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NOTATION

- By \mathbb{R}^n we denote the n -dimensional real Euclidean space.
- By \mathbb{N} we denote the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- By \mathbb{Z} we denote the set of all integer numbers.
- $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, stands for some multi-index, whose length is denote by $|\alpha| = \alpha_1 + \dots + \alpha_n$.
- The partial differential operator of order α is denoted as,

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

- The Euclidean scalar product of $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ is given by $x \cdot y = x_1 y_1 + \dots + x_n y_n$.
- $c_{\alpha, \beta, \dots}$ is a positive constant depending on the indicated parameters α, β, \dots
- The expression $f \lesssim g$ means that $f \leq c g$ for some independent constant c (and non-negative functions f and g).
- $f \approx g$ means $f \lesssim g \lesssim f$.
- The notation $X \hookrightarrow Y$ stands for continuous embeddings from X to Y , where X and Y are quasi-normed spaces.
- For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in \mathbb{R}^n with center x and radius r .

- For any $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to x .
- For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we write $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.
- $\text{supp } f$ is the support of the function f , i.e., the closure of its non-zero set.
- If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E .
- χ_E is the characteristic function of $E \subset \mathbb{R}^n$.
- $\mathcal{D}(\mathbb{R}^n)$ is the set of functions with continuous derivations of all orders and compact support.
- $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of all complex-valued, infinitely differential and rapidly decreasing functions on \mathbb{R}^n . The topology in the complete locally convex space $\mathcal{S}(\mathbb{R}^n)$ is generated by

$$P_N(\varphi) := \sup_{x \in \mathbb{R}^n} (1 + |x|)^N \sum_{|\alpha| \leq N} |\partial^\alpha \varphi(x)|, \quad N = 1, 2, \dots$$

- The dual $\mathcal{S}'(\mathbb{R}^n)$ is the space of all tempered distributions on \mathbb{R}^n .
- A norm on a vector space X is a real-valued function $x \mapsto \|x\|$ on X satisfying the following conditions
 - (1) $\|x\| = 0$ if and only if $x = 0$, and $\|x\| \geq 0$ for all $x \in X$ (Positivity),
 - (2) $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$ and $\lambda \in \mathbb{C}$ (Homogeneity),
 - (3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (The triangle inequality).
- Recall a that quasi-norm on a vector space X is a real-valued function $x \mapsto \|x\|$ on X when the triangle inequality is weakened to
 - (3') $\exists C_X \geq 1 : \|x + y\| \leq C_X (\|x\| + \|y\|)$ for all $x, y \in X$ (The quasi-triangle inequality).

C_X is called the modulus of concavity of $\|\cdot\|$ and independent of x and y .
- A functional $x \mapsto \|x\|$ from a vector space X to a closed interval $[0, \infty]$ shall be called a pseudo-norm if it satisfying the following conditions
 - (a) $\|\lambda x + \mu y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ and for every numbers λ, μ such that $\sup\{|\lambda|, |\mu|\} \leq 1$ (The pseudo-triangle inequality),

(b) $\|\lambda x\| \rightarrow 0$ whenever $\lambda \rightarrow 0$ for every $x \in X$.

If $\|\cdot\|$ has the property that $\|x\| = 0$ implies $x = 0$ it will be called a norm.

- A real quasi-Banach space X is a complete metrizable real vector space whose topology is given by a quasi-norm.
- We say that T is bounded on X if $\|Tx\|_X \leq c\|x\|_X$ for every $x \in X$, where c is independent of x and X .

- A mapping T is said to be sublinear if for every $c \in \mathbb{C}$ and for all measurable functions f and g , we have

(i) $|T(cf)| = |c||T(f)|$,

(ii) $|T(f+g)| \leq |T(f)| + |T(g)|$.

- Let $0 < p \leq \infty$. The classical Lebesgue space $L^p(\mathbb{R}^n)$ is the class of all measurable f such that

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty, \quad 0 < p < \infty$$

and

$$\|f\|_{L^\infty(\mathbb{R}^n)} = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f(x)| < \infty.$$

- By ℓ^q , $0 < q \leq \infty$, we denote the space of all (complex) sequences $(a_k)_{k \in \mathbb{Z}}$ equipped with the quasi-norm

$$\|(a_k)_{k \in \mathbb{Z}}\|_{\ell^q} = \left(\sum_{k=-\infty}^{\infty} |a_k|^q \right)^{\frac{1}{q}}$$

(with the usual modification if $q = \infty$).

- Let $0 < p \leq \infty, 0 < q \leq \infty$ and let $(f_k)_{k \in \mathbb{Z}}$ be a sequence of complex-valued Lebesgue measurable functions on \mathbb{R}^n , we set

$$\|(f_k)_{k \in \mathbb{N}_0}\|_{L^p(\ell^q)} = \left(\int_{\mathbb{R}^n} \left(\sum_{k=-\infty}^{\infty} |f_k(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}$$

and

$$\|(f_k)_{k \in \mathbb{Z}}\|_{\ell^q(L^p)} = \left(\sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}^n} |f_k(x)|^p dx \right)^{\frac{q}{p}} \right)^{\frac{1}{q}},$$

with the usual modification if $p = \infty$ and /or $q = \infty$. The spaces $L^p(\ell^q)$ and $\ell^q(L^p)$ are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$).

- Let $L^1_{loc}(\mathbb{R}^n)$ be the collection of all locally integrable functions on \mathbb{R}^n .
- Given a function $f \in L^1_{loc}(\mathbb{R}^n)$; the Hardy-Littlewood maximal operator is defined by

$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy, \quad x \in \mathbb{R}^n.$$

- We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} f(x) dx, \quad x \in \mathbb{R}^n.$$

Its inverse is denoted by $\mathcal{F}^{-1}(f)$ or \check{f} .

- f its spectrum in $\overline{B}(0, r)$ and $r > 0$; $\text{supp } \hat{f} \subset \overline{B}(0, r)$.
- $f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y) dy$ is the product of the convolution of functions f and g .
- Given $p(\cdot) : \Omega \rightarrow]c; \infty[$, we define the conjugate exponent function $p'(\cdot)$ by the formula

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \Omega,$$

with the convention that $1/\infty = 0$. Since $p(\cdot)$ is a function, the notation $p'(\cdot)$ can be mistaken for the derivative of $p(\cdot)$, but we will never use the symbol ",' in this sense.

- The notation p' will also be used to denote the conjugate of p a constant exponent.
- "i.e." stands simply for "in other words".
- "a.e." stands simply for "almost everywhere".

INTRODUCTION

Function spaces with variable exponents have been intensively studied in the recent years by a huge number of authors. The first article published on this aspect was in the early thirties of the previous century, the Polish mathematician Władysław Orlicz [93] published an article entitled: "Über konjugierte Exponentenfolgen", which included the following question was considered: let (x_i) be a sequence of real numbers and $p_i > 1$ such that $\sum_i x_i^{p_i}$ converges. Which condition on (y_i) are necessary and sufficient for $\sum_i x_i y_i$ to converge? It turns out that the answer is that $\sum_i (\lambda y_i)^{q_i}$ should converge for some $\lambda > 0$ and $\frac{1}{p_i} + \frac{1}{q_i} = 1$. Which are nowadays named Hölder's inequality in the spaces $\ell^{p(\cdot)}$. On the other hand, Orlicz also considered Lebesgue spaces with variable exponents $L^{p(\cdot)}$ on the real line, and also proved Hölder's inequality in the functions spaces $L^{p(\cdot)}$.

In recent years, Herz spaces have played a significant role in harmonic analysis. Recall that the classical Herz spaces were originally introduced by C. S. Herz [53] in 1968. Herz spaces $K_{p(\cdot),q}^{\alpha(\cdot)}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot),q}^{\alpha(\cdot)}(\mathbb{R}^n)$ with a variable exponent p but the remaining exponents $\alpha \in \mathbb{R}$ and $q \in (0, \infty]$ were kept constant, appeared in the literature for the first time already in 2009 and 2010 by M. Izuki [55, 56], in the first paper, where he used the Haar function to obtain wavelet characterization of these function spaces, and in the last paper he studied the boundedness of sublinear operator on such spaces. After that, the most general results date back to A. Almeida and D. Drihem [2], where the variability of α was allowed, and they gave the boundedness results for a wide class of classical operators acting on these function spaces. In 2011, the variable Herz spaces $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$

were first initiated by M. Izuki and T. Noi in [57]. After that, in 2016, D. Drihem and F. Seghiri [28] gave a new norm equivalents of the function spaces $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and they proved the boundedness for a wide class of sublinear operators on such spaces. A huge number of author are interested in the variable of Herz spaces, for instance [25], [26], [100] and [106].

Another proper substitute for the classical Lebesgue spaces is the classical Hardy spaces $H^p(\mathbb{R}^n)$, introduced and developed by E. Stein and G. Weiss [108]. Miyachi [80] introduced the Hardy space on open subset Ω of \mathbb{R}^n via the maximal function, where he has studied the atomic decomposition and the duality theory of this kind of spaces. On the other hand, in 1972, C. Fefferman and E. Stein [43] used a suitable maximal function $G_N f$ to define the Hardy space $H^p(\mathbb{R}^n)$. This function space consists of all tempered distribution f such that $G_N f \in L^p(\mathbb{R}^n)$, nowadays $G_N f$ is named the Fefferman and Stein grand maximal operator. Variable Hardy spaces are a generalization of the classical $H^p(\mathbb{R}^n)$ spaces, replacing the constant exponent p with an exponent function $p(\cdot)$, these spaces appear in the work of many mathematicians. For instance, [19], [89] and [101].

At the end of the eighties of the last century, Z. Chen and S. Lau [17] introduced for the first time the classical Herz-type Hardy spaces $HK_{p,q}^\alpha(\mathbb{R}^n)$ and $H\dot{K}_{p,q}^\alpha(\mathbb{R}^n)$. These function space consists of all tempered distribution f such that $G_N f \in \dot{K}_{p,q}^\alpha(\mathbb{R}^n)$. In 2012, H. Wang and Z. Liu [116] introduced Herz-type Hardy spaces with a variable exponent $HK_{p(\cdot),q}^\alpha(\mathbb{R}^n)$ and $H\dot{K}_{p(\cdot),q}^\alpha(\mathbb{R}^n)$, which is a generalization of classical Herz-type Hardy spaces. After that, the most general results appeared in an article by D. Drihem and F. Seghiri [28], who considered Herz-type Hardy spaces $HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and $H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, where the variabilities of α and q were allowed, they established the atomic decomposition for these spaces. Also, they proved the boundedness of some classical operators on such spaces.

One of the important problems in this field is to prove the boundedness of linear and sublinear operators on generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponents. But, at the beginning, some problems appeared in the direction. For instance, some operators such as Calderón-Zygmund operators and Ricci-Stein oscillatory singular integral operators, are not bounded on $L^1(\mathbb{R}^n)$, but map $L^1(\mathbb{R}^n)$ to $WL^1(\mathbb{R}^n)$. So, the classical weak Lebesgue spaces $WL^p(\mathbb{R}^n)$ play an irreplaceable role in operator theory. Recall that, the classical weak Hardy spaces $WH^1(\mathbb{R}^n)$ was first introduced by Fefferman and Soria [44] who tried to find the biggest spaces X such that the Hilbert transform is bounded from

X to the weak Lebesgue spaces $WL^1(\mathbb{R}^n)$. In the same article, they also established the boundedness of some Calderón-Zygmund operators from $WH^1(\mathbb{R}^n)$ to $WL^1(\mathbb{R}^n)$. Furthermore, Fefferman et al [45] proved that the weak Hardy spaces $WH^p(\mathbb{R}^n)$ is the intermediate space of the real interpolation between the Hardy spaces $H^p(\mathbb{R}^n)$ and the Lebesgue spaces $L^\infty(\mathbb{R}^n)$, which is another main motivation to develop a real-variable theory of $WH^p(\mathbb{R}^n)$. Thus, the classical weak Hardy spaces $WH^p(\mathbb{R}^n)$ play a vital role in the study of the boundedness of operators in this field. Kempka and Vybíral [61] studied the weak Lebesgue spaces $WL^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent. Recently, Yan et al [127] introduced the variable weak Hardy spaces $WH^{p(\cdot)}(\mathbb{R}^n)$ and proved some characterizations and the boundedness of some operators. Subsequently, Zhuo et al [130] further studied the function spaces $WH^{p(\cdot)}(\mathbb{R}^n)$.

The motivation for the increasing interest in such spaces comes not only from theoretical purposes, but also from application to in partial differential equations (PDE) with non-standard growth [51], image processing [16]. On the other hand, due to their wide application in fluid dynamics [99]. Another example of the application, the variable of Lebesgue space proves useful in nonlinear PDE with coefficients of a variable growth. For instance, Kováčik and Rákosník [63] gave an application to Dirichlet boundary value problems for such fields. On the other hand, as a great substitute of the variable Lebesgue space can be found in [42], p.438, Example], where Fan and Zhao proved that Lebesgue space with variable exponents $L^{p(\cdot)}$ are the key tools to control the nonlinear term of PDE. In addition, Scapellato [103] proved that Herz spaces with variable exponents can be used in the study of the regularity of solutions to elliptic equations. It is well known that the variable function spaces such as Herz spaces, Herz-type Hardy spaces and others have been a central topic in many modern mathematical areas such as geometric analysis, harmonic analysis and PDE.

It is well known that weighted spaces of smooth function play a leading role in research related to partial differential equations. In particular, we find that Function spaces of power weights in time play a central role in evolution equations and in order to obtain results for rough initial data.

In the second half of the seventies of the last century, The mathematicians Jaak Peetre [94] and Hans Triebel [111]-[112] invented "paramultiplication" at the same time, indepen-

dent from each other, to investigate the product

$$A_{p_1, q_1}^{s_1} \cdot A_{p_2, q_2}^{s_2} \hookrightarrow A_{p, q}^s$$

in case of $p_1 = p_2 = p$. Here $A_{p, q}^s$ stands for either the Triebel-Lizorkin space $F_{p, q}^s$ or the Besov space $B_{p, q}^s$. Later, many authors followed the same method for these two mathematicians, nowadays this method has come to be called the paramultiplication.

The motivation to study the problem of multiplication on function spaces comes from applications to partial differential equations, see for example [120], where estimates of the product on function spaces are handy in dealing with the quadratic nonlinear term in many partial differential equations, see also, H. Bahouri, J.-Y. Chemin, and R. Danchin [5], V.G. Maz'ya and T.O. Shaposhnikova [79], and Zeidler [128]. Furthermore, estimates of products of functions have played a key role in investigations of composition operators, see [98, Chapter 5].

Concerning earlier contributions to this subject, the paper of Yamazaki [124] deals with the situation where $p_1 = p_2 = p$, whereas Sickel treats the cases with $p_1 = p_2 \neq p$. Hanouzet [52] has investigated $p_1 \neq p_2$ but restricted to Besov spaces. In the a recent work of Sickel and Triebel the case $p_1 \neq p_2$ is also studied and a rather complete set of necessary conditions is given. A new necessary and sufficient conditions are given in the paper of J. Johnsen [58], see also J. Marschall [77] and [78], and Amann [3] for Sobolev and Besov spaces. In [98] the general case is treated in Subsection 4.5.

Concerning the case $F_{p_1, q_1}^{s_1} \cdot B_{p_2, q_2}^{s_2} \hookrightarrow F_{p, q}^s$. This case was studied by Franke [46] and Marschall in [77] and [78] with $p_1 = p$. In recent paper [29], the case $p_1 \neq p$ is also studied where known sufficient conditions for pointwise multiplication have been improved. More close to this contribution is the book of T. Runst, W. Sickel [98], see also V. Maz'ya, T. Shaponiskova [79].

Recently, Meyries and Veraar [82], and Lindemulder [66] considered Besov and Bessel potential spaces $B_{p, q}^s(\mathbb{R}^n, w)$, $H_p^s(\mathbb{R}^n, w)$ with respect to the weight $w(x, t) = |t|^\alpha$, $x \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$. Under some suitable assumptions on s, p, q and α , they observed that the characteristic function of the half space is a pointwise multiplier for $B_{p, q}^s(\mathbb{R}^n, w)$, $H_p^s(\mathbb{R}^n, w)$.

The purpose of this thesis is to study the 2-linear map

$$A_{p_1, q}^s(\mathbb{R}^n, |\cdot|^\alpha) \cdot A_{p_2, q_2}^r(\mathbb{R}^n, |\cdot|^\alpha) \hookrightarrow A_{p, q}^s(\mathbb{R}^n, |\cdot|^\alpha), \quad (1)$$

induced by

$$(f_1, f_2) \longrightarrow f_1 \cdot f_2,$$

with an application to the continuity of pseudodifferential operators on Triebel-Lizorkin spaces of power weight. Here $A_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)$ stands for either the Besov space $B_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)$ or the Triebel-Lizorkin space $F_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)$. Also, we prove the necessity of the majority assumptions on the parameters.

In the rest of this introduction, we talk about the structure of this thesis which is divided into four parts, in the first chapter, we give the most important basic properties of variable exponent spaces and the Mixed variable Lebesgue's-sequence spaces. We also give some key technical lemmas needed in the proofs of the main statements and we have provided many examples in order to better approximate and explain the concepts to the reader.

In the second chapter, we introduce weak Herz spaces $W\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ with variable exponents and the variable weak Herz-type Hardy spaces $WH\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ including the inhomogeneous version where, and we present an equivalent quasi-norm of such spaces. Based on these equivalent quasi-norms and the results of [28] and [70] and we give the atomic decomposition of $H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ which we will use later to prove the boundedness properties of singular operators on $H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, $W\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and $WH\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

In the third chapter, we briefly recall some properties of Littlewood-Paley series, definitions of Besov space $B_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)$, Triebel-Lizorkin space $F_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)$ and we summarize some the most important inequalities which will be used later. But, the main target of this chapter is to investigate the product (1) under some suitable assumptions on the parameters.

In the fourth chapter, we will prove almost all our assumptions are necessary, and we present an application of our results to the continuity of pseudodifferential operators on Triebel-Lizorkin spaces of power weights.

THE MIXED LEBESGUE-SEQUENCE SPACES

In this chapter, we expose the concepts and results used throughout this thesis. We recall some fundamental properties on modular space, variable Lebesgue spaces and mixed Lebesgue-sequence spaces. We also give some key technical lemmas that we will use later.

1.1 Definition and basic properties of modular space

Recall that modular spaces were initiated by Nakano in 1950 [88] in connection with the theory of order spaces and redefined and generalized by Luxemburg and Orlicz in 1959. These spaces were developed following the successful theory of Orlicz spaces, which replaces the particular integral form of the nonlinear functional which controls the growth of members of the space. Orlicz and Birnbaum considered, for instance, function spaces defined as follows:

$$L^\varphi = \left\{ f : \mathbb{R} \longrightarrow \mathbb{R}; \exists \lambda > 0 : \int_{\mathbb{R}} \varphi(\lambda |f(x)|) dx < \infty \right\},$$

where $\varphi : [0, \infty) \longrightarrow [0, \infty)$ was assumed to be a convex function increasing to infinity, that is, the function which to some extent behaves similarly to power functions $\varphi(t) = t^p$. Let us mention two typical examples of such functions: $\varphi_1(t) = e^t - t - 1$ or $\varphi_2(t) = e^{t^2} - 1$. Later named Orlicz spaces, and many applications to differential and integral equations with kernels of non-power types.

J. Musielak and W. Orlicz in [86] and [87] introduced the notions of pseudo-modular, semi-modular and modular spaces, which, with some slight changes.

We refer to the monographs [24] and [85] for an exposition on semi-modular spaces.

We start by recalling about the semi-modular functional space.

Definition 1.1. Let X be a (real or complex) vector space. A function $\varrho : X \longrightarrow [0, +\infty]$ is called a semi-modular on X if it satisfies the following condition:

1. $\varrho(0) = 0$.
2. For all $x \in X$ and all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, we have $\varrho(\lambda x) = \varrho(x)$.
3. ϱ is convex, namely, we have that, for all $x, y \in X$ and all $t \in [0; 1]$,

$$\varrho(tx + (1-t)y) \leq t\varrho(x) + (1-t)\varrho(y).$$

4. ϱ is left-continuous on $[0, +\infty)$ (for all $x \in X$, $\lim_{\lambda \rightarrow 1^-} \varrho(\lambda x) = \varrho(x)$).
5. $\varrho(\lambda x) = 0$ for all $\lambda > 0$ implies $x = 0$.

A semi-modular ϱ is called a modular if, in addition,

6. $\varrho(x) = 0$ implies $x = 0$.

A semi-modular ϱ is called continuous if

7. the mapping $\lambda \longrightarrow \varrho(\lambda x)$ is continuous on $[0, +\infty)$ for all $x \in X$.

Remark 1.2. (a) The modular doesn't satisfy the triangle inequality, i.e.,

$$\varrho(x + y) \leq \varrho(x) + \varrho(y),$$

(b) If we replace (3) by

(3') $\varrho(\delta x + (1-\delta)y) \leq L(\delta\varrho(x) + (1-\delta)\varrho(y))$ for all $x, y \in X$ and all $t \in [0; 1]$; then the semimodular ϱ is called quasi-convex (or L -convex), note that 1-convex is the usual convex.

(c) Let X be a (real or complex) vector space. A function $\varrho : X \longrightarrow [0, +\infty]$ is called a pseudo-modular on X if it satisfies (1), (2) and (3).

(d) Observe that every pseudo-norm is pseudo-modular and every semi-norm is a semi-modular.

Now we give examples of modular functions :

Example 1.3. Let Ω is a Lebesgue measurable subset of \mathbb{R}^n .

(a) If $1 \leq p < \infty$, then

$$\varrho_p(g) = \int_{\Omega} |g(x)|^p dx,$$

defines a continuous modular on the space of all measurable functions on Ω .

(b) If $1 \leq p < \infty$, then

$$\varrho_p((x_j)) = \sum_{j=0}^{\infty} |x_j|^p,$$

defines a continuous modular on \mathbb{R}^n .

(c) Let $\varphi_{\infty}(t) = \infty \cdot \chi_{(1, \infty)}(t)$. Then

$$\varrho_{\infty}(f) = \int_{\Omega} \varphi_{\infty}(|f(x)|) dx,$$

defines a semi-modular on on the space of all measurable functions on Ω which is not continuous.

(d) Let $\omega \in L^1_{loc}(\mathbb{R}^n)$ with $\omega > 0$ almost everywhere and $1 \leq p < \infty$. Then

$$\varrho(f) = \int_{\Omega} |f(x)|\omega(x) dx,$$

defines a continuous modular on the space of all measurable functions on Ω .

Definition 1.4. If ϱ be a semi-modular or modular on X , then

$$X_{\varrho} = \left\{ x \in X : \lim_{\lambda \rightarrow 0} \varrho(\lambda x) = 0 \right\},$$

is called a semi-modular space or modular space, respectively, where the limit $\lambda \rightarrow 0$ takes place in \mathbb{k} .

We will give other defintion of modular function.

Let ϱ be a semimodular or modular on X , then by convexity and non-negative of ϱ and $\varrho(0) = 0$ it follows that $\varrho(\lambda x)$ is non-decreasing on $[0, \infty)$ for every $x \in X$. Also,

$$\begin{aligned} \varrho(\lambda x) &= \varrho(|\lambda| x) \leq c |\lambda| \varrho(x) \quad \text{for all } |\lambda| \leq 1, \\ \varrho(\lambda x) &= \varrho(|\lambda| x) \leq c \frac{1}{|\lambda|} \varrho(x) \quad \text{for all } |\lambda| \leq 1. \end{aligned} \tag{1.1}$$

From (1.1) we can alternatively define X_ϱ by

$$X_\varrho = \{x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

Theorem 1.5. *Let ϱ be a semi-modular. Then X_ϱ is a quasi-normed \mathbb{k} -vector space. The quasi-norm, called the Luxemburg quasi-norm, is defined by*

$$\|x\|_\varrho = \inf \left\{ \lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1 \right\}.$$

Example 1.6. *The Lebesgue space $L^p(\Omega)$ where $1 \leq p < \infty$, is normed space with Luxemburg norm defined by*

$$\|x\|_\varrho = \inf \left\{ \lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1 \right\},$$

with modular function

$$\varrho_p(f) = \int_{\Omega} |f(x)|^p dx$$

on the space of all measurable functions on Ω .

The following result is very useful from the technical point of view since it allows one to skip working with the complicated structure of the quasi-norm directly, in many situations of interest.

Lemma 1.7. *Let ϱ be a semi-modular on X . Then*

- (a) $\|x\|_\varrho \leq 1$ and $\varrho(x) \leq 1$ are equivalent.
- (b) If ϱ is left-continuous then $\|x\|_\varrho < 1$ and $\varrho(x) < 1$ are equivalent.
- (c) If ϱ is left-continuous then $\|x\|_\varrho = 1$ and $\varrho(x) = 1$ are equivalent.

1.2 Variable Lebesgue spaces

Variable Lebesgue spaces as their names implies, generalize classical Lebesgue spaces replacing the constant p with a variable exponent function $p(\cdot)$. In other words, the space $L^{p(\cdot)}$ are special cases of the generalized Orlicz spaces originated by Nakano [88] and developed by Musielak and Orlicz [86], enjoys many properties similar to classical Lebesgue spaces, but it also differs from Lebesgue space on several properties such as Maximal, Poincaré, Young's convolution inequalities, translation and others. The study of these spaces

have been stimulated by problems of nonlinear elastic mechanics, electrorheological fluids or image restoration, calculus of variations, and differential equations with $p(x)$ -growth conditions. For example, in Růžička [99] one can find a model of electrorheological fluids, where the essential part of the energy is given by $\int |\mathbf{Df}(x)|^{p(x)} dx$, where \mathbf{Df} denotes the symmetric part of the gradient $\nabla \mathbf{f}$. The same type of energy also appears in model proposed by Zhikov [129] for another type of fluids, where the stress tensor depends on a distribution of temperature \mathbf{T} . In this section, we recall the definition of Lebesgue spaces with variable exponents $L^{p(\cdot)}(\mathbb{R}^n)$, we also mention that main results on the basic properties of $L^{p(\cdot)}(\mathbb{R}^n)$. The spaces $L^{p(\cdot)}(\mathbb{R}^n)$ fit into the framework of Musielak-Orlicz spaces and are therefore also semi-modular spaces.

1.2.1 Exponent Functions

For the definition of the variable Lebesgue spaces it is necessary to introduce the kind of variable exponents that we are interested in.

Definition 1.8. *The variable exponents that we consider are always measurable functions on \mathbb{R}^n with range in $[c, \infty[$ for some $c > 0$. We denote the set of such functions by $P_0(\mathbb{R}^n)$.*

We define some notation to describe the range of exponent functions. We use the standard notation

$$p^+ = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x), \quad p^- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x).$$

The subset of variable exponents with range $[1, \infty)$ is denoted by $P(\mathbb{R}^n)$. Some examples of exponent functions on \mathbb{R} include $p(x) = p$ for some constant p , or $p(x) = 2 + \sin(x)$

Definition 1.9. *The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is the class of all measurable functions f on \mathbb{R}^n such that the modular*

$$\varrho_{p(\cdot)}(f/\lambda) = \int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx$$

is finite for some $\lambda > 0$. This is a quasi-Banach function space equipped with the quasi-norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \leq 1 \right\}.$$

This functional defines a norm when $p^- \geq 1$.

Furthermore, if $p(x) \equiv p$ is constant and $1 \leq p \leq \infty$, then the definition (1.9) is equivalent to the classical norm on $L^p(\mathbb{R}^n)$; if $p < \infty$ and

$$\int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^p dx = 1,$$

then $\lambda = \|f\|_{L^p(\mathbb{R}^n)}$; the same is true if $p = \infty$.

Remark 1.10. Let $\Omega \subset \mathbb{R}^n$ be open set. We set

$$L^{p(\cdot)}(\Omega) = \{f : \Omega \rightarrow \mathbb{C} \text{ measurable} : f \chi_{\Omega} \in L^{p(\cdot)}(\mathbb{R}^n)\}$$

and

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \|f \chi_{\Omega}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Example 1.11. Suppose that $p(x) = 2\chi_{[0,2]} + (x^2 + 1)\chi_{\mathbb{R} \setminus [0,2]}$ and consider the function $f(x) = \sqrt{2x}e^{x^2/2}\chi_{[0,2]}$. Then

$$\varrho_{p(\cdot)}(f/\lambda) = \int_{\mathbb{R}} \left| \frac{\sqrt{2x}e^{x^2/2}}{\lambda} \right|^{p(x)} dx = \frac{1}{\lambda^2} \int_0^2 2xe^{x^2} dx = \frac{e^4 - 1}{\lambda^2}.$$

Therefore $\|f\|_{p(\cdot)} = \sqrt{e^4 - 1}$, then $f \in L^{p(\cdot)}(\mathbb{R})$.

Another elementary but useful property of the classical Lebesgue norm is that it is homogeneous in the exponent: more precisely, for $1 < s < \infty$, $\| |f|^s \|_p = \|f\|_{sp}^s$.

This property extends to variable Lebesgue spaces.

Proposition 1.12. Let $p \in P(\mathbb{R}^n)$. Then for all $s, \frac{1}{p_-} \leq s < \infty$,

$$\| |f|^s \|_{p(\cdot)} = \|f\|_{sp(\cdot)}^s.$$

Proof. This follows at once from the definition of the norm: if we put $\mu = \lambda^{1/s}$, we find that

$$\begin{aligned} \| |f|^s \|_{p(\cdot)} &= \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|^s}{\lambda} \right)^{p(x)} dx \leq 1 \right\} \\ &= \inf \left\{ \mu^s > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|^s}{\mu} \right)^{sp(x)} dx \leq 1 \right\} = \|f\|_{sp(\cdot)}^s \end{aligned}$$

□

In the development of the variable exponent function spaces, the concept of log-Hölder continuity is the cornerstone, which was introduced in [22, 20].

1.2.2 Logarithmic Hölder continuity

We say that a function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is *locally log-Hölder continuous*, abbreviated $g \in C_{\text{loc}}^{\text{log}}$, if there exists $c_{\text{log}}(g) > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\text{log}}(g)}{\log(e + 1/|x - y|)} \quad (1.2)$$

for all $x, y \in \mathbb{R}^n$. We say that g satisfies the *log-Hölder continuous at the origin* (or has a *log decay at the origin*), if there exist a constant $c_{\text{log}} > 0$ such that

$$|g(x) - g(0)| \leq \frac{c_{\text{log}}}{\log(e + 1/|x|)}$$

for all $x \in \mathbb{R}^n$. We say that g satisfies the *log-Hölder continuous at infinity* (or has a *log decay at infinity*), if there exists $g_\infty \in \mathbb{R}$ and a constant $c_{\text{log}} > 0$ such that

$$|g(x) - g_\infty| \leq \frac{c_{\text{log}}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that g is *globally-log-Hölder continuous*, abbreviated $g \in C^{\text{log}}$, if it is *locally log-Hölder continuous* and satisfies the *log-Hölder decay* condition. The constants $c_{\text{log}}(g)$ and c_{log} are called the *locally log-Hölder constant* and the *log-Hölder decay constant*, respectively. We note that all functions $g \in C_{\text{loc}}^{\text{log}}$ always belong to L^∞ . By $\mathcal{P}_0^{\text{log}}(\mathbb{R}^n)$ and $\mathcal{P}_\infty^{\text{log}}(\mathbb{R}^n)$ we denote the class of all exponents $p \in P(\mathbb{R}^n)$ which have a log decay at the origin and at infinity, respectively. The notation $\mathcal{P}^{\text{log}}(\mathbb{R}^n)$ is used for all those exponents $p \in P(\mathbb{R}^n)$ which are locally log-Hölder continuous and have a log decay at infinity, with $p_\infty := \lim_{|x| \rightarrow \infty} p(x)$. Obviously we have

$$\mathcal{P}^{\text{log}}(\mathbb{R}^n) \subset \mathcal{P}_0^{\text{log}}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\text{log}}(\mathbb{R}^n)$$

Note that $p \in \mathcal{P}^{\text{log}}(\mathbb{R}^n)$ if and only if $p' \in \mathcal{P}^{\text{log}}(\mathbb{R}^n)$, and since $(p')_\infty = (p_\infty)'$ we write only p'_∞ for any of these quantities. We define the following class of variable exponents

$$\mathcal{P}^{\log} = \left\{ p \in \mathcal{P} : \frac{1}{p} \in C^{\log} \right\},$$

were introduced in [23] Section 2]. We define $\frac{1}{p_\infty} := \lim_{|x| \rightarrow \infty} \frac{1}{p(x)}$ and we use the convention $\frac{1}{\infty} = 0$. Note that although $\frac{1}{p}$ is bounded, the variable exponent p itself can be unbounded.

Example 1.13. For all $x \in \mathbb{R}$, we define

$$p(x) = \max\left(1 - e^{-3|x|}, \min\left(6/5, \max(1/2, 3/2 - x^2)\right)\right),$$

then $p \in C^{\log}(\mathbb{R})$.

Remark 1.14. Local log-Hölder continuity was first considered for the variable Lebesgue spaces by Sharapudinov in 1986 [105], was used by L. Diening [22] to show that the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$, when Ω is bounded; log-Hölder continuity at infinity was introduced in 2003 [20]. Various authors have used different names for this condition, e.g., weak Lipschitz, Dini-Lipschitz, and 0-Hölder.

1.2.3 Some important inequalities

The variable Lebesgue space $L^{p(\cdot)}$ share even more results with the classical Lebesgue space like uniform convexity, modified versions of Hölder's inequality, separability, complex interpolation and embedding ($L^{p(\cdot)} \hookrightarrow L^{q(\cdot)}$ for $p \geq q$) hold. But, in contrast, here are some fundamental properties from the classical space which do not hold in the space $L^{p(\cdot)}$ such as Maximal, Poincaré, Sobolev, etc., inequalities do not hold in modular form. For instance, A. Lerner showed that

$$\int_{\mathbb{R}^n} |Mf|^{p(x)} dx \leq c \int_{\mathbb{R}^n} |f|^{p(x)} dx$$

if and only if $p \in (1, \infty)$ is constant (see [24]). For the Poincaré inequality (see [24]). The translation operator for example, is in general not continuous on $L^{p(\cdot)}(\Omega)$ (see examples 1.15 and 1.16). Young's convolution inequality with a function $g \in L^1$ is in general not continuous, i.e. in general $\|f * g\|_{p(\cdot)} \not\leq c \|g\|_1 \|f\|_{p(\cdot)}$ (see example 1.19).

First, before presenting the following two examples, we will introduce the definition of translation operator $\tau_h f$ defined by $\tau_h : L^{p(\cdot)} \rightarrow L^{p(\cdot)}$, $\tau_h f(x) := f(x+h)$ for all $h \in \mathbb{R}^n$ and

measurable function f .

Example 1.15. Let $\Omega = (-1, 1)$ and $1 \leq \alpha < \beta < \infty$. We put

$$p(x) = \beta \chi_{(-1,0)} + \alpha \chi_{[0,1)} \quad \text{and} \quad f(x) = x^{-1/\beta} \chi_{[0,1)}.$$

Then $p \in \mathcal{P}(\Omega)$ and, obviously $f \in L^{p(\cdot)}(\Omega)$. However, given $h \in (0, 1)$ and for every $\lambda > 0$, we have

$$\varrho_{p(\cdot)}\left(\frac{1}{\lambda} \tau_h f\right) \geq \frac{1}{\lambda} \int_{-h}^0 \frac{dx}{x+h} = \infty,$$

so $\tau_h f \notin L^{p(\cdot)}(\Omega)$.

Below, we will give another counter-example about the construction of an $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with $\tau_h f \notin L^{p(\cdot)}(\mathbb{R}^n)$. The idea of this example [1.16](#) is based on: choose $f_v \in L^{p(\cdot)}(\mathbb{R}^n)$ with $f_v \geq 0$, $\|f_v\|_{p(\cdot)} \leq 2^{-v}$ and $\|\tau_h f_v\|_{p(\cdot)} = 1$. Define $f := \sum_{j=1}^{\infty} f_j$ then $\|f\|_{p(\cdot)} \leq 1$ and $\|\tau_h f\|_{p(\cdot)} = \infty$.

Example 1.16. Let $v \in \mathbb{N}$ and (f_v) be a sequence given as follows $f_v(x) = \frac{2^{-v}}{\log^2(2)} x^{-2} \chi_{[1,2]}$, $v \in \mathbb{N}$, consider the functions $p(x) = 4 \chi_{\mathbb{R}^*} + \frac{1}{2} \chi_{\mathbb{R}_+}$. For all $\lambda > 0$, we shall find that

$$\varrho_{p(\cdot)}\left(\frac{1}{\lambda} f_v\right) = \int_{\mathbb{R}} \left| \frac{f_v(x)}{\lambda} \right|^{p(x)} dx = \frac{2^{-v/2}}{\log(2)\sqrt{\lambda}} \int_1^2 \frac{dx}{x} = \frac{2^{-v/2}}{\sqrt{\lambda}}.$$

Therefore, $\|(f_v)_v\|_{p(\cdot)} = 2^{-v}$, so $f_v \in L^{p(\cdot)}(\mathbb{R})$. On the other hand, we find that

$$\varrho_{p(\cdot)}\left(\frac{1}{\lambda} \tau_h f_v\right) = \int_{\mathbb{R}} \left| \frac{\tau_h f_v(x)}{\lambda} \right|^{p(x)} dx = \frac{2^{-v/2}}{\log(2)\sqrt{\lambda}} \int_1^2 \frac{dx}{x+h} = \frac{2^{-v/2}}{\sqrt{\lambda}} \log_2\left(\frac{2+h}{1+h}\right),$$

so $\|(\tau_h f_v)_v\|_{p(\cdot)} = 2^{-v} \log_2^2\left(\frac{2+h}{1+h}\right)$, then $\tau_h f_v \in L^{p(\cdot)}(\mathbb{R})$. Now, we will search for a real number h_v that is not independent of v such that $\|(\tau_{h_v} f_v)_v\|_{p(\cdot)} = 1$, which yields that

$$2^{-v} \log_2^2\left(\frac{2+h_v}{1+h_v}\right) = 1,$$

then

$$h_v = \frac{1}{\exp(2^{v/2} \log(2)) - 1} - 1.$$

We put

$$f(x) = \sum_{v=1}^{\infty} f_v(x).$$

Hence

$$\|f\|_{p(\cdot)} \leq \sum_{v=1}^{\infty} \|f_v\|_{p(\cdot)} = \sum_{v=1}^{\infty} 2^{-v} = 1,$$

then $f \in L^{p(\cdot)}(\mathbb{R})$. But, in contrast, we have

$$\|\tau_{h_v} f\|_{p(\cdot)} = \left\| \sum_{v=1}^{\infty} \tau_{h_v} f_v \right\|_{p(\cdot)} = \infty.$$

Therefore, $\tau_{h_v} f \notin L^{p(\cdot)}(\mathbb{R})$.

The conclusion is from these two previous examples is that the translation operator is in general not bounded on $L^{p(\cdot)}$. Especially, for every $L^{p(\cdot)}(\mathbb{R}^n)$ with p non-constant there exists $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n \setminus \{0\}$ such that the translation $\tau_h f$ is not continuous $L^{p(\cdot)}(\mathbb{R}^n)$.

Remark 1.17. *The question that arises: what are the conditions for which the translation operator $\tau_h f$ is continuous $L^{p(\cdot)}(\mathbb{R}^n)$? Two years ago, Drihem [32] answered this question, where he investigated the inequality*

$$\|\tau_h f\|_{p(\cdot)} \leq C(h) \|f\|_{p(\cdot)}, \quad h \in \mathbb{R}^n,$$

under some suitable assumption on the function f and the variable exponent p , see Theorem below.

Theorem 1.18. *Let $p \in \mathcal{D}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$, $h \in \mathbb{R}^n$ and $k \in \mathbb{N}$. Then for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with $\text{supp } \mathcal{F} f \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2^{v+1}\}$, $v \in \mathbb{N}_0$, we have*

$$\|\tau_h f\|_{p(\cdot)} \leq c \exp((2 + 2^{vnk} |h|^k) c_{\log}(1/p)) \|f\|_{p(\cdot)},$$

where $c > 0$ is independent of h, k and v .

For the proof of this theorem, see [32]

Example 1.19. *Suppose that $p(x) = 2\chi_{\mathbb{R}_-} + 4\chi_{\mathbb{R}_+^*}$ and consider the functions*

$$f(x) = |x+1|^{-12/25} \chi_{(-2,0)} \quad \text{and} \quad g(x) = |x-2|^{-4/5} \chi_{[-3,3]},$$

so that $g \in L^1(\mathbb{R})$. Evidently, $f \in L^{p(\cdot)}(\mathbb{R})$ because f can be integrable on $(-2, 2)$ to the power 2 since $\frac{12}{25} < \frac{1}{2}$ and on the other hand, taking $1 < x < \frac{3}{2}$ with use change of variable $t = -(y+1)$ we find that

$$g * f \geq \int_{x-3}^{-1} |x-y-2|^{-4/5} |y+1|^{-12/25} dy = \int_0^{2-x} t^{-12/25} (x-1+t)^{-4/5} dt.$$

Now, we use change of variable $s = \frac{t}{x-1}$, we get

$$\int_0^{2-x} t^{-12/25} (x-1+t)^{-4/5} dt = (x-1)^{-7/25} \int_0^{\frac{2-x}{x-1}} s^{-12/25} (s+1)^{-4/5} ds.$$

Therefore

$$g * f \geq \frac{c}{(x-1)^{7/25}}$$

with $c = \int_0^{\frac{2-x}{x-1}} s^{-12/25} (s+1)^{-4/5} ds$. Therefore, $g * f \notin L^{p(\cdot)}(\mathbb{R})$ because $g * f$ can not be integrable on $[1, \frac{3}{2}]$ to the power 4 since $\frac{7}{25} > \frac{1}{4}$.

The conclusion is from this previous example is that Young's convolution inequality in the form $\|f * g\|_{p(\cdot)} \leq c \|g\|_1 \|f\|_{p(\cdot)}$ is not valid on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ and a kernel $g \in L^1(\mathbb{R}^n)$.

Theorem 1.20. Let Ω bounded and $p, q \in P(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $1 < q^- \leq q^+ < \infty$. Then the convolution $*$: $(f, g) \mapsto f * g$ is bounded as a mapping from $L^{p(\cdot)}(\Omega) \times L^1(\mathbb{R}^n)$ to $L^{q(\cdot)}(\Omega)$ if and only if $p^- \geq q^+$

For the proof of this theorem, see [24]

Corollary 1.21. Let $p \in P(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. Then

$$\|f * g\|_{p(\cdot)} \leq c \|g\|_1 \|f\|_{p(\cdot)}$$

for some $c > 0$ and all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and all $g \in L^1(\mathbb{R}^n)$ if and only if p is constant.

We also refer to the papers [21] and [22], where various results on maximal function in variable Lebesgue spaces were obtained.

Very often we have to deal with the norm of characteristic functions on balls (or cubes) when studying the behavior of various operators in Harmonic Analysis. In classical L^p spaces the norm of such functions is easily calculated, but this is not the case when we consider variable exponents. Nevertheless, it is known that for $p \in \mathcal{P}^{\log}$ we have

$$\|\chi_B\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)} \approx |B|. \quad (1.3)$$

Also,

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p(x)}}, \quad x \in B \quad (1.4)$$

for small balls $B \subset \mathbb{R}^n$ ($|B| \leq 2^n$), and

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p_\infty}} \quad (1.5)$$

for large balls ($|B| \geq 1$), with constants only depending on the log-Hölder constant of p (see, for example, [24], Section 4.5).

In variable Lebesgue spaces there are some important lemmas as follows.

Lemma 1.22 (norm-modular unit ball property). *If $p \in P(\mathbb{R}^n)$, then $\|f\|_{p(\cdot)} \leq 1$ and $\varrho_{p(\cdot)}(f) \leq 1$ are equivalent. For $f \in L^{p(\cdot)}(\mathbb{R}^n)$ we have*

- (i) *If $\|f\|_{p(\cdot)} \leq 1$, then $\varrho_{p(\cdot)}(f) \leq \|f\|_{p(\cdot)}$.*
- (ii) *If $1 < \|f\|_{p(\cdot)}$, then $\|f\|_{p(\cdot)} \leq \varrho_{p(\cdot)}(f)$.*

For the proof of this Lemma, see [24].

Lemma 1.23 (generalized Hölder's inequality). *Let $p, q, s \in P(\mathbb{R}^n)$ such that*

$$\frac{1}{s(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}.$$

If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{q(\cdot)}(\mathbb{R}^n)$ then $fg \in L^{s(\cdot)}(\mathbb{R}^n)$ and

$$\|fg\|_{s(\cdot)} \leq c \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}.$$

For the proof of this Lemma, see [63].

1.3 The space $\ell^{q(\cdot)}(L^{p(\cdot)})$

The targets of this section are twofold. The first one is to recall the concept of variable mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ originally introduced by Almeida and Hästö in [1]. The second one is to recall some fundamental properties related to $\ell^{q(\cdot)}(L^{p(\cdot)})$.

Definition 1.24. Let $p, q \in P_0(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$ -functions by the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \inf \left\{ \lambda_v > 0 : \varrho_{p(\cdot)} \left(\frac{f_v}{\lambda_v^{1/q(\cdot)}} \right) \leq 1 \right\}.$$

The (quasi)-norm is defined from this as usual:

$$\| (f_v)_v \|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf \left\{ \mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left(\frac{1}{\mu} (f_v)_v \right) \leq 1 \right\}. \quad (1.6)$$

Since $q^+ < \infty$, then we can replace (1.6) by the simpler expression

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \| |f_v|^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Furthermore, if p and q are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p)$.

Example 1.25. Let $v \in \mathbb{N}_0$ and $\{f_v\}$ be a sequence given as follows $f_v(x) = 2^{-v} |x|^{-1/3} \chi_{[1, \infty)}$, consider the functions $q(x) = p(x) = 2\chi_{\mathbb{R}_-} + 4\chi_{\mathbb{R}_+^*}$. For all $\mu > 0$, $\lambda_v > 0$, we shall find that

$$\varrho_{p(\cdot)} \left(\frac{f_v}{\mu \lambda_v^{1/q(\cdot)}} \right) = \int_{\mathbb{R}} \left| \frac{f_v(x)}{\mu \lambda_v^{1/q(x)}} \right|^{p(x)} dx = \frac{2^{-4v}}{\mu^4 \lambda_v} \int_1^{\infty} |x|^{-4/3} dx = \frac{3 \cdot 2^{-4v}}{\mu^4 \lambda_v},$$

on the other hand, we find that

$$\sum_v \inf \left\{ \lambda_v > 0 : \varrho_{p(\cdot)} \left(\frac{f_v}{\mu \lambda_v^{1/q(\cdot)}} \right) \leq 1 \right\} = \sum_v \frac{3}{2^{4v} \mu^4} = \frac{1}{5\mu^4}.$$

Therefore

$$\| (f_v)_v \|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}))} = \inf \left\{ \mu > 0 : \frac{1}{5\mu^4} \leq 1 \right\} = 5^{-1/4} < \infty,$$

then $f_v \in \ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}))$, but it is not in $\ell^{q(\cdot)}(L^{6-p(\cdot)}(\mathbb{R}))$ because $|x|^{-1/3} \chi_{[1, \infty)}$ is not in $L^2(\mathbb{R})$.

Example 1.26. Suppose that $p \equiv \infty$, $\{f_v\}$ be a sequence as follows $f_v(x) = \frac{e^{-x^2}}{v} \chi_{[-1; \infty)}$ for all $v \in \mathbb{N}_0$ and consider the function $q(x) = 2\chi_{[-1; 1]} + 3\chi_{\mathbb{R} \setminus [-1; 1]}$. Then

$$\varrho_{\ell^{q(\cdot)}(L^\infty)}((f_v)) = \sum_v \inf \{ \lambda_v > 0 : \varrho_\infty(f_v / \lambda_v^{1/q(\cdot)}) \leq 1 \}.$$

Now $\varrho_\infty(g) \leq 1$ if and only if $|g| \leq 1$ almost everywhere. Thus $|f_v| / (\lambda_v^{1/q(\cdot)}) \leq 1$ a.e., hence $\lambda_v \geq \text{esssup}_x |f_v(x)|^{q(x)}$. It follows that

$$\varrho_{\ell^{q(\cdot)}(L^\infty)}\left(\frac{1}{\mu}(f_v)\right) = \sum_v \text{esssup}_x \left| \frac{f_v(x)}{\mu} \right|^{q(x)} = \sum_v \text{ess sup}_{-1 \leq x \leq 1} \frac{e^{-2x^2}}{v^2 \mu^2} = \frac{\pi^2}{6\mu^2},$$

and so $\|(f_v)_v\|_{\ell^{q(\cdot)}(L^\infty(\mathbb{R}))} = \frac{\pi}{\sqrt{6}}$, then $f_v \in \ell^{q(\cdot)}(L^\infty(\mathbb{R}))$.

Observed in [1, Proposition 3.5] that $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a modular if $p^+ < \infty$ and it is continuous if $p^+, q^+ < \infty$. Also in [1, Theorem 3.6] it was shown that (1.6) defines a norm in $\ell^{q(\cdot)}(L^{p(\cdot)})$ if $q(\cdot) \geq 1$ is constant almost everywhere (a.e.) on \mathbb{R}^n and $p(\cdot) \geq 1$, or if $\frac{1}{p(x)} + \frac{1}{q(x)} \leq 1$ a.e. More recently, it was observed in [60, Theorem 1] that it also becomes a norm if $1 \leq q(x) \leq p(x) \leq \infty$ a.e. on \mathbb{R}^n . Simple calculations show that

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \|(|f_v|^r)_v\|_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}^{1/r}, \quad r > 0.$$

It is not difficult to verify that $\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty$ implies $(f_v)_v \in \ell^{q(\cdot)}(L^{p(\cdot)})$, which means $f_v \in L^{p(\cdot)}$ for any $v \in \mathbb{N}_0$. On the other hand, we also note that the left-continuity of the semi-modular confirms useful equivalence

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq 1 \quad \text{if and only if} \quad \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) \leq 1 \quad (\text{unit ball property}).$$

WEAK TYPE ESTIMATE OF SINGULAR INTEGRAL OPERATORS ON VARIABLE WEAK HERZ-TYPE HARDY SPACES

In recent years, function spaces with variable exponents have attracted more and more attention. The growing interest in such spaces is strongly stimulated by the treatment of recent problems in PDE, fluid dynamics and others. For example, one of the main application of such spaces appeared in the Dirichlet problem on the second-order elliptic equation in nondivergence form is:

$$\begin{cases} Lu \equiv \sum_{i,j=1}^n a_{ij}(x)u_{x_i x_j} = f & a.e. \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and Ω is a bounded domain of \mathbb{R}^n , the coefficients $\{a_{ij}\}_{i,j=1}^n$ of L are symmetric and uniformly elliptic; that is for some $\nu \geq 1$ and every $\xi \in \mathbb{R}^n$:

$$a_{ij}(x) = a_{ji}(x) \quad \text{and} \quad \nu^{-1}|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i \xi_j \leq \nu|\xi|^2,$$

with a.e. $x \in \Omega$. For more details for this equation, see [118].

Recently, [117] introduced variable weak Herz spaces $W\dot{K}_{p(\cdot),q}^\alpha(\mathbb{R}^n)$ and variable weak Herz-type Hardy spaces $WH\dot{K}_{p(\cdot),q}^\alpha(\mathbb{R}^n)$, including the inhomogeneous version with variable p , but with fixed α and q , and gave the boundedness for a class of singular integral operators on such spaces. The spaces $W\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and $WH\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ including the in-

homogeneous version with variable exponents α, p and q presented for the first time in the article [11], where the authors of this article generalized some of the main results in [17]. For fixed exponents this problem has been provided in [70].

This chapter is concerned with the boundedness properties of singular integral operators on variable weak Herz spaces and variable weak Herz-type Hardy spaces. Allowing our parameters to vary from point to point will raise extra difficulties, which, in general, are overcome by imposing regularity assumptions on these exponents, either at the origin or at infinity. Our results cover the classical results on weak Herz-type Hardy spaces with fixed exponents.

2.1 Technical lemmas

In this section, we need to recall some results that will be used in the proofs of our main results of the present chapter. The proof of the following results are given in [2], where it is a generalization of (1.3), (1.4) and (1.5) to the case of dyadic annuli.

Lemma 2.1. *Let $\alpha \in L^\infty(\mathbb{R}^n)$ and $r_1 > 0$. If α is log-Hölder continuous both at the origin and at infinity, then*

$$r_1^{\alpha(x)} \lesssim r_2^{\alpha(y)} \times \begin{cases} \left(\frac{r_1}{r_2}\right)^{\alpha^+} & \text{if } 0 < r_2 \leq \frac{r_1}{2} \\ 1 & \text{if } \frac{r_1}{2} < r_2 \leq 2r_1 \\ \left(\frac{r_1}{r_2}\right)^{\alpha^-} & \text{if } r_2 > 2r_1 \end{cases}$$

for any $x \in B(0, r_1) \setminus B(0, \frac{r_1}{2})$ and $y \in B(0, r_2) \setminus B(0, \frac{r_2}{2})$, with the implicit constant not depending on x, y, r_1 and r_2 .

Lemma 2.2. *Let $p \in \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ and let $R = B(0, r) \setminus B(0, \frac{r}{2})$. If $|R| \geq 2^{-n}$, then*

$$\|\chi_R\|_{p(\cdot)} \approx |R|^{\frac{1}{p(x)}} \approx |R|^{\frac{1}{p_\infty}}$$

with the implicit constants independent of r and $x \in R$.

The left-hand side equivalence remains true for every $|R| > 0$ if we assume, additionally, $p \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$.

The next lemma is a Hardy-type inequality, see [28] Lemma 2].

Lemma 2.3. *Let $0 < a < 1$ and $0 < q \leq \infty$. Let $(\varepsilon_k)_{k \in \mathbb{Z}}$ be a sequence of positive real numbers, such that*

$$\|(\varepsilon_k)_{k \in \mathbb{Z}}\|_{\ell^q} = I < \infty.$$

Then the sequences $(\delta_k : \delta_k = \sum_{j \leq k} a^{k-j} \varepsilon_j)_{k \in \mathbb{Z}}$ and $(\eta_k : \eta_k = \sum_{j \geq k} a^{j-k} \varepsilon_j)_{k \in \mathbb{Z}}$ belong to ℓ^q and

$$\|(\delta_k)_{k \in \mathbb{Z}}\|_{\ell^q} + \|(\eta_k)_{k \in \mathbb{Z}}\|_{\ell^q} = cI,$$

with $c > 0$ only depending on a and q .

The following lemma is from [27] Lemma 2.11], see also [65] Lemma 2.6]

Lemma 2.4. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$. For any cubes (balls) P and Q , such that $P \subset Q$ we have*

$$C \left(\frac{|Q|}{|P|} \right)^{1/p^+} \leq \frac{\|\chi_Q\|_{p(\cdot)}}{\|\chi_P\|_{p(\cdot)}} \leq c \left(\frac{|Q|}{|P|} \right)^{1/p^-}$$

with $c, C > 0$ are independent of $|Q|$ and $|P|$.

2.2 Sublinear operators on variable weak Herz spaces

In this section we present our first result of this thesis. For convenience, we set

$$B_k = B(0, 2^k), \quad R_k = B_k \setminus B_{k-1} \quad \text{and} \quad \chi_k = \chi_{R_k}, \quad k \in \mathbb{Z}.$$

Now we present the definition of Herz spaces of variable smoothness and integrability, were introduced and investigated in [28].

Definition 2.5. *Let $p, q \in P_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The inhomogeneous Herz space $K_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n)$ such that*

$$\|f\|_{K_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = \|f \chi_{B_0}\|_{p(\cdot)} + \|(2^{k\alpha(\cdot)} f \chi_k)_{k \geq 1}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Similarly, the homogeneous Herz space $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} = \left\| \left(2^{k\alpha(\cdot)} f \chi_k \right)_{k \in \mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

If α and p, q are constant, then $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = K_{p,q}^{\alpha}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = \dot{K}_{p,q}^{\alpha}(\mathbb{R}^n)$ are the classical Herz spaces.

Herz spaces have a long history. In 1968, the classical Herz space was originally introduced by Herz [53] to study the Bernstein theorem on absolutely convergent Fourier transforms, while the study on Herz spaces can be traced back to the work of Beurling [9] in 1964. Indeed, a special Herz space $A^p(\mathbb{R}^n)$, with $p > 1$ which is also called Beurling algebra, was originally introduced by Beurling [9] to study some convolution algebras.

In the mid-eighties, Herz spaces have been generalized and presented lots of applications related to the classical Hardy spaces by Baernstein and Sawyer. In the 1990s, Lu and yang [74], [75] introduced the homogeneous Herz spaces $K_{p,q}^{\alpha}(\mathbb{R}^n)$ and non-homogeneous Herz spaces $\dot{K}_{p,q}^{\alpha}(\mathbb{R}^n)$ and extended the boundedness of a large class of operators to these spaces. Herz spaces $K_{p(\cdot),q}^{\alpha}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot),q}^{\alpha}(\mathbb{R}^n)$ with variable exponent p but fixed $\alpha \in \mathbb{R}$ and q were recently studied by Izuki [55, 56]. These spaces with variable exponents $\alpha(\cdot)$ and $p(\cdot)$ were studied in [2], where they gave the boundedness results for a wide class of classical operators on these function spaces. The spaces $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, were first introduced by Izuki and Noi in [57], where the exponent q is variable as well. Recently, Drihem and Seghiri [28] generalized some of the main results in [2] to the Herz spaces $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

From then on, Herz spaces have been widely explored in the fields of harmonic analysis and partial differential equations. Indeed, Herz-type spaces prove useful in the study related to partial differential equations. For instance, the variable Herz spaces proved to be the key tools in the study of the regularity of solutions to elliptic equations by Scapellato [103], while Drihem [40]-[41] investigated semilinear parabolic equations with the initial data in Herz spaces or Herz-type Triebel–Lizorkin spaces. Also, the Fourier–Herz space is one of the most suitable spaces to study the global stability for fractional Navier–Stokes equations.

Example 2.6. Let f be a function given as follows $f(x) = \frac{1}{x^2} \chi_{[1,\infty)}$, consider the functions $p(x) = 3\chi_{\mathbb{R}^*} + 2\chi_{\mathbb{R}_+}$, $q(x) = 4\chi_{\mathbb{R}^*} + 2\chi_{\mathbb{R}_+}$ and $\alpha(x) = -1$. Then

$$\begin{aligned}\|f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R})} &= \left\| \left(2^{k\alpha(\cdot)} f \chi_k \right)_{k \in \mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &= \left\| \left(2^{-k} f \chi_k \right)_{k \in \mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}\end{aligned}$$

on the other hand, we find that

$$\begin{aligned}\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left(\frac{2^{-k} f \chi_k}{\mu} \right) &= \sum_{k=1}^{\infty} \inf \left\{ \lambda_k > 0 : \varrho_{p(\cdot)} \left(\frac{2^{-k} f \chi_k}{\mu \lambda_k^{1/q(\cdot)}} \right) \leq 1 \right\} \\ &= \sum_{k=1}^{\infty} \inf \left\{ \lambda_k > 0 : \frac{2^{-2k}}{3\lambda_k \mu^2} \leq 1 \right\} = \sum_{k=1}^{\infty} \frac{2^{-2k}}{3\mu^2} = \frac{1}{9\mu^2}.\end{aligned}$$

Therefore $\|f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R})} = \frac{1}{3}$, then $f \in \dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R})$.

Let us denote

$$\|\{g_k\}\|_{\ell_{>}^q(L^{p(\cdot)})} = \left(\sum_{k=0}^{\infty} \|g_k\|_{p(\cdot)}^q \right)^{1/q} \quad \text{and} \quad \|\{g_k\}\|_{\ell_{<}^q(L^{p(\cdot)})} = \left(\sum_{k=-\infty}^{-1} \|g_k\|_{p(\cdot)}^q \right)^{1/q}$$

for sequences $\{g_k\}_{k \in \mathbb{Z}}$ of measurable functions (with the usual modification when $q = \infty$).

The following proposition plays a fundamental role in this thesis.

Proposition 2.7. *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p, q \in P_0(\mathbb{R}^n)$. If α and q are log-Hölder continuous at infinity, then*

$$K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = K_{p(\cdot),q_\infty}^{\alpha_\infty}(\mathbb{R}^n).$$

Additionally, if α and q have a log decay at the origin, then

$$\|f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \approx \left\| \{2^{k\alpha(0)} f \chi_k\} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})} + \left\| \{2^{k\alpha_\infty} f \chi_k\} \right\|_{\ell_{>}^{q_\infty}(L^{p(\cdot)})}.$$

For the proof see [28, Proposition 1]. We refer the reader to the paper [100], where the continuous variable Herz spaces are studied.

Let $k \in \mathbb{Z}$ and $\lambda > 0$. We set

$$A_k(\lambda, f) = \{x \in R_k : |f(x)| > \lambda\} \quad \text{and} \quad \tilde{A}_0(\lambda, f) = \{x \in B(0, 1) : |f(x)| > \lambda\}.$$

Proposition 2.8. *Let f and g be measurable functions. Then for all $\lambda, \mu > 0$ and $k \in \mathbb{Z}$ we have*

- (1) $|g| \leq |f|$ implies that $A_k(\lambda, g) \subseteq A_k(\lambda, f)$;
- (2) $A_k(\lambda, cf) = A_k(\frac{\lambda}{|c|}, f)$ for all $c \in \mathbb{C} \setminus \{0\}$;
- (3) $A_k(\lambda + \mu, f + g) \subseteq A_k(\lambda, f) \cup A_k(\mu, g)$;
- (4) $A_k(\lambda\mu, fg) \subseteq A_k(\lambda, f) \cup A_k(\mu, g)$.

Proposition 2.9. *Let f_1, f_2, \dots, f_n be measurable functions. Then for all $\lambda > 0$ and $k \in \mathbb{Z}$ we have*

$$A_k\left(\lambda, \sum_{i=1}^N f_i\right) \subseteq \bigcup_{i=1}^N A_k(\lambda/N, f_i).$$

Now we will give the definition of the weak Herz space.

Definition 2.10. *Let $p, q \in P_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The inhomogeneous weak Herz space $WK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ consists of all measurable functions f such that*

$$\|f\|_{WK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = \sup_{\lambda > 0} \lambda \left\| \left(2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \right)_{k \geq 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty,$$

where A_0 is replaced by \tilde{A}_0 . Similarly, the homogeneous weak Herz space $W\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ is defined as the set of all measurable functions f such that

$$\|f\|_{W\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = \sup_{\lambda > 0} \lambda \left\| \left(2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \right)_{k \in \mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

If α, p and q are constant, then $WK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)} = WK_{p, q}^\alpha$ and $W\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = W\dot{K}_{p, q}^\alpha(\mathbb{R}^n)$ are the classical weak Herz spaces. Now we recall the variable weak $L^{p(\cdot)}(\mathbb{R}^n)$ spaces.

Definition 2.11. *Let $p \in P_0(\mathbb{R}^n)$. The weak Lebesgue space with variable exponent $L^{p(\cdot), \infty}(\mathbb{R}^n)$ consists of all Lebesgue measurable function f satisfying*

$$\|f\|_{L^{p(\cdot), \infty}} = \sup_{\lambda > 0} \lambda \left\| \chi_{\{x \in \mathbb{R}^n : |f(x)| > \lambda\}} \right\|_{p(\cdot)} < \infty.$$

It is obvious that if p is constant, then $W\dot{K}_{p, p}^0(\mathbb{R}^n) = WK_{p, p}^0(\mathbb{R}^n) = L^{p, \infty}(\mathbb{R}^n)$.

Example 2.12. *Let f be a function given as follows $f(x) = \sqrt{x} \chi_{[0,1]}$, consider the functions $p(x) = 3\chi_{\mathbb{R}^*} + 2\chi_{\mathbb{R}_+}$, $q(x) = 4\chi_{\mathbb{R}^*} + 2\chi_{\mathbb{R}_+}$ and $\alpha(x) = 1$. Then*

$$\begin{aligned}
\|f\|_{W\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R})} &= \sup_{\lambda>0} \lambda \left\| \left(2^{k\alpha(\cdot)} \chi_{A_k(\lambda,f)} \right)_{k \in \mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\
&= \sup_{\lambda>0} \lambda \left\| \left(2^k \chi_{\{x \in R_k: \sqrt{x} \chi_{[0,1]} > \lambda\}} \right)_{k \in \mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\
&= \sup_{0<\lambda<1} \lambda \left\| \left(2^k \chi_{\{x \in R_k: \sqrt{x} > \lambda\}} \right)_{k \in \mathbb{Z} \setminus \mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\
&\lesssim \sup_{0<\lambda<1} \lambda \left\| \left(2^k \chi_{\{x \in [0,1]: x > \lambda^2\}} \right)_{k \in \mathbb{Z} \setminus \mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\
&\lesssim \left\| \left(2^k \chi_{[0,1]} \right)_{k \in \mathbb{Z} \setminus \mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}
\end{aligned}$$

on the other hand, we find that

$$\begin{aligned}
\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left(\frac{2^k \chi_{[0,1]}}{\mu} \right) &= \sum_{k=-\infty}^0 \inf \left\{ \eta_k > 0 : \varrho_{p(\cdot)} \left(\frac{2^k \chi_{[0,1]}}{\mu \eta_k} \right) \leq 1 \right\} \\
&= \sum_{k=-\infty}^0 \inf \left\{ \eta_k > 0 : \frac{2^{-2k}}{\mu^2 \eta_k} \leq 1 \right\} = \sum_{\ell=0}^{\infty} \frac{2^{2\ell}}{\mu^2} = \frac{4}{3\mu^2},
\end{aligned}$$

therefore

$$\|f\|_{W\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R})} \lesssim \left\| \left(2^k \chi_{[0,1]} \right)_{k \in \mathbb{Z} \setminus \mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \frac{2}{\sqrt{3}}$$

then $f \in W\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R})$.

Now, we present the main result of this section.

Proposition 2.13. *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p, q \in P_0(\mathbb{R}^n)$. If α and q are log-Hölder continuous at infinity, then*

$$W K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = W K_{p(\cdot),q_\infty}^{\alpha_\infty}(\mathbb{R}^n).$$

Additionally, if α and q have a log decay at the origin, then

$$\|f\|_{W\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \approx \sup_{\lambda>0} \left(\lambda \left\| \{2^{k\alpha(0)} \chi_{A_k(\lambda,f)}\} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})} + \lambda \left\| \{2^{k\alpha_\infty} \chi_{A_k(\lambda,f)}\} \right\|_{\ell_{>}^{q_\infty}(L^{p(\cdot)})} \right). \quad (2.1)$$

Proof. Our arguments are based on [28, Proposition 1]. We will do the proof in four steps.

Step 1. We will prove that

$$W K_{p(\cdot),q_\infty}^{\alpha_\infty}(\mathbb{R}^n) \leftrightarrow W K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n).$$

We need to prove that

$$\|f\|_{WK_{p^{(\cdot)},q^{(\cdot)}}^{\alpha^{(\cdot)}}} \lesssim \|f\|_{WK_{p^{(\cdot)},q_{\infty}}^{\alpha_{\infty}}}$$

for any $f \in WK_{p^{(\cdot)},q_{\infty}}^{\alpha_{\infty}}(\mathbb{R}^n)$. By the scaling argument, we see that it suffices to consider the case $\|f\|_{WK_{p^{(\cdot)},q_{\infty}}^{\alpha_{\infty}}} = 1$ and show that

$$\sum_{k=1}^{\infty} \left\| \left| c \lambda 2^{k\alpha^{(\cdot)}} \chi_{A_k(\lambda,f)} \right|^{q^{(\cdot)}} \right\|_{\frac{p^{(\cdot)}}{q^{(\cdot)}}} \leq 1 \quad (2.2)$$

for any $\lambda > 0$ and some constant $c > 0$ independent of λ . Since α has logarithmic decay at infinity, then for $k \geq 1$ and $x \in R_k$ we have

$$k|\alpha(x) - \alpha_{\infty}| \lesssim \frac{k}{\log(e + |x|)} \lesssim 1.$$

Therefore, $2^{k\alpha(x)} \approx 2^{k\alpha_{\infty}}$, $x \in R_k$ with constants independent of k and x , and hence

$$\sum_{k=1}^{\infty} \left\| \left| c \lambda 2^{k\alpha^{(\cdot)}} \chi_{A_k(\lambda,f)} \right|^{q^{(\cdot)}} \right\|_{\frac{p^{(\cdot)}}{q^{(\cdot)}}} \approx \sum_{k=1}^{\infty} \left\| \left| c \lambda 2^{k\alpha_{\infty}} \chi_{A_k(\lambda,f)} \right|^{q^{(\cdot)}} \right\|_{\frac{p^{(\cdot)}}{q^{(\cdot)}}}.$$

In addition $\|\lambda \chi_{A_0(\lambda,f)}\|_{p^{(\cdot)}} \leq 1$ for any $\lambda > 0$, which yields that

$$\left\| \left| \lambda \chi_{A_0(\lambda,f)} \right|^{q^{(\cdot)}} \right\|_{\frac{p^{(\cdot)}}{q^{(\cdot)}}} \leq 1$$

for any $\lambda > 0$. Observe that (2.2), clearly follows if we prove the estimate

$$\left\| \left| c \lambda 2^{k\alpha_{\infty}} \chi_{A_k(\lambda,f)} \right|^{q^{(\cdot)}} \right\|_{\frac{p^{(\cdot)}}{q^{(\cdot)}}} \leq \left\| \left| \lambda 2^{k\alpha_{\infty}} \chi_{A_k(\lambda,f)} \right|^{q_{\infty}} \right\|_{p^{(\cdot)}} + 2^{-k} = \delta, \quad k \geq 1. \quad (2.3)$$

This claim can be reformulated as showing that

$$\left\| \delta^{-1} \left| c \lambda 2^{k\alpha_{\infty}} \chi_{A_k(\lambda,f)} \right|^{q^{(\cdot)}} \right\|_{\frac{p^{(\cdot)}}{q^{(\cdot)}}} \leq 1,$$

which is equivalent to

$$\left\| c \lambda \delta^{-\frac{1}{q^{(\cdot)}}} 2^{k\alpha_{\infty}} \chi_{A_k(\lambda,f)} \right\|_{p^{(\cdot)}} \leq 1.$$

Since q has logarithmic decay at infinity, then for $k \geq 1$ and $x \in R_k$ we have

$$\frac{k|q(x)-q_\infty|}{q_\infty q(x)} \leq \frac{k|q(x)-q_\infty|}{q_\infty q^-} \lesssim \frac{k}{\log(e+|x|)} \lesssim 1.$$

Therefore, $2^{k(\frac{1}{q_\infty}-\frac{1}{q(x)})} \approx 1$ with constants independent of k and x . Also, since $1 < 2^k \delta < 2^{k+1}$,

$$(2^k \delta)^{\frac{1}{q_\infty}-\frac{1}{q(x)}} \leq (2^{k+1})^{|\frac{1}{q_\infty}-\frac{1}{q(x)}|} \lesssim 1.$$

Using the fact that

$$\delta^{-\frac{1}{q(x)}} = (2^k \delta)^{\frac{1}{q_\infty}-\frac{1}{q(x)}} 2^{k(\frac{1}{q(x)}-\frac{1}{q_\infty})} \delta^{-\frac{1}{q_\infty}} \quad \text{and} \quad \left\| \lambda 2^{k\alpha_\infty} \chi_{A_k(\lambda, f)} \right\|_{p(\cdot)} \leq \delta^{\frac{1}{q_\infty}},$$

we obtain, with an appropriate choice of $c > 0$

$$\left\| c \lambda \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha_\infty} \chi_{A_k(\lambda, f)} \right\|_{p(\cdot)} \leq \left\| \lambda \delta^{-\frac{1}{q_\infty}} 2^{k\alpha_\infty} \chi_{A_k(\lambda, f)} \right\|_{p(\cdot)} \leq 1.$$

Step 2. In this step we prove that

$$WK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) \hookrightarrow WK_{p(\cdot), q_\infty}^{\alpha_\infty}(\mathbb{R}^n),$$

which is equivalent to

$$\|f\|_{WK_{p(\cdot), q_\infty}^{\alpha_\infty}} \lesssim \|f\|_{WK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}$$

for any $f \in WK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$. Again, by the scaling argument, we see that it suffices to consider the case $\|f\|_{WK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = 1$ and show that

$$\sum_{k=1}^{\infty} \left\| c \lambda 2^{k\alpha_\infty} \chi_{A_k(\lambda, f)} \right\|_{p(\cdot)}^{q_\infty} \lesssim 1 \tag{2.4}$$

for some constant $c > 0$. As before, we have for $k \geq 1$

$$\left\| \lambda 2^{k\alpha_\infty} \chi_{A_k(\lambda, f)} \right\|_{p(\cdot)}^{q_\infty} \lesssim \left\| \lambda 2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \right\|_{p(\cdot)}^{q_\infty}.$$

Now, our estimate [\(2.4\)](#), clearly follows from the inequality

$$\left\| c \lambda 2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \right\|_{p(\cdot)}^{q_\infty} \leq \left\| \left| \lambda 2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^{-k} = \delta. \tag{2.5}$$

This claim can be reformulated as showing that

$$\left\| c \lambda \delta^{-\frac{1}{q_\infty}} 2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \right\|_{p(\cdot)} \leq 1.$$

From above, $\delta^{-\frac{1}{q_\infty}} \lesssim \delta^{-\frac{1}{q(x)}}$, then with an appropriate choice of $c > 0$

$$\left\| c \lambda \delta^{-\frac{1}{q_\infty}} 2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \right\|_{p(\cdot)} \leq \left\| \lambda \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \right\|_{p(\cdot)}.$$

The left-hand side is less than or equal 1 if and only if

$$\left\| \left| \lambda \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1.$$

We see that the right-hand side can be rewritten us

$$\delta^{-1} \left\| \left| \lambda 2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1$$

which follows immediately from the definition of δ . *Step 3.* Let us prove that

$$\left\| \{ \lambda 2^{k\alpha(0)} \chi_{A_k(\lambda, f)} \} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})} + \left\| \{ \lambda 2^{k\alpha_\infty} \chi_{A_k(\lambda, f)} \} \right\|_{\ell_{>}^{q_\infty}(L^{p(\cdot)})} \lesssim \|f\|_{W\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}.$$

for any $\lambda > 0$. We suppose that $\|f\|_{W\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \leq 1$. If, in addition, α has a log decay at the origin, then we also have $2^{k\alpha(x)} \approx 2^{k\alpha(0)}$ for $k < 0$ and $x \in R_k$. Thus

$$\left\| \{ \lambda 2^{k\alpha(0)} \chi_{A_k(\lambda, f)} \} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})} \approx \left\| \{ \lambda 2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})}.$$

As in Step 2 we can prove that

$$\left\| c \lambda 2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \right\|_{p(\cdot)}^{q(0)} \leq \left\| \left| \lambda 2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^k$$

for any $k < 0$ and for some constant $c > 0$. Then

$$\left\| \{ \lambda 2^{k\alpha(0)} \chi_{A_k(\lambda, f)} \} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})} \lesssim 1.$$

Using the estimate (2.5) we obtain

$$\left\| \{ \lambda 2^{k\alpha_\infty} \chi_{A_k(\lambda, f)} \} \right\|_{\ell_{>}^{q_\infty}(L^{p(\cdot)})} \lesssim 1.$$

Therefore,

$$\left\| \{ \lambda 2^{k\alpha(0)} \chi_{A_k(\lambda, f)} \} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})} + \left\| \{ \lambda 2^{k\alpha_\infty} \chi_{A_k(\lambda, f)} \} \right\|_{\ell_{>}^{q_\infty}(L^{p(\cdot)})} \lesssim 1.$$

The desired estimate can be obtained by the scaling argument. *Step 4.* Let

$$\left\| \{ \lambda 2^{k\alpha(0)} \chi_{A_k(\lambda, f)} \} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})} \leq 1$$

and

$$\left\| \{ \lambda 2^{k\alpha_\infty} \chi_{A_k(\lambda, f)} \} \right\|_{\ell_{>}^{q_\infty}(L^{p(\cdot)})} \leq 1.$$

From Step 1 we have for any $k < 0$ and for some constant $c > 0$

$$\left\| \left| c \lambda 2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq \left\| \lambda 2^{k\alpha(0)} \chi_{A_k(\lambda, f)} \right\|_{p(\cdot)}^{q(0)} + 2^k$$

and using (2.3) we obtain

$$\sum_{k=-\infty}^{\infty} \left\| \left| \lambda 2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \lesssim 1.$$

Therefore, $\|f\|_{W\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \lesssim 1$. The desired estimate follows by the scaling argument. \square

We consider sublinear operators satisfying the size condition

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy, \quad x \notin \text{supp } f, \quad (2.6)$$

for integrable and compactly supported functions f . Condition (2.6) was first considered in [109] and it is satisfied by several classical operators in Harmonic Analysis, such as Calderón-Zygmund operators, the Carleson maximal operator and the Hardy-Littlewood maximal operator (see [109], [71]).

We have the following results:

Theorem 2.14. *Let $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ and $q \in P_0(\mathbb{R}^n)$ are log-Hölder continuous at infinity, $p \in \mathcal{P}_\infty^{\text{log}}(\mathbb{R}^n)$, $0 < q^- \leq q^+ < 1$ and $1 < p^- \leq p^+ < \infty$ with*

$$-\frac{n}{p_\infty} < \alpha_\infty < n\left(1 - \frac{1}{p_\infty}\right). \quad (2.7)$$

Suppose that T is a sublinear operator satisfying estimate (2.6). If T is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot),\infty}(\mathbb{R}^n)$, then T is bounded on from $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ to $W\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

For homogeneous spaces we have the following statement:

Theorem 2.15. Let $q \in P_0(\mathbb{R}^n)$, $p \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < 1$. Let $\alpha \in L^\infty(\mathbb{R}^n)$ and q are log-Hölder continuous, both at the origin and at infinity, such that

$$-\frac{n}{p^+} < \alpha^- \leq \alpha^+ < n\left(1 - \frac{1}{p^-}\right). \quad (2.8)$$

Then every sublinear operator T satisfying (2.6) which is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot),\infty}(\mathbb{R}^n)$ is also bounded from $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ to $W\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Proof. By similarity, we prove only Theorem 2.15. Our proofs use partially some decomposition techniques already used in [70] where the constant exponent case was studied. In view of Proposition 2.13 we use the equivalent quasinorm (2.1). We split the operator into

$$|Tf| \leq |T(f\chi_{B_{k-2}})| + |T(f\chi_{\tilde{R}_k})| + |T(f\chi_{\mathbb{R}^n \setminus B_{k+2}})|,$$

with $k \in \mathbb{Z}$, where $\tilde{R}_k := \{x \in \mathbb{R}^n : 2^{k-2} \leq |x| < 2^{k+2}\}$. Estimate of $T(f\chi_{B_{k-2}})$. Let $\lambda > 0$,

$$I_1 = \lambda \left\| \{2^{k\alpha(0)} \chi_{A_k(\lambda, T(f\chi_{B_{k-2}}))}\} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})} \quad \text{and} \quad I_2 = \lambda \left\| \{2^{k\alpha_\infty} \chi_{A_k(\lambda, T(f\chi_{B_{k-2}}))}\} \right\|_{\ell_{>}^{q_\infty}(L^{p(\cdot)})}.$$

Given $k \leq -1$ and $x \in R_k$ we write

$$|T(f\chi_{B_{k-2}})(x)| \lesssim \int_{B_{k-2}} |x-y|^{-n} |f(y)| dy = c \sum_{j=-\infty}^{k-2} \int_{R_j} |x-y|^{-n} |f(y)| dy.$$

Noting that $|x-y| > 2^{k-2}$ for $x \in R_k$ and $y \in R_j$. Hence

$$|T(f\chi_{B_{k-2}})(x)| \lesssim 2^{-kn} \sum_{j=-\infty}^{k-2} \int_{R_j} |f(y)| dy \lesssim 2^{-kn} \sum_{j=-\infty}^{k-2} \|f\chi_j\|_{p(\cdot)} \|\chi_j\|_{p'(\cdot)},$$

by Hölder's inequality. Observe that $2^{j\alpha(x)} \approx 2^{j\alpha(0)}$, $x \in R_j$, which yields that

$$2^{k\alpha(0)} |T(f\chi_{B_{k-2}})(x)| \lesssim 2^{-kn} \sum_{j=-\infty}^{k-2} 2^{(k-j)\alpha(0)} \|2^{j\alpha(\cdot)} f \chi_j\|_{p(\cdot)} \|\chi_j\|_{p'(\cdot)}.$$

Therefore

$$\lambda 2^{k\alpha(0)} \|\chi_{A_k(\lambda, T(f\chi_{B_{k-2}}))}\|_{p(\cdot)}$$

can be estimated by

$$\begin{aligned} & c 2^{-kn} \sum_{j=-\infty}^{k-2} 2^{(k-j)\alpha(0)} \|2^{j\alpha(\cdot)} f \chi_j\|_{p(\cdot)} \|\chi_j\|_{p'(\cdot)} \|\chi_k\|_{p(\cdot)} \\ & \lesssim 2^{-kn} 2^{j\frac{n}{p'(y)}} 2^{k\frac{n}{p(x)}} \sum_{j=-\infty}^{k-2} 2^{(k-j)\alpha(0)} \|2^{j\alpha(\cdot)} f \chi_j\|_{p(\cdot)}, \end{aligned}$$

where the positive constant c is independent of k , $x \in R_k$ and $y \in R_j$. Since p has a log decay at the origin, we also have

$$2^{j\frac{n}{p'(y)}} \approx 2^{j\frac{n}{p'(0)}}, \quad 2^{k\frac{n}{p(x)}} \approx 2^{k\frac{n}{p(0)}}, \quad x \in R_k, y \in R_j.$$

We can estimate the last term by

$$c \sum_{j=-\infty}^{k-2} 2^{(k-j)(\alpha(0)-n+\frac{n}{p(0)})} \|2^{j\alpha(\cdot)} f \chi_j\|_{p(\cdot)}$$

Since $\alpha(0) - n + \frac{n}{p(0)} < 0$, we apply Lemma [2.3](#) we get

$$I_1 \lesssim \|f\|_{\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}},$$

by Proposition [2.7](#). Now we estimate I_2 . In view of the estimation of I_1 , we have

$$\lambda 2^{k\alpha_\infty} \|\chi_{A_k(\lambda, T(f\chi_{B_{k-2}}))}\|_{p(\cdot)}$$

can be estimated by

$$c \sum_{j=-\infty}^{k-2} 2^{(k-j)(\alpha^+ - n)} \|2^{j\alpha(\cdot)} f \chi_j\|_{p(\cdot)} 2^{-j\frac{n}{p'(y)}} 2^{k\frac{n}{p(x)}}, \quad k \geq 0.$$

We can distinguish three cases as follows (here we present the all cases):

- $0 \leq j \leq k-2$: by Lemma 2.2 we get

$$2^{-j \frac{n}{p(y)}} 2^{k \frac{n}{p(x)}} \approx 2^{-j \frac{n}{p_\infty}} 2^{k \frac{n}{p_\infty}} \approx 2^{(k-j) \frac{n}{p_\infty}} \lesssim 2^{(k-j) \frac{n}{p^-}}.$$

- $j < 0 \leq k-2$: in this case we obtain

$$2^{-j \frac{n}{p(y)}} 2^{k \frac{n}{p(x)}} \lesssim 2^{-j \frac{n}{p^-}} 2^{k \frac{n}{p^-}} \lesssim 2^{(k-j) \frac{n}{p^-}}.$$

- $j \leq k-2 < 0$: here we have

$$2^{-j \frac{n}{p(y)}} 2^{k \frac{n}{p(x)}} \approx 2^{\frac{(k-j)n}{p(y)}} 2^{k(\frac{n}{p(x)} - \frac{n}{p(y)})} \lesssim 2^{(k-j) \frac{n}{p^-}}.$$

Indeed, since $|x_k| < 2^k$, $|x_j| < 2^j < 2^k$ we make use of local log-Hölder continuity of p at the origin and get, for $k \leq 0$,

$$(-kn) \left| \frac{1}{p(y)} - \frac{1}{p(x)} \right| \lesssim \frac{-k}{\log(e + \frac{1}{2^k})} \leq c$$

with $c > 0$ independent of k, j, x, y . Therefore

$$\lambda 2^{k\alpha_\infty} \left\| \chi_{A_k(\lambda, T(f \chi_{B_{k-2}}))} \right\|_{p(\cdot)}$$

can be estimated by

$$c \sum_{j=-\infty}^{k-2} 2^{(k-j)(\alpha^+ - n + \frac{n}{p^-})} \left\| 2^{j\alpha(\cdot)} f \chi_j \right\|_{p(\cdot)}, \quad k \geq 0.$$

Since $\alpha^+ - n + \frac{n}{p^-} < 0$, we apply Lemma 2.3 we get $I_2 \lesssim \|f\|_{\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}$, by Proposition 2.7. Estimate of $T(f \chi_{\tilde{R}_k})$. Let $\lambda > 0$,

$$V_1 = \lambda \left\| \{2^{k\alpha(0)} \chi_{A_k(\lambda, T(f \chi_{\tilde{R}_k}))}\} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})}$$

and

$$V_2 = \lambda \left\| \{2^{k\alpha_\infty} \chi_{A_k(\lambda, T(f \chi_{\tilde{R}_k}))}\} \right\|_{\ell_{>}^{q_\infty}(L^{p(\cdot)})}.$$

By similarity, we estimate only V_1 . Since T is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot), \infty}(\mathbb{R}^n)$, it follows that

$$\lambda \left\| \chi_{A_k(\lambda, T(f \chi_{\tilde{R}_k}))} \right\|_{p(\cdot)} \lesssim \left\| f \chi_{\tilde{R}_k} \right\|_{p(\cdot)},$$

where the implicit constant is independent of λ and k . Consequently

$$V_1 \lesssim \left\| \{2^{k\alpha(0)} f \chi_{\tilde{R}_k}\} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})} \lesssim \left\| f \right\|_{\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}},$$

where we used Proposition [2.7](#). *Estimate of $T(f \chi_{\mathbb{R}^n \setminus B_{k+2}})$* . Let $\lambda > 0$

$$J_1 = \lambda \left\| \{2^{k\alpha(0)} \chi_{R_k(\lambda, T(f \chi_{\mathbb{R}^n \setminus B_{k+2}}))}\} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})}$$

and

$$J_2 = \lambda \left\| \{2^{k\alpha_\infty} \chi_{R_k(\lambda, T(f \chi_{\mathbb{R}^n \setminus B_{k+2}}))}\} \right\|_{\ell_{>}^{q_\infty}(L^{p(\cdot)})}$$

By similarity, we estimate only J_1 . Given $k \leq -1$ and $x \in R_k$ we write

$$|T(f \chi_{\mathbb{R}^n \setminus B_{k+2}})(x)| \lesssim \int_{\mathbb{R}^n \setminus B_{k+2}} |x - y|^{-n} |f(y)| dy = c \sum_{j=k+3}^{\infty} \int_{R_j} |x - y|^{-n} |f(y)| dy.$$

Noting that $|x - y| > 2^j$ for $x \in R_k$ and $y \in R_j$, we use successively Hölder's inequality to obtain the estimate

$$2^{k\alpha(0)} |T(f \chi_{\mathbb{R}^n \setminus B_{k+2}})(x)| \lesssim \sum_{j=k+3}^{\infty} 2^{-jn} 2^{(k-j)\alpha^-} \left\| 2^{j\alpha(\cdot)} f \chi_j \right\|_{p(\cdot)} \left\| \chi_j \right\|_{p'(\cdot)}.$$

Therefore

$$\lambda 2^{k\alpha(0)} \left\| \chi_{A_k(\lambda, T(f \chi_{B_{k+2}}))} \right\|_{p(\cdot)}$$

can be estimated by

$$\begin{aligned} & c \sum_{j=k+3}^{\infty} 2^{-jn} 2^{(k-j)\alpha^-} \left\| 2^{j\alpha(\cdot)} f \chi_j \right\|_{p(\cdot)} \left\| \chi_j \right\|_{p'(\cdot)} \left\| \chi_k \right\|_{p(\cdot)} \\ & \lesssim \sum_{j=k+3}^{\infty} 2^{-jn} 2^{(k-j)\alpha^-} \left\| 2^{j\alpha(\cdot)} f \chi_j \right\|_{p(\cdot)} 2^{j \frac{n}{p'(y)}} 2^{k \frac{n}{p(x)}}, \end{aligned}$$

where the positive constant c is independent of k , $x \in R_k$ and $y \in R_j$. Thanks to Lemma [2.4](#), we estimate the last term by

$$c \sum_{j=k+3}^{\infty} 2^{(k-j)(\alpha^- + \frac{n}{p^+})} \|2^{j\alpha(\cdot)} f \chi_j\|_{p(\cdot)}$$

Since $\alpha^- + \frac{n}{p^+} > 0$, we apply Lemma 2.3 we get $J_1 \lesssim \|f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}$, by Proposition 2.7. The proof is complete. \square

Remark 2.16. Theorems 2.14 and 2.15 are proved in [117] in the case of constant $\alpha > 0$. In [117] the author has assumed the boundedness of the maximal operator \mathcal{M} on $L^{p(\cdot)}(\mathbb{R}^n)$ both in the homogeneous and in the inhomogeneous case. Although we assume the slightly stronger assumption $p \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ in the homogeneous case, we stress that only the log-decay of p is assumed when we deal with inhomogeneous spaces. In the classical Herz spaces Theorems 2.14 and 2.15 can be found in [70]. We refer the reader to [97] for the boundedness of Riesz potential operator in continual variable exponents Herz spaces.

Since the Hardy-Littlewood maximal operator \mathcal{M} is sublinear, satisfies the size condition (2.6) and it is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot),\infty}(\mathbb{R}^n)$ if $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $1 \leq p^- \leq p^+ \leq \infty$ (see [24, Theorem 4.3.8]), from Theorems 2.14 and 2.15 we immediately arrive at the following result.

Corollary 2.17. Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 \leq p^- \leq p^+ < \infty$, and $\alpha \in L^\infty(\mathbb{R}^n)$.

- (i) Let $q \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ with $0 < q^- \leq q^+ \leq 1$. If (2.7) holds and α satisfies the log-Hölder decay condition, then \mathcal{M} is bounded from $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ to $W K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.
- (ii) Let $q \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ with $0 < q^- \leq q^+ \leq 1$. If (2.8) holds and α has a log decay, both at the origin and at infinity, then \mathcal{M} is bounded from $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ to $W \dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Remark 2.18. The boundedness of the maximal operator in variable exponent Lebesgue spaces was first proved by Diening [22]. Further details and references on subsequent contributions, can be found in [24, Chapter 4]. We refer the reader to the paper [64] for more results about the boundedness of sublinear operators on Herz spaces of fixed exponents.

2.3 Variable Herz-type Hardy space

In the spirit of the ideas of Orlicz [93] and E. Stein and G. Weiss [108], Nakai and Sawano [89] introduced Hardy spaces with variable exponents on \mathbb{R}^n via the grand maximal function, which can be seen as a generalization of the Hardy spaces and the variable Lebesgue

spaces. Always in the same paper, Nakai and Sawano have investigated several properties for the variable Hardy spaces, in particular, they have obtained the atomic characterization of the Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent, with this decomposition, they have proved the Littlewood–Paley characterization of $H^{p(\cdot)}(\mathbb{R}^n)$. Independently, Cruz-Urbe and Wang [19] studied the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with $p(\cdot)$ satisfying some conditions slightly weaker than those used in [89]. More recently, Liu [67] extended the result of Miyachi [80] to the variable setting, where he introduced the Hardy spaces with variable exponents on domains and studied the atomic decomposition and the duality theory of the variable Hardy space.

Herz-type Hardy spaces were first considered by Chen and Lau [17]. They introduced Hardy spaces associated with the Beurling algebras A^q on the real line with $1 < q \leq 2$. In higher-dimensional and all $1 < q < \infty$ by García-Cuerva [49]. Independently García-Cuerva and Herrero [50] and Lu and Yang [72], [73] generalized the Hardy space theory for Herz spaces. Herz-type Hardy spaces with variable exponent $HK_{p(\cdot),q}^\alpha$ and $\dot{H}K_{p(\cdot),q}^\alpha$ have recently been investigated in [116]. In [28], the authors introduced the variable Herz-type Hardy spaces $HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ and $\dot{H}K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ where all parameters defining the space are variable and established their characterization in terms of atom. Moreover, applying the atomic decomposition, they obtained the boundedness of some singular integral operators on these spaces.

The main goal of this section is to recall an atomic decomposition result for variable Herz-type Hardy space. First we introduce the basic notation. Let $G_N f$ be the grand maximal function of f defined by

$$G_N f(x) = \sup_{\varphi \in \mathcal{A}_N} |\varphi_N^*(f)(x)|,$$

where $\mathcal{A}_N = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha| \leq N, |\beta| \leq N} |x^\alpha \partial^\beta \varphi(x)| \leq 1\}$ and

$$\varphi_N^*(f)(x) = \sup_{t>0} |\varphi_t * f(x)|,$$

with $\varphi_t = t^{-n} \varphi(\frac{\cdot}{t})$.

Definition 2.19. Let $p, q \in P_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$ and $N > n + 1$. The inhomogeneous Herz-type Hardy space $HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$G_N f \in K_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and we define

$$\|f\|_{HK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = \|G_N f\|_{K_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}.$$

Similarly, the homogeneous Herz-type Hardy space $H\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $G_N f \in \dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and we define

$$\|f\|_{H\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = \|G_N f\|_{\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}.$$

We have G_N satisfies the size condition (2.6). Let $q \in P_0(\mathbb{R}^n)$, $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$, and let α and q are log-Hölder continuous at infinity, with $\alpha \in L^\infty(\mathbb{R}^n)$ and $-\frac{n}{p_\infty} < \alpha_\infty < (1 - \frac{1}{p_\infty})$. Then

$$HK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) \cap L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n) = K_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n).$$

Let $q \in P_0(\mathbb{R}^n)$, $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$, and let α and q are log-Hölder continuous, both at the origin and at infinity, such that $\alpha \in L^\infty(\mathbb{R}^n)$ and $-\frac{n}{p^+} < \alpha^- \leq \alpha^+ < n(1 - \frac{1}{p^-})$. Then

$$H\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) \cap L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}) = \dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n).$$

To prove our results we use the atomic decomposition of Herz-type Hardy space.

Definition 2.20. Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p \in P(\mathbb{R}^n)$, $q \in P_0(\mathbb{R}^n)$ and $s \in \mathbb{N}_0$. A function a is said to be a central $(\alpha(\cdot), p(\cdot))$ -atom, if

- (i) $\text{supp } a \subset \overline{B(0, r)} = \{x \in \mathbb{R}^n : |x| \leq r\}$, $r > 0$.
- (ii) $\|a\|_{p(\cdot)} \leq |\overline{B(0, r)}|^{-\alpha(0)/n}$, $0 < r < 1$.
- (iii) $\|a\|_{p(\cdot)} \leq |\overline{B(0, r)}|^{-\alpha_\infty/n}$, $r \geq 1$.
- (iv) $\int_{\mathbb{R}^n} x^\beta a(x) dx = 0$, $|\beta| \leq s$.

A function a on \mathbb{R}^n is said to be a central $(\alpha(\cdot), p(\cdot))$ -atom of restricted type, if it satisfies the conditions (iii), (vi) above and $\text{supp } a \subset B(0, r)$, $r \geq 1$.

Now we come to the atomic decomposition theorems.

Theorem 2.21. Let α and q are log-Hölder continuous at infinity and $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. For any $f \in HK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, we have

$$f = \sum_{k=0}^{\infty} \lambda_k a_k, \quad (2.9)$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$ -atom of restricted type with $\text{supp } a_k \subset B_k$ and

$$\left(\sum_{k=0}^{\infty} |\lambda_k|^{q_{\infty}} \right)^{1/q_{\infty}} \leq c \|f\|_{HK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}.$$

Conversely, if $\alpha_{\infty} \geq n(1 - \frac{1}{p_{\infty}})$ and $s \geq [\alpha_{\infty} + n(\frac{1}{p_{\infty}} - 1)]$, and if (2.9) holds, then $f \in HK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, and

$$\|f\|_{HK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \approx \inf \left\{ \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_{\infty}} \right)^{1/q_{\infty}} \right\},$$

where the infimum is taken over all the decompositions of f as above.

Theorem 2.22. Let α and q are be log-Hölder continuous, both at the origin and at infinity and $p \in \mathcal{D}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. For any $f \in H\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, we have

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, \quad (2.10)$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$ -atom with $\text{supp } a_k \subset B_k$ and

$$\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_{\infty}} \right)^{1/q_{\infty}} \leq c \|f\|_{H\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}.$$

Conversely, if $\alpha(\cdot) \geq n(1 - \frac{1}{p^-})$ and $s \geq [\alpha^+ + n(\frac{1}{p^-} - 1)]$, and if (2.10) holds, then $f \in H\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, and

$$\|f\|_{H\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \approx \inf \left\{ \left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_{\infty}} \right)^{1/q_{\infty}} \right\},$$

where the infimum is taken over all the decompositions of f as above.

For the proof see [28], Theorem 4.

Remark 2.23. In the necessity part of the theorems [2.21]-[2.22], the atoms in the decompositions (2.9) and (2.10) can be taken to be supported in dyadic annuli.

Remark 2.24. We mention that corresponding statements to Theorems 2.21 and 2.22 were proved by Liu and Wang [116], with α and q constants, under the assumption that the maximal operator \mathcal{M} is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (both in the homogeneous and the inhomogeneous situation). Here we are requiring the log-Hölder continuity at two points only (zero and infinity).

Next, we will give the definition of the weak Herz-type Hardy space with variable exponents.

Definition 2.25. Let $p, q \in P_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$ and $N > n + 1$. The inhomogeneous weak Herz-type Hardy space $WHK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $G_N f \in WK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and we define

$$\|f\|_{WHK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = \|G_N f\|_{WK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}.$$

Similarly, the homogeneous weak Herz-type Hardy space $WH\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $G_N f \in W\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and we define

$$\|f\|_{WH\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = \|G_N f\|_{W\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}.$$

2.4 Boundedness of some singular integral operators

Let K be a measurable function defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\}$ that satisfies the size condition

$$|K(x, y) - K(x, 0)| \leq \frac{|y|^\delta}{|x|^{n+\delta}}, \quad (2.11)$$

if $2|y| < |x|$ for some $\delta \in (0, 1]$. Let $T : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ be a linear operator such that for any $f \in L^2(\mathbb{R}^n)$ with compact support and almost everywhere $x \notin \text{supp } f$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy, \quad (2.12)$$

which are more general than the standard Calderón-Zygmund operator.

This section is devoted to the behavior of T on variable Herz type Hardy space.

Theorem 2.26. Let α and q are log-Hölder continuous at infinity and $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < \infty$ such that

$$n\left(1 - \frac{1}{p_\infty}\right) \leq \alpha_\infty < n\left(1 - \frac{1}{p_\infty}\right) + \delta.$$

Then every linear operator T satisfying (2.12) which is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ is also bounded from $HK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ to $K_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

The counterpart for homogeneous Herz spaces runs as follows:

Theorem 2.27. Let α and q be log-Hölder continuous, both at the origin and at infinity and $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < \infty$, such that

$$\alpha(\cdot) \geq n\left(1 - \frac{1}{p^-}\right), \quad \alpha(0) < n\left(1 - \frac{1}{p(0)}\right) + \delta \quad \text{and} \quad \alpha_\infty < n\left(1 - \frac{1}{p_\infty}\right) + \delta.$$

Then every linear operator T satisfying (2.12) which is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ is also bounded from $H\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ to $\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Proof. By similarity, we prove only Theorem 2.27. Let $f \in H\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$. Then

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, \tag{2.13}$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$ -atom with $\text{supp} a_k \subset R_k$ and

$$\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \right)^{1/q_\infty} \leq c \|f\|_{H\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}. \tag{2.14}$$

We will prove that

$$\|Tf\|_{\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \leq c \|f\|_{H\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}.$$

We decompose f into two parts:

$$f = \sum_{k=-\infty}^{j-4} \lambda_k a_k + \sum_{k=j-3}^{\infty} \lambda_k a_k, \quad j \in \mathbb{Z}. \tag{2.15}$$

Define

$$\begin{aligned} I_1 &= \left\| \left\{ 2^{j\alpha(0)} T \left(\sum_{k=-\infty}^{j-4} \lambda_k a_k \right) \chi_j \right\} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})}, \\ I_2 &= \left\| \left\{ 2^{j\alpha_\infty} T \left(\sum_{k=-\infty}^{j-4} \lambda_k a_k \right) \chi_j \right\} \right\|_{\ell_{>}^{q_\infty}(L^{p(\cdot)})}, \\ I_3 &= \left\| \left\{ 2^{j\alpha(0)} T \left(\sum_{k=j-3}^{\infty} \lambda_k a_k \right) \chi_j \right\} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})} \end{aligned}$$

and

$$I_4 = \left\| \left\{ 2^{j\alpha_\infty} T \left(\sum_{k=j-3}^{\infty} \lambda_k a_k \right) \chi_j \right\} \right\|_{\ell_{>}^{q_\infty}(L^{p(\cdot)})}.$$

We will estimate each term separately. *Estimate of I_1 .* Let $x \in R_j$, $j \leq -1$. From (2.11) and Hölder's inequality, we obtain

$$\begin{aligned} |Ta_k(x)| &\leq \int_{\mathbb{R}^n} |k(x, y) - k(x, 0)| |a_k(y)| dy \\ &\leq \int_{R_k} \frac{|y|^\delta}{|x|^{n+\delta}} |a_k(y)| dy \\ &\leq c 2^{k\delta - j(n+\delta)} \|a_k\|_{p(\cdot)} \|\chi_k\|_{p'(\cdot)}, \end{aligned}$$

where the positive constant c independent of x , k and j . Hence, we obtain

$$\begin{aligned} \|(Ta_k)\chi_j\|_{p(\cdot)} &\lesssim 2^{k\delta - j(n+\delta)} \|a_k\|_{p(\cdot)} \|\chi_k\|_{p'(\cdot)} \|\chi_j\|_{p(\cdot)} \\ &\lesssim 2^{(k-j)\delta - jn - k\alpha(0)} \|\chi_k\|_{p'(\cdot)} \|\chi_j\|_{p(\cdot)}. \end{aligned}$$

Observe that

$$2^{-jn} \|\chi_k\|_{p'(\cdot)} \|\chi_j\|_{p(\cdot)} \lesssim 2^{-jn} 2^{k\frac{n}{p'(y)}} 2^{j\frac{n}{p(x)}}, \quad x \in R_j, y \in R_k,$$

where the positive constant c is independent of k , $x \in R_j$ and $y \in R_k$. Since p has a log decay at the origin, we have

$$2^{k\frac{n}{p'(y)}} \approx 2^{k\frac{n}{p'(0)}}, \quad 2^{j\frac{n}{p(x)}} \approx 2^{j\frac{n}{p(0)}}, \quad x \in R_k, y \in R_j.$$

Using the fact that $\alpha(0) < n(1 - \frac{1}{p(0)}) + \delta$, we obtain by Lemma 2.3 that

$$I_1^{q(0)} \lesssim \sum_{j=-\infty}^{-1} \left(\sum_{k=-\infty}^{j-4} |\lambda_k| 2^{(k-j)(\delta+n-\frac{n}{p(0)}-\alpha(0))} \right)^{q(0)} \lesssim \sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \lesssim \|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}^{q(0)}.$$

Estimate of I_2 . We decompose the sum $\sum_{k=-\infty}^{j-4} \lambda_k a_k$ into two parts;

$$\sum_{k=-\infty}^{-1} \lambda_k a_k + \sum_{k=0}^{j-4} \lambda_k a_k, \quad j \in \mathbb{N}_0.$$

We put $\sum_{k=0}^{j-4} \lambda_k a_k = 0$ if $j = 0, 1, 2, 3$. Let $k \leq 0$. From the estimation of I_1 , we get

$$\begin{aligned} \|(Ta_k)\chi_j\|_{p(\cdot)} &\lesssim 2^{(k-j)\delta-jn-k\alpha(0)} \|\chi_k\|_{p'(\cdot)} \|\chi_j\|_{p(\cdot)} \\ &\lesssim 2^{k(\delta+n-\frac{n}{p(0)}-\alpha(0))} 2^{-j(\delta+n-\frac{n}{p_\infty})}, \end{aligned}$$

where we have used Lemma [2.2](#). Therefore

$$\begin{aligned} &\sum_{j=0}^{\infty} 2^{j\alpha_\infty q_\infty} \left(\sum_{k=-\infty}^{-1} |\lambda_k| \|(Ta_k)\chi_j\|_{p(\cdot)} \right)^{q_\infty} \\ &\lesssim \sum_{j=0}^{\infty} 2^{-j(\delta+n-\frac{n}{p_\infty}-\alpha_\infty)q_\infty} \left(\sum_{k=-\infty}^{-1} 2^{k(\delta+n-\frac{n}{p(0)}-\alpha(0))} |\lambda_k| \right)^{q_\infty} \\ &\lesssim \sup_{k \leq 0} |\lambda_k|^{q_\infty} \lesssim \|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}^{q_\infty}. \end{aligned}$$

Now let $k \geq 0$. Again from the estimation of I_1 , we get

$$\begin{aligned} \|(Ta_k)\chi_j\|_{p(\cdot)} &\lesssim 2^{(k-j)\delta-jn-k\alpha_\infty} \|\chi_k\|_{p'(\cdot)} \|\chi_j\|_{p(\cdot)} \\ &\lesssim 2^{k(\delta+n-\frac{n}{p_\infty}-\alpha_\infty)} 2^{-j(\delta+n-\frac{n}{p_\infty})}. \end{aligned}$$

Consequently

$$\begin{aligned} \left(\sum_{j=0}^{\infty} 2^{j\alpha_\infty q_\infty} \left(\sum_{k=0}^{j-4} |\lambda_k| \|(Ta_k)\chi_j\|_{p(\cdot)} \right)^{q_\infty} \right)^{\frac{1}{q_\infty}} &\lesssim \left(\sum_{j=0}^{\infty} \left(\sum_{k=0}^{j-4} 2^{(k-j)(\delta+n-\frac{n}{p_\infty}-\alpha_\infty)} |\lambda_k| \right)^{q_\infty} \right)^{\frac{1}{q_\infty}} \\ &\lesssim \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \right)^{\frac{1}{q_\infty}} \lesssim \|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}, \end{aligned}$$

where the inequality follows by Lemma [2.3](#). Hence $I_2 \lesssim \|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}$. *Estimate of I_3 and I_4 .*

Using the boundedness T on $L^{p(\cdot)}(\mathbb{R}^n)$ we obtain

$$I_3^{q(0)} \leq \sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \left(\sum_{k=j-3}^{\infty} |\lambda_k| \| (Ta_k)\chi_j \|_{p(\cdot)} \right)^{q(0)} \leq \sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \left(\sum_{k=j-3}^{\infty} |\lambda_k| \| a_k \|_{p(\cdot)} \right)^{q(0)}.$$

We write

$$\sum_{k=j-3}^{\infty} |\lambda_k| \| a_k \|_{p(\cdot)} = \sum_{k=j-3}^{-1} |\lambda_k| \| a_k \|_{p(\cdot)} + \sum_{k=0}^{\infty} |\lambda_k| \| a_k \|_{p(\cdot)}.$$

If $k \leq -1$, then $\| a_k \|_{p(\cdot)} \lesssim 2^{-k\alpha(0)}$ and Lemma 2.3 yields

$$\sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \left(\sum_{k=j-3}^{-1} |\lambda_k| \| a_k \|_{p(\cdot)} \right)^{q(0)}$$

is bounded by

$$c \sum_{j=-\infty}^{-1} |\lambda_k|^{q(0)} \lesssim \| f \|_{\dot{H}K_{p(\cdot),q(\cdot)}^{\alpha(0)}}^{q(0)}.$$

Now if $k \geq 0$, then $\| a_k \|_{p(\cdot)} \lesssim 2^{-k\alpha_\infty}$ and

$$\begin{aligned} \sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \left(\sum_{k=0}^{\infty} |\lambda_k| \| a_k \|_{p(\cdot)} \right)^{q(0)} &\lesssim \sup_{k \in \mathbb{N}_0} |\lambda_k|^{q(0)} \sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \\ &\lesssim \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \right)^{q(0)/q_\infty} \\ &\lesssim \| f \|_{\dot{H}K_{p(\cdot),q(\cdot)}^{\alpha(0)}}^{q(0)}. \end{aligned}$$

Finally, we estimate I_4 . We easily obtain

$$I_4^{q_\infty} \lesssim \sum_{j=0}^{\infty} 2^{jq_\infty\alpha_\infty} \sum_{k=j-3}^{\infty} 2^{-kq_\infty\alpha_\infty} |\lambda_k|^{q_\infty} \lesssim \sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \lesssim \| f \|_{\dot{H}K_{p(\cdot),q(\cdot)}^{\alpha(0)}}^{q_\infty}.$$

This completes the proof. □

Theorem 2.28. *Let α and q are be log-Hölder continuous at infinity and $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < 1$, such that*

$$\alpha_\infty = n \left(1 - \frac{1}{p_\infty} \right) + \delta. \quad (2.16)$$

Then every linear operator T satisfying (2.12) which is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ is also bounded from $HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ to $WK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

For homogeneous spaces, we have the following statement:

Theorem 2.29. Let α and q are be log-Hölder continuous, both at the origin and at infinity and $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < 1$ such that

$$\alpha(\cdot) \geq n\left(1 - \frac{1}{p^-}\right), \quad \alpha(0) = n\left(1 - \frac{1}{p(0)}\right) + \delta, \quad \text{and} \quad \alpha_\infty = n\left(1 - \frac{1}{p_\infty}\right) + \delta. \quad (2.17)$$

Then every linear operator T satisfying (2.12) which is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ is also bounded from $H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ to $W\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Proof. By similarity, we prove only Theorem 2.29. Our arguments are based on [70], where the constant exponent case was studied. Let $f \in H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$. Again, thanks to Theorem 2.22, we have the decomposition (4.10) and the estimate (2.14). We will show that

$$\|Tf\|_{W\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \lesssim \|f\|_{H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.$$

for any $f \in H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$. We decompose f as in (2.15). We will use the equivalent quasi-norm (2.1). Let $\lambda > 0$,

$$\begin{aligned} E_1 &= \lambda \left\| \left\{ 2^{j\alpha(0)} \chi_{A_j(\frac{\lambda}{2}, \sum_{k=-\infty}^{j-4} \lambda_k(Ta_k))} \right\} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})}, \\ E_2 &= \lambda \left\| \left\{ 2^{j\alpha_\infty} \chi_{A_j(\frac{\lambda}{2}, \sum_{k=-\infty}^{j-4} \lambda_k(Ta_k))} \right\} \right\|_{\ell_{>}^{q_\infty}(L^{p(\cdot)})}, \\ E_3 &= \lambda \left\| \left\{ 2^{j\alpha(0)} \chi_{A_j(\frac{\lambda}{2}, \sum_{k=j-3}^{\infty} \lambda_k(Ta_k))} \right\} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})} \end{aligned}$$

and

$$E_4 = \lambda \left\| \left\{ 2^{j\alpha_\infty} \chi_{A_j(\frac{\lambda}{2}, \sum_{k=j-3}^{\infty} \lambda_k(Ta_k))} \right\} \right\|_{\ell_{>}^{q_\infty}(L^{p(\cdot)})}.$$

Estimate of E_1 . Let $x \in R_j$ and $k \leq j-4$, $j \leq -1$. Thanks to (2.11) and (2.12) we have the estimate

$$|Ta_k(x)| \leq \int_{R_k} \frac{|y|^\delta}{|x|^{n+\delta}} |a_k(y)| dy.$$

Using Hölder's inequality to estimate the last integral by

$$\begin{aligned} c2^{k\delta-j(n+\delta)}\|a_k\|_{p(\cdot)}\|\chi_k\|_{p'(\cdot)} &\lesssim 2^{k\delta-j(n+\delta)-k\alpha(0)}\|\chi_k\|_{p'(\cdot)} \\ &\lesssim 2^{k(n-\frac{n}{p(0)}+\delta-\alpha(0))}2^{-j(n+\delta)} \\ &\lesssim 2^{-j(n+\delta)}, \end{aligned}$$

where the implicit constant is independent of j and k . Therefore

$$|Ta_k(x)| \leq C2^{-j(n+\delta)} \quad (2.18)$$

for any $x \in R_j$ and any $k \leq j-4$, $j \leq -1$. Observe that (2.18) is true even $j \in \mathbb{N}_0$. Consequently

$$\begin{aligned} \left| \sum_{k=-\infty}^{j-4} \lambda_k T(a_k)(x) \right| &\leq |Ta_k(x)| \sum_{k=-\infty}^{j-4} |\lambda_k| \\ &\leq C2^{-j(n+\delta)} \left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \right)^{\frac{1}{q(0)}} \\ &\leq C2^{-j(n+\delta)} \|f\|_{H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}, \end{aligned}$$

where we used the embedding $\ell_{<}^{q(0)} \hookrightarrow \ell_{<}^1$. Assume that

$$\left\{ x \in R_j : \left| \sum_{k=-\infty}^{j-4} \lambda_k Ta_k(x) \right| > \frac{\lambda}{2} \right\} \neq \emptyset.$$

Then

$$\lambda \leq 2C2^{-j(n+\delta)} \|f\|_{H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}, \quad j \leq -1.$$

For any fixed $\lambda > 0$ we put

$$j_\lambda = \lfloor \frac{1}{n+\delta} \log_2(2C\lambda^{-1} \|f\|_{H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}) \rfloor.$$

The advantage of this choice consists in the fact that

$$\left\{ x \in R_j : \left| \sum_{k=-\infty}^{j-4} \lambda_k Ta_k(x) \right| > \frac{\lambda}{2} \right\} = \emptyset,$$

if $-1 \geq j \geq j_\lambda + 1$. Hence, we obtain

$$\begin{aligned}
E_1 &\leq C\lambda \left(\sum_{j=-\infty}^{j_\lambda} 2^{jq(0)\alpha(0)} \left\| \chi_{\left\{x \in R_j: \left| \sum_{k=-\infty}^{j-4} \lambda_k (Ta_k)(x) \right| > \frac{\lambda}{2}\right\}} \right\|_{p(\cdot)}^{q(0)} \right)^{\frac{1}{q(0)}} \\
&\leq C\lambda \left(\sum_{j=-\infty}^{j_\lambda} 2^{jq(0)\alpha(0)} \left\| \chi_j \right\|_{p(\cdot)}^{q(0)} \right)^{\frac{1}{q(0)}} \\
&\leq C\lambda \left(\sum_{j=-\infty}^{j_\lambda} 2^{jq(0)(\alpha(0) + \frac{n}{p(0)})} \right)^{\frac{1}{q(0)}} \\
&\leq C\lambda 2^{j_\lambda(n+\delta)} \\
&\lesssim \|f\|_{H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.
\end{aligned}$$

Estimate of E_2 . Observe that $\sum_{k=-\infty}^{j-4} |\lambda_k| \lesssim \|f\|_{H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}$, which yields that

$$\left| \sum_{k=-\infty}^{j-4} \lambda_k Ta_k(x) \right| \leq \sum_{k=-\infty}^{j-4} |\lambda_k| |Ta_k(x)| \leq C2^{-j(n+\delta)} \|f\|_{H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}},$$

where we used the estimate (2.18). We have

$$\begin{aligned}
E_2 &\leq C\lambda \left(\sum_{j=0}^{j_\lambda} 2^{jq_\infty \alpha_\infty} \left\| \chi_{\left\{x \in R_j: \left| \sum_{k=-\infty}^{j-4} \lambda_k (Ta_k)(x) \right| > \frac{\lambda}{2}\right\}} \right\|_{p(\cdot)}^{q_\infty} \right)^{\frac{1}{q_\infty}} \\
&\leq C\lambda \left(\sum_{j=0}^{j_\lambda} 2^{jq_\infty \alpha_\infty} \left\| \chi_j \right\|_{p(\cdot)}^{q_\infty} \right)^{\frac{1}{q_\infty}} \\
&\leq C\lambda \left(\sum_{j=0}^{j_\lambda} 2^{jq_\infty (\alpha_\infty + \frac{n}{p_\infty})} \right)^{\frac{1}{q_\infty}} \\
&\lesssim \|f\|_{H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.
\end{aligned}$$

Estimate of E_3 . From the boundedness of T on $L^{p(\cdot)}(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
\sum_{k=j-3}^{\infty} |\lambda_k| \left\| (Ta_k) \chi_j \right\|_{p(\cdot)} &\lesssim \sum_{k=j-3}^{-1} |\lambda_k| \left\| (Ta_k) \chi_j \right\|_{p(\cdot)} + \sum_{k=0}^{\infty} |\lambda_k| \left\| (Ta_k) \chi_j \right\|_{p(\cdot)} \\
&\lesssim \sum_{k=j-3}^{-1} |\lambda_k| \left\| a_k \right\|_{p(\cdot)} + \sum_{k=0}^{\infty} |\lambda_k| \left\| a_k \right\|_{p(\cdot)}
\end{aligned}$$

$$\lesssim \sum_{k=j-3}^{-1} |\lambda_k| 2^{-k\alpha(0)} + \sum_{k=0}^{\infty} |\lambda_k| 2^{-k\alpha_{\infty}}.$$

Hence

$$\sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \left(\sum_{k=j-3}^{-1} |\lambda_k| \|(Ta_k)\chi_j\|_{p(\cdot)} \right)^{q(0)},$$

can be estimated by

$$c \sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \left(\sum_{k=j-3}^{-1} |\lambda_k| 2^{-k\alpha(0)} \right)^{q(0)} \lesssim \sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \lesssim \|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}^{q(0)}.$$

again by Lemma 2.3. Now

$$\begin{aligned} \sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \left(\sum_{k=0}^{\infty} |\lambda_k| \|(Ta_k)\chi_j\|_{p(\cdot)} \right)^{q(0)} &\lesssim \sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \left(\sum_{k=0}^{\infty} |\lambda_k| 2^{-k\alpha_{\infty}} \right)^{q(0)} \\ &\lesssim \sup_{k \in \mathbb{N}_0} |\lambda_k|^{q(0)} \\ &\lesssim \|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}^{q(0)}. \end{aligned}$$

Estimate of E_4 . We easily obtain

$$\begin{aligned} E_4 &\lesssim \left(\sum_{j=0}^{\infty} 2^{jq_{\infty}\alpha_{\infty}} \sum_{k=j-3}^{\infty} 2^{-kq_{\infty}\alpha_{\infty}} |\lambda_k|^{q_{\infty}} \right)^{1/q_{\infty}} \\ &\lesssim \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_{\infty}} \right)^{1/q_{\infty}} \\ &\lesssim \|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}, \end{aligned}$$

by Lemma 2.3. This completes the proof. \square

Remark 2.30. Corresponding statements to Theorems 2.28 and 2.29, with α, p and q constants, can be found in [70], while with α and q constants Theorems 2.28 and 2.29 are proved in [117], under the assumption that the maximal operator \mathcal{M} is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (both in the homogeneous and the inhomogeneous situation). Here we are requiring the log-Hölder continuity at two points only (zero and infinity). We also note that our conditions (2.16) and (2.17) are more explicit than those used in [117], hence allowing better comparison

with the already known constant exponent setting..

If the operator T in Theorems [2.28](#) and [2.29](#) is of convolution type, we then can obtain the following results:

Theorem 2.31. *Let α and q are be log-Hölder continuous at infinity and, $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < 1$, such that*

$$\alpha_\infty = n\left(1 - \frac{1}{p_\infty}\right) + \delta.$$

Let T defined by

$$Tf(x) = \lim_{\varepsilon \rightarrow 0} \int_{|x-y| \geq \varepsilon} K(x-y)f(y) dy,$$

where the kernel K satisfies

$$|K(x-y) - K(x)| \leq \frac{|y|^\delta}{|x|^{n+\delta}},$$

if $2|y| < |x|$ for some $\delta \in (0, 1]$. Assume that T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Then T is also bounded from $HK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ to $WHK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

The counterpart for homogeneous Herz spaces runs as follows:

Theorem 2.32. *Let α and q are be log-Hölder continuous, both at the origin and at infinity and $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < 1$, such that*

$$\alpha(\cdot) \geq n\left(1 - \frac{1}{p^-}\right), \quad \alpha(0) = n\left(1 - \frac{1}{p(0)}\right) + \delta, \quad \text{and} \quad \alpha_\infty = n\left(1 - \frac{1}{p_\infty}\right) + \delta.$$

Let T be as in Theorem [2.31](#). Assume that T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Then T is also bounded from $HK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ to $WHK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Proof. By similarity, we prove only Theorem [2.32](#). Let $f \in HK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$. Then, we have the decomposition [\(4.10\)](#) and the estimate [\(2.14\)](#). As before, we write we decompose f as in [\(2.15\)](#). Let $\lambda > 0$,

$$F_1 = \lambda \left\| \left\{ 2^{j\alpha(0)} \chi_{A_j(\frac{\lambda}{2}, \sum_{k=-\infty}^{j-4} \lambda_k G_N(Ta_k))} \right\} \right\|_{\ell_\zeta^{q(0)}(L^{p(\cdot)})},$$

$$\begin{aligned}
F_2 &= \lambda \left\| \left\{ 2^{j\alpha_\infty} \chi_{A_j(\frac{\lambda}{2}, \sum_{k=-\infty}^{j-4} \lambda_k G_N(Ta_k))} \right\} \right\|_{\ell_{>}^{q_\infty}(L^{p(\cdot)})}, \\
F_3 &= \lambda \left\| \left\{ 2^{j\alpha(0)} \chi_{A_j(\frac{\lambda}{2}, \sum_{k=j-3}^{\infty} \lambda_k G_N(Ta_k))} \right\} \right\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})}
\end{aligned}$$

and

$$F_4 = \lambda \left\| \left\{ 2^{j\alpha_\infty} \chi_{A_j(\frac{\lambda}{2}, \sum_{k=j-3}^{\infty} \lambda_k G_N(Ta_k))} \right\} \right\|_{\ell_{>}^{q_\infty}(L^{p(\cdot)})}.$$

Estimate of F_3 and F_4 . The boundedness of G_N and T on $L^{p(\cdot)}(\mathbb{R}^n)$ yield that

$$\begin{aligned}
F_3 &\leq \left(\sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \left(\sum_{k=j-3}^{\infty} |\lambda_k| \|G_N(Ta_k) \chi_j\|_{p(\cdot)} \right)^{q(0)} \right)^{1/q(0)} \\
&\leq \left(\sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \sum_{k=j-3}^{\infty} |\lambda_k|^{q(0)} \| (Ta_k) \chi_j \|_{p(\cdot)}^{q(0)} \right)^{1/q(0)} \\
&\leq \left(\sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \sum_{k=j-3}^{\infty} |\lambda_k|^{q(0)} \|a_k\|_{p(\cdot)}^{q(0)} \right)^{1/q(0)}. \tag{2.19}
\end{aligned}$$

We decompose the second sum in (2.19) into two parts;

$$\sum_{k=j-3}^{-1} |\lambda_k|^{q(0)} \|a_k\|_{p(\cdot)}^{q(0)} + \sum_{k=0}^{\infty} |\lambda_k|^{q(0)} \|a_k\|_{p(\cdot)}^{q(0)} = P_{1,j} + P_{2,j}.$$

If $k \leq -1$, then $\|a_k\|_{p(\cdot)} \lesssim 2^{-k\alpha(0)}$, and Lemma 2.3, yields

$$\sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} P_{1,j} \lesssim \sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \sum_{k=j-3}^{-1} |\lambda_k|^{q(0)} 2^{-k\alpha(0)q(0)} \lesssim \sum_{j=-\infty}^{-1} |\lambda_k|^{q(0)} \lesssim \|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(0)}}^{q(0)}.$$

Now if $k \geq 0$, then $\|a_k\|_{p(\cdot)} \lesssim 2^{-k\alpha_\infty}$ and

$$\begin{aligned}
\sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} P_{2,j} &\lesssim \sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \sum_{k=0}^{\infty} |\lambda_k|^{q(0)} 2^{-k\alpha_\infty q(0)} \\
&\lesssim \sup_{k \in \mathbb{N}_0} |\lambda_k|^{q(0)} \sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \\
&\lesssim \|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}^{q(0)}.
\end{aligned}$$

We easily obtain that

$$\begin{aligned}
F_4 &\lesssim \left(\sum_{j=0}^{\infty} 2^{jq_{\infty}a_{\infty}} \sum_{k=j-3}^{\infty} 2^{-kq_{\infty}a_{\infty}} |\lambda_k|^{q_{\infty}} \right)^{1/q_{\infty}} \\
&\lesssim \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_{\infty}} \right)^{1/q_{\infty}} \\
&\lesssim \|f\|_{HK_{p(\cdot),q(\cdot)}^{a(\cdot)}},
\end{aligned}$$

by Lemma 2.3. *Estimate of F_1 .* Let $x \in R_j$ and $k \leq j-4$. Let $|x-y| < t$ and $\varphi \in \mathcal{A}_N$. We have

$$\begin{aligned}
(Ta_k * \varphi_t)(y) &= \int_{\mathbb{R}^n} Ta_k(z) (\varphi_t(y-z) - \varphi_t(y)) dz \\
&= t^{-n} \int_{\mathbb{R}^n} Ta_k(z) \left(\varphi\left(\frac{y-z}{t}\right) - \varphi\left(\frac{y}{t}\right) \right) dz \\
&= \int_{|z| < 2^{k+1}} \dots + \int_{2^{k+1} \leq |z| < \frac{|x|}{2}} \dots + \int_{|z| \geq \frac{|x|}{2}} \dots \\
&= P_3(y) + P_4(y) + P_5(y),
\end{aligned}$$

because of $\int_{\mathbb{R}^n} Ta_k(z) dz = 0$. Let us estimate each term separately. By the mean value theorem;

$$\begin{aligned}
\left| \varphi\left(\frac{y-z}{t}\right) - \varphi\left(\frac{y}{t}\right) \right| &\leq t^{-1} |z| \sup_{|\beta|=1} |D^{\beta} \varphi\left(\frac{y}{t} - \theta \frac{z}{t}\right)|, \quad 0 < \theta < 1 \\
&\lesssim t^n |z| (t + |y - \theta z|)^{-n-1}.
\end{aligned} \tag{2.20}$$

Observe that $t + |y - \theta z| \geq |x - y| + |y - \theta z| \geq |x - \theta z| \geq \frac{1}{2}|x|$, which yields

$$\begin{aligned}
|P_3(y)| &\leq C t^{-n} \|Ta_k\|_{p(\cdot)} \left\| \left(\varphi\left(\frac{y-\cdot}{t}\right) - \varphi\left(\frac{y}{t}\right) \right) \chi_{B_k} \right\|_{p'(\cdot)} \\
&\leq C \frac{2^k}{|x|^{n+1}} \|a_k\|_{p(\cdot)} \|\chi_{B_k}\|_{p'(\cdot)} \\
&\leq C \frac{2^{k(1-a(0)+n-\frac{n}{p(0)})}}{|x|^{n+1}} \\
&\leq \frac{C}{|x|^{n+\delta}},
\end{aligned}$$

where the positive constant C is independent of x . Observe that

$$P_4(y) = t^{-n} \int_{2^{k+1} \leq |z| < \frac{|x|}{2}} \int_{R_k} a_k(h)(K(z-h) - K(z)) dh \left(\varphi\left(\frac{y-z}{t}\right) - \varphi\left(\frac{y}{t}\right) \right) dz.$$

and

$$\int_{R_k} |a_k(h)| |K(z-h) - K(z)| dh \leq \int_{R_k} |a_k(h)| dh \frac{|h|^\delta}{|z|^{n+\delta}} \leq \frac{2^{k\delta}}{|z|^{n+\delta}} \|a_k\|_1.$$

Therefore, with the help of (2.20),

$$\begin{aligned} |P_4(y)| &\leq C 2^{k\delta} |x|^{-1-n} \|a_k\|_1 \int_{|z| < \frac{|x|}{2}} \frac{1}{|z|^{n+\delta-1}} dz \\ &\leq C 2^{k\delta} \frac{1}{|x|^{n+\delta}} \|a_k\|_{p(\cdot)} \|\chi_{B_k}\|_{p'(\cdot)} \\ &\leq \frac{C}{|x|^{n+\delta}}, \end{aligned}$$

where the positive constant C is independent of x . Again by the vanishing condition of a_k we have

$$P_5(y) = t^{-n} \int_{|z| \geq \frac{|x|}{2}} \int_{R_k} a_k(h)(K(z-h) - K(z)) dh \left(\varphi\left(\frac{y-z}{t}\right) - \varphi\left(\frac{y}{t}\right) \right) dz.$$

Observe that

$$\begin{aligned} &\int_{|z| \geq \frac{|x|}{2}} |z|^{-n-\delta} \left| \varphi\left(\frac{y-z}{t}\right) - \varphi\left(\frac{y}{t}\right) \right| dz \\ &\leq \int_{|z| \geq \frac{|x|}{2}} |z|^{-n-\delta} \left| \varphi\left(\frac{y-z}{t}\right) \right| dz + \int_{|z| \geq \frac{|x|}{2}} |z|^{-n-\delta} \left| \varphi\left(\frac{y}{t}\right) \right| dz \\ &\leq \frac{C_1 t^n}{|x|^{n+\delta}} + \int_{|z| \geq \frac{|x|}{2}} |z|^{-n-\delta} \left| \varphi\left(\frac{y}{t}\right) \right| dz \\ &\leq \frac{C_1 t^n}{|x|^{n+\delta}} + C_2 t^n \int_{|z| \geq \frac{|x|}{2}} |z|^{-n-\delta} (t + |y|)^{-n} dz \\ &\leq \frac{C_1 t^n}{|x|^{n+\delta}} + C_2 t^n |x|^{-n} \int_{|z| \geq \frac{|x|}{2}} |z|^{-n-\delta} dz \\ &\leq \frac{C t^n}{|x|^{n+\delta}}, \end{aligned}$$

because of $t + |y| \geq |x - y| + |y| \geq |x|$. Hence $|P_5(y)| \leq \frac{C}{|x|^{n+\delta}}$, where the positive constant C is independent of x . Therefore, we obtain

$$\sum_{k=-\infty}^{j-4} \lambda_k G_N(Ta_k)(x) \leq C \sum_{k=-\infty}^{j-4} |\lambda_k| \frac{1}{|x|^{n+\delta}} \leq A2^{-j(n+\delta)} \|f\|_{\dot{H}K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}, \quad x \in R_j.$$

The desired estimate follows by the same arguments as in Theorem [2.29](#). The estimate of F_2 follows as in the estimate of F_1 . This completes the proof. \square

Remark 2.33. Under the assumption that the maximal operator \mathcal{M} is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ spaces, both in the homogeneous and the inhomogeneous situation, Theorems [2.31](#) and [2.32](#) with α and q constants are given in [\[117\]](#). For the classical Herz-type Hardy spaces, see [\[70\]](#).

MULTIPLICATION ON BESOV AND TRIEBEL-LIZORKIN SPACES OF POWER WEIGHTS

The purpose of this chapter is to study the 2-linear map

$$A_{p_1, q}^s(\mathbb{R}^n, |\cdot|^\alpha) \cdot A_{p_2, q_2}^r(\mathbb{R}^n, |\cdot|^\alpha) \hookrightarrow A_{p, q}^s(\mathbb{R}^n, |\cdot|^\alpha), \quad (3.1)$$

induced by

$$(f_1, f_2) \longrightarrow f_1 \cdot f_2.$$

Here, $A_{p, q}^s(\mathbb{R}^n, |\cdot|^\alpha)$ stands for either the Besov space $B_{p, q}^s(\mathbb{R}^n, |\cdot|^\alpha)$ or the Triebel-Lizorkin space $F_{p, q}^s(\mathbb{R}^n, |\cdot|^\alpha)$. But at the beginning of this chapter, we briefly recall some properties of Littlewood-Paley series, definitions of $B_{p, q}^s(\mathbb{R}^n, |\cdot|^\alpha)$, $F_{p, q}^s(\mathbb{R}^n, |\cdot|^\alpha)$ and we summarize some the most important inequalities which will be used later.

3.1 The decomposition of Littlewood-Paley

Littlewood-Paley series play an important role in defining Besov and Triebel-Lizorkin spaces. So, we will recall the definition of the decomposition of Littlewood-Paley of a tempered distribution. Let ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\psi(x) = 1$ for $|x| \leq 1$ and $\psi(x) = 0$ for $|x| \geq \frac{3}{2}$. We put

$$\varphi_0(x) = \psi(x),$$

$$\varphi_1(x) = \psi\left(\frac{x}{2}\right) - \psi(x)$$

and

$$\varphi_j(x) = \varphi_1(2^{-j+1}x) \quad \text{for } j = 2, 3, \dots$$

Then we have $\text{supp} \varphi_j \subset \{x \in \mathbb{R}^n : 2^{j-1} \leq |x| \leq 3 \cdot 2^{j-1}\}$, $\varphi_j(x) = 1$ for $3 \cdot 2^{j-2} \leq |x| \leq 2^j$ and $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$. The system of function $\{\varphi_j\}_{j \in \mathbb{N}}$ is called a smooth dyadic resolution of unity. We define the convolution operators Δ_j as follows

$$\Delta_j f = \mathcal{F}^{-1} \varphi_j * f, \quad j \in \mathbb{N} \quad \text{and} \quad \Delta_0 f = \mathcal{F}^{-1} \psi * f, \quad f \in \mathcal{S}'(\mathbb{R}^n).$$

Also, we define the convolution operators Q_j by

$$Q_j f = \mathcal{F}^{-1}(\psi(2^{-j} \cdot)) * f, \quad j \in \mathbb{N}.$$

We set $\Delta_0 = Q_0$, thus we obtain

$$Q_k f = \sum_{j=0}^k \Delta_j f, \quad k \in \mathbb{N}. \quad (3.2)$$

The decomposition of f of the Littlewood–Paley type given by

$$f = \sum_{j \geq 0} \Delta_j f. \quad (3.3)$$

The serie (3.3) converges in the sense of tempered distributions. The serie (3.3) can be written as

$$f = Q_k f + \sum_{j=k+1}^{\infty} \Delta_j f,$$

this formula is valid for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and $k \in \mathbb{N}$.

3.2 A decomposition of the product $f \cdot g$

For technical reasons it is convenient to replace the pointwise multiplication of functions by a product (defined for every f and g in $\mathcal{S}'(\mathbb{R}^n)$) in the following definition

Definition 3.1. Let $f, g \in \mathcal{S}'(\mathbb{R}^n)$. We define the product $f \cdot g$ by:

$$f \cdot g = \lim_{j \rightarrow \infty} Q_j f \cdot Q_j g, \quad (3.4)$$

whenever this limit exists in $\mathcal{S}'(\mathbb{R}^n)$.

This extends the pointwise product of smooth functions. Observe that $Q_j f \cdot Q_j g$ is well-defined in a pointwise sense the factors have compact Fourier support and are therefore smooth. The definition above has been introduced independently in [59] and [109], but without the ψ -independence. An advantage of the product (3.4) is that it allows the use of para-multiplication which we will know in the following lines. Then (3.2) and (3.3) read as follows:

$$Q_k f = \sum_{j=0}^k \Delta_j f, \quad k \in \mathbb{N} \quad \text{and} \quad f = \sum_{j=0}^{\infty} \Delta_j f \quad (\text{convergence in } \mathcal{S}'(\mathbb{R}^n)),$$

suppose that $f \cdot g$ exists in $\mathcal{S}'(\mathbb{R}^n)$. Then we split

$$\begin{aligned} f \cdot g &= \lim_{j \rightarrow \infty} Q_j f \cdot Q_j g = \lim_{j \rightarrow \infty} \sum_{\ell=0}^j \sum_{k=0}^j \Delta_\ell f \Delta_k g \\ &= \sum_{\ell=0}^{\infty} \sum_{k=0}^{k-2} \Delta_\ell f \Delta_k g + \sum_{k=0}^{\infty} \sum_{\ell=k-1}^{k+1} \Delta_\ell f \Delta_k g + \sum_{\ell=2}^{\infty} \sum_{k=0}^{\ell-2} \Delta_\ell f \Delta_k g \\ &= \Pi_1(f, g) + \Pi_2(f, g) + \Pi_3(f, g), \end{aligned}$$

whenever these three limits $\Pi_1(f, g)$, $\Pi_2(f, g)$ and $\Pi_3(f, g)$ exist $\in \mathcal{S}'(\mathbb{R}^n)$. We may rewrite this into

$$\begin{aligned} \Pi_1(f, g) &= \sum_{k=2}^{\infty} Q_{k-2} g \Delta_k f, \\ \Pi_2(f, g) &= \sum_{k=0}^{\infty} \bar{\Delta}_k g \Delta_k f \end{aligned}$$

and

$$\Pi_3(f, g) = \sum_{k=2}^{\infty} Q_{k-2} f \Delta_k g,$$

with $\bar{\Delta}_k = \sum_{j=k-1}^{k+1} \Delta_j$, $k \in \mathbb{N}_0$. The advantage of the above decomposition consists in

$$\text{supp } \mathcal{F}(Q_{k-2}f \Delta_k g) \subset \{\xi : 2^{k-3} \leq |\xi| \leq 2^{k+1}\}, \quad k \geq 2$$

and

$$\text{supp } \mathcal{F}(\bar{\Delta}_k g \Delta_k f) \subset \{\xi : |\xi| \leq 5 \cdot 2^k\}, \quad k \geq 0.$$

The mapping $\Pi_1(f, g)$, $\Pi_2(f, g)$ and $\Pi_3(f, g)$ are called the para-multiplication (or the para-product). For pointwise multipliers this technique was employed by Peetre [94] and Triebel [110]-[113] in order to treat the case of B - and F -spaces in the full parameter range $p, q \in (0; \infty]$. The para-differential notion was introduced by Bony in [12] and extended in [124] and [125].

3.3 Besov and Triebel-Lizorkin spaces of power weights

Now we give the definition of the spaces $B_{p,q}^s(\mathbb{R}^n)$ and $F_{p,q}^s(\mathbb{R}^n)$.

Definition 3.2. Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$.

(i) The Besov space $B_{p,q}^s(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p,q}^s(\mathbb{R}^n)} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\Delta_j f\|_p^q \right)^{1/q} < \infty,$$

with the obvious modification if $q = \infty$.

(ii) Let $0 < p < \infty$. The Triebel-Lizorkin space $F_{p,q}^s(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F_{p,q}^s(\mathbb{R}^n)} = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\Delta_j f|^q \right)^{1/q} \right\|_p < \infty,$$

with the obvious modification if $q = \infty$.

Besov and Triebel-Lizorkin spaces have a long history. Nikol'skij [92] introduced in 1951 the Nikol'skij-Besov space, which nowadays denote by $B_{p,\infty}^s(\mathbb{R}^n)$. However, he was mentioning that this was based on earlier work of Bernstein [6] ($p = \infty$) and Zygmund [131] (periodic case, $n = 1$, $1 < p < \infty$). Later, Besov [7, 8] introduced in 1959 the Besov space $B_{p,q}^s(\mathbb{R}^n)$ for any given $q \in (0, \infty]$. Around 1970 Triebel [111] and Lizorkin [68, 69]

introduced some new function spaces, nowadays denoted by $F_{p,q}^s(\mathbb{R}^n)$ for any given $p \in (0, \infty)$, $q \in (0, \infty]$, and $s \in (0, \infty)$, which are nowadays named after these two mathematicians. Around 1973–1975 Peetre [96, 95, 94], extended the ranges of s , p , and q to values less than one. Further, we have to mention the contributions of Frazier and Jawerth [48, 47] obtained the molecular and the atomic characterization of $B_{p,q}^s(\mathbb{R}^n)$ and $F_{p,q}^s(\mathbb{R}^n)$. Bownik [14, 13] introduced and developed Besov and Triebel-Lizorkin spaces. Finally, for more information on Besov and Triebel-Lizorkin spaces on \mathbb{R}^n , we refer the reader to the monographs [102, 112, 114, 115].

Let $\alpha \in \mathbb{R}$ and $0 < p < \infty$. The weighted Lebesgue space $L_p(\mathbb{R}^n, |\cdot|^\alpha)$ contains all measurable functions such that

$$\|f\|_{L_p(\mathbb{R}^n, |\cdot|^\alpha)} = \left(\int_{\mathbb{R}^n} |f(x)|^p |x|^\alpha dx \right)^{1/p} < \infty.$$

If $\alpha = 0$, then we put $L_p(\mathbb{R}^n, |\cdot|^0) = L_p(\mathbb{R}^n)$ and $\|f\|_{L_p(\mathbb{R}^n, |\cdot|^0)} = \|f\|_p$.

We are now in a position to state the definition of Besov and Triebel-Lizorkin spaces equipped with power weights.

Definition 3.3. Let $\alpha, s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$.

(i) The Besov space $B_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\Delta_j f\|_{L_p(\mathbb{R}^n, |\cdot|^\alpha)}^q \right)^{1/q} < \infty,$$

with the obvious modification if $q = \infty$.

(ii) The Triebel-Lizorkin space $F_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)} = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\Delta_j f|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n, |\cdot|^\alpha)} < \infty,$$

with the obvious modification if $q = \infty$.

Clearly, for $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$,

$$B_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha) = B_{p,q}^s(\mathbb{R}^n) \quad \text{and} \quad F_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha) = F_{p,q}^s(\mathbb{R}^n) \quad \text{if} \quad \alpha = 0.$$

In addition

$$F_{p,2}^m(\mathbb{R}^n, |\cdot|^\alpha) = W_p^m(\mathbb{R}^n, |\cdot|^\alpha), \quad (\text{Sobolev spaces})$$

where $1 < p < \infty, -n < \alpha < n(p-1)$ and $m \in \mathbb{N}_0$. For more details about Besov and Triebel-Lizorkin spaces equipped with general weights, see [36], [37], [38] and [39].

Remark 3.4. From the compactness of the support of $\varphi_j, j \in \mathbb{N}_0$, the Paley-Wiener-Schwartz theorem tells us that $\Delta_j f$ is an entire analytic function. So the convolution $\mathcal{F}^{-1} \varphi_j * f, j \in \mathbb{N}_0$, make sense. Let $s \in \mathbb{R}, 0 < p < \infty, 0 < q \leq \infty$ and $\alpha > -n$. The spaces $B_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)$ and $F_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_{j \in \mathbb{N}_0}$ (in the sense of equivalent quasi-norms). In particular $B_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)$ and $F_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)$ are quasi-Banach spaces and if $p, q \geq 1$, then $B_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)$ and $F_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)$ are Banach spaces, see [15] and [122]. Furthermore,

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$$

and

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow F_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Before presenting the following examples, we will present definitions for the function ϱ and the set U_q . Let ϱ be a C^∞ function on \mathbb{R} such that $\varrho(x) = 1$ for $x \leq e^{-3}$ and $\varrho(x) = 0$ for $x \geq e^{-2}$. Let $(\lambda, \sigma) \in \mathbb{R}^2$ and U_q be the set of (λ, σ) such that

- $\lambda = 1 - \frac{1}{q}$ and $\sigma > \frac{1}{q}$, or $\lambda < 1 - \frac{1}{q}$, in case $1 < q < \infty$,
- $\lambda = 0$ and $\sigma > 0$, or $\lambda < 0$, in case $q = 1$,
- $\lambda = 1$ and $\sigma \geq 0$, or $\lambda < 1$, in case $q = \infty$.

Example 3.5. Let $(\lambda, \sigma) \in \mathbb{R}^2, 1 \leq p < \infty, 1 \leq q \leq \infty, \alpha > -n$ and

$$(\lambda, \sigma) \in U_q. \tag{3.5}$$

Let $f_{\lambda, \sigma}$ be the function defined by

$$f_{\lambda, \sigma}(x) = |\log|x||^\lambda |\log|\log|x||^{-\sigma} \varrho(|x|).$$

We have $f_{\lambda, \sigma} \in B_{p,q}^{\frac{n+\alpha}{p}}(\mathbb{R}^n, |\cdot|^\alpha)$. In the case $\alpha \geq 0$, the condition (3.5) becomes necessary.

Example 3.6. Let $\beta > 0, 1 \leq p < \infty, -n < \alpha < n(p-1)$,

$$0 < \max\left(\delta + \frac{n}{p}, \delta + \frac{n+\alpha}{p}\right) < 2(\beta+1) \quad \text{and} \quad \sigma = \frac{\delta + \frac{n+\alpha}{p}}{\beta+1}$$

Let $g \in B_{\infty, \infty}^{\gamma}(\mathbb{R})$ for some $\sigma < \gamma$. The function

$$f(x) = |x|^{\delta} g(|x|^{-\beta}) \varrho(|x|)$$

belongs to $B_{p, \infty}^{\sigma}(\mathbb{R}^n, |\cdot|^{\alpha})$.

For more details about the proofs of these two previous examples [3.5](#) and [3.6](#), we refer to [34](#).

3.4 Embeddings

The following theorem gives basic embeddings of the spaces $B_{p, q}^s(\mathbb{R}^n, |\cdot|^{\alpha})$ and $F_{p, q}^s(\mathbb{R}^n, |\cdot|^{\alpha})$, see [121](#).

Theorem 3.7. Let $s \in \mathbb{R}, 0 < p < \infty, 0 < q \leq \infty$ and $\alpha > -n$. Then

$$B_{p, \min(p, q)}^s(\mathbb{R}^n, |\cdot|^{\alpha}) \hookrightarrow F_{p, q}^s(\mathbb{R}^n, |\cdot|^{\alpha}) \hookrightarrow B_{p, \max(p, q)}^s(\mathbb{R}^n, |\cdot|^{\alpha}).$$

Remark 3.8. Theorem [3.7](#) when $\alpha = 0$ generalizes the corresponding results on Besov and Triebel-Lizorkin spaces established in [112](#), Section 2.3.

Theorem 3.9. Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}, 0 < q \leq \infty, \alpha_1 > -n$ and $\alpha_2 > -n$. Let $0 < p \leq s < \infty$ and $\alpha_2 \geq \alpha_1$. We suppose that

$$s_1 - \frac{n + \alpha_1}{s} \leq s_2 - \frac{n + \alpha_2}{p}.$$

Then

$$B_{p, q}^{s_2}(\mathbb{R}^n, |\cdot|^{\alpha_2}) \hookrightarrow B_{s, q}^{s_1}(\mathbb{R}^n, |\cdot|^{\alpha_1})$$

and

$$F_{p, \infty}^{s_2}(\mathbb{R}^n, |\cdot|^{\alpha_2}) \hookrightarrow F_{s, q}^{s_1}(\mathbb{R}^n, |\cdot|^{\alpha_1}).$$

The proof is given in [30](#), [31](#) and [81](#).

Theorem 3.10. Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}, 0 < r \leq \infty, \alpha_1 > -n$ and $\alpha_2 > -n$. Under the following assumptions

$$0 < q < p < \infty, \quad s_1 - \frac{n + \alpha_1}{p} = s_2 - \frac{n + \alpha_2}{q} \quad \text{and} \quad \frac{\alpha_2}{q} \geq \frac{\alpha_1}{p}$$

we have

$$F_{q,\infty}^{s_2}(\mathbb{R}^n, |\cdot|^{\alpha_2}) \hookrightarrow B_{p,q}^{s_1}(\mathbb{R}^n, |\cdot|^{\alpha_1}), \quad (\text{Jawerth embedding}).$$

Theorem 3.11. Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}, 0 < \theta \leq \infty, \alpha_1 > -n$ and $\alpha_2 > -n$. Let

$$0 < q < p < \infty, \quad s_1 - \frac{n + \alpha_1}{p} = s_2 - \frac{n + \alpha_2}{q} \quad \text{and} \quad \frac{\alpha_2}{q} \geq \frac{\alpha_1}{p}.$$

Then

$$B_{q,p}^{s_2}(\mathbb{R}^n, |\cdot|^{\alpha_2}) \hookrightarrow F_{p,\theta}^{s_1}(\mathbb{R}^n, |\cdot|^{\alpha_1}), \quad (\text{Franke embedding})$$

holds.

For the proof, see [33] and [81].

3.5 Basic Inequalities

3.5.1 Classic inequalities

In this section, we will give some estimations such as The weighted Lebesgue space-version of Hölder, Plancherel-Polya-Nikolskij inequalities and others.

Proposition 3.12. Let $\alpha \in \mathbb{R}, 1 \leq p, q \leq \infty$. We suppose that $\alpha > -n, f \in L_p(\mathbb{R}^n, |x|^\alpha)$ and $g \in L_q(\mathbb{R}^n, |x|^\alpha)$. Then

$$\|fg\|_{L_r(\mathbb{R}^n, |x|^\alpha)} \lesssim \|f\|_{L_p(\mathbb{R}^n, |x|^\alpha)} \|g\|_{L_q(\mathbb{R}^n, |x|^\alpha)}, \quad (3.6)$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Proposition 3.13. Let a_j be a sequence of positive reals. we have

- (1) $\left(\sum_{j=1}^{\infty} a_j\right)^\theta \leq \sum_{j=1}^{\infty} a_j^\theta, \quad \text{for any } 0 \leq \theta \leq 1.$
- (2) $\sum_{j=1}^{\infty} a_j^\theta \leq \left(\sum_{j=1}^{\infty} a_j\right)^\theta, \quad \text{for any } 1 \leq \theta < \infty.$

$$(3) \quad \left(\sum_{j=1}^{\infty} a_j \right)^{\theta} \leq N^{\theta-1} \sum_{j=1}^{\infty} a_j^{\theta}, \quad \text{when } 1 \leq \theta < \infty.$$

$$(4) \quad \sum_{j=1}^{\infty} a_j^{\theta} \leq N^{1-\theta} \left(\sum_{j=1}^{\infty} a_j \right)^{\theta}, \quad \text{when } 0 \leq \theta \leq 1.$$

Proposition 3.14. *Let $s \in \mathbb{R}$ and $0 < p < \infty$.*

(i) *We have*

$$\left\| \sup_{j \in \mathbb{N}} |Q_j f| \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)} \leq c \left\| f \right\|_{F_{p,2}^0(\mathbb{R}^n, |x|^\alpha)},$$

for all $f \in F_{p,2}^0(\mathbb{R}^n, |x|^\alpha)$.

(ii) *Let $j \in \mathbb{N}_0$ and $f \in B_{p,\infty}^s(\mathbb{R}^n, |x|^\alpha)$. Then there exists a positive constant c , independent of j , such that*

$$\left\| Q_j f \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)} \leq c \varepsilon_j \left\| f \right\|_{B_{p,\infty}^s(\mathbb{R}^n, |x|^\alpha)},$$

where

$$\varepsilon_j = \begin{cases} 2^{-js}, & \text{if } s < 0, \\ 1, & \text{if } s > 0, \\ (j+1)^{\frac{1}{\min(1,p)}}, & \text{if } s = 0. \end{cases}$$

(iii) *Let $j \in \mathbb{N}_0$, $0 < p \leq p \leq u \leq \infty$ and $f \in B_{p,\infty}^s(\mathbb{R}^n, |x|^\alpha)$. Then*

$$\left\| \Delta_j f \right\|_{L_u(\mathbb{R}^n, |x|^\alpha)} \leq c 2^{\left(\frac{n}{p} - \frac{n}{u} - s\right)j} \left\| f \right\|_{B_{p,\infty}^s(\mathbb{R}^n, |x|^\alpha)},$$

where the positive constant c is independent of j .

(iv) *We have*

$$\left\| Q_j f \right\|_{L_v(\mathbb{R}^n, |x|^\alpha)} \leq c \kappa_j \left\| f \right\|_{B_{p,\infty}^s(\mathbb{R}^n, |x|^\alpha)},$$

for all $j \in \mathbb{N}_0$, all $f \in B_{p,\infty}^s(\mathbb{R}^n, |x|^\alpha)$ and all $p < v \leq \frac{1}{\left(\frac{1}{p} - \frac{s}{n}\right)_+}$, where

$$\kappa_j = \begin{cases} 1, & \text{if } p < v < \frac{1}{\left(\frac{1}{p} - \frac{s}{n}\right)_+}, \\ (j+1)^{\frac{1}{\min(1,v)}}, & \text{if } v = \frac{1}{\left(\frac{1}{p} - \frac{s}{n}\right)_+}. \end{cases}$$

and the positive constant c is independent of j .

For more details about the proof of this previous proposition [3.14](#) refer to [\[35\]](#). The following lemma is the $L_p(\mathbb{R}^n, |x|^\alpha)$ -version of the Plancherel-Polya-Nikolskij inequality,

see [30].

Lemma 3.15. *Let $\alpha \in \mathbb{R}$. We suppose that $\alpha > -n, 0 < p \leq s < \infty$. Then there exists a positive constant $c > 0$ independent of R such that for all $f \in L_p(\mathbb{R}^n, |x|^\alpha) \cap \mathcal{S}'(\mathbb{R}^n)$ with $\text{supp } \mathcal{F}f \subset \{\xi \in \mathbb{R}^n : |\xi| \leq R\}$, we have*

$$\|f\|_{L_s(\mathbb{R}^n, |x|^\alpha)} \leq c R^{\frac{n+\alpha}{p} - \frac{n+\alpha}{s}} \|f\|_{L_p(\mathbb{R}^n, |x|^\alpha)},$$

3.5.2 Maximal inequalities

First, we will give a definition of the Peetre maximal functions

Definition 3.16. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $k \in \mathbb{N}_0$ and $a > 0$. We define the maximal operators associated to the Δ_k and Q_k*

$$\Delta_k^{*,a} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\Delta_k f(x-y)|}{(1+2^k|y|)^a} \quad \text{and} \quad Q_k^{*,a} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|Q_k f(x-y)|}{(1+2^k|y|)^a} \quad (3.7)$$

Clearly, for all $k \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$, $|Q_k f(x)| \leq Q_k^{*,a} f(x)$ ($|\Delta_k f(x)| \leq \Delta_k^{*,a} f(x)$), and we have $Q_k^{*,a} \equiv 0$ if and only if $Q_k f(x) \equiv 0$ ($\Delta_k^{*,a} \equiv 0$ if and only if $\Delta_k f(x) \equiv 0$).

Remark 3.17. (a) *We observe that sometimes $Q_k^{*,a} f$ and $\Delta_k^{*,a} f$ are also defined in definition 3.16 with $(1+2^k|y|)^a$ replaced by $1+|2^k y|^a$, but these two definitions are obviously pointwise equivalent.*

(b) *In definition 3.2 (i) (resp., (ii)), we can replaced $\Delta_k f$ by $\Delta_k^{*,a} f$ with $a > (n/\min(p, q))$ (resp., $a > n/p$), (cf. see [114], Theorem 2.3.2).*

The following lemma means that, for any $0 < p \leq \infty$ and $k \in \mathbb{N}$, $Q_k^{*,a} f$ is bounded by $Q_k f$ in $L^p(\mathbb{R}^n)$.

Lemma 3.18. *Let $0 < p \leq \infty$ and $\frac{n}{p} < a < \infty$. Then there exists a positive constant c such that, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and $k \in \mathbb{N}$,*

$$\|Q_k^{*,a}\|_{L^p(\mathbb{R}^n)} \leq c \|Q_k f\|_{L^p(\mathbb{R}^n)}.$$

The Hardy-Littlewood maximal operator \mathcal{M} is defined on locally integrable functions

by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy, \quad x \in \mathbb{R}^n.$$

Let $0 < t \leq 1$. We put $\mathcal{M}_t f(x) = (\mathcal{M}|f|^t(x))^{\frac{1}{t}}$. In what follows, we need the Marschall's inequalities, see [76, Proposition 1.3] and [126, Proposition 6.1].

Lemma 3.19. *Let $A > 0, R \geq 1$. Let $b \in \mathcal{D}(\mathbb{R}^n)$ and a function $f \in C^\infty(\mathbb{R}^n)$ such that*

$$\text{supp } \mathcal{F}f \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq AR\} \quad \text{and} \quad \text{supp } b \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq A\}.$$

Then

$$|\mathcal{F}^{-1}b * f(x)| \leq c(AR)^{\frac{n}{t}-n} \|b\|_{\dot{B}_{1,t}^{\frac{n}{t}}} \mathcal{M}_t(f)(x)$$

for any $0 < t \leq 1$ and any $x \in \mathbb{R}^n$, where c is independent of A, R, b and f . Here $\dot{B}_{1,t}^{\frac{n}{t}}$ denotes the homogeneous Besov spaces.

Various important results have been proved in the space $L_p(\mathbb{R}^n, |\cdot|^\alpha)$ under some assumptions on α and p . The condition $-n < \alpha < n(p-1), 1 < p < \infty$ is crucial in the study of the boundedness of classical operators in $L_p(\mathbb{R}^n, |\cdot|^\alpha)$ spaces, such as the Hardy-Littlewood maximal operator. One of the main tools of this thesis is based on the following result, which follows since $|\cdot|^\alpha \in \mathcal{A}_p(\mathbb{R}^n)$, Muckenhoupt classes, if and only if $-n < \alpha < n(p-1)$, see [4].

Lemma 3.20. *Let $1 < q \leq \infty$ and $1 < p < \infty$. If $\{f_j\}_{j \in \mathbb{N}_0}$ is a sequence of locally integrable functions on \mathbb{R}^n and $-n < \alpha < n(p-1)$, then*

$$\left\| \left(\sum_{j=0}^{\infty} (\mathcal{M}f_j)^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n, |\cdot|^\alpha)} \lesssim \left\| \left(\sum_{j=0}^{\infty} |f_j|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n, |\cdot|^\alpha)}$$

holds.

The next two lemmas are used in the proof of our result, see [78] for the Besov and Triebel-Lizorkin spaces.

Lemma 3.21. *Let $s \in \mathbb{R}, A, B > 0, 0 < p < \infty, 0 < q \leq \infty$ and $\alpha > -n$. Let $\{f_l\}_{l \in \mathbb{N}_0}$ be a sequence of functions such that*

$$\text{supp } \mathcal{F}f_0 \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq A\}$$

and

$$\text{supp } \mathcal{F} f_l \subseteq \{\xi \in \mathbb{R}^n : B2^{l+1} \leq |\xi| \leq A2^{l+1}\}.$$

There exists a constant $c > 0$ such that the following inequalities

$$\left\| \sum_{l=0}^{\infty} f_l \right\|_{B_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)} \leq c \left(\sum_{l=0}^{\infty} 2^{lsq} \|f_l\|_{L_p(\mathbb{R}^n, |\cdot|^\alpha)}^q \right)^{1/q}$$

and

$$\left\| \sum_{l=0}^{\infty} f_l \right\|_{F_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)} \leq c \left\| \left(\sum_{l=0}^{\infty} 2^{lsq} |f_l|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n, |\cdot|^\alpha)} \quad (3.8)$$

hold.

Proof. By similarity, we prove (3.8). Using the support properties of $\mathcal{F} f_l$ and Lemma 3.19 we easily obtain

$$\left| \sum_{l=0}^{\infty} \Delta_j f_l \right| \leq \sum_{l=j-N_1}^{j+N_2} |\Delta_j f_l| = \sum_{i=-N_1}^{N_2} |\Delta_j f_{j+i}| \lesssim \sum_{i=-N_1}^{N_2} \mathcal{M}_t(f_{j+i}), \quad 0 < t \leq 1, N_1, N_2 \in \mathbb{N},$$

where the implicit constant is independent of $j \in \mathbb{N}_0$. Here we put $f_i = 0$ if $i < 0$. Take $0 < t < \min(1, q, p, \frac{pn}{n+\alpha})$ and using Lemmas 3.23 and 3.20 we get

$$\begin{aligned} \left\| \sum_{l=0}^{\infty} f_l \right\|_{F_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)}^d &\lesssim \sum_{i=-N_1}^{N_2} \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\mathcal{M}_t(f_{j+i})|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n, |\cdot|^\alpha)}^d, \quad d = \min(1, p, q) \\ &\lesssim \left\| \left(\sum_{l=0}^{\infty} 2^{lsq} |f_l|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n, |\cdot|^\alpha)}^d. \end{aligned}$$

The proof is complete. □

Lemma 3.22. Let $A, B > 0, 0 < p < \infty, 0 < q \leq \infty$ and $\alpha > -n$. Let $\{f_l\}_{l \in \mathbb{N}_0}$ be a sequence of functions such that

$$\text{supp } \mathcal{F} f_l \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq A2^{l+1}\}.$$

(i) Let $s > \max(0, \frac{n}{p} - n, \frac{n+\alpha}{p} - n)$. Then it holds that

$$\left\| \sum_{l=0}^{\infty} f_l \right\|_{B_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)} \lesssim \left(\sum_{l=0}^{\infty} 2^{lsq} \|f_l\|_{L_p(\mathbb{R}^n, |\cdot|^\alpha)}^q \right)^{1/q}.$$

(ii) Let $s > \max(0, \frac{n}{p} - n, \frac{n+\alpha}{p} - n, \frac{n}{q} - n)$. Then it holds that

$$\left\| \sum_{l=0}^{\infty} f_l \right\|_{F_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)} \lesssim \left\| \left(\sum_{l=0}^{\infty} 2^{lsq} |f_l|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n, |\cdot|^\alpha)}. \quad (3.9)$$

Proof. By similarity, we prove (3.9). Using again the support properties of $\mathcal{F} f_l$ and Lemma 3.19 obtain

$$2^{js} \left| \sum_{l=0}^{\infty} \Delta_j f_l \right| \leq 2^{js} \sum_{l=j+N_1}^{\infty} |\Delta_j f_l| \lesssim 2^{js} \sum_{l=j+N_1}^{\infty} 2^{(l-j)(\frac{n}{q}-n)} \mathcal{M}_t(f_l), \quad 0 < t \leq 1, N_1, j \in \mathbb{N}_0,$$

where the implicit constant is independent of $j \in \mathbb{N}_0$. Take $\frac{n}{s+n} < t < \min(1, q, p, \frac{pn}{n+\alpha})$ and using Lemma 2.3 we obtain

$$\left(\sum_{j=0}^{\infty} 2^{jsq} \left| \sum_{l=0}^{\infty} \Delta_j f_l \right|^q \right)^{1/q} \lesssim \left(\sum_{j=0}^{\infty} 2^{jsq} |\mathcal{M}_t(f_l)|^q \right)^{1/q}.$$

The desired estimate follows again by Lemma 3.20. \square

We shall also need the following elementary fact.

Lemma 3.23. Let $0 < p \leq \infty, \alpha \in \mathbb{R}$ and $f_k \in L_p^{\text{loc}}(\mathbb{R}^n, |x|^\alpha), k \in \mathbb{N}_0$. Then, for any $0 < \tau \leq \min(1, p)$,

$$\left\| \sum_{k=0}^{\infty} f_k \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)} \leq \left(\sum_{k=0}^{\infty} \|f_k\|_{L_p(\mathbb{R}^n, |x|^\alpha)}^\tau \right)^{\frac{1}{\tau}}.$$

Proposition 3.24. Let $0 < t, b < \infty$ and a function f such that $\text{supp } \widehat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq b\}$, then for any $x \in \mathbb{R}^n$, we have

$$\sup_{y \in \mathbb{R}^n} \frac{|f(x-y)|}{(1+|y|)^{n/t}} \leq c \mathcal{M}_t f(x).$$

3.6 Multiplication on Besov and Triebel-Lizorkin spaces of power weights

Pointwise multiplication has been a central topic in modern analysis, and are now of increasing application in many fields of mathematics. The motivation to study the problem of multiplication on function spaces comes from applications differential equation.

One of the main applications of such results is concerned with global existence results for the two-dimensional (2D) dissipative quasi-geostrophic (QC) equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \kappa (-\Delta)^\alpha \theta = 0, \\ u = (u_1, u_2) = \nabla^\perp \psi, \quad (-\Delta)^{\frac{1}{2}} \psi = \theta. \end{cases}$$

supplemented with the initial condition

$$\theta(x, 0) = \theta_0(x)$$

where $x \in \mathbb{R}^2$, $\kappa > 0$ is the diffusion coefficient and $\alpha \in [0; 1]$ is a parameter, $\theta = \theta(x, t)$ is a scalar representing the temperature, u is the velocity fields, and ψ is the usual stream function. For more details for this equation, see [120].

In this section we present the first results of this thesis. We begin with the sufficient part. Our arguments are based on some techniques used in [29].

Theorem 3.25. *Let $0 < p, p_1, p_2 < \infty, 0 < q \leq \infty, \alpha \geq 0$ and $0 < r < \frac{n+\alpha}{p_2}$. Assume that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{r}{n+\alpha}$$

and

$$-r + \max\left(0, \frac{n+\alpha}{p_1} + \frac{n+\alpha}{p_2} - n\right) < s < \min\left(\frac{n+\alpha}{p_1}, r\right). \quad (3.10)$$

Then

$$F_{p_1, q}^s(\mathbb{R}^n, |x|^\alpha) \cdot B_{p_2, q_2}^r(\mathbb{R}^n, |x|^\alpha) \hookrightarrow F_{p, q}^s(\mathbb{R}^n, |x|^\alpha)$$

holds, with $\frac{1}{q_2} \geq \frac{1}{p_2} - \frac{r}{n+\alpha}$.

Proof. Let $f \in F_{p_1, q}^s(\mathbb{R}^n, |x|^\alpha)$ and $g \in B_{p_2, q_2}^r(\mathbb{R}^n, |x|^\alpha)$. Recall that

$$f \cdot g = \Pi_1(f, g) + \Pi_2(f, g) + \Pi_3(f, g).$$

Hence we need only to estimate

$$\Pi_i(f, g), \quad i = 1, 2, 3,$$

in $F_{p,q}^s(\mathbb{R}^n, |x|^\alpha)$ spaces. **Estimate of $\Pi_1(f, g)$.** Lemma 3.21 yields that

$$\|\Pi_1(f, g)\|_{F_{p,q}^s(\mathbb{R}^n, |x|^\alpha)},$$

is bounded by

$$c \left\| \left(\sum_{k=2}^{\infty} 2^{skq} |Q_{k-2}g \Delta_k f|^q \right)^{\frac{1}{q}} \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)} \lesssim \left\| \sup_{k=2,3,\dots} |Q_{k-2}g| \left(\sum_{k=2}^{\infty} 2^{skq} |\Delta_k f|^q \right)^{\frac{1}{q}} \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)}.$$

By Hölder's inequality we estimate the last term by

$$c \left\| \sup_{k=2,3,\dots} |Q_{k-2}g| \right\|_{L_b(\mathbb{R}^n, |x|^\alpha)} \left\| \left(\sum_{k=2}^{\infty} 2^{skq} |\Delta_k f|^q \right)^{\frac{1}{q}} \right\|_{L_{p_1}(\mathbb{R}^n, |x|^\alpha)},$$

where $\frac{1}{b} = \frac{1}{p_2} - \frac{r}{n+\alpha}$. Recall that

$$\left\| \sup_{k=2,3,\dots} |Q_{k-2}g| \right\|_{L_b(\mathbb{R}^n, |x|^\alpha)} \lesssim \|g\|_{F_{b,2}^0(\mathbb{R}^n, |x|^\alpha)},$$

see [15, Theorem 1.4]. Thanks to the embeddings

$$B_{p_2, q_2}^r(\mathbb{R}^n, |x|^\alpha) \hookrightarrow F_{b,2}^0(\mathbb{R}^n, |x|^\alpha),$$

see Theorem 3.11, we obtain

$$\|\Pi_1(f, g)\|_{F_{p,q}^s(\mathbb{R}^n, |x|^\alpha)} \lesssim \|g\|_{B_{p_2, q_2}^r(\mathbb{R}^n, |x|^\alpha)} \|f\|_{F_{p_1, q}^s(\mathbb{R}^n, |x|^\alpha)}.$$

This is the desired estimate. **Estimate of $\Pi_2(f, g)$.** Let $t \in \mathbb{R}$ be such that

$$\max\left(0, \frac{n+\alpha}{p_1} - r, \frac{n+\alpha}{p_1} - s - r\right) < \frac{n+\alpha}{t} < \frac{n+\alpha}{p_1}. \quad (3.11)$$

Observe that such a choice is always possible. We put

$$\frac{1}{d} = \frac{1}{p_2} + \frac{1}{t}, \quad \eta = s - \frac{n+\alpha}{p} + \frac{n+\alpha}{d} \quad \text{and} \quad \delta = s - \frac{n+\alpha}{p_1} + \frac{n+\alpha}{t}.$$

These guarantee the embedding $B_{d,p}^\eta(\mathbb{R}^n, |x|^\alpha) \hookrightarrow F_{p,q}^s(\mathbb{R}^n, |x|^\alpha)$, see Theorem 3.11. Hence, we need to estimate $\Pi_2(f, g)$ in $B_{d,p}^\eta(\mathbb{R}^n, |x|^\alpha)$ spaces. We claim that

$$\eta > \max\left(0, \frac{n+\alpha}{d} - n\right) \quad \text{and} \quad 0 < d < p. \quad (3.12)$$

From Lemma 3.22 it follows that

$$\|\Pi_2(f, g)\|_{B_{d,p}^\eta(\mathbb{R}^n, |x|^\alpha)} \lesssim \left(\sum_{k=0}^{\infty} 2^{\eta k p} \|\Delta_k f \bar{\Delta}_k g\|_{L_d(\mathbb{R}^n, |x|^\alpha)}^p \right)^{\frac{1}{p}}.$$

Hölder's inequality yields

$$2^{k\eta} \|\Delta_k f \bar{\Delta}_k g\|_{L_d(\mathbb{R}^n, |x|^\alpha)} \lesssim 2^{kr} \|\bar{\Delta}_k g\|_{L_{p_2}(\mathbb{R}^n, |x|^\alpha)} 2^{k\delta} \|\Delta_k f\|_{L_t(\mathbb{R}^n, |x|^\alpha)},$$

where the implicit constant is independent of k . This expression in ℓ^p -quasi-norm can be estimated by

$$\begin{aligned} & c \left(\sum_{k=0}^{\infty} 2^{krb} \|\Delta_k g\|_{L_{p_2}(\mathbb{R}^n, |x|^\alpha)}^b \right)^{\frac{1}{b}} \left(\sum_{k=0}^{\infty} 2^{k\delta p_1} \|\Delta_k f\|_{L_t(\mathbb{R}^n, |x|^\alpha)}^{p_1} \right)^{\frac{1}{p_1}} \\ & \lesssim \|g\|_{B_{p_2, q_2}^r(\mathbb{R}^n, |x|^\alpha)} \|f\|_{B_{t, p_1}^\delta(\mathbb{R}^n, |x|^\alpha)}. \end{aligned}$$

Our desired estimate follows by the embedding

$$F_{p_1, q}^s(\mathbb{R}^n, |x|^\alpha) \hookrightarrow B_{t, p_1}^\delta(\mathbb{R}^n, |x|^\alpha),$$

see Theorem 3.10. To show the claim, by (3.11) it follows that

$$\frac{1}{d} = \frac{1}{p_2} + \frac{1}{t} > \frac{1}{p_2} + \frac{1}{p_1} - \frac{r}{n+\alpha} = \frac{1}{p}, \quad \eta > 0,$$

which together with (3.10) yields that (3.12) holds. Thus, the claim is true. **Estimate of** $\Pi_3(f, g)$. Let $u \in \mathbb{R}$ be such that

$$\max\left(0, \frac{n+\alpha}{p_1} - r\right) < \frac{n+\alpha}{u} < \min\left(\frac{n+\alpha}{p_1}, \frac{n+\alpha}{p_1} - s\right).$$

We put

$$\frac{1}{v} = \frac{1}{p_2} + \frac{1}{u}, \quad \sigma = s - \frac{n+\alpha}{p} + \frac{n+\alpha}{v} \quad \text{and} \quad \varrho = s - \frac{n+\alpha}{p_1} + \frac{n+\alpha}{u}.$$

These guarantee the embeddings

$$B_{v,p}^\sigma(\mathbb{R}^n, |x|^\alpha) \hookrightarrow F_{p,q}^s(\mathbb{R}^n, |x|^\alpha) \quad \text{and} \quad F_{p_1,\infty}^s(\mathbb{R}^n, |x|^\alpha) \hookrightarrow B_{u,p_1}^\varrho(\mathbb{R}^n, |x|^\alpha), \quad (3.13)$$

see Theorems [3.10](#) and [3.11](#). Lemma [3.21](#) yields that

$$\|\Pi_3(f, g)\|_{B_{v,p}^\sigma(\mathbb{R}^n, |x|^\alpha)} \lesssim \left(\sum_{k=2}^{\infty} 2^{\sigma k p} \|Q_{k-2} f \Delta_k g\|_{L_v(\mathbb{R}^n, |x|^\alpha)}^p \right)^{\frac{1}{p}}.$$

Therefore Hölder's inequality leads to

$$2^{k\sigma} \|Q_{k-2} f \Delta_k g\|_{L_v(\mathbb{R}^n, |x|^\alpha)},$$

is bounded by

$$\begin{aligned} & c 2^{k\sigma} \|\Delta_k g\|_{L_{p_2}(\mathbb{R}^n, |x|^\alpha)} \|Q_{k-2} f\|_{L_u(\mathbb{R}^n, |x|^\alpha)} \\ & \lesssim 2^{kr} \|\Delta_k g\|_{L_{p_2}(\mathbb{R}^n, |x|^\alpha)} 2^{k(\sigma-r)} \|Q_{k-2} f\|_{L_u(\mathbb{R}^n, |x|^\alpha)} \end{aligned} \quad (3.14)$$

for any $k = 2, 3, \dots$, where the implicit constant is independent of k . Since $\sigma - r = \varrho < 0$, by Lemmas [3.23](#) and [2.3](#), [\(3.14\)](#) in ℓ^p -quasi-norm is bounded by

$$\begin{aligned} & c \|g\|_{B_{p_2,\infty}^r(\mathbb{R}^n, |x|^\alpha)} \left(\sum_{j=0}^{\infty} 2^{j\varrho p_1} \|\Delta_j f\|_{L_u(\mathbb{R}^n, |x|^\alpha)}^{p_1} \right)^{\frac{1}{p_1}} \\ & \lesssim \|g\|_{B_{p_2,q_2}^r(\mathbb{R}^n, |x|^\alpha)} \|f\|_{B_{u,p_1}^\varrho(\mathbb{R}^n, |x|^\alpha)}. \end{aligned}$$

The proof is complete by the second embedding of [\(3.13\)](#). \square

Remark 3.26. We show below in Theorem [4.1](#) that the assumption $\frac{1}{p} \geq \frac{1}{p_1} + \frac{1}{p_2} - \frac{r}{n+\alpha}$ is optimal and cannot be improved.

Using the embedding $F_{p_2,q_2}^r(\mathbb{R}^n, |x|^\alpha) \hookrightarrow B_{p_2,b}^r(\mathbb{R}^n, |x|^\alpha)$ with $\frac{1}{b} = \frac{1}{p_2} - \frac{r}{n+\alpha}$, see Theorem [3.7](#), we immediately arrive at the following result.

Theorem 3.27. Under the hypotheses of Theorem [3.25](#) we have

$$F_{p_1,q}^s(\mathbb{R}^n, |x|^\alpha) \cdot F_{p_2,q_2}^r(\mathbb{R}^n, |x|^\alpha) \hookrightarrow F_{p,q}^s(\mathbb{R}^n, |x|^\alpha).$$

For the Besov spaces $B_{p,q}^s(\mathbb{R}^n, |x|^\alpha)$ we have the following statement:

Theorem 3.28. *Under the hypotheses of Theorem 3.25 we have*

$$B_{p_1,q}^s(\mathbb{R}^n, |x|^\alpha) \cdot B_{p_2,q_2}^r(\mathbb{R}^n, |x|^\alpha) \hookrightarrow B_{p,q}^s(\mathbb{R}^n, |x|^\alpha).$$

Proof. We employ the same notation as in Theorem 3.25. Using a combination of the arguments used in the Theorem 3.25 with the Sobolev embeddings in $B_{p,q}^s(\mathbb{R}^n, |x|^\alpha)$ spaces instead of Franke-Jawerth embeddings, we obtain the desired result. \square

Remark 3.29. *Corresponding statements to Theorem 3.25 were proved in [29, Theorem 2.1] but with $\alpha = 0$. Theorems 3.27 and 3.28 with $\alpha = 0$ is given in [58] and [98, Chapter 4], see also [29, Corollary 2.2].*

Now, we present the limit case $r = \frac{n+\alpha}{p_2}$.

Theorem 3.30. *Let $0 < p, p_1 < \infty, 0 < q \leq \infty$ and $\alpha \geq 0$. Assume that*

$$-\frac{n+\alpha}{p_1} + \max\left(0, \frac{n+\alpha}{p} + \frac{n+\alpha}{p_1} - n\right) < s < \min\left(\frac{n+\alpha}{p}, \frac{n+\alpha}{p_1}\right).$$

Then

$$F_{p,q}^s(\mathbb{R}^n, |x|^\alpha) \cdot \left(B_{p_1,\infty}^{\frac{n+\alpha}{p_1}}(\mathbb{R}^n, |x|^\alpha) \cap L_\infty(\mathbb{R}^n)\right) \hookrightarrow F_{p,q}^s(\mathbb{R}^n, |x|^\alpha)$$

holds.

Proof. From Theorem 3.25 we need only to estimate $\Pi_1(f, g)$, but

$$|Q_{j-2}g \Delta_j f| \leq \sup_{k=2,3,\dots} (|Q_{k-2}g|) |\Delta_j f| \lesssim \|g\|_\infty |\Delta_j f|, \quad j = 2, 3, \dots$$

where the implicit constant is independent of j . The proof is complete. \square

Theorem 3.31. *Let $0 < p, p_1, p_2 < \infty, 0 < q \leq \infty, \alpha \geq 0$ and $r > 0$. Assume that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

and

$$-r + \max\left(0, \frac{n+\alpha}{p} - n\right) < s < r. \quad (3.15)$$

Then

$$F_{p_1, q}^s(\mathbb{R}^n, |x|^\alpha) \cdot B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha) \hookrightarrow F_{p, q}^s(\mathbb{R}^n, |x|^\alpha)$$

holds.

Proof. We employ the same notation of Theorem [3.25](#). **Estimate of $\Pi_1(f, g)$.** As in Theorem [3.25](#) and Hölder's inequality

$$\begin{aligned} \|\Pi_1(f, g)\|_{F_{p, q}^s(\mathbb{R}^n, |x|^\alpha)} &\lesssim \left\| \left(\sum_{k=0}^{\infty} 2^{skq} |Q_{k-2} g \Delta_k f|^q \right)^{\frac{1}{q}} \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)} \\ &\lesssim \|g\|_{F_{p_2, 2}^0(\mathbb{R}^n, |x|^\alpha)} \|f\|_{F_{p_1, q}^s(\mathbb{R}^n, |x|^\alpha)}. \end{aligned}$$

The desired estimate follows by the embedding

$$B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha) \hookrightarrow F_{p_2, 2}^0(\mathbb{R}^n, |x|^\alpha).$$

Estimate of $\Pi_2(f, g)$. From the embedding

$$B_{p, \min(p, q)}^s(\mathbb{R}^n, |x|^\alpha) \hookrightarrow F_{p, q}^s(\mathbb{R}^n, |x|^\alpha), \quad (3.16)$$

see Theorem [3.7](#), we will estimate $\Pi_2(f, g)$ in the spaces $B_{p, \min(p, q)}^s(\mathbb{R}^n, |x|^\alpha)$. Observe that

$$\sum_{j=0}^{\infty} \Delta_k(\Delta_j f \bar{\Delta}_j g) = \sum_{j=k}^{\infty} \Delta_k(\Delta_j f \bar{\Delta}_j g), \quad k \in \mathbb{N}_0,$$

so we need only to estimate $\{2^{ks} \|\sum_{j=k}^{\infty} \Delta_k(\Delta_j f \bar{\Delta}_j g)\|_{L_p(\mathbb{R}^n, |x|^\alpha)}\}_k$ in $\ell^{\min(p, q)}$ -quasi-norm. From Lemma [3.19](#) it follows that

$$|\Delta_k(\Delta_j f \bar{\Delta}_j g)| \lesssim 2^{(j-k)(\frac{n}{\tau} - n)} \mathcal{M}_\tau(\Delta_j f \bar{\Delta}_j g)$$

for any $0 < \tau \leq 1$ and any $j \geq k$, where the implicit constant is independent of k and j . We choose

$$\max\left(n, \frac{n+\alpha}{p}\right) < \frac{n}{\tau} < n+s+r,$$

which is possible in view of [\(3.15\)](#). Lemma [3.20](#) and Hölder's inequality yield

$$\begin{aligned}
2^{ks} \left\| \Delta_k(\Delta_j f \bar{\Delta}_j g) \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)} &\lesssim 2^{ks} 2^{(j-k)(\frac{n}{r}-n)} \left\| \bar{\Delta}_j g \right\|_{L_{p_2}(\mathbb{R}^n, |x|^\alpha)} \left\| \Delta_j f \right\|_{L_{p_1}(\mathbb{R}^n, |x|^\alpha)} \\
&= c 2^{-kr} 2^{(j-k)(\frac{n}{r}-n-s-r)} 2^{j(s+r)} \left\| \bar{\Delta}_j g \right\|_{L_{p_2}(\mathbb{R}^n, |x|^\alpha)} \left\| \Delta_j f \right\|_{L_{p_1}(\mathbb{R}^n, |x|^\alpha)} \\
&\lesssim 2^{-kr} 2^{(j-k)(\frac{n}{r}-n-s-r)} \left\| g \right\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)} \left\| f \right\|_{F_{p_1, \infty}^s(\mathbb{R}^n, |x|^\alpha)},
\end{aligned}$$

where the positive constant c is independent of j and k . Therefore

$$\begin{aligned}
2^{ks} \left\| \sum_{j=k}^{\infty} \Delta_k(\Delta_j f \bar{\Delta}_j g) \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)} &\lesssim 2^{ks} \left(\sum_{j=k}^{\infty} \left\| \Delta_k(\Delta_j f \bar{\Delta}_j g) \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)}^k \right)^{\frac{1}{k}} \\
&\lesssim \left(\sum_{j=k}^{\infty} 2^{-kr\kappa} 2^{(j-k)(\frac{n}{r}-n-s-r)\kappa} \right)^{\frac{1}{k}} \left\| g \right\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)} \left\| f \right\|_{F_{p_1, \infty}^s(\mathbb{R}^n, |x|^\alpha)}.
\end{aligned}$$

for any $k \in \mathbb{N}_0$, with $\kappa = \min(1, p)$. Since $r > 0$, by Lemma 2.3 the last expression in $\ell^{\min(p, q)}$ -quasi-norm is bounded by

$$c \left\| g \right\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)} \left\| f \right\|_{F_{p_1, \infty}^s(\mathbb{R}^n, |x|^\alpha)}.$$

Estimate of $\Pi_3(f, g)$. By the embedding (3.16), we will estimate $\Pi_3(f, g)$ in the spaces $B_{p, \min(p, q)}^s(\mathbb{R}^n, |x|^\alpha)$. By Lemma 3.21 we obtain

$$\left\| \Pi_3(f, g) \right\|_{B_{p, \min(p, q)}^s(\mathbb{R}^n, |x|^\alpha)} \lesssim \left(\sum_{k=2}^{\infty} 2^{sk \min(p, q)} \left\| \Delta_k g Q_{k-2} f \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)}^{\min(p, q)} \right)^{\frac{1}{\min(p, q)}}.$$

Let $\lambda = \min(1, p_1)$ and $k = 2, 3, \dots$. Hölder's inequality and Lemma 3.23 imply

$$2^{ks} \left\| \Delta_k g Q_{k-2} f \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)}$$

does not exceed

$$\begin{aligned}
&c 2^{kr} \left\| \Delta_k g \right\|_{L_{p_2}(\mathbb{R}^n, |x|^\alpha)} 2^{k(s-r)} \left\| Q_{k-2} f \right\|_{L_{p_1}(\mathbb{R}^n, |x|^\alpha)} \\
&\lesssim 2^{kr} \left\| \Delta_k g \right\|_{L_{p_2}(\mathbb{R}^n, |x|^\alpha)} 2^{k(s-r)} \left(\sum_{j=0}^k 2^{-j(s-r)\lambda} 2^{-jr\lambda} \cdot 2^{js\lambda} \left\| \Delta_j f \right\|_{L_{p_1}(\mathbb{R}^n, |x|^\alpha)}^\lambda \right)^{\frac{1}{\lambda}}
\end{aligned}$$

$$\lesssim \left(2^{k(s-r)} \sum_{j=0}^{k-2} 2^{-j(s-r)\lambda} 2^{-jr\lambda} \right)^{\frac{1}{\lambda}} \|g\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)} \|f\|_{F_{p_1, \infty}^s(\mathbb{R}^n, |x|^\alpha)},$$

where the positive constant c is independent of $k = 2, 3, \dots$. By Lemma 2.3, with help of the hypothesis $s < r$ and $r > 0$, yields the last term in $\ell^{\min(p, q)}$ -quasi-norm is bounded by

$$c \|g\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)} \|f\|_{F_{p_1, \infty}^s(\mathbb{R}^n, |x|^\alpha)}.$$

The proof is complete. □

Remark 3.32. Later on, see Theorem 4.1 below, we prove that the assumption $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$ is optimal and cannot be improved.

Using the embeddings $F_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha) \hookrightarrow B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)$, we immediately arrive at the following result.

Theorem 3.33. Under the hypotheses of Theorem 3.25 we have

$$F_{p_1, q}^s(\mathbb{R}^n, |x|^\alpha) \cdot F_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha) \hookrightarrow F_{p, q}^s(\mathbb{R}^n, |x|^\alpha).$$

For the Besov spaces $B_{p, q}^s(\mathbb{R}^n, |x|^\alpha)$ we have the following statement:

Theorem 3.34. Under the hypotheses of Theorem 3.25 we have

$$B_{p_1, q}^s(\mathbb{R}^n, |x|^\alpha) \cdot B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha) \hookrightarrow B_{p, q}^s(\mathbb{R}^n, |x|^\alpha).$$

The proof is very similar to that of Theorem 3.31.

Remark 3.35. Theorem 3.31 was proved in [29, Theorem 2.1] with $\alpha = 0$, while Theorems 3.33 and 3.34 with $\alpha = 0$ is given in [58] and [98, Chapter 4].

Now we present the much more complicated case, which generalize the corresponding results given in [29], [58] and [98, Chapter 4].

Theorem 3.36. Let $0 < p, p_1, p_2 < \infty, 0 < q \leq \infty, \alpha \geq 0$ and $r > 0$. Assume that

$$\max\left(\frac{1}{p_1}, \frac{s}{n + \alpha}\right) + \max\left(0, \frac{1}{p_2} - \frac{r}{n + \alpha}\right) < \frac{1}{p} < \frac{1}{p_1} + \frac{1}{p_2} \quad (3.17)$$

and

$$-r + \max\left(0, \frac{n+\alpha}{p_1} + \frac{n+\alpha}{p_2} - n\right) < s < r. \quad (3.18)$$

Then

$$F_{p_1, q}^s(\mathbb{R}^n, |x|^\alpha) \cdot B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha) \hookrightarrow F_{p, q}^s(\mathbb{R}^n, |x|^\alpha)$$

holds.

Proof. In comparison with the proof of Theorems [3.25](#) and [3.31](#) we must modify the estimate of $\Pi_2(f, g)$ and $\Pi_3(f, g)$. **Estimate of $\Pi_1(f, g)$.** We put $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{u}$ with

$$\max\left(0, \frac{1}{p_2} - \frac{r}{n+\alpha}\right) < \frac{1}{u} < \frac{1}{p_2}. \quad (3.19)$$

As before, we estimate

$$\|\Pi_1(f, g)\|_{F_{p, q}^s(\mathbb{R}^n, |x|^\alpha)} \lesssim \|g\|_{F_{u, 2}^0(\mathbb{R}^n, |x|^\alpha)} \|f\|_{F_{p_1, q}^s(\mathbb{R}^n, |x|^\alpha)}.$$

We can conclude the desired result by the fact that

$$B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha) \hookrightarrow B_{p_2, u}^{\frac{n+\alpha}{p_2} - \frac{n+\alpha}{u}}(\mathbb{R}^n, |x|^\alpha) \hookrightarrow F_{u, 2}^0(\mathbb{R}^n, |x|^\alpha).$$

Estimate of $\Pi_2(f, g)$. We employ the notations of Theorem [3.25](#), but we replace [\(3.11\)](#) by

$$\max\left(0, \frac{n+\alpha}{p} - \frac{n+\alpha}{p_2}, \frac{n+\alpha}{p_1} - s - r\right) < \frac{n+\alpha}{t} < \frac{n+\alpha}{p_1}. \quad (3.20)$$

Let $\lambda > 0$ be such that

$$\frac{n}{n+r+\delta} < \lambda < \min\left(1, \frac{nd}{n+\alpha}\right), \quad (3.21)$$

which is possible by using [\(3.18\)](#), [\(3.20\)](#) and the fact that

$$\delta + r = s + r - \frac{n+\alpha}{p_1} + \frac{n+\alpha}{t} > \frac{n+\alpha}{p_2} + \frac{n+\alpha}{t} - n = \frac{n+\alpha}{d} - n$$

Thanks to Lemma [3.19](#) it follows that

$$|\Delta_k(\Delta_j f \bar{\Delta}_j g)| \lesssim 2^{(j-k)(\frac{n}{\lambda} - n)} \mathcal{M}_\lambda(\Delta_j f \bar{\Delta}_j g)$$

for any $j \geq k$, $k \in \mathbb{N}$. Lemma 3.20 and Hölder's inequality yield

$$\begin{aligned} & 2^{k\eta} \left\| \Delta_k (\Delta_j f \bar{\Delta}_j g) \right\|_{L_d(\mathbb{R}^n, |x|^\alpha)} \\ & \lesssim 2^{k\eta} 2^{(j-k)(\frac{n}{\lambda} - n)} \left\| \bar{\Delta}_j g \right\|_{L_{p_2}(\mathbb{R}^n, |x|^\alpha)} \left\| \Delta_j f \right\|_{L_t(\mathbb{R}^n, |x|^\alpha)} \\ & \lesssim 2^{k(\eta - \delta - r)} 2^{(j-k)(\frac{n}{\lambda} - n - \delta - r)} \left\| g \right\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)} \left\| f \right\|_{B_{t, \infty}^\delta(\mathbb{R}^n, |x|^\alpha)}, \end{aligned}$$

where the implicit constant is independent of j and k . Therefore

$$\begin{aligned} & 2^{k\eta} \left\| \sum_{j=k}^{\infty} \Delta_k (\Delta_j f \bar{\Delta}_j g) \right\|_{L_d(\mathbb{R}^n, |x|^\alpha)} \\ & \lesssim 2^{k\eta} \left(\sum_{j=k}^{\infty} \left\| \Delta_k (\Delta_j f \bar{\Delta}_j g) \right\|_{L_d(\mathbb{R}^n, |x|^\alpha)}^{\kappa} \right)^{\frac{1}{\kappa}} \\ & \lesssim 2^{k(\eta - \delta - r)} \left(\sum_{j=k}^{\infty} 2^{(j-k)(\frac{n}{\lambda} - n - \delta - r)\kappa} \right)^{\frac{1}{\kappa}} \left\| g \right\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)} \left\| f \right\|_{B_{t, \infty}^\delta(\mathbb{R}^n, |x|^\alpha)} \end{aligned}$$

for any $k \in \mathbb{N}_0$, with $\kappa = \min(1, d)$. Using (3.21) and

$$\eta < s + r - \frac{n + \alpha}{p_1} - \frac{n + \alpha}{p_2} + \frac{n + \alpha}{d} = \delta + r,$$

we get that last expression in ℓ^p -quasi-norm is bounded by

$$c \left\| g \right\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)} \left\| f \right\|_{B_{t, \infty}^\delta(\mathbb{R}^n, |x|^\alpha)} \lesssim \left\| g \right\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)} \left\| f \right\|_{F_{p_1, q}^s(\mathbb{R}^n, |x|^\alpha)}.$$

Estimate of $\Pi_3(f, g)$. The situation is quite different and more complicated. For this reason we are forced to decompose the prove in three separate steps: • *Step 1.* $s < 0$. By the embedding (3.16), we will estimate $\Pi_3(f, g)$ in $B_{p, \min(p, q)}^s(\mathbb{R}^n, |x|^\alpha)$ spaces with the help of Lemma 3.21. Let $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{u}$, with u as in (3.19). Hölder's inequality yields that

$$2^{ks} \left\| \Delta_k g Q_{k-2} f \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)}, \quad (3.22)$$

can be estimated by

$$2^{ks} \left\| \Delta_k g \right\|_{L_u(\mathbb{R}^n, |x|^\alpha)} \left\| Q_{k-2} f \right\|_{L_{p_1}(\mathbb{R}^n, |x|^\alpha)}.$$

Thanks to Lemma 3.15 there exists some positive constant c independent of k such that

$$\begin{aligned}\|\Delta_k g\|_{L_u(\mathbb{R}^n, |x|^\alpha)} &\leq c 2^{\left(\frac{n+\alpha}{p_2} - \frac{n+\alpha}{u}\right)k} \|\Delta_k g\|_{L_{p_2}(\mathbb{R}^n, |x|^\alpha)} \\ &\lesssim 2^{\left(\frac{n+\alpha}{p_2} - \frac{n+\alpha}{u} - r\right)k} \|g\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)}.\end{aligned}$$

By Lemmas 2.3 and 3.23, we obtain

$$\begin{aligned}2^{ks} \|Q_{k-2} f\|_{L_{p_1}(\mathbb{R}^n, |x|^\alpha)} &\lesssim 2^{ks} \left(\sum_{j=0}^{k-2} 2^{-js \min(1, p_1)} 2^{js \min(1, p_1)} \|\Delta_j f\|_{L_{p_1}(\mathbb{R}^n, |x|^\alpha)}^{\min(1, p_1)} \right)^{\frac{1}{\min(1, p_1)}} \\ &\lesssim \|f\|_{F_{p_1, \infty}^s(\mathbb{R}^n, |x|^\alpha)}\end{aligned}$$

for any $k \geq 2$. Since $\frac{n+\alpha}{p_2} - \frac{n+\alpha}{u} - r < 0$, (3.22) in $\ell^{\min(p, q)}$ -quasi-norm is bounded by

$$c \|g\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)} \|f\|_{F_{p_1, \infty}^s(\mathbb{R}^n, |x|^\alpha)}.$$

• *Step 2:* $0 \leq s < \frac{n+\alpha}{p_1}$. We put $\frac{1}{b} = \frac{1}{p_2} + \frac{1}{p_1} - \frac{s}{n+\alpha}$. • *Substep 2.1:* The case $\frac{n+\alpha}{p_2} \geq r$ or ($s \leq \frac{n+\alpha}{p_2} < r$). We distinguish two cases. • *Case* $p \leq b$. We put $\frac{1}{p} = \frac{1}{p_2} + \frac{1}{v}$ with $\frac{1}{p_1} - \frac{s}{n+\alpha} \leq \frac{1}{v} < \frac{1}{p_1}$. Using Hölder's inequality we obtain

$$2^{ks} \|\Delta_k g Q_{k-2} f\|_{L_p(\mathbb{R}^n, |x|^\alpha)} \leq c \|g\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)} 2^{k(s-r)} \|Q_{k-2} f\|_{L_v(\mathbb{R}^n, |x|^\alpha)}.$$

Thanks to Lemmas 3.15 and 3.23 it is obvious that

$$\begin{aligned}\|Q_{k-2} f\|_{L_v(\mathbb{R}^n, |x|^\alpha)} &\lesssim \left(\sum_{j=0}^{k-2} \|\Delta_j f\|_{L_v(\mathbb{R}^n, |x|^\alpha)}^{\min(1, v)} \right)^{\frac{1}{\min(1, v)}} \\ &\lesssim \left(\sum_{j=0}^{k-2} 2^{j\left(\frac{n+\alpha}{p_1} - s - \frac{n+\alpha}{v}\right) \min(1, v)} 2^{js \min(1, v)} \|\Delta_j f\|_{L_{p_1}(\mathbb{R}^n, |x|^\alpha)}^{\min(1, v)} \right)^{\frac{1}{\min(1, v)}} \\ &\lesssim \gamma_k \|f\|_{F_{p_1, \infty}^s(\mathbb{R}^n, |x|^\alpha)},\end{aligned}$$

since $\frac{n+\alpha}{p_1} - s - \frac{n+\alpha}{v} \leq 0$, where

$$\gamma_k = \begin{cases} (k-1)^{\frac{1}{\min(1, v)}}, & \text{if } p = b, \\ 1, & \text{if } p < b. \end{cases}$$

Now since $s < r$, we conclude the desired estimate using the embedding

$$B_{p,\min(p,q)}^s(\mathbb{R}^n, |x|^\alpha) \hookrightarrow F_{p,q}^s(\mathbb{R}^n, |x|^\alpha) \quad (3.23)$$

and Lemma 3.21. • *Case $p > b$.* Let u be a positive number satisfying

$$\max\left(0, \frac{1}{p} - \frac{1}{p_2}\right) < \frac{1}{u} < \frac{1}{p_1} - \frac{s}{n+\alpha}.$$

We put

$$\frac{1}{v} = \frac{1}{p_2} + \frac{1}{u}, \quad \sigma = s - \frac{n+\alpha}{p} + \frac{n+\alpha}{v}, \quad \beta = s - \frac{n+\alpha}{p_1} + \frac{n+\alpha}{u}.$$

Then

$$B_{v,p}^\sigma(\mathbb{R}^n, |x|^\alpha) \hookrightarrow F_{p,q}^s(\mathbb{R}^n, |x|^\alpha). \quad (3.24)$$

We need to estimate $\Pi_3(f, g)$ in $B_{v,p}^\sigma(\mathbb{R}^n, |x|^\alpha)$ spaces. Hölder's inequality yields that

$$2^{k\sigma} \|\Delta_k g Q_{k-2} f\|_{L_v(\mathbb{R}^n, |x|^\alpha)} \lesssim 2^{k\sigma} \|\Delta_k g\|_{L_{p_2}(\mathbb{R}^n, |x|^\alpha)} \|Q_{k-2} f\|_{L_u(\mathbb{R}^n, |x|^\alpha)}.$$

We set $h = \min(1, u)$. Again, by Lemmas 3.15 and 2.3 we obtain

$$2^{k\beta h} \sum_{j=0}^{k-2} \|\Delta_j f\|_{L_u(\mathbb{R}^n, |x|^\alpha)}^h \lesssim 2^{k\beta h} \sum_{j=0}^{k-2} 2^{-j\beta h} 2^{j\beta h} \|\Delta_j f\|_{L_u(\mathbb{R}^n, |x|^\alpha)}^h \lesssim \|f\|_{F_{u,\infty}^\beta(\mathbb{R}^n, |x|^\alpha)}^h,$$

since $\beta < 0$, where the implicit constant is independent of k . Consequently

$$2^{k\sigma} \|\Delta_k g Q_{k-2} f\|_{L_v(\mathbb{R}^n, |x|^\alpha)} \lesssim 2^{k(\sigma-\beta-r)} \|g\|_{B_{p_2,\infty}^r(\mathbb{R}^n, |x|^\alpha)} \|f\|_{F_{u,\infty}^\beta(\mathbb{R}^n, |x|^\alpha)}.$$

Observe that $\sigma - \beta - r < 0$, then the last expression in ℓ^p -quasi-norm is bounded by

$$c \|g\|_{B_{p_2,\infty}^r(\mathbb{R}^n, |x|^\alpha)} \|f\|_{F_{u,\infty}^\beta(\mathbb{R}^n, |x|^\alpha)}.$$

The desired estimate follows by the embeddings (3.24), Lemma 3.21 and

$$F_{p_1,\infty}^s(\mathbb{R}^n, |x|^\alpha) \hookrightarrow F_{u,\infty}^\beta(\mathbb{R}^n, |x|^\alpha).$$

• *Substep 2.2:* $\frac{n+\alpha}{p_2} < s < r$. We have only the case $p < b$ needs to study. As in Substep 2.1

we obtain the desired estimate. • *Step 3: $s \geq \frac{n+\alpha}{p_1}$.* • *Substep 3.1: $s \geq \frac{n+\alpha}{p_1}$ and $p < p_2$.* We put $\frac{1}{p} = \frac{1}{v} + \frac{1}{p_2}$, where $v > p_1$. Observe that

$$2^{ks} \left\| \Delta_k g Q_{k-2} f \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)} \lesssim 2^{k(s-r)} \left\| g \right\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)} \left\| Q_{k-2} f \right\|_{L_v(\mathbb{R}^n, |x|^\alpha)}. \quad (3.25)$$

We set $\mu = \min(1, v)$. From Lemma 3.15, we obtain

$$\begin{aligned} \sum_{j=0}^{k-2} \left\| \Delta_j f \right\|_{L_v(\mathbb{R}^n, |x|^\alpha)}^\mu &\lesssim \sum_{j=0}^{k-2} 2^{j(\frac{n+\alpha}{p_1} - \frac{n+\alpha}{v} - s)\mu} 2^{js\mu} \left\| \Delta_j f \right\|_{L_{p_1}(\mathbb{R}^n, |x|^\alpha)}^\mu \\ &\lesssim \left\| f \right\|_{F_{p_1, \infty}^s(\mathbb{R}^n, |x|^\alpha)}^\mu, \end{aligned}$$

which yields that (3.25) is bounded by

$$c 2^{k(s-r)} \left\| g \right\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)} \left\| f \right\|_{F_{p_1, \infty}^s(\mathbb{R}^n, |x|^\alpha)},$$

where the positive constant c is independent of k . Then, we conclude by the fact that $s < r$, with the help of Lemma 2.3, the embedding (3.23) and Lemma 3.21. • *Substep 3.2: $s > \frac{n+\alpha}{p_1}$ and $p \geq p_2$.* By Lemma 3.15, we obtain

$$\left\| Q_{k-2} f \right\|_\infty \lesssim \sum_{j=0}^{k-2} \left\| \Delta_j f \right\|_\infty \lesssim \sum_{j=0}^{k-2} 2^{j \frac{n+\alpha}{p_1}} \left\| \Delta_j f \right\|_{L_{p_1}(\mathbb{R}^n, |x|^\alpha)} \lesssim \left\| f \right\|_{B_{p_1, 1}^{\frac{n+\alpha}{p_1}}(\mathbb{R}^n, |x|^\alpha)}$$

for any $k = 2, 3, \dots$. Therefore

$$\begin{aligned} 2^{ks} \left\| \Delta_k g Q_{k-2} f \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)} &\lesssim 2^{ks} \left\| \Delta_k g \right\|_{L_p(\mathbb{R}^n, |x|^\alpha)} \left\| Q_{k-2} f \right\|_\infty \\ &\lesssim 2^{k(s + \frac{n+\alpha}{p_2} - r - \frac{n+\alpha}{p})} \left\| g \right\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)} \left\| f \right\|_{B_{p_1, 1}^{\frac{n+\alpha}{p_1}}(\mathbb{R}^n, |x|^\alpha)}. \end{aligned}$$

The desired estimate follows by the embeddings (3.23), Lemma 3.21,

$$F_{p_1, \infty}^s(\mathbb{R}^n, |x|^\alpha) \hookrightarrow B_{p_1, 1}^{\frac{n+\alpha}{p_1}}(\mathbb{R}^n, |x|^\alpha)$$

and the fact that $s + \frac{n+\alpha}{p_2} - r - \frac{n+\alpha}{p} < 0$. • *Substep 3.3: $s = \frac{n+\alpha}{p_1}$ and $p \geq p_2$.* We choose

$$0 < \vartheta < r + \frac{n+\alpha}{p} - \frac{n+\alpha}{p_1} - \frac{n+\alpha}{p_2},$$

which is possible by (3.17). Obviously

$$2^{k\frac{n+\alpha}{p_1}} \|\Delta_k g Q_{k-2} f\|_{L_p(\mathbb{R}^n, |x|^\alpha)} \lesssim 2^{k\frac{n+\alpha}{p_1}} \|\Delta_k g\|_{L_p(\mathbb{R}^n, |x|^\alpha)} \|Q_{k-2} f\|_\infty$$

for any $k = 2, 3, \dots$. As before

$$\|Q_{k-2} f\|_\infty \lesssim \sum_{j=0}^{k-2} 2^{j\vartheta} 2^{j(\frac{n+\alpha}{p_1} - \vartheta)} \|\Delta_j f\|_{L_{p_1}(\mathbb{R}^n, |x|^\alpha)} \lesssim 2^{k\vartheta} \|f\|_{B_{p_1, \infty}^{\frac{n+\alpha}{p_1} - \vartheta}(\mathbb{R}^n, |x|^\alpha)}.$$

Therefore

$$2^{k\frac{n+\alpha}{p_1}} \|\Delta_k g Q_{k-2} f\|_{L_p(\mathbb{R}^n, |x|^\alpha)} \lesssim 2^{k(\frac{n+\alpha}{p_1} + \frac{n+\alpha}{p_2} - r - \frac{n+\alpha}{p} + \vartheta)} \|g\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)} \|f\|_{B_{p_1, \infty}^{\frac{n+\alpha}{p_1} - \vartheta}(\mathbb{R}^n, |x|^\alpha)},$$

where the implicit constant is independent of k . The desired estimate follows by the embeddings (3.23), Lemma 3.21,

$$F_{p_1, \infty}^{\frac{n+\alpha}{p_1}}(\mathbb{R}^n, |x|^\alpha) \hookrightarrow B_{p_1, \infty}^{\frac{n+\alpha}{p_1} - \vartheta}(\mathbb{R}^n, |x|^\alpha)$$

and the fact that $\frac{n+\alpha}{p_1} + \frac{n+\alpha}{p_2} - r - \frac{n+\alpha}{p} + \vartheta < 0$. The proof is complete. \square

Remark 3.37. In Theorem 4.2 below, we prove that $\frac{1}{p} \geq \frac{1}{p_2} - \frac{r}{n+\alpha} + \frac{s}{n+\alpha}$ is optimal and cannot be improved.

OPTIMALITY CONDITIONS AND APPLICATION

The aim of this chapter is shed some light upon the following question:- Which conditions or assumptions on s, r, p_1, p_2, q_1 and q_2 given in the previous chapter 3 are necessary?. When we answer this question, we found that almost all our assumptions are necessary. Historically, A first systematic treatment of necessary conditions in the framework of Besov and Triebel-Lizorkin spaces has been given by J.Franke [46] in 1986. In the mid-nineties, many studies appeared in this direction by many mathematicians, for example, J.Johnsen [58]-[59], J.Marschall [77]-[78], W.Sickel and H.Triebel [107], see the monograph of T.Runst and W.Sickel [98] for more details. In the rest of this chapter, we explain an application to the continuity of pseudo-differential operators with non-regular symbols on Triebel-Lizorkin spaces of power weights.

4.1 Optimality conditions

In this section we prove the necessity of some of our assumptions given in Section 4. Our examples are based on [98, Chapter 5].

Theorem 4.1. *Let $0 < p, p_1, p_2 < \infty, 0 < q, q_1, q_2 \leq \infty, \alpha > -n$ and $s, r \in \mathbb{R}$. Assume that*

$$A_{p_1, q_1}^s(\mathbb{R}^n, |x|^\alpha) \cdot A_{p_2, q_2}^r(\mathbb{R}^n, |x|^\alpha) \hookrightarrow A_{p, q}^s(\mathbb{R}^n, |x|^\alpha). \quad (4.1)$$

Then necessarily we have

$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{r}{n + \alpha} \leq \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}.$$

Proof. We will do the proof into two steps. • *Step 1.* We prove the necessity of $\frac{1}{p} \geq \frac{1}{p_1} + \frac{1}{p_2} - \frac{r}{n+\alpha}$. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\text{supp } \mathcal{F}f \subset \left\{ \xi \in \mathbb{R}^n : \frac{1}{17} < |\xi| < \frac{1}{16} \right\} \quad \text{and} \quad \text{supp } \mathcal{F}g \subset \left\{ \xi \in \mathbb{R}^n : \frac{27}{32} < |\xi| < \frac{14}{16} \right\}.$$

We put $\gamma = s - \frac{n+\alpha}{p_1}$, $b = r - \frac{n+\alpha}{p_2}$,

$$f_k(x) = 2^{-k\gamma} f(2^k x) \quad \text{and} \quad g_k(x) = 2^{-kb} g(2^k x), \quad k \in \mathbb{N}, x \in \mathbb{R}^n.$$

Let $j \in \mathbb{N}$ and $k = 6, 7, \dots$. From the support conditions of $\mathcal{F}f_k$ we easily obtain

$$\Delta_j f_k = 0$$

if $j \leq k-5$ or $j \geq k-2$ and $\Delta_0 f_k = 0$ for any $k = 6, 7, \dots$. Lemma [3.19](#) leads to

$$|\Delta_j f_k| \lesssim \mathcal{M}_\tau(f_k), \quad k-5 < j < k-2$$

for any $0 < \tau < \min(1, p_1, p_2, \frac{np_1}{n+\alpha}, \frac{np_2}{n+\alpha})$. Hence

$$\|f_k\|_{A_{p_1, q_1}^s(\mathbb{R}^n, |x|^\alpha)} \leq c \quad \text{and} \quad \|g_k\|_{A_{p_2, q_2}^r(\mathbb{R}^n, |x|^\alpha)} \leq c,$$

where the positive constant c is independent of k . We have $\text{supp } \mathcal{F}(f_k g_k) \subset \{\xi \in \mathbb{R}^n : 2^k \frac{25}{32} < |\xi| < \frac{15}{16} 2^k\}$. Therefore

$$\Delta_j(f_k g_k) = 0, \quad j \neq k \quad \text{and} \quad \Delta_k(f_k g_k) = f_k g_k.$$

Hence for any $k = 6, 7, \dots$

$$\|f_k g_k\|_{A_{p, q}^s(\mathbb{R}^n, |x|^\alpha)} = 2^{sk} \|f_k g_k\|_{L_p(\mathbb{R}^n, |x|^\alpha)} = c 2^{k(n+\alpha)(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} - \frac{r}{n+\alpha})},$$

where the positive constant c is independent of k . Assume that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} - \frac{r}{n+\alpha} > 0$. Letting k tends to infinity, which contradicts [\(4.1\)](#). • *Step 2.* We prove the necessity of $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$. Define $h_k(x) = f(2^{-k} x)$, $x \in \mathbb{R}^n$, $k \in \mathbb{N}$. We easily obtain

$$\Delta_j h_k = 0 \quad \text{and} \quad \Delta_0 h_k = h_k, \quad j \in \mathbb{N}, k \in \mathbb{N}.$$

Hence

$$\|h_k\|_{A_{p_1, q_1}^s(\mathbb{R}^n, |x|^\alpha)} = \|h_k\|_{L_{p_1}(\mathbb{R}^n, |x|^\alpha)} = c 2^{k \frac{n+\alpha}{p_1}}.$$

Similarly, we obtain

$$\|h_k\|_{A_{p_2, q_2}^r(\mathbb{R}^n, |x|^\alpha)} = c 2^{k \frac{n+\alpha}{p_2}},$$

where the positive constant c is independent of k . Observe that $\text{supp } \mathcal{F}(h_k^2) \subset \{\xi \in \mathbb{R}^n : |\xi| < \frac{1}{8} 2^{-k}\}$. Therefore

$$\Delta_j(h_k^2) = 0 \quad \text{and} \quad \Delta_0(h_k^2) = h_k^2, \quad j \in \mathbb{N}.$$

Consequently,

$$\|h_k^2\|_{A_{p, q}^s(\mathbb{R}^n, |x|^\alpha)} = c 2^{k \frac{n+\alpha}{p}},$$

where $c > 0$ is independent of k . If (4.1) holds, then

$$2^{k(n+\alpha)(\frac{1}{p} - \frac{1}{p_1} - \frac{1}{p_2})} \leq c$$

for some positive constant c is independent of k . Assume that $\frac{1}{p} > \frac{1}{p_1} + \frac{1}{p_2}$. Letting k tends to infinity, which contradicts (4.1). \square

Theorem 4.2. Assume that (4.1). Then necessarily we have

$$0 < p \leq p_1$$

and

$$0 < p \leq p_2, \quad \text{if } r = s.$$

In addition

$$r - \frac{n+\alpha}{p_2} \geq s - \frac{n+\alpha}{p}. \quad (4.2)$$

Proof. By similarity we prove (4.2). Let $v \in \mathbb{R}, 0 < t \leq \infty$ ($0 < t < \infty$ for the F -scale) and $0 < u \leq \infty$. Let $f \in \mathcal{S}(\mathbb{R}^n)$ be such that $\text{supp } f \subset \{x \in \mathbb{R}^n : |x_i| < \frac{1}{4}, i = 1, \dots, n\}$ and

$$\int_{\mathbb{R}^n} x^\beta f(x) dx = 0 \quad \text{for } |\beta| \leq L,$$

with L large enough. Define

$$f^j(x) = f(2^j x), \quad x \in \mathbb{R}^n, j \in \mathbb{N}.$$

Then

$$\|f^j\|_{A_{t,u}^v(\mathbb{R}^n)} \approx 2^{j(v-\frac{n}{t})}, \quad (4.3)$$

see [98, p. 144]. Let us prove that

$$\|f^j\|_{A_{p,q}^s(\mathbb{R}^n, |x|^\alpha)} \approx 2^{j(s-\frac{n+\alpha}{p})}, \quad \alpha > -n. \quad (4.4)$$

Using the atomic decomposition of $A_{p,q}^s(\mathbb{R}^n, |x|^\alpha)$ spaces, see, e.g. [31] and [123], we find that

$$\begin{aligned} \|f^j\|_{A_{p,q}^s(\mathbb{R}^n, |x|^\alpha)} &\lesssim 2^{js} \|\chi_{j,0}\|_{L_p(\mathbb{R}^n, |x|^\alpha)} \\ &\lesssim 2^{j(s-\frac{n+\alpha}{p})}, \end{aligned} \quad (4.5)$$

where the implicit constant is independent of j , with $\chi_{j,0} = \chi_{\{x: 0 \leq x_i < 2^{-j}, i=1, \dots, n\}}$. We prove the opposite inequality of (4.5). Assume that $-n < \alpha < 0$. Let $\alpha_1 > 0$ be such that

$$0 = (1-\theta)\alpha + \theta\alpha_1, \quad 0 < \theta < 1.$$

By Hölder's inequality, (4.3) and (4.5) we obtain

$$\|f^j\|_{A_{p,q}^s(\mathbb{R}^n)} \leq \|f^j\|_{A_{p,q}^s(\mathbb{R}^n, |x|^{\alpha_1})}^\theta \|f^j\|_{A_{p,q}^s(\mathbb{R}^n, |x|^\alpha)}^{1-\theta} \lesssim 2^{j(s-\frac{n+\alpha_1}{p})\theta} \|f^j\|_{A_{p,q}^s(\mathbb{R}^n, |x|^\alpha)}^{1-\theta}.$$

This leads to (4.4) if $-n < \alpha < 0$. Now, assume that $\alpha > 0$. Let $-n < \alpha_2 < 0$ be such that

$$0 = (1-\theta)\alpha + \theta\alpha_2, \quad 0 < \theta < 1.$$

Again, By Hölder's inequality, (4.3) and (4.5) we obtain the desired estimate. Now we prove the necessity of (4.2). Let $g \in \mathcal{S}(\mathbb{R}^n)$ be such that $g \equiv 1$ on $\{x \in \mathbb{R}^n : |x_i| < 1, i = 1, \dots, n\}$. By (4.5) we easily obtain

$$\|f^j\|_{A_{p_2, q_2}^r(\mathbb{R}^n, |x|^\alpha)} \lesssim 2^{j(r-\frac{n+\alpha}{p_2})}.$$

Assume that (4.1) holds. Then

$$\begin{aligned}
\|f^j\|_{A_{p,q}^s(\mathbb{R}^n,|x|^\alpha)} &= \|f^j g\|_{A_{p,q}^s(\mathbb{R}^n,|x|^\alpha)} \\
&\lesssim \|g\|_{A_{p_1,q_1}^s(\mathbb{R}^n,|x|^\alpha)} \|f^j\|_{A_{p_2,q_2}^r(\mathbb{R}^n,|x|^\alpha)} \\
&\lesssim \|f^j\|_{A_{p_2,q_2}^r(\mathbb{R}^n,|x|^\alpha)} \\
&\lesssim 2^{j(r-\frac{n+\alpha}{p_2})},
\end{aligned}$$

which yields that

$$2^{j(s-\frac{n+\alpha}{p})} \lesssim 2^{j(r-\frac{n+\alpha}{p_2})}, \quad j \in \mathbb{N}.$$

This completes the proof of the necessity of (4.2). \square

Theorem 4.3. Assume that (4.1). Then necessarily we have

$$q \geq q_1 \tag{4.6}$$

and

$$q \geq q_2, \quad \text{if } r = s.$$

Proof. By similarity, we prove only that $q \geq q_1$. Let $f \in \mathcal{S}(\mathbb{R}^n)$ be such that $\text{supp } \mathcal{F}f \subset \{\xi \in \mathbb{R}^n : |\xi| < \varepsilon\}$ where ε small enough. Define

$$\theta_k(x) = e^{i\frac{7}{8}2^k x_1} f(x), \quad x \in \mathbb{R}^n, k \geq 3.$$

We obtain $\text{supp } \mathcal{F}\theta_k \subset \{\xi \in \mathbb{R}^n : \frac{7}{8}2^k - \varepsilon \leq |\xi| \leq \frac{7}{8}2^k + \varepsilon\}$ and $\Delta_j \theta_k = 0$, if $j \neq k$, $k \geq 3$. We put

$$\theta_N = \sum_{k=3}^N 2^{-sk} \theta_k, \quad N \in \mathbb{N}.$$

Hence

$$\|\theta_N\|_{A_{p_1,q_1}^s(\mathbb{R}^n,|x|^\alpha)} = c N^{\frac{1}{q_1}},$$

where the positive constant c is independent of N . Now $\text{supp } \mathcal{F}(\theta_k f) \subset \{\xi \in \mathbb{R}^n : 3 \cdot 2^{k-2} \leq |\xi| \leq 2^k\}$ and hence

$$\|\theta_N f\|_{A_{p,q}^s(\mathbb{R}^n,|x|^\alpha)} = c N^{\frac{1}{q}}.$$

Assume that (4.1) holds. Then

$$\|\theta_N f\|_{A_{p,q}^s(\mathbb{R}^n, |x|^\alpha)} \lesssim \|\theta_N\|_{A_{p_1, q_1}^s(\mathbb{R}^n, |x|^\alpha)} \|f\|_{A_{p_2, q_2}^r(\mathbb{R}^n, |x|^\alpha)},$$

which yields $N^{\frac{1}{q} - \frac{1}{q_1}} \lesssim 1$. As above we deduce the necessity of (4.6). \square

Theorem 4.4. *Assume that (4.1). Then necessarily we have*

$$s + r \geq \max\left(0, \frac{n + \alpha}{p_1} + \frac{n + \alpha}{p_2} - n\right).$$

Proof. First let us prove that $s + r \geq 0$ is necessary. Let $\theta_k, k \geq 3$ be as in Theorem 4.3 and

$$\omega_k(x) = e^{-i\frac{7}{8}2^k x_1} f(x), \quad x \in \mathbb{R}^n, k \geq 3.$$

We have

$$\|\theta_k\|_{A_{p_1, q_1}^s(\mathbb{R}^n, |x|^\alpha)} = c2^{ks}, \quad \|\omega_k\|_{A_{p_2, q_2}^r(\mathbb{R}^n, |x|^\alpha)} = c2^{kr},$$

where the positive constant c is independent of k . Assume that (4.1) holds. Then

$$\|\theta_k \omega_k\|_{A_{p,q}^s(\mathbb{R}^n, |x|^\alpha)} \lesssim \|\theta_k\|_{A_{p_1, q_1}^s(\mathbb{R}^n, |x|^\alpha)} \|\omega_k\|_{A_{p_2, q_2}^r(\mathbb{R}^n, |x|^\alpha)} \lesssim 2^{k(s+r)}.$$

Using the fact that $\theta_k \omega_k = f^2, k \geq 3$, we obtain the desired result by letting k tends to infinity. The proof of the necessity of $s + r \geq \frac{n+\alpha}{p_1} + \frac{n+\alpha}{p_2} - n$ can be obtained by the same manner as in [98, p. 145]. \square

4.2 Application

As well known, the theory of classical pseudo-differential operators appeared in the literature for the first time already in 1965 by Kohn and Nirenberg [62] and Hörmander [54]. The notion of pseudo-differential operators have played an important role in various areas of mathematics and mathematical physics such as in quantum mechanics, signal analysis, harmonic analysis and partial differential equations. Since its discovery, it has been found to be one of the most powerful tools in attacking various problems in EDP such as the existence and uniqueness of the boundary value problems [104], regularity of the solutions of the EDP [54], solvability of a general partial differential operator [91], etc.

One of the important problems in harmonic analysis and partial differential equations is boundedness of pseudo-differential operator on function spaces such as on Lebesgue spaces L^p , Sobolev spaces H^s , Hölder Spaces \mathcal{C}^s , Besov spaces $B_{p,q}^s$, Herz-Besov spaces $K_{p,q}^s$, B_β^s and others. Therefore, we find many mathematicians who studied of this operator, for example Hörmander [54] proved that the pseudo-differential operator of order $m = 0$, of symbol $\sigma(x, \xi)$ satisfying

$$|\partial_\xi^\alpha \partial_x^\beta \sigma(x, \xi)| \lesssim (1 + |\xi|)^{m - \eta|\alpha| + \delta|\beta|} \quad (4.7)$$

for $0 \leq \delta < \eta \leq 1$, are bounded on L^2 , and the is true for $\delta = \eta \neq 1$. In the case $\delta = \eta = 1$, A counter-example of Ching [18] in 1972 proved that the continuity of pseudo-differential operator on L^2 generally not insured. In the same year, A. Calderón and R. Vaillancourt have proved the L^2 -continuity of pseudo-differential operator having symbols in $S_{1,\delta}^m$ where $0 \leq \delta < 1$. Another result due to E. Stein, which proved the continuity of such as operator on Sobolev spaces H^s for $s > 0$.

In this section we apply some of our results to the continuity of pseudodifferential operator on $F_{p,q}^s(\mathbb{R}^n, |x|^\alpha)$ spaces. We refer the reader to the monographs [90] and [119] for further details, historical remarks and more references on pseudodifferential operators.

Definition 4.5. Let E be a Banach space of functions in $\mathbb{R}^n \times \mathbb{R}^n$. $S_{1,0}^0(E)$ is the class of symbols a satisfying

$$\|\partial_\xi^\beta a(\cdot, \xi)\|_E \leq c_\beta (1 + |\xi|)^{-|\beta|}, \quad \forall \beta \in \mathbb{N}^n, \xi \in \mathbb{R}^n.$$

Definition 4.6. Let a be a symbol. The pseudo-differential operator Op_a associated to a is defined by

$$Op_a(f)(\xi) = \int_{\mathbb{R}^n} e^{ix\xi} a(x, \xi) \mathcal{F}f(x) dx, \quad f \in \mathcal{S}(\mathbb{R}^n), \xi \in \mathbb{R}^n.$$

Definition 4.7. Let E be a Banach space. We say that $a \in S_{1,0}^0(E)$ is a elementary symbol if

$$a(x, \xi) = \sum_{j=0}^{\infty} a_j(x) \theta_j(\xi),$$

where $\theta_j \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } \theta_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 3\}$, $\text{supp } \theta_j \subset \{\xi \in \mathbb{R}^n : 2^j \leq |\xi| \leq 3 \cdot 2^j\}$, $j \in \mathbb{N}$ and $\sup_{j \geq 0} \|a_j\|_E \leq c$.

Example 4.8. Let $1 \leq p, q < \infty$, $\alpha > -n$ and $s \in \mathbb{R}$. Let $\{\varphi_j\}_{j \in \mathbb{N}_0}$ be a partition of unity and $\{a_j\}_{j \in \mathbb{N}_0} \subset B_{p,q}^s(\mathbb{R}^n, |x|^\alpha)$, such that

$$\sup_{j \in \mathbb{N}_0} \|a_j\|_{B_{p,q}^s(\mathbb{R}^n, |x|^\alpha)} \leq c.$$

Then

$$a(x, \xi) = a_0(x) \varphi_0(\xi) + \sum_{j=1}^{\infty} a_j(x) \varphi(2^{-j} \xi), \quad x, \xi \in \mathbb{R}^n,$$

is an elementary symbol.

Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\text{supp } \mathcal{F}\psi \subset \{\xi \in \mathbb{R}^n : \gamma^{-1} < |\xi| < \gamma\}, \quad \gamma > 1.$$

We put

$$f_j = \psi_j * f, \quad \psi_j = 2^{jn} \psi(2^j \cdot), \quad j \in \mathbb{N}_0.$$

Lemma 4.9. Let $0 < p_1, p_2 < \infty$, $0 < q \leq \infty$ and $\alpha \geq 0$. Assume that

$$-\frac{n+\alpha}{p_2} + \max\left(0, \frac{n+\alpha}{p_1} + \frac{n+\alpha}{p_2} - n\right) < s < \min\left(\frac{n+\alpha}{p_1}, \frac{n+\alpha}{p_2}\right).$$

Let $\{m_j\}_{j \in \mathbb{N}_0} \subset B_{p_2, \infty}^{\frac{n+\alpha}{p_2}}(\mathbb{R}^n, |x|^\alpha) \cap L_\infty(\mathbb{R}^n)$ and $f \in F_{p_1, q}^s(\mathbb{R}^n, |x|^\alpha)$. Then

$$\left\| \sum_{j=0}^{\infty} m_j \cdot f_j \right\|_{F_{p_1, q}^s(\mathbb{R}^n, |x|^\alpha)} \lesssim \sup_{j \in \mathbb{N}_0} \|m_j\|_{B_{p_2, \infty}^{\frac{n+\alpha}{p_2}}(\mathbb{R}^n, |x|^\alpha) \cap L_\infty(\mathbb{R}^n)} \|f\|_{F_{p_1, q}^s(\mathbb{R}^n, |x|^\alpha)}$$

holds.

Proof. Recall that

$$m_j \cdot f_j = \Pi_1(m_j, f_j) + \Pi_2(m_j, f_j) + \Pi_3(m_j, f_j), \quad j \in \mathbb{N}_0.$$

Hence, we need only to estimate

$$\sum_{j=0}^{\infty} \Pi_i(m_j, f_j), \quad i = 1, 2, 3,$$

in $F_{p,q}^s(\mathbb{R}^n, |x|^\alpha)$ spaces. Let $k \in \mathbb{N}_0$. Observe that $\Delta_k f_j \neq 0$ and $Q_{k+1} f_j \neq 0$ if $k - N_1 \leq j \leq k + N_2$ and $j \leq k + N_2$, respectively, where $N_1 = \lfloor \log_2 \gamma \rfloor$ and $N_2 = \lfloor \log_2 3\gamma \rfloor$. **Estimate of** $\sum_{j=0}^{\infty} \Pi_1(m_j, f_j)$. We have

$$\Pi_1(m_j, f_j) = \sum_{i=2}^{\infty} Q_{i-2} m_j \Delta_i f_j = \sum_{i=j-N}^{j+N} Q_{i-2} m_j \Delta_i f_j = \sum_{h=-N}^N Q_{j+h-2} m_j \Delta_{j+h} f_j$$

for some $N \in \mathbb{N}$, where we put $\Delta_v f_j = 0$ and $Q_\tau m_j = 0$ if $\tau, v < 0$. Therefore

$$\begin{aligned} \sum_{j=0}^{\infty} \Delta_k(\Pi_1(m_j, f_j)) &= \sum_{j=k-N_1}^{k+N_1} \Delta_k(\Pi_1(m_j, f_j)) \\ &= \sum_{z=-N_3}^{N_3} \sum_{h=-N}^N \Delta_k(Q_{k+z+h-2} m_{k+z} \Delta_{k+z+h} f_{k+z}) \end{aligned}$$

for some $N_3 \in \mathbb{N}$. By similarity, we estimate only $\Delta_k(Q_{k-2} m_k \Delta_k f_k)$, $k = 2, 3, \dots$. Lemma [3.19](#) yields that

$$|Q_{k-2} m_k \Delta_k f_k| \leq \sup_{j \geq 2} |Q_{j-2} m_j| |\Delta_k f_k| \lesssim \sup_{j \in \mathbb{N}_0} \|m_j\|_{\infty} \mathcal{M}_t(\Delta_k f), \quad k = 2, 3, \dots$$

and

$$\begin{aligned} 2^{ks} |\Delta_k(Q_{k-2} m_k \Delta_k f_k)| &\lesssim 2^{ks} \mathcal{M}_t(Q_{k-2} m_k \Delta_k f_k) \\ &\lesssim 2^{ks} \sup_{j \in \mathbb{N}_0} \|m_j\|_{\infty} \mathcal{M}_t(\mathcal{M}_t(\Delta_k f)) \end{aligned} \quad (4.8)$$

for any $0 < t \leq 1$ and any $k = 2, 3, \dots$. In [\(4.8\)](#) we choose $0 < t < \min(1, \frac{np_1}{n+\alpha})$, the ℓ^q -quasi-norm and then the $L_{p_1}(\mathbb{R}^n, |x|^\alpha)$ -quasi-norm, we obtain that

$$\left\| \left(\sum_{k=0}^{\infty} 2^{skq} |\Delta_k(Q_{k-2} m_k \Delta_k f_k)|^q \right)^{\frac{1}{q}} \right\|_{L_{p_1}(\mathbb{R}^n, |x|^\alpha)},$$

with the help of Lemma [3.20](#), is bounded by

$$c \sup_{j \in \mathbb{N}_0} \|m_j\|_{\infty} \left\| \left(\sum_{k=0}^{\infty} 2^{skq} |\Delta_k f|^q \right)^{\frac{1}{q}} \right\|_{L_{p_1}(\mathbb{R}^n, |x|^\alpha)} = c \sup_{j \in \mathbb{N}_0} \|m_j\|_{\infty} \|f\|_{F_{p_1,q}^s(\mathbb{R}^n, |x|^\alpha)}.$$

This yields the desired estimate. **Estimate of $\sum_{j=0}^{\infty} \Pi_2(m_j, f_j)$.** This follows by the same arguments of Theorem [3.25](#). **Estimate of $\sum_{j=0}^{\infty} \Pi_3(m_j, f_j)$.** Observe that

$$\begin{aligned} \Delta_k(\Pi_3(m_j, f_j)) &= \sum_{i=2}^{\infty} \Delta_k(Q_{i-2} f_j \Delta_i m_j) \\ &= \sum_{i=k-N}^{k+N} \Delta_k(Q_{i-2} f_j \Delta_i m_j) \\ &= \sum_{h=-N}^N \Delta_k(Q_{k+h-2} f_j \Delta_{k+h} m_j) \end{aligned}$$

for some $N \in \mathbb{N}$. Hence

$$\sum_{j=0}^{\infty} \Delta_k(\Pi_3(m_j, f_j)) = \sum_{j=0}^{k+N_1} \Delta_k(\Pi_3(m_j, f_j)) = \sum_{h=-N}^N \sum_{j=0}^{k+N_1} \Delta_k(Q_{k+h-2} f_j \Delta_{k+h} m_j)$$

for some $N_1 \in \mathbb{N}$. We employ the same notation as in the estimation of $\Pi_3(f, g)$ of Theorem [3.25](#) with $p = p_1$. Again by similarity, we estimate

$$2^{k\sigma} \sum_{j=0}^k \Delta_k(Q_{k-2} f_j \Delta_k m_j), \quad k = 2, 3, \dots$$

in $L_\nu(\mathbb{R}^n, |x|^\alpha)$ -quasi-norm and then in ℓ^{p_1} -quasi-norm. Using Lemma [3.19](#) we get

$$|\Delta_k(Q_{k-2} f_j \Delta_k m_j)| \lesssim \mathcal{M}_t(Q_{k-2} f_j \Delta_k m_j)$$

for any $0 < t \leq 1$. We choose $0 < t < \min(1, \frac{n\nu}{n+\alpha})$ and using Lemma [3.20](#), we obtain

$$2^{k\sigma} \left(\sum_{j=0}^k \left\| \Delta_k(Q_{k-2} f_j \Delta_k m_j) \right\|_{L_\nu(\mathbb{R}^n, |x|^\alpha)}^{\min(1, \nu)} \right)^{\frac{1}{\min(1, \nu)}},$$

is bounded by

$$\begin{aligned} &c 2^{k\sigma} \left(\sum_{j=0}^k \left\| Q_{k-2} f_j \Delta_k m_j \right\|_{L_\nu(\mathbb{R}^n, |x|^\alpha)}^{\min(1, \nu)} \right)^{\frac{1}{\min(1, \nu)}} \tag{4.9} \\ &\lesssim \sup_{j \in \mathbb{N}_0} \left\| \Delta_k m_j \right\|_{L_{p_2}(\mathbb{R}^n, |x|^\alpha)} 2^{k\sigma} \left(\sum_{j=0}^k \left\| Q_{k-2} f_j \right\|_{L_u(\mathbb{R}^n, |x|^\alpha)}^{\min(1, \nu)} \right)^{\frac{1}{\min(1, \nu)}} \end{aligned}$$

$$\lesssim \sup_{j \in \mathbb{N}_0} 2^{kr} \|\Delta_k m_j\|_{L_{p_2}(\mathbb{R}^n, |x|^\alpha)} 2^{k(\sigma-r)} \left(\sum_{j=0}^k \|f_j\|_{L_u(\mathbb{R}^n, |x|^\alpha)}^{\min(1, \nu)} \right)^{\frac{1}{\min(1, \nu)}},$$

where we used the Hölder's inequality. Let us prove the last inequality. Observe that

$$Q_{k-2} f_j(x) = 2^{n(k-2)} \int_{\mathbb{R}^n} \mathcal{F}^{-1} \varphi(2^{k-2} y) f_j(x-y) dy, \quad x \in \mathbb{R}^n,$$

which yields that

$$|Q_{k-2} f_j(x)| \lesssim f_j^{*,a}(x) 2^{n(k-2)} \int_{\mathbb{R}^n} \frac{|\mathcal{F}^{-1} \varphi(2^{k-2} y)|}{(1+2^j |y|)^{-a}} dy \lesssim f_j^{*,a}(x)$$

for any $a > \frac{n}{u}$, since $j \leq k$, where

$$f_j^{*,a}(x) = \sup_{y \in \mathbb{R}^n} \frac{|f_j(x-y)|}{(1+2^j |y|)^a}, \quad (\text{Peetre maximal function})$$

and the implicit constant is independent of j and k . Using the well know estimate

$$f_j^{*,a}(x) \lesssim \mathcal{M}_\tau(f_j), \quad 0 < \tau < \min\left(1, \frac{nu}{n+\alpha}\right),$$

see [112, p. 16], we obtain

$$\|Q_{k-2} f_j\|_{L_u(\mathbb{R}^n, |x|^\alpha)} \leq c \|f_j\|_{L_u(\mathbb{R}^n, |x|^\alpha)}$$

for any $0 \leq j \leq k$ and any $k = 2, 3, \dots$. Since $\sigma - r < 0$, by Lemma 2.3, (4.9) in ℓ^{p_1} -quasi-norm is bounded by

$$c \sup_{j \in \mathbb{N}_0} \|m_j\|_{B_{p_2, \infty}^r(\mathbb{R}^n, |x|^\alpha)} \left(\sum_{j=0}^{\infty} 2^{j \varrho p_1} \|f_j\|_{L_u(\mathbb{R}^n, |x|^\alpha)}^{p_1} \right)^{\frac{1}{p_1}}.$$

Using Lemma 3.19 we obtain

$$|f_j| \lesssim \mathcal{M}_\kappa \left(\sum_{i=j-N}^{i=j+N} \Delta_i f \right), \quad j, N \in \mathbb{N}_0, 0 < \kappa \leq 1.$$

Our desired estimate follows by the application of Lemma 3.20 with $0 < \kappa < \min(1, \frac{nu}{n+\alpha})$

and the second embeddings of (3.13). The proof is complete. \square

By using the same arguments as in [83] we obtain the following.

Proposition 4.10. *Let $s \in \mathbb{R}, \alpha > -n$ and $1 \leq p, q \leq \infty$. Let $N \in \mathbb{N}$, even and $N > n$. Let $a \in S_{1,0}^0(L_\infty(\mathbb{R}^n)) \cap S_{1,0}^0(B_{p,q}^s(\mathbb{R}^n, |x|^\alpha))$. Then a allows a representation in the form*

$$a(x, \xi) = \int_{\mathbb{R}^n} (1 + |u|^2)^{-(n+1)/2} a_u(x, \xi) du + \lambda(x, \xi),$$

where $\lambda(x, \xi) = 0$ for any $|\xi| \geq 3$,

$$\left\| \partial_\xi^\beta \lambda(\cdot, \xi) \right\|_{B_{p,q}^s(\mathbb{R}^n, |x|^\alpha) \cap L_\infty(\mathbb{R}^n)} \leq c(a), \quad \forall \xi \in \mathbb{R}^n,$$

and a_u is a elementary symbol;

$$a_u(x, \xi) = \sum_{j \geq 0} m_{j,u}(x) \theta_u(2^{-j} \xi),$$

with

$$\sup_{j \geq 0, u \in \mathbb{R}^n} \left\| m_{j,u} \right\|_{B_{p,q}^s(\mathbb{R}^n, |x|^\alpha) \cap L_\infty(\mathbb{R}^n)} \leq c(a)$$

and

$$\theta_u(\xi) = (2\pi)^{-n} e^{iu\xi} (1 + |u|^2)^{(n+1-N)/2} \theta(\xi),$$

where $\theta \in C^\infty(\mathbb{R}^n)$, $\text{supp } \theta \subset \{\xi \in \mathbb{R}^n : 1 \leq |\xi| \leq 3\}$ and $\left\| \partial^\beta \theta_u \right\|_\infty \leq c, \forall u \in \mathbb{R}^n, |\beta| \leq N - n - 1$.

Now, we formulate the main result of this section.

Theorem 4.11. *Let $1 \leq p, p_1, p_2, q < \infty$ and $\alpha \geq 0$. Assume that $1 \leq p_1 \leq p_2 < \infty$ and*

$$-\frac{n}{p_2} + \max\left(0, \frac{n+\alpha}{p} + \frac{n+\alpha}{p_1} - n\right) < s < \min\left(\frac{n}{p}, \frac{n}{p_2}\right).$$

Assume that $a \in S_{1,0}^0(L^\infty) \cap S_{1,0}^0(B_{p_1, \infty}^{\frac{n+\alpha}{p}}(\mathbb{R}^n, |x|^\alpha))$. Then the pseudodifferential operator Op_a is bounded from $F_{p,q}^s(\mathbb{R}^n, |x|^\alpha) \cap F_{p,q}^s(\mathbb{R}^n)$ to $F_{p,q}^s(\mathbb{R}^n, |x|^\alpha) + F_{p,q}^s(\mathbb{R}^n)$.

Proof. Let $a \in S_{1,0}^0(L^\infty) \cap S_{1,0}^0(B_{p_1,\infty}^{\frac{n+\alpha}{p_1}}(\mathbb{R}^n, |x|^\alpha))$. By Proposition 4.10, a can be rewritten as

$$a(x, \xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} (1 + |u|^2)^{-(n+1)/2} a_u(x, \xi) du + \lambda(x, \xi),$$

where a_u is a elementary symbol. Recall that

$$\|Op_a f\|_{F_{p,q}^s(\mathbb{R}^n, |x|^\alpha) + F_{p,q}^s(\mathbb{R}^n)} = \inf_{Op_a f = g_0 + g_1} (\|g_0\|_{F_{p,q}^s(\mathbb{R}^n, |x|^\alpha)} + \|g_1\|_{F_{p,q}^s(\mathbb{R}^n)}).$$

Applying Lemma 4.9 we obtain

$$\begin{aligned} \|Op_{a_u} f\|_{F_{p,q}^s(\mathbb{R}^n, |x|^\alpha)} &\lesssim \sup_{j \in \mathbb{N}_0, u \in \mathbb{R}^n} \|m_{j,u}\|_{B_{p_1,\infty}^{\frac{n+\alpha}{p_1}}(\mathbb{R}^n, |x|^\alpha) \cap L^\infty(\mathbb{R}^n)} \|f\|_{F_{p,q}^s(\mathbb{R}^n, |x|^\alpha)} \\ &\lesssim \|f\|_{F_{p,q}^s(\mathbb{R}^n, |x|^\alpha) \cap F_{p,q}^s(\mathbb{R}^n)}. \end{aligned}$$

Now, we have $B_{p_1,\infty}^{\frac{n+\alpha}{p_1}}(\mathbb{R}^n, |x|^\alpha) \hookrightarrow B_{p_2,q}^{\frac{n}{p_2}}(\mathbb{R}^n)$, see Theorem 3.9, and

$$Op_\lambda f(x) = \int_{\mathbb{R}^n} (1 + |u|^2)^{-(n+1)/2} \varrho_u(x) f(x+u) du,$$

where

$$\varrho_u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iu\xi} (I - \Delta)^{2n} \lambda(x, \xi) d\xi.$$

Applying Theorem 3.30 we easily obtain

$$\begin{aligned} \|Op_\lambda f\|_{F_{p,q}^s(\mathbb{R}^n)} &\lesssim \sup_{u \in \mathbb{R}^n} \|\varrho_u \cdot f(\cdot + u)\|_{F_{p,q}^s(\mathbb{R}^n)} \\ &\lesssim \sup_{u \in \mathbb{R}^n} (\|\varrho_u\|_{B_{p_1,\infty}^{\frac{n+\alpha}{p_1}}(\mathbb{R}^n, |x|^\alpha) \cap L^\infty(\mathbb{R}^n)}) \|f\|_{F_{p,q}^s(\mathbb{R}^n)} \\ &\lesssim \|f\|_{F_{p,q}^s(\mathbb{R}^n, |x|^\alpha) \cap F_{p,q}^s(\mathbb{R}^n)}. \end{aligned}$$

□

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المخلص: في هذه الأطروحة قمنا بدراسة استمرارية المؤثرات التكاملية الشاذة على فضاءات هارز الضعيفة ذات الدلائل المتغيرة و فضاءات هارز-هاردي الضعيفة ذات الدلائل المتغيرة. من جهة أخرى، تحت شروط معينة قمنا ببرهنة بعض الاحتواءات التي من الشكل $\mathbf{B} \cdot \mathbf{B} \rightarrow \mathbf{B}$ و $\mathbf{F} \cdot \mathbf{B} \rightarrow \mathbf{F}$, $\mathbf{F} \cdot \mathbf{F} \rightarrow \mathbf{F}$ حيث \mathbf{B} و \mathbf{F} ترمز الى فضاءات بيزوف و فضاءات تريبل-ليزوركين ذات الاوزان الطاقية. اضافة الى ذلك، تحصلنا على تطبيق لهذه النتائج حول استمرارية المؤثر شبه التفاضلي في فضاءات تريبل-ليزوركين ذات الاوزان الطاقية و وجدنا ايضا ان كل افتراضاتنا حول الدلائل كانت ضرورية.

الكلمات المفتاحية: فضاءات بيزوف، و فضاءات تريبل-ليزوركين، المؤثر شبه التفاضلي، فضاءات هارز-هاردي، فضاءات هارز الضعيفة، فضاءات هارز-هاردي الضعيفة، الدلائل المتغيرة، الذرة، المؤثرات التكاملية الشاذة.

Résumé : Dans cette thèse, nous présentons la continuité des operateurs intégraux singuliers sur les espaces de Herz faibles avec des exposants variables et sur les espaces de Herz-type Hardy faibles avec des exposants variables. D'autre part, sous certaines hypothèses appropriées, nous prouvons quelques plongements de la form $\mathbf{F} \cdot \mathbf{B} \rightarrow \mathbf{F}$, $\mathbf{F} \cdot \mathbf{F} \rightarrow \mathbf{F}$ et $\mathbf{B} \cdot \mathbf{B} \rightarrow \mathbf{B}$, où \mathbf{B} et \mathbf{F} , dénotra l'espace de Besov des poids de puissance et l'espace de Triebel-Lizorkin des poids de puissance, respectivement, avec une application à la continuité des operateurs pseudo-differentiels sur les espaces de Triebel-Lizorkin des poids de puissance, et nous avons trouvé que presque toutes nos hypothèses sont nécessaires.

Mots-clés : Espace de Besov, espace de Triebel-Lizorkin, poids de puissance, operateurs pseudo-differentiels, espaces de Herz-type Hardy, espaces de Herz faibles, espaces de Herz-type Hardy faibles, atome, operateurs intégraux singuliers.

Abstract: In this thesis we present the boundedness properties of singular integral operators on variable weak Herz spaces and variable weak Herz-type Hardy spaces. On the other hand, under some suitable assumptions, we prove some embeddings of the form $\mathbf{F} \cdot \mathbf{B} \rightarrow \mathbf{F}$, $\mathbf{F} \cdot \mathbf{F} \rightarrow \mathbf{F}$ and $\mathbf{B} \cdot \mathbf{B} \rightarrow \mathbf{B}$, where \mathbf{B} and \mathbf{F} , will denote the Besov space of power weight and the Triebel-Lizorkin space of power weight, respectively. An application to the continuity of pseudo-differential operators on Triebel-Lizorkin spaces of power weights and we find almost all our assumptions are necessary.

Keywords: Besov space, Triebel-Lizorkin space, power weight, pseudo-differential operator, Herz-type Hardy spaces, weak Herz spaces, weak Herz-type Hardy spaces, atom, singular integral operators.