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An inverse coefficient problem for a parabolic equation with of nonlocal boundary and over determination conditions

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Presented by :
M^r *LALDJI* dahbia

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In front of the jury composed of :

Yacine Arioua	PROF,	Mohamed Boudiaf University of M'sila	**President**.
FARID MIHOUBI	MCB,	Mohamed Boudiaf University of M'sila	**Supervisor.**
Belizak Tahar	MCB,	Mohamed Boudiaf University of M'sila	**Examiner.**

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LALDJI dahbia

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Introduction

Over the past decades, significant interest has been devoted to the determination of coefficients in diffusion equations, see [4, 9, 10, 11, 12, 15, 18]. Problems of this type arise in many mathematical models, particularly in heat conduction, radioactive decay processes in a rod, the design of building heating systems, and thermal energy storage in a medium, see [2, 3, 5, 7, 13].

Let $T > 0$ be a fixed number. In the rectangle $\Omega_T = \{(x,t) : 0 < x < 1,\ 0 < t \leq T\}$, we consider the following initial-boundary value problem:

$$u_t(x,t) = a(t)u_{xx}(x,t) + f(x,t), \quad (x,t) \in \Omega_T, \tag{1}$$

$$u(x,0) = \varphi(x), \quad 0 \leq x \leq 1, \tag{2}$$

$$u_x(0,t) = \beta u_x(1,t), \quad 0 \leq t \leq T, \tag{3}$$

$$u(0,t) = u(1,t), \quad 0 \leq t \leq T, \tag{4}$$

where f and φ are given functions, and condition (4) is called a nonlocal boundary condition. If the coefficient $a(t)$, $0 \leq t \leq T$, is known, then problem (1) (4) is called a direct problem.

It is well known that equation (1) models a heat conduction process in a rod undergoing radioactive decay or damage with a heat source. The thermal conductivity, diffusivity, heat capacity, and heat transfer coefficient vary depending on the degree of degradation, which may depend on time, see [3, 12].

The function $\varphi(x)$ represents the initial temperature distribution in the rod at time $t = 0$, while $u(x,t)$ represents the temperature distribution at time t.

The diffusion coefficient $a(t) > 0$ is defined as

$$a(t) = \frac{k(t)}{\rho c_p(t)},$$

where $k(t) > 0$ is the thermal conductivity of the material, ρ is the density, and $c_p(t)$ is the heat capacity, see [3, 12].

The local Neumann condition (3) means that the right end $x = 1$ of the rod is insulated, so that no heat flows through this boundary, while the nonlocal condition (4) means that the temperature at the left end $x = 0$ is equal to the temperature at the right end $x = 1$ for all $t \in [0, T]$.

The problem (1) (4) is well posed in the sense of Hadamard when $a(t) > 0$ is known. However, when $a(t)$ is unknown, the problem becomes ill-posed. In this case, we introduce an additional integral condition:

$$\int_0^1 u(x,t)\, dx = E(t), \quad 0 \leq t \leq T, \tag{5}$$

where $E(t)$ is a given function.

The problem of determining the pair $(u(x,t), a(t))$ satisfying conditions (1)(5) is called an inverse problem. The objective is to determine the diffusion coefficient $a(t)$ that produces the required energy distribution in the rod.

The inverse problem of determining $a(t)$ in equation (1) with nonlocal boundary conditions has been studied in [9, 11, 12].

In this work, we focus on two main aspects:

- **Theoretical aspect:** We study the existence and uniqueness of the solution of the inverse problem (1) (5), as well as the continuous dependence of the solution on the data, using the Fourier method, Schauder fixed point theorem, and a non-self-adjoint spectral problem.

- **Numerical aspect:** We solve numerically the inverse problem (1) (5) using the Crank-Nicolson finite difference scheme combined with a predictorâ€“corrector technique. Numerical examples are presented to validate the efficiency of the method.

This dissertation is organized into three chapters as follows:

In Chapter 1, we recall some fundamental preliminary notions and necessary tools used in this work, including functional spaces, a non-self-adjoint spectral problem, and the Schauder fixed point theorem.

In Chapter 2, we prove the existence of the solution on Ω_T and the uniqueness on Ω_{T_0} with $0 < T_0 < T$ using the Fourier method and Schauder fixed point method. We also establish the continuous dependence of the solution on the data.

In Chapter 3, we introduce a Crank-Nicolson finite difference scheme combined with a predictor-corrector technique to solve the inverse problem numerically. Numerical examples are presented to illustrate the effectiveness of the method.

Finally, we conclude this work with a general conclusion and some perspectives.

CHAPTER 1

RECALL ON MATHEMATICAL TOOLS

In this chapter, we recall some fundamental preliminary notions and the necessary tools used in this work, concerning functional spaces, a non-self-adjoint spectral problem, and the Schauder fixed point method.

1.1 Normed Spaces

Definition 1.1. Let E be a vector space over the field K ($K = \mathbb{R}$ or $\mathbb{C}$). A norm on E is a mapping $\|\cdot\| : E \to \mathbb{R}$ satisfying:

1. $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$,
2. $\|\lambda x\| = |\lambda|\|x\|$, for all $x \in E$ and $\lambda \in K$,
3. $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in E$.

Remark 1.1. A vector space E equipped with a norm is called a normed space, denoted by $(E, \|\cdot\|)$. The following are usual norms on $\mathbb{R}^n$:

$$\|x\|_\infty = \max_{1\leq i\leq n} |x_i|, \quad \|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, \quad p \geq 1.$$

Let Ω be an open subset of $\mathbb{R}^n$. The space of square integrable functions on Ω is:

$$L^2(\Omega) = \left\{ f : \Omega \to \mathbb{R}, \int_\Omega |f(x)|^2 dx < \infty \right\}.$$

The mapping

$$\|f\| = \left(\int_\Omega |f(x)|^2 dx\right)^{\frac{1}{2}}$$

defines a norm on $L^2(\Omega)$.

1.2 Banach Space

Definition 1.2. A Banach space is a complete normed space.

Definition 1.3. Let E be a normed vector space. A sequence (x_n) in E is called a Cauchy sequence if:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n, m \geq N \Rightarrow \|x_n - x_m\| < \varepsilon.$$

Definition 1.4. A normed space E is said to be complete if every Cauchy sequence in E converges in E.

1.3 Hilbert Space

Definition 1.5. Let E be a vector space over K. An inner product on E is a mapping $\langle \cdot, \cdot \rangle$: $E \times E \to K$ such that for all $x, y, x_1, x_2 \in E$ and $\lambda \in K$:

1. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = 0$,
2. $\langle x, y \rangle = \langle y, x \rangle$,
3. $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$,
4. $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$.

Remark 1.2. An inner product induces a norm defined by:

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

In $\mathbb{R}^n$, the standard inner product is:

$$(x, y) = \sum_{i=1}^{n} x_i y_i.$$

1.3.1 L^p Spaces

Let $[a, b]$ be an interval of $\mathbb{R}$.

Definition 1.6. Let $1 < p < \infty$. The space $L^p([a, b])$ is defined by:

$$L^p([a, b]) = \{f : [a, b] \to \mathbb{C},\ f \text{ measurable},\ \|f\|_{L^p} < \infty\},$$

with

$$\|f\|_{L^p} = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}.$$

$L^p([a,b])$ is a Banach space. For $p = 2$, $L^2([a,b])$ is a Hilbert space with inner product:

$$(f,g) = \int_a^b f(x)g(x)\,dx.$$

1.4 Fixed Point Theorem

Fixed-point theorems are the basic mathematical tools that help establish the existence of solutions for various types of equations.

We recall the fixed-point theorems that we will use to obtain various existence results. We begin with the definition of a compact operator.

1.5 Compact Operator

Definition 1.7. Let E and Y be Banach spaces. An operator $P : D(P) \subset E \to Y$ is called compact if:

1. P is continuous,

2. P maps bounded sets into relatively compact sets.

Theorem 1.1 (Schauder Fixed Point Theorem)**.** *Let M be a nonempty, closed, convex subset of a Banach space E, and let $P : M \to M$ be a compact operator. Then P has at least one fixed point.*

Theorem 1.2 (Arzel'â€“Ascoli)**.** *A subset M of $C([0,1], E)$ is relatively compact if:*

1. *M is uniformly bounded,*

2. *M is equicontinuous.*

Corollaire 1.1. *If M is a nonempty, compact, convex subset of a Banach space and $P : M \to M$ is continuous, then P has a fixed point.*

CHAPTER 2

STURM-LIOUVILLE PROBLEM WITH TWO-POINT BOUNDARY CONDITIONS

In this chapter, we study a spectral problem for Sturm-Liouville operator with two-point boundary conditions. We introduce some necessary properties regarding: regular boundary conditions and biorthogonal systems in a Hilbert space. We make a detailed study of the completeness property and the fundamental property of the system of root functions. Finally, we introduce the non-self-adjoint spectral problem that we use in Chapter 2 and which is a key element of a paper published in an international journal [17].

2.1 Sturm-Liouville problem with two-point boundary conditions

We study the linear Sturm-Liouville problem consisting of the equation

$$u''(x) + \lambda u(x) = 0,\ 0 < x < 1,\ \lambda \in, \tag{2.1}$$

and the linear two-point boundary conditions of the general form

$$\begin{cases} B_1(u) = a_{11}u'(0) + a_{12}u'(1) + a_{13}u(0) + a_{14}u(1) = 0, \\ B_2(u) = a_{21}u'(0) + a_{22}u'(1) + a_{23}u(0) + a_{24}u(1) = 0, \end{cases} \tag{2.2}$$

where $B_1(u)$ and $B_2(u)$ are linearly independent forms with arbitrary complex-valued coefficients.

We consider the linear operator $\mathcal{L}u = -u''$ defined on $L^2(0,1)$ with the domain

$$D(\mathcal{L}) := \left\{ u \in L^2(0,1) : B_1(u) = B_2(u) = 0 \right\}.$$

Definition 2.1. [21][Page 193] A number λ_0 is called an eigenvalue of the operator $\mathcal{L}$ if there exists a function $u^0 \in D(\mathcal{L})$ with $u^0 \neq 0$ such that

$$\mathcal{L}u^0 = \lambda_0 u^0. \tag{2.3}$$

The function u^0 is called the eigenfunction, of the operator $\mathcal{L}$, for the eigenvalue λ_0.

Definition 2.2. [21][Page 193] A function u_m is called an associated function of the operator $\mathcal{L}$ of order m $(m = 1, 2, \ldots)$ corresponding to the same eigenvalue λ_0 and the eigenfunction u^0 if satisfies the equation

$$\mathcal{L}u^m = \lambda_0 u^m + u^{m-1}. \tag{2.4}$$

The set of functions $\{u^0, u^1, \ldots\}$ is called the eigen-and associated functions of the problem (2.1)-(2.2).

From the condition (2.2), we can form the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$$

We denote by $A\,(ij)$ the matrix composed of the i-th and j-th columns of A, and denote

$$A_{ij} := detA\,(ij) = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix},\ 1 \le i < j \le 4.$$

Then the general solution of equation (2.1) is given by:

$$u\,(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)\,,\ \lambda = \mu^2,$$

where c_1 and c_2 are arbitrary constants. Substituting the general solution into the boundary conditions (2.2) for finding c_1 and c_2, we obtain the system of equations

$$\begin{cases} \left(-a_{12}\sin\mu + \frac{a_{13}}{\mu} + a_{14}\frac{\cos\mu}{\mu}\right)c_1 + \left(a_{11} + a_{12}\cos\mu + a_{14}\frac{\sin\mu}{\mu}\right)c_2 = 0, \\ \left(-a_{22}\sin\mu + \frac{a_{23}}{\mu} + a_{24}\frac{\cos\mu}{\mu}\right)c_1 + \left(a_{21} + a_{22}\cos\mu + a_{24}\frac{\sin\mu}{\mu}\right)c_2 = 0. \end{cases} \tag{2.5}$$

Hence, the boundary value problem (2.1)-(2.2) has a nonzero solution if and only if the system (2.5) has a nonzero solution. The eigenvalues of the boundary value problem (2.1)-(2.2) are the roots of the characteristic determinant

$$\Delta\,(\mu) = \begin{vmatrix} -a_{12}\sin\mu + \frac{a_{13}}{\mu} + a_{14}\frac{\cos\mu}{\mu} & a_{11} + a_{12}\cos\mu + a_{14}\frac{\sin\mu}{\mu} \\ -a_{22}\sin\mu + \frac{a_{23}}{\mu} + a_{24}\frac{\cos\mu}{\mu} & a_{21} + a_{22}\cos\mu + a_{24}\frac{\sin\mu}{\mu} \end{vmatrix}$$

Simple calculations show that

$$\Delta\,(\mu) = -A_{13} - A_{24} + A_{34}\frac{\sin\mu}{\mu} - (A_{23} + A_{14})\cos\mu + A_{12}\mu\sin\mu. \tag{2.6}$$

Definition 2.3. Two-point boundary conditions (2.2) under one of three conditions

$$\begin{aligned}&(1)\ A_{12} \neq 0,\\&(2)\ A_{12} = 0, A_{14} + A_{23} \neq 0,\\&(3)\ A_{12} = 0, A_{14} + A_{23} = 0, A_{34} \neq 0.\end{aligned} \tag{2.7}$$

are called the non-degenerate boundary conditions. Accordingly, if

$$A_{12} = A_{14} + A_{23} = A_{34} = 0, \tag{2.8}$$

then the two-point boundary conditions (2.2) are called the degenerate boundary conditions.

[[21]] Let the two-point boundary conditions (2.2) be non-degenerate, that is, one of the three conditions (2.7) holds. Then, the problem (2.1)-(2.2) has an infinite countable number of eigenvalues. [[21, page 196]] Consider the Sturm-Liouville problem

$$\begin{cases}-u''(x) = \lambda u(x),\ 0 < x < 1,\\ u'(1) - \alpha u(1) = 0,\ u(0) = 0,\end{cases} \tag{2.9}$$

where $\alpha \in$ is a fixed number. From the boundary conditions of the Sturm-Liouville problem (2.9), we have the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

It is easy to see that for all α the case (2) from (2.7) holds: $A_{12} = 0$, $A_{14} + A_{23} = 1$. Therefore, the boundary conditions of the Sturm-Liouville problem (2.9) are non-degenerate. [[21, page 195]] We consider the Sturm-Liouville problem

$$\begin{cases}-u''(x) = \lambda u(x),\ 0 < x < 1,\\ u'(0) + \alpha u'(1) = 0,\ u(0) - \alpha u(1) = 0,\end{cases} \tag{2.10}$$

where $\alpha \in C$ is a fixed number. From the boundary conditions of the Sturm-Liouville problem (2.10), we have the matrix

$$A = \begin{bmatrix} 1 & \alpha & 0 & 0 \\ 0 & 0 & 1 & -\alpha \end{bmatrix}$$

It is easy to see that for all α, (2.8) holds: $A_{12} = A_{14} + A_{23} = A_{34} = 0$. Therefore, the boundary conditions of the Sturm-Liouville problem (2.10) are degenerate. From (2.10), we have

$$\Delta(\mu) = -A_{13} - A_{24} = -1 + \alpha^2.$$

Then we obtain that for $\alpha^2 \neq 1$ the Sturm-Liouville problem (2.10) does not have eigenvalues, and for $\alpha^2 = 1$ each number $\lambda \in$ is an eigenvalue of this problem.

2.2 Regular boundary conditions

In this section, we present the concept of regular boundary conditions. This concept was first introduced by G. D. Birkhoff in his works in 1908 in [27], for n-th order general ordinary differential operators

$$u^{(n)}(x) + q_2(x)\,u^{(n-2)}(x) + \ldots + q_{n-1}(x)\,u'(x) + q_n(x)\,u(x) = \lambda u(x),\ 0 < x < 1, \tag{2.11}$$

with n linearly independent boundary conditions of the general form

$$B_j(u) = \sum_{s=0}^{n-1} \left(a_{js}u^{(s)}(0) + b_{js}u^{(s)}(1)\right) = 0,\ j = 1, \ldots, n,$$

Rewrite the limit forms $B_j(u)$ by the form

$$a_j u^{(k_j)}(0) + b_j u^{(k_j)}(1) + \sum_{s=0}^{k_j-1} \left(a_{js}u^{(s)}(0) + b_{js}u^{(s)}(1)\right) = 0,\ j = 1, \ldots, n, \tag{2.12}$$

where $|a_j| + |b_j| > 0$, $n - 1 \geq k_1 \geq k_2 \geq \ldots \geq k_n$, $k_j > k_{j+2}$.

Definition 2.4 **([21, page 197]).** We denote by $\varepsilon_j = exp\left(i\frac{2\pi j}{n}\right)$, $j = 1, \ldots, n$, the roots of order n from 1.

☞ In the odd case $n = 2m - 1$, the "normed" boundary conditions (2.12) are called the *regular boundary conditions* if the numbers θ_0 and θ_1 defined by the equality

$$\theta_0 + \theta_1 s = \begin{vmatrix} a_1\varepsilon_1^{k_1} & \ldots & a_1\varepsilon_{m-1}^{k_1} & (a_1 + sb_1)\,\varepsilon_m^{k_1} & b_1\varepsilon_{m+1}^{k_1} & \ldots & b_1\varepsilon_n^{k_1} \\ a_2\varepsilon_1^{k_2} & \ldots & a_2\varepsilon_{m-1}^{k_2} & (a_2 + sb_2)\,\varepsilon_m^{k_2} & b_2\varepsilon_{m+1}^{k_2} & \ldots & b_2\varepsilon_n^{k_2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n\varepsilon_1^{k_n} & \ldots & a_n\varepsilon_{m-1}^{k_n} & (a_n + sb_n)\,\varepsilon_m^{k_n} & b_n\varepsilon_{m+1}^{k_n} & \ldots & b_n\varepsilon_n^{k_n} \end{vmatrix}$$

are different from zero.

☞ In the even case $n = 2m$, the "normed" boundary conditions (2.12) are called the *regular boundary conditions* if the numbers θ_1 and θ_2 defined by the equality

$$\theta_0 + \theta_1 s + \frac{\theta_2}{s} + = \begin{vmatrix} a_1\varepsilon_1^{k_1} & \ldots & a_1\varepsilon_{m-1}^{k_1} & (a_1 + sb_1)\,\varepsilon_m^{k_1} & \left(a_1 + \frac{b_1}{s}\right)\varepsilon_{m+1}^{k_1} & b_1\varepsilon_{m+2}^{k_1} & \ldots & b_1\varepsilon_n^{k_1} \\ a_2\varepsilon_1^{k_2} & \ldots & a_2\varepsilon_{m-1}^{k_2} & (a_2 + sb_2)\,\varepsilon_m^{k_2} & \left(a_2 + \frac{b_2}{s}\right)\varepsilon_{m+1}^{k_2} & b_2\varepsilon_{m+2}^{k_2} & \ldots & b_2\varepsilon_n^{k_2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n\varepsilon_1^{k_n} & \ldots & a_n\varepsilon_{m-1}^{k_n} & (a_n + sb_n)\,\varepsilon_m^{k_n} & \left(a_n + \frac{b_n}{s}\right)\varepsilon_{m+1}^{k_n} & b_n\varepsilon_{m+2}^{k_n} & \ldots & b_n\varepsilon_n^{k_n} \end{vmatrix}$$

are different from zero.

An important subclass of the regular boundary conditions, the so-called *strengthened regular boundary conditions* was defined.

Definition 2.5. ☞ In the odd case $n = 2m - 1$ of equation (2.11) all the regular boundary conditions are strengthened regular.

☞ In the even case $n = 2m$ of equation (2.11), we have

1. If $\theta_0^2 - 4\theta_1\theta_2 \neq 0$, the regular boundary conditions (2.12) are called strengthened regular.
2. If $\theta_0^2 - 4\theta_1\theta_2 = 0$, the regular boundary conditions (2.12) are not strengthened regular.

For the case of the Sturm-Liouville problem (2.1)-(2.2), we have $n = 2$, $m = 1$, $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$.

Let first $A_{12} \neq 0$. In this case the boundary conditions (2.2) have the normed form. we have $a_1 = a_{11}$, $b_1 = a_{12}$, $a_2 = a_{21}$, $b_2 = a_{22}$, $k_1 = k_2 = 1$. We calculate the determinant

$$\theta_0 + \theta_1 s + \frac{\theta_2}{s} = \begin{vmatrix} (a_1 + sb_1)\,\varepsilon_1^{k_1} & \left(a_1 + \frac{b_1}{s}\right)\varepsilon_2^{k_1} \\ (a_2 + sb_2)\,\varepsilon_1^{k_2} & \left(a_2 + \frac{b_2}{s}\right)\varepsilon_2^{k_2} \end{vmatrix} = \begin{vmatrix} -(a_{11} + sa_{12}) & \left(a_{11} + \frac{a_{12}}{s}\right) \\ -(a_{21} + sa_{22}) & \left(a_{21} + \frac{a_{22}}{s}\right) \end{vmatrix} = A_{12}\left(s - \frac{1}{s}\right).$$

Then $\theta_0 = 0$, $\theta_1 = A_{12}$, $\theta_2 = -A_{12}$. In this case the boundary conditions (2.2) are regular. Since $\theta_0^2 - 4\theta_1\theta_2 = 4A_{12}^2 \neq 0$, in this case the conditions are strengthened regular.

Let now $A_{12} = 0$, and $|a_{11}| + |a_{12}| > 0$. Then the boundary conditions (2.2) can be reduced to the normed form

$$\begin{cases} a_{11}u'(0) + a_{12}u'(1) + & a_{13}u(0) + a_{14}u(1) = 0, \\ & a_{23}u(0) + a_{24}u(1) = 0. \end{cases} \tag{2.13}$$

we have $a_1 = a_{11}$, $b_1 = a_{12}$, $a_2 = a_{23}$, $b_2 = a_{24}$, $k_1 = 1$, $k_2 = 0$. We calculate the determinant

$$\theta_0 + \theta_1 s + \frac{\theta_2}{s} = \begin{vmatrix} -(a_{11} + sa_{12}) & a_{11} + \frac{a_{12}}{s} \\ a_{23} + sa_{24} & a_{23} + \frac{a_{24}}{s} \end{vmatrix} = -\left(s + \frac{1}{s}\right)(A_{14} + A_{23}) - 2\,(A_{13} + A_{24}).$$

Then, $\theta_1 = \theta_2 = -(A_{14} + A_{23})$, $\theta_0 = -2\,(A_{13} + A_{24})$. That is, in this case the boundary conditions (2.13) are regular under the additional condition $A_{14} + A_{23} \neq 0$. The condition of the strengthened regularity will be written in the form:

$$\theta_0^2 - 4\theta_1\theta_2 = (A_{13} + A_{24})^2 - (A_{14} + A_{23})^2 \neq 0.$$

Consider the remaining case $A_{12} = 0$, with $a_{11} = a_{12} = 0$. Then the boundary conditions (2.2) can be reduced to the normed form

$$\begin{cases} a_{13}u(0) + a_{14}u(1) = 0, \\ a_{23}u(0) + a_{24}u(1) = 0. \end{cases} \tag{2.14}$$

We have $a_1 = a_{13}$, $b_1 = a_{14}$, $a_2 = a_{23}$, $b_2 = a_{24}$, $k_1 = k_2 = 0$. We calculate the determinant

$$\theta_0 + \theta_1 s + \frac{\theta_2}{s} = \begin{vmatrix} a_{13} + sa_{14} & a_{13} + \frac{a_{14}}{s} \\ a_{23} + sa_{24} & a_{23} + \frac{a_{24}}{s} \end{vmatrix} = A_{34}\left(\frac{1}{s} - s\right).$$

then, $\theta_2 = -\theta_1 = A_{34}$, $\theta_0 = 0$. The inequality $A_{34} \neq 0$ is satisfied in view of the linear independence of the boundary conditions (2.14). Hence, in this case the boundary conditions (2.14) are regular. Since $\theta_0^2 - 4\theta_1\theta_2 = 4A_{34}^2$, these boundary conditions are strengthened regular.

Theorem 2.1 ([21, Theorem 3.105]). *The boundary conditions* (2.2) *are regular in the following three cases:*

$$\begin{aligned} &(1)\ A_{12} \neq 0, \\ &(2)\ A_{12} = 0,\ A_{14} + A_{23} \neq 0, \\ &(3)\ A_{12} = A_{13} = A_{14} = A_{23} = A_{24} = 0,\ A_{34} \neq 0. \end{aligned} \tag{2.15}$$

Here, the boundary conditions will be strengthened regular in the cases (1) *and* (3), *and in the case* (2) *under the additional condition*

$$A_{13} + A_{24} \neq \pm (A_{14} + A_{23}). \tag{2.16}$$

Corollaire 2.1 ([21, Corollary 3.106]). *1. For the case of the Sturm-Liouville equation* (2.1), *all the regular boundary conditions* (2.2) *are non-degenerate.*

2. Here, the boundary conditions can be non-degenerate and simultaneously irregular in the case when

$$A_{12} = A_{14} + A_{23} = 0,\ A_{34} \neq 0,\ \textit{and}\ |A_{13}| + |A_{14}| + |A_{23}| + |A_{24}| > 0$$

[[21, Example 3.107]] Consider the spectral problem

$$\begin{cases} -u''(x) = \lambda u(x),\ 0 < x < 1, \\ u'(0) - \alpha u(1) = 0,\ u(0) = 0, \end{cases} \tag{2.17}$$

where $\alpha \in C$ is a fixed number. It is easy to see that for all $\alpha \neq 0$ in case (3) from (2.7) holds: $A_{12} = 0$, $A_{14} + A_{23} = 0$, $A_{34} = \alpha \neq 0$. Therefore, the boundary conditions of the problem (2.17) are non-degenerate.

Here, the determinant $A_{13} = 1$ is not equal to zero since condition (3) from (2.15) does not hold. Hence, the boundary conditions of the problem (2.17) are not regular. For convenience of use we reformulate Theorem 2.1 in terms of coefficients of the boundary conditions (2.2).

Theorem 2.2 ([21, Theorem 3.108]). *The boundary conditions* (2.2) *are regular, if one of the following three conditions holds:*

$$\begin{aligned} &(1)\ a_{11}a_{22} - a_{12}a_{21} \neq 0, \\ &(2)\ a_{11}a_{22} - a_{12}a_{21} = 0,\ |a_{11}| + |a_{12}| > 0,\ a_{11}a_{24} + a_{12}a_{23} \neq 0, \\ &(3)\ a_{11} = a_{12} = a_{21} = a_{22} = 0,\ a_{13}a_{24} - a_{14}a_{23} \neq 0. \end{aligned} \tag{2.18}$$

The regular boundary conditions are strengthened regular in the first and third cases, and in the second case under the additional condition

$$a_{11}a_{23} + a_{12}a_{24} \neq a_{11}a_{24} + a_{12}a_{23} \tag{2.19}$$

2.3 Regular but not strengthened regular boundary conditions

From Theorem 2.1 and (2.19), the boundary conditions (2.2) are called regular but not strengthened regular boundary conditions, if the following conditions are hold:

$$A_{12} = 0,\ A_{14} + A_{23} \neq 0,\ A_{13} + A_{24} = \pm \left(A_{14} + A_{23}\right).$$

From Theorem 2.2, the regular but not strengthened regular boundary conditions (2.2) can be written in the form

$$\begin{cases} a_{11}u'(0) + a_{12}u'(1) + \quad a_{13}u(0) + a_{14}u(1) = 0, \\ \qquad\qquad\qquad\qquad\quad a_{23}u(0) + a_{24}u(1) = 0, \end{cases} \tag{2.20}$$

when $|a_{11}| + |a_{22}| > 0$ and two conditions

$$a_{11}a_{24} + a_{12}a_{23} \neq 0, \tag{2.21}$$

$$a_{11}a_{23} + a_{12}a_{24} = \pm \left(a_{11}a_{24} + a_{12}a_{23}\right), \tag{2.22}$$

simultaneously hold. Indeed, condition (2.22) can be written in the form:

$$\left(a_{11} \pm a_{12}\right)\left(a_{23} \pm a_{24}\right) = 0.$$

Theorem 2.3 ([?]). *If the boundary conditions* (2.2) *are regular but not strengthened regular, they can be always reduced to the form* (2.20) *(with* $|a_{11}| + |a_{22}| > 0$*) of one of the following four types:*

$$\begin{aligned} &(1)\ a_{11} = a_{12}, \quad a_{23} \neq -a_{24};\\ &(2)\ a_{11} = -a_{12}, \quad a_{23} \neq a_{24};\\ &(3)\ a_{23} = a_{24}, \quad a_{11} \neq -a_{12};\\ &(4)\ a_{23} = -a_{24}, \quad a_{11} \neq a_{12} \end{aligned} \tag{2.23}$$

Corollaire 2.2 ([21, Corollary 3.110]). *All regular, but not strengthened regular boundary conditions can be reduced to one of the four forms:*

$$\begin{cases} u'(0) - u'(1) + & au(0) + bu(1) = 0, \\ & u(0) + \alpha u(1) = 0, \end{cases} \quad \begin{cases} u'(0) + u'(1) + & au(0) + bu(1) = 0, \\ & u(0) - \alpha u(1) = 0, \end{cases}$$

$$\begin{cases} u'(0) + \alpha u'(1) + & au(0) + bu(1) = 0, \\ & u(0) - u(1) = 0, \end{cases} \quad \begin{cases} u'(0) - \alpha u'(1) + & au(0) + bu(1) = 0, \\ & u(0) + u(1) = 0, \end{cases}$$

where $\alpha \neq 1$*, and the coefficients* a *and* b *can be arbitrary. For* $\alpha = 1$*, these boundary conditions are degenerate and consequently, are not regular.*

2.4 Biorthogonal systems in Hilbert spaces

Definition 2.6. Let H be a Hilbert space. Two systems of elements (x_k) and (y_k) are said to be biorthogonal systems in H if the relation

$$\langle x_i, y_j \rangle = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \tag{2.24}$$

holds for all values of the indices i and j. Here δ_{ij} is the Kronecker delta.

Definition 2.7. A system of elements of a Hilbert space H is said to be a *complete system* if any vector orthogonal to all vectors of this system is equal to zero.

Definition 2.8. A system of elements of a Hilbert space H is said to be a Riesz basis in H if there exist two constants $m, M > 0$ such that for any $f \in H$, the following inequality holds:

$$m \|f\|_H^2 \leq \sum_{i=0}^{+\infty} f_i^2 \leq M \|f\|_H^2 \tag{2.25}$$

2.5 A non-self-adjoint boundary value problem

In this section we consider main properties of eigen- and associated functions of a non-self-adjoint boundary value problem for a second-order ordinary differential operator.

In $L^2(0,1)$, we consider the operator $\mathcal{L}$ given by:

$$\mathcal{L}X = -X''(x) = \lambda X(x),\ 0 < x < 1, \tag{2.26}$$

and nonlocal boundary conditions:

$$\begin{cases} B_1(X) = X(0) - X(1) = 0, \\ B_2(X) = \beta X'(0) - X'(1) = 0,\ |\beta| < 1. \end{cases} \tag{2.27}$$

where $B_1(X)$ and $B_2(X)$ are linearly independent forms. It is easy to justify that the operator $\mathcal{L}$ is a linear operator on $L^2(0,1)$ defined by (2.26) with the domain

$$D(\mathcal{L}) = \left\{X \in L^2(0,1) : B_1(X) = B_2(X) = 0\right\}.$$

Remark 2.1. From case (3) in Theorem 2.3, the boundary conditions in (2.27) are regular but not strengthened regular.

The adjoint problem of the boundary value problem (2.26)-(2.27) is given by:

$$\mathcal{L}^*Y = -Y''(x) = \lambda Y(x),\ 0 < x < 1, \tag{2.28}$$

with nonlocal boundary conditions

$$\begin{cases} B_1^*(Y) = Y(0) - \beta Y(1) = 0,\ |\beta| < 1, \\ B_2^*(Y) = Y'(x) - Y'(1) = 0. \end{cases} \tag{2.29}$$

Proof. The operator $\mathcal{L}^*$ defined by (2.28) is a linear operator on $L^2(0,1)$ with the domain

$$D(\mathcal{L}^*) = \left\{X \in L^2(0,1) : B_1^*(X) = B_2^*(X) = 0\right\}.$$

Let $X \in D(\mathcal{L})$ and $Y \in D(\mathcal{L}^*)$, by integration by parts twice, we obtain

$$\begin{aligned} \langle \mathcal{L}X, Y\rangle &= -\int_0^1 X''(x)\,Y(x)\,dx \\ &= X'(0)\left[Y(0) - \beta Y(1)\right] + X(0)\left[Y'(1) - Y'(0)\right] - \int_0^1 X(x)\,Y''(x)\,dx \\ &= \langle X, \mathcal{L}^*Y\rangle, \end{aligned}$$

then, $\mathcal{L}^*$ is the adjoint operator of the operator $\mathcal{L}$. Since, the problem (2.28)-(2.29) is the adjoint problem of the problem (2.26)-(2.27). $\square$

Remark 2.2. Obviously $D(\mathcal{L}) \neq D(\mathcal{L}^*)$ then, the spectral problems (2.26)-(2.27) and (2.28)-(2.29) are not self-adjoint.

We have

1. The two spectral problem (2.26)-(2.27) and (2.28)-(2.29) have the same double eigenvalues $\lambda_k = (2\pi k)^2$ (except for the first $\lambda_0 = 0$). The set of eigenfunctions of the problems (2.26)-(2.27) and (2.28)-(2.29) are the following:

$$X_0(x) = b_0;\ X_{2k-1}(x) = A_k \cos(2\pi kx),\ k \in^*,\ b_0,\ A_k \in, \tag{2.30}$$

$$Y_0(x) = a_0'\left(x + \frac{\beta}{1-\beta}\right);\ Y_{2k}(x) = B_k' \sin(2\pi kx),\ k \in^*,\ a_0',\ B_k' \in . \tag{2.31}$$

2. The sets of eigenfunctions $\{X_0(x), X_{2k-1}(x)\}$ and $\{Y_0(x), Y_{2k}(x)\}$, $k \in^*$, are not complete in the space $L^2(0,1)$.

Proof. 1. We have:

(a) If $\lambda = 0$ in (2.26) and (2.28), then $X_0(x) = a_0 x + b_0$ and $Y_0(x) = a_0' x + b_0'$. From (2.27), we obtain:

$$\begin{cases} X(0) = X(1), \\ \beta X'(0) = X'(1) \end{cases} \Leftrightarrow \begin{cases} b_0 = a_0 + b_0 \\ \beta a_0 = \beta a_0 \end{cases} \Leftrightarrow a_0 = 0,$$

then $X_0(0) = b_0$. On other hand,

$$\begin{cases} Y(0) = \beta Y(1), \\ Y'(0) = Y'(1) \end{cases} \Leftrightarrow \begin{cases} b_0' = \beta(a_0' + b_0') \\ a_0' = a_0' \end{cases} \Leftrightarrow b_0' = \frac{\beta a_0'}{1-\beta},$$

then, $Y_0(x) = a_0'\left(x + \frac{\beta}{1-\beta}\right)$.

(b) If $\lambda = \mu^2$, the general solutions of equations (2.26) and (2.28) are given by :

$$X_{2k-1}(x) = A_k \cos(\mu x) + B_k \sin(\mu x),\ Y_{2k}(x) = A_k' \cos(\mu x) + B_k' \sin(\mu x),\quad A_k, B_k, A_k', B_k' \in \mathbb{R}.$$

From boundary conditions (2.27), we have:

$$\begin{cases} X_{2k-1}(0) = X_{2k-1}(1), \\ \beta X_{2k-1}'(0) = X_{2k-1}'(1). \end{cases} \Leftrightarrow \begin{cases} A_k(\cos\mu - 1) + B_k \sin\mu = 0, \\ -A_k \sin\mu + B_k(\cos\mu - \beta) = 0. \end{cases} \tag{2.32}$$

The system (2.32) admits a non-trivial solution, then the determinant of this system is zero. Therefore, we have:

$$\Delta(\mu) = \begin{vmatrix} \cos\mu - 1 & \sin\mu \\ -\sin\mu & \cos\mu - \beta \end{vmatrix} = 2(\beta+1)\sin^2(\mu/2) = 0 \Leftrightarrow \mu_k = 2\pi k,\ k \in^*,$$

then $\lambda_k = (2\pi k)^2$ are multiple eigenvalues and From (2.32), we obtain $B_k = 0$. Then, $X_{2k-1} = A_k \cos(2\pi k x)$ are eigenfunctions.

From boundary conditions (2.29), we have:

$$\begin{cases} Y_{2k}(0) = \beta Y_{2k}(1), \\ Y'_{2k}(0) = Y'_{2k}(1). \end{cases} \Leftrightarrow \begin{cases} A'_k(\beta\cos\mu - 1) + B'_k \sin\mu = 0, \\ -A'_k \sin\mu + B'_k(\cos\mu - 1) = 0. \end{cases} \tag{2.33}$$

The system (2.33) admits a non-trivial solution, then the determinant of this system is zero. Therefore, we have:

$$\Delta(\mu) = \begin{vmatrix} \beta\cos\mu - 1 & \sin\mu \\ -\sin\mu & \cos\mu - 1 \end{vmatrix} = 2(2-(\beta-1)\cos\mu)\sin^2(\mu/2) = 0 \Leftrightarrow \mu_k = 2\pi k,\ k \in,$$

then $\lambda_k = (2\pi k)^2$ are multiple eigenvalues and from (2.33), we obtain $A'_k = 0$. Then, $Y_{2k} = B'_k \sin(2\pi k x)$ are eigenfunctions.

(c) If $\lambda = -\mu^2$, with $\mu \neq 0$, the general solutions of equations (2.26) and (2.28) are given by :

$$X_{2k}(x) = C_k e^{\mu x} + D_k e^{-\mu x},\ Y_{2k-1} = C'_k e^{\mu x} + D'_k e^{-\mu x},\ C_k, D_k, C'_k, D'_k \in .$$

From boundary conditions (2.27), we have:

$$\begin{cases} X_{2k}(0) = X_{2k}(1), \\ \beta X'_{2k}(0) = X'_{2k}(1). \end{cases} \Leftrightarrow \begin{cases} (e^{\mu} - 1) C_k + (e^{-\mu} - 1) D_k = 0, \\ (e^{\mu} - \beta) C_k + (\beta - e^{-\mu}) D_k = 0. \end{cases} \tag{2.34}$$

The system (2.34) admits a non-trivial solution, then the determinant of this system is zero. Therefore, we have:

$$\Delta(\mu) = \begin{vmatrix} e^{\mu} - 1 & e^{-\mu} - 1 \\ e^{\mu} - \beta & \beta - e^{-\mu} \end{vmatrix} = 2(\beta+1)(\cosh\mu - 1) = 0,$$

then, $\mu = 0$ (impossible).

From boundary conditions (2.29), we have:

$$\begin{cases} Y_{2k-1}(0) = \beta Y_{2k-1}(1), \\ Y'_{2k-1}(0) = Y'_{2k-1}(1). \end{cases} \Leftrightarrow \begin{cases} (\beta e^{\mu} - 1) C'_k + (\beta e^{-\mu} - 1) D'_k = 0, \\ (e^{\mu} - 1) C'_k + (1 - e^{-\mu}) D'_k = 0. \end{cases} \tag{2.35}$$

The system (2.35) admits a non-trivial solution, then the determinant of this system is zero. Therefore, we have:

$$\Delta(\mu) = \begin{vmatrix} \beta e^{\mu} - 1 & \beta e^{-\mu} - 1 \\ e^{\mu} - 1 & 1 - e^{-\mu} \end{vmatrix} = 2(\beta + 1)(\cosh \mu - 1) = 0,$$

then, $\mu = 0$ (impossible).

2. Let $f(x) = 2\sin(2\pi kx)$ and $g(x) = 2\cos(2\pi kx)$. We have:

$$\langle X_{2k-1}(x), f(x)\rangle = \int_0^1 2a_k \cos(2\pi kx)\sin(2\pi kx)\,dx = \int_0^1 a_k \sin(4\pi kx)\,dx = \frac{a_k}{4\pi k}\left[\cos(4\pi kx)\right]_0^1 = 0,$$

and

$$\langle Y_{2k}(x), g(x)\rangle = \int_0^1 2a'_k \sin(2\pi kx)\cos(2\pi kx)\,dx = \int_0^1 a'_k \sin(4\pi kx)\,dx = \frac{a'_k}{4\pi k}\left[\cos(4\pi k)\right]_0^1 = 0,$$

then, sets of eigenfunctions $\{X_0, X_{2k-1}(x)\}$ and $\{Y_0(x), Y_{2k}(x)\}$, $k \in^{*}$, are not complete in $L^2(0,1)$.

$\square$

To make the set $\{X_0(x), X_{2k-1}(x)\}$, $k \in^{*}$, a complete set on $L^2(0,1)$, we have to look for the associated eigenfunctions $X_{2k}(x)$. According to Definition 2.2, we have to solve the following spectral problem:

$$\begin{cases} -X''_{2k}(x) = \lambda_k X_{2k}(x) + X_{2k-1}(x),\ 0 < x < 1, \\ X_{2k}(0) = X_{2k}(1),\ \beta X'_{2k}(0) = X'_{2k}(1). \end{cases} \tag{2.36}$$

The solution of spectral problem (2.36) is given by:

$$X_{2k}(x) = \left(C_k + \frac{A_k}{16\pi^2 k^2}\right)\cos(2\pi kx) + \frac{A_k}{4\pi k}\left(\frac{1}{\beta - 1} + x\right)\sin(2\pi kx),$$

where $A_k, C_k \in$ and $k \in^{*}$. Similarly for the set $\{Y_0(x), Y_{2k}(x)\}$, $k \in^{*}$, we have to solve the following spectral problem:

$$\begin{cases} -Y''_{2k-1}(x) = \lambda_k Y_{2k-1}(x) + Y_{2k}(x),\ 0 < x < 1, \\ Y_{2k-1}(0) = \beta Y_{2k-1}(1),\ Y'_{2k-1}(0) = Y'_{2k-1}(1). \end{cases} \tag{2.37}$$

The solution of spectral problem (2.37) is given by:

$$Y_{2k-1}(x) = \frac{B_k}{4\pi k}\left(\frac{\beta}{\beta - 1} - x\right)\cos(2\pi kx) + \left(D_k + \frac{B_k}{16\pi^2 k^2}\right)\sin(2\pi kx),$$

where $B_k, D_k \in$ and $k \in^{*}$.

To show that the two systems of functions $\{X_0(x), X_{2k-1}(x), X_{2k}(x)\}$ and $\{Y_0(x), Y_{2k-1}(x), Y_{2k}(x)\}$ are biorthogonal, we must explicitly determine the coefficients b_0, a_0, A_k, B_k, C_k and D_k. According the biorthogonality condition:

$$\langle X_0, Y_0\rangle = 1,\ \langle X_{2k-1}, Y_{2k-1}\rangle = 1,\ \langle X_{2k}, Y_{2k}\rangle = 1,\ \langle X_{2k}, Y_{2k-1}\rangle = 0. \tag{2.38}$$

From the first condition we get

$$1 = \langle X_0, Y_0\rangle = b_0 a_0' \int_0^1 \left(x + \frac{\beta}{1-\beta}\right) dx = b_0 a_0' \left[\frac{x^2}{2} + \frac{\beta x}{1-\beta}\right]_0^1 = b_0 a_0' \frac{(1+\beta)}{2(1-\beta)}.$$

Then, $b_0 = 2$ and $a_0' = \frac{1-\beta}{1+\beta}$. In this case, we have:

$$X_0(x) = 2,\ Y_0(x) = ax + b, \tag{2.39}$$

where $a = \frac{1-\beta}{1+\beta}$ and $b = \frac{\beta}{1+\beta}$.

We use the second condition from (2.38):

$$\begin{aligned}
1 = \langle X_{2k-1}, Y_{2k-1}\rangle &= \frac{A_k B_k}{4\pi k} \int_0^1 \left(\frac{\beta}{\beta-1} + x\right) \cos^2(2\pi k x)\, dx + \frac{A_k}{2}\left(D_k + \frac{B_k}{16\pi^2 k^2}\right) \int_0^1 \sin(4\pi k x)\, dx \\
&= \frac{A_k B_k}{8\pi k} \int_0^1 \left(\frac{\beta}{\beta-1} - x\right) dx + \frac{A_k B_k}{8\pi k} \int_0^1 \left(\frac{\beta}{\beta-1} - x\right) \cos(4\pi k x)\, dx \\
&= \frac{A_k B_k}{8\pi k} \left[\frac{\beta x}{\beta-1} - \frac{x^2}{2}\right]_0^1 + \frac{A_k B_k}{8\pi k} \left[\left(\frac{\beta}{\beta-1} - x\right) \frac{\sin(4\pi k x)}{4\pi k}\Big|_0^1 + \int_0^1 \frac{\sin(4\pi k x)}{4\pi k} dx\right] \\
&= \frac{A_k B_k (\beta+1)}{16\pi k(\beta-1)}.
\end{aligned}$$

Then, $A_k = 4$ and $B_k = \frac{4\pi k(\beta-1)}{1+\beta}$. In this case, we have:

$$X_{2k-1}(x) = 4\cos(2\pi k x),\ Y_{2k-1}(x) = (ax+b)\cos(2\pi k x) + \left(D_k - \frac{a}{4\pi k}\right) \sin(2\pi k x), \tag{2.40}$$

where $D_k \in$. We use the third condition from (2.38):

$$\begin{aligned}
1 = \langle X_{2k}, Y_{2k}\rangle &= \frac{B_k'}{2}\left(C_k + \frac{A_k}{16\pi^2 k^2}\right) \int_0^1 \sin(4\pi k x)\, dx + \frac{A_k B_k'}{4\pi k} \int_0^1 \left(\frac{1}{\beta-1} + x\right) \sin^2(2\pi k x)\, dx \\
&= \frac{A_k B_k'}{8\pi k} \int_0^1 \left(\frac{1}{\beta-1} + x\right) dx - \frac{A_k B_k'}{8\pi k} \int_0^1 \left(\frac{1}{\beta-1} + x\right) \cos(4\pi k x)\, dx \\
&= \frac{A_k B_k'}{8\pi k} \left[\frac{x}{\beta-1} + \frac{x^2}{2}\right]_0^1 \\
&= \frac{A_k B_k'}{16\pi k} \frac{\beta+1}{\beta-1}.
\end{aligned}$$

Then, $B'_k = 1$ and $A_k = \frac{16\pi k(\beta-1)}{\beta+1}$. In this case, we have:

$$X_{2k}(x) = \left(C_k - \frac{a}{\pi k}\right)\cos(2\pi kx) + 4(1-b-ax)\sin(2\pi kx),\ Y_{2k}(x) = \sin(2\pi kx), \tag{2.41}$$

where $C_k \in$. We use the fourth condition from (2.38), and from (2.40), (2.41) we have:

$$\begin{aligned}
0 = \langle X_{2k}, Y_{2k-1}\rangle &= \left(C_k - \frac{a}{\pi k}\right)\int_0^1 (ax+b)\cos^2(2\pi kx)\,dx + 4\left(D_k - \frac{a}{4\pi k}\right)\int_0^1 (1-b-ax)\sin^2(2\pi kx)\,dx \\
&= \left(\frac{C_k}{2} - \frac{a}{2\pi k}\right)\left[\int_0^1 (ax+b)\,dx + \int_0^1 (ax+b)\cos(4\pi kx)\,dx\right] \\
&+ 2\left(D_k - \frac{a}{4\pi k}\right)\left[\int_0^1 (1-b-ax)\,dx - \int_0^1 (1-b-ax)\cos(4\pi kx)\,dx\right] \\
&= a\left(\frac{C_k}{2} - \frac{a}{2\pi k}\right) - 2a\left(D_k - \frac{a}{4\pi k}\right) \\
&= a\left(\frac{C_k}{2} - 2D_k\right).
\end{aligned}$$

Then, we have:

$$C_k = 4D_k. \tag{2.42}$$

Thus, from (2.39)-(2.42), the system given by:

$$X_0(x) = 2,\ X_{2k-1}(x) = 4\cos(2\pi kx),\ X_{2k}(x) = 4\left(D_k - \frac{a}{4\pi k}\right)\cos(2\pi kx) + 4(1-b-ax)\sin(2\pi kx) \tag{2.43}$$

will be biorthogonal to the system

$$Y_0(x) = ax+b,\ Y_{2k-1}(x) = (ax+b)\cos(2\pi kx) + \left(D_k - \frac{a}{4\pi k}\right)\sin(2\pi kx),\ Y_{2k}(x) = \sin(2\pi kx). \tag{2.44}$$

We calculate the norms of elements of the biorthogonal systems (2.43) and (2.44).

$$\|X_0\| = 2,\ \|X_{2k-1}\| = 2\sqrt{2},\ \|X_{2k}\|^2 = 8\left(D_k - \frac{a}{4\pi k}\right)^2 + \frac{4}{\pi ka}\left(D_k - \frac{a}{4\pi k}\right) - \frac{8}{3a}\left(2b^3 - 3b^2 + 3b - 1\right),$$
$$\|Y_0\| = \sqrt{\frac{a^2+3b^2+3ab}{3}},\ \|Y_{2k-1}\|^2 = \frac{1}{2}\left(D_k - \frac{a}{4\pi k}\right)^2 - \frac{a}{4\pi k}\left(D_k - \frac{a}{4\pi k}\right) + \frac{a^2}{8\pi^2k^2} + \frac{1}{6}\left(a^2+3ab+3b^2\right),$$
$$\|Y_{2k}\| = \frac{\sqrt{2}}{2}.$$

Using [21, Theorem 3.151], we check for which of the constants D_k the necessary and sufficient condition of the unconditional basis holds.

If the sequence (D_k) is unbounded. Then,

$$\lim_{k\to+\infty}\|X_{2k}\|\,\|Y_{2k}\| = +\infty,\ \lim_{k\to+\infty}\|X_{2k-1}\|\,\|Y_{2k-1}\| = +\infty.$$

Hence, the requirement of the criterion in [20, Theorem 3.151] for an unconditional basis does not hold. Therefore, the systems (2.43) and (2.44) does not give an unconditional basis in $L^2(0,1)$.

In the case when the sequence D_k is bounded, we get

$$\lim_{k\to+\infty} \|X_{2k}\| \|Y_{2k}\| < +\infty, \lim_{k\to+\infty} \|X_{2k-1}\| \|Y_{2k-1}\| < +\infty.$$

Then, we obtain the following result for the systems (2.43) and (2.44). If $D_k = \frac{a}{4\pi k}$ with $k \in^*$, then the systems

$$X_0(x) = 2, \ X_{2k-1}(x) = 4\cos(2\pi kx), \ X_{2k} = 4(1-b-ax)\sin(2\pi kx), \tag{2.45}$$

and

$$Y_0 = ax+b, \ Y_{2k-1}(x) = (ax+b)\cos(2\pi kx), \ Y_{2k}(x) = \sin(2\pi kx), \tag{2.46}$$

are bi-orthonormal in $L^2(0,1)$.

Proof. It is easy to show that the systems (2.45) and (2.46) form a bi-orthogonal system on $[0,1]$, i.e.

$$\langle X_i, Y_j \rangle = \int_0^1 X_i(x) Y_j(x)\, dx = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

□

The systems of functions (2.45) and (2.46) are complete in $L^2(0,1)$.

Proof. Let $f \in L^2(0,1)$ be orthogonal with the system of functions (2.45). $f(x)$ can be presented by the series

$$f(x) = \sum_{n=1}^{+\infty} B_n \sin(2\pi nx), \tag{2.47}$$

which converges in $L^2(0,1)$. Since $f(x)$ is orthogonal with (2.45), we have

$$\begin{aligned} 0 &= \int_0^1 4(1-b-ax) f(x)\sin(2\pi kx)\, dx \\ &= \sum_{n=1}^{+\infty} B_n \int_0^1 4(1-b-ax)\sin(2\pi kx)\sin(2\pi nx)\, dx = B_k, \ k \in \mathbb{N}^*. \end{aligned}$$

From (2.47), $f(x) = 0$. Then, (2.45) is complete in $L^2(0,1)$. □

The following theorem is valid:

Theorem 2.4. *The system of functions* (2.45) *forms a Riesz basis in* $L^2(0,1)$.

Proof. From [21, page 211], the system (2.45) is a Riesz basis in $L^2(0,1)$ if there exist two constants $m, M > 0$ such that for any $f \in L^2(0,1)$, the following inequality holds:

$$m\|f\|^2_{L^2(0,1)} \le \sum_{i=0}^{+\infty} f_i^2 \le M\|f\|^2_{L^2(0,1)},$$

where

$$f_i = \langle f, Y_i\rangle = \int_0^1 f(x)\,Y_i(x)\,dx \text{ and } \bar{f}_i = \langle f, X_i\rangle = \int_0^1 f(x)\,X_i(x)\,dx. \tag{2.48}$$

For $i = 0$, and using the Cauchy-Schwarz inequality we have

$$\begin{aligned} f_0^2 = \langle f, Y_0\rangle^2 = \left[\int_0^1 Y_0(x)\,f(x)\,dx\right]^2 &\le \int_0^1 Y_0^2(x)\,dx \int_0^1 f^2(x)\,dx \\ &\le \frac{1+\beta+\beta^2}{3(1+\beta)^2}\|f\|^2_{L^2(0,1)}. \end{aligned} \tag{2.49}$$

For $i = 2k-1$, and using the Bessel inequality we obtain:

$$\begin{aligned} \sum_{k=1}^{+\infty} f_{2k-1}^2 = \sum_{k=1}^{+\infty} \langle f, Y_{2k-1}\rangle^2 &\le \|Y_{2k-1}\|^2_{L^2(0,1)}\|f\|^2_{L^2(0,1)} \\ &\le \frac{7-11\beta+7\beta^2}{6(1+\beta)^2}\|f\|^2_{L^2(0,1)}. \end{aligned} \tag{2.50}$$

For $i = 2k$, and using the Bessel inequality we obtain:

$$\sum_{k=1}^{+\infty} f_{2k}^2 = \sum_{k=1}^{+\infty} \langle f, Y_{2k}\rangle^2 \le \|Y_{2k}\|^2_{L^2(0,1)}\|f\|^2_{L^2(0,1)} \le \frac{1}{2}\|f\|^2_{L^2(0,1)}. \tag{2.51}$$

From (2.49)-(2.51), we have

$$\sum_{i=0}^{+\infty} f_i^2 = f_0^2 + \sum_{k=1}^{+\infty} f_{2k-1}^2 + \sum_{k=1}^{+\infty} f_{2k}^2 \le M\|f\|^2_{L^2(0,1)}, \tag{2.52}$$

where $M = \frac{4-\beta+4\beta^2}{2(1+\beta)^2}$.

On the other hand we have:

$$\bar{f}_0^2 = \langle f, X_0\rangle^2 = \left[\int_0^1 X_0(x)\,f(x)\,dx\right]^2 \le 4\|f\|^2_{L^2(0,1)}. \tag{2.53}$$

Using the Bessel inequality, we obtain:

$$\sum_{k=1}^{+\infty} \bar{f}_{2k-1}^2 = \sum_{k=1}^{+\infty} \langle f, X_{2k-1}\rangle^2 \leq 8 \|f\|_{L^2(0,1)}^2 , \tag{2.54}$$

$$\sum_{k=1}^{+\infty} \bar{f}_{2k}^2 = \sum_{k=1}^{+\infty} \langle f, X_{2k}\rangle^2 \leq \frac{8\left(1+\beta+\beta^2\right)}{3\left(1+\beta\right)^2} \|f\|_{L^2(0,1)}^2 . \tag{2.55}$$

Then, from (2.53)-(2.55) we have:

$$\sum_{i=0}^{+\infty} \bar{f}_i^2 \leq \frac{44+80\beta+44\beta^2}{3\left(1+\beta\right)^2} \|f\|_{L^2(0,1)}^2 . \tag{2.56}$$

Using the Cauchy-Schwarz inequality and (2.56), we get

$$\begin{aligned} \|f\|_{L^2(0,1)}^2 &= \langle f, f\rangle \\ &= \sum_{i=0}^{+\infty} \bar{f}_i f_i \\ &\leq \left[\sum_{i=0}^{+\infty} \bar{f}_i^2\right]^{1/2} \left[\sum_{i=0}^{+\infty} f_i^2\right]^{1/2} \\ &\leq \left[\frac{44+80\beta+44\beta^2}{3\left(1+\beta\right)^2}\right]^{1/2} \|f\|_{L^2(0,1)} \left[\sum_{i=0}^{+\infty} f_i^2\right]^{1/2} . \end{aligned}$$

Consequently, we have:

$$m \|f\|_{L^2(0,1)}^2 \leq \sum_{i=0}^{+\infty} f_i^2, \ m = \frac{3\left(1+\beta\right)^2}{44+80\beta+44\beta^2}. \tag{2.57}$$

From (2.52) and (2.57), the system (2.45) is a Riesz basis in $L^2(0,1)$. □

Corollaire 2.3. *From Lemma 2.5 and Theorem 2.4, the systems* (2.45) *and* (2.46) *are equivalent bases in* $L^2(0,1)$.

CHAPTER 3

AN INVERSE COEFFICIENT PROBLEM FOR A PARABOLIC EQUATION IN THE CASE OF NONLOCAL BOUNDARY AND OVER DETERMINATION CONDITIONS

In this chapter, we study the existence and uniqueness of the solution for the inverse problem associated with a parabolic equation with nonlocal boundary conditions.

We consider the following problem:

3.1 Problem Formulation

$$u_t(x,t) = a(t)u_{xx}(x,t) + f(x,t), \quad (x,t) \in (0,1) \times (0,T] \tag{3.1}$$

with initial and boundary conditions:

$$u(x,0) = \varphi(x), \tag{3.2}$$

$$u_x(1,t) = \beta u_x(0,t), \tag{3.3}$$

$$u(0,t) = u(1,t), \tag{3.4}$$

and the additional condition:

$$\int_0^1 u(x,t)\,dx = E(t) \tag{3.5}$$

The aim is to determine the pair $(u(x,t), a(t))$.

3.1.1 Existence, uniqueness and continuous dependence results

We have the following hypotheses on the problem data 3.1:

(A_1) $E(T) \in C^1[0,T], E'(t) < 0 \quad \forall t \in [0,T]$.

(A_2) $\varphi(x) \in C^4[0,1]$;

(1) $\varphi(0) = \varphi(1); \varphi'(1) = \beta\varphi'(0); \varphi''(0) = \varphi''(1); \int_0^1 \varphi(x)dx = E(0)$;

(2) $\varphi_{2k} \geq 0, k = 1, 2, ..$

(A_3) $F(x,t) \in C(\overline{Q_T}), F(x,t) \in C^4[0,1]$ for fixed arbitrary $\forall t \in [0,T]$;

(1) $F(0,t) = F(1,t); F_x(1,t) = 0; F_{xx}(0,t) = F_{xx}(1,t)$;

(2) $F_0(t) \geq 0; F_{2k}(t) \geq 0; \forall t \in [0,T]$;

(3) $F_{2k}(t) \geq 0, k = 0, 1, 2, ..., \int_0^t E'(t)dt + \sum_{k=1}^{\infty} \frac{2}{\pi k}\varphi_{2k} - 2\int_0^T F_0(t)dt > 0$

Where $\varphi_k = \int_0^1 \varphi(x)Y_k(x)dx, F_k(t) = \int_0^1 F(x,t)Y_k(x)dx, k = 0, 1, 2, ...$ Let hypotheses (A_1) – (A_3) be satisfied. Then the following statements are true:

(1) the inverse problem 3.1has a solution in φ_T

(2) the solution of inverse problem 3.1-**??** is unique in φ_{T_0}; where the number T_0 $(0 < T_0 < T)$ is determined by the problem data.

Proof. Any solution of equation 3.1 can be represented as:

$$U(x,t) = \sum_{k=0}^{\infty} U_k(t)X_k(x) \tag{3.6}$$

Where the functions $U_k(t), k = 0, 1, 2, ...$ satisfy the following system of equations:

$$U(x,t) = U_0(t)X_0(x) + \sum_{k=1}^{\infty}[U_{2k}(t)X_{2k}(x) + U_{2k-1}(t)X_{2k-1}(x)] \tag{3.7}$$

By using the Fourier method, we can easily see that $U_k(t), k = 0, 1, 2, ...$ satisfies the following system of many linear equations:

$$\begin{cases} U_k'(t) + a(t)\lambda_k U_k(t) = F_k(t) \\ U_k(0) = \varphi_k \end{cases} \tag{3.8}$$

Where $\lambda_k, k = 0, 1...$ are the eigenvalues of

Following the same work, we will look for the solution of the equation in the form

$$\begin{cases} U_{2k-1}'(t) + (2\pi k)^2 a(t)U_{2k-1}(t) + 4\pi k U_{2k}(t) = F_{2k-1}(t) \\ U_{2k-1}(0) = \varphi_{2k-1} \end{cases}$$

We look for the solution of the equation under the second term (homogeneous equation)
We have $U_{2k-1}'(t) + (2\pi k)^2 a(t)U_{2k-1}(t) + 4\pi k U_{2k}(t) = 0$, a separable variable equation
We have

$$U_{2k-1}'(t) + (2\pi k)^2 a(t)U_{2k-1}(t) = -4\pi k U_{2k}(t)$$

$$\Rightarrow U_{2k-1}(t) = C_{2k-1}e^{-(2\pi k)^2 \int_0^t a(s)ds}$$

For $t = 0$, then $U_{2k-1}(0) = C_{2k-1} = \varphi_{2k-1}$

$$\Rightarrow \varphi_{2k-1}(t) = \varphi_{2k-1}e^{-(2\pi k)^2 \int_0^t a(s)ds}$$

We will look for the solution of the equation in the form

$$U_{2k-1}(t) = \varphi_{2k-1}(t)e^{-(2\pi k)^2 A(t)}$$

Differentiating the function U_{2k-1} with respect to t and substituting, we find

$$\varphi'_{2k-1}(t)e^{-(2\pi k)^2 A(t)} = -4\pi k\varphi_{2k}e^{-(2\pi k)^2 A(t)}$$

$$\Rightarrow \varphi'_{2k-1}(t) = -4\pi k\varphi_{2k}$$

By integration

$$\varphi_{2k-1}(t) = -4\pi k\varphi_{2k}t + C_{2k-1}, \text{ with } \varphi_{2k-1}(0) = C_{2k-1} = \varphi_{2k-1}$$

$$\Rightarrow \varphi_{2k-1}(t) = \varphi_{2k-1} - 4\pi k\varphi_{2k}t$$

From where

$$\Rightarrow U_{2k-1}(t) = (\varphi_{2k-1} - 4\pi k\varphi_{2k}t)e^{-(2\pi k)^2 \int_0^t a(s)ds}$$

2^{nd} step: using the same method, we find the last part

$$\sum_{k=1}^{\infty}\left[\int_0^t (F_{2k-1}(\tau) - 4\pi kF_{2k}(\tau)(t-\tau))e^{-(2\pi k)^2 \int_\tau^t a(s)ds}d\tau\right]$$

By substituting the solution of the system of equations and initial condition 3.6 into 3.7, we obtain the solution of the problem **??-??** in the form:

$$\begin{aligned} U(x,t) = &\left[\varphi_0 + \int_0^t F_0(\tau)d\tau\right]X_0(x) \\ &+ \sum_{k=1}^{\infty}\left[\varphi_{2k}e^{-(2\pi k)^2 \int_0^t a(s)ds} + \int_0^t F_{2k}(\tau)e^{-(2\pi k)^2 \int_\tau^t a(s)ds}d\tau\right]X_{2k}(x) \\ &+ \sum_{k=1}^{\infty}\left[(\varphi_{2k-1} - 4\pi k\varphi_{2k}t)e^{-(2\pi k)^2 \int_0^t a(s)ds}\right]X_{2k-1}(x) \\ &+ \sum_{k=1}^{\infty}\left[\int_0^t (F_{2k-1}(\tau) - 4\pi kF_{2k}(\tau)(t-\tau))e^{-(2\pi k)^2 \int_\tau^t a(s)ds}d\tau\right]X_{2k-1}(x) \end{aligned} \tag{3.9}$$

Under condition (1) of (A_2) and (A_3) of the series 3.9 and of $\frac{\partial U}{\partial x}$ are uniformly convergent in $\overline{\Omega_T}$ because their majorant sums are absolutely convergent. Consequently, their $U(x,t)$ and $U_x(x,t)$ are continuous in $\overline{\Omega_T}$. Moreover, $\frac{\partial U}{\partial t}$ and the series $\frac{\partial^2 U}{\partial x^2}$ of the second order are uniformly convergent in φ_T. Thus we have $U(x,t) \in C^2(\Omega_T) \cap C^{1,0}(\overline{\Omega_T})$. Furthermore, $U_t(x,t)$ is continuous in $\overline{\Omega_T}$.

By differentiating condition **??**, under hypothesis (A_1), we obtain. We have from **??**

$$\int_0^1 U(x,t)dx = E(t) \Rightarrow \frac{d}{dt}\int_0^1 U(x,t)dx = \frac{d}{dt}E(t)$$

So

$$\int_0^1 U_t(x,t)dx = E'(t), \quad 0 \leq t \leq T \tag{3.10}$$

This gives

$$a(t) = P[a(t)]. \tag{3.11}$$

From equation **??** we have

$$U_t = a(t)U_{xx} + F(x,t)$$

Then by integration over [0, 1]

$$\int_0^1 U_t(x,t)dx = a(t)\int_0^1 U_{xx}dx + \int_0^1 F(x,t)dx$$

From 3.9 we find

$$E'(t) = a(t)[\beta U_x(0,t) - U_x(1,t)] + \int_0^1 F(x,t)dx$$

With the condition of **??**, we have

$$(-\beta+1)a(t)U_x(1,t) = \int_0^1 F(x,t)dx - E'(t) \tag{3.12}$$

We will look for $U_x(0,t)$. From the series 3.7 we differentiate with respect to x We have

$$\frac{d}{dx}U(x,t) = [4(1-x)\sin(2\pi kx) + 8\pi k(1-x)\cos(2\pi kx)]U_{2k}(t) - [8\pi k\sin(2\pi kx)]U_{2k-1}(t)$$

Then

$$U_x(0,t) = \sum_{k=1}^{\infty} 8\pi k(1-b)\left[\varphi_{2k}e^{-(2\pi k)^2\int_0^t a(s)ds} + \int_0^t F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^t a(s)ds}d\tau\right]$$

From where equation 3.12, we have

$$a(t) = \frac{\int_0^1 F(x,t) - E'(t)}{(1-\beta)U_x(0,t)}$$

It is clear that the function $F(x,t)$ is written in the form

$$F(x,t) = F_0(t)X_0(x) + \sum_{k=1}^{\infty} F_{2k-1}(t)X_{2k-1}(x) + \sum_{k=1}^{\infty} F_{2k}(t)X_{2k}(x)$$

By integrating over [0, 1] and integration by parts, we obtain $\boxed{\text{Calculation of } \int_0^1 F(x,t)\,dx}$

$$\begin{aligned}
\int_0^1 F(x,t)\,dx &= 2F_0(t)\int_0^1 dx + \sum_{k=1}^{\infty} 4F_{2k-1}(t)\int_0^1 \cos(2\pi kx)\,dx \\
&\quad + \sum_{k=1}^{\infty} 4F_{2k}(t)\int_0^1 (1-b-ax)\sin(2\pi kx)\,dx \\
&= 2F_0(t) + \sum_{k=1}^{\infty} 4F_{2k}(t)\int_0^1 (1-b-ax)\sin(2\pi kx)\,dx \\
&= 2F_0(t) - a\sum_{k=1}^{\infty} 4F_{2k}(t)\int_0^1 x\sin(2\pi kx)\,dx \\
&= 2F_0(t) + \frac{2a}{\pi}\sum_{k=1}^{\infty} \frac{F_{2k}(t)}{k}.
\end{aligned}$$

$$\boxed{\int_0^1 F(x,t)\,dx = 2F_0(t) + \frac{2a}{\pi}\sum_{k=1}^{\infty} \frac{F_{2k}(t)}{k}}$$

From where we obtain:

$$P[a(t)] = \frac{2F_0(t) + \frac{2a}{\pi}\sum_{k=1}^{\infty}\frac{F_{2k}(t)}{k} - E'(t)}{\sum_{k=1}^{\infty} 8\pi k\left[\varphi_{2k}e^{-(2\pi k)^2\int_0^t a(s)ds} + \int_0^t F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^t a(s)ds}d\tau\right]} \tag{3.13}$$

We note

$$C_0 = 2\min_{t\in[0,T]} F_0(t) + \min_{t\in[0,T]}\left(\frac{2a}{\pi}\sum_{k=1}^{\infty}\frac{F_{2k}(t)}{k}\right) - \max_{t\in[0,T]} E'(t)$$

$$C_1 = 2\max_{t\in[0,T]} F_0(t) + \max_{t\in[0,T]}\left(\frac{2a}{\pi}\sum_{k=1}^{\infty}\frac{F_{2k}(t)}{k}\right) - \min_{t\in[0,T]} E'(t)$$

$$C_2 = \int_0^T E'(t)dt + \sum_{k=1}^{\infty}\frac{2}{\pi k}\varphi_{2k} - 2\int_0^T F_0(t)$$

$$C_3 = \sum_{k=1}^{\infty} 8\pi k\left(\varphi_{2k} + \int_0^T F_{2k}(\tau)d\tau\right)$$

It is easy to verify that $C_k > 0, k = 0,1,2,3$ and $C_2 \leq C_3$. From 3.9 by integration over [0, T]

$$\int_0^T \left(\int_0^1 U_t(x,t)dx\right)dt = \int_0^T E'(t)dt$$

We will look for $\int_0^1 U_t(x,t)dx$. We have

$$U_t(x,t) = U_0'(t)X_0(x) + \sum_{k=1}^{\infty} U_{2k-1}'(t)X_{2k-1}(x) + \sum_{k=1}^{\infty} U_{2k}'(t)X_{2k}(x)$$

$$\begin{aligned}
\int_0^1 U_t(x,t)\,dx &= 2U_0'(t)\int_0^1 dx + \sum_{k=1}^{\infty} U_{2k-1}'(t)\int_0^1 X_{2k-1}(x)\,dx + \sum_{k=1}^{\infty} U_{2k}'(t)\int_0^1 X_{2k}(x)\,dx \\
&= 2U_0'(t) + \sum_{k=1}^{\infty} 4U_{2k-1}'(t)\int_0^1 \cos(2\pi kx)\,dx \\
&\quad + \sum_{k=1}^{\infty} 4U_{2k}'(t)\int_0^1 (1-b-ax)\sin(2\pi kx)\,dx.
\end{aligned}$$

Since

$$\int_0^1 \cos(2\pi kx)\,dx = \left[\frac{\sin(2\pi kx)}{2\pi k}\right]_0^1 = 0,$$

and integrating by parts,

$$\int_0^1 (1-b-ax)\sin(2\pi kx)\,dx = \frac{a}{2\pi k},$$

we obtain

$$\int_0^1 U_t(x,t)\,dx = 2U_0'(t) + \sum_{k=1}^{\infty} \frac{2a}{\pi k}U_{2k}'(t).$$

Using

$$U_0'(t) = F_0(t),$$

it follows that

$$\boxed{\int_0^1 U_t(x,t)\,dx = 2F_0(t) + \sum_{k=1}^{\infty} \frac{2a}{\pi k}U_{2k}'(t)}.$$

Integrating over $[0,T]$ gives

$$\begin{aligned}
\int_0^T \left(\int_0^1 U_t(x,t)\,dx\right) dt &= 2\int_0^T F_0(t)\,dt + \sum_{k=1}^{\infty} \frac{2a}{\pi k}\int_0^T U_{2k}'(t)\,dt \\
&= 2\int_0^T F_0(t)\,dt + \sum_{k=1}^{\infty} \frac{2a}{\pi k}\Big(U_{2k}(T) - U_{2k}(0)\Big).
\end{aligned}$$

Since

$$U_{2k}(t) = \varphi_{2k}e^{-(2\pi k)^2\int_0^t a(s)\,ds} + \int_0^t F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^t a(s)\,ds}\,d\tau,$$

we have

$$\int_0^T U_{2k}'(t)\,dt = \varphi_{2k}e^{-(2\pi k)^2\int_0^T a(s)\,ds} + \int_0^T F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^T a(s)\,ds}\,d\tau - \varphi_{2k}.$$

Therefore

$$\int_0^T E'(t)\,dt = 2\int_0^T F_0(t)\,dt$$
$$+\sum_{k=1}^{\infty}\frac{2a}{\pi k}\left[\varphi_{2k}e^{-(2\pi k)^2\int_0^T a(s)\,ds}+\int_0^T F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^T a(s)\,ds}\,d\tau-\varphi_{2k}\right].$$

Hence

$$\int_0^T E'(t)\,dt+\sum_{k=1}^{\infty}\frac{2a}{\pi k}\varphi_{2k}-2\int_0^T F_0(t)\,dt=\sum_{k=1}^{\infty}\frac{2a}{\pi k}\left[\varphi_{2k}e^{-(2\pi k)^2\int_0^T a(s)\,ds}+\int_0^T F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^T a(s)\,ds}\,d\tau\right].$$

Since

$$0<e^{-(2\pi k)^2\int_0^t a(s)\,ds}\le 1,$$

we obtain

$$\int_0^T E'(t)\,dt+\sum_{k=1}^{\infty}\frac{2a}{\pi k}\varphi_{2k}-2\int_0^T F_0(t)\,dt$$
$$\le\sum_{k=1}^{\infty}\frac{2a}{\pi k}\left[\varphi_{2k}+\int_0^T F_{2k}(\tau)\,d\tau\right].$$

Using the following estimation: **Lemma 2.1.** For $t_1\le t\le t_2$ where $t_1,t_2>0$ we have

$$\left|e^{-(2\pi k)^2\int_0^{t_1}a(s)ds}-e^{-(2\pi k)^2\int_0^{t_2}a(s)ds}\right|\le(2\pi k)^2|t_1-t_2|\max_{[0,T]}a(t)$$

where $0<\tau<t$, and $\|a(t)\|_{C[0,T]}$ denotes the Chebyshev norm defined by

$$\|a\|_{C[0,T]}=\max_{0\le t\le T}|a(t)|$$

Proof. We define $g(t,\tau)=e^{(2\pi k)^2\int_\tau^t a(x)dx}$, $t\in[t_1,t_2]$, $0\le\tau\le T$. Where $g(.,\tau)\in C[t_1,t_2]$ and $g(0,\tau)$ is differentiable on (t_1,t_2), by using the mean value theorem

$$|g(t_1,\tau)-g(t_2,\tau)|=|t_1-t_2|g'(t,\tau)$$

Moreover

$$|g(t_1,\tau)-g(t_2,\tau)|\le(2\pi k)^2|t_1-t_2|\max_{C[0,T]}|a|$$

Therefore

$$\left|e^{-(2\pi k)^2\int_0^{t_1}a(x)dx}-e^{-(2\pi k)^2\int_0^{t_2}a(x)dx}\right|\le(2\pi k)^2|t_1-t_2|\max_{[0,T]}a(t),$$

And on the other hand we have

$$|N(t_1)-N(t_2)|\le\left|\sum_{k=1}^{\infty}8\pi k\varphi_{2k}\left(e^{-(2\pi k)^2\int_0^{t_1}a(x)dx}-e^{-(2\pi k)^2\int_0^{t_2}a(x)dx}\right)\right|$$

$$+\left|\sum_{k=1}^{\infty} 8\pi k\left[\int_0^{t_1} F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^{t_1}a(s)ds}-\int_0^{t_2} F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^{t_2}a(s)ds}\right]\right|$$

With $0<t_1<t_2<T$

$$|N(t_1)-N(t_2)|\le \sum_{k=1}^{\infty} 8\pi k(2\pi k)^2\varphi_{2k}|t_1-t_2|\max_{[0,T]} a(t)$$

$$+\left|\sum_{k=1}^{\infty} 8\pi k\left[\int_0^{t_1} F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^{t_1}a(s)ds}-\int_0^{t_1} F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^{t_2}a(s)ds}-\int_{t_1}^{t_2} F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^{t_2}a(s)ds}\right]\right|$$

$$\le \sum_{k=1}^{\infty}4(2\pi k)^3\varphi_{2k}|t_1-t_2|\max_{[0,T]} a(t)+\sum_{k=1}^{\infty} 8\pi k\int_0^{t_1} F_{2k}(\tau)\left|e^{-(2\pi k)^2\int_\tau^{t_1}a(s)ds}-e^{-(2\pi k)^2\int_\tau^{t_2}a(s)ds}\right|$$

$$+\sum_{k=1}^{\infty} 8\pi k\int_{t_1}^{t_2} F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^{t_2}a(s)ds}$$

Since $e^{-(2\pi k)^2\int_\tau^{t_2}a(s)ds}\le 1$, $k=0,1,2$ we obtain

$$\le \sum_{k=1}^{\infty}4(2\pi k)^3\varphi_{2k}|t_1-t_2|\max_{[0,T]} a(t)+\sum_{k=1}^{\infty}4(2\pi k)^3\int_0^{t_1} F_{2k}(\tau)d\tau\max_{[0,T]} a(t)$$

$$+\sum_{k=1}^{\infty} 8\pi k\int_{t_1}^{t_2} F_{2k}(\tau)d\tau$$

We set

$$C_4=\sum_{k=1}^{\infty}4(2\pi k)^3\varphi_{2k},\quad C_5=\int_0^T\sum_{k=1}^{\infty}4(2\pi k)^3F_{2k}(\tau)d\tau,\quad C_6=\max_{[0,T]}\left(\sum_{k=1}^{\infty}8\pi kF_{2k}(t)\right).$$

Then

$$|N(t_1)-N(t_2)|\le\left[(C_4+C_5)\max_{[0,T]} a(t)+C_6\right]|t_1-t_2|$$

From hypothesis $(A_2)_2$ and $(A_3)_3$

$$0<a(t)\le\frac{C_1}{C_2}<\infty,\quad 0\le t\le T$$

We obtain

$$|N(t_1)-N(t_2)|\le\left[(C_4+C_5)\frac{C_1}{C_2}+C_6\right]|t_1-t_2| \tag{3.14}$$

Notice that from the definition of C_1, we obtain

$$|k(t)|\le C_1;\quad N(t)\ge C_2\quad t\in[0,T]$$

Since $K(t)$ is a continuous function, then for a given $\epsilon > 0, \exists \delta_1 > 0$ Such that $\delta_1 = \delta_1(\epsilon), \forall t_1, t_2 \in [0,T](|t_1 - t_2| < \delta)$ From where

$$|K(t_1) - K(t_2)| < \frac{C_2 \epsilon}{2} \text{ whenever } |t_1 - t_2| < \delta_1 \tag{3.15}$$

Let

$$\delta = \min \left\{ \delta_1(\epsilon), \frac{C_2^2 \epsilon}{2\left((C_4 + C_5)C_1 + C_2 C_6\right) C_1} \right\}$$

From 3.14 for $|t_1 - t_2| < \epsilon$ we obtain

$$|N(t_1) - N(t_2)| \leq \frac{[(C_4 + C_5)C_1 + C_2 C_6]}{C_2} \times \frac{C_2^3 \epsilon}{2\left[(C_4 + C_5)C_1 + C_2 C_6\right] C_1}$$

From where

$$|N(t_1) - N(t_2)| \leq \frac{C_2^2 \epsilon}{2C_1} \tag{3.16}$$

And on the other part

$$|K(t_1)| \leq 2 \max_{t \in [0,T]} F_0(t) + \max_{t \in [0,T]} \left(\sum_{k=1}^{\infty} \frac{2}{\pi k} F_{2k}(t) \right) - \min_{t \in [0,T]} E'(t)$$

Then

$$|K(t_1)| \leq C_1$$

And

$$N(t_2) \leq C_3 \text{ such that } C_2 \leq C_3 \text{ then } N(t_2) \leq C_2$$

And also

$$N(t_1)N(t_2) \leq C_2^2$$

By substituting 3.15 and 3.16 into 3.14, gives

$$|P[a(t_1)] - P[a(t_2)]| \leq \frac{C_2 \epsilon}{2C_2} + \frac{C_1 C_2^2 \epsilon}{2C_1 C_2^2}$$

Then

$$|P[a(t_1)] - P[a(t_2)]| < \epsilon$$

Thus, the set is equi-continuous. Then $P(M_1)$ is a compact set and the operator is compact and the application of set M to itself. Using the Schauder fixed point theorem, we have a solution $a(t) \in C[0,T]$ of equation 3.11.

3.1.2 Uniqueness of the solution

Let us now show that there exists $\Omega_{T_0}(0 < T_0 \leq T)$ for which the solution (a, U) of problem 3.1 3.22 is unique in Ω_{T_0}. Suppose that $(U(x,t), a(t))$ and $(V(x,t), b(t))$ are two pairs of solutions to the problem 3.13.22. Then, from the representations 3.9 and 3.11 of the solution

$$
\begin{aligned}
U(x,t) - V(x,t) &= \sum_{k=1}^{\infty} \varphi_{2k} \left(e^{-(2\pi k)^2 \int_0^t a(s)ds} - e^{-(2\pi k)^2 \int_0^t b(s)ds} \right) X_{2k}(x) \\
&+ \sum_{k=1}^{\infty} \left(\int_0^t F_{2k}(\tau) \left(e^{-(2\pi k)^2 \int_\tau^t a(s)ds} - e^{-(2\pi k)^2 \int_\tau^t b(s)ds} \right) d\tau \right) X_{2k}(x) \\
&+ \sum_{k=1}^{\infty} (\varphi_{2k-1} - 4\pi k \varphi_{2k} t) \left(e^{-(2\pi k)^2 \int_0^t a(s)ds} - e^{-(2\pi k)^2 \int_0^t b(s)ds} \right) X_{2k-1}(x) \\
&+ \sum_{k=1}^{\infty} \left(\int_0^t F_{2k-1}(\tau) \left(e^{-(2\pi k)^2 \int_\tau^t a(s)ds} - e^{-(2\pi k)^2 \int_\tau^t b(s)ds} \right) d\tau \right) X_{2k-1}(x) \\
&- \sum_{k=1}^{\infty} \left(4\pi k \int_0^t F_{2k}(\tau)(t-\tau) \left(e^{-(2\pi k)^2 \int_\tau^t a(s)ds} - e^{-(2\pi k)^2 \int_\tau^t b(s)ds} \right) d\tau \right) X_{2k-1}(x)
\end{aligned}
$$

$$a(t) - b(t) = P[a(t)] - P[b(t)].$$

Where

$$
\begin{aligned}
P[a(t)] - P[b(t)] = &\frac{2F_0(t) + \sum_{k=1}^{\infty} \frac{2a}{\pi k} U'_{2k}(t) - E'(t)}{(1-\beta)(1-b) \sum_{k=1}^{\infty} 8\pi k \left(\varphi_{2k} e^{-(2\pi k)^2 \int_0^t a(s)\, ds} + \int_0^t F_{2k}(\tau) e^{-(2\pi k)^2 \int_\tau^t a(s)\, ds} d\tau \right)} \\
&- \frac{2F_0(t) + \sum_{k=1}^{\infty} \frac{2a}{\pi k} U'_{2k}(t) - E'(t)}{\sum_{k=1}^{\infty} 8\pi k \left(\varphi_{2k} e^{-(2\pi k)^2 \int_0^t b(s)\, ds} + \int_0^t F_{2k}(\tau) e^{-(2\pi k)^2 \int_\tau^t b(s)\, ds} d\tau \right)}.
\end{aligned}
$$

The following estimate is true:

$$\sum_{k=1}^{\infty} 8\pi k \left(\varphi_{2k} + \int_0^t F_{2k}(\tau)\, d\tau \right) \geq C_2.$$

Then

$$|P[a(t)] - P[b(t)]| \le \frac{2F_0(t) + \sum_{k=1}^{\infty} \frac{2a}{\pi k} U'_{2k}(t) + |E'(t)|}{(1-\beta)(1-b)C_2^2}$$
$$\times \left(\sum_{k=1}^{\infty} 8\pi k \, \varphi_{2k} \left| e^{-(2\pi k)^2 \int_0^t a(s)\,ds} - e^{-(2\pi k)^2 \int_0^t b(s)\,ds} \right| \right.$$
$$\left. + \sum_{k=1}^{\infty} 8\pi k \int_0^t |F_{2k}(\tau)| \left| e^{-(2\pi k)^2 \int_\tau^t a(s)\,ds} - e^{-(2\pi k)^2 \int_\tau^t b(s)\,ds} \right| d\tau \right).$$

Using the estimates

$$\left| e^{-(2\pi k)^2 \int_0^t a(s)\,ds} - e^{-(2\pi k)^2 \int_0^t b(s)\,ds} \right| \le (2\pi k)^2 T \max_{0\le t\le T} |a(t) - b(t)|, \tag{3.17}$$

$$\left| e^{-(2\pi k)^2 \int_\tau^t a(s)\,ds} - e^{-(2\pi k)^2 \int_\tau^t b(s)\,ds} \right| \le (2\pi k)^2 T \max_{0\le t\le T} |a(t) - b(t)|, \tag{3.18}$$

we obtain

$$|P[a(t)] - P[b(t)]| \le \frac{2F_0(t) + \sum_{k=1}^{\infty} \frac{2a}{\pi k} U'_{2k}(t) + |E'(t)|}{(1-\beta)(1-b)C_2^2}$$
$$\times \left[\sum_{k=1}^{\infty} 32\pi^3 k^3 \varphi_{2k} + \sum_{k=1}^{\infty} 32\pi^3 k^3 \int_0^T |F_{2k}(\tau)|\,d\tau \right] T \max_{0\le t\le T} |a(t) - b(t)|.$$

Let

$$C_4 = \sum_{k=1}^{\infty} 32\pi^3 k^3 \varphi_{2k}, \qquad C_5 = \sum_{k=1}^{\infty} 32\pi^3 k^3 \int_0^T |F_{2k}(\tau)|\,d\tau.$$

Then

$$|P[a(t)] - P[b(t)]| \le \frac{2F_0(t) + \sum_{k=1}^{\infty} \frac{2a}{\pi k} U'_{2k}(t) + |E'(t)|}{(1-\beta)(1-b)C_2^2} (C_4 + C_5) T \max_{0\le t\le T} |a(t) - b(t)|.$$

Taking the maximum over $[0,T]$, we obtain

$$\max_{0\le t\le T} |P[a(t)] - P[b(t)]| \le \frac{2 \max_{0\le t\le T} |F_0(t)| + \max_{0\le t\le T} \left(\sum_{k=1}^{\infty} \frac{2}{\pi k} |F_{2k}(t)| \right) + \max_{0\le t\le T} |E'(t)|}{(1-\beta)(1-b)C_2^2} (C_4 + C_5) T \max_{0\le t\le T} |a(t) - b(t)|.$$

By the definition of C_1,

$$2 \max_{0\le t\le T} |F_0(t)| + \max_{0\le t\le T} \left(\sum_{k=1}^{\infty} \frac{2}{\pi k} |F_{2k}(t)| \right) + \max_{0\le t\le T} |E'(t)| \le C_1.$$

Hence

$$\max_{0\le t\le T} |P[a(t)] - P[b(t)]| \le \frac{C_1 (C_4 + C_5)}{(1-\beta)(1-b)C_2^2} T \max_{0\le t\le T} |a(t) - b(t)|.$$

Set

$$\alpha = \frac{C_1(C_4 + C_5)}{(1-\beta)(1-b)C_2^2} T.$$

Then

$$\max_{0\le t\le T} |P[a(t)] - P[b(t)]| \le \alpha \max_{0\le t\le T} |a(t) - b(t)|.$$

Let $\alpha \in (0,1)$ and choose T_0, $0 < T_0 \le T$, such that

$$\frac{C_1(C_4 + C_5)}{(1-\beta)(1-b)C_2^2} T_0 \le \alpha.$$

Then from (**??**) we obtain

$$\|a - b\|_{C[0,T_0]} \le \alpha \|a - b\|_{C[0,T_0]}.$$

Since $\alpha < 1$, it follows that

$$(1-\alpha)\|a - b\|_{C[0,T_0]} \le 0.$$

Therefore,

$$a(t) = b(t), \qquad t \in [0, T_0].$$

Substituting $a = b$ into (**??**), we obtain

$$U(x,t) - V(x,t) = 0,$$

hence

$$U(x,t) = V(x,t).$$

Therefore

$$U(x,t) = V(x,t), \qquad a(t) = b(t), \qquad t \in [0, T].$$

3.2 Continuous dependence on data

Theorem 2.2. Under hypotheses $(A_1) - (A_3)$ the solution (a, u) of problem 3.1-3.22 depends continuously on the data for small T.

Proof. Let $\phi = \{\varphi, F, E\}$ and $\overline{\phi} = \{\overline{\varphi}, \overline{F}, \overline{E}\}$ be two sets of data, which satisfy the hypotheses $(A_1) - (A_3)$. Then there exist positive constants $M_i, i = 1, 2, 3$ such that □

$$\|\phi\|_{C^4[0,1]} \le M_1, \quad \|F\|_{C^{4,0}(\overline{\Omega_T})} \le M_2, \quad \|E\|_{C^4[0,1]} \le M_3,$$

$$\|\overline{\phi}\|_{C^4[0,1]} \le M_1, \quad \|\overline{F}\|_{C^{4,0}(\overline{\Omega_T})} \le M_2, \quad \|\overline{E}\|_{C^4[0,1]} \le M_3,$$

Let (a, U) and $(\overline{a}, \overline{U})$ be the solution of the inverse problem 3.1 3.22 corresponding to the data ϕ and $\overline{\phi}$, respectively, according to the equation 3.11 we have

$$a(t) = \frac{2F_0(t) + \sum_{k=1}^{\infty} \frac{2}{\pi k} F_{2k}(t) - E'(t)}{\sum_{k=1}^{\infty} 8\pi k \left[\varphi_{2k} e^{-(2\pi k)^2 \int_0^t a(s)ds} + \int_0^t F_{2k}(\tau) e^{-(2\pi k)^2 \int_\tau^t a(s)ds} d\tau\right]}, \tag{3.19}$$

$$\overline{a}(t) = \frac{2\overline{F}_0(t) + \sum_{k=1}^{\infty} \frac{2}{\pi k} \overline{F}_{2k}(t) - \overline{E}'(t)}{\sum_{k=1}^{\infty} 8\pi k \left[\overline{\varphi}_{2k} e^{-(2\pi k)^2 \int_0^t \overline{a}(s)ds} + \int_0^t \overline{F}_{2k}(\tau) e^{-(2\pi k)^2 \int_\tau^t \overline{a}(s)ds} d\tau\right]}. \tag{3.20}$$

First, let us estimate $a - \overline{a}$ to calculate that

$$\begin{aligned}
&\left| F_0(t) \sum_{k=1}^{\infty} 8\pi k \overline{\varphi}_{2k} e^{-(2\pi k)^2 \int_0^t \overline{a}(s)ds} - \overline{F}_0(t) \sum_{k=1}^{\infty} 8\pi k \varphi_{2k} e^{-(2\pi k)^2 \int_0^t a(s)ds} \right| \\
&= \left| F_0(t) \sum_{k=1}^{\infty} 8\pi k \overline{\varphi}_{2k} \left(e^{-(2\pi k)^2 \int_0^t \overline{a}(s)ds} - e^{-(2\pi k)^2 \int_0^t a(s)ds} \right) \right| + \left| \sum_{k=1}^{\infty} 8\pi k e^{-(2\pi k)^2 \int_0^t a(s)ds} (\overline{\varphi}_{2k} - \varphi_{2k}) \right| \\
&+ \left| \sum_{k=1}^{\infty} 8\pi k \varphi_{2k} e^{-(2\pi k)^2 \int_0^t a(s)ds} (F_0(t) - \overline{F}_0(t)) \right|
\end{aligned}$$

By using **??** and the theorem hypotheses, we have the following for the first estimation:

$$\begin{aligned}
&\left| F_0(t) \sum_{k=1}^{\infty} 8\pi k \overline{\varphi}_{2k} e^{-(2\pi k)^2 \int_0^t \overline{a}(s)ds} - F_0(t) \sum_{k=1}^{\infty} 8\pi k \varphi_{2k} e^{-(2\pi k)^2 \int_0^t a(s)ds} \right| \\
&\leq \max \left(\sum_{k=1}^{\infty} 4(2\pi k)^3 |F_0(t)| |\overline{\varphi}_{2k}(t)| \right) T \max |a(t) - \overline{a}(t)| + \sum_{k=1}^{\infty} 8\pi k \max |\varphi_{2k} - \overline{\varphi}_{2k}| \\
&+ \max \left(\sum_{k=1}^{\infty} 8\pi k |\varphi_{2k}| \right) \max |F(t) - \overline{F}(t)| \\
&\leq \sum_{k=1}^{\infty} 4(2\pi k)^3 \|F\|_{C^{4,0}(\overline{\Omega_T})} \|\varphi\|_{C^4[0,1]} \|a - \overline{a}\|_{C[0,T]} + \sum_{k=1}^{\infty} 8\pi k \|\varphi - \overline{\varphi}\|_{C^4[0,1]} + \sum_{k=1}^{\infty} \|\varphi\|_{C^4[0,1]} \|F - \overline{F}\|_{C^{4,0}(Q_T)} \\
&\leq \sum_{k=1}^{\infty} 4(2\pi k)^3 T M_1 M_2 \|a - \overline{a}\|_{C[0,T]} + \sum_{k=1}^{\infty} 8\pi k \|\varphi - \overline{\varphi}\|_{C^4[0,1]} + \sum_{k=1}^{\infty} 8\pi k M_1 \|F - \overline{F}\|_{C^{4,0}(Q_T)}
\end{aligned}$$

In the same way, we have:

$$\begin{aligned}
&\left| F_0(t) \sum_{k=1}^{\infty} \int_0^t F_{2k}(\tau) e^{-(2\pi k)^2 \int_\tau^t \overline{a}(s)ds} d\tau - \overline{F}_0(t) \sum_{k=1}^{\infty} 8\pi k \int_0^t F_{2k}(\tau) e^{-(2\pi k)^2 \int_\tau^t a(s)ds} d\tau \right| \\
&\leq \sum_{k=1}^{\infty} 4(2\pi k)^3 T^2 M_2^2 \|a - \overline{a}\|_{C[0,T]} + \sum_{k=1}^{\infty} 16\pi k M_2 T \|F - \overline{F}\|_{C^{4,0}(Q_T)}
\end{aligned}$$

$$\left|\sum_{k=1}^{\infty}\frac{2}{\pi k}F_{2k}(t)\sum_{k=1}^{\infty}8\pi k\overline{\varphi}_{2k}e^{-(2\pi k)^2\int_0^t\overline{a}(s)ds}-\sum_{k=1}^{\infty}\frac{2}{\pi k}F_{2k}(t)\sum_{k=1}^{\infty}8\pi k\varphi_{2k}e^{-(2\pi k)^2\int_0^t a(s)ds}\right|$$
$$\leq\sum_{k=1}^{\infty}16(2\pi k)^2TM_1M_2\|a-\overline{a}\|_{C[0,T]}+16M_2\|\varphi-\overline{\varphi}\|_{C^4[0,1]}+16M_1\|F-\overline{F}\|_{C^{4,0}(Q_T)}$$

$$\left|\sum_{k=1}^{\infty}\frac{2}{\pi k}F_{2k}(t)\sum_{k=1}^{\infty}8\pi k\int_0^t F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^t a(s)ds}d\tau-\sum_{k=1}^{\infty}\frac{2}{\pi k}\overline{F}_{2k}(t)\sum_{k=1}^{\infty}8\pi k\int_0^t\overline{F}_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^t\overline{a}(s)ds}d\tau\right|$$
$$\leq\sum_{k=1}^{\infty}16(2\pi k)^2M_2^2T^2\|a-\overline{a}\|_{C[0,T]}+32M_2T\|F-\overline{F}\|_{C^{4,0}(Q_T)}$$

$$\left|E'(t)\sum_{k=1}^{\infty}8\pi k\overline{\varphi}_{2k}e^{-(2\pi k)^2\int_0^t\overline{a}(s)ds}-\overline{E}'(t)\sum_{k=1}^{\infty}8\pi k\varphi_{2k}e^{-(2\pi k)^2\int_0^t a(s)ds}\right|$$
$$\leq\sum_{k=1}^{\infty}4(2\pi k)^3TM_1M_3\|a-\overline{a}\|_{C[0,T]}+\sum_{k=1}^{\infty}8\pi k\|\varphi-\overline{\varphi}\|_{C^4[0,1]}+\sum_{k=1}^{\infty}8\pi kM_1\|E-\overline{E}\|_{C^{4,0}(Q_T)}$$

$$\left|E'(t)\sum_{k=1}^{\infty}8\pi k\int_0^t F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^t a(s)ds}d\tau-\overline{E}'(t)\sum_{k=1}^{\infty}8\pi k\int_0^t F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^t\overline{a}(s)ds}d\tau\right|$$
$$\leq\sum_{k=1}^{\infty}4(2\pi k)^3T^2M_2M_3\|a-\overline{a}\|_{C[0,T]}+\sum_{k=1}^{\infty}8\pi kT\|F-\overline{F}\|_{C^{4,0}(Q_T)}+\sum_{k=1}^{\infty}8\pi kTM_2\|E-\overline{E}\|_{C^{4,0}(Q_T)}$$

Where $M_k, k=1,\ldots,7$ are some constants. We have:

$$M_1=\sum_{k=1}^{\infty}4(2\pi k)^3TM_1M_2+\sum_{k=1}^{\infty}4(2\pi k)^3T^2M_2^2+\sum_{k=1}^{\infty}16(2\pi k)^2TM_1M_2+\sum_{k=1}^{\infty}16(2\pi k)^2M_2^2T^2$$
$$+\sum_{k=1}^{\infty}4(2\pi k)^3TM_1M_3$$
$$M_2=\sum_{k=1}^{\infty}16\pi k+16M_2$$
$$M_3=\sum_{k=1}^{\infty}8\pi kM_1+\sum_{k=1}^{\infty}16\pi kTM_2+16M_1+32TM_2+\sum_{k=1}^{\infty}8\pi k$$
$$M_4=\sum_{k=1}^{\infty}8\pi kM_1+\sum_{k=1}^{\infty}TM_2$$

After using these estimations in $a-\overline{a}$, we obtain:

$$(1-M_1)\|a-\overline{a}\|_{C[0,T]} \leq M_5\left(\|E-\overline{E}\|_{C^{4,0}(Q_T)}+\|\varphi-\overline{\varphi}\|_{C^4[0,1]}+\|F-\overline{F}\|_{C^{4,0}(Q_T)}\right)$$

Where $M_5=\max\{M_2,M_3,M_4\}$. With the inequality $M_1<1$ by taking small T, we finally obtain:

$$\|a-\overline{a}\|_{C[0,T]} \leq \frac{M_5}{(1-M_1)}\|\phi-\overline{\phi}\|$$

If we set $M_6=\frac{M_5}{(1-M_1)}$, then:

$$\|\phi-\overline{\phi}\|=\|E-\overline{E}\|_{C^{4,0}(Q_T)}+\|\varphi-\overline{\varphi}\|_{C^4[0,1]}+\|F-\overline{F}\|_{C^{4,0}(Q_T)}$$

From equation 3.9, a similar estimation is also obtained for the difference $U-\overline{U}$ as:

$$\|U-\overline{U}\|_{C(\overline{Q}_T)} \leq M_6\|\phi-\overline{\phi}\|$$

$\square$

Then we have

$$|a(t)-\overline{a}(t)| \leq \frac{M_1\|a-\overline{a}\|_{C[0,T]} + M_2\|\varphi-\overline{\varphi}\|_{C^4[0,1]} + M_3\|F-\overline{F}\|_{C^{4,0}(Q_T)} + M_4\|E-\overline{E}\|_{C^{4,0}(Q_T)}}{C_2} \tag{3.21}$$

Where C_2 is a constant defined previously. Then

$$\begin{aligned}(C_2-M_1)\|a-\overline{a}\|_{C[0,T]} &\leq M_2\|\varphi-\overline{\varphi}\|_{C^4[0,1]} + M_3\|F-\overline{F}\|_{C^{4,0}(Q_T)} + M_4\|E-\overline{E}\|_{C^{4,0}(Q_T)} \\ &\leq M_5\left(\|\varphi-\overline{\varphi}\|_{C^4[0,1]} + \|F-\overline{F}\|_{C^{4,0}(Q_T)} + \|E-\overline{E}\|_{C^{4,0}(Q_T)}\right)\end{aligned}$$

Where $M_5 = \max\{M_2, M_3, M_4\}$. For a small T such that $C_2 - M_1 > 0$, we obtain

$$\|a-\overline{a}\|_{C[0,T]} \leq \frac{M_5}{C_2-M_1}\|\phi-\overline{\phi}\| \tag{3.22}$$

Where $\|\phi-\overline{\phi}\| = \|\varphi-\overline{\varphi}\|_{C^4[0,1]} + \|F-\overline{F}\|_{C^{4,0}(Q_T)} + \|E-\overline{E}\|_{C^{4,0}(Q_T)}$. Let $M_6 = \frac{M_5}{C_2-M_1}$, then

$$\|a-\overline{a}\|_{C[0,T]} \leq M_6\|\phi-\overline{\phi}\| \tag{3.23}$$

Now, let's estimate $U(x,t)-\overline{U}(x,t)$ using the representation 3.9. From 3.9, we have

$$\begin{aligned}|U(x,t)-\overline{U}(x,t)| \leq |U_0(t)-\overline{U}_0(t)||X_0(x)| &+ \sum_{k=1}^{\infty}|U_{2k}(t)-\overline{U}_{2k}(t)||X_{2k}(x)| \\ &+ \sum_{k=1}^{\infty}|U_{2k-1}(t)-\overline{U}_{2k-1}(t)||X_{2k-1}(x)|\end{aligned}$$

Using the properties of the basis functions $X_k(x)$ and the previous estimations for $a(t)-\overline{a}(t)$, we obtain

$$\|U-\overline{U}\|_{C(\overline{\Omega}_T)} \leq M_7\|\phi-\overline{\phi}\| \tag{3.24}$$

Where M_7 is a constant depending on the data and T. This completes the proof of Theorem 2.2 regarding the continuous dependence on the data.

Chapter 4

Numerical approximation of the solution

In this chapter, we present a numerical method to solve the inverse problem **??**-**??**. Since the problem is non-linear and coupled, we use an iterative algorithm based on the fixed-point method described in Chapter 3.

4.1 Numerical Approximation

Let $N_1, N_2 \in \mathbb{N}$ and define

$$t_j = j\tau, \qquad j = 0, 1, \ldots, N_1,$$

$$x_i = ih, \qquad i = 0, 1, \ldots, N_2,$$

where

$$\tau = \frac{T}{N_1}, \qquad h = \frac{1}{N_2}.$$

Let U_i^j be the numerical approximation of $U(x_i, t_j)$ and a^j the approximation of $a(t_j)$. The finite difference approximations are given as follows.

1. The approximation of the time derivative:

$$U_t(x_i, t_j) = \frac{U_i^{j+1} - U_i^j}{\tau} + O(\tau). \tag{4.1}$$

2. The Crank–Nicolson approximation for the second derivative in space:

$$U_{xx}(x_i, t_j) = \frac{1}{2h^2}\left[(U_{i+1}^j - 2U_i^j + U_{i-1}^j) + (U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1})\right] + O(h^2). \tag{4.2}$$

3. The initial condition:

$$U(x_i, 0) = \varphi(x_i) \quad \Longrightarrow \quad U_i^0 = \varphi_i. \tag{4.3}$$

4. The nonlocal boundary conditions become

$$U(0,t_j) = U(1,t_j) \quad \Longrightarrow \quad U_0^j = U_{N_2}^j, \tag{4.4}$$

and

$$U_x(0,t_j) = \beta U_x(1,t_j) \quad \Longrightarrow \quad \frac{U_1^j - U_0^j}{h} = \beta \frac{U_{N_2}^j - U_{N_2-1}^j}{h}. \tag{4.5}$$

Hence,

$$U_1^j - U_0^j = \beta(U_{N_2}^j - U_{N_2-1}^j). \tag{4.6}$$

Finite Difference Scheme

Using equations (3.1)–(3.2), the Crank–Nicolson discretization of the problem becomes

$$\frac{U_i^{j+1} - U_i^j}{\tau} = \frac{a^{j+1} + a^j}{4h^2} \left[(U_{i+1}^j - 2U_i^j + U_{i-1}^j) + (U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1}) \right] + \frac{F_i^{j+1} + F_i^j}{2}. \tag{4.7}$$

Let

$$R = \frac{(a^{j+1} + a^j)\tau}{4h^2}.$$

Then equation (3.7) can be written as

$$-RU_{i-1}^{j+1} + 2(1+R)U_i^{j+1} - RU_{i+1}^{j+1} = RU_{i-1}^j + 2(1-R)U_i^j + RU_{i+1}^j + \frac{\tau}{2}(F_i^{j+1} + F_i^j). \tag{4.8}$$

for $i = 1, 2, \ldots, N_2 - 1$.
The boundary relations are

$$U_0^j = U_{N_2}^j, \tag{4.9}$$

and

$$U_1^j - U_0^j = \beta(U_{N_2}^j - U_{N_2-1}^j). \tag{4.10}$$

Matrix Formulation

The numerical scheme can be written in the matrix form

$$AU^{j+1} = BU^j + F^j, \tag{4.11}$$

where

$$U^j = \begin{pmatrix} U_1^j \\ U_2^j \\ \vdots \\ U_{N_2-1}^j \end{pmatrix}.$$

The matrices A and B are tridiagonal matrices given respectively by

$$A = \begin{pmatrix} 2(1+R) & -R & 0 & \cdots & 0 \\ -R & 2(1+R) & -R & \cdots & 0 \\ 0 & -R & 2(1+R) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -R & 2(1+R) \end{pmatrix},$$

and

$$B = \begin{pmatrix} 2(1-R) & R & 0 & \cdots & 0 \\ R & 2(1-R) & R & \cdots & 0 \\ 0 & R & 2(1-R) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & R & 2(1-R) \end{pmatrix}.$$

Moreover,

$$F^j = \frac{\tau}{2} \begin{pmatrix} F_1^{j+1} + F_1^j \\ F_2^{j+1} + F_2^j \\ \vdots \\ F_{N_2-1}^{j+1} + F_{N_2-1}^j \end{pmatrix}.$$

Remark 3.1. The matrices A and B are strictly diagonally dominant. Therefore, the matrix A is invertible and the discrete system admits a unique solution.

Predictor–Corrector Iterative Method

Integrating the equation over $(0, 1)$ and using the nonlocal boundary conditions, we obtain

$$a(t) = \frac{-E'(t) + \displaystyle\int_0^1 F(x,t)dx}{(1-\beta)U_x(0,t)}. \tag{4.12}$$

Using finite differences,

$$a^j = \frac{\left[-\dfrac{E^{j+1}-E^j}{\tau} + (F_{in})^j\right] h}{(1-\beta)(U_1^j - U_0^j)}. \tag{4.13}$$

At each iteration step $(s+1)$, we compute

$$a_{j+1}^{(s+1)} = \frac{\left[-\dfrac{E^{j+2}-E^{j+1}}{\tau} + (F_{in})^{j+1}\right] h}{(1-\beta)(U_1^{j+1(s)} - U_0^{j+1(s)})}. \tag{4.14}$$

Then the corrected solution is obtained from

$$\begin{aligned} \frac{U_i^{j+1(s+1)} - U_i^{j+1(s)}}{\tau} = & \frac{1}{2h^2}\left(\frac{a_{j+1}^{(s+1)} + a_{j+1}^{(s)}}{2}\right) \\ & \times \Big[(U_{i+1}^{j+1(s+1)} - 2U_i^{j+1(s+1)} + U_{i-1}^{j+1(s+1)}) \\ & + (U_{i+1}^{j+1(s)} - 2U_i^{j+1(s)} + U_{i-1}^{j+1(s)})\Big] \\ & + \frac{1}{2}(F_i^{j+1} + F_i^j). \end{aligned} \tag{4.15}$$

The iterative process is stopped when the difference between two successive iterations satisfies a prescribed tolerance.

4.1.1 Numerical Example

We consider the inverse problem (3.1), with

$$F(x,t) = e^{-t}\left[(4\pi^2 t^2 - 1)(1-b-ax)\sin(2\pi x) + 4\pi a\, t^2\cos(2\pi x)\right],$$

$$\varphi(x) = (1-b-ax)\sin(2\pi x),$$

$$E(t) == \frac{a}{2\pi}e^{-t}, \qquad x \in [0,1],\ t \in [0,T].$$

It is easy to verify that the analytical solution of problem (3.1)–(**??**) is given by

$$\{a(t), U(x,t)\} = \left\{t^2,\ (1-b-ax)\sin(2\pi x)e^{-t}\right\}. \tag{4.16}$$

For $a = \frac{1-\beta}{1+\beta}$ and $b = \frac{\beta}{1+\beta}$

Applying the numerical scheme described in the previous section with the mesh sizes

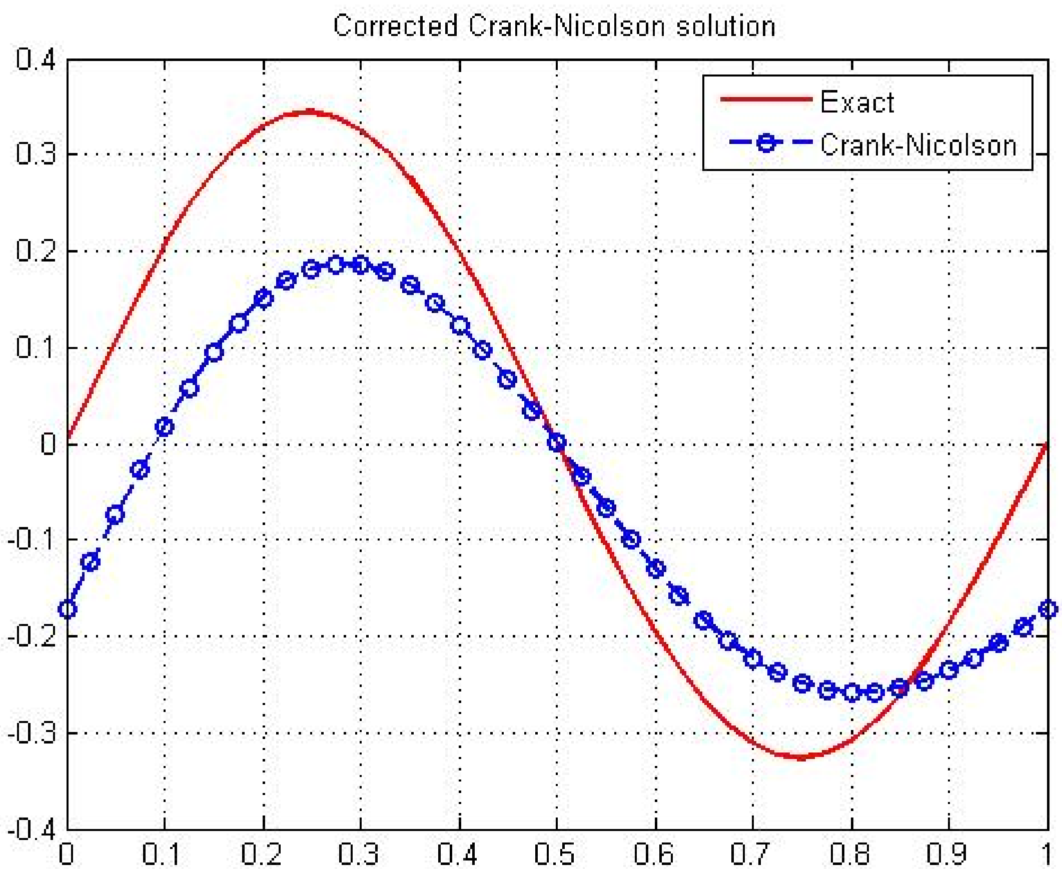
$$h = \frac{1}{N_x}, \qquad \tau = \frac{1}{N_t},$$

where $N_x, N_t \in \mathbb{N}$, we observe that the scheme becomes unstable when T exceeds a critical value. In particular, for the step size

$$h = \frac{1}{40}, \qquad \tau = 0.005,$$

We consider the inverse problem with $\beta = 9$:

Hence, the nonlocal boundary conditions are satisfied.



Conclusion

In this thesis, we have studied an inverse problem for a parabolic equation with an unknown coefficient $a(t)$. We established the existence and uniqueness of the solution using the Schauder fixed-point theorem and proved the continuous dependence on the data. Finally, a numerical method was implemented to approximate the solution, showing good agreement with exact cases.

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الملخص

في هذه المذكرة، قمنا بدراسة مسألة عكسية لتحديد معامل الانتشار الحراري المعتمد على الزمن وتوزيع درجة الحرارة في معادلة انتشار أحادية البعد مع شرط تكاملي، وذلك عند حدوث تبادل حراري عبر حدود المادة. وقد أثبتنا أن المسألة العكسية جيدة الوضع تحت بعض شروط الانتظام المفروضة على معطيات المسألة. كما تمت دراسة المسألة العكسية عددياً باستعمال مخطط الفروق المنتهية لكرانك–نيكولسون مقترناً بتقنية المتنبئ–المصحح. وقد قدمنا أمثلة عددية للتحقق من فعالية هذا المخطط وكفاءته.

الكلمات المفتاحية: طريقة فورييه، طريقة النقطة الثابتة لشاودر، مسألة طيفية غير ذاتية الترافق، مخطط كرانك–نيكولسون.

Abstract

In this memoir, we have studied an inverse problem for the identification of a coefficient of thermal diffusivity as a function of time and the distribution of temperature in a one-dimensional diffusion equation with an integral condition when a heat exchange takes place through the material border. We have shown that the inverse problem is well posed under certain regularity conditions on the data of the problem. The inverse problem is also studied numerically, using the Crankâ€"Nicolson finite difference scheme combined with the predictor-corrector technique. We have presented numerical examples to validate the effectiveness of this scheme.
Keywords : Fourier method, Schauder fixed point method, Non-self-adjoint spectral problem,

Crank-Nicolson scheme.