



## ON THE NUMERICAL SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND

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### Abstract

In this work we concern with the approximate solution of the linear equation  $Af = f$  where  $A$  is injective and compact operator, this equation admits a unique solution in direct sense or in the least square sense provided the right-hand side  $f$  is in  $R(A)$  or in  $R(A)+R(A)^\perp$ , respectively. Due to the nonclosed range  $R(A)$  the solution is not stable. Besides, if  $A$  is positive definite we can replace the original equation by the auxiliary one  $\alpha\varphi_\alpha + A\varphi_\alpha = f$  where its solution  $\varphi_\alpha$  exist, stable and converges to the exact solution  $\varphi$  of the original equation as  $\alpha$  tends to zero.

### 1. Introduction

The inverse problem takes a considerable part in the domain of differential and partial differential equations. As prominent example of the ill posed problem we find integral equation of the first kind where the most problems of engineering and mathematical physics can be modelled in this equation, many methods study its approximation solution and stability. In [1] the authors solved the integral equation of the first kind by Chebyshev wavelet constructed in the bounded interval and used the Galerkin technical

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in order to reduce the integral equation into an algebraic linear system. On the other hand, in [3, 4] the collocation method with Legendre wavelets are used directly for this equation and convert it to an algebraic linear system.

The use of the sine basis functions method for solving the first kind integral equation was found in [5]. The method proposed by authors in [10] is to use the Hermite polynomial with least square method in order to solve integral equation of the first kind with degenerate kernel supported by Galerkin and collocation methods.

Let  $A$  be a linear compact operator defined from Hilbert space  $H$  to itself over the field  $\mathbb{R}$ . We explicit the linear inverse problem of a first kind by

$$A\varphi = f, \quad (1)$$

where  $f$  is the data function and  $\varphi$  the unknown potential one, suppose that  $A$  is injective, then the equation (1) admits a unique solution in direct sense or in the least square sense provided the right-hand side  $f$  is in  $R(A)$  or in  $R(A) + R(A)^\perp$ , respectively. Due to the nonclosed range  $R(A)$  the solution is not stable. Besides, if  $A$  is positive definite we can replace the original equation by the auxiliary one

$$\alpha\varphi_\alpha + A\varphi_\alpha = f, \quad (2)$$

where we add the term  $\alpha\varphi$  to the operator  $A\varphi$  for  $\alpha$  positive and small, the equation (2) admits a stable solution  $\varphi_\alpha$ . Noting that the function  $\varphi_\alpha$  converges to the exact solution  $\varphi$  of equation (1) as  $\alpha$  tends to zero [8].

### Lavrentiev method

The Lavrentiev method for the equation (1) is to replace the equation by the following one  $A\varphi = f_\delta$ , with  $\|f - f_\delta\| \leq \delta$ , if the right-hand side  $f_\delta$  is not in the range  $R(A)$ , Lavrentiev changes the equation  $A\varphi = f_\delta$  by its auxiliary equation

$$\alpha\varphi_{\alpha\delta} + A\varphi_{\alpha\delta} = f_\delta, \quad \alpha > 0. \quad (3)$$

It is clear that, if the operator  $A$  is positive definite, the problem of the second kind (3) is well-posed. Lavrentiev proves that the solution  $\varphi_{\alpha\delta}$  of the equation

(3) tends to the exact solution  $\varphi$  of the equation (1) with conditions  $\alpha$  and  $\delta$  tend to zero.

### Tikhonov Regularization Method

The Tikhonov regularization of the equation (1) corresponds to the regularization operators

$$R_\alpha = (\alpha I + A^* A)^{-1} = A^*, \text{ for } \alpha > 0, \quad (4)$$

which approximates the unbounded operator  $A^{-1}$  on  $R(A)$ . Noting that, the solution  $\varphi_\alpha = R_\alpha f$  represents the unique solution of the equation

$$\alpha \varphi_\alpha + A^* A \varphi_\alpha = A^* f, \quad (5)$$

and depends continuously on  $f$ , for all  $f \in H$  and  $\alpha > 0$ . Also this solution is the unique minimum of the Tikhonov functional

$$J_\alpha(X) = \|A\varphi - f\|^2 + \alpha \|\varphi\|^2, \text{ for } \varphi \in H \text{ and } \alpha > 0.$$

## 2. Main Results

In this work we focus our study to the Fredholm integral equations of the first kind

$$A\varphi(x) = \int_a^b k(x, t)\varphi(t)dt = f(x), a \leq x \leq b$$

where  $k(x, t)$  and  $f$  are given continuous functions and  $\varphi(x) \in H([a, b])$  is the unknown potential function to be determined.

**Lemma 1** [9]. *The problem (2) is well posed with the norm  $\|(\alpha I + A)^{-1}\| = O\left(\frac{1}{\sqrt{\alpha}}\right)$  provided  $A$  is injective and positive definite operator.*

**Proposition.** *The injectivity and the positivity of the compact operator  $A$  lead to the existence and uniqueness of the solution of the auxiliary problem (2)*

$$\alpha \varphi_\alpha + A \varphi_\alpha = f, \alpha > 0.$$

*Besides, the solution  $\varphi_\alpha$  converges to the exact solution  $\varphi$  of the initial problem (1), as  $\alpha$  goes to zero, say*

$$\lim_{\alpha \rightarrow 0} \|\varphi - \varphi_\alpha\| = 0.$$

**Proof.**

Indeed,

$$\begin{aligned} \varphi - \varphi_\alpha &= \varphi - (\alpha I + A)^{-1} f \\ &= \varphi - (\alpha I + A)^{-1} A \varphi \\ &= \varphi (\alpha I + A)^{-1} \varphi. \end{aligned}$$

Therefore

$$\begin{aligned} \|\varphi - \varphi_\alpha\| &\leq \alpha \|(\alpha I + A)^{-1}\| \|\varphi\| \\ \|\varphi - \varphi_\alpha\| &= O(\sqrt{\alpha}). \end{aligned}$$

### Nyström method

Using the quadrature rule to approximate  $\int_a^b k(x, t)\varphi(t)dt$  say

$$A\varphi(x) = \int_a^b k(x, t)\varphi(t)dt \cong \sum_{j=1}^n w_j k(x, t_j)\varphi(t_j).$$

So, the equation (1) can be replaced by

$$A_n\varphi(x) = \sum_{j=1}^n w_j k(x, t_j)\varphi(t_j) = f(x), \quad 0 \leq x \leq 1. \quad (6)$$

In the collocation method the values of  $\varphi(t_j)$ ,  $j = 1, 2, \dots, n$  are found so that the equation (6) is verified for all points  $x_1, x_2, \dots, x_m$ , in  $[0, 1]$ . It is not necessary to take  $m = n$ , but often  $m$  and  $n$  are chosen to be equal, and  $x_i$  is chosen as  $x_i = t_i$ ,  $i = 1, 2, \dots, n$ .

$$\sum_{j=1}^n w_j k(x, t_j)\varphi(t_j) = f(x_i), \quad i = 1, 2, \dots, n. \quad (7)$$

Taking  $\mathcal{A}=(a_{ij})$  the  $n \times n$  matrix such that  $a_{ij}=w_jk(x_i,t_j)$  for  $1 \leq i, j \leq n$ , the unknown vector  $\vec{\Phi} = (\varphi(t_1), \varphi(t_2), \dots, \varphi(t_n))^T = (\varphi_1, \varphi_2, \dots, \varphi_n)^T$  and the right-hand side vector  $\vec{F} = (f(x_1), f(t_2), \dots, f(x_n))^T = (f_1, f_2, \dots, f_n)^T$ , hen the auxiliary equation (2) can be approximated by the matrix equation

$$(\alpha I + \mathcal{A})\vec{\Phi}_\alpha = \vec{F}. \quad (8)$$

The algebraic system (8) admits a unique solution  $\vec{\Phi}_\alpha$  converges to the solution  $\vec{\Phi}$  of the system  $\mathcal{A}\vec{\Phi} = \vec{F}$  as  $\alpha \rightarrow 0$ .

**Lemma 2.** *The norm  $\|(\alpha I + A)^{-1}\| \leq \frac{1}{\alpha}$  provided  $A$  is positive definite operator. Indeed,  $A$  is positive and injective, it follows*

$$\begin{aligned} \langle \varphi, \varphi \rangle &= \langle (\alpha I + A)(\alpha I + A)^{-1} \varphi, (\alpha I + A)(\alpha I + A)^{-1} \varphi \rangle \\ &= \langle \alpha(\alpha I + A)^{-1} \varphi + A(\alpha I + A)^{-1} \varphi, \alpha(\alpha I + A)^{-1} \varphi + A(\alpha I + A)^{-1} \varphi \rangle \\ &= \|\varphi\|^2 = \alpha^2 \|(\alpha I + A)^{-1} \varphi\|^2 + 2\alpha \langle A(\alpha I + A)^{-1} \varphi, (\alpha I + A)^{-1} \varphi \rangle \\ &\quad + \|(\alpha I + A)^{-1} \varphi\|^2, \|\varphi\|^2 \geq \alpha^2 \|(\alpha I + A)^{-1} \varphi\|^2. \end{aligned}$$

Therefore, we obtain

$$\|(\alpha I + A)^{-1}\| \leq \frac{1}{\alpha}.$$

**Lemma 3.** *The operator  $(\alpha I + A_n)$  is invertible on the Hilbert space  $H$  to itself if  $\|(A - A_n)A_n\| \leq \alpha^2$ . Besides, if  $(\alpha I + A_n)\varphi_n = f$  and  $(\alpha I + A)\varphi = f$ , then*

$$\|\varphi - \varphi_n\| = O(\alpha).$$

Indeed,

$$\frac{1}{\alpha}[I - (\alpha I + A)^{-1}A_n](\alpha I + A_n) = I - (\alpha I + A)^{-1}A_n + \frac{1}{\alpha}[I - (\alpha I + A)^{-1}A_n]A_n$$

$$\begin{aligned}
&= I - (\alpha I + A)^{-1} A_n + \frac{1}{\alpha} [(\alpha I + A)^{-1} ((\alpha I + A) - A_n)] A_n \\
&= I - (\alpha I + A)^{-1} A_n + \frac{1}{\alpha} (\alpha I + A)^{-1} (\alpha I + A - A_n) A_n \\
&= \left( I + \frac{1}{\alpha} (\alpha I + A)^{-1} (A - A_n) A_n \right).
\end{aligned}$$

Noting that, the right side of the last expression is invertible, for

$$\frac{1}{\alpha} \| (\alpha I + A)^{-1} (A - A_n) A_n \| \leq \frac{1}{\alpha} \| (\alpha I + A)^{-1} \| \| (A - A_n) A_n \| < 1.$$

The injectivity of the right side involves the one of  $(\alpha I + A_n)$  and so its bijectivity. For the error, we get

$$\begin{aligned}
\| \varphi_n - \varphi \| &= \| ((\alpha I + A_n)^{-1} - (\alpha I + A)^{-1}) f \| \\
&= \| ((\alpha I + A)^{-1} (A_n - A) (\alpha I + A_n)^{-1}) f \| \\
&= \| ((\alpha I + A)^{-1} (A - A_n) \varphi_n \| \\
&\leq \| (\alpha I + A)^{-1} \| \| (A - A_n) \| \| \varphi_n \| \\
&= O(\alpha).
\end{aligned}$$

### 3. Explanatory Examples

**Example 1.** Consider the first-kind integral equations of Fredholm

$$\int_0^1 \exp(xt) \varphi(t) dt = \frac{\exp(x+1) - 1}{x+1},$$

where  $0 \leq t, x \leq 1$ , and the function  $f(x)$  is chosen so that the exact solution is given by

$$\varphi(x) = \exp(x).$$

**Table 1.** We present the exact solution  $\varphi$  and its approximate one  $\varphi_\alpha$  as well as the absolute error  $|\varphi - \varphi_\alpha|$  of the example 1 in some arbitrary points

for  $N = 10$  and  $\alpha = 10^{-6}$ , the error is compared with the Chebyshev Wavelet Method [1] and the Haar wavelets method [6].

Val of $x$	Exact sol $\varphi$	App sol $\varphi_\alpha$	$ \varphi - \varphi_\alpha $	Error [1]	Error [6]
0.000	1.00e+00	9.99e-01	1.21e-06	1.46e -05	7.85e-03
0.200	1.22e+00	1.22e+00	8.22e-07	1.73e -05	5.69e-03
0.400	1.49e+00	1.49e+00	3.09e-07	1.57e -05	2.31e-03
0.600	1.82e+00	1.82e+00	1.05e-06	1.30e -06	2.86e-03
0.800	2.22e+00	2.22e+00	1.80e-06	1.52e -05	1.04e-02
1.000	2.71e+00	2.71e+00	7.95e-06	1.04e -05	4.98e-03

**Example 2.** Consider the first-kind integral equations of Fredholm

$$\int_0^1 \exp(t \sin x) \varphi(t) dt = \frac{1}{1 + \sin^2 x} (\exp(\sin x) (\cos 1 \sin x + \sin 1) - \sin x),$$

where  $0 \leq t, x \leq 1$ , and the function  $f(x)$  is chosen so that the exact solution is given by  $\varphi(x) = \cos x$ .

**Table 2.** We present the exact solution  $\varphi$  and its approximate one  $\varphi_\alpha$  as well as the absolute error  $|\varphi - \varphi_\alpha|$  of the example 2 in some arbitrary points for  $N = 10$  and  $\alpha = 10^{-6}$ , the error is compared with Legendre wavelets collocation method [3].

Val of $x$	Exact sol $\varphi$	App sol $\varphi_\alpha$	$ \varphi - \varphi_\alpha $	Error [3]
0.000	1.000e+00	9.991e-01	8.828e-04	1.488e-03
0.200	9.800e-01	9.803e-01	2.667e-04	2.531e-03
0.400	9.210e-01	9.207e-01	3.320e-04	4.740e-03
0.600	8.253e-01	8.255e-01	2.007e-04	2.920e-03
0.800	6.967e-01	6.966e-01	4.073e-05	4.951e-03
0.900	5.816e-01	5.815e-01	9.163e-05	1.239e-03

**Example 3.** Consider the first-kind integral equations of Fredholm

$$\int_0^1 (x^2 + 2tx + t^2)\varphi(t)dt = \frac{x^2}{2} + \frac{2x}{3} + \frac{1}{4},$$

where  $0 \leq t \leq x \leq 1$ , and the function  $f(x)$  is chosen so that the exact solution is given by  $\varphi(x) = x$ .

**Table 3.** We present the exact solution  $\varphi$  and its approximate one  $\varphi_\alpha$  as well as the absolute error  $|\varphi - \varphi_\alpha|$  of the example 3 in some arbitrary points for  $N = 10$  and  $\alpha = 10^{-6}$ , the error is compared with Hermite polynomial method [9].

Val of $x$	Exact sol $\varphi$	App sol $\varphi_\alpha$	$ \varphi - \varphi_\alpha $	Error [10]
0.000	0.000e+00	1.781e-07	1.781e-07	1.6e-03
0.200	2.000e-01	2.000e-01	2.158e-08	6.0e-04
0.400	4.000e-01	4.000e-01	4.687e-08	5.0e-04
0.600	6.000e-01	5.999e-01	8.356e-08	5.0e-04
0.800	8.000e-01	8.000e-01	7.016e-09	4.0e-04
1.000	1.000e+00	1.000e+00	1.140e-07	1.7e-03

**Example 4.** Consider the first-kind integral equations of Fredholm

$$\int_0^1 \cos(x-t)\varphi(t)dt = \frac{1}{4}\cos x + \frac{1}{2}\sin x - \frac{1}{4}\cos(x-2),$$

where  $0 \leq t, x \leq 1$ , and the function  $f(x)$  is chosen so that the exact solution is given by  $\varphi(x) = \sin x$ .

**Table 4.** We present the exact solution  $\varphi$  and its approximate one  $\varphi_\alpha$  as well as the absolute error  $|\varphi - \varphi_\alpha|$  of the example 4 in some arbitrary points, the error is calculated for  $N = 10$  and  $\alpha = 10^{-6}$ ,

Val of $x$	Exact sol $\varphi$	App sol $\varphi_\alpha$	$ \varphi - \varphi_\alpha $
0.000	0.000e+00	1.384e-08	1.384e-08
0.200	1.986e-01	1.986e-01	8.067e-09
0.400	3.894e-01	3.894e-01	5.170e-09
0.600	5.646e-01	5.646e-01	1.680e-08
0.800	7.173e-01	7.173e-01	1.941e-08
1.000	8.414e-01	8.414e-01	3.261e-08

**Example 5.** Consider the first-kind integral equations of Fredholm

$$\int_0^1 \sinh(x-t)\varphi(t)dt = -\frac{1}{8} \exp(x)(\exp(-2) - 3 + \exp(-2x) + \exp(2)\exp(-2x)),$$

where  $0 \leq t, x \leq 1$ , and the function  $f(x)$  is chosen so that the exact solution is given by  $\varphi(x) \cosh x$ .

**Table 5.** We present the exact solution  $\varphi$  and its approximate one  $\varphi_\alpha$  as well as the absolute error  $|\varphi - \varphi_\alpha|$  of the example 5 in some arbitrary points, the error is calculated for  $N = 10$  and  $\alpha = 10^{-6}$ ,

Val of $x$	Exact sol $\varphi$	App sol $\varphi_\alpha$	$ \varphi - \varphi_\alpha $
0.000	1.000e+00	9.999e-01	5.193e-08
0.200	1.020e+00	1.020e+00	3.413e-08
0.400	1.081e+00	1.081e+00	2.062e-08
0.600	1.185e+00	1.185e+00	1.546e-09
0.800	1.337e+00	1.337e+00	2.289e-08
1.000	1.543e+00	1.543e+00	2.962e-08

#### 4. Conclusion

This numerical technique for solving Fredholm integral equations of first kind, concentrated on the few modification of Lavrentiev classical method

supported by the modified Simpson approximation [7], the approximate solution  $\varphi_\alpha$  is measurably close to the exact solution  $\varphi$  of the given equation on the whole interval  $[0, 1]$ . This method is tested by solving some examples for which the exact solution is known and proves its efficiency compared with other methods.

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