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Title

Some Properties Of Function Spaces Of Besov-Type And Applications

Defended before the jury composed of :

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DEDICATE

To my `mother` **and** `father`

To my `wife` **and** `child`

To my `brothers` **and** `sisters`

To all my `friends`.....

And to all my `teachers` **....**

CONTENTS

INTRODUCTION

Function spaces have been a central topic in both theoretical and applied mathematics and of great interest to researchers over the last century. Their importance have been growing up because of their role in analysis and applications to fluid dynamics, image processing, modern theory of partial differential equations (PDE), the calculus of variation and many other fields. Since the thirties more sophisticated function spaces have been introduced and developed, in the first place the Hölder spaces , Lebesgue spaces L^p, Orlics spaces and Sobolev spaces. Later on many new spaces were created and investigated, among these spaces were Besov spaces $B^{\alpha}_{p,q}$ and Triebel-Lizorkin spaces $F^{\alpha}_{p,q}$ in the 1960's and 1970's which cover many well-known classical spaces such as (fractional)Sobolev spaces, Holder-Zygmund spaces and (inhomogeneous)Hardy spaces.

Function spaces have proven to be very efficient in the study of evolution equations such as transport and heat equations, wave equations, Navier–Stokes, Euler equations, and the Schrödinger equation, for a detailed presentation of harmonic analysis tools that are used to solve partial differential equations and references on the this topic see [8].

Simultaneously to this development, the need to generalize classical spaces such as Lebesgue and Sobolev spaces to the case with either variable integrability or variable smoothness function spaces have been arousing, especially when problems required more generilized spaces appeared. the first steps of the generalization can be traced back to 1931 by W. Orlicz [10], but the modern age began with the paper [30] of Kováčik and Rákosník in 1991. where several basic properties of variable Lebesgue spaces $L^{p(\cdot)}$ and generalized Sobolev spaces $W^{k,p(\cdot)}$ were investigated and proved.

Since the 1990s the field of variable exponent function spaces has grown explosively because of their use in a variety of applications and generalizing Besov spaces $B^{\alpha}_{p,q}$ and Triebel-Lizorkin spaces $F^{\alpha}_{p,q}$ to the variable indices $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ has taken great part of researchers work. The interest in these generalized spaces comes not only from theoretical reasons but also from their applications to several classical problems in analysis such as applications to fluid dynamics [4, 5, 41], PDE and the calculus of variation [24, 43] image processing [14] . For further considerations of PDEs, we refer to [19] and references therein.

In 2009, L. Diening, P. Hästö and S. Roudenko defined and studied a generalized Triebel –Lizorkin spaces with variable smoothness, $\alpha(x)$, and variable indices of integrability, $p(x)$ and $q(x)$ where several basic properties of these spaces were proved [20], in 2011, Besov spaces of variable smoothness and integrability $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ initially appeared in the paper of A. Almeida and P. Hästö [3], where several basic properties were shown, such as the Fourier analytical characterization. Later these space was characterized by local means and established the atomic characterization in [21]. After that, Kempka and Vybíral [31] characterized these spaces by ball means of differences and also by local means. The duality of these function spaces is given in [28] and [39].

The aim from this thesis is achieving new results, properties and applications about the structure of Besov spaces of variable smoothness and integrability. Our thesis consists of four chapters. In the first chapter, we provide a brief introduction to variable Lebesgue spaces, Besov and Triebel–Lizorkin spaces with variable smoothness integrability, where we give their definitions and basic properties and the material that we need in other chapters, we also recount some of their history, and very briefly sketch some of the motivations for their study.

In the second chapter, we introduce new equivalent quasi-norms of Besov spaces of variable smoothness integrability and characterize these spaces by continuous local mean.

In the third chapter, we study an important commutator on Triebel-Lizorkin spaces of variable smoothness integrability and prove certain estimates under no vanishing assumptions on the divergence of vector fields . Such commutator estimates are motivated by the study of well-posedness results for some models in incompressible fluid mechanics.

In the forth chapter, we introduce new equivalent quasi-norms of Besov-type spaces of variable smoothness and integrability

The second chapter was accepted to be published by the Ukrainian mathematical journal but not published yet. The third chapter is a published paper, see [9], and the forth chapter is a paper in preparation with the advisor.

NOTATIONS

Here we give general notations, throughout the thesis other notations will be introduced whenever needed.

- $\mathbb{R}^n$ is the n-dimensional real Euclidean space,
- $\mathbb{N}$ is the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of all non-negative integers.
- The notation $f \lesssim g$ means that $f \leqslant c\, g$ for some independent positive constant c of f and g (and non-negative functions f and g),
- and $f \approx g$ means that $f \lesssim g \lesssim f$. For $x \in \mathbb{R}$,
- $\lfloor x \rfloor$ stands for the largest integer smaller than or equal to x.
- If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the Lebesgue measure of E and χ_E denotes its characteristic function.
- By c we denote generic positive constants, which may have different values at different occurrences. Although the exact values of the constants are usually irrelevant for our purposes, sometimes we emphasize their dependence on certain parameters (e.g., $c(p)$ means that c depends on p, etc.).
- For a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| := \sum_{i=1}^n \alpha_i$ and $D^\alpha := D^{\alpha_1} \cdots D^{\alpha_n}$.
- The Euclidean scalar product of $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ is given by $x \cdot y = x_1 y_1 + ... + x_n y_n$.
- For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in $\mathbb{R}^n$ with center x and radius r.

- By supp f we denote the support of the function f , i.e., the closure of its non-zero set.
- For $a, b \in \mathbb{R}$, $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$.
- $\mathbb{R}^{n+1}_{+} := \mathbb{R}^n \times [0, \infty)$.
- For any $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$ denote by Q_{jk} the dyadic cube $2^{-j}([0,1)^n + k)$ and $\ell(Q_{jk})$ its side length.
- $\mathscr{Q} := \{Q_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ and $j_Q := -\log_2 \ell(Q)$ for any $Q \in \mathscr{Q}$.
- For all dyadic cubes $Q \in \mathscr{Q}$ and $r > 0$, rQ is the cube concentric with Q having the side length $r\ell(Q)$.
- The symbol $\mathscr{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions on $\mathbb{R}^n$.
- We define the Fourier transform of a function $f \in \mathscr{S}(\mathbb{R}^n)$ by

  $$\mathscr{F}(f)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx, \quad \xi \in \mathbb{R}^n.$$

  and its inverse by $\mathscr{F}^{-1}(f)(x) := \mathscr{F}(f)(-x)$.
- We denote by $\mathscr{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$.

CHAPTER 1

BASIC SPACES OF FUNCTIONS

1.1 Modular spaces

The theory of semi-modular spaces sometimes called Nakano spaces, is a generalization of the theory of normed vector spaces was founded by Hidegoro Nakano, in 1950s who developed an extensive theory of such spaces in [35, 36, 37].

In this subsection we give some preliminaries, where we fix some notations and recall some basic facts on semi-modular spaces.

Definition 1.1. *Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A function $\varrho : X \mapsto [0;\infty]$ is called a semi-modular on X if the following four properties hold:*

(1) $\varrho(0)=0$

(2) $\varrho(\lambda x)=\varrho(x)$ *for all* $x \in X$ *and* $|\lambda|=1$

(3) $\varrho(\lambda x)=0$ *for every* $\lambda>0$ *implies* $x=0$

(4) $\lambda \longmapsto \varrho(\lambda x)$ *is left-continuous on* $[0,\infty[$ *for every* $x \in X$.

A semi-modular is called modular if

(5) $\varrho(x)=0$ *implies* $x=0$,

and called continuous if:

(6) $\lambda \longmapsto \varrho(\lambda x)$ *is continuous on* $[0,\infty[$ *for every* $x \in X$.

A semi-modular is called quasi-convex if there exist $c \geqslant 1$ such that

$$\varrho(\alpha x+(1-\alpha)y) \leqslant c[\alpha\varrho(x)+(1-\alpha)\varrho(y)] \tag{1.1}$$

for all $x, y \in X$ and $\alpha \in [0;1]$, if $c=1$ it's called convex.

Once we have a semi-modular on X we obtain a normed space in the following way

Definition 1.2. *If ϱ is a semi-modular on X, then*

$$X_\varrho := \{x \in X : \exists \lambda > 0, \varrho(\lambda x) < \infty\}$$

is called a semi-modular space.

The norm on X_ϱ is given by

Theorem 1.3. *If ϱ is a quasi-convex semi-modular on X, then X_ϱ is a quasi-normed vector space with the Luxemburg quasi-norm given by*

$$\|x\|_{X_\varrho} := \inf\left\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leqslant 1\right\}$$

An important and useful result of left-continuity is that: $\|x\|_{X_\varrho} \leqslant 1$ if and only if $\varrho(x) \leqslant 1$.

Remark 1.4.

(i) An important and useful result of left-continuity is that: $\|x\|_{X_\varrho} \leqslant 1$ if and only if $\varrho(x) \leqslant 1$.

(ii) If ϱ is a quasi-convex semi-modular on a vector space X and $x \in X_\varrho$ is such that $\varrho(x) \leqslant M$ where $M \geqslant 1$, then

$$\varrho(\frac{1}{cM}x) = \varrho(\frac{1}{cM}x + (1 - \frac{1}{cM})0) \leqslant c[\frac{1}{cM}\varrho(x) + (1 - \frac{1}{cM})\varrho(0)] \leqslant 1,$$

where c is the constant from (1.1), which implies that $\|x\|_{X_\varrho} \leqslant cM$, so, to prove that $\|x\|_{X_\varrho} \lesssim 1$ it suffices to prove that $\varrho(x) \leqslant M$ for some $M \geqslant 1$.

(iii) If $\|x\|_{X_\varrho} \leqslant 1$ then $\varrho(x) \leqslant \|x\|_{X_\varrho}$.

Example 1.5.

- *If $1 \leqslant p < \infty$, then*

  $$\varrho(f) := \int_{\mathbb{R}^n} |f(x)|^p \, dx$$

  defines a continuous modular on The set of all measurable functions over $\mathbb{R}^n$, $X_\varrho = L^p(\mathbb{R}^n)$ and $\|f\|_{X_\varrho} = \|f\|_{L^p(\mathbb{R}^n)}$

- *Given a function $p(\cdot) : \mathbb{N} \to]0; +\infty[$*

  $$\varrho(\{x_i\}) := \sum_{i=0}^{\infty} |x_i|^{p_i}$$

  The sequence spaces ℓ^p the generalized discrete version of the variable Lebesgue spaces.

- *Let Ω be a measurable subset of $\mathbb{R}^n$, $\varphi : \Omega \times [0, \infty) \to [0, \infty]$ a function such that* *(i)* *for every* $x \in \mathbb{R}^n$, $\varphi(x,\cdot)$ *is a convex, left-continuous function,* $\varphi(0)=0$, $\lim_{t\to 0^+} \varphi(x,t)=\varphi(x,0)$ *and* $\lim_{t\to\infty} \varphi(x,t)=\infty$ *for every* $x \in \mathbb{R}^n$.

(ii) $\varphi(\cdot,t)$ *is measurable for every* $t \geqslant 0$.

then ϱ_φ *defined on the set of all measurable functions on* Ω *by*

$$\varrho_\varphi(f)=\int_\Omega \varphi(x,|f(x)|)\,dx$$

is a semi-modular, if $\varphi(x,t)>0$ *for all* $x \in \mathbb{R}^n$ *and* $t \in (0,\infty)$ *then* ϱ_φ *is a modular. The space induced by the semi-modular* ϱ_φ *via Definition 1.2 is called Musielak-Orlicz space or generalized Orlicz space, see [19, Chapter 2], [34] for more details and references therein, for an independent theory of Orlicz spaces of the theory of modular spaces see [29].*

1.2 Variable Exponent Lebesgue Spaces

In this subsection we give the definition of Lebesgue spaces with variable exponents $L^{p(\cdot)}$ and their basic properties and results. Lebesgue spaces with variable exponents are special case of semi-modular spaces, so, the techniques of semi-modular spaces are used to establish many of theirs properties.

The origin of the variable Lebesgue spaces goes back to 1931 were they were studied by Orlicz for the first time. Two decades later H.Nakano developed the theory of modular spaces and introduced the variable Lebesgue spaces as specific examples of modular spaces where he developed further properties and results. Variable exponent Lebesgue spaces were independently developed by Russian researchers. They considered and solved The problem of minimizing the integral

$$\int_0^1 |f(x)-\phi(x)|^{p(x)}\,dx$$

where f is continuous and ϕ is a polynomial of fixed degree. In 1979, Sharapudinov developed variable Lebesgue spaces theory on intervals [44], introducing the Luxemburg norm the local log-Hölder continuity condition throughout many papers. In 1986 Zhikov began applying the variable Lebesgue spaces to problems in the calculus of variations , see [12, section 1.2] and references therein. The interest in these spaces increased since the 1990s because of their use in a variety of applications. Foremost among these is the mathematical modeling of electrorheological fluids.

In 1991 Kováčik and Rákosník published a paper that is considered the beginning of the modern period in the study of variable Lebesgue spaces, they established many of the basic properties of variable Lebesgue spaces in this work.

The variable exponents that we consider are always measurable functions p on $\mathbb{R}^n$ with range in $(0,\infty]$. We denote by $\mathscr{P}_0(\mathbb{R}^n)$ the set of such functions bounded away from the origin (i.e., $p^- > 0$). The subset of variable exponents with range in $[1,\infty]$ is denoted by $\mathscr{P}(\mathbb{R}^n)$. We use the standard notation:

$$p^- := \operatorname*{ess-inf}_{x\in\mathbb{R}^n} p(x) \quad \text{and} \quad p^+ := \operatorname*{ess-sup}_{x\in\mathbb{R}^n} p(x).$$

We put

$$\omega_p(t) = \begin{cases} t^p & \text{if} \quad p \in (0,\infty) \text{ and } t \in (0,\infty), \\ 0 & \text{if} \quad p = \infty \text{ and } t \in (0,1], \\ \infty & \text{if} \quad p = \infty \text{ and } t \in (1,\infty). \end{cases}$$

The variable exponent semi-modular is defined by

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \omega_{p(x)}(|f(x)|)\, dx,$$

if $p \in \mathscr{P}(\mathbb{R}^n)$ then $\varrho_{p(\cdot)}$ is a continuous convex modular. The variable exponent Lebesgue space $L^{p(\cdot)}$ consists of measurable functions f on $\mathbb{R}^n$ such that $\varrho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$. We define the Luxemburg (quasi)-norm on this space by the formula

$$\left\|f\right\|_{p(\cdot)} := \inf\left\{\lambda > 0 : \varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leqslant 1\right\}.$$

If $p \in \mathscr{P}(\mathbb{R}^n)$ then $L^{p(\cdot)}$ is a normed vector space. As is known, the following inequalities hold , see Remark 1.4/*(i)*

$$\left\|f\right\|_{p(\cdot)} \leqslant 1 \Leftrightarrow \varrho_{p(\cdot)}(f) \leqslant 1. \tag{1.2}$$

Now we introduce the most important condition on the exponent to study variable exponent spaces, the log-Hölder continuity condition.

Definition 1.6. *We say that a real valued-function* g *on* $\mathbb{R}^n$ *is locally log-Hölder continuous on* $\mathbb{R}^n$*, abbreviated* $g \in C^{\log}_{loc}(\mathbb{R}^n)$*, if there exists a constant* $c_{\log}(g) > 0$ *such that*

$$\left|g(x) - g(y)\right| \leqslant \frac{c_{\log}(g)}{\log(e + 1/\left|x - y\right|)} \tag{1.3}$$

for all $x, y \in \mathbb{R}^n$*. We say that* g *satisfies the log-Hölder decay condition, if there exist two* *constants $g_\infty \in \mathbb{R}$ and $c_{\log} > 0$ such that*

$$\left|g(x) - g_\infty\right| \leqslant \frac{c_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that g is globally log-Hölder continuous on $\mathbb{R}^n$, abbreviated $g \in C^{\log}(\mathbb{R}^n)$, if it is locally log-Hölder continuous on $\mathbb{R}^n$ and satisfies the log-Hölder decay condition. The constants $c_{\log}(g)$ and $c_{\log}$ are called the locally log-Hölder constant and the log-Hölder decay constant, respectively.

The local log-Hölder condition was first used in the variable exponent context by Zhikov [56]. We note that any function $g \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ always belongs to L^∞.

Definition 1.7. *We define the following class of variable exponents:*

$$\mathscr{P}_0^{\log}(\mathbb{R}^n) := \left\{ p \in \mathscr{P}_0(\mathbb{R}^n) : \frac{1}{p} \in C^{\log}(\mathbb{R}^n) \right\},$$

which is introduced in [18, Section 2]. The class $\mathscr{P}^{\log}(\mathbb{R}^n)$ is defined analogously.

We define

$$\frac{1}{p_\infty} := \lim_{|x|\to\infty} \frac{1}{p(x)},$$

and we use the convention $\frac{1}{\infty} = 0$. Note that although $\frac{1}{p}$ is bounded, the variable exponent p itself can be unbounded.

Let $\varphi \in L^1$, Define the radial majorant of φ by

$$\Psi(x) := \sup_{|y| \geqslant |x|} \left|\varphi\left(y\right)\right|, \quad x \in \mathbb{R}^n.$$

The function Φ is radial and decreasing as $|x|$ increases, we note that even if $\varphi \in L^1$, the function Φ need not be integrable. We suppose that $\Phi \in L^1$. Then it was proved in [19, Lemma 4.6.3] that if $p \in \mathscr{P}^{\log}(\mathbb{R}^n)$, then

$$\|\varphi_\varepsilon * f\|_{p(\cdot)} \leqslant c\|\Psi\|_1 \|f\|_{p(\cdot)}$$

for all $f \in L^{p(\cdot)}$, where

$$\varphi_\varepsilon := \frac{1}{\varepsilon^n} \varphi\left(\frac{\cdot}{\varepsilon}\right), \quad \varepsilon > 0.$$

We put

$$\eta_{t,m}(x) := t^{-n}(1 + t^{-1}|x|)^{-m}$$

for any $x \in \mathbb{R}^n$, $t > 0$ and $m > 0$. Note that $\eta_{t,m} \in L^1$ when $m > n$ and that $\left\|\eta_{t,m}\right\|_1 = c(m)$ is independent of t. If $t = 2^{-\nu}$, $\nu \in \mathbb{N}_0$ then we put $\eta_{\nu,m} := \eta_{2^{-\nu},m}$.

The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined on L^1_{loc} by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|f(y)\right| dy.$$

It was shown in [19], Theorem 4.3.8 that $\mathcal{M} : L^{p(\cdot)} \to L^{p(\cdot)}$ is bounded if $p \in \mathcal{P}^{\log}$ and $p^- > 1$. Note that if $p \in \mathcal{P}$ and $p^+ < \infty$, then $p \in \mathcal{P}^{\log}$ if and only if $p \in C^{\log}_{\mathrm{loc}}$.

We also refer to the papers [13], [12] and [17], where various results on maximal function in variable Lebesgue spaces were obtained.

We end with the generalization of Hölder's inequality on variable Lebesgue space. As usual, if $p \in \mathcal{P}$, then p' denotes the conjugate exponent of p given by $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$ with the convention that $1/\infty = 0$.

Theorem 1.8 (Hölder Inequality)**.** *If $p \in \mathcal{P}(\mathbb{R}^n)$ then for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \leqslant c\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$

We refer to the recent monographs [12, 19] for further properties, historical remarks and references on variable exponent spaces.

1.3 The mixed Lebesgue-sequence spaces

1.3.1 The space $\ell^{q(\cdot)}(L^{p(\cdot)})$

Definition 1.9. *Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the semi-modular*

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_\nu)_{\nu\in\mathbb{N}_0}) := \sum_{\nu\in\mathbb{N}_0} \inf\left\{\lambda_\nu > 0 : \varrho_{p(\cdot)}\left(\frac{f_\nu}{\lambda_\nu^{1/q(\cdot)}}\right) \leqslant 1\right\},$$

we use the convention $\lambda^{\frac{1}{\infty}} = 1$. The quasi-norm is defined from this as usual:

$$\left\|(f_\nu)_{\nu\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf\left\{\mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\frac{1}{\mu}(f_\nu)_\nu\right) \leqslant 1\right\}. \tag{1.4}$$

If $q^+ < \infty$, then

$$\inf\left\{\lambda > 0 : \varrho_{p(\cdot)}\left(\frac{f}{\lambda^{1/q(\cdot)}}\right) \leqslant 1\right\} = \left\|\left|f\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}},$$

since the right-hand side expression is much simpler, we often replace (4.3) by a simpler expression even when $q^+ = \infty$, this expression is given as follows:

$$\left\|(f_\nu)_{\nu\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \sum_{\nu=0}^{\infty} \left\||f_\nu|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

As special cases if the exponent q is a constant from $(0,\infty)$ then the semi-modular $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_\nu)_{\nu\in\mathbb{N}_0}) = \sum_{\nu=0}^{\infty} \left\|f_\nu\right\|_{p(\cdot)}^{q}$ and $\left\|(f_\nu)_{\nu\in\mathbb{N}_0}\right\|_{\ell^{q}(L^{p(\cdot)})} = \left(\sum_{\nu=0}^{\infty} \left\|f_\nu\right\|_{p(\cdot)}^{q}\right)^{\frac{1}{q}}$, and if $q = \infty$ then $\left\|(f_\nu)_{\nu\in\mathbb{N}_0}\right\|_{\ell^{\infty}(L^{p(\cdot)})} = \sup_{\nu\in\mathbb{N}_0} \left\|f_\nu\right\|_{p(\cdot)}$.

Now, we give some important properties of the semi-modular of Definition 1.9 and the quasi-norm (4.3). The first theorem ensures that indeed Definition 1.9 gives a semi-modular, see [3].

Theorem 1.10. *If $p, q \in \mathscr{P}_0$, then $\ell^{q(\cdot)}(L^{p(\cdot)})$ is a semi-modular. Additionally,*

(i) it is a modular if $p^+ < \infty$; and

(ii) it is continuous if $p^+, q^+ < \infty$.

Next, we give sufficient conditions on p and q that make $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ a quasi-norm or a norm, we begin with results from [3].

Theorem 1.11. *Let $p, q \in \mathscr{P}_0$, then $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a quasi-norm on $\ell^{q(\cdot)}(L^{p(\cdot)})$.*

Theorem 1.12. *Let $p, q \in \mathscr{P}$. If either $\frac{1}{p} + \frac{1}{q} \leqslant 1$ point-wise, or q is a constant, then $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a norm.*

In [32, Theorem 1] another sufficient condition was given

Theorem 1.13. *Let $p, q \in \mathscr{P}$. If $q(x) \leqslant p(x) \leqslant \infty$ for almost every $x \in \mathbb{R}^n$, then $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ defines a norm.*

Remark 1.14. *Let $p, q \in \mathscr{P}_0(\mathbb{R}^n)$,*

(i) for any $r > 0$ and any sequence $(f_\nu)_{\nu\in\mathbb{N}_0}$ of measurable functions on $\mathbb{R}^n$, $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_\nu)_{\nu\in\mathbb{N}_0}) = \varrho_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}((|f_\nu|^r)_{\nu\in\mathbb{N}_0})$ and $\|(f_\nu)_{\nu\in\mathbb{N}_0}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \|(|f_\nu|^r)_{\nu\in\mathbb{N}_0}\|_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}^{1/r}$

(ii) $\||f|^{q(\cdot)}\|_{p(\cdot)/q(\cdot)} \leqslant 1$ if and only if $\|f\|_{p(\cdot)} \leqslant 1$.

1.3.2 The space $L^{p(\cdot)}(\ell^{q(\cdot)})$

Let $p, q \in \mathscr{P}_0(\mathbb{R}^n)$ and q is with real values. For a family of functions $f_\nu : \mathbb{R}^n \to \mathbb{R}$, $\nu \in \mathbb{N}_0$, we define for any $x \in \mathbb{R}^n$

$$\left\|(f_\nu)_{\nu\in\mathbb{N}_0}\right\|_{\ell^{q(x)}} = \left(\sum_{\nu\in\mathbb{N}_0} |f_\nu(x)|^{q(x)}\right)^{\frac{1}{q(x)}},$$

The mapping $x \to \left\|(f_\nu)_{\nu\in\mathbb{N}_0}\right\|_{\ell^{q(x)}}$ is a function of x and can be measured in $L^{p(\cdot)}$. The space $L^{p(\cdot)}(\ell^{q(\cdot)})$ is defined to be the set of all sequences $\left(f_\nu\right)_{\nu\in\mathbb{N}_0}$ of functions such that

$$\left\|(f_\nu)_{\nu\in\mathbb{N}_0}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} = \left\|\left\|(f_\nu)_{\nu\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}}\right\|_{p(\cdot)} < \infty.$$

It is easy to show that $L^{p(\cdot)}(\ell^{q(\cdot)})$ is always a quasi-normed space and if $p,q \in \mathscr{P}(\mathbb{R}^n)$ then it's a normed space.

1.3.3 The maximal operator

The maximal operator has proved to be very useful in analysis, but it is not well suited to the mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ and $L^{p(\cdot)}(\ell^{q(\cdot)})$. It was found in [20, Section 5] that the maximal operator is not bounded on $L^{p(\cdot)}(\ell^{q(\cdot)})$ when q is non-constant. In [3] A. Almeida and P. Hästö showed by example [3, Example 4.1] that the maximal operator is in general not bounded on $\ell^{q(\cdot)}(L^{p(\cdot)})$. We use instead the η-functions when we work in variable exponents scales $\ell^{q(\cdot)}(L^{p(\cdot)})$ and $L^{p(\cdot)}(\ell^{q(\cdot)})$ to replace the role of the maximal operator in $\ell^q(L^p)$ and $L^p(\ell^q)$ with fixed p and q. The following statement are from [3, Lemma 4.7] and [20, Section 5] respectively.

Lemma 1.15. *Let $p \in \mathscr{P}^{\log}(\mathbb{R}^n)$ and $q \in \mathscr{P}_0(\mathbb{R}^n)$ with $\frac{1}{q} \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$. For $m > n + c_{\log}(1/q)$, there exists $c > 0$ such that*

$$\left\|\left(\eta_{\nu,m} * f_\nu\right)_\nu\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c \left\|\left(f_\nu\right)_\nu\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Lemma 1.16. *Let $p,q \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$ and $1 < q^- \leqslant q^+ < \infty$. For $m > n$, there exists $c > 0$ such that*

$$\left\|(\eta_{\nu,m} * f_\nu)_{\nu\in\mathbb{N}_0}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leqslant c \left\|(f_\nu)_{\nu\in\mathbb{N}_0}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}.$$

1.4 The space $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$

Let us first introduce the concept of a smooth dyadic resolution of unity or dyadic decomposition of unity, see also [46, Section 2.3.1].

Let Ψ be a function in $\mathscr{S}(\mathbb{R}^n)$ satisfying $\Psi(x) = 1$ for $|x| \leqslant 1$ and $\Psi(x) = 0$ for $|x| \geqslant 2$. We define ψ_0 and ψ by $\mathscr{F}\psi_0 = \Psi$ and $\mathscr{F}\psi(x) = \Psi(x) - \Psi(2x)$ and

$$\mathscr{F}\psi_\nu(x) = \mathscr{F}\psi(2^{-\nu}x) \quad \textit{for} \quad \nu \in \mathbb{N}.$$

Then $\{\mathscr{F}\psi_\nu\}_{\nu\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity, that is *(i)* $\operatorname{supp}\mathscr{F}\psi_0 \subset \{x \in \mathbb{R}^n : |x| \leqslant 2\}$;

(ii) $\operatorname{supp}\mathscr{F}\psi \subset \left\{x \in \mathbb{R}^n : 1/2 \leqslant |x| \leqslant 2\right\}$; and

(iii) $\sum_{\nu=0}^{\infty}\mathscr{F}\psi_\nu(x) = 1$ for all $x \in \mathbb{R}^n$,

any system of functions $\{\psi_\nu, \nu \in \mathbb{N}_0\} \subset \mathscr{S}(\mathbb{R}^n)$ satisfies*(i),(ii)* and *(iii)* is called smooth dyadic resolution of unity. Thus we obtain the Littlewood-Paley decomposition

$$f = \sum_{\nu=0}^{\infty} \psi_\nu * f$$

for all $f \in \mathscr{S}'(\mathbb{R}^n)$ (convergence in $\mathscr{S}'(\mathbb{R}^n)$).

Remark 1.17. *We denote by $\{\mathscr{F}\Psi, \mathscr{F}\psi\}$ every smooth dyadic resolution of unity $\{\psi_\nu, \nu \in \mathbb{N}_0\} \subset \mathscr{S}(\mathbb{R}^n)$ generated by Ψ and ψ with $\psi_0 = \Psi$ and $\mathscr{F}\psi_\nu(x) = \mathscr{F}\psi(2^{-\nu}x)$ for $\nu \in \mathbb{N}$.*

We are now in a position to state the definition of the spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$.

Definition 1.18. *Let $(\mathscr{F}\psi_\nu)_{\nu\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity. For $\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $p, q \in \mathscr{P}_0$ where $p, q < \infty$, the Triebel-Lizorkin space $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ consists of all distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ such that*

$$\left\|f\right\|_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \left\|(2^{\nu\alpha(\cdot)}\psi_\nu * f)_{\nu\in\mathbb{N}_0}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} < \infty.$$

Using the system $(\varphi_\nu)_{\nu\in\mathbb{N}_0}$ we can define the quasi-norm

$$\left\|f\right\|_{F^{\alpha}_{p,q}} = \left\|(2^{\nu\alpha}\varphi_\nu * f)_{\nu\in\mathbb{N}_0}\right\|_{L^{p}(\ell^{q})}$$

for constants $\alpha \in \mathbb{R}$, $p \in [1, \infty)$ and $q \in [1, \infty]$. The Triebel-Lizorkin-type space $F^{\alpha}_{p,q}$ consist of all distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ for which $\left\|f\right\|_{F^{\alpha}_{p,q}} < \infty$. It is well-known that these spaces do not depend on the choice of the system $(\varphi_\nu)_{\nu\in\mathbb{N}_0}$ (up to equivalence of norms). Further details on the classical theory of these spaces can be found in [46] and [47], see also [48] for recent developments. One recognizes immediately that if s, p and q are constants, then $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} = F^{\alpha}_{p,q}$.

For any $p, q \in \mathscr{P}_0^{\log}$ with $p^+ < \infty, q^+ < \infty$, and $\alpha \in C^{\log}_{\text{loc}}$, the space $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ does not depend on the chosen of system $(\psi_\nu)_{\nu\in\mathbb{N}_0}$ (in the sense of equivalent norms). They are Banach spaces, and

$$\mathscr{S}(\mathbb{R}^n) \hookrightarrow F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathscr{S}'(\mathbb{R}^n).$$

In particular if $p \in C^{\log}_{\text{loc}}$, $1 < p^- \leqslant p^+ < \infty$, and $m \in \mathbb{N}_0$, then

$$F^{m}_{p(\cdot),2} = W^{m,p(\cdot)},$$

see [20]. The full treatment of the spaces $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ can be found in [20] and [19]. We refer to the paper [51], for further results on the variable Triebel-Lizorkin spaces $F^{\alpha(\cdot)}_{p(\cdot),q}$ (only the case of constant q was considered). We also mention the papers [6] and [7], for further results on the variable Bessel potentials spaces and variable Sobolev spaces.

1.5 The spaces $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$

We state the definition of the spaces $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, which was introduced and investigated in [3].

Definition 1.19. *Let* $\{\mathcal{F}\psi_v\}_{v\in\mathbb{N}_0}$ *be a resolution of unity,* $\alpha : \mathbb{R}^n \to \mathbb{R}$ *and* $p, q \in \mathcal{P}_0(\mathbb{R}^n)$*. The Besov space* $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ *consists of all distributions* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that*

$$\|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left\|(2^{v\alpha(\cdot)}\psi_v * f)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Taking $\alpha \in \mathbb{R}$ and $q \in (0,\infty]$ as constants we derive the spaces $B^{\alpha}_{p(\cdot),q}$ studied by Xu in [51]. We refer the reader to the recent papers [1], [2], [21] and [31] for further details, historical remarks and more references on these function spaces. For any $p, q \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $s \in C^{\log}_{\text{loc}}$, the space $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ does not depend on the chosen smooth dyadic resolution of unity $\{\mathcal{F}\psi_v\}_{v\in\mathbb{N}_0}$ (in the sense of equivalent quasi-norms) and

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{s(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Moreover, if p, q, s are constants, we re-obtain the usual Besov spaces $B^{\alpha}_{p,q}$, studied in detail in [46] and [47].

The following theorem gives basic embeddings between Besov spaces and Triebel- Lizorkin spaces

Theorem 1.20. *Let* $\alpha, \alpha_0, \alpha_1 \in L^\infty$ *and* $p, q_0, q_1 \in \mathcal{P}_0$.

- *If* $q_0 \leqslant q_1$*, then*
$$B^{\alpha(\cdot)}_{p(\cdot),q_0(\cdot)} \hookrightarrow B^{\alpha(\cdot)}_{p(\cdot),q_1(\cdot)}$$
- *If* $(\alpha_0 - \alpha_1)^- > 0$*, then*
$$B^{\alpha_0(\cdot)}_{p(\cdot),q_0(\cdot)} \hookrightarrow B^{\alpha_1(\cdot)}_{p(\cdot),q_1(\cdot)}$$
- *If* $p^+, q^+ < \infty$*, then*

$$B^{\alpha(\cdot)}_{p(\cdot),\min(p(\cdot),q(\cdot))} \hookrightarrow F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow B^{\alpha(\cdot)}_{p(\cdot),\max(p(\cdot),q(\cdot))}$$

Now, we consider embeddings of Sobolev-type

Theorem 1.21 (Sobolev inequality). *Let* $p_0, p_1, q \in \mathscr{P}$ *and* $\alpha_0, \alpha_1 \in L^\infty$ *with* $\alpha_0 \geqslant \alpha_1$. *If* $1/q$ *and*

$$\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)}$$

are in $C^{log}_{loc}(\mathbb{R}^n)$, *then*

$$B^{\alpha_0(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow B^{\alpha_1(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

We end the chapter with some results which are useful for us. The first Lemma is [19, Lemma A.3] while the second lemma is from [20, Lemma 6.1], see also [31, Lemma 19].

Lemma 1.22. *for* $v_1, v_2 \in \mathbb{N}_0$ *and* $m > 0$, *we have*

$$\eta_{v_1,m} * \eta_{v_2,m} \approx \eta_{\min\{v_1,v_2\},m}.$$

Lemma 1.23. *Let* $\alpha \in C^{\log}_{\text{loc}}(\mathbb{R}^n)$, $m \in \mathbb{N}_0$ *and let* $R \geqslant c_{\log}(\alpha)$, *where* $c_{\log}(\alpha)$ *is the constant from* (1.3). *Then there exists a constant* $c > 0$ *such that*

$$t^{-\alpha(x)} \eta_{t,m+R}(x-y) \leqslant c\ t^{-\alpha(y)} \eta_{t,m}(x-y)$$

for any $0 < t \leqslant 1$ *and* $x, y \in \mathbb{R}^n$.

The previous lemma allows us to treat the variable smoothness in many cases as if it were not variable at all. Namely, we can move the factor $t^{-\alpha(x)}$ inside the convolution as follows:

$$t^{-\alpha(x)} \eta_{t,m+R} * f(x) \leqslant c\ \eta_{t,m} * (t^{-\alpha(\cdot)} f)(x).$$

The proof of this lemma is given in [23, Lemma 2.2].

The following is [31, Lemma 9].

Lemma 1.24. *Let* $\varrho, \mu \in \mathscr{S}(\mathbb{R}^n)$, *and* $M \geqslant -1$ *an integer such that*

$$\int_{\mathbb{R}^n} x^\alpha \mu(x) dx = 0 \tag{1.5}$$

for all $|\alpha| \leqslant M$. *Then for any* $N > 0$, *there is a constant* $c(N) > 0$ *such that*

$$\sup_{z \in \mathbb{R}^n} \left| t^{-n} \mu(t^{-1} \cdot) * \varrho(z) \right| (1 + |z|)^N \leqslant c(N)\ t^{M+1}.$$

Remark 1.25. *Since* $\int_{\mathbb{R}^n} x^\alpha \mu(x) dx = i^{|\alpha|} D^\alpha (\mathscr{F}\mu)(0)$ *the assumption (1.5) is equivalent to*

$$D^\alpha(\mathscr{F}\mu)(0) = 0$$

for all $|\alpha| \leqslant M$.

CHAPTER 2

CONTINUOUS CHARACTERIZATION OF BESOV SPACES OF VARIABLE SMOOTHNESS AND INTEGRABILITY

2.1 Introduction

The main aim of this chapter is to present new equivalent quasi-norms of Besov Spaces of variable smoothness and integrability $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$. Based on the continuous version of Calderón reproducing formula, we define new family of function spaces and prove their basic properties such as the independence of the chosen resolution of unity and the characterization by the so-called continuous Peetre maximal functions. Later, under some suitable assumptions on the parameters we prove that these function spaces are just the Besov spaces of variable smoothness and integrability of Almeida and Hästö. Finally we characterize these function spaces in terms of continuous local means.

There are lots of difficulties we need to over come when working in variable Besov spaces and other variable scales generally, many techniques used in classical fail with variable scales or must be adapted and generalized, other difficulty is the definition of norms in variable scales which, in general ad additional difficulties and most of time we have to use the associated semi-modulars rather than quasi-norms.

2.2 Basic tools

In this section we present some useful results. The following lemma is from [54, Lemma 3.14], it's valid even when $q^+ = \infty$.

Lemma 2.1. *Let $p, q \in \mathscr{P}_0(\mathbb{R}^n)$. Let f be a measurable function on $\mathbb{R}^n$. If*

$$\||f|^{q(\cdot)}\|_{\frac{p(\cdot)}{q(\cdot)}} \geqslant 1$$

then

$$\|f\|_{p(\cdot)}^{q^-} \leqslant \||f|^{q(\cdot)}\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

The next lemma is a Hardy type inequalitie, see [33].

Lemma 2.2. *Let $s > 0$ and $(\varepsilon_t)_{0<t\leqslant 1}$ be a sequence of positive measurable functions when t is a continuous variable. Let*

$$\eta_t = t^s \int_t^1 \tau^{-s} \varepsilon_\tau \frac{d\tau}{\tau} \quad \text{and} \quad \delta_t = t^{-s} \int_0^t \tau^s \varepsilon_\tau \frac{d\tau}{\tau}.$$

Then there exists a constant $c > 0$ depending only on s such that

$$\int_0^1 \eta_t \frac{dt}{t} + \int_0^1 \delta_t \frac{dt}{t} \leqslant c \int_0^1 \varepsilon_t \frac{dt}{t}.$$

Lemma 2.3. *Let $r, N > 0$, $m > n$ and $\theta, \omega \in \mathscr{S}(\mathbb{R}^n)$ with $\operatorname{supp} \mathscr{F}\omega \subset \overline{B(0,1)}$. Then there exists a constant $c = c(r, m, n) > 0$ such that for all $g \in \mathscr{S}'(\mathbb{R}^n)$, we have*

$$\left|\theta_N * \omega_N * g(x)\right| \leqslant c(\eta_{N,m} * \left|\omega_N * g\right|^r (x))^{1/r}, \quad x \in \mathbb{R}^n,$$

where $\theta_N(\cdot) := N^n \theta(N\cdot)$, $\omega_N(\cdot) := N^n \omega(N\cdot)$ and $\eta_{N,m} := N^n(1 + N|\cdot|)^{-m}$.

The proof of this lemma is given in [23, Lemma 2.2]. We begin with a very important Lemma

Lemma 2.4. *Let $\alpha \in (0,1), \beta \in (1,\infty), p \in \mathscr{P}^{\log}(\mathbb{R}^n)$ and $q \in \mathscr{P}(\mathbb{R}^n)$ with $\frac{1}{q} \in C^{\log}_{\text{loc}}(\mathbb{R}^n)$. Let*

$$g_t(x) := \int_{\alpha t}^{\min\{1,\beta t\}} \eta_{\tau,m} * f_\tau(x) \frac{d\tau}{\tau}, \quad t \in (0,1], x \in \mathbb{R}^n.$$

(i) *The inequality*

$$\Big\||c g_t|^{q(\cdot)}\Big\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant \int_{\alpha t}^{\min\{1,\beta t\}} \Big\||f_\tau|^{q(\cdot)}\Big\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{d\tau}{\tau} + t, \quad t \in (0,1]$$

holds for every sequence of functions $(f_t)_{0<t\leqslant 1}$ *and constant* $m > n + c_{\log}(\frac{1}{q})$ *such that the first term on the right-hand side is at most one, where the constant* c *independent of* t.

(ii) *The inequality*

$$\left\|(g_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}} \leqslant c \left\|(f_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}$$

holds for every sequence of functions $(f_t)_{0<t\leqslant 1}$ *and constant* $m > n + c_{\log}(\frac{1}{q})$ *such that the right-hand side is finite.*

Proof. First let us prove (i). The claim can be reformulated as showing that

$$J := \left\|c_1\, \delta^{-\frac{1}{q(\cdot)}} g_t\right\|_{p(\cdot)} \leqslant 1 + \log\frac{\beta}{\alpha}, \quad t \in (0,1],$$

where $c_1 > 0$ and $\delta := \int_{\alpha t}^{\min\{1,\beta t\}} \left\||f_\tau|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{d\tau}{\tau} + t$. Applying Lemma 1.23, with an appropriate choice of c_1, we get

$$\begin{aligned} J &\leqslant \int_{\alpha t}^{\beta t} \left\|c_1\, \delta^{-\frac{1}{q(\cdot)}}(\eta_{\tau,m} * f_\tau)\right\|_{p(\cdot)} \frac{d\tau}{\tau} \\ &\leqslant c\, c_1 \int_{\alpha t}^{\beta t} \left\|\eta_{t,m-c_{\log}(\frac{1}{q})} * \delta^{-\frac{1}{q(\cdot)}} |f_\tau|\right\|_{p(\cdot)} \frac{d\tau}{\tau}, \quad m > n + c_{\log}(\frac{1}{q}) \\ &\leqslant \int_{\alpha t}^{\beta t} \left\|\delta^{-\frac{1}{q(\cdot)}} f_\tau\right\|_{p(\cdot)} \frac{d\tau}{\tau}, \end{aligned}$$

since $\delta \in (t, 1+t]$ and that the convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}$, since $m > n + c_{\log}(\frac{1}{q})$. Write

$$\begin{aligned} \int_{\alpha t}^{\beta t} \left\|\delta^{-\frac{1}{q(\cdot)}} f_\tau\right\|_{p(\cdot)} \frac{d\tau}{\tau} &= \int_{(\alpha t, \beta t] \cap B} \cdots \frac{d\tau}{\tau} + \int_{(\alpha t, \beta t] \cap B^c} \cdots \frac{d\tau}{\tau} \\ &= J_{1,t} + J_{2,t}, \end{aligned}$$

where

$$B := \left\{\tau > 0 : \left\||\delta^{-\frac{1}{q(\cdot)}} f_\tau|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \geqslant 1\right\}.$$

By Lemma 2.1,

$$J_{1,t} \leqslant \int_{(\alpha t, \beta t] \cap B} \left\||\delta^{-\frac{1}{q(\cdot)}} f_\tau|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}^{\frac{1}{q^-}} \frac{d\tau}{\tau} \leqslant \delta^{-1} \int_{\alpha t}^{\beta t} \left\||f_\tau|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{d\tau}{\tau} \leqslant 1$$

and

$$J_{2,t} \leqslant \int_{\alpha t}^{\beta t} \frac{d\tau}{\tau} = \log\frac{\beta}{\alpha}.$$

Now we prove (ii). By the scaling argument, it suffices to consider the case

$$\left\|(f_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}} = 1$$

and show that the modular of f on the left-hand side is bounded. By rewriting

$$\left\| |c g_t|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \lesssim t \int_t^1 \tau^{-1} \varepsilon_\tau \frac{d\tau}{\tau} + t^{-1} \int_0^t \tau\, \varepsilon_\tau \frac{d\tau}{\tau} + t.$$

where $\varepsilon_\tau = \left\| |f_\tau|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}$ and the constant c is from (i). Applying Hardy inequality, Lemma 2.2, we get

$$\int_0^1 \left\| |c g_t|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{dt}{t} \leqslant C$$

for some positive constant C, which proves the desired result. □

Lemma 2.5. *Let* $0 < m, r < \infty$ *such that* $mr > n$. *Let* $\{\mathscr{F}\Phi, \mathscr{F}\varphi\}$ *be a resolution of unity:*

$$\mathscr{F}\Phi(\xi) + \int_0^1 \mathscr{F}\varphi(t\xi) \frac{dt}{t} = 1, \quad \xi \in \mathbb{R}^n.$$

(i) *Let* $\theta \in \mathscr{S}(\mathbb{R}^n)$ *be such that* $\operatorname{supp}\mathscr{F}\theta \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 2\}$. *There exists a constant* $c > 0$ *such that*

$$|\theta * f|^r \leqslant c\ \eta_{1,mr} * |\Phi * f|^r + c \int_{1/4}^1 \eta_{1,mr} * |\varphi_\tau * f|^r \frac{d\tau}{\tau}$$

for any $f \in \mathscr{S}'(\mathbb{R}^n)$, *where* $\varphi_\tau = \tau^{-n}\varphi(\frac{\cdot}{\tau})$.
(ii) *Let* $\omega \in \mathscr{S}(\mathbb{R}^n)$ *be such that* $\operatorname{supp}\mathscr{F}\omega \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \leqslant |\xi| \leqslant 2\}$. *There exists a constant* $c > 0$ *such that*

$$|\omega_t * f|^r \leqslant c\ \eta_{1,mr} * |\Phi * f|^r + c \int_{t/4}^{\min(1,4t)} \eta_{\tau,mr} * |\varphi_\tau * f|^r \frac{d\tau}{\tau}$$

for any $f \in \mathscr{S}'(\mathbb{R}^n)$ *and any* $0 < t \leqslant 1$ *(if* $t \in (0, 1/4]$, *the first term on the right hand side is null), where* $\omega_t = t^{-n}\omega(\frac{\cdot}{t})$.

Proof. We split the proof into two steps. First the case $1 \leqslant r < \infty$ follows by the Hölder inequality.
Step 1. Proof of (i). Since $\{\mathscr{F}\Phi, \mathscr{F}\varphi\}$ is a resolutions of unity, it follows that

$$\theta * f = \Phi * \theta * f + \int_{1/4}^1 \theta * \varphi_\tau * f \frac{d\tau}{\tau}.$$

First recall the elementary inequality

$$d^n \eta_{d,m}(y-z) \leqslant d^{2n} \eta_{d,-m}(y-x)\eta_{d,m}(x-z), \quad d>0, x,y,z \in \mathbb{R}^n, \tag{2.1}$$

which together with Lemma 2.3 implies that

$$\begin{aligned} |\Phi * \theta * f(y)|^r &\lesssim \eta_{1,mr} * |\Phi * f|^r(y) \\ &= c \int_{\mathbb{R}^n} \eta_{1,mr}(y-z)|\Phi * f(z)|^r dz \\ &\lesssim \eta_{1,-mr}(y-x)\eta_{1,mr} * |\Phi * f|^r(x) \end{aligned}$$

for any $x \in \mathbb{R}^n$ and any $m > \frac{n}{r}$. Furthermore,

$$\begin{aligned} |\Phi * \theta * f(y)| &\leqslant \int_{\mathbb{R}^n} \eta_{1,N}(y-z)|\theta * f(z)| dz \\ &\leqslant \eta_{1,-m}(y-x)\theta_1^{*,m} f(x) \int_{\mathbb{R}^n} \eta_{1,N-m}(y-z) dz \\ &\lesssim \eta_{1,-m}(y-x)\theta_1^{*,m} f(x) \end{aligned}$$

for any $N > m+n$, where

$$\theta_1^{*,m} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\theta * f(y)|}{(1+|y-x|)^m}, \quad x \in \mathbb{R}^n.$$

Therefore,

$$|\Phi * \theta * f(y)| \lesssim \eta_{1,-m}(y-x)\left(\theta_1^{*,m} f(x)\right)^{1-r} \eta_{1,mr} * |\Phi * f|^r(x)$$

for any $x \in \mathbb{R}^n$ with $mr > n$. Again from Lemma 2.3 we conclude

$$\begin{aligned} |\theta * \varphi_\tau * f(y)|^r &\lesssim \eta_{1,mr} * |\varphi_\tau * f|^r(y) \\ &\lesssim (1+|y-x|)^{mr} \eta_{1,mr} * |\varphi_\tau * f|^r(x) \end{aligned}$$

and

$$\begin{aligned} |\theta * \varphi_\tau * f(y)| &\lesssim \int_{\mathbb{R}^n} \eta_{\tau,N}(y-z)|\theta * f(z)| dz, \quad \frac{1}{4} \leqslant \tau \leqslant 1 \\ &\lesssim (1+|y-x|)^m \theta_1^{*,m} f(x) \end{aligned}$$

for any $x \in \mathbb{R}^n$, any $m > n$ and any $N > m + n$. Consequently

$$\theta_1^{*,m} f(x) \leqslant c \left(\theta_1^{*,m} f(x)\right)^{1-r} \left(\eta_{1,mr} * |\Phi * f|^r(x) + \int_{1/4}^1 \eta_{1,mr} * |\varphi_\tau * f|^r(x) \frac{d\tau}{\tau} \right), \tag{2.2}$$

which implies that

$$|\theta * f(x)|^r \leqslant c\ \eta_{1,mr} * |\Phi * f|^r(x) + c \int_{1/4}^1 \eta_{1,mr} * |\varphi_\tau * f|^r(x) \frac{d\tau}{\tau} \tag{2.3}$$

when $\theta_1^{*,m} f(x) < \infty$, which is true if $m \geqslant \frac{n}{r} + N_0$ (order of distribution). We will use the Strömberg and Torchinsky idea [45]. Observe that the right-hand side of (2.3) decreases as m increases. Therefore, we have (2.3) for all $m > \frac{n}{r}$ but with $c = c(f)$ depending on f. We can easily check that if the right-hand side of (2.3), with $c = c(f)$, is finite imply that $\theta_1^{*,m} f(x) < \infty$, otherwise, there is nothing to prove. Returning to (2.2) and having in mind that now $\theta_1^{*,m} f(x) < \infty$, we obtain the desired estimate (2.3).
Step 2. Proof of (ii). We have

$$\omega_t * f = \int_{t/4}^{\min(1,4t)} \omega_t * \varphi_\tau * f \frac{d\tau}{\tau} + \begin{cases} 0, & \text{if } 0 < t < \frac{1}{4}; \\ \omega_t * \Phi * f, & \text{if } \frac{1}{4} \leqslant t \leqslant 1. \end{cases}$$

Let

$$g_t(y) = \int_{t/4}^{\min(1,4t)} \omega_t * \varphi_\tau * f(y) \frac{d\tau}{\tau}, \quad y \in \mathbb{R}^n, 0 < t \leqslant 1.$$

It follows from Lemma 2.3 that

$$\begin{aligned} |\omega_t * \varphi_\tau * f(y)|^r &\lesssim \eta_{t,mr} * |\varphi_\tau * f|^r(y) \\ &\lesssim \eta_{\tau,mr} * |\varphi_\tau * f|^r(y) \\ &= c \int_{\mathbb{R}^n} \eta_{\tau,mr}(y-z) |\varphi_\tau * f(z)|^r dz \\ &\lesssim (1 + \tau^{-1}|y-x|)^{mr} \eta_{\tau,mr} * |\varphi_\tau * f|^r(x) \end{aligned}$$

and

$$\begin{aligned} |\omega_t * \varphi_\tau * f(y)| &\lesssim \int_{\mathbb{R}^n} \eta_{\tau,N}(y-z) |\omega_t * f(z)| dz \\ &\lesssim \omega_{t,m}^* f(y) \int_{\mathbb{R}^n} \eta_{\tau,N}(y-z)(1 + t^{-1}|y-z|)^m dz \\ &\lesssim \omega_t^{*,m} f(y) \\ &\lesssim (1 + t^{-1}|y-x|)^m \omega_t^{*,m} f(x) \end{aligned}$$

for any $x, y \in \mathbb{R}^n$, any $t/4 \leqslant \tau \leqslant \min(1,4t), 0 < t \leqslant 1$ and any $N > m+n$, where

$$\omega_t^{*,m} f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\omega_t * f(y)|}{(1+t^{-1}|y-x|)^m}, \quad x, y \in \mathbb{R}^n, 0 < t \leqslant 1.$$

Therefore, $|g_t(y)|$ can be estimated from above by

$$\begin{aligned} & c\left(\omega_t^{*,m} f(x)\right)^{1-r} (1+t^{-1}|y-x|)^{m(1-r)} \\ & \times \int_{t/4}^{\min(1,4t)} (1+\tau^{-1}|y-x|)^{mr} \eta_{\tau,mr} * |\varphi_\tau * f|^r(x) \frac{d\tau}{\tau} \\ \lesssim \ & (1+t^{-1}|y-x|)^m \left(\omega_t^{*,m} f(x)\right)^{1-r} \int_{t/4}^{\min(1,4t)} \eta_{\tau,mr} * |\varphi_\tau * f|^r(x) \frac{d\tau}{\tau}, \end{aligned}$$

if $0 < t \leqslant 1$. Now if $\frac{1}{4} \leqslant t \leqslant 1$, we easily obtain

$$\begin{aligned} |\omega_t * \Phi * f(y)| &= |\omega_t * \Phi * f(y)|^{1-r} |\omega_t * \Phi * f(y)|^r \\ &\lesssim (1+t^{-1}|y-x|)^{m(1-r)} \left(\omega_t^{*,m} f(x)\right)^{1-r} \eta_{1,mr} * |\Phi * f|^r(y) \\ &\lesssim (1+t^{-1}|y-x|)^m \left(\omega_t^{*,m} f(x)\right)^{1-r} \eta_{1,mr} * |\Phi * f|^r(x), \end{aligned}$$

which yields that

$$\sup_{y\in\mathbb{R}^n} \frac{|\omega_t * \Phi * f(y)|}{(1+t^{-1}|y-x|)^m} \lesssim \left(\omega_t^{*,m} f(x)\right)^{1-r} \eta_{1,mr} * |\Phi * f|^r(x).$$

Consequently

$$|\omega_t * f(x)|^r \lesssim \left(\omega_t^{*,m} f(x)\right)^r \lesssim \eta_{1,mr} * |\Phi * f|^r(x) + \int_{t/4}^{\min(1,4t)} \eta_{\tau,mr} * |\varphi_\tau * f|^r(x) \frac{d\tau}{\tau}. \qquad (2.4)$$

when $\omega_t^{*,m} f(x) < \infty, 0 < t \leqslant 1$ and $x \in \mathbb{R}^n$. Using a combination of the arguments used in (i), we arrive at the desired estimate. The proof is complete. □

The following lemma is [42, Lemma 1]

Lemma 2.6. *Let $p, q \in \mathscr{P}_0(\mathbb{R}^n)$ and $\delta > 0$. Let $(g_k)_{k\in\mathbb{N}_0}$ be a sequence of non-negative measurable functions on $\mathbb{R}^n$ and denote*

$$G_\nu(x) = \sum_{k\in\mathbb{N}_0} 2^{-\delta|k-\nu|} g_k(x), \quad x \in \mathbb{R}^n, \ \nu \in \mathbb{N}_0.$$

Then there exists a positive constant c depending on p, q and δ such that

$$\left\|(G_v)_{v\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c\left\|(g_k)_{k\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

2.3 Variable Besov spaces

2.3.1 The space $\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}$

Let $(f_t)_{t\in(0,1]}$ be a sequence of functions $f_t(x)$, $t\in(0,1]$, $x\in\mathbb{R}^n$, measurable on $(0,1]\times\mathbb{R}^n$, We set

$$\varrho_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}((f_t)_{0<t\leqslant 1}) := \int_0^1 \inf\left\{\lambda_t>0 : \varrho_{p(\cdot)}\left(\frac{f_t}{\lambda_t^{1/q(\cdot)}}\right)\leqslant 1\right\}\frac{dt}{t}.$$

we use the convention $\lambda^{\frac{1}{\infty}}=1$. The quasi-norm is defined from this as usual:

$$\left\|(f_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}} := \inf\left\{\mu>0 : \varrho_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}\Big(\frac{1}{\mu}(f_t)_{0<t\leqslant 1}\Big)\leqslant 1\right\}.$$

Now, we give two cases where it is possible to simplify the expression of the previous quasi-norm.

Example 2.7.

(i) *If the exponent q is a constant from $(0,\infty)$ then the semi-modular*

$$\varrho_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}((f_t)_{0<t\leqslant 1}) = \int_0^1 \left\|f_t\right\|_{p(\cdot)}^q\frac{dt}{t} \text{ and } \left\|(f_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}} = \left(\int_0^1 \left\|f_t\right\|_{p(\cdot)}^q\frac{dt}{t}\right)^{1/q}.$$

(ii) *If $q=\infty$, let $\mu = \underset{t\in(0,1]}{\text{ess-sup}}\,\|f_t\|_{p(\cdot)}<\infty$, then for all $t\in(0,1]$, $\lambda_t>0$ we have $\varrho_{p(\cdot)}\Big(\frac{f_t/\mu}{\lambda_t^{1/\infty}}\Big) = \varrho_{p(\cdot)}\big(f_t/\mu\big)\leqslant 1$ then $\inf\Big\{\lambda_t>0 : \varrho_{p(\cdot)}\Big(\frac{f_t/\mu}{\lambda_t^{1/\infty}}\Big)\leqslant 1\Big\}=0$, implies that $\varrho_{\widetilde{\ell^{\infty}(L^{p(\cdot)})}}((f_t/\mu)_{0<t\leqslant 1})=0$, therefore $\left\|(f_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{\infty}(L^{p(\cdot)})}}\leqslant\mu$, now, we prove the reverse inequality, let $m\in(0,\mu)$ then there exists a subset $I\subset(0,1]$ where $|I|>0$ and $m<\|f_t\|_{p(\cdot)}$ on I, then for all $t\in I$, $\lambda_t>0$ we have $\varrho_{p(\cdot)}\Big(\frac{f_t/m}{\lambda_t^{1/\infty}}\Big)=\varrho_{p(\cdot)}\big(f_t/m\big)>1$, then $\inf\Big\{\lambda_t>0 : \varrho_{p(\cdot)}\Big(\frac{f_t/m}{\lambda_t^{1/\infty}}\Big)\leqslant 1\Big\} = \inf\emptyset = +\infty$ for all $t\in I$, therefore $\varrho_{\widetilde{\ell^{\infty}(L^{p(\cdot)})}}((f_t/m)_{0<t\leqslant 1})=+\infty$ for any $m\in(0,\mu)$, hence*

$$\left\|(f_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{\infty}(L^{p(\cdot)})}} = \underset{t\in(0,1]}{\text{ess-sup}}\|f_t\|_{p(\cdot)}.$$

If $\underset{t\in(0,1]}{\text{ess-sup}}\,\|f_t\|_{p(\cdot)}=\infty$ then for any positive m there exist a measurable subset $I\subset(0,1]$ of positive measure such that $m<\|f_t\|_{p(\cdot)}$ on I, it follows that for any positive m and $t\in$

I, $\inf\left\{\lambda_t>0:\varrho_{p(\cdot)}\left(\frac{f_t/m}{\lambda_t^{1/\infty}}\right)\leqslant 1\right\}=\inf\emptyset=+\infty$, and then $\varrho_{\widetilde{\ell^{\infty}(L^{p(\cdot)})}}((f_t/m)_{0<t\leqslant 1})=+\infty$, therefore,

$$\left\|(f_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}:=\inf\left\{m>0:\varrho_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}\left((f_t/m)_{0<t\leqslant 1}\right)\leqslant 1\right\}=\inf\emptyset=+\infty.$$

Theorem 2.8. *Let $p,q\in\mathscr{P}_0(\mathbb{R}^n)$. For every function $f(t,x)$ measurable on $(0,1]\times\mathbb{R}^n$ the function defined by*

$$\Theta_f(t):=\inf\left\{\lambda_t>0:\varrho_{p(\cdot)}\left(\frac{f_t}{\lambda_t^{1/q(\cdot)}}\right)\leqslant 1\right\},\qquad t\in(0,1],$$

is measurable on $(0,1]$.

Proof. First, we prove that if g is a non-negative function null outside a measurable subset of $(0,1]\times\mathbb{R}^n$ of finite measure , then Θ_g is measurable on $(0,1]$. For such a function there exists an increasing sequence of step functions g_k that converges pointwise to g. namely, $g_k(t,x)\leqslant g_{k+1}(t,x)$ for all $k\in\mathbb{N}_0$ and $(t,x)\in(0,1]\times\mathbb{R}^n$, $\lim_{k\to\infty}g_k(t,x)=g(t,x)$, where

$$g_k(t,x)=\sum_{i=0}^{N_k}\alpha_i^k\chi_{R_i^k}$$

where each rectangle R_i^n is a product of $n+1$ one-dimensional bounded intervals,

$$R_i^k=\prod_{j=1}^{n+1}[a_j^{i,k},b_j^{i,k}[$$

with $[a_1^{i,k},b_1^{i,k}[\subset(0,1]$, and $\prod_{j=2}^{n+1}[a_j^{i,k},b_j^{i,k}[\subset\mathbb{R}^n$, without loss of the generality we can relabel the numbers $a_1^{i,k},b_1^{i,k},0\leqslant i\leqslant N_k$ as $x_1<x_2<\cdots<x_{2N_k}$(the number of the x_i's may be less than $2N_k$), so, we can rewrite

$$g_k(t,x)=\sum_{i=0}^{2N_k}\alpha_i^k\chi_{B_i^k}$$

where the first side of each rectangle B_i^k is one of intervals $[x_i,x_{i+1}[$, the advantage of this new representation is that $g_k(t_1,\cdot)=g_k(t_2,\cdot)$ for any $t_1,t_2\in[x_i,x_{i+1}[$, it follows that Θ_{g_k} is constant or equals ∞ on every $[x_i,x_{i+1}[$ and null outside $\bigcup[x_i,x_{i+1}[$, then, Θ_{g_k} is measurable for any $k\in\mathbb{N}_0$. We have $\Theta_{g_k}\leqslant\Theta_{g_{k+1}}\leqslant\Theta_g$ since $g_k\leqslant g_{k+1}\leqslant g$ for all $k\in\mathbb{N}_0$. The sequence of functions $(\Theta_{g_k})_{k\in\mathbb{N}_0}$ converges to a measurable function we denote by Λ, then $\Lambda\leqslant\Theta_g$. We have for any $t\in(0,1]$, if $\Lambda_t=\infty$ then $\Theta_g(t)=\infty$ and if $\Lambda_t<\infty$ then the sequence of functions $\left(g_k(t,\cdot)/\Lambda_t^{1/q(\cdot)}\right)^{p(\cdot)}$ is increasing and converges pointwise to

$\left(g(t,\cdot)/\Lambda_t^{1/q(\cdot)}\right)^{p(\cdot)}$, then we get

$$\varrho_{p(\cdot)}\left(\frac{g_t}{\Lambda_t^{1/q(\cdot)}}\right)=\int_{\mathbb{R}^n}\left(\frac{g(t,x)}{\Lambda_t^{1/q(x)}}\right)^{p(x)}dx=\lim_k\int_{\mathbb{R}^n}\left(\frac{g_k(t,x)}{\Lambda_t^{1/q(x)}}\right)^{p(x)}dx\leqslant 1$$

which implies that $\Lambda=\Theta_g$, then, Θ_g is measurable.

Now, let f be an arbitrary measurable function defined on $(0,1]\times\mathbb{R}^n$, since $\Theta_f=\Theta_{|f|}$ it suffices to suppose that f is non-negative. Then, there exists an increasing sequence of non-negative simple functions f_n that converges pointwise to f , namely,

$f_n(t,x)\leqslant f_{n+1}(t,x)$ for all $n\in\mathbb{N}_0$ and $(t,x)\in(0,1]\times\mathbb{R}^n$, $\lim_{n\to\infty}f_n(t,x)=f(t,x)$, where

$$f_n(t,x)=\sum_{i=0}^{N_n}\alpha_i^n\chi_{A_i^n}$$

where the A_i^n are disjoint measurable subsets of $(0,1]\times\mathbb{R}^n$ and of finite measure. since f_n is null outside $\bigcup A_i^n$ and $\bigcup A_i^n$ is of finite measure, by the first part, Θ_{f_n} is measurable for any $n\in\mathbb{N}_0$, by similar arguments as in the first part we prove that Θ_f is measurable.

□

Theorem 2.9. *If $p,q\in\mathscr{P}_0$, then $\varrho_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}$ is a semi-modular.*

Proof. To show that $\varrho_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}$ is a semi-modular we need to check properties(1)-(4) of Definition 1.1, Properties (1) and (2) are clear. To prove (3), suppose that $\varrho_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}((\lambda f_t)_{0<t\leqslant 1})=0$ for all $\lambda>0$ then $\left\||\lambda f_t|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}=0$ for all $\lambda>0$, since $\left\||\lambda f_t|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}=\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((\lambda f_t,0,0,\cdots))$ and $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a semi-modular we deduce that $f_t=0$ for all $t\in(0,1]$.

To prove the left-continuity we see that it suffices to show that the sequence $\varrho_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}((\mu_n f_t)_{0<t\leqslant 1})$ converges to $\varrho_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}((f_t)_{0<t\leqslant 1})$ for every non-decreasing sequence μ_n of elements in $(0,1)$ that converges to 1, since the sequence of function $(\||\mu_n f_t|^{q(\cdot)}\|_{\frac{p(\cdot)}{q(\cdot)}})_n$ is non-decreasing we have

$$\lim_n\int_0^1\left\||\mu_n f_t|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\frac{dt}{t}=\int_0^1\lim_n\left\||\mu_n f_t|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\frac{dt}{t}=\int_0^1\left\||f_t|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\frac{dt}{t}$$

where we have used $\lim_n\||\mu_n f_t|^{q(\cdot)}\|_{\frac{p(\cdot)}{q(\cdot)}}=\||f_t|^{q(\cdot)}\|_{\frac{p(\cdot)}{q(\cdot)}}$ which follows from the left continuity of $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}$. □

By Definition 1.2, the space $\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is the set of all sequences $(f_t(x))_{t\in(0,1]}$ of measurable functions f_t on $(0,1]\times\mathbb{R}^n$ such that $\varrho_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}((\lambda f_t)_{0<t\leqslant 1})$ for some $\lambda>0$. It still to prove that $\left\|(f_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}$ is indeed a quasi-norm, we follow the approach used in [3].

Theorem 2.10. *Let $p,q\in\mathscr{P}$. If $\frac{1}{p}+\frac{1}{q}\leqslant 1$ point-wise, then $\left\|(f_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}$ is a norm.*

Proof. Let $(f_t)_{0<t\leqslant 1} \in \widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}$. If $\left\|(f_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}} = 0$ then $\int_0^1 \||f_t|^{q(\cdot)}\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{dt}{t} \leqslant \mu$ for all $\mu \in (0,1]$ which implies $\int_0^1 \||f_t|^{q(\cdot)}\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{dt}{t} = 0$ and then $f_t = 0$ for all $t \in (0,1]$. it's clear that $\left\|(\lambda f_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}} = |\lambda| \left\|(f_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}$ for all $\lambda \in \mathbb{C}$.

Now, we prove that

$$\left\|(f_t+g_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}} \leqslant \left\|(f_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}} + \left\|(f_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}$$

for all $(f_t)_{0<t\leqslant 1}, (g_t)_{0<t\leqslant 1} \in \widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}$. By [3, page 1636], we have for all $\lambda, \mu > 0$,

$$\left\|\left|\frac{f_t+g_t}{\lambda+\mu}\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant \frac{\lambda}{\lambda+\mu}\left\|\left|\frac{f_t}{\lambda}\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} + \frac{\mu}{\lambda+\mu}\left\|\left|\frac{g_t}{\mu}\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Now, taking $\lambda > \left\|(f_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}$, $\mu > \left\|(g_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}$ and the integral $\int_0^1 \cdot\, dt/t$ on both sides, we get

$$\left\|(f_t+g_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}} \leqslant \lambda+\mu,$$

which implies the desired result, The proof is complete. □

Theorem 2.11. *Let $p, q \in \mathscr{P}_0$, then $\|\cdot\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}$ is a quasi-norm on $\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}$.*

2.3.2 The space $\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$

In this subsection we present our version of Besov spaces of variable smoothness and integrability, and prove the basic properties in analogy to the case of fixed exponents. Select a pair of Schwartz functions Φ and φ satisfying

$$\operatorname{supp}\mathscr{F}\Phi \subset \{x \in \mathbb{R}^n : |x| \leqslant 2\}, \quad \operatorname{supp}\mathscr{F}\varphi \subset \left\{x \in \mathbb{R}^n : 1/2 \leqslant |x| \leqslant 2\right\} \tag{2.5}$$

and

$$\mathscr{F}\Phi(\xi) + \int_0^1 \mathscr{F}\varphi(t\xi)\frac{dt}{t} = 1, \quad \xi \in \mathbb{R}^n. \tag{2.6}$$

Such a resolution (2.5) and (2.6) of unity can be constructed as follows. Let $\mu \in \mathscr{S}(\mathbb{R}^n)$ be such that $\left|\mathscr{F}\mu(\xi)\right| > 0$ for $1/2 < |\xi| < 2$. There exists $\eta \in \mathscr{S}(\mathbb{R}^n)$ with

$$\operatorname{supp}\mathscr{F}\eta \subset \left\{x \in \mathbb{R}^n : 1/2 < |x| < 2\right\}$$

such that

$$\int_0^\infty \mathscr{F}\mu(t\xi)\mathscr{F}\eta(t\xi)\frac{dt}{t} = 1, \quad \xi \neq 0,$$

see [16], [26] and [25]. We set $\mathscr{F}\varphi = \mathscr{F}\mu\,\mathscr{F}\eta$ and

$$\mathscr{F}\Phi(\xi) = \begin{cases} \displaystyle\int_1^{\infty} \mathscr{F}\varphi(t\xi)\frac{dt}{t} & \text{if} \quad \xi \neq 0, \\ 1 & \text{if} \quad \xi = 0. \end{cases}$$

Then $\mathscr{F}\Phi \in \mathscr{S}(\mathbb{R}^n)$, and as $\mathscr{F}\eta$ is supported in $\{x \in \mathbb{R}^n : 1/2 \leqslant |x| \leqslant 2\}$, we see that supp $\mathscr{F}\Phi \subset \{x \in \mathbb{R}^n : |x| \leqslant 2\}$.

Now we define the spaces under consideration.

Definition 2.12. *Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $p, q \in \mathscr{P}_0(\mathbb{R}^n)$. Let $\{\mathscr{F}\Phi, \mathscr{F}\varphi\}$ be a resolution of unity and we put $\varphi_t = t^{-n}\varphi(\frac{\cdot}{t})$, $0 < t \leqslant 1$. The Besov space $\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that*

$$\left\|f\right\|^{\Phi,\varphi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|\Phi * f\|_{p(\cdot)} + \left\|(t^{-\alpha(\cdot)}\varphi_t * f)_{0<t\leqslant 1}\right\|_{\ell^{\widetilde{q(\cdot)}(L^{p(\cdot)})}} < \infty.$$

When $q = \infty$, the Besov space $\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),\infty}$ consist of all distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\left\|f\right\|^{\Phi,\varphi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),\infty}} := \left\|\Phi * f\right\|_{p(\cdot)} + \sup_{t\in[0,1]} \left\|t^{-\alpha(\cdot)}(\varphi_t * f)\right\|_{p(\cdot)} < \infty.$$

One recognizes immediately that $\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ is a quasi-normed space and if α, p and q are constants, then

$$\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} = B^{\alpha}_{p,q},$$

where $B^{\alpha}_{p,q}$ is the usual Besov spaces.

Now, we are ready to show that the definition of these function spaces is independent of the chosen resolution $\{\mathscr{F}\Phi, \mathscr{F}\varphi\}$ of unity. This justifies our omission of the subscript Φ and φ in the sequel.

Theorem 2.13. *Let $\{\mathscr{F}\Phi, \mathscr{F}\varphi\}$ and $\left\{\mathscr{F}\Psi, \mathscr{F}\psi\right\}$ be two resolutions of unity. Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $p, q \in \mathscr{P}_0(\mathbb{R}^n)$. Assume that $p \in \mathscr{P}_0^{\log}(\mathbb{R}^n)$ and $\alpha, \frac{1}{q} \in C^{\log}_{\text{loc}}(\mathbb{R}^n)$. Then*

$$\left\|f\right\|^{\Phi,\varphi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \left\|f\right\|^{\Psi,\psi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Proof. It is sufficient to show that there exists a constant $C > 0$ such that for all $f \in \mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ we have

$$\left\|f\right\|^{\Phi,\varphi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant C\left\|f\right\|^{\Psi,\psi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}, \tag{2.7}$$

interchanging the roles of $\big(\Psi,\psi\big)$ and $\big(\Phi,\varphi\big)$ we obtain the desired result. By Lemma 2.5/(ii)

$$|\varphi_t * f|^r \leqslant c\int_{t/4}^{\min(1,4t)} \eta_{\tau,N} * |\psi_\tau * f|^r \frac{d\tau}{\tau} + \begin{cases} 0, & \text{if } 0<t<\frac{1}{4};\\ c\ \eta_{1,N} * |\Psi * f|^r, & \text{if } \frac{1}{4}\leqslant t\leqslant 1.\end{cases}$$

where $r=\min\{p^-,q^-\}$ and N arbitrary large. For all $t\in(0,1]$ and $t/4\leqslant\tau\leqslant\min(1,4t)$ we have $1/4\leqslant\tau/t\leqslant 4$, since $\alpha\in C^{\log}_{\text{loc}}(\mathbb{R}^n)$ then α is bounded, hence, by Lemma 1.23

$$|t^{-\alpha}\varphi_t * f|^r \leqslant c\,h(t)+c\,g(t)$$

where

$$g_t := \int_{t/4}^{\min(1,4t)} \eta_{\tau,N'} * |\tau^{-\alpha}\psi_\tau * f|^r \frac{d\tau}{\tau}$$

and

$$h_t := \begin{cases} 0, & \text{if } 0<t<\frac{1}{4};\\ \eta_{1,N'} * |\Psi * f|^r, & \text{if } \frac{1}{4}\leqslant t\leqslant 1,\end{cases}$$

where $N' = N - \text{c}_{\log}(r\alpha)$ still can be taken arbitrary large, remark that $\eta_{1,N'}\leqslant\eta_{1,N}$. Since $p/r\in\mathscr{P}^{\log}(\mathbb{R}^n)$ and $\frac{1}{q/r}\in C^{\log}_{\text{loc}}(\mathbb{R}^n)$, by taking N' large such that $N'>n+r\ \text{c}_{\log}(1/q(\cdot))$ Lemma 2.4 yields

$$\begin{aligned}\left\|(g_t)_{0<t\leqslant 1}\right\|_{\widehat{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}} &\leqslant c\left\|(|t^{-\alpha}\psi_t * f|^r)_{0<t\leqslant 1}\right\|_{\widehat{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}}\\ &\leqslant c\left\|(t^{-\alpha}\psi_t * f)_{0<t\leqslant 1}\right\|^r_{\widehat{\ell^{q(\cdot)}(L^{p(\cdot)})}}.\end{aligned}$$

Now, we estimate $\left\|(h_t)_{0<t\leqslant 1}\right\|_{\widehat{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}}$: if $\left\|\Psi * f\right\|_{p(\cdot)}\leqslant 1$, since $N'>n$, $p/r\in\mathscr{P}^{\log}(\mathbb{R}^n)$ and the convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)/r}$, it follows, $\left\|\eta_{1,N'} * |\Psi * f|^r\right\|_{p(\cdot)/r}\lesssim\left\|\Psi * f\right\|^r_{p(\cdot)}\lesssim 1$, hence there exists $c_1>0$ such that $\left\|c_1\eta_{1,N'} * |\Psi * f|^r\right\|_{p(\cdot)/r}\leqslant 1$, it follows that $\left\|\left[c_1\,\eta_{1,N'} * |\Psi * f|^r\right]^{(q(\cdot)/r)}\right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}}\leqslant 1$, therefore

$$\varrho_{\widehat{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}}((c_1 h_t)_{t\in(0,1]}) = \int_{1/4}^{1}\left\|\left[c_1\,\eta_{1,N'} * |\Psi * f|^r\right]^{(q(\cdot)/r)}\right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}}\frac{dt}{t}\leqslant c,$$

then, $\left\|(h_t)_{0<t\leqslant 1}\right\|_{\widehat{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}}\leqslant c$, by scaling arguments we get,

$$\left\|(h_t)_{0<t\leqslant 1}\right\|_{\widehat{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}}\leqslant c\left\|\Psi * f\right\|^r_{p(\cdot)},$$

therefore

$$\begin{aligned}\left\|(|t^{-\alpha(\cdot)}\varphi_t * f|^r)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}} &\leqslant c\left\|(g_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}} + c\left\|(h_t)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}}\\ &\leqslant c\left\|(t^{-\alpha}\psi_t * f)_{0<t\leqslant 1}\right\|^r_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}} + c\left\|\Psi * f\right\|^r_{p(\cdot)}\\ &\leqslant 2c\left(\left\|f\right\|^{\Psi,\psi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}\right)^r,\end{aligned}$$

this implies that

$$\left\|(t^{-\alpha(\cdot)}\varphi_t * f)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}} \lesssim \left\|f\right\|^{\Psi,\psi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}. \tag{2.8}$$

With similar arguments, Lemma 2.5/(i)

$$\begin{aligned}|\Phi * f|^r &\leqslant c\ \eta_{1,N} * |\Psi * f|^r + c\int_{1/4}^1 \eta_{1,N} * |\psi_\tau * f|^r \frac{d\tau}{\tau}\\ &\leqslant c\ \eta_{1,N} * |\Psi * f|^r + c\int_{1/4}^1 \eta_{\tau,N} * |\tau^{-\alpha}\psi_\tau * f|^r \frac{d\tau}{\tau}\end{aligned}$$

let $k := \int_{1/4}^1 \eta_{\tau,N} * |\tau^{-\alpha}\psi_\tau * f|^r \frac{d\tau}{\tau}$. If $\left\|f\right\|^{\Psi,\psi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant 1$ then

$$\int_{1/4}^1 \left\|\left[|\tau^{-\alpha}\psi_\tau * f|^r\right]^{(q(\cdot)/r)}\right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}} \frac{dt}{t} \leqslant 1,$$

remark that $\|[|\cdot|]^{(q(\cdot))}\|_{\frac{p(\cdot)}{q(\cdot)}} = \|(|\cdot|^r)^{(q(\cdot)/r)}\|_{\frac{p(\cdot)/r}{q(\cdot)/r}}$, by taking N large enough, Lemma 2.4/(i) ensures that there exists a positive constant c such that $\left\||c\,k|^{q(\cdot)/r}\right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}} \leqslant 1$ which implies that $\|c\,k\|_{p(\cdot)/r} \leqslant 1$, it follows that $\left\||\Phi * f|^r\right\|_{p(\cdot)/r} \leqslant c$, by scaling arguments we get

$$\left\|\Phi * f\right\|_{p(\cdot)} \lesssim \left\|f\right\|^{\Psi,\psi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},$$

this with (2.8) proves (2.7), the proof of Theorem 2.13 is complete. □

2.4 Relation between $\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$

In this section we present the coincidence between the above function spaces and the variable Besov spaces of Almeida and Hästö..

Theorem 2.14. *Let* $\alpha : \mathbb{R}^n \to \mathbb{R}$ *and* $p, q \in \mathscr{P}_0(\mathbb{R}^n)$. *Assume that* $p \in \mathscr{P}_0^{\log}(\mathbb{R}^n)$ *and* $\alpha, \frac{1}{q} \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$. *Then*

$$\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} = B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)},$$

in the sense of equivalent quasi-norms.

Proof. Step 1. We will prove that

$$\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}.$$

Let $\{\mathscr{F}\Phi, \mathscr{F}\varphi\}$ and $\left\{\mathscr{F}\psi_j\right\}_{j\in\mathbb{N}_0}$ be two resolutions of unity and let $f \in \mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ with

$$\left\| f \right\|_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant 1. \tag{2.9}$$

We have, $\mathscr{F}\psi_0$ is supported in $\{\xi \in \mathbb{R}^n : |\xi| \leqslant 2\}$, and $\operatorname{supp} \mathscr{F}\psi_1 \subset \{\xi \in \mathbb{R}^n : 1/2 \leqslant |\xi| \leqslant 2\}$. from Lemma 2.5 for all $\nu \in \mathbb{N}_0$ we have

$$|2^{\nu\alpha(\cdot)}\psi_\nu * f|^r \lesssim \int_{2^{-\nu-2}}^{\min(1,2^{2-\nu})} \eta_{\tau,N} * |\tau^{-\alpha(\cdot)}\varphi_\tau * f|^r \frac{d\tau}{\tau} + \begin{cases} 0, & \text{if} \quad \nu \geqslant 2; \\ \eta_{1,N} * |\Phi * f|^r, & \text{if} \quad \nu = 0,1. \end{cases}$$

where $r = \min\{p^-, q^-\}$ and N can be taken arbitrary large. For any $\nu \in \mathbb{N}_0$ denote the first term on the right hand side by g_ν. Since $p/r \in \mathscr{P}^{\log}(\mathbb{R}^n)$ the convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)/r}$, we obtain

$$\left\| \left| c_1 \eta_{1,N} * |\Phi * f|^r \right|^{q(\cdot)/r} \right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}} \leqslant 1 \tag{2.10}$$

for some suitable positive constant c_1. By (2.9) we have $\varrho_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}((t^{-\alpha(\cdot)}\varphi_t * f)_{0<t\leqslant 1}) \leqslant 1$ then

$$\int_{2^{-\nu-2}}^{\min(1,2^{2-\nu})} \left\| \left(|\tau^{-\alpha(\cdot)}\varphi_\tau * f|^r \right)^{q(\cdot)/r} \right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}} \frac{d\tau}{\tau} \leqslant 1,$$

for any $\nu \in \mathbb{N}_0$. Applying Lemma 2.4, we obtain

$$\left\| |c_2 g_\nu|^{q(\cdot)/r} \right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}} \leqslant \int_{2^{-\nu-2}}^{\min(1,2^{2-\nu})} \left\| |\tau^{-\alpha(\cdot)}\varphi_\tau * f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{d\tau}{\tau} + 2^{-\nu}, \quad \nu \in \mathbb{N}_0, \tag{2.11}$$

with an appropriate choice of c_2(remark that $\left\| |\cdot|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} = \left\| |(|\cdot|^r)^{q(\cdot)/r} \right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}}$). For the positive constant $C = 1/2 \min\{c_1, c_2\}$ we get

$$\begin{aligned} \left\| |C|2^{\nu\alpha(\cdot)}\psi_\nu * f|^r|^{q(\cdot)/r} \right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}} \leqslant \left\| |c_2 g_\nu|^{q(\cdot)/r} \right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}} + \\ \begin{cases} 0, & \text{if} \quad \nu \geqslant 2; \\ \left\| \left| c_1 \eta_{1,N} * |\Phi * f|^r \right|^{q(\cdot)/r} \right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}}, & \text{if} \quad \nu = 0,1, \end{cases} \end{aligned}$$

(we remark that $\left\||\frac{1}{2}(f+g)|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant \left\||f|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} + \left\||g|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}$ if $p \in \mathscr{P}(\mathbb{R}^n)$). Taking the sum over $v \in \mathbb{N}_0$ using (2.10) and (2.11) we obtain

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((c2^{v\alpha(\cdot)}\psi_v * f)_v) \lesssim 1$$

which implies that $\left\|f\right\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim 1$.

Step 2. We will prove that

$$B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}.$$

Let $\{\mathscr{F}\Phi, \mathscr{F}\varphi\}$ and $\left\{\mathscr{F}\psi_v\right\}_{v\in\mathbb{N}_0}$ be two resolutions of unity and let $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ with

$$\left\|f\right\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant 1.$$

We have

$$\begin{aligned} \varphi_t * f &= \sum_{v=0}^{\infty} \varphi_t * \psi_v * f \\ &= \sum_{v=[\log_2(\frac{1}{2t})]}^{[\log_2(\frac{4}{t})]+1} \varphi_t * \psi_v * f + \begin{cases} 0, & \text{if } \ 0 < t \leqslant \frac{1}{4}; \\ \varphi_t * \psi_0 * f, & \text{if } \quad t > \frac{1}{4} \end{cases} \end{aligned}$$

and

$$\Phi * f = \sum_{v=0}^{1} \Phi * \psi_v * f.$$

Notice that if $v \leqslant 0$ then we put $\psi_v * f = 0$(in the last sum when $t \in [1/2, 1]$).

Let $0 < r < \min\{1, p^-\}$, $t \in]2^{-i}, 2^{-i+1}]$, $i \in \mathbb{N}$, denote $K_i = \{v \in \mathbb{N}, i-2 \leqslant v \leqslant i+3\}$, we have

$$|\varphi_t * f| \leqslant \sum_{v\in K_i} |\varphi_t * \psi_v * f| + \begin{cases} 0, & \text{if } \quad i \geqslant 3; \\ |\varphi_t * \psi_0 * f|, & \text{if } \ i = 1, 2, \end{cases}$$

we have $\eta_{t,N} \approx \eta_{v,N}$ for all $t \in]2^{-i}, 2^{-i+1}]$ and $v \in K_i$, then, $|\varphi_t| \lesssim \eta_{t,N/r} \lesssim \eta_{v,N/r}$ for any large positive N. By Lemma 2.3, for any large N, $|\psi_v * f| \leqslant c(\eta_{v,N} * |\psi_v * f|^r)^{1/r}$, it follows that for any $t \in]2^{-i}, 2^{-i+1}]$ and $v \in K_i$

$$|\varphi_t * \psi_v * f| \leqslant c\eta_{v,N/r} * (\eta_{v,N} * |\psi_v * f|^r)^{1/r},$$

by Minkowski's integral inequality (with exponent $1/r > 1$)

$$\begin{aligned} |\varphi_t * \psi_\nu * f|^r &\leqslant c[\eta_{\nu,N/r} * \eta_{\nu,N}^{1/r}]^r * |\psi_\nu * f|^r \\ &\leqslant c\eta_{\nu,N} * |\psi_\nu * f|^r \end{aligned}$$

where we used $\eta_{\nu,N/r} * \eta_{\nu,N/r} \approx \eta_{\nu,N/r}$ which follows by Lemma 1.22, also we have when $i = 1, 2$ and $t \in]2^{-i}, 2^{-i+1}]$

$$\begin{aligned} |\varphi_t * \psi_0 * f| &\leqslant c\eta_{i,N/r} * (\eta_{1,N} * |\psi_0 * f|^r)^{1/r} \\ &\leqslant c\eta_{1,N/r} * (\eta_{1,N} * |\psi_0 * f|^r)^{1/r}, \end{aligned}$$

then we get

$$|\varphi_t * \psi_0 * f|^r \leqslant c\eta_{1,N} * |\psi_0 * f|^r,$$

it follows that

$$|\varphi_t * f|^r \lesssim \sum_{\nu \in K_i} \eta_{\nu,N} * |\psi_\nu * f|^r + \begin{cases} 0, & \text{if } \quad i \geqslant 3; \\ \eta_{1,N} * |\psi_0 * f|^r, & \text{if } \quad i = 1, 2, \end{cases} \tag{2.12}$$

since $\alpha \in C^{\log}_{\text{loc}}(\mathbb{R}^n)$ then α is bounded, hence, for all $t \in]2^{-i}, 2^{-i+1}]$ and $\nu \in K_i$, $t^{-\alpha(\cdot)} \approx 2^{\nu\alpha(\cdot)}$, and if $i = 1, 2$ then $t^{-\alpha(\cdot)} \leqslant c$, by Lemma 1.23, for all $t \in]2^{-i}, 2^{-i+1}]$ and $i \in \mathbb{N}$

$$|t^{-\alpha(\cdot)}\varphi_t * f|^r \leqslant c_1 \sum_{\nu \in K_i} \eta_{\nu,N'} * |2^{\nu\alpha(\cdot)}\psi_\nu * f|^r + \begin{cases} 0, & \text{if } \quad i \geqslant 3; \\ c_1\eta_{1,N'} * |\psi_0 * f|^r, & \text{if } \quad i = 1, 2, \end{cases}$$

where $N' = N - c_{\log}(r\alpha)$ still can be taken large(remark $\eta_{1,N} \leqslant \eta_{1,N'}$). Now observe that for the constant c where $c^r = 1/6c_1^{-1}c_2$(the number of terms in the last estimate is always $\leqslant 6$) and c_2 will be determined later, we have

$$\begin{aligned} \int_0^1 \left\| |c\ t^{-\alpha(\cdot)}\varphi_t * f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{dt}{t} &= \sum_{i=1}^{\infty} \int_{2^{-i}}^{2^{1-i}} \left\| \left(|c\,t^{-\alpha(\cdot)}\varphi_t * f|^r\right)^{q(\cdot)/r} \right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}} \frac{dt}{t} \\ &\leqslant c \sum_{i=1}^{\infty} \sum_{\nu \in K_i} \left\| \left(c_2\ \eta_{\nu,N'} * |2^{\nu\alpha(\cdot)}\psi_\nu * f|^r\right)^{q(\cdot)/r} \right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}} + A \\ &= c \sum_{j=-2}^{3} \sum_{i=3}^{\infty} \left\| \left(c_2\ \eta_{i+j,N'} * |2^{(i+j)\alpha(\cdot)}\psi_{i+j} * f|^r\right)^{q(\cdot)/r} \right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}} + B \\ &\leqslant c\ \varrho_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}((c_2\ \eta_{\nu,N'} * |2^{\nu\alpha(\cdot)}\psi_\nu * f|^r)_\nu), \end{aligned}$$

where

$$A = c \left\| \left(c_2\ \eta_{1,N'} * |\psi_0 * f|^r\right)^{q(\cdot)/r} \right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}}$$

and

$$B = A + c\sum_{\nu\in K_1}\left\|\left(c_2\,\eta_{\nu,N'} * |2^{\nu\alpha(\cdot)}\psi_\nu * f|^r\right)^{q(\cdot)/r}\right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}} + c\sum_{\nu\in K_2}\cdots,$$

by taking $N' > n + c_{\log}(1/(q/r))$ Lemma 3.1 yields

$$\begin{aligned}\left\|\left(\eta_{\nu,N'} * |2^{\nu\alpha(\cdot)}\psi_\nu * f|^r\right)_\nu\right\|_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})} &\leqslant c\left\|\left(|2^{\nu\alpha(\cdot)}\psi_\nu * f|^r\right)_\nu\right\|_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}\\ &\leqslant c\left\|f\right\|^r_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}\\ &\leqslant c\end{aligned}$$

therefore, there exists a positive constant c such that

$$\varrho_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}((c\ \eta_{\nu,N'} * |2^{\nu\alpha(\cdot)}\psi_\nu * f|^r)_\nu) \leqslant 1$$

taking c_2 to be this constant we find

$$\int_0^1 \left\||c\ t^{-\alpha(\cdot)}\varphi_t * f|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\frac{dt}{t} \lesssim 1. \tag{2.13}$$

Similarly, we have for large N

$$|\Phi * f|^r \leqslant c\sum_{\nu=0}^{1}\eta_{1,N} * |\psi_\nu * f|^r, \tag{2.14}$$

then, we get

$$\begin{aligned}\left\||c\Phi * f|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} &\leqslant \sum_{\nu=0}^{1}\|(c\eta_{\nu,N} * |2^{\nu\alpha(\cdot)}\psi_\nu * f|^r)^{q(\cdot)/r}\|_{\frac{p(\cdot)/r}{q(\cdot)/r}}\\ &\leqslant \varrho_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}((c\ \eta_{\nu,N} * |2^{\nu\alpha(\cdot)}\psi_\nu * f|^r)_\nu)\\ &\leqslant 1\end{aligned}$$

for some suitable positive constant c, hence, $\left\|\Phi * f\right\|_{p(\cdot)} \leqslant c$, this with (2.13) proves that $\left\|\Phi * f\right\|_{p(\cdot)} + \left\|(t^{-\alpha(\cdot)}\varphi_t * f)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}} \leqslant c$. The proof is complete. □

2.5 Characterization of $\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$

Let $a > 0$, $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $f \in \mathscr{S}'(\mathbb{R}^n)$. Then we define the Peetre maximal function as follows:

$$\varphi_t^{*,a}t^{-\alpha(\cdot)}f(x) := \sup_{y\in\mathbb{R}^n}\frac{t^{-\alpha(y)}\left|\varphi_t * f(y)\right|}{\left(1 + t^{-1}\left|x-y\right|\right)^a}, \qquad t > 0$$

and

$$\Phi^{*,a} f(x) := \sup_{y\in\mathbb{R}^n} \frac{\left|\Phi * f(y)\right|}{\left(1+\left|x-y\right|\right)^a}.$$

The next theorem is from [21].

Theorem 2.15. *Let $p,q\in\mathscr{P}_0^{\log}(\mathbb{R}^n)$, $\alpha\in C_{\mathrm{loc}}^{\log}(\mathbb{R}^n)$, and $a>\frac{n}{p^-}$. Then*

$$\left\|\left(\psi_\nu^{*,a}2^{\nu\alpha(\cdot)}f\right)_{\nu\geqslant 0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$$

is an equivalent quasi-norm in $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, where

$$\psi_\nu^{*,a}2^{\nu\alpha(\cdot)}f(x) := \sup_{y\in\mathbb{R}^n} \frac{2^{\nu\alpha(y)}\left|\psi_\nu * f(y)\right|}{\left(1+2^\nu\left|x-y\right|\right)^a}, \qquad \nu\in\mathbb{N}_0.$$

We now present a fundamental characterization of the spaces under consideration.

Theorem 2.16. *Let $p,q\in\mathscr{P}_0^{\log}(\mathbb{R}^n)$, and $\alpha:\mathbb{R}^n\to\mathbb{R}$ such that $\alpha\in C_{\mathrm{loc}}^{\log}(\mathbb{R}^n)$, and $a>\frac{n}{p^-}$. Then*

$$\left\|f\right\|^{\blacktriangledown}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left\|\Phi^{*,a}f\right\|_{p(\cdot)} + \left\|(\varphi_t^{*,a}t^{-\alpha(\cdot)}f)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}$$

is an equivalent quasi-norm in $\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$.

Proof. It is easy to see that for any $f\in\mathscr{S}'(\mathbb{R}^n)$ with $\left\|f\right\|^{\blacktriangledown}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}<\infty$ and any $x\in\mathbb{R}^n$ we have

$$t^{-\alpha(x)}\left|\varphi_t * f(x)\right| \leqslant \varphi_t^{*,a}t^{-\alpha(\cdot)}f(x).$$

This shows that $\left\|f\right\|_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant \left\|f\right\|^{\blacktriangledown}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$. We will prove that there is a constant $C>0$ such that for every $f\in\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$

$$\left\|f\right\|^{\blacktriangledown}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant C\left\|f\right\|_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}. \tag{2.15}$$

First we will prove that

$$\left\|f\right\|^{\blacktriangledown}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant c\left\|\left(\psi_\nu^{*,a}2^{\nu\alpha(\cdot)}f\right)_{\nu\geqslant 0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \tag{2.16}$$

By (2.12) , for all $t\in[2^{-i},2^{-i+1}]$, $i\in\mathbb{N}$,

$$|\varphi_t^{*,a}t^{-\alpha(\cdot)}f|^r \lesssim \sum_{\nu\in K_i}|\psi_\nu^{*,a}2^{\nu\alpha(\cdot)}f|^r + \begin{cases} 0, & \text{if} \quad i\geqslant 3; \\ |\psi_0^{*,a}f|^r, & \text{if} \quad i=1,2, \end{cases}$$

and by (2.14)

$$|\Phi^{*,a}f|^r \leqslant c\sum_{\nu=0}^{1}|\psi_\nu^{*,a}f|^r,$$

with similar arguments used to the prove Step 2 of the proof of theorem 2.14 we get (2.16).

Now, we have the following chain of estimates

$$\left\| \left(\psi_\nu^{*,a} 2^{\nu\alpha(\cdot)} f\right)_{\nu\geqslant 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \lesssim \left\| (2^{\nu\alpha(\cdot)} \psi_\nu * f)_\nu \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \lesssim \left\| f \right\|_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$$

where the first estimate follows by Theorem 2.15 and the second by Theorem 2.14, this proves (2.15). the proof is complete. □

2.6 Local mean characterization of $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$

In order to formulate the main result of this section, let us consider $k_0, k \in \mathscr{S}(\mathbb{R}^n)$ and $S \geqslant -1$ an integer such that for an $\varepsilon > 0$

$$|\mathscr{F} k_0(\xi)| > 0 \quad \text{for} \quad |\xi| < 2, \tag{2.17}$$

$$|\mathscr{F} k(\xi)| > 0 \quad \text{for} \quad \frac{1}{2} < |\xi| < 2 \tag{2.18}$$

and

$$\int_{\mathbb{R}^n} x^\alpha k(x) dx = 0 \quad \text{for any} \quad |\alpha| \leqslant S. \tag{2.19}$$

where, when $S = -1$, the above last requirement disappears automatically. Here (2.17) and (2.18) are Tauberian conditions, while (2.19) states that moment conditions on k. We recall the notation

$$k_t(x) := t^{-n} k(t^{-1}x) \quad \text{for} \quad t > 0.$$

For any $a > 0$, $f \in \mathscr{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we denote

$$k_t^{*,a} t^{-\alpha(\cdot)} f(x) := \sup_{y\in\mathbb{R}^n} \frac{t^{-\alpha(y)} \left| k_t * f(y) \right|}{\left(1 + t^{-1} \left| x - y \right|\right)^a}, \quad t > 0.$$

The next theorem is from [21].

Theorem 2.17. *Let $p, q \in \mathscr{P}_0^{\log}(\mathbb{R}^n)$ with $q^+ < \infty$, $\alpha \in C^{\log}_{\text{loc}}(\mathbb{R}^n)$ with $\alpha^+ < S + 1$, and $a > \frac{n}{p^-}$. Then*

$$\left\| \left(k_\nu^{*,a} 2^{\nu\alpha(\cdot)} f\right)_{\nu\geqslant 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$$

is an equivalent quasi-norm in $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$.

We are now able to state the so called local mean characterization of $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ spaces.

Theorem 2.18. *Let* $p, q \in \mathscr{P}_0^{\log}(\mathbb{R}^n)$ *with* $q^+ < \infty$, $\alpha \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$, $a > \frac{n}{p^-}$ *and* $\alpha^+ < S+1$. *Then*

$$\left\|f\right\|'_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left\|k_0^{*,a} f\right\|_{p(\cdot)} + \left\|(k_t^{*,a}\, t^{-\alpha(\cdot)} f)_{0<t\leqslant 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}$$

is an equivalent quasi-norm on $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$.

Proof. The idea of the proof is from V. S. Rychkov [42]. The proof is divided into three steps. *Step* 1. Take any pair of functions φ_0 and $\varphi \in \mathscr{S}(\mathbb{R}^n)$ such that

$$\begin{aligned} \left|\mathscr{F}\varphi_0(\xi)\right| > 0 \quad &\text{for} \quad |\xi| < 2\,, \\ \left|\mathscr{F}\varphi(\xi)\right| > 0 \quad &\text{for} \quad \frac{1}{2} < |\xi| < 2. \end{aligned}$$

We prove that there is a constant $c > 0$ such that for any $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$

$$\left\|f\right\|'_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant c \left\| \left(\varphi_j^{*,a} 2^{j\alpha(\cdot)} f\right)_{j\geqslant 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}. \tag{2.20}$$

Let Λ, $\lambda \in \mathscr{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp} \mathscr{F}\Lambda \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\}, \quad \operatorname{supp} \mathscr{F}\lambda \subset \{\xi \in \mathbb{R}^n : 1/2 < |\xi| < 2\}$$

and

$$\mathscr{F}\Lambda(\xi)\mathscr{F}\varphi_0(\xi) + \sum_{j=1}^{\infty} \mathscr{F}\lambda(2^{-j}\xi)\mathscr{F}\varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n.$$

In particular, for any $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ the following identity is true:

$$f = \Lambda * \varphi_0 * f + \sum_{j=1}^{\infty} \lambda_j * \varphi_j * f,$$

where

$$\varphi_j := 2^{jn}\varphi(2^j\cdot) \quad \text{and} \quad \lambda_j := 2^{jn}\lambda(2^j\cdot), \quad j \in \mathbb{N}.$$

Hence we can write

$$k_t * f = k_t * \Lambda * \varphi_0 * f + \sum_{j=1}^{\infty} k_t * \lambda_j * \varphi_j * f, \quad t \in (0,1].$$

Let $t \in]2^{-i}, 2^{1-i}]$, $i \in \mathbb{N}$. First, let $j \leqslant i$, $j \in \mathbb{N}$. Writing for any $z \in \mathbb{R}^n$

$$k_t * \lambda_j(z) = 2^{jn} k_{2^j t} * \lambda(2^j z),$$

we deduce from Lemma 1.24 that for any $N > 0$ there is a constant $c > 0$ independent of t and j such that

$$\left|k_t * \lambda_j(z)\right| \leqslant c\ \left(2^j t\right)^{S+1} \eta_{j,N}(z), \quad z \in \mathbb{R}^n.$$

This together with Lemma 1.23 yield that

$$t^{-\alpha(y)}\left|k_t * \lambda_j * \varphi_j * f(y)\right|,$$

can be estimated from above by

$$c\,2^{(j-i)(S+1-\alpha^+)}\varphi_j^{*,a}2^{j\alpha(\cdot)}f(y)\int_{\mathbb{R}^n}\eta_{j,N-c_{\log}(\alpha)-a}(y-z)dz \lesssim 2^{(j-i)(S+1-\alpha^+)}\varphi_j^{*,a}2^{j\alpha(\cdot)}f(y)$$

for any $N > n + a + c_{\log}(\alpha)$ any $y \in \mathbb{R}^n$ and any $j \leqslant i$. Next, let $j > i$. Then, again by Lemma 1.24, we have for any $z \in \mathbb{R}^n$ and any $L > 0$

$$\left|k_t * \lambda_j(z)\right| = t^{-n}\left|k * \lambda_{\frac{1}{2^j t}}\left(\frac{z}{t}\right)\right| \leqslant c\left(\frac{1}{2^j t}\right)^{M+1}\eta_{t,L}(z),$$

where an integer $M \geqslant -1$ is taken arbitrarily large, since $D^{\beta}\mathscr{F}\lambda(0) = 0$ for all β. Hence, again with Lemma 1.23,

$$\begin{aligned}
& t^{-\alpha(y)}\left|k_t * \lambda_j * \varphi_j * f(y)\right| \\
\leqslant\ & t^{-\alpha(y)}\int_{\mathbb{R}^n}\left|k_t * \lambda_j(y-z)\right|\left|\varphi_j * f(z)\right| dz \\
\lesssim\ & 2^{(i-j)(M+1+\alpha^-)}\varphi_j^{*,a}2^{j\alpha(\cdot)}f(y)\int_{\mathbb{R}^n}\left(1+2^j\left|y-z\right|\right)^{c_{\log}(\alpha)+a}\eta_{i,L}(y-z)dz.
\end{aligned}$$

We have for any $j > i$

$$\left(1+2^j\left|z\right|\right)^{c_{\log}(\alpha)+a} \leqslant 2^{(j-i)(c_{\log}(\alpha)+a)}\left(1+2^i\left|z\right|\right)^{c_{\log}(\alpha)+a}.$$

Then, by taking $L > n + a + c_{\log}(\alpha)$,

$$t^{-\alpha(y)}\left|k_t * \lambda_j * \varphi_j * f(y)\right| \lesssim 2^{(i-j)(M+1+\alpha^- - c_{\log}(\alpha)-a)}\varphi_j^{*,a}2^{j\alpha(\cdot)}f(y).$$

Let us take $M > c_{\log}(\alpha) - \alpha^- + 2a$ to estimate the last expression by

$$c\ 2^{(i-j)(a+1)}\varphi_j^{*,a}2^{j\alpha(\cdot)}f(y),$$

where $c>0$ is independent of i, j and f. Using the fact that for any $z\in\mathbb{R}^n$ and any $N>0$

$$|k_t*\Lambda(z)|\leqslant c\ t^{S+1}\eta_{1,N}(z),$$

we obtain by similar arguments that for any $t\in]2^{-i},2^{1-i}]$, $i\in\mathbb{N}$

$$\sup_{y\in\mathbb{R}^n}\frac{t^{-\alpha(y)}\left|k_t*\Lambda*\varphi_0*f(y)\right|}{(1+t^{-1}\left|x-y\right|)^a}\leqslant C\ 2^{-i(S+1-\alpha^+)}\varphi_0^{*,a}f(x).$$

Further, note that for all $x,y\in\mathbb{R}^n$ all $t\in]2^{-i},2^{1-i}]$, $i\in\mathbb{N}$ and any $j\in\mathbb{N}$

$$\begin{aligned}\varphi_j^{*,a}2^{j\alpha(\cdot)}f(y) &\leqslant \varphi_j^{*,a}2^{j\alpha(\cdot)}f(x)(1+2^j\left|x-y\right|)^a\\ &\leqslant \varphi_j^{*,a}2^{j\alpha(\cdot)}f(x)\max(1,2^{(j-i)a})(1+2^i\left|x-y\right|)^a.\end{aligned}$$

Hence

$$\sup_{y\in\mathbb{R}^n}\frac{t^{-\alpha(y)}\left|k_t*\lambda_j*\varphi_j*f(y)\right|}{(1+t^{-1}\left|x-y\right|)^a}\leqslant C\ \varphi_j^{*,a}2^{j\alpha(\cdot)}f(x)\times\begin{cases}2^{(j-i)(S+1-\alpha^+)} & \text{if}\quad 1\leqslant j\leqslant i,\\ 2^{i-j} & \text{if}\quad j>i.\end{cases}$$

Therefore for all $f\in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, any $x\in\mathbb{R}^n$ and any $t\in]2^{-i},2^{1-i}]$, $i\in\mathbb{N}$, we get

$$\begin{aligned}&k_t^{*,a}t^{-\alpha(\cdot)}f(x)\\ \lesssim\ & 2^{-i(S+1-\alpha^+)}\varphi_0^{*,a}f(x)+C\sum_{j=1}^{\infty}\min\left(2^{(j-i)(S+1-\alpha^+)},2^{i-j}\right)\varphi_j^{*,a}2^{j\alpha(\cdot)}f(x)\\ =\ & C\sum_{j=0}^{\infty}\min\left(2^{(j-i)(S+1-\alpha^+)},2^{(i-j)}\right)\varphi_j^{*,a}2^{j\alpha(\cdot)}f(x)\\ =\ & C\Psi_i(x).\end{aligned}$$

Assume that the right hand side of (2.20) is less than or equal one. We have

$$\begin{aligned}\int_0^1\left\||k_t^{*,a}t^{-\alpha(\cdot)}f|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\frac{dt}{t} &= \sum_{i=1}^{\infty}\int_{2^{-i}}^{2^{1-i}}\left\||k_t^{*,a}t^{-\alpha(\cdot)}f|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\frac{dt}{t}\\ &\lesssim \sum_{i=1}^{\infty}\left\||c\Psi_i|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\end{aligned}$$

for some positive constant c. The last term on the right hand side is less than or equal one if and only if

$$\left\|(c_1\Psi_i)_i\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}\leqslant 1$$

for some suitable positive constant c_1, which follows by Lemma 2.6 of and the fact that

$\alpha^+ < S+1$. Also we have for any $z \in \mathbb{R}^n$, any $N>0$ and any integer $M \geqslant -1$

$$\left|k_0 * \lambda_j(z)\right| \leqslant c\, 2^{-j(M+1)} \eta_{j,N}(z) \quad \text{and} \quad |k_0 * \Lambda(z)| \leqslant c\ \eta_{1,N}(z).$$

As before, we get for any $x \in \mathbb{R}^n$

$$k_0^{*,a} f(x) \leqslant C \varphi_0^{*,a} f(x) + C \sum_{j=1}^{\infty} 2^{-j} \varphi_j^{*,a} 2^{j\alpha(\cdot)} f(x). \tag{2.21}$$

In (2.21) taking the $L^{p(\cdot)}$-norm and using the embedding $\ell^{q(\cdot)}(L^{p(\cdot)}) \hookrightarrow \ell^{\infty}(L^{p(\cdot)})$ we get (2.20).
Step 2. Let $\left\{\mathscr{F}\varphi_j\right\}_{j\in\mathbb{N}_0} \subset \mathscr{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp} \mathscr{F}\varphi \subset \left\{\xi \in \mathbb{R}^n : 1/2 \leqslant |\xi| \leqslant 2\right\}$$

and

$$\operatorname{supp} \mathscr{F}\varphi_0 \subset \left\{\xi \in \mathbb{R}^n : |\xi| \leqslant 2\right\},$$

with $\varphi_j = 2^{jn}\varphi(2^j\cdot)$, $j \in \mathbb{N}$. We will prove that

$$\left\| \left(2^{j\alpha(\cdot)}(\varphi_j * f)\right)_{j\geqslant 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \lesssim \left\|f\right\|'_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \tag{2.22}$$

Let Λ, $\lambda \in \mathscr{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp} \mathscr{F}\Lambda \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\varepsilon\}, \quad \operatorname{supp} \mathscr{F}\lambda \subset \{\xi \in \mathbb{R}^n : \varepsilon/2 < |\xi| < 2\varepsilon\},$$

$$\mathscr{F}\Lambda(\xi)\mathscr{F}k_0(\xi) + \int_0^1 \mathscr{F}\lambda(\tau\xi)\mathscr{F}k(\tau\xi)\frac{d\tau}{\tau} = 1, \quad \xi \in \mathbb{R}^n.$$

In particular, for any $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ the following identity is true:

$$f = \Lambda * k_0 * f + \int_0^1 \lambda_\tau * k_\tau * f \frac{d\tau}{\tau}.$$

Hence we can write

$$\varphi_j * f = \int_{2^{-j-2}}^{\min\{1,2^{-j+2}\}} \varphi_j * \lambda_\tau * k_\tau * f \frac{d\tau}{\tau} + \left\{ \begin{array}{ccc} 0, & \text{if} & j \geqslant 2; \\ \varphi_j * \Lambda * k_0 * f, & \text{if} & j = 0,1. \end{array} \right. .$$

Using the fact that

$$\max(|k_\tau * \lambda_\tau(z)|, |\varphi_j * \lambda_\tau(z)|) \lesssim \eta_{j,N}(z), \quad z \in \mathbb{R}^n, 2^{-j-2} \leqslant \tau \leqslant \min\{1, 2^{-j+2}\}, j \in \mathbb{N},$$

and Lemma 1.23, with $N > 0$ large enough, we easily obtain

$$2^{j\alpha(y)}\left|\varphi_j * \lambda_\tau * k_\tau * f(y)\right| \lesssim \min(k^{*,a}_\tau \tau^{-\alpha(\cdot)} f(y), \varphi^{*,a}_j 2^{j\alpha(y)} f(y))$$

for all $y \in \mathbb{R}^n$, $2^{-j+2} \leqslant \tau \leqslant \min\{1, 2^{-j+2}\}$ and $j \in \mathbb{N}$, if $j = 1$

$$2^{j\alpha(y)}\left|\varphi_j * \Lambda * k_0 * f(y)\right| \lesssim \min(k^{*,a}_0 f(y), \varphi^{*,a}_j 2^{j\alpha(y)} f(y))$$

Therefore for any $0 < r < 1$

$$2^{j\alpha(y)}|\varphi_j * f(y)| \lesssim \left(\varphi^{*,a}_j 2^{j\alpha(\cdot)} f(y)\right)^{1-r} \Bigg(\int_{2^{-j-2}}^{\min\{1,2^{-j+2}\}} \left(k^{*,a}_\tau \tau^{-\alpha(\cdot)} f(y)\right)^r \frac{d\tau}{\tau} + \begin{cases} 0, & \text{if} \quad j \geqslant 2; \\ (k^{*,a}_0 f(y))^r, & \text{if} \quad j = 1. \end{cases} \Bigg)$$

which yields that

$$\varphi^{*,a}_j 2^{j\alpha(\cdot)} f(x) \lesssim \left(\varphi^{*,a}_j 2^{j\alpha(\cdot)} f(x)\right)^{1-r} \Bigg(\int_{2^{-j-2}}^{\min\{1,2^{-j+2}\}} \left(k^{*,a}_\tau \tau^{-\alpha(\cdot)} f(x)\right)^r \frac{d\tau}{\tau} + \begin{cases} 0, & \text{if} \quad j \geqslant 2; \\ (k^{*,a}_0 f(x))^r, & \text{if} \quad j = 1. \end{cases} \Bigg)$$

This estimate gives

$$\left(\varphi^{*,a}_j 2^{j\alpha(\cdot)} f(x)\right)^r \lesssim \int_{2^{-j-2}}^{\min\{1,2^{-j+2}\}} \left(k^{*,a}_\tau \tau^{-\alpha(\cdot)} f(x)\right)^r \frac{d\tau}{\tau} + \begin{cases} 0, & \text{if} \quad j \geqslant 2; \\ (k^{*,a}_0 f(x))^r, & \text{if} \quad j = 1. \end{cases}$$

and

$$2^{j\alpha(x)r}|\varphi_j * f(x)|^r \lesssim \int_{2^{-j-2}}^{\min\{1,2^{-j+2}\}} \left(k^{*,a}_\tau \tau^{-\alpha(\cdot)} f(x)\right)^r \frac{d\tau}{\tau} + \begin{cases} 0, & \text{if} \quad j \geqslant 2; \\ (k^{*,a}_0 f(x))^r, & \text{if} \quad j = 1. \end{cases} \tag{2.23}$$

but if $\varphi^{*,a}_j 2^{j\alpha(\cdot)} f(x) < \infty$. Using a combination of the arguments used in Lemma 2.5, we get (2.23) for all $0 < r < 1$, $a > 0$ and all $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. Similarly we obtain

$$|\varphi_0 * f(x)|^r \lesssim \left(k^{*,a}_0 f(x)\right)^r + \int_{\frac{1}{4}}^{1} \left(k^{*,a}_\tau \tau^{-\alpha(\cdot)} f(x)\right)^r \frac{d\tau}{\tau}. \tag{2.24}$$

Let $g_j = \int_{2^{-j-2}}^{\min\{1,2^{-j+2}\}} \left(k^{*,a}_\tau \tau^{-\alpha(\cdot)} f(x)\right)^r \frac{d\tau}{\tau}$ and $0 < r < \min\{1, p^-, q^-\}$, suppose that the right hand side of (2.22) is less than or equal one , with similar arguments used to prove *(i)* of

Lemma 2.4 we prove that there exists a positive constant c such that for all $j \in \mathbb{N}$

$$\left\| |c g_j|^{q(\cdot)/r} \right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}} \leqslant \int_{2^{-j-2}}^{\min\{1,2^{-j+2}\}} \left\| |k^{*,a}_{\tau} \tau^{-\alpha(\cdot)} f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{d\tau}{\tau},$$

also we have

$$\left\| \left| c_1 (k^{*,a}_0 f)^r \right|^{q(\cdot)/r} \right\|_{\frac{p(\cdot)/r}{q(\cdot)/r}} \leqslant 1.$$

Following similar arguments as in Step 1 of the proof of Theorem 2.14 we prove that

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((c 2^{j\alpha(\cdot)} \psi_j * f)_{\nu \geqslant 1}) \lesssim 1,$$

by (2.24) we prove that

$$\left\| \left| c(\varphi_0 * f) \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1,$$

which implies that $\left\| \left(2^{j\alpha(\cdot)}(\varphi_j * f)\right)_{j\geqslant 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c$, this proves (2.22) whenever its right hand side is less than or equal one, by scaling arguments we get (2.22).

Step 3. We will prove in this step that for all $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ the following estimates are true:

$$\|f\|'_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|'_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}. \tag{2.25}$$

Let $\varphi_0, \varphi \in \mathscr{S}(\mathbb{R}^n)$ satisfy (2.17), (2.18) and (2.19). The first inequality follows by the chain of the estimates

$$\|f\|'_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \left\| \left(\varphi^{*,a}_j 2^{j\alpha(\cdot)} f\right)_{j\geqslant 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \lesssim \|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},$$

where the first inequality is (2.20), see *Step 1*, and the second inequality follows by Theorem 2.17. Now the second inequality in (2.25) can be obtained by the following chain of the estimates

$$\|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \left\| \left(2^{j\alpha(\cdot)}(\varphi_j * f)\right)_{j\geqslant 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \lesssim \|f\|'_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},$$

where the first inequality is obvious and the second inequality is (2.22), see Step 2. Thus, Theorem 2.18 is proved. □

CHAPTER 3

COMMUTATOR ESTIMATES FOR VECTOR FIELDS ON VARIABLE TRIEBEL-LIZORKIN SPACES

3.1 Introduction

We aim in this chapter to estimate the commutator

$$[V\cdot\nabla,\Delta_j]f=\sum_{k=1}^{n}V_k\partial_k\Delta_j f-\Delta_j(V_k\partial_k f), \tag{3.1}$$

in variable Triebel-Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$. In (3.1), $\Delta_j f=\varphi_j*f, j\in\mathbb{N}_0$, $V=(V_1,\dots,V_n)$ is a smooth vector field in $\mathbb{R}^n$ and $(\mathscr{F}\varphi_j)_{j\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity, see Section 3.2.

The motivation to do such an estimate is that the estimation of the commutator (3.1) is one of the main tools to study the Euler equations over different function spaces such as Lebesgue spaces, Sobolev spaces, Besov spaces, and Triebel-Lizorkin spaces. Euler equations for the homogeneous incompressible fluid flows is:

$$\begin{cases} \frac{\partial u}{\partial t}+(u\cdot\nabla u)u=-\nabla p, & (x,t)\in\mathbb{R}^n\times(0,\infty),\\ \operatorname{div} u=0, & (x,t)\in\mathbb{R}^n\times(0,\infty),\\ u(x,0)=u_0(x), & x\in\mathbb{R}^n, \end{cases}$$

where $u=(u_1,\dots,u_n)$ is the velocity of the fluid flows, p is the scalar pressure, and u_0 is the given initial velocity satisfying $\operatorname{div} u_0 = 0$. For the Euler equations, V in (3.1) can be taken the velocity of the fluid flows, see for example [8, Lemma 2.100] and [11, p. 663].

The theory of Euler equation in function spaces has been developed in detail in [8] but has a longer history already including many contributors.

Allowing s, p and q to be functions over $\mathbb{R}^n$ will raise extra difficulties which, in general, are overcome by imposing some regularity assumptions on these exponents. Recently, in [27] were presented new estimates of (3.1) in weighted and variable exponent Lebesgue, Triebel-Lizorkin $F^s_{p(\cdot),q}$, and Besov spaces $B^s_{p(\cdot),q}$, but with s and q constant. These estimates are obtained under no vanishing assumptions on the divergence of the vector field. which is based on the boundedness of the maximal function on the spaces $L^{p(\cdot)}(\ell^q)$ and $\ell^q(L^{p(\cdot)})$. Since the maximal operator is in general not bounded on $L^{p(\cdot)}(\ell^{q(\cdot)})$, to estimate (3.1) on $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ we are forced to introduce some new methods. In the case of constant exponents we recover the results of [27].

The chapter is organized as follows. First we give some preliminaries where we fix some notation and recall some basic facts on the variable Triebel-Lizorkin spaces. Also we give the key technical lemmas needed in the proofs of the main statements. The main statements and their proofs are formulated in Section 3.2.

3.2 The results and their proofs

Lemma 3.1. *Let $p, q \in \mathscr{P}^{\log}$ with $1 < p^- \leqslant p^+ < \infty$ and $1 < q^- \leqslant q^+ < \infty$. For $m > n$, there exists $c > 0$ such that*

$$\left\|(\eta_{j,m} * f_j)_{j\in\mathbb{N}_0}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leqslant c \left\|(f_j)_{j\in\mathbb{N}_0}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}.$$

The proof of this lemma is given in [20, Section 5]. The next lemma is a Hardy-type inequality which is easy to prove.

Lemma 3.2. *Let $0 < a < 1$ and $1 \leqslant q \leqslant \infty$. Let $(\varepsilon_k)_{k\in\mathbb{N}_0}$ be a sequence of positive real numbers and denote*

$$\delta_k = \sum_{j=0}^{k} a^{k-j}\varepsilon_j, \quad \text{and} \quad \eta_k = \sum_{j=k}^{\infty} a^{j-k}\varepsilon_j, \quad k \in \mathbb{N}_0.$$

Then there exists a constant $c > 0$ depending only on a and q such that

$$\Big(\sum_{k=0}^{\infty}\delta_k^q\Big)^{\frac{1}{q}} + \Big(\sum_{k=0}^{\infty}\eta_k^q\Big)^{\frac{1}{q}} \leqslant c\,\Big(\sum_{k=0}^{\infty}\varepsilon_k^q\Big)^{\frac{1}{q}}.$$

Lemma 3.3. *Let $p, q \in \mathscr{P}$ with $1 < p^- \leqslant p^+ < \infty, 1 < q^- \leqslant q^+ < \infty$ and $\big(f_j\big)_{j\in\mathbb{N}_0} \in L^{p(\cdot)}(\ell^{q(\cdot)})$. Then*

$$\big\|(f_j)_{j\in\mathbb{N}_0}\big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \approx \sup \int_{\mathbb{R}^n} \sum_{j=0}^{\infty} |f_j(x)||g_j(x)| dx,$$

where the supremum is taken over all sequences of functions $\big(g_j\big)_{j\in\mathbb{N}_0} \in L^{p'(\cdot)}(\ell^{q'(\cdot)})$ such that

$$\big\|(g_j)_{j\in\mathbb{N}_0}\big\|_{L^{p'(\cdot)}(\ell^{q'(\cdot)})} \leqslant 1.$$

Proof. Let

$$\big\|(f_j)_{j\in\mathbb{N}_0}\big\|^{\bullet}_{L^{p(\cdot)}(\ell^{q(\cdot)})} = \sup \int_{\mathbb{R}^n} \sum_{j=0}^{\infty} |f_j(x)||g_j(x)| dx.$$

Since $\frac{1}{q(\cdot)} + \frac{1}{q'(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$, by Hölder's inequality,

$$\big\|(f_j)_{j\in\mathbb{N}_0}\big\|^{\bullet}_{L^{p(\cdot)}(\ell^{q(\cdot)})} \lesssim \big\|(f_j)_{j\in\mathbb{N}_0}\big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}.$$

Now let us prove the converse inequality. By the scaling argument, it suffices to consider the case $\big\|(f_j)_{j\in\mathbb{N}_0}\big\|^{\bullet}_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leqslant 1$ and prove that $\big\|(f_j)_{j\in\mathbb{N}_0}\big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leqslant 1$. Since $(f_j)_{j\in\mathbb{N}_0} \in L^{p(\cdot)}(\ell^{q(\cdot)})$ we have

$$\int_{\mathbb{R}^n} \big\|(f_j(x))_{j\in\mathbb{N}_0}\big\|^{p(x)}_{\ell^{q(x)}} dx < \infty.$$

Assume, for the sake of contradiction, that

$$\int_{\mathbb{R}^n} \big\|\big(f_j(x)\big)_{j\in\mathbb{N}_0}\big\|^{p(x)}_{\ell^{q(x)}} dx > 1.$$

Then by the continuity of the modular there exists $d > 1$ such that

$$\int_{\mathbb{R}^n} \big\|(d^{-1} f_j(x))_{j\in\mathbb{N}_0}\big\|^{p(x)}_{\ell^{q(x)}} dx = 1.$$

Define

$$g_j(x) = \frac{|\frac{f_j(x)}{d}|^{q(x)-1}}{\big(\sum_{v=0}^{\infty} |\frac{f_v(x)}{d}|^{q(x)}\big)^{1-\frac{p(x)}{q(x)}}}, \quad j \in \mathbb{N}_0, x \in \mathbb{R}^n,$$

which leads to

$$\big\|(f_j)_{j\in\mathbb{N}_0}\big\|^{\bullet}_{L^{p(\cdot)}(\ell^{q(\cdot)})} \geqslant \int_{\mathbb{R}^n} \sum_{j=0}^{\infty} |f_j(x)||g_j(x)| dx = d \int_{\mathbb{R}^n} \big\|(d^{-1} f_j(x))_{j\in\mathbb{N}_0}\big\|^{p(x)}_{\ell^{q(x)}} dx > 1,$$

which contradicts our assumption. The proof is completed by applying the unit ball property (1.2). □

Let $(\mathscr{F}\varphi_j)_{j\in\mathbb{N}_0}$ be a smooth dyadic resolution of unity. Let $\Psi\in\mathscr{S}(\mathbb{R}^n)$ and

$$\Lambda_{j,m}(f,g)(x)=\int_{\mathbb{R}^{2n}}\varphi_j(x-y)(\Psi_m(x-z)-\Psi_m(y-z))f(y)\varphi_m*g(z)dydz,$$

where $j,m\in\mathbb{N}_0$ and $\Psi_m=2^m\Psi(2^m\cdot)$.

Lemma 3.4. *Let $s\in C^{\log}_{\mathrm{loc}}, a\in\mathbb{R}, p,p_1,p_2,q\in\mathscr{P}^{\log}$ with $1<p^-\leqslant p^+<\infty, 1<p_1^-\leqslant p_1^+\leqslant\infty, 1<p_2^-\leqslant p_2^+<\infty$ and $1<q^-\leqslant q^+<\infty$. Assume that $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and $(s+a)^->0$. Then*

$$\sum_{j=0}^{\infty}\sum_{m=j}^{\infty}\int_{\mathbb{R}^n}2^{ja}|\Lambda_{j,m}(f,g)(x)||h_j(x)|dx\lesssim\|f\|_{p_1(\cdot)}\|g\|_{F^{s(\cdot)+a}_{p_2(\cdot),q(\cdot)}}$$

holds for any $(h_j)_{j\in\mathbb{N}_0}\subset\mathscr{S}(\mathbb{R}^n)$ such that

$$\Big\|\big\|(2^{-js(\cdot)}h_j)_{j\in\mathbb{N}_0}\big\|_{\ell^{q'(\cdot)}}\Big\|_{p'(\cdot)}\leqslant 1.$$

Proof. We use some decomposition techniques of [27]. Since $\varphi,\Psi\in\mathscr{S}(\mathbb{R}^n)$, we have

$$|\varphi_j|\leqslant c\eta_{j,N}\quad\text{and}\quad|\Psi_m|\leqslant c\eta_{m,N},\quad j,m\in\mathbb{N}_0,N>n,$$

where the positive constant c is independent of j and m. Therefore $|\Lambda_{j,m}(f,g)(x)|$ can be estimated by

$$\begin{aligned}
&c\int_{\mathbb{R}^{2n}}\eta_{j,N}(x-y)\eta_{m,N}(x-z)|f(y)||\varphi_m*g(z)|dydz\\
&+c\int_{\mathbb{R}^{2n}}\eta_{j,N}(x-y)\eta_{m,N}(y-z)|f(y)||\varphi_m*g(z)|dydz\\
\leqslant\ & c(\eta_{j,N}*|f|(x))(\eta_{m,N}*|\varphi_m*g|(x))+c\ I_{j,m}(x)\\
=\ & c\ H_{j,m}(x)+c\ I_{j,m}(x),
\end{aligned}$$

with

$$I_{j,m}(x)=\eta_{j,N}*(|f|\eta_{m,N}*|\varphi_m*g|)(x),\quad x\in\mathbb{R}^n,j,m\in\mathbb{N}_0,N>n.\tag{3.2}$$

Let us estimate each term separately. Let $(h_j)_{j\in\mathbb{N}_0}$ be a sequence of Schwartz functions with

$$\Big\|\big\|(2^{-s(\cdot)j}h_j)_{j\in\mathbb{N}_0}\big\|_{\ell^{q'(\cdot)}}\Big\|_{p'(\cdot)}\leqslant 1.$$

Using the well-known estimate

$$\eta_{j,N}*|f|\lesssim\mathcal{M}(f)\tag{3.3}$$

and Lemma 1.23 we can estimate

$$\sum_{j=0}^{\infty}\sum_{m=j}^{\infty}\int_{\mathbb{R}^n} 2^{ja} H_{j,m}(x)|h_j(x)|dx \tag{3.4}$$

by

$$\begin{aligned}
& c\int_{\mathbb{R}^n} \mathcal{M}(f)(x)\sum_{j=0}^{\infty}\sum_{m=j}^{\infty} 2^{j(a+s(x))}\eta_{m,N} * |\varphi_m * g|(x) 2^{-js(x)}|h_j(x)|dx \\
\lesssim \; & \int_{\mathbb{R}^n} \mathcal{M}(f)(x)\sum_{j=0}^{\infty}\sum_{m=j}^{\infty} 2^{(j-m)(a+s(x))}\eta_{m,N_1} * (2^{m(a+s(\cdot))}|\varphi_m * g|)(x) 2^{-js(x)}|h_j(x)|dx
\end{aligned}$$

for some $N_1 > n$. Now

$$\sum_{j=0}^{\infty}\sum_{m=j}^{\infty} 2^{(j-m)(a+s(x))}\eta_{m,N_1} * (2^{m(a+s(\cdot))}|\varphi_m * g|)(x) 2^{-js(x)}|h_j(x)|$$

can be estimated by

$$\begin{aligned}
& \Big\|\Big(\sum_{m=j}^{\infty} 2^{(j-m)(a+s(x))}\eta_{m,N_1} * (2^{m(a+s(\cdot))}|\varphi_m * g|)(x)\Big)_{j\in\mathbb{N}_0}\Big\|_{\ell^{q(x)}} \\
& \times\Big\|\big(2^{-js(x)}h_j(x)\big)_{j\in\mathbb{N}_0}\Big\|_{\ell^{q'(x)}} \\
\lesssim \; & \Big\|\big(\eta_{j,N_1} * (2^{j(a+s(\cdot))}|\varphi_j * g|)(x)\big)_{j\in\mathbb{N}_0}\Big\|_{\ell^{q(x)}}\Big\|\big(2^{-js(x)}h_j(x)\big)_{j\in\mathbb{N}_0}\Big\|_{\ell^{q'(x)}},
\end{aligned}$$

where we have used Hölder's inequality and Lemma 3.2. Hence (3.4) does not exceed

$$\begin{aligned}
& c\int_{\mathbb{R}^n} \mathcal{M}(f)(x)\big\|(\eta_{j,N_1} * (2^{j(a+s(\cdot))}|\varphi_j * g|)(x))_{j\in\mathbb{N}_0}\big\|_{\ell^{q(x)}}\big\|(2^{-s(\cdot)j}h_j(x))_{j\in\mathbb{N}_0}\big\|_{\ell^{q'(x)}}dx \\
\lesssim \; & \Big\|\big\|\mathcal{M}(f)(\eta_{j,N_1} * 2^{j(a+s(\cdot))}|\varphi_j * g|)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\Big\|_{p(\cdot)}\Big\|\big\|(2^{-s(\cdot)j}h_j)_{j\in\mathbb{N}_0}\big\|_{\ell^{q'(\cdot)}}\Big\|_{p'(\cdot)} \\
\lesssim \; & \big\|\mathcal{M}(f)\big\|_{p_1(\cdot)}\Big\|\big\|(\eta_{j,N_1} * 2^{j(a+s(\cdot))}|\varphi_j * g|)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\Big\|_{p_2(\cdot)} \\
\lesssim \; & \big\|f\big\|_{p_1(\cdot)}\Big\|\big\|(2^{j(a+s(\cdot))}\varphi_j * g)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\Big\|_{p_2(\cdot)} \\
\lesssim \; & \big\|f\big\|_{p_1(\cdot)}\big\|g\big\|_{F^{s(\cdot)+a}_{p_2(\cdot),q(\cdot)}}.
\end{aligned}$$

In the first estimate we used Hölder's inequality, while the second follows by Lemma 1.16 and the boundedness of maximal function on variable Lebesgue spaces. We estimate the term (3.2). Observe that

$$\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{m=j}^{\infty}\int_{\mathbb{R}^n} 2^{ja} I_{j,m}(x)|h_j(x)|dx \\
&= \int_{\mathbb{R}^n}\sum_{j=0}^{\infty} 2^{-js(x)}|h_j(x)| 2^{j(a+s(x))}\eta_{j,N} * \sum_{m=j}^{\infty}(|f|\eta_{m,N} * |\varphi_m * g|)(x)dx \\
&= \int_{\mathbb{R}^n}\sum_{j=0}^{\infty} 2^{-js(x)}|h_j(x)| 2^{j(a+s(x))}\eta_{j,N} * \kappa_j(x)dx,
\end{aligned}$$

where

$$\kappa_j = \sum_{m=j}^{\infty}(|f|\eta_{m,N} * |\varphi_m * g|), \quad j \in \mathbb{N}_0.$$

To continue we apply Hölder's inequality and obtain

$$\sum_{j=0}^{\infty} 2^{-js(x)}|h_j(x)| 2^{j(a+s(\cdot))}\eta_{j,N} * \kappa_j(x)$$

is bounded by

$$\left\|(2^{j(a+s(x))}\eta_{j,N} * \kappa_j(x))_{j\in\mathbb{N}_0}\right\|_{\ell^{q(x)}}\left\|(2^{-s(x)j} h_j(x))_{j\in\mathbb{N}_0}\right\|_{\ell^{q'(x)}}.$$

Taking the L^1-norm and using the Hölder inequality, we get

$$\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{m=j}^{\infty}\int_{\mathbb{R}^n} 2^{ja} I_{j,m}(x)|h_j(x)|dx \\
&\lesssim \left\|\left\|(2^{j(a+s(\cdot))}\eta_{j,N} * \kappa_j)_{j\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}}\right\|_{p(\cdot)}\left\|\left\|(2^{-s(\cdot)j} h_j)_{j\in\mathbb{N}_0}\right\|_{\ell^{q'(\cdot)}}\right\|_{p'(\cdot)} \\
&\lesssim \left\|\left\|(\eta_{j,N_1} * 2^{j(a+s(\cdot))}\kappa_j)_{j\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}}\right\|_{p(\cdot)} \\
&\lesssim \left\|\left\|(2^{j(a+s(\cdot))}\kappa_j)_{j\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}}\right\|_{p(\cdot)} \\
&\lesssim \left\|\left\|f(2^{j(a+s(\cdot))}\eta_{j,N} * |\varphi_j * g|)_{j\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}}\right\|_{p(\cdot)},
\end{aligned}$$

by Lemmas 1.23 and 1.16, and Lemma 3.2, where $N_1 = N - c_{\log}(s) > n$. Using again Hölder's inequality and Lemmas 1.23-1.16 we estimate the last expression by

$$\begin{aligned}
&c\left\|f\right\|_{p_1(\cdot)}\left\|\left\|(2^{j(a+s(\cdot))}\eta_{j,N} * |\varphi_j * g|)_{j\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}}\right\|_{p_2(\cdot)} \\
&\lesssim \left\|f\right\|_{p_1(\cdot)}\left\|\left\|(2^{j(a+s(\cdot))}\varphi_j * g)_{j\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}}\right\|_{p_2(\cdot)} \\
&\lesssim \left\|f\right\|_{p_1(\cdot)}\left\|g\right\|_{F^{s(\cdot)+a}_{p_2(\cdot),q(\cdot)}}.
\end{aligned}$$

The proof is completed. □

For $0 \leqslant m \leqslant j$, $j, m \in \mathbb{N}_0$, $x \in \mathbb{R}^n$ and $K \in \mathbb{N}$, we set

$$\begin{aligned} & E_{j,m,K}(f,g)(x) \\ = \; & 2^{(m-j)K} \int_{\mathbb{R}^{2n}} \eta_{j,N}(x-y)\eta_{m,N}(x-z)|f(y)||\varphi_m * g(z)|dydz \\ & +2^{(m-j)K} \int_{\mathbb{R}^{2n}} \eta_{j,N}(x-y)\eta_{m,N}(y-z)|f(y)||\varphi_m * g(z)|dydz, \end{aligned}$$

where $N > n$ large enough.

Lemma 3.5. *Let* $s \in C^{\log}_{\mathrm{loc}}, a \in \mathbb{R}, K \in \mathbb{N}, p, p_1, p_2, q \in \mathscr{P}^{\log}$ *with* $1 < p^- \leqslant p^+ < \infty, 1 < p_1^- \leqslant p_1^+ \leqslant \infty, 1 < p_2^- \leqslant p_2^+ < \infty$ *and* $1 < q^- \leqslant q^+ < \infty$. *Assume that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $(s+a)^+ < K$. *Then*

$$\sum_{j=0}^{\infty}\sum_{m=0}^{j}\int_{\mathbb{R}^n} 2^{ja} E_{j,m,K}(f,g)(x)|h_j(x)|dx \lesssim \big\|f\big\|_{p_1(\cdot)}\big\|g\big\|_{F^{s(\cdot)+a}_{p_2(\cdot),q(\cdot)}}$$

holds for any $(h_j)_{j\in\mathbb{N}_0} \subset \mathscr{S}(\mathbb{R}^n)$ *such that*

$$\Big\|\big\|(2^{-js(\cdot)}h_j)_{j\in\mathbb{N}_0}\big\|_{\ell^{q'(\cdot)}}\Big\|_{p'(\cdot)} \leqslant 1. \tag{3.5}$$

Proof. We employ the same notation as in Lemma 3.4. Let $0 \leqslant m \leqslant j$. We have

$$\begin{aligned} 2^{ja}E_{j,m,K}(f,g)(x) & \lesssim 2^{(m-j)K+aj}\big(\eta_{j,N} * |f|(x)\eta_{m,N} * |\varphi_m * g|(x) + I_{j,m}(x)\big) \\ & = 2^{(m-j)K+aj}\big(H_{j,m}(x) + I_{j,m}(x)\big). \end{aligned}$$

where N large enough and $x \in \mathbb{R}^n$. The estimate (3.3) and Lemma 1.23, yield that

$$\sum_{j=0}^{\infty}\sum_{m=0}^{j} 2^{(m-j)K+aj}H_{j,m}(x)$$

is bounded by

$$\begin{aligned} & c\mathscr{M}(f)(x)\sum_{j=0}^{\infty}2^{-js(x)}\sum_{m=0}^{j}2^{(m-j)(K-s(x)-a)}\eta_{m,N_1} * (2^{m(s(\cdot)+a)}|\varphi_m * g|)(x) \\ = \; & c\mathscr{M}(f)(x)\sum_{j=0}^{\infty}2^{-js(x)}\vartheta_j(x), \end{aligned}$$

where N_1 large enough and the constant c is independent of x. Observe that

$$\sum_{m=0}^{j}2^{(m-j)K+ja}I_{j,m}(x) = \eta_{j,N} * \omega_j,$$

with

$$\omega_j = \sum_{m=0}^{j} 2^{(m-j)K+ja}(|f|\eta_{m,N} * |\varphi_m * g|), \quad j \in \mathbb{N}_0.$$

From Lemma 1.23 we derive

$$2^{js(x)}\eta_{j,N} * \omega_j \lesssim \eta_{j,N_2} * 2^{js(\cdot)}\omega_j \lesssim \eta_{j,N_2} * \mu_j,$$

where the implicit constant is independent of x, N_2 large enough and

$$\mu_j = \sum_{m=0}^{j} 2^{(m-j)(K-s(\cdot)-a)}(|f|\eta_{m,N_3} * 2^{m(s(\cdot)+a)}|\varphi_m * g|), \quad N_3 > n.$$

Consequently

$$\begin{aligned}
& \int_{\mathbb{R}^n} \sum_{j=0}^{\infty}\sum_{m=0}^{j} 2^{(m-j)K+aj} H_{j,m}(x)|h_j(x)|dx \\
\lesssim\ & \int_{\mathbb{R}^n} \mathcal{M}(f)(x)\sum_{j=0}^{\infty}|h_j(x)|2^{-js(x)}\vartheta_j(x)dx \\
\lesssim\ & \|f\|_{p_1(\cdot)}\Big\|\big\|\big(\vartheta_j\big)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\Big\|_{p_2(\cdot)},
\end{aligned}$$

where we used Hölder's inequality, (3.5) and the boundedness of maximal function on variable Lebesgue spaces. Lemmas 3.2 and 1.16 yield

$$\begin{aligned}
\Big\|\big\|\big(\vartheta_j\big)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\Big\|_{p_2(\cdot)} &\lesssim \Big\|\big\|(\eta_{j,N_1} * 2^{j(s(\cdot)+a)}|\varphi_j * g|)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\Big\|_{p_2(\cdot)} \\
&\lesssim \|g\|_{F^{s(\cdot)+a}_{p_2(\cdot),q(\cdot)}}.
\end{aligned}$$

Again by Hölder's inequality and (3.5) we derive

$$\begin{aligned}
\int_{\mathbb{R}^n} \sum_{j=0}^{\infty}\sum_{m=0}^{j} 2^{(m-j)K+aj} I_{j,m}(x)|h_j(x)|dx &\lesssim \int_{\mathbb{R}^n} \sum_{j=0}^{\infty} 2^{-js(x)}|h_j(x)|\eta_{j,N_2} * \mu_j(x)dx \\
&\lesssim \Big\|\big\|(\eta_{j,N_2} * \mu_j)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\Big\|_{p(\cdot)} \\
&\lesssim \Big\|\big\|\big(\mu_j\big)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\Big\|_{p(\cdot)},
\end{aligned}$$

where the last term follows by Lemma 1.16. Applying Lemmas 3.2, 1.23 and 1.16 we obtain

$$\begin{aligned}
\Big\|\Big\|\big(\mu_j\big)_{j\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}}\Big\|_{p(\cdot)} &\lesssim \Big\|\Big\|(|f|\eta_{j,N_3} * 2^{j(s(\cdot)+a)}|\varphi_m * g|)_{j\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}}\Big\|_{p(\cdot)} \\
&\lesssim \big\|f\big\|_{p_1(\cdot)}\Big\|\Big\|(\eta_{j,N_3} * 2^{j(s(\cdot)+a)}|\varphi_m * g|)_{j\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}}\Big\|_{p_2(\cdot)} \\
&\lesssim \big\|f\big\|_{p_1(\cdot)}\Big\|\Big\|(2^{j(s(\cdot)+a)}(\varphi_m * g))_{j\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}}\Big\|_{p_2(\cdot)} \\
&\lesssim \big\|f\big\|_{p_1(\cdot)}\big\|g\big\|_{F^{s(\cdot)+a}_{p_2(\cdot),q(\cdot)}},
\end{aligned}$$

which is the desired estimate. □

In this section we present our main results of this chapter. Let $\mathbf{f} = (f_1, ..., f_n) \in X^n$ for some normed space X. Then we put

$$\big\|\mathbf{f}\big\|_X = \sum_{i=1}^{n}\big\|f_i\big\|_X.$$

We have the following result:

Theorem 3.6. *Let $s \in C^{\log}_{\mathrm{loc}}, s^- > 0, p, p_1, p_2, q \in \mathscr{P}^{\log}$ with $1 < p^- \leqslant p^+ < \infty, 1 < p_1^- \leqslant p_1^+ \leqslant \infty, 1 < p_2^- \leqslant p_2^+ < \infty$, and $1 < q^- \leqslant q^+ < \infty$. Assume that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Let $V = (V_1, ..., V_n) \in (\mathscr{S}(\mathbb{R}^n))^n$ be vector field. Then for any $f \in \mathscr{S}(\mathbb{R}^n)$*

$$\Big\|\Big\|(2^{j(s(\cdot)}[V\cdot\nabla, \Delta_j]f)_{j\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}}\Big\|_{p(\cdot)} \lesssim \big\|\nabla f\big\|_{p_1(\cdot)}\big\|V\big\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}} + A \tag{3.6}$$

and

$$\begin{aligned}
&\Big\|\Big\|(2^{js(\cdot)}[V\cdot\nabla, \Delta_j]f)_{j\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}}\Big\|_{p(\cdot)} \\
\lesssim\ & \|f\,div(V)\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} + \big\|\nabla V\big\|_{p_1(\cdot)}\big\|f\big\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}} + \big\|f\big\|_{p_1(\cdot)}\big\|V\big\|_{F^{s(\cdot)+1}_{p_2(\cdot),q(\cdot)}},
\end{aligned} \tag{3.7}$$

where

$$A = \big\|\nabla V\big\|_{p_1(\cdot)}\big\|f\big\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}} \quad or \quad A = \big\|V\big\|_{p_1(\cdot)}\big\|\nabla f\big\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}}.$$

Proof. The proof follows the ideas in [27, p. 1217].

Step 1. *Preparation.* Let $V = (V_1, ..., V_n) \in (\mathscr{S}(\mathbb{R}^n))^n$ and $f \in \mathscr{S}(\mathbb{R}^n)$. From Lemma 3.3 we need to estimate

$$\int_{\mathbb{R}^n}\sum_{j=0}^{\infty}|[V\cdot\nabla, \Delta_j]f(x)h_j(x)|dx$$

for any $(h_j)_{j\in\mathbb{N}_0} \subset \mathscr{S}(\mathbb{R}^n)$ such that

$$\Big\|\Big\|(2^{-js(\cdot)}h_j)_{j\in\mathbb{N}_0}\Big\|_{\ell^{q'(\cdot)}}\Big\|_{p'(\cdot)} \leqslant 1. \tag{3.8}$$

We have

$$\begin{aligned}[V\cdot\nabla,\Delta_j]f(x) &= \sum_{k=1}^{n} V_k(x)\partial_k\Delta_j f(x)-\Delta_j(V_k\partial_k f)(x)\\ &= \sum_{k=1}^{n}\int_{\mathbb{R}^n}\varphi_j(x-y)(V_k(x)-V_k(y))\partial_k f(y)dy.\end{aligned}$$

Let $(\mathscr{F}\varphi_j)_{j\in\mathbb{N}_0}$ be a smooth dyadic resolution of unity. Then

$$\begin{aligned}[V\cdot\nabla,\Delta_j]f(x) &= \sum_{m=0}^{\infty}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\varphi_j(x-y)(\varphi_m*V_k(x)-\varphi_m*V_k(y))\partial_k f(y)dy\\ &= \sum_{m=0}^{\infty}\sum_{k=1}^{n}\Pi_{j,m,k}(\partial_k f,V_k)(x)\\ &= \sum_{m=0}^{j}\cdots+\sum_{m=j+1}^{\infty}\cdots\end{aligned}$$

for any $x\in\mathbb{R}^n$ and any $j\in\mathbb{N}_0$. Therefore we need only to estimate

$$\Big(\sum_{m=0}^{j}\sum_{k=1}^{n}\Pi_{j,m,k}(\partial_k f,V_k)\Big)_{j\in\mathbb{N}_0}\quad\text{and}\quad\Big(\sum_{m=j+1}^{\infty}\sum_{k=1}^{n}\Pi_{j,m,k}(\partial_k f,V_k)\Big)_{j\in\mathbb{N}_0},\tag{3.9}$$

in $L^{p(\cdot)}(\ell^{q(\cdot)})$-norm.

Step 2. In this step we prove (3.6). From the support properties of $(\mathscr{F}\varphi_j)_{j\in\mathbb{N}_0}$ we have

$$\Pi_{j,m,k}(\partial_k f,V_k)(x)=\Lambda_{j,m}(\partial_k f,V_k)(x),\quad x\in\mathbb{R}^n,j,m\in\mathbb{N}_0,k\in\{1,...,n\},$$

with $\Psi_m=\sum_{i=-2}^{2}\varphi_{m+i}$, $m\in\mathbb{N}$ and $\Psi_0=\Phi+\varphi_1$. For simplicity, we use φ_m instead of Ψ_m, $m\in\mathbb{N}_0$. Applying Lemmas 3.3 and 3.4, with the help of (3.8), we estimate the second term of (3.9) in $L^{p(\cdot)}(\ell^{q(\cdot)})$-norm by

$$\big\|\nabla f\big\|_{p_1(\cdot)}\big\|V\big\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}}.$$

Let $K\in\mathbb{N}$ be such that $0<s^-\leqslant s^+<K$. From [27, Lemma 3.1] we derive

$$\begin{aligned}\Pi_{j,m,k}(\partial_k f,V_k) &= \sum_{1\leqslant|\alpha|<K}2^{|\alpha|(m-j)}(\theta_{j,\alpha}*\partial_k f)(\partial^\alpha\varphi)_m*\varphi_m*V_k+\Upsilon_{j,m,K,k}(\partial_k f,V_k),\\ &= \sum_{1\leqslant|\alpha|<K}I_{1,j,m,|\alpha|,k}+\Upsilon_{j,m,K,k}(\partial_k f,V_k),\end{aligned}\tag{3.10}$$

where

$$\Upsilon_{j,m,K,k}(\partial_k f, V_k)(x) = \int_{\mathbb{R}^{2n}} \varphi_j(x-y)\Big(\sum_{|\alpha|=K} \frac{1}{\alpha!}(\partial^\alpha \varphi_m)(\xi_\alpha)(y-x)^\alpha\Big)\partial_k f(y)\varphi_m * V_k(z)\,dy\,dz,$$

ξ_α is on the line segment joining $y-z$ and $x-z$ and

$$\theta_{j,\alpha}(x) = \frac{(-1)^{|\alpha|}}{\alpha!}(2^j x)^\alpha \varphi_j(x), \quad x \in \mathbb{R}^n, j \in \mathbb{N}_0.$$

When $K=1$, the sum on the right-hand side of (3.10) is interpreted as zero. Again from [27, Lemma 3.1],

$$|\Upsilon_{j,m,K,k}(\partial_k f, V_k)| \lesssim E_{j,m,K}(\partial_k f, V_k), \quad 0 \leqslant m \leqslant j,$$

which yields by Lemma 3.5 with $a=0$ that

$$\sum_{j=0}^{\infty}\sum_{m=0}^{j}\int_{\mathbb{R}^n} |\Upsilon_{j,m,K,k}(\partial_k f, V_k)(x)||h_j(x)|dx \lesssim \big\|\partial_k f\big\|_{p_1(\cdot)}\big\|V_k\big\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}}$$

for any $k \in \{1,...,n\}$. From the support properties of $(\mathscr{F}\varphi_j)_{j\in\mathbb{N}_0}$, we have $\theta_{j,\alpha} * \partial_k f = \theta_{j,\alpha} * \tilde{\varphi}_j * \partial_k f$, where $\tilde{\varphi}_j = \sum_{l=-2}^{l=2}\varphi_{j+l}$. Here we put $\varphi_v = 0$ if $v<0$. Hence

$$\begin{aligned} |I_{1,j,m,1,k}| &\lesssim 2^{m-j}(\eta_{j,N} * |\tilde{\varphi}_j * \partial_k f|)(\eta_{m,N} * |\varphi_m * V_k|) \\ &\lesssim 2^{m-j}(\eta_{j,N} * |\tilde{\varphi}_j * \partial_k f|)\mathscr{M}(V_k) \end{aligned}$$

and then

$$\sum_{j=0}^{\infty}\sum_{m=0}^{j}\int_{\mathbb{R}^n} |I_{1,j,m,1,k}(x)||h_j(x)|dx \tag{3.11}$$

does not exceed

$$\begin{aligned} &c\int_{\mathbb{R}^n} \mathscr{M}(V_k)(x)\sum_{j=0}^{\infty}(\eta_{j,N} * |\tilde{\varphi}_j * \partial_k f|)|h_j(x)|dx \\ \lesssim &\int_{\mathbb{R}^n} \mathscr{M}(V_k)(x)\big\|(2^{-js(x)}h_j(x))_{j\in\mathbb{N}_0}\big\|_{\ell^{q'(x)}}\big\|(2^{js(x)}\eta_{j,N} * \tilde{\varphi}_j * \partial_k f(x))_{j\in\mathbb{N}_0}\big\|_{\ell^{q(x)}}dx, \end{aligned}$$

which is bounded by

$$\begin{aligned} &c\big\|\mathscr{M}(V_k)\big\|(2^{js(\cdot)}\eta_{j,N} * \tilde{\varphi}_j * \partial_k f(x))_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p(\cdot)} \\ \lesssim &\big\|\mathscr{M}(V_k)\big\|_{p_1(\cdot)}\Big\|\big\|\big(2^{js(\cdot)}\eta_{j,N} * \tilde{\varphi}_j * \partial_k f\big)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\Big\|_{p_2(\cdot)}. \end{aligned}$$

Finally the boundedness of maximal function on variable Lebesgue spaces and Lemmas

1.23 and 1.16 guarantee that the last term can be estimated by

$$c\big\|V_k\big\|_{p_1(\cdot)}\big\|\nabla f\big\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}}.$$

We have

$$\sum_{m=0}^{j}\sum_{1\leqslant|\alpha|<K} I_{1,j,m,|\alpha|,k}=\sum_{m=0}^{j}\sum_{1\leqslant|\alpha|<K}2^{|\alpha|(m-j)}(\theta_{j,\alpha}*\tilde{\varphi}_j*\partial_k f)(\partial^{\alpha}\varphi)_m*\varphi_m*V_k.$$

Recall that

$$\theta_{j,\alpha}(x)=2^{j|\alpha|}\frac{(-1)^{|\alpha|}}{\alpha!}x^{\alpha}\varphi_j(x),$$

which yields that

$$|\theta_{j,\alpha}*\tilde{\varphi}_j*\partial_k f|\lesssim 2^j\eta_{j,L}*|\tilde{\varphi}_j*f|,\quad L>n.$$

The arguments of [27, p. 1218] yield that

$$\begin{aligned}&\Big|\sum_{m=0}^{j}\sum_{1\leqslant|\alpha|<K}2^{|\alpha|(m-j)}I_{1,j,m,|\alpha|,k}\Big|\\ \lesssim\;& \eta_{j,L}*|\tilde{\varphi}_j*f|\sum_{m=0}^{j}\sum_{1<|\alpha|<K}2^{(|\alpha|-1)(m-j)}\eta_{m,L}*|\nabla V_k|+(\eta_{j,L}*|\tilde{\varphi}_j*f|)(\eta_{j,L}*|\nabla V_k|)\\ \lesssim\;& \mathcal{M}(\nabla V_k)(\eta_{j,L}*|\tilde{\varphi}_j*f|).\end{aligned}$$

Therefore (3.11) with $I_{1,j,m,|\alpha|,k}$ in place of $I_{1,j,m,1,k}$ can be estimated by

$$c\big\|\nabla V\big\|_{p_1(\cdot)}\big\|f\big\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}}.$$

Step 3. In this step we prove (3.7). As in [27, p. 1219]

$$\Pi_{j,m,k}(\partial_k f,V_k)(x)=J^1_{j,m,k}(\partial_k f,V_k)(x)+J^2_{j,m,k}(\partial_k f,V_k)(x),$$

where

$$J^1_{j,m,k}(\partial_k f,V_k)(x)=\int_{\mathbb{R}^{2n}}\varphi_j(x-y)(\partial_k\Psi_m)(y-z)f(y)\varphi_m*V_k(z)dydz$$

and

$$J^2_{j,m,k}(\partial_k f,V_k)(x)=\int_{\mathbb{R}^{2n}}2^j(\partial_k\varphi)_j(x-y)(\Psi_m(x-z)-\Psi_m(y-z))f(y)\varphi_m*V_k(z)dydz.$$

Similarly as in [27, p. 1219] we obtain

$$\Big\|\Big\|\Big(2^{js(\cdot)}\sum_{m=0}^{\infty}\sum_{k=1}^{n} J^1_{j,m,k}(\partial_k f, V_k)\Big)_{j\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}}\Big\|_{p(\cdot)} = \big\|f\,\mathrm{div}(V)\big\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Observe that $J^2_{j,m,k}(\partial_k f, V_k)$ is just $2^j\Lambda_{j,m}(f, V_k)$ but with $(\partial_k\varphi)_j$ in place of φ_j. Using the same type of arguments as in Step 2 it is easy to see that

$$\begin{aligned}&\Big\|\Big\|\Big(2^{j(s(\cdot)}\sum_{m=0}^{\infty}\sum_{k=1}^{n} J^2_{j,m,k}(\partial_k f, V_k)\Big)_{j\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}}\Big\|_{p(\cdot)}\\ \lesssim\ & \big\|\nabla V\big\|_{p_1(\cdot)}\big\|f\big\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}} + \big\|f\big\|_{p_1(\cdot)}\big\|V\big\|_{F^{s(\cdot)+1}_{p_2(\cdot),q(\cdot)}}.\end{aligned}$$

The proof is completed. □

Remark 3.7. *Corresponding statements to Theorem 3.6 were proved in [27, Theorem 1.1] under the assumption* s *and* q *are constants and based on the fact that the maximal operator* $\mathcal{M}$ *is bounded on* $L^{p(\cdot)}(\ell^q)$*, which in general is not bounded on* $L^{p(\cdot)}(\ell^{q(\cdot)})$*.*
Optimal inequality in (3.6) *is much more complicated even in the case of constant exponents. Probably* (3.6) *can be improved in some particular case, see Theorem 3.8 below.*

The last statement of this paper is the following theorem; the case where s and q are constants it is given in [27, Theorem 1.2].

Theorem 3.8. *Let* $s \in C^{\log}_{\mathrm{loc}}, p, p_1, p_2, q \in \mathcal{P}^{\log}$ *with* $1 < p^- \leqslant p^+ < \infty, 1 < p_1^- \leqslant p_1^+ \leqslant \infty, 1 < p_2^- \leqslant p_2^+ < \infty$*, and* $1 < q^- \leqslant q^+ < \infty$*. Assume that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$*. Let* $V = (V_1, ..., V_n) \in (\mathcal{S}(\mathbb{R}^n))^n$ *be vector field. Then for any* $f \in \mathcal{S}(\mathbb{R}^n)$

$$\Big\|\Big\|\big(2^{j(s(\cdot)}[V\cdot\nabla, \Delta_j]f\big)_{j\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}}\Big\|_{p(\cdot)} \lesssim \big\|\nabla f\big\|_{p_1(\cdot)}\big\|V\big\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}}, \quad 0 < s^- \leqslant s^+ < 1$$

and

$$\Big\|\Big\|\big(2^{j(s(\cdot)}[V\cdot\nabla, \Delta_j]f\big)_{j\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}}\Big\|_{p(\cdot)} \lesssim \big\|f\,div(V)\big\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} + \big\|f\big\|_{p_1(\cdot)}\big\|V\big\|_{F^{s(\cdot)+1}_{p_2(\cdot),q(\cdot)}},$$

with $-1 < s^- \leqslant s^+ < 0$.

Proof. The first estimate follows by Steps 1-2 of Theorem 3.6, with $K = 1$ and $a = 0$, while the second one follows by the same arguments of Step 3 in Theorem 3.6. □

Remark 3.9. *In Theorem 3.8 we present an improvement of* (3.6) *with* $0 < s^- \leqslant s^+ < 1$ *and of* (3.7) *with* $-1 < s^- \leqslant s^+ < 0$*. An extension of Theorem 3.8 to general case* $s \in C^{\log}_{\mathrm{loc}}$ *is still open.*

CHAPTER 4

LOCAL MEAN CHARACTERIZATION OF VARIABLE-BESOV-TYPE SPACES

4.1 Introduction

In this chapter, we aim to establish new characterizations of the recently introduced Variable Besov-type spaces in terms of local means, Peetre maximal function of local means. The motivation is that this characterizations were used to study Fourier multipliers on Besov-type and Triebel–Lizorkin-type spaces, and to present some equivalent quasi-norms of these spaces in terms of derivatives.

4.2 Variable Besov-type space

Following [38, 54], we denote $\mathcal{G}(\mathbb{R}_+^{n+1})$ the set of all measurable functions $\phi : \mathbb{R}_+^{n+1} \to (0,\infty)$ having the following properties: there exist positive constants c_1 and c_2 such that, for all $x \in \mathbb{R}^n$ and $r > 0$

$$c_1^{-1}\phi(x,2r) \leqslant \phi(x,r) \leqslant c_1\phi(x,2r) \tag{S1}$$

and, for all $x, y \in \mathbb{R}^n$ and $r > 0$ with $\|x-y\| \leqslant r$,

$$c_2^{-1}\phi(y,r) \leqslant \phi(x,r) \leqslant c_2\phi(y,r). \tag{S2}$$

We point out that (S1) and (S2) are called the doubling condition and the compatibility condition, respectively.

In what follows, for $\phi \in \mathscr{G}(\mathbb{R}^{n+1}_+)$ and all cubes $Q := Q(x, r)$ with center $x \in \mathbb{R}^n$ and radius $r > 0$, define $\phi(Q) := \phi(Q(x, r)) := \phi(x, r)$.

Now we give some examples.

Example 4.1.

(i) if $\phi(P) := |P|^\tau$ with $\tau \in [0, \infty)$ for all cubes P. Then, Φ satisfies the conditions (S1) and (S2), and $c_1 = 2^{n\tau}$ and $c_2 = 1$.

(ii) Let $\tau(\cdot) \in \mathscr{P}_0^{\log}(\mathbb{R}^n)$ and Φ defined for all cubes $P \in \mathscr{Q}$,

$$\phi(P) := \frac{\|\chi_P\|_{L^{\tau(\cdot)}(\mathbb{R}^n)}}{|P|},$$

ϕ satisfies the conditions (S1), (S2).

For more examples and information and references see [54, 38, 53].

Remark 4.2. *Let $\phi \in \mathscr{G}(\mathbb{R}^{n+1}_+)$,*

(i) for all $x \in \mathbb{R}^n$, any $r > 0$ and any $j \in \mathbb{Z}$, $c_1^{-|j|}\phi(x, r) \leqslant \phi(x, 2^j r) \leqslant c_1^{|j|}\phi(x, r)$, where c_1 is the constant from (S1).

(ii) The constant $c_1 \geqslant 1$. For any $P \in \mathscr{Q}$ of center x and radius r, we have $r = \frac{\sqrt{n}}{2}\ell(P)$, hence, $\phi(P) = \phi(x, \frac{\sqrt{n}}{2}\ell(P))$.

(iii) for any constant $r \in (0, \infty)$, $\phi^r \in \mathscr{G}(\mathbb{R}^{n+1}_+)$, if $c_1(\phi)$ and $c_1(\phi^r)$ are the constant from (S1) of ϕ and ϕ^r respectively then $c_1(\phi^r) = c_1^r(\phi)$.

Definition 4.3. *Let $p, q \in \mathscr{P}_0(\mathbb{R}^n)$ and $P \in \mathscr{Q}$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)}_\phi(P), I)$, $I \subset \mathbb{N}_0$ is defined on sequences to be the set of all sequences $F := (f_v)_{v \in I}$ of measurable functions on $\mathbb{R}^n$ such that*

$$\|F\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_\phi(P), I)} := \frac{1}{\phi(P)}\|(f_v \chi_P)_{v \in I}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty \tag{4.1}$$

The space $\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)}_\phi(P), J)}$, $J \subset (0, 1]$, denotes the set of all measurable functions $F := f_t(x)$, $x \in \mathbb{R}^n$, $t \in J$ measurable on $\mathbb{R}^n \times J$ such that

$$\|F\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)}_\phi(P), J)}} := \frac{1}{\phi(P)}\|(f_t \chi_P)_{t \in J}\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}} < \infty \tag{4.2}$$

The space $\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(P))$ is defined by $\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(P)) := \ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(P), I)$ where $I = \{\nu \in \mathbb{N}_0;\ \nu \geqslant j_P \vee 0\}$. Therefore for any sequence $F := (f_\nu)_{\nu \geqslant j_P \vee 0} \in \ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(P))$ of measurable functions on $\mathbb{R}^n$ we have

$$\|F\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(P))} = \frac{1}{\phi(P)} \inf\Big\{\mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}\Big(\frac{1}{\mu}(f_\nu \chi_P)_{\nu \geqslant j_P \vee 0}\Big) \leqslant 1\Big\}. \tag{4.3}$$

with

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}\big((f_\nu \chi_P)_{\nu \geqslant j_P \vee 0}\big) = \sum_{\nu \geqslant j_P \vee 0} \inf\Big\{\lambda_\nu > 0 : \varrho_{p(\cdot)}\Big(\frac{f_\nu \chi_P}{\lambda_\nu^{1/q(\cdot)}}\Big) \leqslant 1\Big\}.$$

The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(\mathbb{R}^n))$ is defined on sequences to be the set of all sequences $(f_\nu)_{\nu \in \mathbb{N}_0}$ of measurable functions on $\mathbb{R}^n$ such that

$$\Big\|(f_\nu)_{\nu \in \mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(\mathbb{R}^n))} := \sup_{P \in \mathcal{Q}} \Big\|(f_\nu)_{\nu \geqslant j_P \vee 0}\Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(P))} < \infty.$$

The space $\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(P))}$ is defined by $\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(P))} := \widetilde{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(P), J)}$, where $J = (0, \ell(P) \wedge 1]$. Therefore for any sequence $F := (f_t)_{t \in (0, \ell(P) \wedge 1]}$ of functions $f_t(x)$, $x \in \mathbb{R}^n$, $t \in (0, \ell(P) \wedge 1]$ measurable on $\mathbb{R}^n \times (0, \ell(P) \wedge 1]$, we have

$$\|F\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(P))}} = \frac{1}{\phi(P)} \inf\Big\{\mu > 0 : \varrho_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}\Big(\frac{1}{\mu}(f_t(\cdot)\chi_P)_{t \in (0, \ell(P) \wedge 1]}\Big) \leqslant 1\Big\} \tag{4.4}$$

with

$$\varrho_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}}\big((f_t \chi_P)_{t \in (0, \ell(P) \wedge 1]}\big) = \int_0^{\ell(P) \wedge 1} \inf\Big\{\lambda_t > 0 : \varrho_{p(\cdot)}\Big(\frac{f_t \chi_P}{\lambda_t^{1/q(\cdot)}}\Big) \leqslant 1\Big\} \frac{dt}{t}.$$

The space $\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(\mathbb{R}^n))}$ is defined to be the set of all sequences $f_t(x)$, $x \in \mathbb{R}^n$, $t \in (0,1]$ measurable on $\mathbb{R}^n \times (0,1]$ such that

$$\Big\|(f_t)_{0 < t \leqslant 1}\Big\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(\mathbb{R}^n))}} := \sup_{P \in \mathcal{Q}} \Big\|(f_t)_{0 < t \leqslant \ell(P) \wedge 1}\Big\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(P))}} < \infty.$$

For $p, q \in \mathcal{P}_0(\mathbb{R}^n)$, the $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(P), I)}$ is a quasi-norm on $\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(P), I)$, $I \subset \mathbb{N}_0$

Remark 4.4.

(i) Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $P \in \mathcal{Q}$. For any sequence $(f_\nu)_{\nu \in I} \in \ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(P), I)$, $I \subset \mathbb{N}_0$ of measurable functions on $\mathbb{R}^n$ and $r > 0$ we have

$$\Big\|(f_\nu)_{\nu \in I}\Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(P), I)} = \Big\|(|f_\nu|^r)_{\nu \in I}\Big\|^{1/r}_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r}_{\phi^r}(P), I)},$$

and for any $(f_\nu)_{\nu\in\mathbb{N}_0}\in\ell^{q(\cdot)}(L^{p(\cdot)}_\phi(\mathbb{R}^n))$ *and* $r>0$*,*

$$\left\|(f_\nu)_{\nu\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_\phi(\mathbb{R}^n))}=\left\|(|f_\nu|^r)_{\nu\in\mathbb{N}_0}\right\|^{1/r}_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r}_{\phi^r}(\mathbb{R}^n))}.$$

(ii) For any sequence $(f_\nu)_{\nu\geqslant j_P\vee 0}$ *of measurable functions on* $\mathbb{R}^n$*, we have* $\left\|(f_\nu)_{\nu\geqslant j_P\vee 0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_\phi(P))}\leqslant 1$ *if and only if* $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_\nu\chi_P/\phi(P))_{\nu\geqslant j_P\vee 0})\leqslant 1$ *which is equivalent to* $\sum_{\nu\geqslant j_P\vee 0}\inf\left\{\lambda_\nu>0:\varrho_{p(\cdot)}\left(\frac{f_\nu\chi_P}{\phi(P)\lambda_\nu^{1/q(\cdot)}}\right)\leqslant 1\right\}\leqslant 1.$

Definition 4.5. *Let* $\{\varphi_\nu\}_{\nu\in\mathbb{N}_0}$ *be a resolution of unity,* $p,q\in\mathscr{P}_0(\mathbb{R}^n)$*,* $s:\mathbb{R}^n\to\mathbb{R}$ *and* $\phi\in\mathscr{G}(\mathbb{R}^{n+1}_+)$*, Then the Besov-type space with variable smoothness and integrability,* $B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}$*, is defined to be the set of all* $f\in\mathscr{S}'(\mathbb{R}^n)$ *such that*

$$\|f\|_{B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}}:=\sup_{P\in\mathcal{Q}}\left\|\left(2^{\nu\alpha(\cdot)}\varphi_\nu*f_\nu\right)_{\nu\geqslant j_P\vee 0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_\phi(P))}<\infty \tag{4.5}$$

where, when $j=0$*,* φ_0 *is replaced by* Φ*, and the supremum is taken over all dyadic cubes* P *in* $\mathbb{R}^n$*.*

If $\phi(P)=1$ for all cubes $P\in\mathcal{Q}(\mathbb{R}^n)$, then $B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}=B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, where $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ is the Besov space with variable smoothness and integrability introduced in [3], and if p,q,τ are constant exponents and $\phi(P)=|P|^\tau$ with $\tau\in[0,\infty)$ for all cubes $P\in\mathcal{Q}(\mathbb{R}^n)$, then $B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}=B^{s,\tau}_{p,q}$, which was investigated in [55].

If $p,q\in C^{\log}(\mathbb{R}^n)$ and $s\in C^{\log}_{\text{loc}}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ Then the Besov-type space with variable smoothness and integrability is independent of the choice of the resolution of unity.

4.3 Characterizations of the space $B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}$

Lemma 4.6. *Let* $p,q\in\mathscr{P}_0(\mathbb{R}^n)$*,* $\phi\in\mathscr{G}(\mathbb{R}^{n+1}_+)$*,* $\delta>\log_2(c_1)$ *where* c_1 *is the constant from (S1) and* $\{g_m\}_{m\in\mathbb{N}_0}\in\ell^{q(\cdot)}(L^{p(\cdot)}_\phi(\mathbb{R}^n))$ *. For any* $\nu\in\mathbb{N}_0$ *and* $x\in\mathbb{R}^n$*, let* $G_\nu(x)=\sum_{m\in\mathbb{N}_0}2^{-|m-\nu|\delta}g_m(x)$ *then,*

$$\left\|(G_\nu)_{\nu\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_\phi(\mathbb{R}^n))}\lesssim\left\|(g_m)_{m\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_\phi(\mathbb{R}^n))}.$$

Proof. By scaling arguments it suffices to suppose that $\left\|(g_m)_{m\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_\phi(\mathbb{R}^n))}\leqslant 1$. We prove,

$$\left\|(G_\nu)_{\nu\geqslant j_P\vee 0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_\phi(P))}\lesssim 1,\quad\forall P\in\mathcal{Q}.$$

Since $\left\|(g_m)_{m\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(\mathbb{R}^n))} \leqslant 1$ then, for every $P\in\mathcal{Q}$, $\left\|(g_m)_{m\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(P))} \leqslant 1$, therefore,

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}((g_m\chi_P/\phi(P))_{m\geqslant j_P\vee 0}) \leqslant 1, \quad \forall P\in\mathcal{Q}. \tag{4.6}$$

By definition, for any $P\in\mathcal{Q}$, for any $\varepsilon>0$ there exists $\{\lambda_m>0, m\geqslant j_P\vee 0\}$, such that

$$\varrho_{p(\cdot)}\left(\frac{g_m\chi_P}{\phi(P)\lambda_m^{1/q(\cdot)}}\right) \leqslant 1, \qquad \sum_{m\geqslant j_P\vee 0}\lambda_m \leqslant \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}((g_m\chi_P/\phi(P))_{m\geqslant j_P\vee 0})+\varepsilon, \tag{4.7}$$

Fix $P\in\mathcal{Q}$ with $j_P>0$ and set $\varepsilon=1$. For any $m<j_P$, $m\in\mathbb{N}_0$, let $P_m\in\mathcal{Q}$ be such that $P\subset P_m$, P and P_m cocentric, and $\ell(P_m)=2^{-m}$, then $j_{P_m}=m$, by (4.6) and (4.7) (with $\varepsilon=1$) there exist $\lambda^*_m>0$ such that

$$\varrho_{p(\cdot)}\left(\frac{g_m\chi_{P_m}}{\phi(P_m)(\lambda^*_m)^{1/q(\cdot)}}\right) \leqslant 1, \qquad \lambda^*_m \leqslant 2. \tag{4.8}$$

Since $\delta-\log_2(c_1)>0$ there exist $\gamma>0$ such that $\delta-\log_2(c_1)>\gamma 1/q^-$, for any $\nu\geqslant j_P$ set $\beta_\nu := \sum_{m<j_P} 2^{-|m-\nu|\gamma}\lambda^*_m + \sum_{m\geqslant j_P} 2^{-|m-\nu|\gamma}\lambda_m$, where λ_m, $m\geqslant j_P$ are from (4.7) of the fixed P, it's clear that $\beta_\nu<\infty$, let $r:=\min\{1,p^-\}$ and $\tilde{p}(\cdot):=p(\cdot)/r$, we have

$$|G_\nu|^r \leqslant \sum_{m\in\mathbb{N}_0} 2^{-|m-\nu|r\delta}|g_m|^r$$

then

$$|G_\nu|^r \leqslant \sum_{m<j_P} 2^{-|m-\nu|r\delta}|g_m|^r + \sum_{m\geqslant j_P} 2^{-|m-\nu|r\delta}|g_m|^r.$$

By Remark 4.2,

$$\phi(P)=\phi(x,c2^{-j_P})=\phi(x,2^{-j_P+m}c2^{-m})\geqslant c_1^{-|-j_P+m|}\phi(x,c2^{-m})=c_1^{-|-j_P+m|}\phi(P_M),$$

where x is the center of P (and P_m), hence,

$$\begin{aligned}\sum_{0\leqslant m<j_P} 2^{-|m-\nu|r\delta}(\phi(P))^{-r}|g_m|^r &\leqslant \sum_{0\leqslant m<j_P} 2^{-|m-\nu|r\delta}c_1^{r|-j_P+m|}(\phi(P_m))^{-r}|g_m|^r\\ &= \sum_{0\leqslant m<j_P} 2^{-|m-\nu|r\delta}2^{r(j_P-m)\log_2(c_1)}(\phi(P_m))^{-r}|g_m|^r,\end{aligned}$$

since $\sum_{k\in\mathbb{N}_0} 2^{kr(-\delta+\gamma 1/q^-+\log_2(c_1))}$ is finite and $2^{-|m-\nu|\gamma}\lambda^*_m/\beta_\nu \leqslant 1$, $0\leqslant m<j_P$, by (4.8) and

Remark 1.1(i), it follows,

$$\begin{aligned}
\left\| \frac{\sum_{m<j_P} 2^{-|m-\nu| r\delta} |g_m|^r \chi_P}{(\phi(P))^r \beta_\nu^{r/q(\cdot)}} \right\|_{\tilde{p}(\cdot)} &\leqslant \Bigg\| \sum_{m<j_P} 2^{|m-\nu| r(-\delta+\gamma 1/q^-)} 2^{r(j_P-m)\log_2(c_1)} \\
&\quad \times \frac{(2^{-|m-\nu|\gamma} \lambda_m^*)^{r/q(\cdot)}}{\beta_\nu^{r/q(\cdot)}} \left(\frac{|g_m(x)|\chi_{P_m}}{\phi(P_m)(\lambda_m^*)^{q(\cdot)}} \right)^r \Bigg\|_{\tilde{p}(\cdot)} \\
&\leqslant \sum_{m<j_P} 2^{(j_P-m)r(-\delta+\gamma 1/q^- + \log_2(c_1))} \\
&\leqslant \sum_{k\in\mathbb{N}_0} 2^{kr(-\delta+\gamma 1/q^- + \log_2(c_1))} \\
&\leqslant c,
\end{aligned}$$

since $\sum_{k\in\mathbb{N}_0} 2^{kr(-\delta+\gamma 1/q^-)}$ is finite and $2^{-|m-\nu|\gamma}\lambda_m/\beta_\nu \leqslant 1$, $j_P \leqslant m$, by (4.7) and Remark 1.1(i),

$$\begin{aligned}
\left\| \frac{\sum_{m\geqslant j_P} 2^{-|m-\nu| r\delta} |g_m(x)|^r \chi_P}{\phi(P) \beta_\nu^{r/q(\cdot)}} \right\|_{\tilde{p}(\cdot)} &\leqslant \left\| \sum_{m\geqslant j_P} 2^{-|m-\nu| r\delta} \frac{\lambda_m^{r/q(\cdot)}}{\beta_\nu^{r/q(\cdot)}} \left(\frac{g_m \chi_P}{\phi(P)\lambda_m^{1/q(\cdot)}} \right)^r \right\|_{\tilde{p}(\cdot)} \\
&\leqslant \sum_{m\geqslant j_P} 2^{|m-\nu| r(-\delta+\gamma 1/q^-)} \\
&\leqslant 2 \sum_{k\in\mathbb{N}_0} 2^{kr(-\delta+\gamma 1/q^-)} \\
&\leqslant c,
\end{aligned}$$

therefore, there exists $c > 0$ independent of ν such that $\varrho_{p(\cdot)}\left(\frac{cG_\nu \chi_P}{\phi(P)\beta_\nu^{1/q(\cdot)}} \right) \leqslant 1$.

It follows

$$\begin{aligned}
\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}\left((cG_\nu \chi_P/\phi(P))_{\nu\geqslant j_P} \right) &\leqslant \sum_{\nu\in\geqslant j_P} \beta_\nu \\
&\leqslant \sum_{\nu\geqslant j_P\vee 0} \sum_{m<j_P} 2^{-|m-\nu|\gamma} \lambda_m^* + \sum_{\nu\geqslant j_P} \sum_{m\geqslant j_P\vee 0} 2^{-|m-\nu|\gamma} \lambda_m
\end{aligned}$$

we have if $m < j_P \leqslant \nu$ then $|m-\nu| = |m-j_P| + |j_P-\nu|$, since $\lambda_m^* \leqslant 2$, it follows

$$\begin{aligned}
\sum_{\nu\geqslant j_P} \sum_{m<j_P} 2^{-|m-\nu|\gamma} \lambda_m^* &= \sum_{\nu\geqslant j_P} \sum_{m<j_P} 2^{-|m-j_P|\gamma} 2^{-|\nu-j_P|\gamma} \lambda_m^* \\
&\leqslant 2 \sum_{\nu\geqslant j_P} 2^{-|\nu-j_P|\gamma} \sum_{m<j_P} 2^{-|m-j_P|\gamma} \\
&\leqslant c,
\end{aligned}$$

also,

$$\sum_{\nu\geqslant j_P}\sum_{m\geqslant j_P}2^{-|m-\nu|\gamma}\lambda_m=\sum_{m\geqslant j_P}\lambda_m\sum_{\nu\geqslant j_P}2^{-|m-\nu|\gamma}\leqslant c\sum_{m\geqslant j_P}\lambda_m.$$

therefore

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((cG_\nu\chi_P/\phi(P))_{\nu\geqslant j_P}\big)\lesssim 1+\sum_{m\geqslant j_P}\lambda_m.$$

If $P\in\mathcal{Q}$ with $j_P\leqslant 0$, then (4.6) and (4.7) holds true with $m\geqslant 0$, for any $\nu\geqslant 0$ set $\beta_\nu:=\sum_{m\geqslant 0}2^{-|m-\nu|\gamma}\lambda_m$, by similar arguments, there exists $c>0$ independent of ν such that $\varrho_{p(\cdot)}\left(\frac{cG_\nu\chi_P}{\phi(P)\beta_\nu^{1/q(\cdot)}}\right)\leqslant 1$ and $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((cG_\nu\chi_P/\phi(P))_{\nu\geqslant 0}\big)\lesssim 1+\sum_{m\geqslant 0}\lambda_m$.

Then, by (4.7) (with $\varepsilon=1$) and (4.6), for every $P\in\mathcal{Q}$,

$$\begin{aligned}\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((cG_\nu\chi_P/\phi(P))_{\nu\geqslant j_P\vee 0}\big)&\lesssim 1+\sum_{m\geqslant \widehat{j_P}}\lambda_m\\&\lesssim 1+\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((g_m\chi_P/\phi(P))_{m\geqslant j_P\vee 0}\big)\qquad(4.9)\\&\lesssim 1\end{aligned}$$

implies for every $P\in\mathcal{Q}$, $\big\|(G_\nu)_{\nu\geqslant j_P\vee 0}\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_\phi(P))}\lesssim 1$, which ends the proof. □

Remark 4.7. *If $\phi(P)=|P|^\tau$, then $c_1=2^{n\tau}$, so, the previous Lemma holds true if $\delta>\log_2(c_1)=n\tau$, which coincides with [52, Lemma 2.3].*

Let $R\in\mathbb{N}_0\cup\{-1\}$ and $\Phi_0,\Phi\in\mathcal{S}(\mathbb{R}^n)$ satisfy,

$$\mathcal{F}\Phi_0(\xi)>0 \text{ on } \{\xi\in\mathbb{R}^n:|\xi|<2\}\tag{4.10}$$

and

$$\mathcal{F}\Phi(\xi)>0 \text{ on } \{\xi\in\mathbb{R}^n:2<|\xi|<2\} \text{ and } D^\alpha(\mathcal{F}\Phi)(0)=0 \text{ for } |\alpha|\leqslant R,\tag{4.11}$$

where, when $R=-1$, the above last requirement disappears automatically. For any $a\in(0,\infty)$ and $\varphi\in\mathcal{S}(\mathbb{R}^n)$, any distribution f such that $\varphi*f$ makes sense, and $x\in\mathbb{R}^n$, we let

$$(\varphi_t^*f)_a(x):=\sup_{y\in\mathbb{R}^n}\frac{|\varphi_t*f(x+y)|}{(1+t^{-1}|y|)^a} \text{ and } (\varphi_k^*f)_a(x):=\sup_{y\in\mathbb{R}^n}\frac{|\varphi_k*f(x+y)|}{(1+2^k|y|)^a}$$

which are called the *Peetre-type* maximal functions, where, for any $k\in\mathbb{N}$ and $t\in(0,1)$

$$\varphi_t(x):=t^{-n}\varphi(\frac{x}{t}) \text{ and } \varphi_k(x):=2^{kn}\varphi(2^kx),\ \forall x\in\mathbb{R}^n$$

In view of the above notation, $\varphi_k=\varphi_{2^{-k}}$ and $(\varphi_k^*)_a=(\varphi_{2^{-k}}^*)_a$ for any $k\in\mathbb{N}$. Recall that Φ_t*f

for any $t \in (0,1)$ is usually called the *local means*

Lemma 4.8. *Let* Φ_0 *and* Ψ_0 *satisfy (4.10),* Φ *and* Ψ *satisfy (4.11) and* $a > 0$*, then for all large* $L \in \mathbb{N}$, $\nu \in \mathbb{N}_0$, $s, t \in [1,2]$ *and* $x \in \mathbb{R}^n$

$$|(\Psi^*_{2^{-\nu}s})_a f(x)| \lesssim 2^{-\nu(R+1)} \left(\Phi^*_0 f\right)_a (x) + \\ \sum_{1 \leqslant j \leqslant \nu} 2^{(j-\nu)(R+1)} (\Phi^*_{2^{-j}t})_a (x) + \sum_{j \geqslant \nu+1} 2^{(\nu-j)(L-a)} (\Phi^*_{2^{-j}t})_a (x),$$

where when $\nu = 0$ *we replace* Ψ *with* Ψ_0.

Proof. Let Φ_0, Φ satisfy (4.10) and (4.11) respectively, there exist Λ_0, Λ such that, $\operatorname{supp} \mathscr{F}\Lambda_0 \subset \{\xi \in \mathbb{R}^n / |\xi| \leqslant 2\}$, $\operatorname{supp} \mathscr{F}\Lambda \subset \{\xi \in \mathbb{R}^n / 1/2 \leqslant |\xi| \leqslant 2\}$ and

$$\mathscr{F}\Phi_0(\xi)\mathscr{F}\Lambda_0(\xi) + \sum_{j \geqslant 1} \mathscr{F}\Phi(2^{-j}\xi)\mathscr{F}\Lambda(2^{-j}\xi) = 1 \tag{4.12}$$

then, for all $\xi \in \mathbb{R}^n$ and $t \in [1,2]$

$$\mathscr{F}\Phi_0(t\xi)\mathscr{F}\Lambda_0(t\xi) + \sum_{j \geqslant 1} \mathscr{F}\Phi(2^{-j}t\xi)\mathscr{F}\Lambda(2^{-j}t\xi) = 1$$

therefore, for all $g \in \mathscr{S}'(\mathbb{R}^n)$

$$g = (\Phi_0)_t * (\Lambda_0)_t * g + \sum_{j \geqslant 1} (\Phi_j)_t * (\Lambda_j)_t * g.$$

Let (Ψ_0, Ψ) be a pair of functions satisfying (4.10) and (4.11) respectively, we put $g = (\Psi_\nu)_s * f$ where $f \in \mathscr{S}'(\mathbb{R}^n)$, $\nu \in \mathbb{N}_0$ and $s \in [1,2]$, it follows that

$$(\Psi_\nu)_s * f = (\Psi_\nu)_s * (\Lambda_0)_t * (\Phi_0)_t * f + \sum_{j \geqslant 1} (\Psi_\nu)_s * (\Lambda_j)_t * (\Phi_j)_t * f.$$

by (4.12) and since $\operatorname{supp} \mathscr{F}(\Lambda_0)_t \subset \{\xi \in \mathbb{R}^n / |\xi| \leqslant 2\}$ for any $t \in [1,2]$ we have

$$(\Lambda_0)_t * f = (\Lambda_0)_t * \Phi_0 * \Lambda_0 * f$$

then

$$\begin{aligned}(\Psi_\nu)_s * (\Phi_0)_t * (\Lambda_0)_t * f &= (\Psi_\nu)_s * (\Phi_0)_t * (\Lambda_0)_t * \Phi_0 * \Lambda_0 * f \\ &= (\Psi_\nu)_s * W_t * \Lambda_0 * \Phi_0 * f\end{aligned}$$

where $W_t = (\Phi_0)_t * (\Lambda_0)_t = (\Phi_0 * \Lambda_0)_t$ For all $\nu \in \mathbb{N}_0$ we have

$$(\Psi_\nu)_s * W_t(x) = \begin{cases} t^{-n}(\Psi_0)_{s/t} * W(\frac{x}{t}), & \text{if} \quad \nu = 0; \\ t^{-n}\Psi_{\frac{2^{-\nu}s}{t}} * W(\frac{x}{t}), & \text{if} \quad \nu \in \mathbb{N}, \end{cases}$$

by Lemma 1.24 with $M = -1$ for Ψ_0 and $M = R$ for Ψ and since $t, s \in [1,2]$ we obtain

$$|(\Psi_\nu)_s * W_t(x)| \leqslant C(N) \begin{cases} \eta_{t,N}(x), & \text{if} \quad \nu = 0; \\ 2^{-\nu(R+1)}\eta_{t,N}(x), & \text{if} \quad \nu \in \mathbb{N}, \end{cases}$$

for all $N > 0$ and $t \in [1,2]$.

We have $\eta_{t,N} \lesssim \eta_{1,N}$ for any $t \in [1,2]$ and $|\Lambda_0 * \Phi_0 * f| \lesssim \eta_{1,N} * |\Phi_0 * f|$, it follows that for all $\nu \in \mathbb{N}_0$ and $t, s \in [1,2]$

$$\begin{aligned} |(\Psi_\nu)_s * W_t * \Lambda_0 * \Phi_0 * f| &\lesssim 2^{-\nu(R+1)}\eta_{1,N} * \eta_{1,N} * |\Phi_0 * f| \\ &\lesssim 2^{-\nu(R+1)}\eta_{1,N} * |\Phi_0 * f|, \end{aligned}$$

therefore, for all $\nu \in \mathbb{N}_0$, $t, s \in [1,2]$

$$|(\Psi_\nu)_s * (\Phi_0)_t * (\Lambda_0)_t * f| \lesssim 2^{-\nu(R+1)}\eta_{1,N} * |\Phi_0 * f|.$$

For all $\nu \in \mathbb{N}_0$, $j \in \mathbb{N}$ we have

$$(\Psi_\nu)_s * (\Lambda_j)_t = \begin{cases} s^{-n}\Lambda_{\frac{2^{-j}t}{s}} * \Psi_0(\frac{x}{s}), & \text{if} \quad \nu = 0; \\ (2^{-j}t)^{-n}(\Psi)_{\frac{2^{-\nu}s}{2^{-j}t}} * \Lambda(\frac{x}{2^{-j}t}), & \text{if} \quad j \leqslant \nu; \\ (2^{-\nu}s)^{-n}\Lambda_{\frac{2^{-j}t}{2^{-\nu}s}} * \Psi(\frac{x}{2^{-\nu}s}), & \text{if} \quad j > \nu, \end{cases}$$

again, by Lemma 1.24, with $M = R$ for Ψ and $M = T$ for Λ where T can be chosen arbitrary positive integer since $D^\beta \mathscr{F}\Lambda(0) = 0$ for all multi-indice β because $\mathscr{F}\Lambda$ is supported away from zero, we aobtain, for all $N > 0$, $\nu \in \mathbb{N}_0$, $j \in \mathbb{N}$ and $s, t \in [1,2]$

$$|(\Psi_\nu)_s * (\Lambda_j)_t(x)| \leqslant C(N) \begin{cases} 2^{-jT}\eta_{s,N}(x), & \text{if} \quad \nu = 0; \\ 2^{(j-\nu)(R+1)}\eta_{2^{-j}t,N}(x), & \text{if} \quad 1 \leqslant j \leqslant \nu; \\ 2^{(\nu-j)T}\eta_{2^{-\nu}s,N}(x), & \text{if} \quad j > \nu. \end{cases}$$

For all $\nu \in \mathbb{N}_0$, $j \in \mathbb{N}$ where $j > \nu$ and any $s \in [1,2]$ we have $\eta_{2^{-\nu}s,N} \lesssim 2^{(\nu-j)(n-N)}\eta_{j,N}$ then, for all $\nu\in\mathbb{N}_0$ and $j\in\mathbb{N}$

$$|(\Psi_\nu)_s*(\Lambda_j)_t(x)|\lesssim\begin{cases}2^{(j-\nu)(R+1)}\eta_{2^{-j}t,N}(x), & \text{if}\quad 1\leqslant j\leqslant\nu;\\ 2^{(\nu-j)(T+n-N)}\eta_{2^{-j}t,N}(x), & \text{if}\quad j>\nu.\end{cases}$$

It follows, for any $\nu\in\mathbb{N}_0$

$$\begin{aligned}|(\Psi_\nu)_s*(\Lambda_0)_t*(\Phi_0)_t*f|(y)&\leqslant c2^{-\nu(R+1)}\eta_{1,N}*|\Phi_0*f|\\&=c2^{-\nu(R+1)}\int_{\mathbb{R}^n}\eta_{1,N-a}(y-z)\frac{|\Phi_0*f(z)|}{(1+|y-z|)^a}\,dz\\&\leqslant c2^{-\nu(R+1)}\left(\Phi_0^*f\right)_a(y)\int_{\mathbb{R}^n}\eta_{1,N-a}(x)\,dx\\&\leqslant c2^{-\nu(R+1)}\left(\Phi_0^*f\right)_a(y)\\&\leqslant c2^{-\nu(R+1)}\left(\Phi_0^*f\right)_a(x)(1+(2^{-\nu}s)^{-1}|x-y|)^a\end{aligned}$$

By taking N large such that $N-a>n$, where we used in the last estimate the following

$$(\Phi_0^*f)_a(y)\lesssim(\Phi_0^*f)_a(x)(1+|x-y|)^a\lesssim(\Phi_0^*f)_a(x)(1+(2^{-\nu}s)^{-1}|x-y|)^a$$

for all $x,y\in\mathbb{R}^n$ and $s,t\in[1,2]$. Similarly, or all $\nu\in\mathbb{N}_0$ and $j\in\mathbb{N}$ and $s,t\in[1,2]$

$$|(\Psi_\nu)_s*(\Lambda_j)_t*(\Phi_j)_t*f(y)|\lesssim$$
$$\begin{cases}2^{(j-\nu)(R+1)}(\Phi^*_{2^{-j}t})_a(x)(1+(2^{-\nu}s)^{-1}|x-y|)^a, & \text{if}\quad 1\leqslant j\leqslant\nu;\\ 2^{(\nu-j)(T+n-N-a)}(\Phi^*_{2^{-j}t})_a(x)(1+(2^{-\nu}s)^{-1}|x-y|)^a, & \text{if}\quad j>\nu.\end{cases}$$

Therefore, for all $\nu\in\mathbb{N}_0$ and $s,t\in[1,2]$

$$\frac{|(\Psi_\nu)_s*f(y)|}{(1+(2^{-\nu}s)^{-1}|x-y|)^a}\lesssim2^{-\nu(R+1)}\left(\Phi_0^*f\right)_a(x)+\sum_{1\leqslant j\leqslant\nu}2^{(j-\nu)(R+1)}(\Phi^*_{2^{-j}t})_a(x)+\sum_{j\geqslant\nu+1}2^{(\nu-j)(L-a)}(\Phi^*_{2^{-j}t})_a(x)$$

where $L=T+n-N$ can be taking arbitrary large. Taking the suppermum we obtain, for all $\nu\in\mathbb{N}_0$ and $s,t\in[1,2]$

$$|(\Psi^*_{2^{-\nu}s})_af(x)|\lesssim2^{-\nu(R+1)}\left(\Phi_0^*f\right)_a(x)+\sum_{1\leqslant j\leqslant\nu}2^{(j-\nu)(R+1)}(\Phi^*_{2^{-j}t})_a(x)+\sum_{j\geqslant\nu+1}2^{(\nu-j)(L-a)}(\Phi^*_{2^{-j}t})_a(x).$$

where when $\nu=0$ we replace Ψ with Ψ_0. The proof is complete. □

Theorem 4.9. *Let* $p \in \mathscr{P}_0^{\log}(\mathbb{R}^n)$, $q \in \mathscr{P}_0(\mathbb{R}^n)$ *and* $\phi \in \mathscr{G}(\mathbb{R}^{n+1}_+)$ *with* $\frac{1}{q} \in C^{\log}_{\text{loc}}(\mathbb{R}^n)$ *and* $q^+ < \infty$*,* $R \in \mathbb{N}_0 \cup \{-1\}$ *be such that* $R + 1 > \log_2(c_1) + \alpha^+$ *where* c_1 *is the constant from (S1),* $a > \frac{n}{p^-} + c_{log}(\alpha) + c_{log}(1/q) + \log_2(c_1)$*, then the space* $B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is characterized by*

$$B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)} = \left\{ f \in \mathscr{S}'(\mathbb{R}^n) : \left\| f \left| B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right. \right\|_i < \infty \right\}, \forall i \in \{1, \cdots, 3\},$$

where

$$\left\| f \left| B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)} \right. \right\|_1 := \sup_{p \in \mathscr{Q}} \frac{1}{\phi(P)} \left\| (\Phi_0^*)_a f \right\|_{L^{p(\cdot)}(P)} + \left\| t^{-\alpha} (\Phi_t^* f)_a \right\|_{\ell^{q(\cdot)} \widetilde{(L^{p(\cdot)}_\phi(\mathbb{R}^n))}},$$
$$\left\| f \left| B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)} \right. \right\|_2 := \left\| (2^{-\nu\alpha(\cdot)} (\Phi_\nu^* f)_a)_{\nu \geqslant 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_\phi(\mathbb{R}^n))}$$

and

$$\left\| f \left| B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)} \right. \right\|_3 := \left\| (2^{-\nu\alpha(\cdot)} \Phi_\nu * f)_{\nu \geqslant 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_\phi(\mathbb{R}^n))}.$$

Furthermore, any $\left\| \cdot \left| B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)} \right. \right\|_i$, $i \in \{1, 2, 3\}$ *is an equivalent quasi-norm on* $B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}$.

Proof. Step 1: In this step we will prove that $\left\| f \left| B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)} \right. \right\|_2$ and $\left\| f \left| B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)} \right. \right\|_3$ are equivalent, First we prove that, for any

$$\left\| f \left| B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)} \right. \right\|_2^{\Psi} \leqslant C \left\| f \left| B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)} \right. \right\|_1^{\Phi} \tag{4.13}$$

where $\|f|B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}\|_2^{\Psi}$ denotes $\|\cdot|B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}\|_2$ defined in Theorem 4.9 where Φ_0 and Φ are replaced by Ψ_0 and Ψ satisfying (4.10) and (4.11) respectively. By Lemma 4.8 with $s = 1$, we have, for all $\nu \in \mathbb{N}_0$ and $t \in [1, 2]$

$$(\Psi_\nu^* f)_a(x) \lesssim 2^{-\nu(R+1)} (\Phi_0^* f)_a(x) + \sum_{k=1}^{\nu} 2^{(k-\nu)(R+1)} (\Phi_{2^{-k}t}^* f)_a(x)$$
$$+ \sum_{k=\nu+1}^{\infty} 2^{(\nu-k)(L-a)} (\Phi_{2^{-k}t}^* f)_a(x),$$

where $L \in \mathbb{N}$ can be chosen large enough, therefore

$$2^{\nu\alpha(x)} (\Psi_\nu^* f)_a(x) \lesssim 2^{-\nu(R+1-\alpha^+)} (\Phi_0^* f)_a(x) + \sum_{k=1}^{\nu} 2^{(k-\nu)(R+1-\alpha^+)} 2^{k\alpha(x)} (\Phi_{2^{-k}t}^* f)_a(x)$$
$$+ \sum_{k=\nu+1}^{\infty} 2^{(\nu-k)(L-a+\alpha^-)} 2^{k\alpha(x)} (\Phi_{2^{-k}t}^* f)_a(x).$$

Choosing L large such that $L-a+\alpha^->\delta$ where $\delta:=R+1-\alpha^+$, then

$$\begin{aligned}2^{\nu\alpha(x)}(\Psi_\nu^*f)_a(x)&\lesssim 2^{-\nu\delta}(\Phi_0^*f)_a(x)+\sum_{k=1}^{\infty}2^{-|k-\nu|\delta}2^{k\alpha(\cdot)}(\Phi^*_{2^{-k}t}f)_a(x)\\&\lesssim 2^{-\nu\delta}(\Phi_0^*f)_a(x)+\sum_{k=1}^{\infty}2^{-|k-\nu|\delta}(2^{-k}t)^{-\alpha(x)}(\Phi^*_{2^{-k}t}f)_a(x)\end{aligned}$$

since $\alpha\in L^\infty$ and $t\in[1;2]$. Let $0<r<1$, it follows that

$$|2^{\nu\alpha(x)}(\Psi_\nu^*f)_a(x)|^r\lesssim 2^{-\nu r\delta}|(\Phi_0^*f)_a(x)|^r+\sum_{k=1}^{\infty}2^{-|k-\nu|r\delta}|(2^{-k}t)^{-\alpha(x)}(\Phi^*_{2^{-k}t}f)_a(x)|^r, \tag{4.14}$$

taking the integral $\int_1^2\cdots\frac{dt}{t}$ on both sides, gives

$$|2^{\nu\alpha(x)}(\Psi_\nu^*f)_a(x)|^r\lesssim 2^{-\nu r\delta}|(\Phi_0^*f)_a(x)|^r+\sum_{k=1}^{\infty}2^{-|k-\nu|r\delta}\int_{2^{-k}}^{2^{-k+1}}|t^{-\alpha(x)}(\Phi_t^*f)_a(x)|^r\frac{dt}{t}$$

set $g_0=|(\Phi_0^*f)_a|^r$ and $g_k:=\int_{2^{-k}}^{2^{-k+1}}|t^{-\alpha(\cdot)}(\Phi_t^*f)_a|^r\frac{dt}{t}$, $k\in\mathbb{N}$, $\tilde{p}:=p/r,\tilde{q}:=q/r$, since $\delta>\log_2(c_1)$ then by Remark 4.2/(iii), $r\delta>\log_2(c_1^r)=\log_2(c_1(\phi^r))$ where $c_1(\phi^r)$ is the constant of ϕ^r from (S1), now, Lemma 4.6 gives

$$\left\|(|2^{\nu\alpha(\cdot)}(\Psi_\nu^*f)_a|^r)_{\nu\in\mathbb{N}_0}\right\|_{\ell^{\tilde{q}(\cdot)}(L^{\tilde{p}(\cdot)}_{\phi^r}(\mathbb{R}^n))}\lesssim\left\|(g_k)_{k\in\mathbb{N}_0}\right\|_{\ell^{\tilde{q}(\cdot)}(L^{\tilde{p}(\cdot)}_{\phi^r}(\mathbb{R}^n))}, \tag{4.15}$$

then

$$\left\|(2^{\nu\alpha(\cdot)}(\Psi_\nu^*f)_a)_{\nu\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(\mathbb{R}^n))}\lesssim\left\|\left(g_k^{1/r}\right)_{k\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(\mathbb{R}^n))}, \tag{4.16}$$

next, we prove that

$$\left\|\left(g_k^{1/r}\right)_{k\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_{\phi}(\mathbb{R}^n))}\lesssim C\left\|f\,\middle|\,B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}\right\|_1^{\Phi}, \tag{4.17}$$

Suppose that the right hand of (4.17) is less than or equal one. for any $P\in\mathcal{Q}$ we have

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((g_k^{1/r}\chi_P/\Phi(P))_{k\geqslant j_P\vee 0})=\sum_{\nu\geqslant j_P\vee 0}\left\|\left|g_k^{1/r}\chi_P/\Phi(P)\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Let r and γ be such that $\max\{1,\frac{q^+}{p^-}\}<\gamma<\frac{q^-}{r}$, by Hölder's inequality for any $x\in P$

$$|g_k(x)/\Phi^r(P)|^{\frac{\tilde{q}(x)}{\gamma}}\leqslant C\int_{2^{-k}}^{2^{-k+1}}|t^{-\alpha(\cdot)}(\Phi_t^*f)_a(x)/\Phi(P)|^{r\frac{\tilde{q}(x)}{\gamma}}\frac{dt}{t},$$

by Minkowski's integral inequality

$$\begin{aligned}\left\| |g_k\chi_P/\Phi^r(P)|^{\frac{\tilde{q}(\cdot)}{\gamma}}\right\|_{\gamma\frac{\tilde{p}(\cdot)}{\tilde{q}(\cdot)}} &\leqslant C\int_{2^{-k-1}}^{2^{-k}}\left\||t^{-\alpha(\cdot)}(\Phi_t^*f)_a\chi_P/\Phi(P)|^{r\frac{\tilde{q}}{\gamma}}\right\|_{\gamma\frac{\tilde{p}(\cdot)}{\tilde{q}(\cdot)}}\frac{dt}{t}\\ &= C\int_{2^{-k-1}}^{2^{-k}}\left\|t^{-\alpha(\cdot)}(\Phi_t^*f)_a\chi_P/\Phi(P)|^{r\tilde{q}(\cdot)}\right\|^{1/\gamma}_{\frac{\tilde{p}(\cdot)}{\tilde{q}(\cdot)}}\frac{dt}{t}\\ &\leqslant C\left(\int_{2^{-k-1}}^{2^{-k}}\left\|t^{-\alpha(\cdot)}(\Phi_t^*f)_a\chi_P/\Phi(P)|^{r\tilde{q}(\cdot)}\right\|_{\frac{\tilde{p}(\cdot)}{\tilde{q}(\cdot)}}\frac{dt}{t}\right)^{1/\gamma},\end{aligned}$$

it follows that

$$\left\||g_k\chi_P/\Phi^r(P)|^{\tilde{q}(\cdot)}\right\|_{\frac{\tilde{p}(\cdot)}{\tilde{q}(\cdot)}} \leqslant C\int_{2^{-k-1}}^{2^{-k}}\left\|t^{-\alpha(\cdot)}(\Phi_t^*f)_a\chi_P/\Phi(P)|^{r\tilde{q}(\cdot)}\right\|_{\frac{\tilde{p}(\cdot)}{\tilde{q}(\cdot)}}\frac{dt}{t},$$

which means that

$$\left\||g_k^{1/r}\chi_P/\Phi(P)|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant C\int_{2^{-k}}^{2^{-k+1}}\left\|t^{-\alpha(\cdot)}(\Phi_t^*f)_a\chi_P/\Phi(P)|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\frac{dt}{t},$$

for $P \in \mathcal{Q}$ with $j_P > 0$, taking the sum on both sides we get

$$\begin{aligned}\sum_{\nu\geqslant j_P}\left\||g_k^{1/r}\chi_P/\Phi(P)|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} &\leqslant C\int_0^{2\ell(P)}\left\||t^{-\alpha(\cdot)}(\Phi_t^*f)_a\chi_P/\Phi(P)|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\frac{dt}{t}\\ &\leqslant C\int_0^{\ell(2P)\wedge 1}\left\||t^{-\alpha(\cdot)}(\Phi_t^*f)_a\chi_{2P}/\Phi(2P)|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\frac{dt}{t}\\ &\leqslant C,\end{aligned}$$

for $P \in \mathcal{Q}$ with $j_P \leqslant 0$ we have

$$\begin{aligned}\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}((g_k\chi_P/\Phi^r(P))_{k\geqslant j_P\vee 0}) &= \sum_{\nu\geqslant 0}\left\||g_k^{1/r}\chi_P/\Phi(P)|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\\ &\lesssim \left\|\left|(\Phi_0^*f)_a\chi_P/\Phi(P)\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\\ &\quad+\sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{-k+1}}\left\|t^{-\alpha(\cdot)}(\Phi_t^*f)_a\chi_P/\Phi(P)|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\frac{dt}{t}, \qquad (4.18)\end{aligned}$$

if the right hand of (4.17) is less than or equal one then $\sup_{P\in\mathcal{Q}}\frac{1}{\phi(P)}\left\|(\Phi_0^*)_af\right\|_{L^{p(\cdot)}(P)} \leqslant 1$ which implies that $\sup_{P\in\mathcal{Q}}\left\|\left|(\Phi_0^*f)_a\chi_P/\Phi(P)\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1$, for the second term on the right hand side of (4.18) we have

$$\begin{aligned}\sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{-k+1}}\left\|t^{-\alpha(\cdot)}(\Phi_t^*f)_a\chi_P/\Phi(P)|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\frac{dt}{t} &= \int_0^1\left\|t^{-\alpha(\cdot)}(\Phi_t^*f)_a\chi_P/\Phi(P)|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\frac{dt}{t}\\ &= \int_0^{\ell(P)\wedge 1}\left\|t^{-\alpha(\cdot)}(\Phi_t^*f)_a\chi_P/\Phi(P)|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\frac{dt}{t}\\ &\leqslant 1,\end{aligned}$$

therefore, we get $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}((g_k\chi_P/\Phi^r(P))_{k\geqslant j_P\vee 0})\leqslant C$ for all $P\in\mathcal{Q}$, so, we have

$$\left\|\left(g_k^{1/r}\right)_{k\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}(L_\phi^{p(\cdot)}(\mathbb{R}^n))}\leqslant C$$

whenever $\left\|f\left|B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}\right.\right\|_1^{\Phi}\leqslant 1$, by scaling arguments we get (4.17).

Next, we prove that

$$\left\|f\left|B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}\right.\right\|_1^{\Psi}\leqslant C\left\|f\left|B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}\right.\right\|_2^{\Phi}\tag{4.19}$$

by Lemma 4.8 with $t=1$, for any $\nu\in\mathbb{N}$ and $s\in[1,2]$,

$$((\Psi_\nu)_s^*f)_a(x)\lesssim\sum_{k=0}^{\nu}2^{(k-\nu)(R+1)}(\Phi_k^*f)_a(x)+\sum_{k=\nu+1}^{\infty}2^{(\nu-k)(L-a)}(\Phi_k^*f)_a(x)$$

and

$$(\Psi_0^*f)_a(x)\lesssim\sum_{k=0}^{\infty}2^{-k(L-a)}(\Phi_k^*f)_a(x),$$

hence

$$(2^{-\nu}s)^{-\alpha(\cdot)}(\Psi_{2^{-\nu}s}^*f)_a(x)\lesssim\sum_{k=0}^{\infty}2^{-|k-\nu|\delta}2^{k\alpha(\cdot)}(\Phi_k^*f)_a(x)$$

and

$$(\Psi_0^*f)_a(x)\lesssim\sum_{k=0}^{\infty}2^{-k\delta}2^{k\alpha(\cdot)}(\Phi_k^*f)_a(x),\tag{4.20}$$

denote $G_0=0$ and $G_\nu=(2^{-\nu}s)^{-\alpha(\cdot)}(\Psi_{2^{-\nu}s}^*f)_a$, $\nu\in\mathbb{N}_0$, when $\|f|B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}\|_2^{\Phi}\leqslant 1$ then by (4.9) we have,

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((c\,G_\nu \chi_P/\phi(P))_{\nu\geqslant j_P\vee 0}\big)\leqslant C,\quad \forall P\in\mathcal{Q}.$$

where C is independent of s. For any $P\in\mathcal{Q}$, with an appropriate choice of some constant c, we get

$$\begin{aligned}\int_0^{\ell(P)\wedge 1}\Big\||c\,s^{-\alpha(\cdot)}(\Phi_s^* f)_a\chi_P/\Psi(P)|^{q(\cdot)}\Big\|_{\frac{p(\cdot)}{q(\cdot)}}\frac{ds}{s} &= \int_1^2\sum_{\nu=j_P\vee 0+1}^{\infty}\Big\||c(2^{-\nu}s)^{-\alpha(\cdot)}(\Psi_{2^{-\nu}s}^* f)_a\chi_P/\Phi(P)|^{q(\cdot)}\Big\|_{\frac{p(\cdot)}{q(\cdot)}}\frac{ds}{s}\\ &\lesssim \int_1^2\sum_{\nu=j_P\vee 0}^{\infty}\Big\||c\,G_\nu\chi_P/\Phi(P)|^{q(\cdot)}\Big\|_{\frac{p(\cdot)}{q(\cdot)}}\frac{ds}{s}\\ &= c\int_1^2\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((c\,G_\nu\chi_P/\phi(P))_{\nu\geqslant j_P\vee 0}\big)\frac{ds}{s}\\ &\leqslant C.\end{aligned}$$

therefore

$$\Big\|\big(s^{-\alpha}(\Phi_s^* f)_a\big)_{s\in[0,1]}\Big\|_{\widetilde{\ell^{q(\cdot)}(L_\phi^{p(\cdot)}(\mathbb{R}^n))}}\leqslant C. \tag{4.21}$$

Now, we prove

$$\sup_{p\in\mathcal{Q}}\frac{1}{\phi(P)}\Big\|(\Psi_0^* f)_a\chi(P)\Big\|_{p(\cdot)}\leqslant C, \tag{4.22}$$

by (4.20) with $r=\min\{1,p^-\}$ we have

$$|(\Psi_0^* f)_a\chi_P/\phi(P)|^r\lesssim\sum_{k=0}^{\infty}2^{-kr\delta}|2^{k\alpha(\cdot)}(\Phi_k^* f)_a\chi_P/\phi(P)|^r,$$

taking the norm $\|\cdot\|_{p(\cdot)/r}$ we get

$$\||(\Psi_0^* f)_a\chi_P/\phi(P)|^r\|_{p(\cdot)/r}\lesssim\sup_{k\in\mathbb{N}_0}\||2^{k\alpha(\cdot)}(\Phi_k^* f)_a\chi_P/\phi(P)|^r\|_{p(\cdot)/r},$$

when $\|f|B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}\|_2^{\Phi}\leqslant 1$, we have $\|2^{k\alpha(\cdot)}(\Phi_k^* f)_a\chi_P/\phi(P)\|_{p(\cdot)}\leqslant 1$ for any $P\in\mathcal{Q}$, hence, $\|(\Psi_0^* f)_a \chi_P/\phi(P)\|_{p(\cdot)}\leqslant c$, which proves (4.22). it follows that

$$\sup_{p\in\mathcal{Q}}\frac{1}{\phi(P)}\Big\|(\Psi_0^* f)_a\chi(P)\Big\|_{p(\cdot)}+\Big\|\big(s^{-\alpha}(\Phi_s^* f)_a\big)_{s\in[0,1]}\Big\|_{\widetilde{\ell^{q(\cdot)}(L_\phi^{p(\cdot)}(\mathbb{R}^n))}}\leqslant C,$$

whenever $\|f|B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}\|_2^{\Phi}\leqslant 1$, by scaling arguments we get (4.19). □

Conclution

In chapters two, three and four we introduce new methods and tools that help us to work in variable Besov spaces, variable Triebel-Lizorkin spaces and variable-Besov-type spaces to overcome many difficulties that rise. Chapter four is a part of a paper in preparation which we aim to finish presenting new valuable results.

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ملخص:

في هذه المذكرة نقدم توصيفات متكافئة لفضاءات بيزوف $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ ، $B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}$ و فضاءات تريبل ليزوركين $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ حيث ϕ,α,q,p توابع على $\mathbb{R}^n$. يتم الاستعانة بصيغة كالديرون لاثبات بعض التوصيفات. كما يتم التطرق لدراسة المؤثر $[V\cdot\nabla,\Delta_j](f)$ على فضاءات تريبل ليزوركين $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ حيث V حقل شعاعي تباعده $\mathrm{Div}(V)$ لا يشترط أن يكون معدوما.

كلمات مفتاحية:

فضاء بيزوف، فضاء تريبل ليزوركن، الأس المتغير، صيغة كالديرون، المؤثر، حقل شعاعي.

Abstract:

In this thesis we introduce new equivalent characterizations and certain commutator estimations on Besov spaces, Besov-type spaces and Triebel-Lizorkin spaces of variable smoothness and integrability.

We prove the continuous characterization of the space $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and characterization via Peetre-type maximal functions of local means of variable Besov spaces $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ based on the continuous version of Calderón reproducing formula and and of variable Besov-type spaces $B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}$. We prove certain estimates on Triebel-Lizorkin spaces of variable smoothness and integrability $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ for the commutator $[V\cdot\nabla,\Delta_j](f)$, these estimates are obtained under no vanishing assumptions on the divergence of the vector field.

Key words:

Besov space, Triebel-Lizorkin space, variable exponent, Calderón reproducing formula, Commutator, vector field.

Résumé

Dans cette thèse, nous présentonts des nouvelles caractérisations équivalentes pour les espaces de Besov $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ à l'aide de la formule de Calderón et pour les espaces de type de Besov $B^{\alpha(\cdot),\phi}_{p(\cdot),q(\cdot)}$ avec des exposants variable, et estimons le commutateur $[V\cdot\nabla,\Delta_j](f)$ dans les espace de Triebel-Lizorkin avec des exposants variable.

Mots clés: espaces de Beso, espace de Triebel-Lizorkin, exposants variable, formule de Calderón, commutateur, champ vectoriel.