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Traveling profiles solutions to heat equation with a power-law nonlinearity

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Abstract. We propose in this work a new approach to obtain a general form of self similar solutions to heat equation with a power-law nonlinearity. We use the "traveling profiles method". This method is inspired of the traveling wavelets method which belongs to the category of the particular methods. The traveling profiles method enables us to obtain many explicit exact solutions to this equation.

Keywords: Heat equation with a power-law nonlinearity; Traveling profiles method.

Mathematics Subject Classification 2010: 35K05, 35C06.

1 Introduction

In general, the partial differential equations does not admit an exact solutions, particularly when we imposed initial /boundary conditions. But for some classes of PDEs which enjoy certain symmetries we can find their exact solutions for many particular cases [1, 4, 11, 12] . With some finite or infinite transformations, these partial differential equations becomes invariant and are exactly reduced to ordinary differential equations which can be integrated in a closed form. These solutions are called "self similar solutions" [3, 5, 9, 15].

The heat equation with a power-law nonlinearity called also "porous medium equation" is one of class of equations which admits these properties of similarity. This equation is written in the form:

$$\left\{ \frac{\partial u}{\partial t} = \left(u^{n-1} u_x\right)_x \right. , \tag{1.1}$$

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where $x \in \mathbb{R},\ t > 0$, and $n > 1$, is a fixed real number.

This equation occurs in nonlinear problems of heat and mass transfer and flows in porous media [2, 3, 16]. For $n = 1$, equation (1.1) is the classical equation of heat conduction.

In this work, we seek a solution to equation (1.1), called "traveling profiles solutions", which are written in the form:

$$u\,(x,t) = c\,(t)\,\psi\left[\frac{x-b(t)}{a\,(t)}\right],\ a,\ c,\ b \in \mathbb{R}^{+}, \tag{1.2}$$

where $a(t)$, $c(t), b(t)$ and the profile ψ are to be determined.

For $c\,(t) = t^{\alpha}$, $a\,(t) = t^{-\beta}$, and $b(t) = 0$, we find the "classical" self similarity solutions :

$$u(x,t) = t^{\alpha} f(\eta), \qquad \eta = xt^{-\beta}, \tag{1.3}$$

where α and β (constants) and the profile f are to be determined.

2 Traveling profile solutions to heat equation with a power-law nonlinearity

We want to find a general form of self similar solutions called "traveling profiles solutions" of (1.1), with the form (1.2), written as follow:

$$u\,(x,t) = c\,(t)\,\psi(\xi),\ \text{with}\ \xi = \frac{x-b(t)}{a\,(t)},$$

if we replace this form of solutions in equation (1.1) we find

$$\alpha\frac{\dot{c}}{c}\psi - \beta\frac{\dot{a}}{a}\xi\psi'_{\xi} - \gamma\frac{\dot{b}}{b}\psi'_{\xi} = \frac{c^{n}}{a^{2}}(\psi^{n-1}\psi'_{\xi})'_{\xi} \tag{2.1}$$

this equation depends of many unknown parameters and our aim is to determine the coefficients a, c, b and the profile ψ, we use new approach called the "traveling profiles method" (TPM) [6, 7].

The principle of TPM is to determine firstly the coefficients $a\,(t)\,, c\,(t)$ and $b(t)$ by resolving the following minimizition problem:

$$\min_{\dot{c},\dot{a},\dot{b}} \int_{-\infty}^{+\infty} \left|\frac{\partial u}{\partial t} - \left(u^{n-1}u_{x}\right)_{x}\right|^{2} dx, \tag{2.2}$$

therefore, we obtain three orthogonal equations which are read

$$\begin{cases} \langle \frac{\partial u}{\partial t} - \left(u^{n-1}u_{x}\right)_{x}, \psi\rangle = 0 \\ \langle \frac{\partial u}{\partial t} - \left(u^{n-1}u_{x}\right)_{x}, \xi\psi'_{\xi}\rangle = 0 \\ \langle \frac{\partial u}{\partial t} - \left(u^{n-1}u_{x}\right)_{x}, \psi'_{\xi}\rangle = 0 \end{cases} \tag{2.3}$$

where $\langle .,. \rangle$ is the inner product in L^2 space.
Then the PDE (1.1) is transformed into a set of three coupled ODE's :

$$\begin{cases} \frac{\dot{c}}{c}\langle \psi, \psi\rangle - \frac{\dot{a}}{a}\langle \xi\psi'_\xi, \psi\rangle - \frac{\dot{b}}{a}\langle \psi', \psi\rangle = \frac{c^n}{a^2}\langle (\psi^{n-1}\psi'_\xi)'_\xi, \psi\rangle \\ \frac{\dot{c}}{c}\langle \xi\psi'_\xi, \psi\rangle - \frac{\dot{a}}{a}\langle \xi\psi'_\xi, \xi\psi'_\xi\rangle - \frac{\dot{b}}{a}\langle \xi\psi'_\xi, \psi'_\xi\rangle = \frac{c^n}{a^2}\langle (\psi^{n-1}\psi'_\xi)'_\xi, \xi\psi'_\xi\rangle \\ \frac{\dot{c}}{c}\langle \psi'_\xi, \psi\rangle - \frac{\dot{a}}{a}\langle \psi'_\xi, \xi\psi'_\xi\rangle - \frac{\dot{b}}{a}\langle \psi'_\xi, \psi'_\xi\rangle = \frac{c^n}{a^2}\langle (\psi^{n-1}\psi'_\xi)'_\xi, \xi\psi'_\xi\rangle \end{cases} \tag{2.4}$$

to finding exact solution in the form (2.1) we based on the following main theorem

Theorem 2.1. The function

$$u(x,t) = c(t)\,\psi(\xi), \text{ with } \xi = \frac{x - b(t)}{a(t)}, \quad x \in \mathbb{R},\ t > 0, \text{and } a(t),\ c(t),\ b(t) > 0.$$

is an exact solution of problem (1.1), if the "based profile" ψ is a solution of following differential equation

$$(\psi^{n-1}\psi'_\xi)'_\xi = \alpha\psi + \beta\xi\psi'_\xi + \gamma\psi'_\xi, \qquad \text{where } \alpha, \beta, \gamma \in \mathbb{R}, \tag{2.5}$$

in this case, the coefficients $c(t)$, $a(t)$ and $b(t)$ are given by :

$$\begin{cases} a(t) = (1 - A\beta t)^{\frac{1}{A}} \\ c(t) = (1 - A\beta t)^{\frac{-\alpha}{\beta A}} \\ b(t) = \frac{\gamma}{\beta}(1 - A\beta t)^{\frac{1}{A}} - \frac{\gamma}{\beta} \end{cases}, \ 0 < t < T, \tag{2.6}$$

for

$$2\beta + (n-1)\alpha > 0 \tag{2.7}$$

with

$$A = 2 + \frac{\alpha}{\beta}(n-1), \tag{2.8}$$

The finite time T is equal to

$$T = \frac{1}{2\beta + \alpha(n-1)} > 0. \tag{2.9}$$

For

$$2\beta + (n-1)\alpha = 0 \tag{2.10}$$

we have

$$\begin{cases} a(t) = \exp(-\beta t) \\ c(t) = \exp(\alpha t) \\ b(t) = \frac{\gamma}{\beta}\exp(-\beta t) - \frac{\gamma}{\beta} \end{cases}, \ 0 < t < \infty, \tag{2.11}$$

Proof. The proof of this theorem is based on the result shown in [7] for an evolution differential operator, in follow we recall the result but in special case where the differential operator is in the form (1.1).

Let

$$V_t = \{\psi, \psi'_\xi, \xi\psi'_\xi, \},$$

the subspace of L^2 generated by associated functions to ψ at the moment t.
From relations (2.3), it is deduced that $\frac{\partial u}{\partial t} - (\psi^{n-1}\psi'_\xi)'_\xi$ is orthogonal to subspace V_t.
In particular we have $\frac{\partial u}{\partial t} \in V_t$, then $\langle \frac{\partial u}{\partial t} - (\psi^{n-1}\psi'_\xi)'_\xi, \frac{\partial u}{\partial t}\rangle = 0$, thus if also $(\psi^{n-1}\psi'_\xi)'_\xi$ belongs to V_t then the method provides us a weakly exact solution, which is written under the form

$$u(x,t) = c(t)\,\psi\left[\frac{x - b(t)}{a(t)}\right].$$

According to the principle estimates of this method, if $\left(u^{n-1}u_x\right)_x = \frac{c^n}{a^2}(\psi^{n-1}\psi'_\xi)'_\xi$ belongs to the subspace V_t, then the function $u(x,t) = c(t)\,\psi\left[\frac{x-b(t)}{a(t)}\right]$ is an exact solution of equation (1.1), in this case the term $(\psi^{n-1}\psi'_\xi)'_\xi$ can be expressed as a linear combination of functions $\psi, \xi\psi'_\xi$, and ψ'_ξ, thus

$$(\psi^{n-1}\psi'_\xi)'_\xi = \alpha\psi + \beta\xi\psi'_\xi + \gamma\psi'_\xi, \text{ for } \alpha, \beta, \gamma \in \mathbb{R}.$$

The coefficients $c(t), a(t), b(t)$ are obtained as follow:
When one replaces $(\psi^{n-1}\psi'_\xi)'_\xi$ by the combination $\alpha\psi + \beta\xi\psi'_\xi + \gamma\psi'_\xi$ in (2.4), we obtain the system:

$$MX = \frac{c^{n-1}}{a^2}MF \tag{2.11}$$

with

$$M = \begin{pmatrix} \langle\psi,\psi\rangle & \langle\xi\psi'_\xi,\psi\rangle & \langle\psi'_\xi,\psi\rangle \\ \langle\xi\psi'_\xi,\psi\rangle & \langle\xi\psi'_\xi,\xi\psi'_\xi\rangle & \langle\xi\psi'_\xi,\psi'_\xi\rangle \\ \langle\psi'_\xi,\psi\rangle & \langle\xi\psi'_\xi,\psi'_\xi\rangle & \langle\psi'_\xi,\psi'_\xi\rangle \end{pmatrix}, X = \begin{pmatrix} \frac{\dot{c}}{c} \\ -\frac{\dot{a}}{a} \\ -\frac{\dot{b}}{a} \end{pmatrix} \text{ and } F = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

where $\langle .,. \rangle$ is the inner product in L^2.
The matrix M in system (2.11) is symmetric and invertible, then (2.11) can be written under the form:

$$\begin{cases} \dot{c} &= \frac{c^n}{a^2}\alpha \\ \dot{a} &= -\frac{c^{n-1}}{a}\beta \\ \dot{b} &= -\frac{c^{n-1}}{a}\gamma \end{cases} \tag{2.12}$$

At the boundaries, we impose the lateral boundary conditions $a(0) = 1$, $c(0) = 1$, $b(0) = 0$.
We can see that from (2.12) we have

$$\begin{cases} c(t) = K_0.a(t)^{\frac{-\alpha}{\beta}} \\ b(t) = \frac{\gamma}{\beta}a(t) + K_2 \end{cases}, \tag{2.13}$$

if we replace (2.13) in (2.12) then we deduct (2.6)

$$\begin{cases} a(t) = (1 - A\beta t)^{\frac{1}{A}}, \\ c(t) = (1 - A\beta t)^{\frac{-\alpha}{\beta A}} \\ b(t) = \frac{\gamma}{\beta}(1 - A\beta t)^{\frac{1}{A}} - \frac{\gamma}{\beta} \end{cases}, \ 0 < t < T,$$

for $(n-1)\alpha + 2\beta > 0$ with $A = 2 + \frac{\alpha}{\beta}(n-1)$ and $T = \frac{1}{2\beta + \alpha(n-1)}$.
For $(n-1)\alpha + 2\beta = 0$, we have (2.11)

$$\begin{cases} a(t) = \exp(-\beta t) \\ c(t) = \exp(\alpha t) \\ b(t) = \frac{\gamma}{\beta}\exp(-\beta t) - \frac{\gamma}{\beta} \end{cases}, \ 0 < t < \infty,$$

We notice that from this theorem we have two time behaviors of coefficients $c(t)$, $a(t)$ and $b(t)$, there behaviors depends of parameters of similarity α, β and γ. □

3 New explicit exact solution

For $\alpha = \beta > 0$, we can find explicitly a new exact solution to equation (1.1). In fact, in this case, equation (2.5) becomes

$$(\psi^{n-1}\psi'_{\xi})'_{\xi} = [(\alpha\xi + \gamma)\psi]'_{\xi}, \qquad \text{where } \gamma \in \mathbb{R}, \tag{3.1}$$

after integration, we obtain

$$\psi^{n-1}\psi'_{\xi} = (\alpha\xi + \gamma)\psi + k, \quad \text{with} \ \ k \text{ constant.}$$

if we put for example $\psi(0) = 0$, this implies $k = 0$, then we obtain

$$\psi^{n-2}\psi'_{\xi} = (\alpha\xi + \gamma),$$

finally the solution of (3.1) is written under the form

$$\psi(\xi) = \left[(n-2)^2\left(\frac{\alpha}{2}\xi^2 + \gamma\xi\right)_+\right]^{\frac{1}{n-1}}$$

the coefficients $c(t)$, $a(t)$ and $b(t)$ are given by :

$$\begin{cases} a(t) = (1 - (n+1)\alpha t)^{\frac{1}{n+1}}, \\ c(t) = (1 - (n+1)\alpha t)^{\frac{-1}{n+1}} \\ b(t) = \frac{\gamma}{\alpha}(1 - (n+1)\alpha t)^{\frac{1}{n+1}} - \frac{\gamma}{\alpha} \end{cases}, \ 0 < t < T = \frac{1}{(n+1)\alpha}. \tag{3.2}$$

Then the solution of (1.1) is written under the form

$$u(x,t) = c(t)\left[(n-2)^2\left(\frac{\alpha}{2}\left(\frac{x - b(t)}{a(t)}\right)^2 + \gamma\left(\frac{x - b(t)}{a(t)}\right)\right)_+\right]^{\frac{1}{n-1}}.$$

where $a(t)$, $b(t)$ and $a(t)$ are given by (3.2).
This particular explicit solution is new in literature.

4 Conclusion

In this work we have find a new solution of heat equation with a power-law nonlinearity in a general form of self similar solutions. The method used is called "traveling profiles method" [7]. This method enables us to obtain exact solutions to large classes of nonlinear PDEs. It gives us the possibility to obtain a varied choice of classes of exact solutions.

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