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Banach spaces of Lipschitz functions

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Dedications

In the name of ALLAH the most gracious the most merciful

I dedicate this work to :

The deare

their virtue

My dear sister and my brothers

All my friends and family

All the student

2019/2020.

Acknowledgments

First, I am so grateful to " **Allah** " for giving us protection and ability to start and finish this work.

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Contents

Notations

$d(.,.)$	Distance .
(X,d)	Metric space .
(X,d,e)	Pointed metric space .
$\mathcal{M}_0(X)$	Class of complete pointed metric spaces .
$\mathbb{R}$	Reel space .
$\mathrm{Lip}(X)$	Space of bounded Lipschitz functions from X into $\mathbb{R}$.
$\Vert . \Vert_L$	Norm of Lip space .
$\mathrm{Lip}_0(X, Y)$	Space of all Lipschitz functions between X and Y that vanish at e .
$X^{\#} = \mathrm{Lip}_0(X,\mathbb{R}) = \mathrm{Lip}_0(X)$	Lipschitz dual of the pointed metric space X .
$\mathrm{Lip}_0(.)$	Norm of Lip_0 space .
X^*	Topological dual of X.
$\hat{X}$	Set of pairs (x,y) in X^2 such that $x \neq y$.
$\mathcal{B}_X$	Unit ball of X.
ω^*	Weak * topology .
$\mathcal{T}_p$	Topology of pointwise convergence .
$\mathcal{M}(X)$	Linear space of all molecules on the metric space X.
$m_{xx'}$	Molecule defined by $m_{xx'} = \mathbf{1}_{\{x\}} - \mathbf{1}_{\{x'\}}$ for $x, x' \in X$.
Æ(X)	Arens-Eells space of X.
$\mathcal{F}(X)$	Lipschitz free space of X.
$\mathrm{lip}_0(X)$	Little Lipschitz space of X.

Introduction

The Lipschitz function is the natural morphism between metric spaces like linear operator between Banach spaces. In mathematical analysis, Lipschitz continuity, named after Rudolf Lipschitz, is a strong form of uniform continuity for functions. So a map $f : (X, d_X) \longrightarrow (Y, d_Y)$ between two metric spaces is called Lipschitz if there is a positive constant C such that that

$$\forall x, y \in X, d_Y(f(x), f(y)) \leq C d_X(x, y).$$

For a Lipschitz map f, we define its Lipschitz constant by

$$\mathrm{Lip}(f) := \sup_{x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} = \inf\{C : \text{verifying above definition }\}.$$

We can be define the space $\mathrm{Lip}(X)$ of all bounded real-valued Lipschitz functions on X, with the norm

$$\|f\|_{\mathrm{L}} = \max\{\|f\|_\infty, \mathrm{Lip}(f)\}.$$

Also, we define the Lipschitz space $\mathrm{Lip}_0(X) = X^{\#}$ is the set of all real-valued Lipschitz functions on X which vanish e (e is a distinguished element of X). With the norm $\mathrm{Lip}(.)$.

So, The space $\mathrm{Lip}_0(X)$ is a Banach space. It is also a dual space. The first chapter of Lipschitz spaces we shall see the concepts Preliminairs on metric space and the propriety of Lipschitz functions for defining the space $\mathrm{Lip}_0(X)$ and it is a Banach space, Also, we mention of retraction space foresee the projection in the linear case on the non-linear case (Lipschitz case). Moreover we see two important concepts the unit ball $\mathcal{B}_{X^{\#}}$ is compact and the conjugate space, for proof the space $\mathrm{Lip}_0(X)$ is a dual.

The second chapter called of "Pestov's theorem" we shall see two predual of the space $\mathrm{Lip}_0(X)$ are Arens-Eells space ($Æ(X)$) and free-Banah space ($\mathcal{F}(X)$) , also another construction of the last space , all this in more details.

The third chapter is denoted to $\mathrm{lip}_0(X)$. We shall give some properties concerning this space. We show that $\mathrm{lip}_0(X)$ is a predual of $Æ(X)$ if X is compact.

The last chapter , we introduce the class of B-Lipschitz summing operators.

CHAPTER 1

LIPSCHITZ SPACES

In this chapter we recall, some properties about metric space, and Lipschitz functions in order to prove that $\mathrm{Lip}_0(X)$ is Banach space. In addition to that, we prove the compactness of the unit ball $\mathcal{B}_{X^\#}$ which has a significant role as well as the concept of conjugate space to prove that $\mathrm{Lip}_0(X)$ is dual. Moreover , all of these concepts can be found in the references [5], [14] for more details.

1.1 Preliminaries on metric spaces

Metric spaces as the natural framework for the study of Lipschitz functions, we present some notions and results used in the sequel. They are treated in many books on topology, mathematical analysis, functional analysis, etc. As dedicated exclusively to metric spaces we mention the books [8], [12] and a good reference for this is the book of Weaver [14].

Definition 1.1.1. Let X be a non empty set . We say that d is a distance on X if d is an application from X^2 into $\mathbb{R}_+$ such that for all x , y , z in X we have

(i) $d(x,y) = 0 \Leftrightarrow x = y$ (separation),

(ii) $d(x,y) = d(y,x)$ (symmetry),

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ (triangular inequality) .

The space X equipped with d is called metric space (X, d)

Definition 1.1.2. A metric space (X, d) is called discrete if there exists a constant $\delta > 0$ such that

$$\forall x_1, x_2 : (x_1 \neq x_2) \in X; d(x_1, x_2) > \delta.$$

Definition 1.1.3. A pointed metric space (X, d, e), is a metric space (X, d) with a distinguish element $e \in X$.

Definition 1.1.4. Let $\{(X_i, d_{X_i}, e_i), i \in I\}$ be a family of metric spaces in $\mathcal{M}_0$, we can define by $(\prod^\infty X_i, d, e)$ the set of elements $x = (x_i)$ such that $\sup_{i\in I} d_{X_i}(x_i, e_i) < \infty$, with the metric

$$d(x,y) = \sup_{i\in I} d_{X_i}(x_i, e_i)$$

and the distinguished point $e = (e_i)_{i\in I}$.

This definition is well defined. Indeed, We have

$$\begin{aligned} d(x,y) &= \sup_{i\in I} d_{X_i}(x_i,y_i) \\ &\leq \sup_{i\in I} \left(d_{X_i}(x_i,e_i) + d_{X_i}(e_i,y_i) \right) \\ &\leq \sup_{i\in I} d_{X_i}(x_i,e_i) + \sup_{i\in I} d_{X_i}(e_i,y_i) \end{aligned}$$

also, we have $(\prod^{\infty} X_i, d, e) \in \mathcal{M}_0$, as we have by Definition 1.1.4, $\sup_{i\in I} d_{X_i}(x_i,e_i) < \infty$. So $d(x,y) \leq \infty$.

1.2 Lipschitz spaces

1.2.1 Lipschitz functions and proprieties

Lipschitz functions are natural morphisms between metric spaces like linear operator between Banach spaces. In mathematical analysis, Lipschitz continuity, named after Rudulf Lipschitz, is a strong from of uniform continuity for functions. For more informations in the book [14] .

Definition 1.2.1. Let $(X,d_X),(Y,d_Y)$ be two metric spaces . A map $f:(X,d_X)\longrightarrow(Y,d_Y)$ is called Lipschitz if there is a positive constant C such that

$$\forall x,y\in X, d_Y(f(x),f(y)) \leq C d_X(x,y). \tag{1.1}$$

If $C=1$, the map is called nonexpansive (and contraction if $C<1$).

For a Lipschitz map f, we define its Lipschitz constant by

$$\mathrm{Lip}(f) := \sup_{x\neq y} \frac{d_Y(f(x),f(y))}{d_X(x,y)} = \inf\{C : \text{verifying } (1.1)\}.$$

Let $(X,d_X,e_X),(Y,d_Y,e_Y)$ be pointed metric spaces. We say a map $f:(X,d_X)\longrightarrow(Y,d_Y)$ preserves distinguished point if $f(e_X)=e_Y$.

Definition 1.2.2. $(X,d_X),(Y,d_Y)$ be two metric spaces. A map $f:(X,d_X)\longrightarrow(Y,d_Y)$ is called bi-Lipschitz or quasi-isometry, if f is bijective (one-to-one = injective and into = surjective) and both f,f^{-1} are Lipschitz.

In this case X and Y are called

(i) Lipschitz isomorphie or Lipschitz homeomorphic (Nigel Kalton).

Or

(ii) Quasi-isomorphic (Nik Weaver).

A bi Lipschitz function f is an isometry if

$$\forall x, y \in X, d_Y\left(f(x), f(y)\right) = d_X(x, y).$$

We give some properties and the proof for certain, for the aisance of the reader.

Proposition 1.2.1. *Let X, Y and Z be metric spaces and let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be Lipschitz functions , then $g \circ f : X \longrightarrow Z$ is Lipschitz with*

$$\operatorname{Lip}(g \circ f) \leq \operatorname{Lip}(g) . \operatorname{Lip}(f).$$

Theorem 1.2.1. *Let X_0, Y_0 be metric spaces and let X, Y be their completions . Let $f_0 : X_0 \longrightarrow Y_0$ be a Lipschitz function. Then f has a unique Lipschitz extension $f : X \longrightarrow Y$ such that*

$$\operatorname{Lip}(f) = \operatorname{Lip}(f_0).$$

Proposition 1.2.2. *Let (X, d) be a metric space. For Lipschitz functions $f, g : (X, d) \longrightarrow \mathbb{R}$ and scalar $\alpha \in \mathbb{R}$, the Lipschitz constant has the properties*

(i) $\operatorname{Lip}(f + g) \leq \operatorname{Lip}(f) + \operatorname{Lip}(g)$.

(ii) $\operatorname{Lip}(\alpha f) = |\alpha| \operatorname{Lip}(f)$.

(iii) $\operatorname{Lip}(\min\{f, g\}, \text{or} \max\{f, g\}) \leq \max\{\operatorname{Lip}(f), \operatorname{Lip}(g)\}$.

Where $\min(f, g)$ (resp, $\max(f, g)$) denotes the pointwise minimum (resp , maximum) of the functions f and g.

Proposition 1.2.3. *Let X, Y be metric spaces and let f and $\{f_n\}_{n\in\mathbb{N}}$ be Lipschitz functions from X to Y . Suppose that $f_n \longrightarrow f$ pointwise . Then*

$$\operatorname{Lip}(f) \leq \sup_n \operatorname{Lip}(f_n).$$

Proof. Let x, y be in X. We have

$$\begin{aligned} d_Y(f(x), f(y)) &= \lim_{n \longrightarrow \infty} d_Y(f_n(x), f_n(y)) \\ \frac{d_Y(f(x), f(y))}{d_X(x,y)} &= \lim_{n \longrightarrow \infty} \frac{d_Y(f_n(x), f_n(y))}{d_X(x,y)} \\ \sup_{x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x,y)} &= \sup_{x \neq y} \lim_{n \longrightarrow \infty} \frac{d_Y(f_n(x), f_n(y))}{d_X(x,y)} \\ &\leq \sup_{x \neq y} \sup_{n} \frac{d_Y(f_n(x), f_n(y))}{d_X(x,y)} \end{aligned}$$

by permitting the $\sup$, we obtain the result. □

Corollary 1.2.1. *If* $\sum_{n \geq 0} f_n$ *converges pointwise then*

$$\mathrm{Lip}\left(\sum_{n \geq 0} f_n\right) \leq \sum_{n \geq 0} \mathrm{Lip}(f_n).$$

Proposition 1.2.4. *Let* X *be a metric space and let* $f, g : X \longrightarrow \mathbb{R}$ *be Lipschitz map , then*

(i) $\mathrm{Lip}(f.g) \leq \|f\|_\infty \mathrm{Lip}(g) + \|g\|_\infty \mathrm{Lip}(f)$.

(ii) $\mathrm{Lip}\left(\frac{1}{f}\right) \leq \frac{\mathrm{Lip}(f)}{\epsilon^2}$, *if* $|f(x)| > \epsilon > 0$ *for all* $x \in X$.

Proposition 1.2.5. *Let* $(X, d_X), (X_i, d_{X_i}), (i \in I)$ *be metric spaces in* $\mathcal{M}_0$. *For each* i *in* I, *let* $f_i : X \longrightarrow X_i$ *be a Lipschitz map which preserver distinguished point. Suppose that* $\sup_{i \in I} \mathrm{Lip}(f_i) < \infty$. *Then , the product map* $f : X \longrightarrow \prod^\infty X_i$ *satisfies*

$$\mathrm{Lip}(f) := \sup_{i \in I} \mathrm{Lip}(f_i).$$

Proof. Let x be in X. We prove that $(f_i(x)) \in \prod^\infty X_i$. We have

$$\begin{aligned} \sup_{i \in I} d_{X_i}(f_i(x), e_i) &= \sup_{i \in I} d_{X_i}(f_i(x), f_i(e)) \\ \left(d = \sup_{i \in I} d_{X_i}\right) &\leq \sup_{i \in I} \mathrm{Lip}(f_i)\, d(x, e) \\ &< \infty. \end{aligned}$$

For x, y in X. We have by definition

$$\frac{d(f(x), f(y))}{d(x,y)} = \sup_{i \in I} \frac{d_{X_i}(f_i(x), f_i(y))}{d(x,y)}$$

and hence

$$\begin{aligned}\sup_{x\neq y}\frac{d\left(f\left(x\right),f\left(y\right)\right)}{d\left(x,y\right)} &= \sup_{x\neq y}\sup_{i\in I}\frac{d_{X_i}\left(f_i\left(x\right),f_i\left(y\right)\right)}{d\left(x,y\right)}\\ &= \sup_{i\in I}\sup_{x\neq y}\frac{d_{X_i}\left(f_i\left(x\right),f_i\left(y\right)\right)}{d\left(x,y\right)}\\ &= \sup_{i\in I}\mathrm{Lip}\left(f_i\right).\end{aligned}$$

This implies that $\mathrm{Lip}\left(f\right) := \sup_{i\in I}\mathrm{Lip}\left(f_i\right)$; and we obtain the result. □

We give the non linear Hahn Banach theorem [14].

Theorem 1.2.2. ***(Non linear Hahn-Banach theorem)***

Let E be a subset of a metric space (X,d) and let $f : E \longrightarrow \mathbb{R}$ be a Lipschitz function. Then f can be extended to a Lipshitz function $\tilde{f} : X \longrightarrow \mathbb{R}$ with he same Lipschitz constant

$$\mathrm{Lip}\left(\tilde{f}\right) = \mathrm{Lip}\left(f\right).$$

$$\begin{array}{ccc} X & & \\ & \tilde{f} & \\ i\uparrow & \searrow & \\ E & \overset{f}{\longrightarrow} & \mathbb{R}\end{array}$$

$$\left(i \quad \textit{is the canonicol injection from } E \quad \textit{to} \quad X \quad \textit{and} \quad f = \tilde{f}\circ i\right)$$

Proof. Fix z in $X - E$. We must find a value for $\tilde{f}\left(x\right)$ such that for all x in E.

$$\left|\tilde{f}\left(z\right) - f\left(x\right)\right| \leq \mathrm{Lip}\left(f\right)d\left(x,z\right), \forall x\in E$$

or equivalently

$$f\left(y\right) - \mathrm{Lip}\left(f\right)d\left(y,z\right) \leq \tilde{f}\left(z\right) \leq f\left(x\right) + \mathrm{Lip}\left(f\right)d\left(x,z\right), \forall y\in E$$

hence

$$\sup_{y\in E}\left(f\left(y\right) - \mathrm{Lip}\left(f\right)d\left(y,z\right)\right) \leq \tilde{f}\left(z\right) \leq \inf_{x\in E}\left(f\left(x\right) + \mathrm{Lip}\left(f\right)d\left(x,z\right)\right).$$

It is possible because for all x,y in E, we have

$$\begin{aligned}f\left(x\right) - f\left(y\right) &\leq \mathrm{Lip}\left(f\right)d\left(x,y\right)\\ &\leq \mathrm{Lip}\left(f\right)\left(d\left(x,z\right) + d\left(z,y\right)\right).\end{aligned}$$

Define the function $\tilde{f} : X \longrightarrow \mathbb{R}$ by the formula

$$\tilde{f} = \inf_{x \in E} (f(x) + \mathrm{Lip}(f) d(x, z))$$

To see that this function satisfies the results, fix an arbitrary $x_0 \in E$.

Then, for any $x \in E$

$$\begin{aligned} f(x_0) - f(x) &\leq \mathrm{Lip}(f)(d(x_0, x)) \\ &\leq \mathrm{Lip}(f)(d(x_0, z) + d(z, x)). \end{aligned}$$

This implies (that $f(x) + \mathrm{Lip}(f) d(x, z)$ is bounded belew)

$$f(x_0) - \mathrm{Lip}(f) d(x_0, z) \leq f(x) + \mathrm{Lip}(f) d(x, z).$$

So $\tilde{f}(z)$ is well-defined. Also , if $z \in E$, the above shows that $\tilde{f}(z) = f(z)$. Finally (by definition of the $\inf$), for $x, y \in X$ and $\epsilon > 0$, choose $x_z \in E$ such that

$$\tilde{f}(z) \geq f(x_z) + \mathrm{Lip}(f) d(z, x_z) - \epsilon$$

$$-\tilde{f}(z) \leq -f(x_z) - \mathrm{Lip}(f) d(z, x_z) + \epsilon$$

Then

$$\begin{aligned} \tilde{f}(y) - \tilde{f}(z) &\leq f(x_z) + \mathrm{Lip}(f) d(y, x_z) - f(x_z) - \mathrm{Lip}(f) d(z, x_z) + \epsilon \\ &\leq \mathrm{Lip}(f) d(y, z) + \epsilon. \end{aligned}$$

Thus , we see that $\tilde{f}$ is indeed $\mathrm{Lip}(f)$ – Lipschitz $i.e.$, $\mathrm{Lip}\left(\tilde{f}\right) \leq \mathrm{Lip}(f)$. We have $f = \tilde{f} \circ i$ and

$$\begin{aligned} \mathrm{Lip}(f) = \mathrm{Lip}(f \circ i) &\leq \mathrm{Lip}\left(\tilde{f}\right) \mathrm{Lip}(i) \\ &\leq \mathrm{Lip}\left(\tilde{f}\right), \text{because} \quad \mathrm{Lip}(i) = 1. \end{aligned}$$

Conclusion

$$\mathrm{Lip}(f) = \mathrm{Lip}\left(\tilde{f}\right)$$

and this proves the theorem. □

1.2.2 Lipschitz spaces

Definition 1.2.3. Let (X, d) be a metric space , then $\mathrm{Lip}\,(X)$ is the space of all bounded scalar valued Lipschitz function on X with the norm

$$\|f\|_{\mathrm{L}} = \max\{\|f\|_{\infty}, \mathrm{Lip}\,(f)\}.$$

Let now (X, d, e) be a pointed metric space with a distinguished " base point " e which is fixed in advance . We denote by $\mathrm{Lip}_0\,(X)$ the space of all bounded scalar valued Lipchitz mapping on X vanishing at e by

$$\mathrm{Lip}_0\,(X) = \{f : X \longrightarrow \mathbb{R} \quad \text{Lipschitz such that } f\,(e) = 0\}$$

with the norm

$$\mathrm{Lip}\,(f) := \sup_{x \neq y} \frac{|f\,(x) - f\,(y)|}{d_X\,(x, y)}$$

The spaces $\mathrm{Lip}\,(X)$ and $\mathrm{Lip}_0\,(X, \mathbb{R})$ become Banach spaces, we put

$$X^{\#} = \mathrm{Lip}_0\,(X) = \mathrm{Lip}_0\,(X, \mathbb{R}).$$

This Banach space of Lipschitz functions is called also Lipschitz dual of X .

Proposition 1.2.6. *Let (X, d, e) be a pointed metric space . The space $(\mathrm{Lip}_0\,(X), \mathrm{Lip}\,(.))$ is a Banach space.*

Proof. One verify that $\mathrm{Lip}\,(.)$ is a norm on $\mathrm{Lip}_0\,(X)$. Let f be in $\mathrm{Lip}_0\,(X)$, we have

$$\begin{aligned} \mathrm{Lip}\,(f) = 0 &\Leftrightarrow \forall\,(x, y) \in \tilde{X}, \frac{|f\,(x) - f\,(y)|}{d\,(x, y)} = 0 \\ &\Leftrightarrow \forall\,(x, y) \in X, f\,(x) = f\,(y). \end{aligned}$$

This implies that f is constant, as $f\,(e) = 0$, thus $f \equiv 0$.

Consider f, g in $\mathrm{Lip}_0(X)$. We have

$$\begin{aligned}
\mathrm{Lip}(f+g) &= \sup_{x\neq y} \frac{|f(x)+g(x)-(f(y)\,g(y))|}{d(x,y)} \\
&\leq \sup_{x\neq y} \frac{|f(x)-f(x)|+|(g(x)-g(y))|}{d(x,y)} \\
&\leq \sup_{x\neq y} \frac{|f(x)-f(x)|}{d(x,y)} + \frac{|g(x)-g(x)|}{d(x,y)} \\
&\leq \sup_{x\neq y} \frac{|f(x)-f(x)|}{d(x,y)} + \sup_{x\neq y} \frac{|g(x)-g(x)|}{d(x,y)} \\
&\leq \mathrm{Lip}(f)+\mathrm{Lip}(g)\,.
\end{aligned}$$

Let f be in $\mathrm{Lip}_0(X)$ and λ be in $\mathbb{R}$. One have

$$\begin{aligned}
\mathrm{Lip}(\lambda f) &= \sup_{x\neq y} \frac{|\lambda f(x)-\lambda f(y)|}{d(x,y)} \\
&= \sup_{x\neq y} \frac{|\lambda|\,|f(x)-f(x)|}{d(x,y)} \\
&= \lambda \mathrm{Lip}(f)\,.
\end{aligned}$$

This means that $(\mathrm{Lip}_0(X), \mathrm{Lip}(.))$ is a normed space.

We prove now that $(\mathrm{Lip}_0(X), \mathrm{Lip}(.))$ is a Banach space. We use this : a normed vector space is complete if, and only if, every absolutey convergent sequence (a sequence (f_n) in a normed vector space is said to converge absolutely if $\sum \|f_n\|$ converges) converges. Indeed, the forward direction of this is easy. To prove the reverse direction, let (g_n) be any Cauchy sequence, we must show that it converges.

Passing to sub-sequence, we may assume that $g_{n+1} - g_n < \frac{1}{2^{-n}}$ for all n . Then define $f_1 = g_1$ and, for $n > 1, f_n = g_n - g_{n-1}$. Evidently (f_n) is absolutely convergent, and since its nth partial sum is just g_n , the implication " absolutely convergent implies convergent " now entails that (g_n) converges.

Let (f_n) be a sequence in $\mathrm{Lip}_0(X)$ such that $\sum_{n=1}^{\infty} \mathrm{Lip}(f_n) < \infty$. For any $x \in X$ we have $|f_n| \leq \mathrm{Lip}(f_n)\,d(x,e) < \infty$. Thus (f_n) converges pointwise, and the sum f is Lipschitz by Proposition

1.2.3 . Letting $g_n = \sum_{k=1}^{n} f_k$ be the nth partial sum, we have

$$\mathrm{Lip}\,(f - g_n) = \mathrm{Lip}\left(\sum_{k=n+1}^{\infty} f_k\right) \leq \left(\sum_{k=n+1}^{\infty} \mathrm{Lip}\,(f_k)\right) \longrightarrow 0.$$

This shows that the series (f_n) converges to f in $\mathrm{Lip}_0\,(X)$. By the above, we conclude that $\mathrm{Lip}_0\,(X)$ is complete.

Let $(f_n)_{n\in\mathbb{N}}$ a Cauchy sequence in $\mathrm{Lip}_0\,(X)$. We have

$$\forall \epsilon > 0 \quad \exists n_0 \in \mathbb{N} : \forall m, n \geq n_0; \quad \mathrm{Lip}\,(f_m - f_n) < \epsilon \tag{1.2}$$

$$\mathrm{Lip}\,(f_m - f_n) = \sup_{x \neq y} \frac{|f_m(x) - f_m(y) - (f_n(x) - f_n(y))|}{d(x, y)} < \epsilon.$$

So , for every $x \in X$ $(f_m(x) - (f_n(x)))$ is a Cauchy in $\mathbb{R}$ and hence converges. Let $f(x)$ be its limit. We have

(a) $f(0) = \lim_{n\longrightarrow\infty} f_n(0) = 0.$

(b) Let x, y be in X . We have

$$\begin{aligned} |f(x) - f(y)| &= \lim_{n\longrightarrow\infty} |f_n(x) - f_n(y)| \\ &\leq \lim_{n\longrightarrow\infty} \mathrm{Lip}\,(f_n)\, d(x, y) \leq K d(x, y). \end{aligned}$$

Where $K = \mathrm{Lip}\,(f_n)$. Indeed, by (1.2)

$$|\mathrm{Lip}\,(f_n) - \mathrm{Lip}\,(f_m)| \leq \mathrm{Lip}\,(f_n - f_m) \leq \epsilon.$$

Hence $(\mathrm{Lip}\,(f_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and thus converges to K , so $f \in \mathrm{Lip}_0\,(X)$.

(c) (f_n) converges to f .

Consider $n \geq n_0$. We have $\mathrm{Lip}\,(f_n - f) = \lim_{m\longrightarrow\infty} \mathrm{Lip}\,(f_n - f_m) \leq \epsilon$ and hence $(f_n)_{n\in\mathbb{N}}$ converges to f.

This ends the proof. □

Remark 1.2.1. The space $\mathrm{Lip}_0(X)$ does not depend on the choice of base point.

If e_1 and e_2 are to different distinguished elements, then the linear map is a surjective isometry

$$\begin{array}{cccc} u: & \mathrm{Lip}_0(X,e_1) & \longrightarrow & \mathrm{Lip}_0(X,e_2) \\ & f & \longmapsto & f-f(e_2) \end{array}$$

- u well defined

$$\begin{aligned} u(f)(e_2) &= f(e_2)-f(e_2) \\ &= 0. \end{aligned}$$

  So $u(f)\in \mathrm{Lip}_0(X,e_2)$.

- u linear

$$\begin{aligned} u(f+g) &= f+g-(f(e_1)+g(e_2)) \\ &= f-f(e_1)+g-g(e_2) \\ &= u(f)+u(g) \\ u(\lambda f) &= \lambda f-\lambda f(e_2) \\ &= \lambda(f-f(e_2)) \\ &= \lambda u(f) \end{aligned}$$

- u isometry

$$\begin{aligned} \mathrm{Lip}(u(f)) &= \sup_{x\neq y}\frac{|u(f)(x)-u(f)(y)|}{d(x,y)} \\ &= \sup_{x\neq y}\frac{|f(x)-f(e_2)-(f(y)-f(e_2))|}{d(x,y)} \\ &= \sup_{x\neq y}\frac{|f(x)-f(y)|}{d(x,y)} \\ &= \mathrm{Lip}(f). \end{aligned}$$

- u surjective

  Consider g in $\mathrm{Lip}_0(X,e_2)$, the reciprocal image $g-g(e_2)=u(g)$ so $g-g(e_2)=g$.

Definition 1.2.4. Consider X, Y in $\mathcal{M}_0$ and let $T : X \longrightarrow Y$ be a Lipschitz map which preserver base point. We define $T^{\#} : \mathrm{Lip}_0(Y) \longrightarrow \mathrm{Lip}_0(X)$ by

$$T^{\#}(g)(x) = (g \circ T)(x) = g(T(x)).$$

The definition make sens by the property of composition maps.

Proposition 1.2.7. *Consider X, Y in $\mathcal{M}_0$ and let $T : X \longrightarrow Y$ be a Lipschitz map which preserver base point. Then $T^{\#}$ is a bounded linear map and $\|T^{\#}\| = \mathrm{Lip}(T)$. The map $T^{\#}$ is compatible with products and preserver order.*

Proof. We have

$$\mathrm{Lip}\left(T^{\#}(g)\right) = \mathrm{Lip}(g \circ T) \leq \mathrm{Lip}(g)\,\mathrm{Lip}(T)$$

so $\|T^{\#}\| \leq \mathrm{Lip}(T)$. For the converse inequality, fix $p, q \in Y$. Let $g = d_Y(., p) - d_Y(e_Y, q)$ then $\mathrm{Lip}(g) = 1$ and

$$\begin{aligned}
\|T^{\#}\| &\geq \mathrm{Lip}\left(T^{\#}(g)\right) \\
&\geq \frac{\left|T^{\#}(g)(x) - T^{\#}(g)(y)\right|}{d_X(x,y)} \\
&\geq \frac{|gT(x) - gT(y)|}{d_X(x,y)} \\
&\geq \frac{|gT(x) - gT(y)|}{d_Y(T(x), T(y))} \frac{d_Y(T(x), T(y))}{d_X(x,y)}.
\end{aligned}$$

Taking the supremum over x and y, we find $\|T^{\#}\| \geq \mathrm{Lip}(T)$. □

1.3 Retract spaces

The notion of Lipschitz retract in metric spaces is like linear projection in Banach spaces.

Definition 1.3.1. Let X be a metric space and let E be a subspace of X. A Lipschitz map $p : X \longrightarrow E$ is called a Lipschitz retraction if $p/E = Id$. In this case we say that E is a Lipschitz retract of X.

A metric space E is called an absolute Lipschitz retract if it is a Lipschitz retract of every metric space containing it.

(E absolute Lipschitz retract $\Leftrightarrow \forall X \supset E : \exists \quad p : X \longrightarrow E$ Lipschitz retraction)

$$\begin{array}{ccc} X & & \\ & p & \\ i\uparrow & \searrow & \\ E & \xrightarrow{f} & E \end{array}$$

Proposition 1.3.1. *Let Y be a metric space. Then, the following properties are equivalent*

(i) *The space Y is an absolute retract space.*

(ii) *For every metric space X , for every subset $E \subset X$ and every Lipschitz function $f : E \longrightarrow Y$ can be extended to a Lipschitz function $\tilde{f} : X \longrightarrow Y$.*

$$\begin{array}{ccc} X & & \\ & \widetilde{f} & \\ i\uparrow & \searrow & \\ E & \xrightarrow{f} & Y \end{array}$$

(iii) *For every metric space Z containing Y and for every metric space F , then every Lipschitz function $f : Y \longrightarrow F$ can be extended to a Lipschitz function $\tilde{f} : Z \longrightarrow F$.*

$$\begin{array}{ccc} Z & & \\ & \widetilde{f} & \\ i\uparrow & \searrow & \\ Y & \xrightarrow{f} & F \end{array}$$

1.4 The unit ball $\mathcal{B}_{X^\#}$ is compact

1.4.1 Product Topology

Let $(X_i, \mathcal{T}_i)_{i\in I}$ be a net of topological spaces. We note by

$$X = \prod_{i\in I} X_i.$$

The product topology of X noted $\mathcal{T}$ is the least fine topology making projections continuous

$$\begin{array}{cccc} p_i : & X & \longrightarrow & X_i \\ & (x_i)_{i\in I} & \longmapsto & x_i \end{array}$$

The least fine,$i.e.$, having the fewest openings. The elementary openings of the product topology are of the form

$$\bigcap_{j\in J} p_j^{-1}\left(\mathcal{U}_j\right), \quad J \text{ (finite)} \subset I.$$

Remark 1.4.1. (i) The projection p_i is an open application.

(ii) An application $f:(Y,\sum)\longrightarrow(X,\mathcal{T})$ is continuous if, and only if, $p_i\circ f$ is continuous for every i in I.

The celebrate theorem in the product topology is the theorem of Tychonov(ff).

Theorem 1.4.1 (Tychonov)**.** *A product space product $X = \prod\limits_{i\in I} X_i$ is compact if, and only if, X_i is compact for all i in I. In other words, the topological product of any number of compact spaces is compact.*

Pointwise convergence is the same as convergence in the product topology on the space Y^X, where X is the domain and Y is the codomain. If the codomain Y is compact, then, by Tychonov's theorem, the space Y^X is also compact.

Let (X,d,e) be a pointed metric space. The topology $\mathcal{T}_p$ of pointwise convergence is the topology induced by the product $\mathbb{R}^X$ and determinates by the condition

$$f_i \xrightarrow{\mathcal{T}_p} f \Longleftrightarrow \forall x\in X, \quad f_i\left(x\right)\longrightarrow f\left(x\right)$$

for any net $(f_i)_{i\in I}$in $\mathbb{R}^X$ and $f\in\mathbb{R}^X$.

Let now giving the analog of the Aloaglu (1940 for every Banach spaces)-Banach (1932 for separable Banach spaces) theorem for the unit bull $\mathcal{B}_{X^\#}$ of $\mathrm{Lip}_0\left(X\right)$.

1.4.2 Compactness of $\mathcal{B}_{X^\#}$ is compact

Proposition 1.4.1. *The unit bull $\mathcal{B}_{X^\#}$ is compact for the topology $\mathcal{T}_p$.*

Proof. Observe that $\mathcal{B}_{X^\#}$ is closed in $\mathbb{R}^X$ with respect the topology $\mathcal{T}_p$. Indeed, consider a net $(f_i)_{i\in I}$in $\mathbb{R}^X$ such that

$$f_i \xrightarrow{\mathcal{T}_p} f \in \mathbb{R}^X.$$

For x, y in X, the inequality

$$|f_i(x) - f_i(y)| \leq d(x, y)$$

implies that

$$|f(x) - f(y)| \leq d(x, y)$$

and consequently $f \in \mathcal{B}_{X^\#}$.

Let now $f \in \mathcal{B}_{X^\#}$. We have

$$|f(x)| \leq d(x, e), \quad \forall x \in X.$$

This shows that

$$f \in \prod_{x\in X} [0, d(x, e)]$$

and this implies

$$\mathcal{B}_{X^\#} \subset \prod_{x\in X} [0, d(x, e)].$$

The space $\prod_{x\in X} [0, d(x, e)]$ is compact by Tychonov's theorem and $\mathcal{B}_{X^\#}$is closed so it is compact (closed of compact is compact). □

1.5 Conjugate space

For more details on this section, we can consult [9].

Let E be a Banach space. We say that E is a conjugate space if there exists a Banach space B

such that B^* is isometrically isomorphic to E (i.e., $B^* \equiv E$). We now give a simple sufficient condition to generate that space B exists.

Let us recall that a family of seminorms on a linear space generates a locally convex topology in the following sense.

Theorem 1.5.1. *Let $\{p_i : i \in I\}$ be a family of seminorms on the linear space E. Let $\mathcal{U}$ be the class of all finite intersections of sets of the form*

$$\{x \in E : p_j(x) < r_j\}$$

where $j \in J$ (finite) $\subset I$, $r_j > 0$. Then $\mathcal{U}$ is a local base for a topology $\mathcal{J}$ that makes E a locally convex topological vector space. This topology is the weakest making all the p_i continuous, and for a net $\{x_\alpha\} \subset E$, $x_\alpha \to x$ in J if, and only if, $p_i(x_\alpha - x) \to 0$ for each $i \in I$.

Theorem 1.5.2. ***(Dixmier-Ng theorem)***

Let E be a Banach space. Suppose that there is a (Hausdorff) locally convex topology σ on E such that $\mathcal{B}_E$ is σ -compact. Then E is a conjugate space.

Proof. Let $B = \left\{\varphi \in E' \varphi|\mathcal{B}_E \text{ is } \sigma - \text{continuous}\right\} \left(E' = \text{algebraic conjugate space}\right)$. Then B is a closed linear subspace of E^*, and is therefor a Banach space. (To see that $B \subset E^*$ observe that for any $\varphi \in B$ the image $\varphi\left(\mathcal{B}_E\right)$ is compact hence bounded set of scalar; that is, $\|\varphi\|$ is finit and so $\varphi \in E^*.B$ is closed in E^* because convergence in E^* entails uniform convergence on $\mathcal{B}_E$) We now bring in the (canonical embedding) operator $J_{E,B} : E \longrightarrow B^*$ defined by

$$< \varphi, J_{E,B}\left(x\right) >= \varphi\left(x\right)$$

this operator assigns to each $x \in X$ the functional " evaluation at x " in B^* , we clearly have $\|J_{E,B}\left(x\right)\| \leq 1$. The proof will be completed by showing that $J_{E,B}\left(x\right)$ is an isomorphic isometry between E and B^*. We do this by showing that $J_{E,B}\left(x\right)$ is injective and that it maps $\mathcal{B}_E$ onto $\mathcal{B}_{B^*}$. The first assertion follows because B is total. Indeed B contains the dual space E; which certainly separates the points of E. The second assertion follows from the fact (evident by definition of B) that $J_{E,B}$ is continuous from the σ-topology on E into the weak*-topology on B^*. This means in particular that $J_{E,B}(\mathcal{B}_E)$ is ω^*-compact in B^*. But, by the Goldstine-Weston density lemma, this image is also weak*-dense in $\mathcal{B}_{B^*}$. □

Example 1.5.1. Consider the space $\mathrm{Lip}(X, d, \mathbb{R})$ of bounded Lipschitz functions defined on the metric space (X, d) and normed by

$$\|.\|_{\mathrm{L}} = \max\{\|.\|_\infty, \mathrm{Lip}(.)\}.$$

Let σ be the topology of pointwise convergence on X , which we denote by $\sigma(\mathrm{Lip}(X, d, \mathbb{R}), X)$, then $\mathcal{B}_X$ is certainly a $\sigma(\mathrm{Lip}(X, d, \mathbb{R}), X)$ -closed subset of X . We have

$$\mathcal{B}_{\mathrm{Lip}(X,d,\mathbb{R})} \subset [-1, 1]^X.$$

Since $[-1, 1]$ is compact by Tychonov's theorem we have $[-1, 1]^X$. consequently , $\mathcal{B}_{\mathrm{Lip}(X,d,\mathbb{R})}$ is $\sigma(\mathrm{Lip}(X, d, \mathbb{R}), X)$ compact and so X is a conjugate space .
Let $X^\# = \mathrm{Lip}_0(X)$. Let τ_p be the topology of pointwise convergence on $X^\# i.e., \tau_p = \sigma\left(X^\#, X\right)$. Then $\mathcal{B}_{X^\#}$ is certainly a τ_p -closed subset of $X^\#$. (And this what have we seen the previous title) so $X^\#$ is a conjugate space .

1.6 $\mathrm{Lip}_0(X)$ is dual

We have seen that the unit ball $\mathcal{B}_{X^\#}$ is $\mathcal{T}_p$ -compact and according to " Dixmier-Ng theorem " $\mathrm{Lip}_0(X)$ is a dual space , for every $X \in \mathcal{M}_0$.

Theorem 1.6.1. *The space $X^\#$ is a dual space.*

Proof. By Dixmier-Ng's theorem, it suffices to prove that $\mathcal{T}_p$ is Hausdorff locally convex.

(i) The topology $\mathcal{T}_p$ is locally convex.

(ii) The topology $\mathcal{T}_p$ is separating.

(i) Define

$$p_x(f) = |f(x)|, \quad x \in E \text{ and } f \in \mathcal{B}_{X^\#}$$

and put $P = \{p_x\}_{x \in E}$. By the precedent theorem, the topology defined by P is locally convex and it is exactly the topology of pointwise convergence $\mathcal{T}_p$.

(ii) The topology $\mathcal{T}_p$ is a Hausdorff topology if, and only if, the family $\{p_x\}_{x \in E}$ is separating, $i.e.$, given $f \neq 0$, there exists $x \in E$ such that $p_x(f) \neq 0$.

This is the case and this ends the proof. □

CHAPTER 2

PESTOV'S THEOREM

In this chapter, we will introduce the predual of dual Banach space $\mathrm{Lip}_0(X)$ is Arens-Eells space denoted by $\text{Æ}(X)$, due to "Pestov's theorem", we will also introduce Free Banach space and a main principals of this space to proof that they are isometrically isomorphic to $\mathrm{Lip}_0(X)$, as well as another construction, all of those concepts can be found in the references [1], [11], [14], [10], [13], [7], [5] for more details.

2.1 Arens-Eells space

This space was first introduced by Arens and Eelles [1] in 1956 .Originally, the basic idea goes back to Kantorovich [11]. The terminology Arens-Eelles $Æ(X,d)$ is due to Weaver [14].

Let (X,d,e) be a pointed a metric space . A molecule on X is a real valued function m on X with finite support ($i.e.$, the set where m has non-zero values) and satisfies

$$\sum_{x\in supp(m)} m\left(x\right)=0$$

Denote by $\mathcal{M}\left(X\right)$ the real linear space of molecules on X . We can write

$$\begin{aligned} m &= \sum_{x\in supp(m)} m\left(x\right)\mathbf{1}_{\{x\}} \\ &= \sum_{i=1}^{n} m\left(x_i\right)\mathbf{1}_{\{x_i\}} \end{aligned}$$

Where $supp\left(m\right)=\{x_1,x_2,.....,x_n\}$ and $\mathbf{1}_{\{x\}}$ denotes the characteristic function of the set $\{x\}$. For $x,y\in X$ we define the basic molecule $m_{x_1x_2}=\mathbf{1}_{\{x_1\}}-\mathbf{1}_{\{x_2\}}$ (with $x_1,x_2\in X$ are called atoms). It is easy to see that every molecule m can be written as a (non unique) finite linear combination of basic molecule (the condition $\sum_{i=1}^{n} m\left(x_i\right)=0$ insures that such representations of m exist $m=\lambda_1 m_{x_1x_2}+\left(\lambda_1+\lambda_2\right)m_{x_2x_3}+....+\left(\lambda_1+....+\lambda_{n-1}\right)m_{x_{n-1}x_n}$). We have

$$\begin{aligned} m &= \sum_{j=1}^{l}\alpha_j\left(\mathbf{1}_{\{x_j\}}-\mathbf{1}_{\{y_j\}}\right) \\ &= \sum_{j=1}^{l}\alpha_j m_{x_jy_j}. \end{aligned}$$

Example 2.1.1. Consider $m:\mathbb{R}\longrightarrow\mathbb{R}$ such that

$$\begin{cases} m\left(0\right)=-4; \\ m\left(1\right)=1; \\ m\left(2\right)=3; \\ 0 \quad otehwise. \end{cases}$$

$$\begin{aligned} m &= -4.\mathbf{1}_{\{0\}}+1.\mathbf{1}_{\{1\}}+3.\mathbf{1}_{\{2\}} \\ &= -3.\mathbf{1}_{\{0\}}-1.\mathbf{1}_{\{0\}}+1.\mathbf{1}_{\{1\}}+3.\mathbf{1}_{\{2\}} \\ &= 1\left(\mathbf{1}_{\{1\}}-\mathbf{1}_{\{0\}}\right)+3\left(\mathbf{1}_{\{2\}}-\mathbf{1}_{\{0\}}\right). \end{aligned}$$

Put now

$$\|m\|_{\mathcal{M}(X)} = \inf\left\{\sum_{j=1}^{l} |\alpha_j| \, d\left(x_j, y_j\right)\right\}$$

over all representation of $m = \sum_{j=1}^{l} \alpha_j \left(\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{y_j\}}\right)$.

If follows that $\|.\|_{\mathcal{M}(X)}$ is a norm on the vector space $\mathcal{M}\left(X\right)$. Denote by $Æ(X, d_X)$ the completion of the normed space $\left(\mathcal{M}\left(X\right), \|.\|_{\mathcal{M}(X)}\right)$.

Remark 2.1.1. Every molecule m is uniquely expressible in the form

$$\sum_{j=1}^{l} \alpha_j \left(\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{e\}}\right)$$

where the points x_j are all distinct and none equals to e .

Indeed , suppose that there is two representations

$$\sum_{j=1}^{l} \lambda_j \left(\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{e\}}\right) = \sum_{j=1}^{l} \alpha_j \left(\mathbf{1}_{\{y_j\}} - \mathbf{1}_{\{e\}}\right)$$

where $x_i \neq x_j \neq e$.

We now prove that $(Æ(X))^* \overset{isometrically}{\equiv} \mathrm{Lip}_0\left(X\right)$.

Theorem 2.1.1. $(Æ(X))^*$ *is isometrically isomorphic to* $\mathrm{Lip}_0\left(X\right)$.

Proof. Define

$$S : Æ^*(X, d) \longrightarrow \mathrm{Lip}_0\left(X\right)$$

by

$$\left(S\varphi\right)\left(x\right) = \varphi\left(\mathbf{1}_{\{x\}} - \mathbf{1}_{\{e\}}\right).$$

Since $\left\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{x'\}}\right\|_{Æ(X,d)} = d\left(x, x'\right)$ for all $x, x' \in X$, we have

$$\begin{aligned}
\left|\left(S\varphi\right)\left(x\right) - \left(S\varphi\right)\left(x'\right)\right| &= \left|\varphi\left(\left(\mathbf{1}_{\{x\}} - \mathbf{1}_{\{e\}}\right)\right) - \varphi\left(\left(\mathbf{1}_{\{x'\}} - \mathbf{1}_{\{e\}}\right)\right)\right| \\
&= \left|\varphi\left(\left(\mathbf{1}_{\{x\}} - \mathbf{1}_{\{x'\}}\right)\right)\right| \\
&\leq \|\varphi\| \, d\left(x, x'\right).
\end{aligned}$$

Also $(S\varphi)(e) = \varphi(0)$, so indeed $S\varphi \in \mathrm{Lip}_0(X)$. It follows that S is a non expansive linear mapping from $Æ^*(X,d)$ to $\mathrm{Lip}_0(X)$ $i.e.$, $\mathrm{Lip}(S\varphi) \leq \|\varphi\|_{Æ^*}$.

Define now $R : \mathrm{Lip}_0(X) \longrightarrow Æ^*(X,d)$ by

$$(Rf)(m) = \sum_x m(x) f(x)$$

for $f \in \mathrm{Lip}_0(X)$ and m a molecule. If $m = \sum_{j=1}^{l} \lambda_j \left(\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{x'_j\}}\right)$, we have

$$\begin{aligned}
|(Rf)(m)| &= \left|\left(\sum_x m(x) f(x)\right)\right| \\
&\leq \left|\sum_{j=1}^{l} \lambda_j f(x_i) - f\left(x'_j\right)\right| \\
&\leq \sum_{j=1}^{l} |\lambda_j| \left|f(x_j) - f\left(x'_j\right)\right| \\
&\leq \mathrm{Lip}(f) \sum_{j=1}^{l} |\lambda_j| \, d\left(x_j, x'_j\right).
\end{aligned}$$

Hence $|(Rf)(m)| \leq \mathrm{Lip}(f) \|m\|_{\mathcal{M}(X)}$, which uniquely extends to a continuous linear functional on the completion $Æ(X,d)$ of $\mathcal{M}(X)$, denoted by the same symbol Rf . Thus $Rf \in Æ^*(X,d)$, and $\|Rf\| \leq \mathrm{Lip}(f)$.

Straightforward calculations show that R and S are inverses . Indeed , for all $x \in X$

$$\begin{aligned}
(S \circ R)(f)(x) &= S(R(f))(x) \\
&= R(f)\left(\mathbf{1}_{\{x\}} - \mathbf{1}_{\{e\}}\right) \\
&= f(x)
\end{aligned}$$

and for all $m \in \mathcal{M}(X)$

$$\begin{aligned}(R \circ S)(\varphi)(m) &= R(S(\varphi))(m) \\ &= \sum_{x} m(x) S(\varphi)(x) \\ &= \sum_{j=1} \lambda_j \left(S(\varphi)(x_j) - S(\varphi)(x'_j)\right) \\ &= \sum_{j=1}^{l} \lambda_j \varphi \left(\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{x'_j\}}\right) \\ &= \varphi(m).\end{aligned}$$

The operators R, S are non expansive and $R \circ S = S \circ R = Id$, so S is isometric ($\|x\| = \|(R \circ S)(x)\| \leq \|R\| \|S(x)\| \leq \|S(x)\|$) and hence $\mathrm{Lip}_0(X)$ is isomorphic to $Æ^*(X, d)$ (for more information see [4]). □

Proposition 2.1.1. *Let (X, e, d) be pointed metric space.*

(i) *For any molecule m we have*

$$\|m\|_{Æ(X,d)} = \sup \left\{ |\langle m, f \rangle| = \left| \sum_{x \in X} m(x) f(x) \right| : f \in \mathcal{B}_{X^\#} \right\}$$

and there exists $f \in \mathcal{B}_{X^\#}$ such that $\langle m, f \rangle = \|m\|_{Æ(X,d_X)}$.

(ii) $\|.\|_{Æ(X,d_X)}$ *is a norm on $\mathcal{M}(X)$ and $\left\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\right\|_{Æ} = d(x, y)$ for all x, y in X .*

(iii) $\|.\|_{Æ(X,d)}$ *is the largest semi norm on $\mathcal{M}(X)$ which satisfies for all x, y in $X, \mathcal{M}(X)$ and $\left\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\right\|_{Æ} = d(x, y)$.*

Proof. (i) This follows from the identification of $\mathrm{Lip}_0(X, d)$ with $Æ(X, d)^*$ and the Hahn-Banach theorem.

(ii) The inequality $\left\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\right\|_{Æ} \leq d(x, y)$ follows from the definition. Conversly, fix x in X and define

$$f_x(y) = d(x, y) - d(x, e).$$

We have $f_x \in \mathcal{B}_{\mathrm{Lip}_0(X,d)}$ because $f_x(e) = 0$ and $\mathrm{Lip}(f_x) = 1$. Indeed

$$\begin{aligned}\mathrm{Lip}(f_x) &= \sup_{y_1 \neq y_2} \frac{|f_x(y_1) - f_x(y_2)|}{d(y_1, y_2)} \\ &\geq \sup_{x \neq y} \frac{|f_x(y) - f_x(x)|}{d(x, y)} \\ &\geq \frac{d(x, y)}{d(x, y)} = 1\end{aligned}$$

and

$$\begin{aligned}\mathrm{Lip}(f_x) &= \sup_{y_1 \neq y_2} \frac{|f_x(y_1) - f_x(y_2)|}{d(y_1, y_2)} \\ &= \sup_{y_1 \neq y_2} \frac{|d(x, y_1) - d(x, y_2)|}{d(y_1, y_2)} \\ &= \frac{d(y_1 y_2)}{d(y_1, y_2)} = 1.\end{aligned}$$

By part (i) , we have

$$\begin{aligned}\left\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\right\|_{Æ} &\geq |\langle m_{xy}, f_x\rangle| \\ &\geq |m_{xy}(x) f_x(x) + m_{xy}(y) f_x(y)| \\ &\geq |-m_{xy}(x) d(x, e) + m_{xy}(y) d(x, y) + m_{xy}(y) d(x, e)| \\ &\geq |m_{xy}(y) d(x, y)| \\ &\geq d(x, y).\end{aligned}$$

(iii) Let $\|.\|_0$ be any semi norm such that

$$\left\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\right\|_0 \leq d(x, y)$$

for all $x, y \in X$. Let $m = \sum_{i=1}^{n} \alpha_i m_{x_i y_i}$ be a molecule. We have

$$\begin{aligned}\|m\|_0 &= \left\|\sum_{i=1}^{n} \alpha_i m_{x_i y_i}\right\|_0 \\ &\leq \sum_{i=1}^{n} |\alpha_i| \, \|m_{x_i y_i}\|_0 \\ &\leq \sum_{i=1}^{n} |\alpha_i| \, d(x_i, y_i).\end{aligned}$$

Taking the infimum of all such representation of m yields $\|m\|_0 \leq \|m\|_{\AE}$.

This ends the proof. □

Corollary 2.1.1. *The application* $i_X : X \longrightarrow \AE(X, d)$ *defined by*

$$i_X(x) = \mathbf{1}_{\{x\}} - \mathbf{1}_{\{e\}} = m_{xe}$$

is an isometric embedding of X *into* $\AE(X, d_X)$ *.*

Proof. We have by (ii) in the precedent proposition

$$\|i_X(x) - i_X(y)\|_{\AE} = \left\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\right\|_{\AE} = d(x, y)$$

for all $x, y \in X$. So i_X is an isometry. □

The following theorem is known as the linearization of Lipschitz operators

Theorem 2.1.2. *Let* (X, d, e) *be a pointed metric space. Let* E *be a Banach space and let* $T : X \longrightarrow E$ *be a Lipschitz map which preserves base points (i.e.,* $T(e) = 0$*) . Then there is a unique bounded linear operator* $u : \AE(X) \longrightarrow E$ *such that* $T = u \circ i_X$ *and* $\|u\| = \operatorname{Lip}(T)$ $(i_X : X \longrightarrow \AE(X))$*.*

$$\begin{array}{ccc} \AE(X) & & \\ & u & \\ i_X \uparrow & \searrow & \\ X & \overset{T}{\longrightarrow} & E \end{array}$$

The operator u *is denoted by* T_L*.*

2.2 Free Banach spaces

There are a different notation and appelation was used in [10] by Godefroy and Kalton. It is the Lipschitz-free space denoted by $\mathcal{F}(X)$ which we introduce in the sequel. Similar space was introduced by Pestov in [13] under the name free Banach space.

2.2.1 Banach Free spaces

The following theorem was independently proved by Flood in [7], Pestov in [13] and Weaver in [14].

Theorem 2.2.1. *Let (X, d, e) be a pointed metric space, then there exist a unique up to an isomorphism a Banach space $B(X)$ over the field $\mathbb{F}$ and an isometric embedding*
$i_X : X \longrightarrow B(X)$ such that

(i) *The linear span of $i_X(X)$ is dense in $B(X)$.*

(ii) *Every map T in $\mathrm{Lip}_0(X, E)$ can be extended to a continuous linear operator $T^L : B(X) \longrightarrow E$ such that $\|T^L\| = \mathrm{Lip}(T)$ for any arbitrary normed space E .*

2.2.2 Lipschitz free spaces

It is proved without any reference to molecules that the closed linear subspace of $\left(X^{\#}\right)^*$ spanned by the evaluation function $\delta_x : X \longrightarrow \mathbb{K}$, given by

$$\delta_x(f) = f(x) ; x \in X$$

is a predual of $X^{\#}$ (we note that any weak*-closed linear subspace B of a conjugate space E^* is itself a conjugate space. This follows from the observation that $\mathcal{B}_B$ is compact in the (relative) weak*-topology). This space was called Lipschitz-free space and denoted $\mathcal{F}(X)$ by Godefroy and Kalton in [10].

Definition 2.2.1. The Lipschitz free space on X is

$$\mathcal{F}(X, d_X) = \overline{span\{\delta_x, x \in X\}}^{\mathrm{Lip}_0\{X)^*}.$$

We say that $\gamma \in \mathcal{F}(X, d_X)$ is finitely supported if

$$\gamma \in span\{\delta_x, x \in X\}.$$

Then, the support of such a γ (denoted $supp\gamma$) is the smallest subset F of X which contains e and such that $\gamma \in span\{\delta_x, x \in X\}$.

Remark 2.2.1. By applying the bipolar theorem , we give a precise description of $\mathcal{B}_{\mathcal{F}(X)}$ by means of the Lipschitz evaluation functionals $\delta_{(x,y)} = \dfrac{\delta_x - \delta_y}{d(x,y)}$ defined on $X^{\#}$, where (x, y) runs through $\hat{X} = \{(x, y) \in X^2 : x \neq y\}$

(i) The closed unit ball of $\mathcal{F}(X)$ is the closed, convex, balanced hull of the set $\left\{\delta_{(x,y)} : (x,y) \in \hat{X}\right\}$ in $\left(X^{\#}\right)^{*}$.

(ii) The space $\mathcal{F}(X)$ is the closed linear hull of the set $\{\delta_x : x \in X\}$ in $\left(X^{\#}\right)^{*}$.

(iii) From (i) , we deduce that $\mathcal{F}(X)$ is the closed linear hull in $\left(X^{\#}\right)^{*}$ this set $\left\{\delta_{(x,y)} : (x,y) \in \hat{X}\right\}$. Then (ii) follows since the linear hulls of this set and the set $\{\delta_x : x \in X\}$ coincide. Notice that $\delta_x = \delta_x - \delta_0 = d(x,0)\,\delta_{(x,0)}\,(x \in X, x \neq 0)$.

Proposition 2.2.1. *Define*

$$\begin{array}{cccc} \delta & : & X & \longrightarrow & \left(X^{\#}\right)^{*} \\ & & x & \longmapsto & \delta_x \end{array}$$

The application δ is an isometry, i.e., for every x_1, x_2 in X , one have $\|\delta_{x_1} - \delta_{x_2}\| = d(x_1, x_2)$ (thies implies that $\|\delta_x\| = d(x,0)$.)

Proof. For $x_1, x_2 \in X$, we have in the first part

$$\begin{aligned} \|\delta_{x_1} - \delta_{x_2}\| &= \sup_{\mathrm{Lip}(f)=1} |\delta_{x_1}(f) - \delta_{x_2}(f)| \\ &= \sup_{\mathrm{Lip}(f)=1} |f(x_1) - f(x_2)| \\ &\leq d(x_1, x_2). \end{aligned}$$

In the second part, for a fixed $x_0 \in X$, let $g \in \mathcal{B}_{X^{\#}}$ defined by

$$g(x) = d(x, x_1) - d(x_0, x_1).$$

We have

$$\begin{aligned} \|\delta_{x_1} - \delta_{x_2}\| &\geq g(x_1) - g(x_2) \\ &\geq d(x_1, x_2) \end{aligned}$$

and this ends the proof. □

Proposition 2.2.2. *For any metric space $X, \mathcal{F}(X, d_X)^{*} \equiv \mathrm{Lip}_0(X)$.*

Proof. We define a linear surjective isometry J on $\mathrm{Lip}_0(X)$ with values in $\mathcal{F}(X, d_X)^*$ by $J(f)(\delta_x) = f(x)$ and we extend by continuity to $\mathcal{F}(X, d_X)$. Consider f in $\mathrm{Lip}_0(X)$ and m in *span* $\{\delta_x, x \in X\}$ such that $m = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$. $J(f)(m) = \sum_{i=1}^{n} \alpha_i f(x_i)$. We show that J is a surjective isometry.

$a)$ Consider f in $\mathrm{Lip}_0(X)$ and m in $\mathcal{F}(X, d_X)$. We have

$$|J(f)(m)| = \left|\langle f, m\rangle_{(\mathrm{Lip}_0(X), \mathcal{F}(X))}\right| = \left|\langle f, m\rangle_{(\mathrm{Lip}_0(X), \mathrm{Lip}_0(X)^*)}\right| \leq \mathrm{Lip}(f)\, \|m\|_{\mathcal{F}(X)}$$

and we obtain $\|J(f)\| \leq \mathrm{Lip}(f)$.

$b)$ Let (x, y) be in $\tilde{X}$ and put $m = \dfrac{\delta_x - \delta_y}{d(x, y)}$. We have $\|m\|_{\mathcal{F}(X)} = 1$ because δ is an isometry see Proposition 2.2.1 above and

$$\begin{aligned} \|J(f)\|_{\mathcal{F}(X,d)^*} &\geq |J(f)(m)| \\ &\geq \left|\frac{f(x) - f(y)}{d(x, y)}\right| \\ (\text{ we take the } \sup) &\geq \mathrm{Lip}(f). \end{aligned}$$

$c)$ Consider $\varphi \in \mathcal{F}(X, d)^*$. Then φ is determinated by δ_x for every x in X, we put for every x in X, $f(x) = \varphi(\delta_x)$ and we prove that f is Lipschitz and $J(f) = \varphi$.

(i) We show that $f \in \mathrm{Lip}_0(X)$.

- $f(0) = \varphi(\delta_0) = \varphi(0) = 0.$

- Let x, y be in X

$$\begin{aligned} |f(x) - f(y)| &= |\varphi(\delta_x) - \varphi(\delta_y)| \\ &= |\langle \varphi, \delta_x - \delta_y\rangle| \\ &\leq \|\varphi\|_{\mathcal{F}(X,d)^*} \|\|\|_{(\mathrm{Lip}_0(X))^*} \\ &\leq \|\varphi\|_{\mathcal{F}(X,d)^*}\, d(x, y). \end{aligned}$$

(ii) Let $m = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ be in *span* $\{\delta_x : x \in X\}$. Then $\varphi(m) = \sum_{i=1}^{n} \alpha_i f(x_i) = J(f)(m)$

and ends the proof. □

Remark 2.2.2. By Theorem 2.2.1 of Pestov, the predual of $X^{\#}$ provided by the Dixmier-Ng theorem coincides with the Lipschitz-free space of X, *i.e.*, $\mathcal{F}(X,d)$ is isometrically isomorphic to Æ(X,d).

Theorem 2.2.2. *Let (X,d,e) be a pointed metric space and let E be a Banach space. Let $T: X \longrightarrow E$ be a Lipschitz map such that $T(e)=0$. Then, there is a unique linear map u (noted T_L) : $\mathcal{F} \longrightarrow E$ with $\|T_L\| = \mathrm{Lip}(T)$ and such that the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ \delta_X \downarrow & \nearrow & \\ & T_L & \\ \mathcal{F}(X) & & \end{array}$$

2.3 Another construction

We can regard that $\mathcal{F}(X)$ as the completion of the set of Borel measures μ on X For more information on this, you can consult [10]

2.3.1 Another interpretation of $\mathcal{F}(X)$

Let (X,d,e) be a pointed metric space. A functional

$$\sum_{i=1}^{n} \alpha_i \delta_{x_i}$$

from $span\{\delta_x : x \in X\}$ can be viewed as a Borel measure μ on X with finite support $\{x_1,...,x_n\}$ acting by the rule $\mu(x_i)=\alpha_i$ and $\mu(Y)=0$ for any Borel subset Y of X with $Y \cap \{x_1,...,x_n\} = \emptyset$. Then

$$\int_X f d\mu = \sum_{i=1}^{n} \alpha_i f(x_i) = \left(\sum_{i=1}^{n} \alpha_i \delta_{x_i}\right)(f)$$

for every $f \in \mathrm{Lip}_0(X)$. So the space $\mathcal{F}(X)$ will be exactly the completion of the set of Borel measures μ on X with finite support under the norm

$$\|\mu\| = \sup\left\{\left|\int_X f d\mu\right| : f \in \mathcal{B}_{X^\#}\right\}$$

It now follows that any finite Borel measure μ supported on a compact subset K of X can be identified with a member of $\mathcal{F}(X)$. Indeed,

$$\mu = \int_K \delta(x)\, d\mu(x)$$

and

$$\mu(f) = \int_K f d\mu, \quad f \in \mathrm{Lip}_0(X).$$

2.3.2 Properties

Lemma 2.3.1. *Let X, Y be two Banach spces. If $T : Y^* \longrightarrow X^*$is an opeartor $w^* - w^*$ continuous, then there is an operator $S : X \longrightarrow Y$ such that $T = S^*$.*

Theorem 2.3.1. *Let X and Y be Banach spaces and suppose $T : X \to Y$ is a Lipschitz map such that $T(0) = 0$. There exists a unique linear map $\widehat{T} : \mathcal{F}(X) \to \mathcal{F}(Y)$ such that $\widehat{T}\delta_X = \delta_Y T$, i.e., the following diagram commutes:*

$$\begin{array}{ccc} X & \overset{T}{\longrightarrow} & Y \\ \delta_X \downarrow & & \downarrow \delta_Y \\ \mathcal{F}(X) & \overset{\widehat{T}}{\longrightarrow} & \mathcal{F}(Y) \end{array}$$

and $\left\|\widehat{T}\right\| = \mathrm{Lip}(T)$.

Proof. The linear map $T^\# : \mathrm{Lip}(Y) \longrightarrow \mathrm{Lip}(X)$ defined by $T^\#(F) = F \circ T$ is pointwise-to-pointwise continuous, hence by previous lemma there is a linear map $\widehat{T}$ between the preduals such that $\left(\widehat{T}\right)^* = T^\#$. It is clear that $\|T^\#\| = \mathrm{Lip}(T)$, and $\left\|\left(\widehat{T}\right)^*\right\| = \left\|\widehat{T}\right\| = \|T^\#\|$. The other assertions are clear. □

If μ is a measure of finite support on X we can define its Brycenter $\beta(\mu) = \beta_X(\mu) \in X$ by

$$beta(\mu) = \int_X x d\mu.$$

Note that if $x^* \in X^*$, we have

$$|\langle beta\,(\mu)\,, x^*\rangle| \leq \|x^*\|\,\|\mu\|_{\mathcal{F}(X)}\,.$$

and so β extends is a bounded linear operator

$$\beta : \mathcal{F}\,(X) \longrightarrow X.$$

Lemma 2.3.2. *Let X be a Banach space. Then β is a linear quotient map and is a left measure of $\delta, i.e., \beta\delta = Id_X$.*

If μ is a measure of finite support on X, we can define its If we define $\overline{T} = \beta_Y \widehat{T}$, we deduce from the precedent theorem the corollary.

Corollary 2.3.1. *Let T be a Lipschitz map from a Banach space X to a Banach space Y such that $T(0) = 0$. There exists a unique linear map $T : \mathcal{F}\,(X) \rightarrow Y$ such that $\overline{T}\delta_X = T$, and $\left\|\overline{T}\right\| = \mathrm{Lip}\,(T)$.*

CHAPTER 3

THE LITTLE LIPSCHITZ SPACE

In this chapter we interest to define the little Lipschitz space $\mathrm{lip}_0\,(X)$ and its some properties, in order to prove that space is predual of Æ(X). For more details and informations see [5], [14].

3.1 Definition

Let (X, d, e) be a pointed metric space. A Lipschitz function $f : X \longrightarrow \mathbb{R}$ is called little Lipschitz if

$$\lim_{\delta \longrightarrow 0^{+}} \sup \{ |f(x) - f(y)| / d(x, y) : 0 < d(x, y) \leq \delta \} = 0. \tag{3.1}$$

This means that for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$,

$$d(x, y) \leq \delta \Rightarrow |f(x) - f(y)| \leq \epsilon d(x, y).$$

Condition (3.1) can be written also in the equivalent form

$$\lim_{d(x,y) \longrightarrow 0} \frac{|f(x) - f(y)|}{d(x, y)} = 0$$

The space of little Lipschitz functions is denoted by $\mathrm{lip}_0(X)$, that of little Lipschitz functions vanishing at e by $\mathrm{lip}_0(X)$ $(i.e., : f(e) = 0)$.

3.2 De Leeuw's Map

This is a map considered first by de Leeuw [6] in the study of spaces of Lipschitz functions. For a pointed metric space (X, d, e) let $\hat{X} = \{(x, y) \in X^2 : x \neq y\}$. For a function $f : X \longrightarrow \mathbb{R}$, let $\Phi f : \hat{X} \longrightarrow \mathbb{R}$ be giving by

$$(\Phi f)(x, y) = \frac{f(x) - f(y)}{d(x, y)}, (x, y) \in \hat{X}.$$

It is obvious that f is Lipschitz if and only if Φf is bounded and is this case

$$\mathrm{Lip}(f) = \|\Phi f\|_{\infty}.$$

3.3 Properties the space $\mathrm{lip}_0(X)$

In this section (X, d, e) will be a pointed compact metric space. We shall present the basic properties of the spaces of little Lipschitz functions.

Proposition 3.3.1. *The space* $\mathrm{lip}_0(X)$ *is a closed subspace of* $\mathrm{Lip}_0(X)$.

Example 3.3.1. If $f \in \mathrm{lip}_0(X)$, so f is differentiable and $f' \equiv 0$. Moreover $f(0) = 0$, so $\mathrm{lip}_0([0,1]) = \{0\}$.

Proposition 3.3.2. *Let* (X, d, e) *be pointed compact metric space and* $f \in \mathrm{Lip}_0(X)$. *Then*

$$f \in \mathrm{lip}_0(X) \Leftrightarrow \Phi f \in C_0\left(\hat{X}\right).$$

The subspace $\mathrm{lip}_0(X) = \Phi^{-1}\left(C_0\left(\hat{X}\right)\right)$ *is closed in* $\mathrm{Lip}_0(X)$, *so it is Banach space (with respect to the Lipschitz norm).*

Remark 3.3.1. The space $\mathrm{lip}_0(X)$ is a Banach algebra and Banach lattice, that is,

(i) $f, g \in \mathrm{lip}_0(X) \Rightarrow f.g \in \mathrm{lip}_0(X)$.

(ii) $f, g \in \mathrm{lip}_0(X) \Rightarrow \max\{f, g\}, \min\{f, g\} \in \mathrm{lip}_0(X)$.

Definition 3.3.1. Let (X, d, e) be a pointed compact metric space. One says that $\mathrm{lip}_0(X)$ separates points uniformly if there exists $\alpha > 1$ such that for every $x, y \in X$ there exists $f \in \mathrm{lip}_0(X)$ with

$$\mathrm{Lip}(f) \leq \alpha \text{ and } |(x) - f(y)| = d(x, y).$$

Theorem 3.3.1. *Let* (X, d, e) *be a pointed compact metric space. If* $\mathrm{lip}_0(X)$ *separates points uniformly, then* $\mathrm{lip}_0(X)^* \cong \text{Æ}(X)$ *and* $\mathrm{lip}_0(X)^{**} \cong \mathrm{Lip}_0(X)$.
The same is true in the case of the space $\mathcal{F}(X)$.

Proof (Sketch). One defines $\Gamma : \text{Æ}(X) \longrightarrow \mathrm{lip}_0(X)^*$ by

$$(\Gamma m)(f) = \langle f, m \rangle = \sum_i \alpha_i (f(x_i) - f(y_i)), \quad f \in \mathrm{lip}_0(X).$$

for every molecule m in $\text{Æ}(X)$. One shows that Γ is a linear isometry that extends to an isometric isomorphism of $\text{Æ}(X)$ onto $\mathrm{lip}_0(X)^*$. But then the conjugate operator Γ^* will be an isomorphism of $\mathrm{lip}_0(X)^{**}$ onto $\text{Æ}(X)^* \cong \mathrm{Lip}_0(X)$. In proving the surjectivity of Γ one appeals to the isometric isomorphism Φ between $\mathrm{lip}_0(X)$ and $C_0\left(\hat{X}\right)$ from proposition 3.3.2 and to the representation of continuous linear functionals on $C_0\left(\hat{X}\right)$ as Radon measures.

In this case of the space $\mathcal{F}(X)$ one can consider the evaluation functionals $\tilde{\delta_x}, x \in X$, acting on $\mathrm{lip}_0(X)$ by the rule $\tilde{\delta_x}(f) = f(x), f \in \mathrm{lip}_0(X)$. In fact these are the restriction of the evaluation functionals $\delta_x \in \mathrm{lip}_0(X)^*$ considered in subsection 2.2.2 to the closed subspace $\mathrm{lip}_0(X)$ of $\mathrm{Lip}_0(X)$. Put $X_\delta := \{\delta_x : x \in X\} \subseteq \mathrm{Lip}_0(X)^*$ and $X_{\tilde{\delta_x}} := \left\{\tilde{\delta_x} : x \in X\right\} \subseteq \mathrm{lip}_0(X)^*$ and consider the mapping $\Gamma : span(X_\delta) \longrightarrow span\left(X_{\tilde{\delta_x}}\right) \subseteq \mathrm{lip}_0(X)^*$ given by

$$\Gamma\left(\sum_i \alpha_i \delta_{x_i}\right) = \sum_i \alpha_i \tilde{\delta_{x_i}}$$

for $\sum_i \alpha_i \delta_{x_i} \in span(X_\delta)$. This correspondence is linear and isometric, so that it extends to an isometric linear mapping from $\mathcal{F}(X)$ to $\mathrm{lip}_0(X)^*$. One shows that this mapping is also onto, and so it is an isometric isomorphism.

Consequently, $\Gamma(\mathcal{F}(X)) = \overline{span}\left(X_{\tilde{\delta_x}}\right) = \mathrm{lip}_0(X)^*$.

The space $\mathrm{Lip}_0(X)$ is isometrically isomorphism to the space $\mathrm{lip}_0(X)^{**} \cong \mathcal{F}(X)^*$, the isomorphism $\Psi : \mathrm{Lip}_0(X) \longrightarrow \mathrm{lip}_0(X)^{**}$ determined by the condition

$$\Psi(f)\left(\tilde{\delta_x}\right) = f(x), x \in X, f \in \mathrm{Lip}_0(X).$$

Its inverse Ψ^{-1} satisfies

$$\Psi^{-1}(\varphi)(x) = \varphi\left(\tilde{\delta_x}\right), x \in X, \varphi \in \mathrm{lip}_0(X)^{**}.$$

This ends the proof. □

CHAPTER 4

APPLICATION

In this chapter, we introduce the notion of B-Lipschitz summing operators which is the generalization of the class of B-Lipschitz summing operators.

4.1 B-Lipschitz summing operators

We introduce and study a new class class of Lipschitz operators which we call "B-*Lipschitz summing operators*". In the linear case, it was studied by F. Baur [2] . Let $(\Omega, \sum, \mu)$ be a probability space such that $L_2(\Omega, \sum, \mu)$ is infinite and $\mathrm{B} \subset L_2(\Omega, \sum, \mu)$ denotes an infinite orthonormal system.

Definition 4.1.1. Let X be a pointed metric space and E be a Banach space. A Lipschitz operator $T : X \longrightarrow E$ is said to be B-Lipschitz summing operator if there exists a constant $C > 0$ such that for all finite sequences $(b_i)_{1\leq i\leq n} \subset \mathrm{B}$, $\{x_i\}_{1\leq i\leq n}, \{y_i\}_{1\leq i\leq n}$ in X and all $\{a_i\}_{1\leq i\leq n} \subset \mathbb{R}^+$, we have

$$\left(\int_\Omega \left\|\sum_{i=1}^n a_i^{\frac{1}{2}} b_i(\omega)(T(x_i) - T(y_i))\right\|^2 d\mu(\omega)\right)^{\frac{1}{2}} \leq C \sup_{f\in \mathcal{B}_{X^\#}} \left(\sum_{i=1}^n a_i |f(x_i) - f(y_i)|^2\right)^{\frac{1}{2}} \tag{4.1}$$

We denote by $\Pi_{\mathrm{B}}^L(X, E)$ the space of all B-Lipschitz summing operators from X into E and by $\pi_{\mathrm{B}}^L(.)$ the smallest constant C satisfying (4.1) which is a norm on $\Pi_{\mathrm{B}}^L(X, E)$. the space $\Pi_{\mathrm{B}}^L(X, Y)$ equipped with the norm $\pi_{\mathrm{B}}^L(.)$ is a Banach space.

Proposition 4.1.1. *Let X, Y be metric spaces and E, F be Banach spaces. Let $W : E \longrightarrow F$ be a bounded linear operator and $V : Y \longrightarrow X$ be Lipschitz function. Consider $T : X \longrightarrow E$ be a* B*-Lipschitz summing operator. Then WTV is* B*-Lipschitz summing and $\pi_B^L(WTV) \leq \|W\|)\pi_B^L(T)\,\mathrm{Lip}(V)$.*

Proof. Consider $\{x_i\}_{1\leq i\leq n}, \{y_i\}_{1\leq i\leq n}$ in X and $\{a_i\}_{1\leq i\leq n} \subset \mathbb{R}^+$. We have

$$\begin{aligned}
&\left(\int_\Omega \left\|\sum_{i=1}^n a_i^{\frac{1}{2}} b_i(\omega)\left(WTV(x_i) - WTV(y_i)\right)\right\|^2 d\mu(\omega)\right)^{\frac{1}{2}} \\
\leq\ & \|W\| \left(\int_\Omega \left\|\sum_{i=1}^n a_i^{\frac{1}{2}} b_i(\omega)\left(TV(x_i) - TV(y_i)\right)\right\|^2 d\mu(\omega)\right)^{\frac{1}{2}} \\
\leq\ & \|W\| \pi_{\mathrm{B}}^L(T) \sup_{f\in B_{X\#}} \left(\sum_{i=1}^n a_i \left|fV(x_i) - fV(y_i)\right|^2\right)^{\frac{1}{2}} \\
\leq\ & \|W\| \pi_{\mathrm{B}}^L(T) \operatorname{Lip}(V) \sup_{f\in \mathcal{B}_{X\#}} \left(\sum_{i=1}^n a_i \left|\frac{f\circ V}{\operatorname{Lip}(V)}(x_i) - \frac{f\circ V}{\operatorname{Lip}(V)}(y_i)\right|^2\right)^{\frac{1}{2}} \\
\leq\ & \|W\| \pi_{\mathrm{B}}^L(T) \operatorname{Lip}(V) \sup_{g\in \mathcal{B}_{Y\#}} \sum_{i=1}^n a_i |g(x_i) - g(y_i)|^p.
\end{aligned}$$

This implies that WTV is Lipschitz B-summing operator and

$\pi_{\mathrm{B}}^L(WTV) \leq \|W\| \pi_{\mathrm{B}}^L(T) \operatorname{Lip}(V)$. □

Remark 4.1.1. If T is linear then $\pi_{\mathrm{B}}^L(T) \leq \pi_{\mathrm{B}}(T)$.

The linear case was study by F. Baur in [2]. The definition is the following: a linear operator $u : E \longrightarrow F$ between Banach spaces is said to belong to the class $\Pi_{\mathrm{B}}(E, F)$ of B-summing operators if there exists a constant $C > 0$ such that for all finite sequences $\{x_i\}_{1\leq i\leq n}$ in E, we have

$$\left(\int_\Omega \left\|\sum_{i=1}^n b_i(\omega) u(x_i)\right\|^2 d\mu(\omega)\right)^{\frac{1}{2}} \leq C \sup_{f\in \mathcal{B}_{X\#}} \left(\sum_{i=1}^n |\xi(x_i)|^2\right)^{\frac{1}{2}} \tag{4.2}$$

We denote by $\pi_{\mathrm{B}}(u)$ the infimim of C such that the inquality (4.2) holds. $\pi_{\mathrm{B}}(u)$ is norm on $\Pi_{\mathrm{B}}(E, F)$ which becomes a Banach space.

Probleme 4.1.1. Now we try to generalize some properties in linear case to Lipschitz case.

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Abstract

في هذا العمل، سنهتم بالفضاء الليبشيزي $\mathrm{Lip}_0(X)$ حيث X فضاء متري نقطي، و كونه ثنوي.
سندرس الفضاءات ماقبل ثنوي $Æ(X)$ و $\mathcal{F}(X)$ التي هي متساوية القياس و خصائصها. ثم
سندرس الفضاء $\mathrm{lip}_0(X)$ الذي يكون تحت بعض الشروط على X ليكون ماقبل ثنوي ل $Æ(X)$.
لننهي هذا العمل بتعريف المؤثرات B-جمعيية ليبشيزية.

كلمات مفتاحية

الفضاء المتري، الفضاء الليبشيزي، فضاء آرنس آل، فضاء بناخ الحر، الفضاء المصغر الليبشيزي، المؤثرات B-جميعية ليبشيزية.

Dans ce travail, on s'intéresse à l'espace $\mathrm{Lip}_0(X)$ où X est un espace métrique pointé et qui est un dual, on étudie les préduaux $Æ(X)$ et $\mathcal{F}(X)$ qui sont isométriques et leurs propriétés. Puis on étudie l'espace $\mathrm{lip}_0(X)$ qui est sous certaine conditions de X un prédual de $Æ(X)$. On termine ce travail, par introduire les opérateurs B-Lipschitz sommants.

Mots-Clés: Espace métrique, espace de Lipschitz, espace Arens-Eells, espace de Banach libre, le petit espace de Lipschitz, opérateurs B-Lipschitz sommants.

In this work, we tackle Lipschitz space $\mathrm{Lip}_0(X)$, that X is a pointed metric space, and it is dual. We study the preduals $Æ(X)$ and $\mathcal{F}(X)$, that are isometrics in addition to its properties. Then we study the space $\mathrm{lip}_0(X)$, which is under certain conditions of X is a predual of $Æ(X)$. We end this work, by introducing B-Lipschitz summing operators.

Key-words: Metric space, Lipschitz space, Arens-Eells space, Free Banach space, The little Lipschitz space, B-Lipschitz summing operators.