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Faculty of Mathematics and Computer Science
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Domaine : Mathematics and Computer Science
Track : Mathematics
Option : Algebra and Discrete Mathematics

Theme

The problem of isomorphism of two graphs and counting the number of nonisomorphic graphs

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Doumi Sara

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Notations

$\mid G \mid$	*Order of group*
$M_{n\times n}(\mathbb{R})$	*The set of* $n \times n$ matrices over $\mathbb{R}$
$GL_n(\mathbb{R})$	*The set of* $n \times n$ *invertible matrices over* $\mathbb{R}$
G/H	*The set of all left cosets of a subgroup* H *in a group* G
H/G	*The set of all right cosets of a subgroup* H *in a group* G
$[G : H]$	*The index of a subgroup* H *in a group* G
$\curvearrowright$	*an action*
$cl(x)$	*Equivalence class of an element*
σ	*A permutation*
S_n	*The symmetric group of the first* n *natural numbers*
C_n	*The cyclic group of order* n
D_n	*The* n*th dihedral group*
O_x	*The orbit of an element*
$Stab(x)$	*The stabilizer of an element*
$Fix(g)$	*The invariant of an element*
$\mathrm{Ci}\, m(\sigma)$	*Cycle index monomial of a permutation*
P_G	*Cycle index polynomial associated with a group*
w_c	*The weight of an element*
$W(f)$	*The weight of a function*
PI_G	*Pattern inventory under a group* G

Introduction

This memory is about the study of an isomorphism problem between two graphs and counting the number of nonisomorphic graphs on n vertices.

The isomophism between two graphs G and G' is a bijection between the vertices of G and G' which preserve the adjacency. Since, $\mid G \mid = \mid G' \mid = n$. There is $n!$ bijection between them. The problem is then: how to find this bijection, if there is one, from the set of the $n!$ permutations.

For instance, if there is $n = 50$ vertices in the graphs, then there is $n! = 50! =$ 30410932017133780436126081660647688443776415689605120000000000000 permutations to check if the graphs are isomorphic or not and this number is very large and we say that $NP - hard$. The Pólya's enumeration theorem can be used to calculate the number of non-isomorphic graphs with a fixed the number of vertices. Let V and $V^{'}$ be a set of n vertices. Two graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there is a permutation $\pi \in S_V$ such that $(x, y) \in E \Leftrightarrow (\pi(x), \pi(y)) \in E'$. To apply the Pólya's enumeration theorem we identify each graph $G = (V, E)$ with a function f from the set X of all edges to the set $Y = \{black, white\}$, So the set of all graphs is identify with the set of all functions from X to Y. Hence, the problem of finding the number of nonisomorphic graphs is equivalent to find the number of distinct colorings of the edges of the complete simple graph on n vertices using two colors black and white. In this case, the method of finding the group of permutations (denoted by S_n^2) and its cycle index polynomial is given in this memory.
This memory is organized as follows:

• In chapter 01; we present the group theory through examples and a connection with symmetry.

• In chapter 02; we study the concept of an action of a group G on a non empty set X and we given the Pólya's enumeration theorem and use it to solve some counting problems like the counting of the number of nonisomorphic graphs with n vertices.

• In chapter 03; we study some proprieties of the graphs and counting the number of nonisomorphic graphs with n vertices, then we present the method of finding the group of permutations S_n^2 and its cycle index polynomial.

Chapter 1

General concepts of group theory

1.1 Group

Definition 1.1.1 ***(Group)****A group is a nonempty set G together with an operation "$*$" on G such that each of the following axioms are satisfied:*
1)Associativity:
$a*(b*c)=(a*b)*c$, *for all* $a,b,c \in G$
2)Existence of an identity element:
There is an element $e \in G$ *such that* $a*e=e*a=a$, *for each* $a \in G$.
3)Existence of inverse element:
for each $a \in G$, *there is an element* $b \in G$ *such that* $a*b=b*a=e$.

Definition 1.1.2 *A group* $(G,*)$ *is said to be abelian if the group operation is commutative i.e.,*
$\forall a,b \in G,\ a*b=b*a$.

Example 1.1.1

1. *The sets* $\mathbb{Z}$,$\mathbb{Q}$,$\mathbb{R}$,$\mathbb{C}$ *with addition are a groups.*
2. *The sets* $\mathbb{Q}^*, \mathbb{R}^*, \mathbb{C}^*$ with multiplication are a groups.

Example 1.1.2

1. *The groups* $(\mathbb{Z},+)$,$(\mathbb{Q}^*,\times)$,$(M_n(\mathbb{C}),+)$ *are abelian.*

2. *The set* $\mathbb{C}^*$ *of the numbers complexes with the multiplication is an abelian group.*

3. *The set* $GL_n(\mathbb{R})$ *of all the* $n\times n$ *matrices over* $\mathbb{R}$ *with non-zero determinants is a not an abelian group under the matrix multiplication. This group is called the general linear group.*

Theorem 1.1.1 *Assume that* G *together with* $*$ *is a group*

1. The identity element of G *is unique.*

2. Each element in a group has a unique inverse.

Proof. a) The identity element of G is unique, that is if e and f are elements of G such that,

$e*a=a*e=a$ for each $a\in G$ and $f*a=a*f=a$ for each $a\in G$

Then, $e=f$.

b)If a,x and y are elements of G

$a*x=x*a=e$ and $a*y=y*a=e$ with a,x and y as stated, write

$$\begin{aligned} x &= x*e && (e \text{ is the identity}) \\ &= x*(a*y) && (a*y=e) \\ &= (x*a)*y && (\text{associativity}) \\ &= e*y && (x*a=e) \\ &= y && (e \text{ is the identity}) \ \blacksquare \end{aligned}$$

1.1.1 Subgroups

Definition 1.1.3 *A subset* H *of a group* G *is a subgroup of* G *if* H *is itself a group with respect to the operation on* G *(denoted* $H\leq G$*).*

Notice that if G *is a group with operation* $*$, H *is a subgroup of* G, *and* a,$b\in H$, *then* $a*b\in H$. *That is,* H *must be closed with* $a*a\in H$ *for each* $a\in H$.

Lemma 1.1.1 *Let* G *be a group with operation* $*$ *and let* H *be a subgroup of* G.

1. If f *is the identity of* H *and* e *is the identity of* G, *then* $f=e$.

2. If $a\in H$, *then the inverse of* a *in* H *is the same as the inverse of* a *in* G.

Theorem 1.1.2 *Let G be a group with operation $*$, and let H be a subset of G. Then H is a subgroup of G if and only if:*

1. H is nonempty .

2. If $a \in H$ and $b \in H$, then $a * b \in H$.

3. If $a \in H$, then $a^{-1} \in H$.

1.The sets $\mathbb{Z}$,$\mathbb{Q}$ and $\mathbb{R}$ are subgroups of the group $\mathbb{C}$ with addition.

2.The set $\{-1,1\}$ is a subgroup of $(\mathbb{R}^*,\times)$.

1.1.2 Cyclic groups

Definition 1.1.4 *A group G is called cyclic if there is an element a in G such that $G = \{a^n/n \in \mathbb{Z}\}$ such an element a is called a generator of G, we may indicate that G is a cyclic group generated by a, by writing $G = \prec a \succ$.*

Example 1.1.3 *The subgroup $(\{1,-1,i,-i\},\times)$ of the group $(\mathbb{C},\times)$ is a cyclic group of order 4 generated by i because $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$*
Hence, The group can be written as $(\{1, i, i^2, i^3\},\times)$.

Example 1.1.4 *1.The group $(G,+)$ is cyclic group if $G = \{ng/n \in \mathbb{Z}\}$ for some $g \in G$.*

2.The group $(\mathbb{Z},+)$ is cyclic group with generator 1(or -1).

Theorem 1.1.3 *Generators of cyclic groups*
Let $G = \prec a \succ$ be a cyclic group of order n. Then $G = \prec a^k \succ$ if and only if the $\gcd(k,n) = 1$.

Proof. If the $\gcd(k,n) = 1$, we may write $1 = ku + nv$ for some integers u and v. Then,

$$\begin{aligned} a &= a^{ku+nv} \\ &= a^{ku}.a^{nv} \\ &= a^{ku} \end{aligned}$$

Thus, a belong to $\prec a^k \succ$ and therfore all powers of a belong to $\prec a^k \succ$. So, $G = \prec a^k \succ$ and a^k is a generator of G. ■

1.1.3 Isomorphisms

Definition 1.1.5 *Let G and H be two groups. We say that G and H are* ***isomorphic*** *if there is a bijective map $\phi : G \longrightarrow H$, which respects the group structure. That is to say, for every g and h in G,*

$$\phi(gh) = \phi(g)\phi(h).$$

The map ϕ is called an isomorphism.

Definition 1.1.6 *Let G be a group. An isomorphism of G with itself is called an* ***automorphism.***

Lemma 1.1.2 *Let G be a group and let $a \in G$ be an element of G. Define a map ϕ, $\phi : G \longrightarrow G$ by the rule $\phi(x) = axa^{-1}$. Then ϕ is an automorphism of G .*

Proof. We first check that ϕ is a bijection. Define a map $\psi : G \longrightarrow G$ by the rule $\psi(x) = a^{-1}xa$.
Then,

$$\begin{aligned} \psi(\phi(x)) &= \psi(axa^{-1}) \\ &= a^{-1}(axa^{-1})a \\ &= (a^{-1}a)x(a^{-1}a) \\ &= x \end{aligned}$$

Thus the composition of ϕ and ψ is the identity map. Similary, the composition of ψ and ϕ is the identity map. In particular, ϕ is a bijection.
Now, we check that ϕ is an isomorphism

$$\begin{aligned} \phi(x)\phi(y) &= (axa^{-1})(aya^{-1}) \\ &= a(xy)a^{-1} \\ &= \phi(xy) \end{aligned}$$

Thus, ϕ is an isomorphism. ■

1.1.4 Cosets and Lagrange's theorem

Definition 1.1.7 *Let $H \leq G$ and $a \in G$. The set $aH = \{ah/h \in H\}$ is called a* ***left coset*** *of H in G. Similary, the set $Ha = \{ha/h \in H\}$ is called a* ***right coset*** *of H in G. The set* *of all left cosets of H in G is denoted by $G\backslash H$ and the set of all right cosets of H in G is denoted by $H\backslash G$.*

Example 1.1.5 *Let $H = \{0, 3\}$ be a subgroup of $\mathbb{Z}_6$. The left cosets of H in $\mathbb{Z}_6$ are*

$$1 + H = \{1, 4\}$$

$$2 + H = \{2, 5\}$$

$$3 + H = \{0, 3\}$$

$$4 + H = \{1, 4\}$$

$$5 + H = \{2, 5\}$$

Definition 1.1.8 ***(Equivalence relation)****Let X be a set. The relation $\sim$ on X is an* ***equivalence relation*** *if it satisfies all of the following conditions:*
*1.(**Reflexivity**) For all $x \in X$, $x \sim x$.*
*2.(**Symmetry**) For all x,$y \in X$, if $x \sim y$, then $y \sim x$.*
*3.(**Transitivity**) For all x,y,$z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.*

Definition 1.1.9 *Let $\sim$ be an equivalence relation on a set X. For $x \in X$, the* ***equivalence class*** *of x induced by $\sim$, denoted by $cl(x)$, is the set*

$$cl(x) = \{y \in X \setminus x \sim y\}$$

Example 1.1.6 *Let n be a fixed natural number. We define the relation $\sim$ on $\mathbb{Z}$ by :*

$$a \sim b \Leftrightarrow a - b$$

is divisible by n
Then $\sim$ is an equivalence relation and for every $x \in \mathbb{Z}$, we have :

$$cl(x) = x + n\mathbb{Z}$$

Proposition 1.1.1 *Let x and y be two elements of a group G and $H \leq G$. Then*

$$xH = yH \Leftrightarrow y^{-1}x \in H$$

Proof. •(⇒) Suppose that $xH = yH$. Since $e \in H$, $xe = x \in xH$, and therfore $x \in yH$. So that there exists $h \in H$ such that $x = yh$. That is, H contains $h = y^{-1}x$.
•(⇐) Let x and y be two elements of G such that $y^{-1}x \in H$. There exists $h \in H$ such that $y^{-1}x = h$, then there exists $h \in H$ such that $x = yh$.
Let $z \in xH$, there exists $h' \in H$ such that:

$$z = xh' = (yh)h' = y(hh') = yh''.$$

This implies that $z \in yH$. Likewise, every element in yH is in xH. So $xH = yH$. ∎

Theorem 1.1.4 *Let H be a subgroup of a group G.*
The number of left cosets of H in G is the same as the number of right cosets of H in G.

Proof. Define a map $f : G/H \to H/G$ by $f(gh) = Hg^{-1}$
We need to show that the map is well defined and bijective.
• f is well define:

$$\begin{aligned} xH = yH &\Rightarrow y^{-1}xH = H \\ &\Rightarrow y^{-1}x \in H \\ &\Rightarrow Hy^{-1}x = H \\ &\Rightarrow Hy^{-1} = Hx^{-1} \\ &\Rightarrow f(xH) = f(yH) \end{aligned}$$

•f is injective: suppose that $f(xH) = f(yH)$

$$\begin{aligned} f(xH) = f(yH) &\Rightarrow Hx^{-1} = Hy^{-1} \\ &\Rightarrow H = Hy^{-1}x \\ &\Rightarrow y^{-1}x \in H \end{aligned}$$

$$\Rightarrow xH = yH$$

• f is surjective: Clearly that f is surjective, because for every $Hy \in H/G$ there is $xH \in G/H$ $(x = y^{-1})$ such that $f(xH) = Hy$. ■

Proposition 1.1.2 *Let H be a subgroup of a group G and $g \in G$.*
The number of elements in H is the same as the number of elements in gH.

Proof. Let $H = \{h_1, h_2, ..., h_k\}$ and $g \in G$. Then $gH = \{gh_1, gh_2, ..., gh_k\}$. The elements of gH must be distinct, because for $gh_i = gh_j$ imply $h_i = h_j$. Hence, $\mid gH \mid = k$. ■

Proposition 1.1.3 *The left cosets of a subgroup H in a group G constitutes a partition of the group.*

Proof. Let G be a group and H be a subgroup of G .We define a relation $\sim$ in G by:

$$x \sim y \Leftrightarrow x^{-1}y \in H$$

We need to show that $\sim$ is an equivalence ralation whose equivalence classes are the left cosets of H in G

• $\sim$ is reflexive: For every $x \in G$, $x^{-1}x = e \in H$, then $x \sim x$.

• $\sim$ is symmetric: Let x,$y \in G$ such that $x \sim y$. So $x^{-1}y \in H$, then $(x^{-1}y)^{-1} \in H$, so $y^{-1}x \in H$, then $y \sim x$.

• $\sim$ is transitive: Let x,y,$z \in G$ such that $x \sim y$ and $y \sim z$. So $x^{-1}y \in H$ and $y^{-1}z \in H$, then $x^{-1}yy^{-1}z \in H$, then $x^{-1}z \in H$, then $x \sim z$.

Now, we show that for every $x \in G$ we have $cl(x) = xH$.

• Let $y \in cl(x)$ then,

$$\begin{aligned} y \in cl(x) &\Rightarrow x^{-1}y \in H \\ &\Rightarrow x(x^{-1}y) \in xH \\ &\Rightarrow y \in xH \\ &\Rightarrow cl(x) \subseteq xH \end{aligned}$$

• Let $y \in xH$ then

$$\begin{aligned} y \in xH \Rightarrow y = xh \ \ \textit{for same } h \in H \\ \Rightarrow x^{-1}y = h \in H \\ \Rightarrow x \sim y \\ \Rightarrow y \in cl(x) \\ \Rightarrow xH \subseteq cl(x). \end{aligned}$$

■

Definition 1.1.10 *Let G be a group, the* ***index*** *of H in G, denoted by $[G : H]$, is equal to the number of left coset of H in G.*

Example 1.1.7 *Let $G = \mathbb{Z}_6$ and $H = \{0, 3\}$. Then $[G : H] = 3$.*

Theorem 1.1.5 *The order of any subgroup of a finite group divides the order of the group.*

Proof. Let G be a group of order n and $H \leq G$ of order k, have r distinct left cosets. From proposition 1.1.1 and proposition 1.1.3 we have :

$$\begin{aligned} n &= \mid G \mid \\ &= \mid \bigcup_{i=1}^{r} g_i H \mid \\ &= \sum_{i=1}^{r} \mid g_i H \mid \\ &= \sum_{i=1}^{r} k \\ &= rk \end{aligned}$$

so, $\mid G \mid$ is divisible by $\mid H \mid$. ■

Corollary 1.1.1 *Let g be an element of a finite group G. Then the order of g divides the order of G.*

Remark 1.1.1 *The converse of* ***Lagrange's Theorem*** *is not always true. That is, if a natural number m divides the order of the group G, we are not guaranted that G has a subgroup of order m.*

1.2 Symmetry groups

Definition 1.2.1 (permutation) *A* ***permutation*** *of a nonempty set X is a bijective function on X. The set of all permutations of X is denoted by S_X.*

Lemma 1.2.1 *Let X be a set and σ_1 and σ_2 be two permutations of X. The composition of σ_1 and σ_2 is a permutation of X and the inverse of σ_1 is a permutation of X.*

Theorem 1.2.1 *The set S_X form a group under the composition of permutations called the* ***permutation group*** *of X. If X is the set of the first n natural numbers, the permutation group of X is called* ***symmetric group*** *and denoted by S_n.*

Proof. It follows from Lemma 1.2.1 that the composition of two permutations is a permutation and the inverse of a permutation is a permutation, so S_X is closed under composition and closed under inverses. The composition of functions is always associative, and the identity of S_X is the identity bijection of X. Therefore, $(S_X, \circ)$ satisfies all the axioms for a group. ■

Theorem 1.2.2 (Cayley's theorem) *Every finite group is isomorphic to a group of permutations.*

Corollary 1.2.1 *If G is a finite group of order n, then G is isomorphic to a subgroup of S_n.*

Example 1.2.1 *If $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $\rho = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ are two elements of S_3 calculate $\pi \circ \rho$ and $\rho \circ \pi$.*

$$\begin{aligned} \pi \circ \rho &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \end{aligned}$$

In similar way we can show that ,

$$\rho \circ \pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Not that $\pi \circ \rho \neq \rho \circ \pi$ *and so* S_3 *is not commutative.*

How to represent a permutation *? There are two ways to represent* $\sigma \in S_n$.

The first is by a matrix of type $2 \times n$, *that is*

$$\sigma = \begin{pmatrix} 1 & 2 & ... & n \\ \sigma(1) & \sigma(2) & ... & \sigma(n) \end{pmatrix}$$

The second way is called the ***cycle notation,*** *if* $a_1, a_2, ..., a_k$ *are distinct elements of the set* $\{1, 2, ..., n\}$, *the permutation* σ, *defind by*

$$\sigma(a_1) = a_2, \sigma(a_2) = a_3, ..., \sigma(a_{k-1}) = a_k, \sigma(a_k) = a_1$$

and $\sigma(a) = a$ *for all* $a \notin \{a_1, a_2, ..., a_k\}$, *is called a* ***cycle*** *of length* r *or an* r-***cycle,*** *we denoted it by* $(a_1 a_2 ... a_k)$.

A 2-cycle is called ***transposition .***

Definition 1.2.2 (Conjugation) *Let* g *and* h *be two elements of a group* G.
The element ghg^{-1} *is called the conjugate of* h *by* g.

Definition 1.2.3 (Disjoint cycles) *Two cycles* $\sigma = (a_1 a_2 ... a_k)$ *and* $\rho = (b_1 b_2 ... b_m)$ *are disjoint if* $a_i \neq b_j$ *for all i,j.*

Theorem 1.2.3 *Every permutation* σ *in* S_n *is a product of disjoint cycles. This product is unique and called the* ***cycle decomposition*** *of* σ.

Example 1.2.2 *Let* $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 4 & 1 & 2 & 5 & 3 & 8 & 9 & 7 \end{pmatrix}$ *into a product of disjoint cycles.*
$\sigma(1) = 6$, *so* σ *is begin by* (1 6;$\sigma(6) = 3$, σ *continues* (1 6 3; *since* $\sigma(3) = 1$, *the parentheses*

close, and σ *is begin by the cycle* $(1\ 6\ 3)$. *The smallest integer not having appeard is* 2; *write* $(1\ 6\ 3)(2$, $\sigma(2) = 4$, *then* $(1\ 6\ 3)(2\ 4$; *continuing is this way, we obtained*

$$\sigma = (163)(24)(5)(789)$$

Corollary 1.2.2 *Every permutation* σ *in* S_n *is a product of transposition.*

1.3 Symmetries of geometric figures

1.3.1 Symmetries in two dimentions

Definition 1.3.1 ***(cyclic group)***
Let n *be a naturel number. The* ***cyclic group*** *of order* n, *denoted by* C_n, *is the set of all ratations of a regular* n*-gon.*

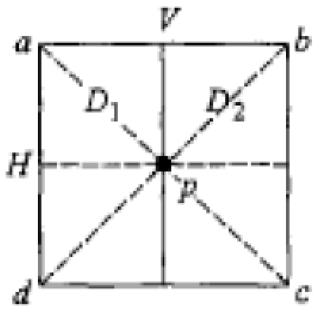
Definition 1.3.2 ***(Dihedral group)***
Let n *be a natural number. The* n*-th* ***dihedral group***, *denoted by* D_n, *is the set of all ratations and reflexive symmetries of a regular* n*-gon, its order is* $2n$.

Chapter 2

Group actions on a set and Polya's Enumeration Theorem

2.1 Group actions

We begin by recalling the group of symmetries of a square:



Group of symmetries of a square

μ_0=*identity pennutation*

μ_{90}=*rotation* 90° *clockwise around* ρ

μ_{180}=*rotation* 180° *clockwise around* ρ

μ_{270}=*rotation* 270° *clockwise around* ρ

ρ_H=*reflection through* H

ρ_V=*reflection through* V

ρ_1=*reflection through* D_1

ρ_2=*reflection through* D_2.

Definition 2.1.1 ***(The action)***

An action of a group G on a nonempty set X is a function $f : G \times X \longrightarrow X$, denoted by $(g, x) \longmapsto g.x$, for all $g \in G$, $x \in X$ such that:
1) For any $x \in X$, $1.x = x$, where 1 is the identity element of G.
2) For any g_1,$g_2 \in G$, and each $x \in X$, $g_1.(g_2.x) = (g_1 g_2).x$.

Definition 2.1.2 ***(Representation map)***

Let G be a group and X a nonempty set. The group G is said to be acts on the set X if there is a morphism λ from G to S_X. The morphism $\lambda : G \longrightarrow S_X$ is called the representation map of G on X.

Proposition 2.1.1 *The action of a group G on a set X is equivalent to a group homomorphism from G to S_X the group of all permutations on X.*

Proof. Let G be a group acts on a set X. Define a map $\lambda_g : X \longrightarrow X$ by $\lambda_g(x) = g.x$, and we show that λ_g is a permutation for all $g \in G$.
• Clearly that for element $y \in X$, there is an element $x \in X$ $(x = g^{-1}.y)$
such that $\lambda_g(x) = y$.

$$\begin{aligned}\lambda_g(x) &= \lambda_g(g^{-1}.y)\\ &= g.(g^{-1}.y)\\ &= (gg^{-1})y\\ &= 1.y\\ &= y\end{aligned}$$

Then λ_g is surjective.
• Let x,$y \in X$ such that $\lambda_g(x) = \lambda_g(y)$.

$$\begin{aligned}\lambda_g(x) = \lambda_g(y) &\Rightarrow g^{-1}.\lambda_g(x) = g^{-1}.\lambda_g(y)\\ &\Rightarrow g^{-1}.(g.x) = g^{-1}.(g.y)\\ &\Rightarrow (g^{-1}g).x = (g^{-1}g).y\\ &\Rightarrow 1.x = 1.y\\ &\Rightarrow x = y\end{aligned}$$

Then λ_g is injective.

Now we show that the map $\lambda : G \to S_X$ defined by $\lambda(g) = \lambda_g$ is a group homomorphism. For all $g, h \in G$ and $x \in X$, we have

$$\begin{aligned}(\lambda_g \circ \lambda_h)(x) &= \lambda_g(\lambda_h(x)) \\ &= \lambda_g(h.x) \\ &= g.(h.x) \\ &= (gh).x \\ &= \lambda_{gh}(x)\end{aligned}$$

So, λ is a group homomorphism.

Let λ be a group homomorphism from G to S_X, we can define an action of G on X by $g.x = \lambda(g)(x)$ for all $g \in G$ and $x \in X$.

•$g.(h.x) = g.(\lambda(h)(x)) = \lambda(g)(\lambda(h)(x)) = (\lambda(g) \circ \lambda(h))(x) = \lambda(gh)(x) = (gh)(x)$

•$1.x = \lambda(1)(x) = Id_X(x) = x$

Then $g.x = \lambda(g)(x)$ define an action of G on X. ■

Example 2.1.1 *Let $G = D_4$, the symmetry group of a square and $X = \{1, 2, 3, 4\}$ the set of vertices of the square. D_4 consist of the following permutations:*
$\{(1), (13), (24), (1432), (12)(34), (14)(23), (13)(24)\}$
The elements of D_4 act on X as functions. For example, the permutation $(13)(24)$ acts on vertex 1 by sending it to vertex 3, on vertex 2 by sending it to vertex 4, and so on. It is easy to see that the axioms of a group action are satisfied.
In general, if X be a nonempty set and G a subgroup of S_X. Then X is a G-set under the group action $(\sigma, x) \longrightarrow \sigma(x)$ for $\sigma \in G$ and $x \in X$.

Lemma 2.1.1 *Let G be a group that acts on a set X.*
Then for each $g \in G$, and all x,$y \in X$, $g.x = y \Leftrightarrow g^{-1}.y = x$.

Proof. Suppose that $g.x = y$. Then, $g^{-1}.y = g^{-1}.(g.x) = (g^{-1}g).x$ (by condition 2))

$$e.x = x \text{ (by condition1))}$$

The converse implication :

$$g^{-1}.y = x \Rightarrow g.x = y$$

Suppose that $g^{-1}.y = x$

Then,

$$g.x = g.(g^{-1}.y) = (gg^{-1}).x \text{ (by condition 2))}$$

$$e.y = y \text{ (bycondition1))}$$

■

2.2 Orbits and Stabilizers

Definition 2.2.1 ***(Stabilizers)*** *Let X and let G act on X, If $x \in X$ then the stabilizer of x is :*

$$Stab(x) = \{g \in G/g.x = x\}$$

Example 2.2.1 *Let $X = \{1, 2, 3, 4, 5, 6\}$ and suppose that G is the permutation group given by the permutations*

$$e = (1)(2)(3)(4)(5)(6)$$
$$\sigma_1 = (12)(3456)$$
$$\sigma_2 = (1)(2)(35)(46)$$
$$\sigma_3 = (12)(3654)$$

Find the stabilizer of each element in X under the action of G on X.

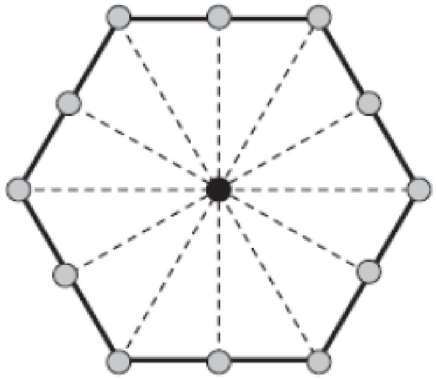
Solution: *The stabilizer are given in the following table:*

The elements	*The stabilizers*
1	$\{e, \sigma_2\}$
2	$\{e, \sigma_2\}$
3	$\{e\}$
4	$\{e\}$
5	$\{e\}$
6	$\{e\}$

Lemma 2.2.1 *For any $G \curvearrowright X$, and any $x \in X$, $Stab(x)$ is a subgroup of G.*

Definition 2.2.2 *Let $G \curvearrowright X$, if for some $x \in X$, $Stab(x) = \{e\}$, then x is said to be moved freely by the action of G, the action $G \curvearrowright X$ is free if for every $x \in X$, $Stab(x) = \{e\}$.*

Example 2.2.2 *The dihedral group D_n acts on a regular n-gon, if x is a vertex of this n-gon or the exact middle of any edge, then $Stab(x) \approx \mathbb{Z}_n$.*
Notice, however, that if x and x' are two vertices that are not directly opposite each other, then $Stab(x) \neq Stab(x')$. If x is the center of the n-gon then $Stab(x) \approx D_n$. If x is any point that is not a vertex, the center of an edge, or an a line connecting the center of the polygon to a vertex or the center of an edge, then $Stab(x) = \{e\}$.



Definition 2.2.3 ***(Orbits)*** *Let $G \curvearrowright X$ and let $x \in X$, then the orbit of x is*

$$Orb(x) = \{g.x/g \in G\}.$$

Example 2.2.3 *Let G be the permutation group defined by*
$G = \{(1), (123), (132), (45), (123)(45), (132)(45)\}$
and $X = \{1, 2, 3, 4, 5\}$. Then X is a G-set and the orbits are $O_1 = O_2 = O_3 = \{1, 2, 3\}$ and $O_4 = O_5 = \{4, 5\}$.

Theorem 2.2.1 *Given an action $G \curvearrowright X$ and any $x \in X$, there is a bijective correspondence between the set $Orb(x)$ and left cosets of $Stab(x)$, given by $g.x \leftrightarrow g.Stab(x)$.*

Definition 2.2.4 ***(Kernel of an action)*** *Let G be a group and X be a set. The* ***kernel*** *of the action of G on X is the kernel of the homomorphisme $\lambda : G \longrightarrow S_X$, denoted by $Ker(\lambda)$.*

Theorem 2.2.2 ***(Orbit-Stabilizer Theorem)*** *Let G be a finite group acting on X, then for any $x \in X$,*

$$| G |=| Stab(x) | . | Orb(x) |$$

Proof. The order of G is the product of the order of $Stab(x)$ and the index of $Stab(x)$ in G, But the index of $Stab(x)$ is the number of left cosets of $Stab(x)$, which by theorem 2.2.1 is the size of the orbit of x . ■

Corollary 2.2.1 *Let $G \curvearrowright X$. For any $x \in X$, where $Stab(x) = e$, there is a one-to-one correspondence between the elements of G and the elements of $Orb(x)$.*

Theorem 2.2.3 ***(Class Formula)*** *Let X be a finite G-set. Then*

$$| X |= \sum_{x \in X} \frac{| G |}{| Stab(x) |}$$

Proof. The orbits form a partition of X , So $X = \bigcup_{x \in X} O_x$, then $| X |= \sum_{x \in X} | O_x |$.
$| O_x |= \frac{|G|}{|Stab(x)|}$, then $| X |= \sum_{x \in X} \frac{|G|}{|Stab(x)|}$. ■

Definition 2.2.5 ***(Invariant)*** *Let G be a group act on a set X and $g \in G$. The* ***invariant*** *of g, denoted by $Fix(g)$, is the set of all elements of X fixed by g, that is*

$$Fix(g) = \{x \in X/g.x = x\}$$

Example 2.2.4 *Let $X = \{1, 2, 3, 4, 5, 6\}$ and suppose that G is the permutation group given by the permutations*

$$e = (1)(2)(3)(4)(5)(6)$$
$$\sigma_1 = (12)(3456)$$
$$\sigma_2 = (1)(2)(35)(46)$$

$$\sigma_3 = (12)(3654)$$

The invariants are given in the following table:

The permutations	*The invariants*
e	X
σ_1	$\emptyset$
σ_2	$\{1, 2\}$
σ_3	$\emptyset$

Remark 2.2.1 *It is important to remember that $Fix(g) \subset X$ and $Stab(x) \subset G$.*

Theorem 2.2.4 ***(Burnside's Formula)*** *If a group G acts on a set X, then the number of orbits is*

$$\frac{1}{\mid G \mid} \sum_{g \in G} \psi(g)$$

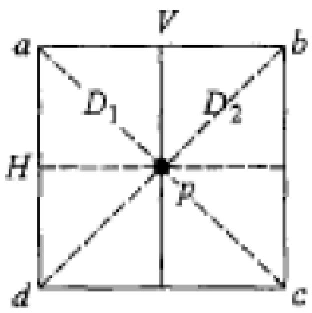
Example 2.2.5 *Consider the group $\{(1), (12), (34), (12)(34)\}$ acting on $\{1, 2, 3, 4\}$. Then, for example, the permutation (12) leaves only 3 and 4 invariant, so that $\psi((12)) = 2$. Here are all values $\psi(g)$:*

$\psi((1)) = 4$ $\qquad$ $\psi((12)) = 2$

$\psi((34)) = 2$ $\qquad$ $\psi((12)(34)) = 0$

The formula in Burnside's Theorem gives $\frac{1}{4}(4 + 2 + 2 + 0) = 2$. The two orbits are $\{1, 2\}$ and $\{3, 4\}$.

Example 2.2.6 *How many distinguishable ways can the four edges of a square be painted with four diffrent colors if there is no restriction on the number of times each color can be used, and two ways are considered indistinguishable if one can be obtained from the other by an isometry in the group of symmetries of the square?*



Solution: *The appropriate set X in this case is the set of $4^4 = 256$ ways of painting the edgeswithout regard to equivalence. If ρ is a group element then $\psi(\rho) = 4^k$, where k is the number of independant choices to be made in painting the edges so as to have invariance under ρ*

$\psi(\mu_0) = 4^4$ *(always $\psi(i) =\mid X \mid$ if, is the identity– four choices).*

$\psi(\mu_{90}) = 4$ *(for invariance under μ_{90}, all edges must be of the same color– one choices).*

$\psi(\mu_{180}) = 4^2$ *(pairs of opposite edges must be of the same color– two choices).*

$\psi(\mu_{270}) = 4$ *(like μ_{90}– one choice)*

$\psi(\rho_H) = 4^3$ *(ab and dc must be of the same color, ab and bc are independant– three choices).*

$\psi(\rho_V) = 4^3$ *(like ρ_H – three choices).*

$\psi(\rho_1) = 4^2$ *(ab and ad must be of the same color, cb and cd must be of the same color– two choices).*

$\psi(\rho_2) = 4^2$ *(like ρ_1– two choices).*

Therfore, by Burnside's Theorem, the number of distinguishable ways is:

$$\frac{1}{8}(4^4 + 4 + 4^2 + 4 + 4^3 + 4^3 + 4^2 + 4^2) = 55.$$

2.3 Types of actions

Definition 2.3.1 *(**Transitive action**) An action of a group G on a set X is said to be **transitive** if it has only one orbit.*

Example 2.3.1 *The action of the cyclic group C_n on the set $X = \{1, 2, .., n\}$ is transitive.*

Proposition 2.3.1 *An action of a group G on a set X is transitive if and only if for any x and y in X, there is some $g \in G$ such that $y = g.x$.*

Proof. Let X be a G-set

• $\Rightarrow$) Suppose that the action is transitive, so there is one orbit. For any $x \in X$, its orbits is X, so every element $y \in X$ has the from $g \cdot x$ for some $g \in G$.

• $\Leftarrow$) Conversely suppose that for any $x, y \in X$ we can write $y = g \cdot x$ for some $g \in G$. Fix $x \in X$. Because every $y \in X$ has the form $g \cdot x$ for some $g \in G$, every y is in the orbit of x. Thus X has only one orbit. ■

Definition 2.3.2 ***(Faithful action)*** *The action of G on X is called* ***faithful*** *if its kernel is trivial.*

Example 2.3.2 *The action of a symmetric group S_n on the set $\{1, 2, ..., n\}$ is faithful.*

Proposition 2.3.2 *An action of G on X is faithful if and only if the homomorphism $\lambda : G \longrightarrow S_X$ is injective.*

Proof. Let G be a group acts on a set X.
• $\Rightarrow$) : Suppose that the action is faithful, so $Ker(\lambda) = e$. Let $x, y \in G$ such that $\lambda(x) = \lambda(y)$. We have

$$\begin{aligned}\lambda(x)\lambda^{-1}(x) = e &\Rightarrow \lambda(y)\lambda(x^{-1}) = e\\ &\Rightarrow \lambda(yx^{-1}) = e\\ &\Rightarrow yx^{-1} \in Ker(\lambda)\\ &\Rightarrow yx^{-1} = e\\ &\Rightarrow y = x\end{aligned}$$

Then the homomorphism λ is injective.
• $\Leftarrow$) : Suppose that λ is injective. Let $g \in Ker(\lambda)$. So $\lambda(g) = e = \lambda(e)$, then $g = e_G$. So, the kernel of λ is trivial, then the action is faithful. ■

Definition 2.3.3 ***(Free action)*** *Let X be a G-set. The action of G on X is said to be* ***free*** *if for every $x \in X$, $Stab(x) = \{e\}$, where e is the identity of G.*

Example 2.3.3 *The action of a group G on itself by left multiplication is free.*

Example 2.3.4 *The action of a group G on itself by left conjugation is not free.*

Definition 2.3.4 ***(Regular action)*** *Let X be a G-set. The action of G on X is* ***regular*** *if it is transitive and free. In this case, we also say that G is* ***regular*** *on X.*

Example 2.3.5 *The action of a group G on itself by left multiplication is regular.*

2.4 Pólya's Enumeration Theorems

2.4.1 Cycles indexes

Definition 2.4.1 ***(Cycle index monomial)*** *Let G be a subgroup of S_n and $\pi \in G$. If π is a permutation with j_k cycle of length k, then the* ***cycle index monomial*** *associated with the permutation π is*

$$\mathrm{Ci}\, m(\pi) = \prod_{i=1}^{n} x_i^{k_i}$$

How to determinte the monomiale associated to a permutations? *To determine the monomial associated to a permutation, we need to write the permutation in cycle notation and then determine the monomial based on the number of cycles of each length. Specifically, if π is a permutation of $[n]$ with j_k cycles of length k for $1 \leq k \leq n$, then the monomial associated to π is $x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}$. Note that $j_1 + 2j_2 + \ldots + nj_n = n$.*

Example 2.4.1 *The permutation $\pi_1 = (1234)$ is associated with the monomial x_4^1 since it consists of a single cycle of length 4.*

Example 2.4.2 *The permutation $\pi_2 = (14)(2)(3)$ we have two 1-cycles and one 2-cycle, yielding the monomial $x_1^2 x_2^1$.*

Example 2.4.3 *Determine the cycle index monomial for each element of C_5.*
Solution: *The following table gives the cycle index monomial of each $\sigma \in C_5$.*

permutation	*cycle decomposition*	*cycle index monomial*
e	(1)(2)(3)(4)(5)	x_1^5
σ_1	(12345)	x_5
σ_2	(13524)	x_5
σ_3	(14253)	x_5
σ_4	(15432)	x_5

Definition 2.4.2 ***(cycle index polynomial)***
Let G be a subgroup of S_n. The cycle index poly.nomial of G is a polynomial of n variables

x_1,x_2,...,x_n , *denoted by* P_G*, that is*

$$P_G(x_1, x_2, ..., x_n) = \frac{1}{| G |} \sum_{\pi \in G} \text{Ci}\, m(\pi)$$

Example 2.4.4 *Find the cycle index polynomial associated with* C_5*.*

$$P_{C_5}(x_1, x_2, x_3, x_4, x_5) = \frac{1}{5}(x_1^5 + 4x_5)$$

Example 2.4.5 *Find the cycle index polynomial associated with* D_8*.*

Solution: D_8 *is the dihderal group of a square. Now we find the cycle index polynomial associated with* D_8*, see the following table:*

permutations	*Monomial*	*Fixed coloring*
$i = (1)(2)(3)(4)$	x_1^4	16
$r_1 = (1234)$	x_4^1	2
$r_2 = (13)(24)$	x_2^2	4
$r_3 = (1432)$	x_4^1	2
$v = (12)(34)$	x_2^2	4
$h = (14)(23)$	x_2^2	4
$p = (14)(2)(3)$	$x_1^2x_2^1$	8
$n = (1)(24)(3)$	$x_1^2x_2^1$	8

we find:

$$P_{D_8}(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 2x_1^2x_2^1 + 3x_2^2 + 2x_4^1)$$

Definition 2.4.3 ***(The weight of a function)***

Let f *be a function from* X *to* Y*. We can assign a weight* w_y *to each* $y \in Y$*. The weight of* f *denoted* $w(f)$*, is the product of the weights of the elements of* Y *used in* f*, that is*

$$w(f) = \prod_{i=1}^{n} w_f(x_i)$$

Example 2.4.6 *Let* f *be a coloring of a square (function from the set of the vertices* $\{1, 2, 3, 4\}$ *to the set of colors {black,white}) defined by*

$f(1) = black$ $\quad$ $f(2) = white$ $\quad$ $f(3) = white$ $\quad$ $f(4) = black$

Suppose that the colors have the following weights:

$w_{black} = b$ $\qquad\qquad$ $w_{white} = w$

So, the weight of the coloring f is

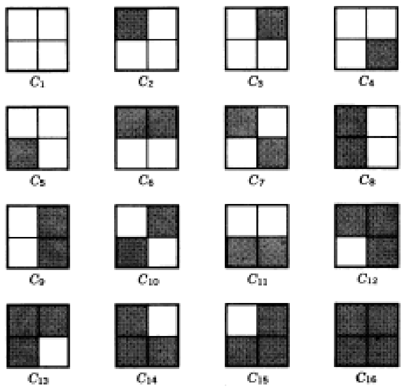
$$W(f) = \prod_{i=1}^{4} w_{f(i)} = b^2 w^2$$

2.4.2 Pólya's Enumeration Theorems

Definition 2.4.4 ***(Patterns)***

Let G be a group acts on a set X and Y be a nonempty set. The orbits under the extended action of G on the set Y^X are called ***patterns.***

Example 2.4.7 *How many patterns (ways) are there to color the squares of 2×2 chessboard using only the black and white colors?*



2 coloring C_i,C_j are the same if one can be obtained how the other by rotation or reflection, for example: C_1

$C_2 \sim C_3 \sim C_4 \sim C_5$

$C_6 \sim C_8 \sim C_9 \sim C_7$

$C_{10} \sim C_{11}$

$C_{12} \sim C_{13} \sim C_{14} \sim C_{15}$

C_{16}

Hence, we have six differents colorings patterns $\{C_1\}$,$\{C_2, C_3, C_4, C_5\}$,$\{C_6, C_7, C_8, C_9\}$,$\{C_{10}$,$C_{11}\}$, $\{C_{12}, C_{13}, C_{14}, C_{15}\}$,$\{C_{16}\}$.

Theorem 2.4.1 ***(Pólya's First Enumeration Theorem)***
Let X be a set with $\mid X \mid = n$ and C the set of colorings of X usings the colors $c_1, ..., c_m$. Let a permutations group G act on X to induce an equivalent ralation on C. Then,

$$P_G(\sum_{i=1}^{m} c_i, \sum_{i=1}^{m} c_i^2, ..., \sum_{i=1}^{m} c_i^n)$$

is the generating function for the number of nonequivalent colorings of X in C.

Example 2.4.8 *If $G = \{e\}$ then any 2 colorings are distinguishable, so that the number of patterns is the number of colorings. Because*

$$P_G(x_1, x_2, ..., x_n) = P_e(x_1, x_2, ..., x_n)$$
$$= x_1^n$$

Example 2.4.9 *How many different colorings of the sides of a regular pentagon using three colors under rotations and reflexions.*
Solution:*In this case $X = \{1, 2, 3, 4, 5\}$, $Y = \{c_1, c_2, c_3\}$ and $G = D_5$*
The cycle index polynomial of D_5 is

$$\frac{1}{10}(x_1^5 + 5x_1x_2^2 + 4x_5)$$

The number of different colorings of the sides of a regular pentagon using three colors under rotations and reflexions is $P_{D_5}(3, 3, 3, 3, 3) = \frac{1}{10}(3^5 + 5 \times 3 \times 3^2 + 4 \times 3) = 39$
We can also using the Burnside's Lemma to count the number of patterns because the patterns are orbits under an appropriate group action.

Definition 2.4.5 ***(weight of pattern)***
Let P be a pattern. The Weight of P, denoted by $W(P)$, is the common weight of all functions in P.

Theorem 2.4.2 ***(Pólya's Second Enumeration Theorem)***
Let C be the set of all functions from an n-set X to an m-set Y and G be a subgroup of S_n with cycle index polynomial $P_G(x_1, x_2, ..., x_n)$. The pattern inventory under G is

$$PI_G(\sum_{i=1}^{m} w_{y_i}, \sum_{i=1}^{m} w_{y_i}^2, ..., \sum_{i=1}^{m} w_{y_i}^n)$$

Example 2.4.10 *Continuing with the 2-coloring of a square with the weight 'b' and 'w'. the pattern inventory is given by*

$$PI_G(\sum_{i=1}^{m} w_{y_i}, \sum_{i=1}^{m} w_{y_i}^2, \sum_{i=1}^{m} w_{y_i}^3, \sum_{i=1}^{m} w_{y_i}^4) = P_{C_4}(b+w, b^2+w^2, b^3+w^3, b^4+w^4)$$

$$= \frac{1}{4}((b+w)^4 + (b^2+w^2)^2 + 2(b^4+w^4))$$

$$= \frac{1}{4}(4b^4 + 4b^3w + 8b^2w^2 + 4bw^3 + 4w$$

$$= b^4 + b^3w + 2b^2w^2 + bw^3 + w^4.$$

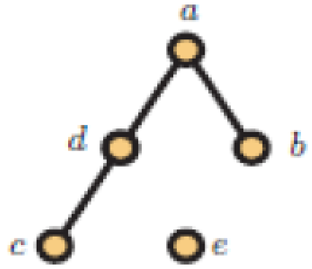
Chapter 3

The problem of isomorphism of two graphs

3.1 The graphs

Definition 3.1.1 *A graph G is an ordered pair (V, E), consisting of a vertex set V and an edge set $E \subseteq [V]^2$. Vertices are also called points or nodes. Edges are also called lines or arcs. In our definition of graph there are no loops or multiple edges. In a drawing of a graph, two vertices x and y are joined by a line if and only if $\{x, y\} \in E$. Two vertices joined by a line are said to be adjacent; If they are not joined by a line, they are nonadjacent.*

Example 3.1.1 *we could define a graph $G = (V, E)$ with vertex set $V = \{a, b, c, d\}$ and edge set $E = \{\{a, b\}, \{a, d\}, \{c, d\}\}$.*

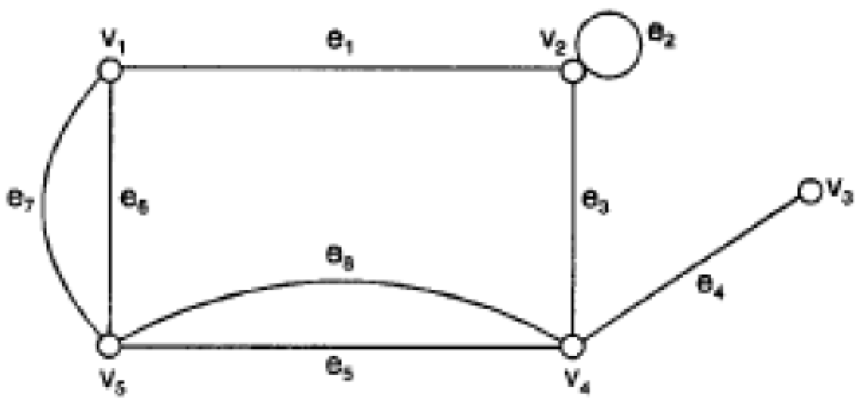


Notice that no edge is incident to e, which is perfectly permissible based on our definition. It is quite common to identify a graph with a visualization in which we draw a point for each *vertex and a line connecting two vertices if they are adjacent. It's important to remember that while a drawing of a graph is a helpful too, it is not the same as the graph. We could draw G in any of several different ways without changing what it is as a graph.*

1. Let $V = \{1,2,3,4\}$ *and* $E = \{\{1,2\},\{1,3\},\{3,2\},\{4,4\}\}$ *then* $G(V,E)$ *is a graph*

2. Let $V = \{1,2,3,4\}$ *and* $E = \{\{1,5\},\{2,3\}\}$ *then* $G(V,E)$ *is not a graph as* 5 *is not in* V*.*

3.



A graph with 5-vertices and 8-edges is called a (5,8) graph.

3.2 Graphs isomorphism

Definition 3.2.1 *Let* $G_1 = \{V_1, E_1\}$ *and* $G_2 = \{V_2, E_2\}$ *be two graphs. A function* $f : V_1 \longrightarrow V_1$ *is called a graphs isomorphism if:*

i) f *is one-to-one and into.*

ii) for all $x, y \in V_1, \{x,y\} \in E_1$ *if and only if* $\{f(x), f(y)\} \in E_2$ *when such a function exists, then* G_1 *and* G_2 *are called isomorphic graphs and is written as* $G_1 \cong G_2$*.*

In other words, two graphs G_1 *and* G_2 *are said to be isomorphic to each other if there is a one-to-one correspondence between their vertices and between edges such that incidence relationship is preserved written as* $G_1 \cong G_2$*.*

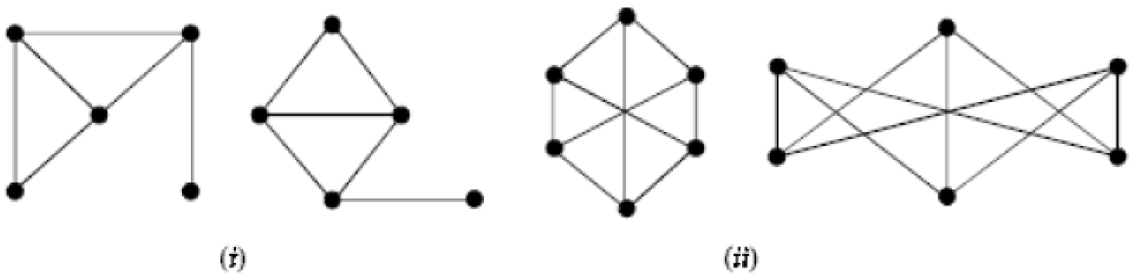
The necessery conditions *for two graphs to be isomorphic are :*

1-Both must have ***the same number of vertices.***

2-Both must have ***the same number of edges.***

3-Both must have ***equal number of vertices*** *with* ***the same degree.***

4-They must have the same degree sequence and same cycle vector $(c_1, ..., c_n)$*, where* c_i *is the number of cycles of length* i*.*



(i),(ii),(iii) Isomorphic pair of graphs

Two graphs that are not isomorphic.

Example 3.2.1 *Show that two graphs are isomorphic*

Solution*:*

Here $V(G_1) = \{1, 2, 3, 4\}$, $V(G_2) = \{a, b, c, d\}$

$E(G_1) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$ *and* $E(G_2) = \{\{a, b\}, \{b, d\}, \{d, c\}\}$

Define a function $f : V(G_1) \longrightarrow V(G_2)$ *as* $f(1) = a$*,* $f(2) = b$*,* $f(3) = d$*,* $f(4) = c$*,*

f *is clearly one-to-one and onto, Hence an isomorphism.*

Further, $\{1, 2\} \in E(G_1)$ *and* $\{f(1), f(2)\} = \{a, b\} \in E(G_2)$

$\{2, 3\} \in E(G_1)$ *and* $\{f(2), f(3)\} = \{b, d\} \in E(G_2)$

$\{3, 4\} \in E(G_1)$ *and* $\{f(3), f(4)\} = \{d, c\} \in E(G_2)$

$\{1, 3\} \notin E(G_1)$ *and* $\{f(1), f(3)\} = \{a, d\} \notin E(G_2)$

$\{1, 4\} \notin E(G_1)$ *and* $\{f(1), f(4)\} = \{a, c\} \notin E(G_2)$

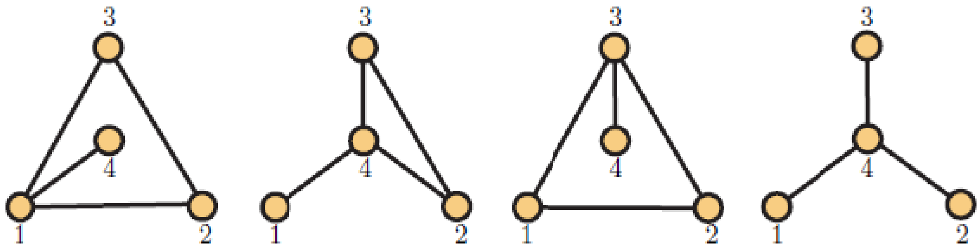
$\{2, 4\} \notin E(G_1)$ *and* $\{f(2), f(4)\} = \{b, c\} \notin E(G_2)$

Hence, f *preserves adjacency as well as no adjacency of the vertices.*

Therfore, G_1 *and* G_2 *are isomorphic.*

3.3 Counting nonisomorphic graphs

Counting the graphs with vertex set $[n]$ *is not difficult. There are* $C(n,2)$ *possible edges, each of which can be included or excluded. Thus, there are* $2^{C(n,2)}$ *labled graphs on* n *vertices. It's only a bit of extra thought to determine that if you only want to count the labled graphs on* n *vertices with* k *edges, you simply must choose a* k*-element subset of the set of all* $C(n,2)$ *possible edges. Thus, there are* $\binom{\binom{n}{2}}{k}$ *graphs with vertex set* $[n]$ *and exactly* k *edges see* [22] *for more information about the subject.*



We show four different labeled graphs on four vertices. The first three graphs shown there, however, are isomorphic to each other. Thus, only two nonisomorphic graphs on four vertices are illustrated in the figure.
*To account for isomorphisms, we need to bring Pólya's enumeration theorem into play***:** *we need to determine the cycle index of the appropriate group action. A graph* $G = (V, E)$ *may be identified with a function* $f : [V]^2 \longrightarrow \{0,1\}$*, where* $f(\{x,y\}) = 1$ *or* 0 *according to whether or not* $\{x,y\}$ *is a member of* E*. We say that two graphs* G_1 *and* G_2 *are isomorphic if there is a bijection between their vertex sets* V_1 *to* V_2 *that preserves adjacency. We normally ragard two isomorphic graphs as the same graphs. Two graphs are isomorphic if the corresponding functions are aqual uo to a permutation of* V*. As any permutation of* V *is allowed, the group acting on* V *is* S_n*, where* $n = \mid V \mid$*. Our goal is to calculate the cycle index* $P_{S_4^{(2)}}$*.*

How to find the cycle index :

Since, any vertex can be mapped to any other vertex, the symmetrie group S_4 *acts on the vertices. When we were working with colorings of the vertices of the square, we realized that all the vertices appearing in the same cycle of a permutation* π *had to be colored by the same color. Since we're concerned with edges here and not vertex colorings, what we really need* *for a permutation to fix a graph is that every edge be sent to an edge and every non-edge be sent to a non-edge. To be specific, if* $\{1, 2\}$ *is an edge of some* G *and* $\pi \in S_4$ *fixes* G*, then* $\{\pi(1), \pi(2)\}$ *must also be an edge of* G*. Similary, if vertices* 3 *and* 4 *are not adjacent in* G*, then* $\pi(3)$ *and* $\pi(4)$ *must also be nonadjacent in* G*.*

Case 1*:To account for edges, we move from the symmetric group* S_4 *to its pair group* $S_4^{(2)}$*. The objects that* $S_4^{(2)}$ *permutes are the* 2*-element subsets of* $\{1, 2, 3, 4\}$*. For ease of notation, we will denote the* 2*-element subset* $\{i, j\}$ *by* e_{ij} *. To find the permutations in* $S_4^{(2)}$*, we consider the vertex permutations in* S_4 *and see how they permute the* e_{ij} *. The identity permutation* $I = (1)(2)(3)(4)$ *of* S_4 *corresponds to the identity permutation* $I = (e_{12})(e_{13})(e_{14})(e_{23})(e_{24})(e_{34})$ *of* $S_4^{(2)}$*.*

Case 2: *Now let's consider the permutation*$(12)(3)(4)$*. It fixes* e_{12} *since it sends* 1 *to* 2 *and* 2 *to* 1*. It also fixeds* e_{34} *by fixing* 3 *and* 4*. However, it interchanges* e_{13} *with* e_{23} *(*3 *is fixed and* 1 *is swapped with* 2*) and* e_{14} *with* e_{24} *(*1 *is sent to* 2 *and* 4 *is fixed). Thus, the corresponding permutation of pairs is* $(e_{12})(e_{13}e_{23})(e_{14}e_{24})(e_{34})$*.*

For another example, consider the permutation $(123)(4)$*. It corresponds to the permutation* $(e_{12}e_{23}e_{13})(e_{14}e_{24}e_{34})$ *in* $S_4^{(2)}$*.*

Case 3*:Since we're only after the cycle index of* $S_4^{(2)}$*, we don't need to find all* 24 *permutations in the pair group. However, we do need to know the types of those permutations in terms of cycle lengths so we can associate the appropriate monomials. For the three examples we've considered, the cycle structure of the permutation in the pair group doesn't depend on the original permutation in* S_4 *other than for its cycle structure. Any permutation in* S_4 *consisting of a* 2*-cycle and two* 1*-cycles will correspond to a permutation with two* 2*-cycles and two* 1*-cycles in* $S_4^{(2)}$*. A permutation in* S_4 *with one* 3*-cycle and one* 1*-cycle will correspond to a permutation with two* 3*-cycles in the pair group. By considering an example of a permutation in* S_4 *consisting of a single* 4*-cycle, we find that the corresponding permutation in the pair group has a* 4*-cycle and a* 2*-cycle. Finally, a permutation of* S_4 *consisting of two* 2*-cycles corresponds to a permutation in* $S_4^{(2)}$ *having two* 2*-cycles and two* 1*-cycles.*

Case 4*:Now that we know the cycle structure of the permutations in* $S_4^{(2)}$ *, the only task remaining before we can find its cycle index of is to determine how many permutations have each of the possible cycle structures. For this, we again refer back to permutations of the* *symmetric group S_4. A permutation consisting of a single 4-cycle begins with 1 and then has 2, 3, and 4 in any of the $3! = 6$ possible orders, so there are 6 such permutations. For permutations consisting of a 1-cycle and a 3-cycle, there are 4 ways to choose the element for the 1-cycle and then 2 ways to arrange the other three as a 3-cycle. (Remember the smallest of them must be placed first, so there are then 2 ways to arrange the remaining two.) Thus, there are 8 such permutations. For a permutation consisting of two 1-cycles and a 2-cycle, there are $C(4; 2) = 6$ ways to choose the two elements for the 2-cycle. Thus, there are 6 such permutations. For a permutation to consist of two 2- cycles, there are $C(4; 2) = 6$ ways to choose two elements for the first 2-cycle. The other two are then put in the second 2-cycle. However, this counts each permutation twice, once for when the first 2-cycle is the chosen pair and once for when it is the "other two". Thus, there are 3 permutations consisting of two 2-cycles. Finally, only consists of four 1-cycles.*

We apply Polya's theorem to the cycle index $P_{S_4^{(2)}}$ to enumerate nonisomorphic graphs of order 4 by number of edges:

$$P_{S_4^{(2)}}(x_1, ..., x_6) = \frac{1}{24}(x_6^1 + 9x_1^2x_2^2 + 8x_3^2 + 6x_2x_4).$$

To use this to enumerate graphs, we can now make the substitution $x_i = 1+x_i$ for $1 \leq i \leq 6$. This allows us to account for the two options of an edge not being present or being present. In doing so, we find

$$P_{S_4^{(2)}}(1 + x, ..., 1 + x^6) = 1 + x + 2x^2 + 3x^3 + 2x^4 + x^5 + x^6.$$

is the generating function for the number of 4-vertex graphs with m edges, $0 \leq m \leq 6$. To find the total number of nonisomorphic graphs on four vertices, we substitute $x = 1$ into this polynomial. This allows us to conclude there are 11 nonisomorphic graphs on four vertices, a marked reduction from the 64 labeled graphs.

The techniques of this subsection can be used, given enough computing power, to find the number of nonisomorphic graphs on any number of vertices.

Example 3.3.1

Order	*Number of graphs*
1	1
2	2
3	4
4	1
5	34
6	156
7	1044
8	12346
9	274668
10	12005168

Lemma 3.3.1 *The number of nonisomorphic graphs of order n is $P_{S_n^{(2)}}$.*

Example 3.3.2 *For 30 vertices, there are*

334494316309257669249439569928080028956631479935393064329967834887217734534

880582749030521599504384 $\simeq 3,3 \times 10^{98}$

nonisomorphic graphs, as compared to $2^{435} \simeq 8,9\times\ 10^{130}$ *labeled graphs on 30 vertices. And the number of nonisomorphic graphs with precisely 200 edges is*

313382480997072627625877247573364018544676703365501785583608267705079

9699893512219821910360979601 $\simeq 3,1\times\ 10^{96}$.

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Abstract

In this memory we have given some examples of isomorphic and non isomorphic graphs. Also, we have shown how to use the Polya enumeration theorem to count the number of non isomorphic graphs on n vertices. Since, if n=50 vertices in the graphs, then there is n!=50!=304109320171337804361260 8166064768844377641568960512000000000000 permutations to check if the graphs are isomorphic or not and this number is very large and we say that NP-hard problem.

Keywords: Polya's theorem, permutation, isomorphic graphs, non isomorphic graphs.

Résumé

Dans ce mémoire on a donné quelques exemples de graphs isomorphes et non isomorphes. De plus, nous avons montré comment utiliser le théorème d'énumération de Polya pour compter le nombre de graphs non isomorphes sur n sommets. Puisque, si n=50 sommets dans les graphes, alors il y a n !=50 !=30410932017133780436126081660647688443776415689605120 000000000000 permutations pour vérifier si les graphes sont isomorphes ou non et ce nombre est très grand et on dit que le problème est NP-hard.

Les mots clés : Théorème de Polya, permutation, graphes isomorphes, graphes non isomorphes

ملخص

في هذه المذكرة أعطينا بعض الأمثلة على الرسومات البيانية المتماثلة. أيضا وضحنا كيفية استخدام نظرية تعداد بوليا لحساب عدد الرسومات البيانية غير المتماثلة على n رأس. لأنه إذا كان $n = 50$ رأسا في الرسومات البيانية.لدينا :

$$n! = 50! = 30410932017133780436126081660647688443776415689605120000000000000$$

تبديلة للتحقق مما إذا كانت الرسومات البيانية متماثلة أم لا و هذا العد كبير جدا و نقول على هذا المشكل انه صعب جدا

الكلمات المفتاحية: نظرية بوليا، تبديلة، رسومات بيانية متماثلة، رسومات بيانية غير متماثلةم.