|\nabla u|^{p-2} \nabla u}{(1+|u|)} \right) + e(x)|u|^{p-2}u = f & \text{on } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded, open subset of \mathbb{R}^N , $N > 2$. $p \geq 1$, f belongs to $L^1(\Omega)$ and $a(x)$ is such that

$$a(x) \in W^{1,p}(\Omega) : a(x) \geq \alpha > 0. \quad (3.1)$$

for some α in \mathbb{R} and the function $0 \leq e(x) \in L^1(\Omega)$. Even if $f \in L^1(\Omega)$, the assumption

$$\text{there exists } k_0 > 0 \text{ such that } |f(x)| \leq k_0 \cdot e(x). \quad (3.2)$$

implies the existence of a weak solution u belonging to $W_0^{1,p}(\Omega)$ and to $L^\infty(\Omega)$.

3.1.1 Weak solutions

Definition 3.1. *says that u is a weak solution to problem (P), if $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and*

$$\int_{\Omega} a(x) \frac{|\nabla u|^{p-2} \nabla u}{(1+|u|)} \nabla \varphi \, dx + \int_{\Omega} e(x)|u|^{p-2}u \varphi \, dx = \int_{\Omega} f \varphi \, dx, \quad \forall \varphi \in W_0^{1,\infty}(\Omega). \quad (3.3)$$

Now, The result proved in this chapter is the following theorem

Theorem 3.1. *Let $p \geq 1$ and let $f \in L^1(\Omega)$. Then, the problem (P) has at least one weak solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ in the sense of Definition 3.1.*

Remark 3.1. *the differential operator*

$$u \mapsto -\operatorname{div} \left(a(x) \frac{|\nabla u|^{p-2} \nabla u}{(1 + |u|)} \right)$$

is not coercive on $W_0^{1,p}(\Omega)$ despite being well-defined between $W_0^{1,p}(\Omega)$ and its dual $W^{-1,p'}(\Omega)$. Degenerate coercivity implies that as $|u|$ becomes large, $\frac{1}{1+|u|}$ tends to zero.

This shows that classical methods for elliptic operators cannot be applied. To solve this issue, we are going to approximate the operator by employing truncations in $\frac{1}{1+|u|}$ to obtain a coercive differential operator. Next, we will prove some a priori estimates on the sequence of approximate solutions, and we shall finally pass to the limit in the approximate problems to establish the existence of a weak solution for the problem (P).

3.1.2 the truncation functions

Lemma 3.1 (Stampacchia). *Let $T : \mathbb{R} \rightarrow \mathbb{R}$ be a globally lipschitz function, i.e.*

$$\exists C > 0 \quad \text{such that} \quad |T(s) - T(t)| \leq C|s - t|, \quad \forall s, t \in \mathbb{R},$$

such that $T(0) = 0$.

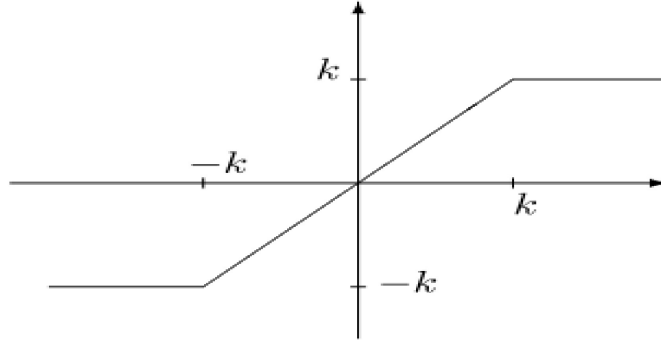
Then, $\forall v \in W_0^{1,p}(\Omega)$ with $1 \leq p \leq \infty$ we have :

$$T(v) \in W_0^{1,p}(\Omega) \quad \text{and} \quad \nabla T(v) = T'(v) \nabla v \quad \text{in} \quad \mathcal{D}'(\Omega) \quad \text{and} \quad \text{almost everywhere in } \Omega.$$

Example 3.1. *Let $k > 0$. The truncation at levels $-k$ and k is defined by the function T_k of \mathbb{R} from \mathbb{R} given by*

$$T_k(s) = \begin{cases} k, & \text{if } s \geq k, \\ s, & \text{if } |s| < k, \\ -k, & \text{if } s \leq -k. \end{cases}$$

It can be verified that the function T_k is a globally Lipschitz function, The graph of T_k is:



We can verify that the function T_k is a globally Lipschitz function satisfying $|T_k(s)| \leq k$ and $|T_k(s)| \leq |s|$.

Example 3.2. Let $k > 0$. We will use the following function defined for $r \in \mathbb{R}$ by

$$G_k(s) = s - T_k(s) = \begin{cases} s - k, & \text{if } s > k, \\ 0, & \text{if } |s| \leq k, \\ s + k, & \text{if } s < -k. \end{cases}$$

so,

$$G_k(u_n) = u_n - T_k(u_n) = \begin{cases} u_n - k, & \text{if } u_n > k, \\ 0, & \text{if } |u_n| \leq k, \\ u_n + k, & \text{if } u_n < -k. \end{cases}$$

and

$$\nabla G_k(u_n) = \begin{cases} \nabla u_n, & \text{if } |u_n| > k, \\ 0, & \text{if } |u_n| \leq k. \end{cases}$$

3.2 Approximation of (P)

Consider the following approximated problem

$$(P_n). \quad \begin{cases} -\operatorname{div} \left(a_n(x) \frac{|\nabla u_n|^{p-2} \nabla u_n}{1 + |T_n(u_n)|} \right) + e_n(x) |u_n|^{p-2} u_n = f_n(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

i.e., satisfying

$$\int_{\Omega} a_n(x) \frac{|\nabla u_n|^{p-2} \nabla u_n}{1 + |T_n(u_n)|} \nabla \varphi dx + \int_{\Omega} e_n(x) |u_n|^{p-2} u_n \varphi dx = \int_{\Omega} f_n \varphi dx, \quad (3.4)$$

for every $\varphi \in W_0^{1,\infty}(\Omega)$, with f_n, e_n be a sequence of bounded functions defined in Ω defined by

$$e_n(x) = \frac{e(x)}{1 + \frac{k_0 e(x)}{n}}, \quad f_n(x) = \frac{f(x)}{1 + \frac{|f(x)|}{n}}. \quad (3.5)$$

Note that the definition of e_n and f_n in (3.5) and since $\psi(s) = s(1 + \frac{s}{n})^{-1}$ is increasing, we deduce by (3.2) that

$$\begin{cases} \|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}, & \|e_n\|_{L^1(\Omega)} \leq \|e\|_{L^1(\Omega)}, \\ |f_n| \leq n, & |e_n| \leq n. \end{cases}$$

and

$$|f_n(x)| = \frac{f(x)}{1 + \frac{|f(x)|}{n}} \leq \frac{k_0 e(x)}{1 + \frac{k_0 e(x)}{n}} = k_0 e_n(x) \quad (3.6)$$

and $a_n(x)$ be a sequence of bounded functions in $W^{1,p}(\Omega)$, such that

$$a_n(x) = \frac{a(x)}{1 + \frac{a(x)}{n}}. \quad (3.7)$$

Note that we have, thanks to (3.1), that

$$C_\alpha = \frac{\alpha}{1 + \alpha} \leq a_n(x) \leq n. \quad (3.8)$$

Using that $|T_n(u_n)| \leq n$, we obtain

$$\frac{1}{1 + |T_n(u_n)|} \geq \frac{1}{(1 + n)}, \quad (3.9)$$

so that the operator $B_n : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ defined by

$$\int_{\Omega} \langle B_n u, \varphi \rangle dx = \int_{\Omega} a_n(x) \frac{|\nabla u_n|^{p-2} \nabla u_n}{1 + |T_n(u_n)|} \nabla \varphi dx, \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

is coercive due to the fact that $a_n(x) \geq C_\alpha = \frac{\alpha}{1+\alpha} > 0$ and is pseudo-monotone on $W_0^{1,p}(\Omega)$.

Thus, the existence of the approximate solution u_n is proved as in [8].

3.3 Uniform Estimates of Approximate solutions

Lemma 3.2. *Let u_n be the solutions to problems (P_n) . Then, there exists a positive constant C such that*

$$\|u_n\|_{L^\infty(\Omega)} \leq C, \quad \|u_n\|_{W_0^{1,p}(\Omega)} \leq C, \quad \forall n \in \mathbb{N}.$$

Proof. We choose $G_R(u_n)$ as test function in (P_n) with $R = k_0^{\frac{1}{p-1}}$, we get

$$\begin{aligned} \int_{\Omega} a_n(x) \frac{|\nabla u_n|^{p-2} \nabla u_n \nabla(G_R(u_n))}{1 + |T_n(u_n)|} dx \\ + \int_{\Omega} e_n(x) |u_n|^{p-2} u_n G_R(u_n) dx = \int_{\Omega} f_n G_R(u_n) dx. \end{aligned} \quad (3.10)$$

Since $|\nabla u_n|^{p-2} \nabla u_n \nabla(G_R(u_n)) = |\nabla G_R(u_n)|^p$ and the fact that $|T_n(u_n)| \leq |u_n|$, we obtain

$$\begin{aligned} C_{\alpha} \int_{\Omega} \frac{1}{1 + |u_n|} |\nabla G_R(u_n)|^p dx + \int_{\Omega} e_n(x) |u_n|^{p-2} u_n G_R(u_n) dx \\ \leq \int_{\Omega} |f_n| |G_R(u_n)| dx. \end{aligned} \quad (3.11)$$

Using that $u_n G_R(u_n) \geq 0$ and (3.2), we obtain

$$C_{\alpha} \int_{\Omega} \frac{1}{1 + |u_n|} |\nabla G_R(u_n)|^p dx + \int_{\Omega} e_n(x) (|u_n|^{p-1} - k_0) |G_R(u_n)| dx \leq 0.$$

We remark that the integrand in the second integral is zero if $|u_n(x)| \leq k_0^{\frac{1}{p-1}} = R$, so we deduce that

$$C_{\alpha} \int_{\Omega} \frac{1}{(1 + |u_n|)} |\nabla G_R(u_n)|^p dx = 0,$$

which implies

$$|u_n| \leq R. \quad (3.12)$$

This implies that $(u_n)_n$ is bounded in $L^{\infty}(\Omega)$.

Now, to show that the sequence $\{u_n\}_n$ is bounded in $W_0^{1,p}(\Omega)$, inserting u_n in (P_n) , and from (3.12), we get

$$C_{\alpha} \frac{1}{(1 + R)} \int_{\Omega} |\nabla u_n|^p dx + \int_{\Omega} e_n(x) |u_n|^p dx \leq \int_{\Omega} |f_n| |u_n| dx,$$

dropping nonnegative terms, we can rewrite the above inequality as follows

$$C_{\alpha} \frac{1}{(1 + R)} \int_{\Omega} |\nabla u_n|^p dx \leq R \|f\|_{L^1(\Omega)}.$$

Because that $f \in L^1(\Omega)$, we find

$$\int_{\Omega} |\nabla u_n|^p dx \leq C, \quad C = \frac{R(1 + R)}{C_{\alpha}} \|f\|_{L^1(\Omega)}.$$

Thus,

$$\|u_n\|_{W_0^{1,p}(\Omega)} \leq C. \quad (3.13)$$

This implies that $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$. \square

Lemma 3.3. *Let $p \geq 1$ and u_n be the solutions to problems (P_n) . Then we have for all $k > 0$*

$$\int_{\Omega} |\nabla T_k(u_n)|^p dx \leq \frac{k(1+k)}{C_{\alpha}} \|f\|_{L^1(\Omega)}. \quad (3.14)$$

Proof. choosing $T_k(u_n)$ as test function in (P_n) , we obtain

$$\begin{aligned} \int_{\Omega} a_n(x) \frac{|\nabla u_n|^{p-2} \nabla u_n}{1 + |T_n(u_n)|} \nabla(T_k(u_n)) dx \\ + \int_{\Omega} e_n(x) |u_n|^{p-2} u_n T_k(u_n) dx = \int_{\Omega} f_n T_k(u_n) dx. \end{aligned} \quad (3.15)$$

Using that $u_n T_k(u_n) \geq 0$, so after dropping non-negative terms, we obtain

$$\int_{\Omega} a_n(x) \frac{|\nabla T_k(u_n)|^p}{1 + |T_n(u_n)|} dx \leq \int_{\Omega} |f_n| |T_k(u_n)| dx.$$

According to the conditions (3.8) and for $n > k > 0$, we get

$$\frac{C_{\alpha}}{1+k} \int_{\Omega} |\nabla T_k(u_n)|^p dx \leq k \|f\|_{L^1(\Omega)}. \quad (3.16)$$

Therefore (3.14) is established. Moreover, by Fatous Lemma, we have

$$\int_{\Omega} |\nabla T_k(u)|^p dx \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} |\nabla T_k(u_n)|^p dx \leq C.$$

\square

3.4 Passage to the limite

By Lemma 3.2, the sequence of approximated solutions u_n is bounded in $W_0^{1,p}(\Omega)$. Therefore, there exists a function u belongs to $W_0^{1,p}(\Omega)$ such that, up to subsequences,

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega), \quad (3.17)$$

$$u_n \longrightarrow u \quad \text{strongly in } L^p(\Omega). \quad (3.18)$$

Therefore

$$u_n \longrightarrow u \quad \text{a.e in } \Omega, \quad (3.19)$$

Now, we are going to prove the almost everywhere convergence of ∇u_n to ∇u in Ω .

Lemma 3.4. *Let $p > 1$. Let (g_n) be a sequence of functions in $L^1(\Omega)$, and let (a_n) be a sequence of functions in $W^{1,p}(\Omega)$ such that:*

1. *the sequence (g_n) is bounded in $L^1(\Omega)$;*
2. *the sequence (a_n) is bounded in $W^{1,p}(\Omega)$;*
3. *for every n in \mathbb{N} there exists $\lambda > 0$ and $\beta_n > 0$ such that*

$$\lambda \leq a_n(x) \leq \beta_n. \quad (3.20)$$

4. *for every n in \mathbb{N} there exists a weak solution u_n in $W_0^{1,p}(\Omega)$ of*

$$(*) \quad \begin{cases} -\operatorname{div} \left(a_n(x) \frac{|\nabla u_n|^{p-2} \nabla u_n}{1 + |T_n(u_n)|} \right) = g_n, & \text{in } \Omega \\ u_n = 0, & \text{on } \partial\Omega. \end{cases}$$

5. *the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$.*

Then we have

$$\int_{\Omega} \frac{|\nabla u_n|^{p-2} \nabla u_n}{1 + |T_n(u_n)|} \nabla \varphi = \int_{\Omega} G_n \varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad (3.21)$$

where G_n is bonded in $L^1(\Omega)$. Furthermore,

$$\text{the sequence } (\nabla u_n) \text{ converges almost everywhere to } \nabla u, \quad (3.22)$$

where u is the weak limit of the sequence (u_n) in $W_0^{1,p}(\Omega)$.

Proof. Thanks to Lemma 2.1 applied to the equations solved by u_n , we have that (3.21) holds, with (see (2.9))

$$G_n(x) = \frac{1}{a_n(x)} \left(\frac{|\nabla u_n|^{p-2} \nabla u_n}{1 + |T_n(u_n)|} \nabla a_n + g_n \right). \quad (3.23)$$

Thanks to (3.8), the assumptions (2) of Lemma 3.4 holds, and the sequence $\{\frac{1}{a_n(x)}\}_n$ is bounded by $\frac{1}{C_\alpha}$. furthermore, since

$$|\nabla a_n(x)| \leq |\nabla a(x)|,$$

and the fact that $1 + |T_n(v_n)| \geq 1$, we have

$$\begin{aligned} \int_{\Omega} |G_n(x)| dx &\leq \frac{1}{C_\alpha} \int_{\Omega} |\nabla u_n|^{p-2} |\nabla u_n| |\nabla a(x)| dx + \frac{1}{C_\alpha} \int_{\Omega} |g_n| dx \\ &\leq C_1 \int_{\Omega} |\nabla u_n|^{p-1} |\nabla a(x)| dx + C_1 \int_{\Omega} |g_n| dx. \end{aligned} \quad (3.24)$$

By application of Young's inequality, we have

$$\int_{\Omega} |G_n(x)| dx \leq \frac{C_1}{p'} \int_{\Omega} |\nabla u_n|^p dx + \frac{C_1}{p} \int_{\Omega} |\nabla a(x)|^p dx + C_1 \int_{\Omega} |g_n| dx.$$

By assumption (2), the $L^p(\Omega)$ -bound on ∇u_n (since(5)) and the fact that the sequence $\{g_n\}_n$ is bounded in $L^1(\Omega)$ (since (1)), we conclude that G_n is bounded in $L^1(\Omega)$. Now, to show that

$$\nabla u_n \longrightarrow \nabla u \quad \text{almost everywhere in } \Omega,$$

Let $k > 0$ and let us take

$$w_n = T_{2k}(T_k(u_n) - T_k(u)),$$

as test function in (*), we have

$$\int_{\Omega} \frac{a_n(x)}{1 + |T_n(u_n)|} |\nabla u_n|^{p-2} \nabla u_n \nabla w_n dx = \int_{\Omega} g_n w_n dx. \quad (3.25)$$

For the first integral in (3.25) and the fact that $\frac{1}{1+|T_n(u_n)|} \geq \frac{1}{1+k}$ on the set $\{|u_n| \leq k\}$, we

can write it as

$$\begin{aligned} & \int_{\Omega} \frac{a_n(x)}{1 + |T_n(u_n)|} |\nabla u_n|^{p-2} \nabla u_n \nabla w_n dx \\ & \geq \frac{C_{\alpha}}{(1+k)} \int_{\{|u_n| \leq k\}} |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) \nabla (T_k(u_n) - T_k(u)) dx \\ & \quad - \int_{\{|u_n| > k\}} \frac{a_n(x)}{1 + |T_n(u_n)|} |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) \nabla T_k(u) dx \end{aligned}$$

Using that $\frac{1}{1+|T_n(u_n)|} \leq 1$, we have

$$\begin{aligned} & - \int_{\{|u_n| > k\}} \frac{a_n(x)}{1 + |T_n(u_n)|} |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) \nabla T_k(u) dx \\ & \geq - \int_{\{|u_n| > k\}} a_n(x) |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) \nabla T_k(u) dx \end{aligned}$$

So,

$$\begin{aligned} & \int_{\Omega} \frac{a_n(x)}{1 + |T_n(u_n)|} |\nabla u_n|^{p-2} \nabla u_n \nabla w_n dx \\ & \geq \frac{C_{\alpha}}{(1+k)} \int_{\{|u_n| \leq k\}} |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) \nabla (T_k(u_n) - T_k(u)) dx \\ & \quad - \int_{\{|u_n| > k\}} a_n(x) |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) \nabla T_k(u) dx. \end{aligned}$$

It follows from the above inequality that

$$\begin{aligned} & \frac{C_{\alpha}}{(1+k)} \int_{\Omega} |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) \nabla (T_k(u_n) - T_k(u)) dx \\ & \leq \int_{\{|u_n| > k\}} a_n(x) |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) \nabla T_k(u) dx + \int_{\Omega} g_n w_n dx. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} & \frac{C_{\alpha}}{(1+k)} \int_{\Omega} [|\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) - |\nabla T_k(u)|^{p-2} \nabla T_k(u)] \nabla (T_k(u_n) - T_k(u)) dx \\ & \leq \int_{\{|u_n| > k\}} a_n(x) |\nabla T_k(u_n)|^{p-1} |\nabla T_k(u)| dx \\ & \quad - \frac{C_{\alpha}}{(1+k)} \int_{\Omega} |\nabla T_k(u)|^{p-2} \nabla T_k(u) \nabla (T_k(u_n) - T_k(u)) dx + \int_{\Omega} g_n w_n dx \\ & = I_1 + I_2 + I_3. \end{aligned} \tag{3.26}$$

Limit of I_1 . The fact that $|\nabla T_k(u_n)|^{p-1}$ is bounded in $L^{p'}(\Omega)$ and by the dominated convergence theorem

$$\chi_{\{|u_n| > k\}} a_n(x) |\nabla T_k(u_n)| \longrightarrow \chi_{\{|u| > k\}} a(x) |\nabla T_k(u)| \quad \text{strongly in } L^p(\Omega),$$

which is zero, as $n \rightarrow +\infty$. Thus, we obtain

$$\lim_{n \rightarrow +\infty} I_1 = 0. \quad (3.27)$$

Limit of I_2 . Since (3.16), we obtain $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ in $L^p(\Omega)$ and the bounded of $|\nabla T_k(u)|^{p-2} \nabla T_k(u)$ in $L^{p'}(\Omega)$, we have

$$\lim_{n \rightarrow +\infty} I_2 = 0. \quad (3.28)$$

Limit of I_3 . Notice that

$$\begin{aligned} \int_{\Omega} |g_n - g| |w_n| dx + \int_{\Omega} |g w_n| dx \\ \leq 2k \int_{\Omega} |g_n - g| dx + \int_{\Omega} |g T_{2k}(T_k(u_n) - T_k(u))| dx. \end{aligned}$$

Since $g_n \rightarrow g$ strongly in $L^1(\Omega)$, using (3.19) and the Lebesgue dominated convergence theorem, we have

$$\lim_{n \rightarrow +\infty} I_3 \leq \lim_{n \rightarrow +\infty} \int_{\Omega} |g T_{2k}(T_k(u_n) - T_k(u))| dx = 0. \quad (3.29)$$

Now, passing to the limits in (3.26) as n tend to infinity, and by (3.27)-(3.29), we deduce that

$$\lim_{n \rightarrow +\infty} I_n^k = 0,$$

where

$$I_n^k = \int_{\Omega} [|\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) - |\nabla T_k(u)|^{p-2} \nabla T_k(u)] \nabla (T_k(u_n) - T_k(u)) dx.$$

Using the following well-known inequalities that hold for any two real vectors ξ, η and a real $p \geq 2$:

$$(|\xi|^{p-2} \xi - |\eta|^{p-2} \eta)(\xi - \eta) \geq \begin{cases} 2^{2-p} |\xi - \eta|^p, & \text{if } p \geq 2, \\ (p-1) \frac{|\xi - \eta|^2}{(|\xi| + |\eta|)^{2-p}}, & \text{if } 1 < p < 2, \end{cases} \quad (3.30)$$

we find,

$$2^{2-p} \int_{\{x \in \Omega, p \geq 2\}} |\nabla (T_k(u_n) - T_k(u))|^p dx \leq I_n^k.$$

Now, on the set $E = \{x \in \Omega, 1 < p < 2\}$, we have

$$\begin{aligned}
 \int_E |\nabla(T_k(u_n) - T_k(u))|^p dx &= \int_E \frac{|\nabla(T_k(u_n) - T_k(u))|^p}{(|\nabla T_k(u_n)| + |\nabla T_k(u)|)^{\frac{p(2-p)}{2}}} (|\nabla T_k(u_n)| + |\nabla T_k(u)|)^{\frac{p(2-p)}{2}} dx \\
 &\leq \left\| \frac{|\nabla(T_k(u_n) - T_k(u))|^p}{(|\nabla T_k(u_n)| + |\nabla T_k(u)|)^{\frac{p(2-p)}{2}}} \right\|_{L^{\frac{2}{p}}(E)} \left\| (|\nabla T_k(u_n)| + |\nabla T_k(u)|)^{\frac{p(2-p)}{2}} \right\|_{L^{\frac{2}{2-p}}(E)} \\
 &\leq \left(\int_E \frac{|\nabla(T_k(u_n) - T_k(u))|^2}{(|\nabla T_k(u_n)| + |\nabla T_k(u)|)^{2-p}} dx \right)^{\frac{p}{2}} \left(\int_E (|\nabla T_k(u_n)| + |\nabla T_k(u)|)^p dx \right)^{\frac{2-p}{2}} \\
 &\leq (p-1)^{-\frac{p}{2}} (I_n^k)^{\frac{p}{2}} \left(\int_E (|\nabla T_k(u_n)| + |\nabla T_k(u)|)^p dx \right)^{\frac{2-p}{2}}.
 \end{aligned} \tag{3.31}$$

Since $I_n^k \rightarrow 0$ as $n \rightarrow +\infty$ and $(T_k(u_n))_n$ is bounded in $W_0^{1,p}(\Omega)$ (see lemma 3.3), we conclude that for every $k > 0$,

$$\nabla T_k(u_n) \rightarrow \nabla T_k(u) \text{ strongly in } L^p(\Omega).$$

From this result, we deduce that (up to subsequences)

$$\nabla u_n \rightarrow \nabla u \text{ almost everywhere in } \Omega. \tag{3.32}$$

□

3.4.1 The end of the proof of Theorem 3.1

For $\varphi \in W_0^{1,\infty}(\Omega)$ (See (3.4)), we have

$$\int_{\Omega} a_n(x) \frac{|\nabla u_n|^{p-2} \nabla u_n}{1 + |T_n(u)|} \nabla \varphi dx + \int_{\Omega} e_n(x) |u_n|^{p-2} u_n \varphi dx = \int_{\Omega} f_n \varphi dx,$$

1) Passage to the limit in $\int_{\Omega} f_n \varphi dx$

Using that $f_n \rightarrow f$ strongly in $L^1(\Omega)$, we have $\forall \varphi \in W_0^{1,\infty}(\Omega)$

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f_n \varphi dx = \int_{\Omega} f \varphi dx.$$

2) Passage to the limit in $\int_{\Omega} e_n(x) |u_n|^{p-2} u_n \varphi dx$.

Thanks to (3.12), we have

$$\int_{\Omega} e_n(x) |u_n|^{p-2} u_n dx \leq \int_{\Omega} e_n(x) |u_n|^{p-1} dx \leq k_0 \int_{\Omega} e_n(x) dx. \tag{3.33}$$

Since $e_n(x) \in L^\infty(\Omega)$, the sequence $\{e_n(x)|u_n|^{p-2}u_n\}_n$ is bounded in $L^1(\Omega)$ and thanks to (3.19), we have that

$$e_n(x)|u_n|^{p-2}u_n \rightarrow e(x)|u|^{p-2}u \quad \text{strongly in } L^1(\Omega). \quad (3.34)$$

So, we have

$$\begin{aligned} \left| \int_{\Omega} e_n(x)|u_n|^{p-2}u_n\varphi - \int_{\Omega} e_n(x)|u|^{p-2}u\varphi \right| &= \left| \int_{\Omega} (e_n(x)|u_n|^{p-2}u_n - e(x)|u|^{p-2}u)\varphi \right| \\ &\leq C \|e_n(x)|u_n|^{p-2}u_n - e(x)|u|^{p-2}u\|_{L^1(\Omega)} \rightarrow 0, \\ &\text{as } n \rightarrow +\infty \end{aligned}$$

ensure that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} e_n(x)|u_n|^{p-2}u_n\varphi dx = \int_{\Omega} e(x)|u|^{p-2}u\varphi dx.$$

3) Passage to the limit in $\int_{\Omega} a_n(x) \frac{|\nabla u_n|^{p-2}\nabla u_n}{1+|T_n(u_n)|} \nabla \varphi dx$.

It is clear that assumption (1) in the Lemma 3.4 is satisfied because that the sequence $\{g_n\}_n$ where

$$g_n(x) = f_n(x) - e_n(x)|u_n|^{p-2}u_n,$$

is bounded in $L^1(\Omega)$. We can apply the Lemma 3.4 to have that $\{\nabla u_n\}$ converges almost everywhere to ∇u in Ω . Indeed, thanks to the boundedness of the sequence $(u_n)_n$ in $W_0^{1,p}(\Omega)$, we have that

$$|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla u|^{p-2}\nabla u \quad \text{in } L^{p'}(\Omega). \quad (3.35)$$

Since the sequence $(a_n)_n$ strongly converges to $a(x)$ in $L^p(\Omega)$ (being strongly convergent in $W^{1,p}(\Omega)$), we can use (3.35) and by the dominated convergence Theorem, we get

$$a_n(x)|\nabla u_n|^{p-2}\nabla u_n \rightarrow a(x)|\nabla u|^{p-2}\nabla u \quad \text{in } L^1(\Omega).$$

and using (3.19), we have that

$$\begin{cases} \frac{1}{1+|T_n(u_n)|} \nabla \varphi \rightarrow \frac{1}{1+|u|} \nabla \varphi, \text{ a.e in } \Omega, \\ \left| \frac{1}{1+|T_n(u_n)|} \nabla \varphi \right| \leq |\nabla \varphi| \in L^\infty(\Omega), \end{cases} .$$

So,

$$\lim_{n \rightarrow +\infty} \int_{\Omega} a_n(x) \frac{|\nabla u_n|^{p-2} \nabla u_n}{1 + |T_n(u_n)|} \nabla \varphi dx = \int_{\Omega} a(x) \frac{|\nabla u|^{p-2} \nabla u}{1 + |u|} \nabla \varphi dx.$$

Finally, we pass to the limit in the formulation (3.4), we obtain

$$\int_{\Omega} a(x) \frac{|\nabla u|^{p-2} \nabla u}{1 + |u|} \nabla \varphi dx + \int_{\Omega} e(x) |u|^{p-2} u \varphi dx = \int_{\Omega} f \varphi dx,$$

for every $\varphi \in W_0^{1,\infty}(\Omega)$. Therefore, we have to prove that u is a solution to problem (P).

This finishes the proof of theorem 3.1.

Conclusion

In this thesis, we have focused our attention on a class of nonlinear elliptic equations with degenerate coercivity of the form:

$$(P) \quad \begin{cases} Au + e(x)|\nabla u|^{p-2}\nabla u = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the differential operator:

$$Au = -\operatorname{div} \left(a(x) \frac{|\nabla u|^{p-2}\nabla u}{1 + |u|} \right), \quad 1 < p < \infty$$

and the right-hand f in $L^1(\Omega)$. I would like to apply the ideas and results mentioned in [1] to the case of nonlinear weighted elliptic equations with degenerate coercivity. The method used to prove the existence of solutions $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ for the problem (P) is the same as used in [1]. We are going to prove the existence of the weak solution of the problem (P). To do this, we approximate the problem (P) by a sequence of approximate problems (P_n) given in L^∞ whose existence of the solution approximate is guaranteed (See [8]). Then we will prove some estimates uniform on the sequence of solutions of these problems (P_n) and their partial derivatives. Once this is done, the linearity of the operator with respect to the gradient as well as the boundedness and the continuity of the function $a = |\nabla u|^{p-2}\nabla u$ will make it possible to pass to the limit, thus finding the solution.

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Abstract

In this work, we prove the existence of a weak solution of elliptic problem (P) defined by:

$$(P) \quad \begin{cases} -\operatorname{div}(a(x) \frac{|\nabla u|^{p-2} \nabla u}{1+|u|}) + e(x)|u|^{p-2}u = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f \in L^1(\Omega)$ and the operator $Au = -\operatorname{div}(a(x) \frac{|\nabla u|^{p-2} \nabla u}{1+|u|})$, $1 < p < \infty$ is not coercive on $W_0^{1,p}(\Omega)$ despite being well-defined between $W_0^{1,p}(\Omega)$ and its dual $W^{-1,p'}(\Omega)$. Degenerate coercivity implies that as $|u|$ becomes large, $\frac{1}{1+|u|}$ tends to zero. To solve this issue, we are going to approximate the operator by employing truncations in $\frac{1}{1+|u|}$ to obtain a coercive differential operator. Next, we will prove some a priori estimates on the sequence of approximate solutions, and we shall finally pass to the limit in the approximate problems to establish the existence of a weak solution for the problem (P) .

keywords: weighted Sobolev spaces, pseudo-monotone, operator nonlinear, elliptic equation, weak solution.

Résumé

Dans ce travail, nous prouvons l'existence d'une solution faible d'un problème elliptique non régulière définie par:

$$(P) \quad \begin{cases} -\operatorname{div}(a(x) \frac{|\nabla u|^{p-2} \nabla u}{1+|u|}) + e(x)|u|^{p-2}u = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

avec $f \in L^1(\Omega)$ L'opérateur $Au = -\operatorname{div}(a(x) \frac{|\nabla u|^{p-2} \nabla u}{1+|u|})$, $1 < p < \infty$ n'est pas coercitif sur $W_0^{1,p}(\Omega)$ bien qu'il soit bien défini entre $W_0^{1,p}(\Omega)$ et son dual $W^{-1,p'}(\Omega)$. La coercitivité dégénérée implique que lorsque $|u|$ devient grand, $\frac{1}{1+|u|}$ tend vers zéro. Pour résoudre ce problème, nous allons approximer l'opérateur en employant des troncatures dans $\frac{1}{1+|u|}$ pour obtenir un opérateur différentiel coercitif. Ensuite, nous prouverons quelques estimations

a priori sur la suite des solutions approchées, et nous passerons enfin à la limite dans les problèmes approximatifs pour établir l'existence d'une solution faible pour le problème (P) .

mots-clés: Espace de Sobolev avec poids, pseudo-monotone, opérateur non linéaire, équation elliptique, solution faible .

ملخص

في هذه المذكرة نسعى الى اثبات وجود الحلول الضعيفة للمعادلة التناقصية وغير الخطية المعرفة كما يلي :

$$\begin{cases} -\operatorname{div}(a(x)\frac{|\nabla u|^{p-2}\nabla u}{1+|u|}) + e(x)|u|^{p-2}u = f & \Omega; \\ u = 0 & \partial\Omega, \end{cases} \quad (P)$$

تحت الشرط التالي $u = 0$ على الحافة حيث ان الطرف الايمن f ينتمي الى الفضاء $L^1(\Omega)$ المؤثر $1 < p < \infty$ $Au = -\operatorname{div}(a(x)\frac{|\nabla u|^{p-2}\nabla u}{1+|u|})$ معرف بشكل جيد بين الفضاء $W_0^{1,p}$ وفضاءه الثنوي، الخطوات الرئيسية للاثبات تتمحور فيمايلي:

نقرب الجملة (P) بمتتالية جمل (P_n) ثم نبرهن ان الجملة المقربة تتمتع بحل ضعيف محدود u_n ثم الحصول على التقديرات لمتتالية الحلول التقريبية u_n و ∇u_n . وفي الاخير نمر بالنهاية في المسألة التقريبية للحصول على حل للجملة (P) .
كلمات مفتاحية: معادلات تناقصية غير خطية ، مؤثر غير خطي، فضاء سوبولوف، حل ضعيف