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Existence results for a problems involving Hardy Potentials

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CONTENTS

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NOTATION

We introduce the necessary notations and definitions that are used

Symbol	**Signification**
$\mathbb{N}$	Set of natural numbers.
$\mathbb{N}^*$	Set of nonzero natural numbers.
$\mathbb{R}$	Set of real numbers.
$\mathbb{R}_+$	Set of positive real numbers.
$\mathbb{R}^N$	Euclidean space of dimension N.
Ω	Open of $\mathbb{R}^N$ equipped with the measure of Lebesgue.
u	Defined measurable function of Ω in $\mathbb{R}$.
∇u	gradient of u, $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_N}\right)$.
Δu	Laplacian of u, $\nabla u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_N^2}$.
$C(\Omega)$	Space of continuous functions on Ω.
$C^k(\Omega)$	space of continuous functions on Ω whose partial derivatives of order k are continuous on , Ω, k positive integer.
$C^1(\Omega, \mathbb{R})$	The set of differentiable functions and the derivative is continuous.
$C^\infty(\Omega)$	space $C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega)$.
$L^p(\Omega)$	$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}^N; \text{f is measurable and } \int_\Omega \lvert f(x) \rvert^p \, dx < \infty \right\} (1 < p < \infty, constant)$.
$L^\infty(\Omega)$	$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R}^N; f\text{is measurable and } \lvert f(x) \rvert \leq c \text{ a.e.on } \Omega \right\}$.
$\mathcal{D}(\Omega)$	Space of indefinitely differentiable functions in Ω, with compact support in Ω.

$W^{1.p}(\Omega)$	Sobolev space of functions of $L^p(\Omega)$ whose derivatives partial in the sense of first-order distributions are also in $L^p(\Omega)$, provided with the norm $\Vert u\Vert_{W^{1.p}}=\sum_{i=1}^{N}\Vert\partial_i u\Vert_{L^p}$.
$W_0^{1.p}(\Omega)$	The closing of $\mathcal{D}(\Omega)$ in $W^{1.p}(\Omega)$, $W_0^{1.p}(\Omega)=\overline{\mathcal{D}(\Omega)}^{W^{1.p}}$.
$H^1(\Omega)$	$u\in L^2(\Omega)$ et $\nabla u\in L^2(\Omega)$.
$H_0^1(\Omega)$	is $\overline{\mathcal{D}(\Omega)^{H^1(\Omega)}}$.
$\rightharpoonup$	Weakly converges.
$\to$	Strongly converges .
$\cdot$	the scalar product.
$\langle\cdot,\cdot\rangle$	the dual product.
$E\hookrightarrow F$	E is continuously injected into F.
$E\hookrightarrow_c F$	E is injected in a compact way into F.
p^*	The sobolev exponent, such that $p^*=\frac{Np}{N-p}$.
p'	Hölder's conjugate of p, $p'=\frac{p}{p-1}$, if $p>1$ and $p'=\infty$ if $p=1$.
E^*	dual topology .
$D(A)$	domain of definition of an unbounded operator A.
$R(A)$	image of A which is also noted by ImA.
$\mathcal{L}(E,F)$	set of continuous linear applications.
a.e	Almost everywhere.
i.e.	that is to say.
$(P.S)$	Palais-Smale condition.
$(P.S)_c$	Palais-Smale condition at the level c.
$\sum_{i=1}^{N}$	Sum for i various between 1 and N.
$[\cdot]^+,[\cdot]^-$	the positive and the negative parts, respectively, i.e. $t^+=max\{t,0\}$ and $t^-=min\{-t,0\}$.
$\Vert\cdot\Vert$	a norm on a Banach space.
$\vert\cdot\vert$	a semi-norm on a Banach space, or an Euclidean norm in $\mathbb{R}^N$.

GENERAL INTRODUCTION

Hardy inequality has a long history and many variants. Together with the Sobolev inequalities, it is one of the most frequently used inequalities in analysis. In this work, we present some of its extensions and applications. The story started in **1915**-as **Godfrey Harold Hardy** needed an estimate for the arithmetic means, more precisely, an inequality of the form

$$\sum_{n=1}^{\infty}\left|\frac{1}{n}A_n\right|^2 \leq C_2\sum_{n=1}^{\infty}|a_n|^2 \tag{1}$$

with $A_n = \sum_{k=1}^{n} a_k$ and $a = \{a_k\}$ a sequence. Hardy's original motivation was obviously to find an elementary proof of another famous inequality, namely the Hilbert inequality (see [2]or Appendix A). This inequality (the discrete version) was then extended to

$$\sum_{n=1}^{\infty}\left|\frac{1}{n}A_n\right|^p \leq C_p\sum_{n=1}^{\infty}|a_n|^p, \quad 1 < p < \infty, \tag{2}$$

with the continuous (=integral) counterpart

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p\,dx \leq C_p\int_0^{\infty} f(x)^p\,dx, \quad f \geq 0. \tag{3}$$

This inequality was finally stated and proved by Hardy in his famous **1925** (see [7]).
While the version I used was **1998** published by **J.P Garcıa Azorero** and **I. Peral Alonso** (see[12])

$$\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^p}\,dx \leq C_{N,p}\int_{\mathbb{R}^N}|\nabla u|^p\,dx. \tag{4}$$

(The reader may notice many forms that have the same name as Hardy). we apply (4) inequality in article [1].

The objective of our work is to study some elliptic problems using Critical Point Theory, this techniques are to proving the existence of solutions. It is fortunate (for some of us) that many differential equation problems can be handled by variational techniques, in other words, by considering an associated real-valued function $J : E \to \mathbb{R}^N$ and by looking for points of minimum, maximum or mini-max of J, so that our given problem reads $J'(u) = 0$. through

the application of variational methods and critical point theory. Semi-linear elliptic equations arise in a variety of contexts in geometry, physics, mechanics, engineering and, more recently, in life sciences. None of these fields can be investigated without taking into account nonlinear phenomena, and variational methods, as a branch or an evolution of the Calculus of Variations, are almost entirely concerned with non-linearity.
This work is divided into three chapters that we will present.
In the first chapter, we have collected some of the concepts that we need to study the problems in the next two chapters, and we mentioned the concept of critical points and some spaces such as Lebesgue space and Sobolev space, and some related properties. Here, we focus on the following references [8, 10, 11, 13, 15, 17].

In the second chapter, we present Lemma of Hardy inequality which proved by **J.P Garcıa Azorero** and **I. Peral Alonso** , we refer to [12], then we present some auxiliary results on Weighted Lebesgue Space) in article [9] published in (**2013**) and the famous Theorem which called Mountain Pass Theorem that we need it in study two elliptic problems. we try to apply Hardy inequality to the following elliptic linear problem

$$\begin{cases} -\Delta u = \gamma \frac{u}{x^2} + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

by considering $f \in L^2(\Omega)$, $0 \in \Omega$, γ is a real parameter.
Then consider a quasilinear elliptic problem involving Hardy potential. the difficulty in this problem is the nonlinear therm $f(x, u)$, we can not apply Lax Milligram Theorem. so we study the existence at least non-trivial weak solution using Mountain-Pass Theorem,
more precisely, we consider our problem is a particular case of a problems studied in [1] which published in (**2022**).

$$\begin{cases} -\Delta u = \gamma \frac{u}{|x|^2} + f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ be open and bounded, $0 \in \Omega$, γ is a real parameter.
we introduce conditions for solve problem. the more important articles and books that used [1, 6, 9, 10, 12, 15].

For the third chapter, By variational method deals multiplicity of solutions for the following Kirchhoff type problem with Hardy type potentials that were presented in the article [14], published in (**2014**)

$$\begin{cases} -M(\int_\Omega |\nabla u|^2 \ dx)\Delta u = \mu \ a(x)\frac{u}{|x|^2} + \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is bounded domain with smooth boundary $\partial\Omega$, $0 \in \Omega$, $M : \mathbb{R}_0^+ \to \mathbb{R}$ is continuous and increasing function with $\mathbb{R}_0^+ := [0, +\infty)$, the function $a : \Omega \to \mathbb{R}$ may change sign, λ is positive parameter, $0 \leq \mu < \frac{1}{C_{N,2}}$, where $C_{N,2} = \left(\frac{2}{N-2}\right)^2$ is optimal constant in the

Hardy Inequality. we prove Theorem 3.1 by using The proofs based a recent result on the existence of at least three critical points which proved by **G. Bonanno** we refer to [5]. see the text in Proposition 3.1. In this section, we focus mainly on [4, 5, 14].

CHAPTER 1

SOME PRELIMINARIES

In this chapter we give a series of notions that will be useful for the rest of this thesis. We recall some definitions and properties used in the remainder of the these thesis, this to allow a more rapid assimilation of our work. These consist essentially of functional analysis results. We thus find some properties for convergence, continuity and differentiability of operators in Banach space, Sobolev Embeddings and Inequalities, critical points theory and Some Theorems used.

1.1 Operators on Banach Spaces.

Let E and F be two Banach space.

Definition 1.1 *[17]**(Linear Bounded Operator)** Let A be a linear operator such that $D(A) = E$ et $R(A) \subset F$. We say that A is bounded if it is bounded on the unit ball $\overline{B}(0,1)$, i.e. If the set*

$$\{\|Ax\|_F : \quad x \in E, \|x\|_E \leq 1\}$$

is bounded.
According to this definition, if A is bounded, there exists a constant $c > 0$ such that for all x where $\|x\|_E \leq 1$ we have the inequality

$$\|Ax\|_F \leq c.$$

1.1.1 Some Notion of Convergence

Definition 1.2 *(**Convergence of a sequence**) We say that a sequence $\{u_j\}_j$ of E converges (in norm), such that $u \in E$ if*

$$\lim_{j \to +\infty} \|u_j - u\|_E = 0, \qquad \text{and we write} \quad u_j \to u as\ j \to +\infty.$$

Definition 1.3 ***(Weak Convergence)*** *We say that a sequence $\{u_j\}_j$ of E weakly convergence such that $u \in E$ if*

$$\forall \varphi \in E^*, \langle \varphi, u_j \rangle \longrightarrow \langle \varphi, u \rangle$$

and we write $u_j \rightharpoonup u$.

Noted E^* dual space and E^{**} the bi-dual of E.

Definition 1.4 *[17]**(Dual Space)** The set of continuous linear functionals, defined on a normed vector space, constitutes a vector space. It is called dual of the space E and we denote E^*.*
with the norm

$$\|f\|_{E^*} = \sup_{x \in E, \|x\|_E \leq 1} |\langle f, x \rangle|, \quad \forall f \in E^*$$

The bi-dual E^{**} is the dual of E^* with norm

$$\|\xi\|_{E^{**}} = \sup_{f \in E^*, \|f\|_{E^*} \leq 1} |\langle \xi, f \rangle| \quad (\xi \in E^{**}).$$

1.1.2 Reflexive Spaces

Definition 1.5 *[8](**Canonical Injection**) There is canonical injection $J : E \to E^{**}$ defined as follows: given $x \in E$, the map $f \mapsto \langle f, x \rangle$ is a continuous linear functional on E^*; thus it is an element of E^{**}, which we denote by Jx. We have*

$$\langle Jx, f \rangle_{E^{**}, E^*} = \langle f, x \rangle_{E^*, E} \quad \forall x \in E, \quad \forall f \in E^*.$$

It is clear that J is linear and that J is an isometry, that is, $\|Jx\|_{E^{**}} = \|x\|_E$.

Definition 1.6 *[8](**Reflexive Space**) Let E be a Banach space and let $J : E \to E^{**}$ be the canonical injection from E into E^{**}. The space E is said to be reflexive if J is surjective , i.e, $J(E) = E^{**}$.*

Remark 1.1 *When E is reflexive the bi-dual E^{**} is usually identified with E.*

Theorem 1.1 *[8] Assume that E is reflexive Banach space. Let $\{u_j\}_{j \in \mathbb{N}}$ be bounded sequence in E. then, there exists a sub-sequence $\{u_{jk}\}$ that convergence weakly to some $x \in E$.*

Proposition 1.1 *[11] Let $\{u_j\}$ be a sequence of E. We have*

1. *If $u_j \longrightarrow u$, so $u_j \rightharpoonup u$.*
2. *If $u_j \rightharpoonup u$, so (u_j) is bounded and $\|u\|_E \leq \liminf_{j \longrightarrow \infty} \|u_j\|_E$.*

corollary 1.1 *Let $(E, \|\cdot\|_E)$ be a reflexive Banach space. let $\{u_j\}_j$ and u be in E such that as $n \to \infty$*

1. *$u_j \rightharpoonup u$ in E.*
2. *$\|u_j\|_E \to \|u\|_E$. Then $u_j \to u$ in E as $j \to \infty$*

1.1.3 Separable Spaces

Definition 1.7 *[8] We say that a metric space E is separable if there exists a subset $D \subset E$ that is countable and dense.*

1.1.4 Uniformly Convex Spaces

Definition 1.8 *[8] A Banach space is said to be uniformly convex if*

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; such \;\; that$$

$$[x, y \in E, \|x\| \leq 1, \|y\| \leq 1 \;\; and \;\; \|x-y\| > \epsilon] \Rightarrow \left[\left\|\frac{x+y}{2}\right\| < 1-\delta\right].$$

The uniform convexity is a geometric property of the unit ball: if we slide a rule of length $\epsilon > 0$ in the unit ball, then it is midpoint must stay within a ball of radius $(1-\delta)$ for some $\delta > 0$. In particular, the unit sphere must be round and cannot include any line segment.

Theorem 1.2 *[8] Every uniformly convex Banach space is reflexive.*

1.1.5 Continuity of Operators

We will consider operators T from E into F and we will give a definition concerning the properties of the continuity of T.
The easy notion is the following

Definition 1.9 *The operator $T : E \longrightarrow F$ is said to be continuous at u, if for any sequence $\{u_j\} \subset E$ which converges to u, $\{T(u_j)\}_j$ Converges to $T(u)$.*
T is called continuous on $\Omega \subset E$ if T is continuous at any point $u \in \Omega$.

Definition 1.10 *An operator $T : E \longrightarrow F$ is said to be compact if it is continuous and has the property, for any $\{u_j\}$ bounded in E, the sequence $\{T(u_j)\}$ admits a convergent sub-sequence.*

1.2 Functional Spaces

1.2.1 L^p Spaces

Definition 1.11 *[8] Let $p \in \mathbb{R}$ with $1 < p < \infty$, we set*

$$L^p(\Omega) = \left\{f : \Omega \to \mathbb{R}^N; f \text{ is measurable and } \int_\Omega |f(x)|^p \, dx < \infty\right\}.$$

with

$$\|f\|_{L^p} = \|f\|_p = \left[\int_\Omega |f(x)|^p \, dx\right]^{\frac{1}{p}}.$$

Definition 1.12 *[8] We set*

$$L^\infty(\Omega) = \left\{f : \Omega \to \mathbb{R}^N; f \text{is measurable and } |f(x)| \leq c \;\; a.e.on \;\; \Omega\right\},$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = inf\,\{c : |f(x)| \leq c \;\; a.e.on \;\; \Omega\},$$

with c is a constant.

Theorem 1.3 *[8] $L^p(\Omega)$ is a Banach space, for any $1 \leq p \leq \infty$, and reflexive for $1 \leq p < \infty$, and separable for $1 < p < \infty$.*

Notation 1.4 *.Let $1 \leq p \leq \infty$, we denote by p' the conjugate exponent*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 1.5 *[8](Hölder's inequality) Assume that $f \in L^p$ and $g \in L^p$ with $1 \leq p \leq \infty$, then $fg \in L^1$ and*

$$\int |fg| \leq \|f\|_p \|g\|_{p'} .$$

1.2.2 $W^{1.p}(\Omega)$ Spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.
We notice $\|u\|_{L^p} = \|u\|_p$.

Definition 1.13 *[8] We denote by $\mathcal{D}(\Omega)$ the set of function of class $\mathcal{C}^\infty(\Omega)$ with support compact include in Ω. The Sobolev space $W^{1.p}(\Omega)$ is defined by*

$$W^{1.p}(\Omega) = \left\{ u \in L^p(\Omega), \exists g_i \in L^p(\Omega) \textit{such that:} \int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega g_i \varphi dx, \forall \varphi \in \mathcal{D}(\Omega), \forall i = 1, 2, ..., N \right\}.$$

We set

$$H^1(\Omega) = W^{1.2}(\Omega).$$

For $u \in W^{1.p}(\Omega)$ we set $\frac{\partial u}{\partial x_i} = g_i$, and we write

$$\nabla u = grad\ u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N}).$$

The space $W^{1.p}$ is equipped with the norm $\|u\|_{W^{1.P}} = \|u\|_P + \|\nabla u\|_P$, or sometimes with the equivalent norm $\|u\|_{W^{1.P}} = (\|u\|_p^p + \|\nabla u\|_p^p)^{\frac{1}{p}}$ if $(1 \leq p < \infty)$.
The space $H^1(\Omega)$ is equipped with the scalar product

$$\int_\Omega uv dx + \int_\Omega \nabla u \nabla v dx.$$

The associated norm $\|u\|_{H^1} = (\|u\|_2^2 + \|\nabla u\|_2^2)^{\frac{1}{2}}$

Proposition 1.2 *[8] $W^{1.p}(\Omega)$ is a Banach space for $1 \leq p \leq \infty$, reflexive for $1 < p < \infty$, and separable for $1 \leq p < \infty$.*
In particular $H^1(\Omega)$ is reflexive, separable and Hilbert space.

1.2.3 Sobolev Embeddings and Inequalities

Let $(E, \|.\|_E), (F, \|.\|_F)$ Banach spaces

Notation 1.6

1. E is injected continuously into F, means that the canonical embedding $j : E \to F$ is continuous i.e. $\ni c > 0, \forall x \in E : \|x\|_F \leq c\|x\|_E$, and we denote by $E \hookrightarrow F$.
2. E is injected in compact into F means that the canonical embedding $j : E \to F$ is compact i.e. for all sequence bounded u_j in E we can extract sub-sequence $\{u_{jk}\}$ convergent in F, and we denote by $E \hookrightarrow_c F$.
   If $1 \leq p < \infty$, the sobolev exponent of p defined by $p^* = \frac{Np}{N-p}$ or $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.

Theorem 1.7 *[8] Let $1 \leq p \leq \infty$, we suppose that Ω is on open set of class $\mathcal{C}^1$ a bounded frontier, and we take $\Omega = \mathbb{R}^N_+$*

1. $W^{1.p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [1, p^*[\quad$ *if* $p < N$.
2. $W^{1.p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [p, \infty[\quad$ *if* $p = N$.
3. $W^{1.p}(\Omega) \hookrightarrow L^\infty(\Omega)$ *if* $p > N$.

Theorem 1.8 *[8](**Rellich-Kondrachon**) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $\mathcal{C}^1$*

1. $W^{1.p}(\Omega) \hookrightarrow_c L^q(\Omega) \quad \forall q \in [1.p^*[\quad$ *if* $p < N$.
2. $W^{1.p}(\Omega) \hookrightarrow_c L^q(\Omega) \quad \forall q \in [p.\infty[\quad$ *if* $p = N$.
3. $W^{1.p}(\Omega) \subset \mathcal{C}\left(\overline{\Omega}\right)$ *if* $p > N$.

1.2.4 $W_0^{1.p}(\Omega)$ Space

Definition 1.14 *Let $1 \leq p \leq \infty$, $W_0^{1.p}$ means the closing of $\mathcal{D}(\Omega)$ in $W^{1.p}$, we notice*

$$\begin{aligned} W_0^{1.p}(\Omega) &= \overline{\mathcal{D}(\Omega)}^{W^{1.p}} \\ &= \left\{u \in W^{1.P}(\Omega) : u = 0 \text{ on } \partial\Omega\right\}, \end{aligned}$$

and

$$H_0^1(\Omega) = W_0^{1.2}(\Omega).$$

The space $W_0^{1.p}(\Omega)$ provided with norm induced by $W^{1.p}$, H_0^1 is a Hilbert space for the scalar product of H^1.

Remark 1.2 *when $\Omega = \mathbb{R}^N$, we know that $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{1.p}(\mathbb{R}^N)$, and there for*

$$W_0^{1.p}(\mathbb{R}^N) = W^{1.p}(\mathbb{R}^N).$$

Proposition 1.3 *[8](**Poincaré's inequality**)Let $\Omega \subset \mathbb{R}^N$ on open set, Then there exists a constant C such that*

$$\|u\|_{W^{1.p}(\Omega)} \leq c\|\nabla u\|_p \quad \forall u \in W_0^{1.p}(\Omega).$$

In other words, on $W_0^{1.p}$, the quantity $\|\nabla u\|_p$ is a norm equivalent to the $W^{1.p}$ norm.

Theorem 1.9 *(**Young's inequality**) For $a, b \geq 0$ and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

1.3 Differentiability of Functionals and Critical Points

In what follows, we introduce some notion of derivatives for functions defined on Banach spaces, we start with the directional derivative.

1.3.1 Derivative in the sense of Gateau

Definition 1.15 *Let E be a Banach space, $\Omega \subseteq E$ an open set, and let $I : \Omega \longrightarrow \mathbb{R}$ a functional, we say that I is ad differentiable in the sense of Gateau (G-differentiable) at $u \in \Omega$, if there exists $A \in E'$ (A linear and continuous), denoted by $I'_G(u)$ such that, for all $v \in E$, where $I(u+tv)$ exists for $t > 0$ small enough, the directional derivative $DI(u)$ exists i.e:*

$$\lim_{t \longrightarrow 0} \frac{I(u+tv) - I(u)}{t} = \langle A, v \rangle$$

if I is differentiable in the sense of Gateau in u, there exists only one verified linear functional.

Example 1.1 *Let $I : L^p(\Omega) \to \mathbb{R}$, $I(u) = \int_\Omega |u|^p dx$. I is G-differentiable and we have $\langle I'_G(u), v\rangle = p \int_\Omega |u|^{p-2} uv dx$, indeed, let $x \in \Omega$, t sufficient small fixed and we define $g_{u.v}(s) = |u+tv|^p$. $s \in [0,t]$ let $g_{u.v} : [0,t] \to \mathbb{R}$, we have $g_{u.v}$ is continuous on the closed interval $[0,t]$ and differentiable on the interval open $]0,t[$, then according to the mean value theorem, there exists a real $c_t \in]0,t[$ and when $t \to 0$, we have $c_t \to 0$*

$$\begin{aligned} \lim_{t\to 0} \frac{|u+tv|^p - |u|^p}{t} &= \lim_{c_t \longrightarrow 0} p|u+c_t v|^{p-2}(u+c_t v)v \\ &= p|u|^{p-2}uv. \end{aligned}$$

According to Lebesgue's Convergence Theorem, we have

$$\langle I'(u), v\rangle = p\int_\Omega |u|^{p-2} uv dx.$$

and we define $A : L^p \to \mathbb{R}$

$$v \to A(v) = p\int_\Omega |u|^{p-2} uv dx$$

we prove that $A \in (L^p(\Omega))' = L^{p'}(\Omega)$, i.e A is linear and continuous
A is linear, in effect, let $v_1, v_2 \in L^p(\Omega)$ and let $\alpha.\beta \in \mathbb{R}$

$$\begin{aligned} A(\alpha v_1 + \beta v_2) &= p\int_\Omega |u|^{p-2} u(\alpha v_1 + \beta v_2) dx \\ &= p[\int_\Omega |u|^{p-2} u\alpha v_1 dx + \int_\Omega |u|^{p-2} u\beta v_2 dx] \\ &= \alpha p\int_\Omega |u|^{p-2} uv_1 dx + \beta p\int_\Omega |u|^{p-2} uv_2 dx \\ &= \alpha A(v_1) + \beta A(v_2). \end{aligned}$$

A is linear. A is continuous, in effect, let $u, v \in L^p(\Omega)$

$$\begin{aligned} |p\int_\Omega |u|^{p-2} uv dx| &\leq p\int_\Omega |u|^{p-1}|v| dx \\ &\leq p(\int_\Omega |u|^{(p-1)p'} dx)^{\frac{1}{p'}} (\int_\Omega |v|^p dx)^{\frac{1}{p}} \\ &\leq p(\int_\Omega |u|^{(p-1)(\frac{p}{p-1})} dx)^{\frac{p-1}{p}} (\int_\Omega |v|^p dx)^{\frac{1}{p}} \\ &= \|u\|_{L^p}^{p-1} \|v\|_{L^p}. \end{aligned}$$

Then A is a continuous, so the function is G-differentiable.

1.3.2 Derivative in the sense of Frechet

Definition 1.16 *Let E be a Banach space ,$\Omega \subseteq E$ an open set and let $I : \Omega \longrightarrow \mathbb{R}$ a functional, we say that I is ad differentiable in the sense of Frechet, at $u \in \Omega$, if there exists $A \in E'$ such as*

$$\lim_{\|v\|\to 0} \frac{I(u+tv)-I(v)-Av}{\|v\|} = 0$$

or

$$I(u+tv)-I(u) = Av + o(\|v\|).$$

if I is differentiable, then A is unique and we denote $I'(u) = A$, the set of differentiable function, we will be denote $\mathcal{C}^1(\Omega.\mathbb{R})$.

Proposition 1.4 *Suppose that $\Omega \subseteq E$ is on open set, such that I G-differentiable in Ω and that I'_G is continuous at $\Omega \in E$, then I is also differentiable at u, and of cours $I'_G = I'(u)$.*

Remark 1.3 *The importance of Proposition 1.4 reside in the fact that it is often technically easier to calculate the derivative in the sense of Gateau an then to prove that it is continuous, rather than to directly prove the differentiability in the sense of Frechet.*

Example 1.2 *We prove that the function $J : L^p(]0,1[) \to \mathbb{R} \quad (p > 2)$ and by using Holder inequality we get*

$$u \to J(u) = \int_0^1 |u|^p dx.$$

is a Frechet differentiable on $L^2(]0,1[)$.
we already prove that J is G-differentiable, with

$$J'_G : L^p(\Omega) \to (L^p(\Omega))' = L^q(\Omega)$$

$$u \to J'_G(u) : L^p(\Omega) \to \mathbb{R}$$

$$v \to \langle J'(u), v\rangle = p\int_0^1 |u|^{p-1}uvdx$$

it remains to prove that $J'_G : L^p(\Omega) \to (L^p(\Omega))' = L^q(\Omega)$

$$u \to J'_G(u)$$

is continuous.
Let $(u_j) \subset L^p(\Omega)$ such as $u_j \to u$ in $L^p(\Omega)$ we prove that $J'_G(u_j) \to J'_G(u)$ in $(L^p(\Omega))'$

$$\|J'_G(u_j) - J'_G(u)\|_{(L^p(\Omega))'} = \sup_{v\in L^p(\Omega)} |\langle J'_G(u_j) - J'_G(u), v\rangle|$$

let $v \in L^p(\Omega)$, such as $\|v\|_{L^p} = 1$, we have

$$\begin{aligned} |\,\langle J'_G(u_j) - J'_G(u), v\rangle\,| &= |\,\langle J'_G(u_j), v\rangle - \langle J'_G(u), v\rangle\,| \\ &= p|\int_0^1 |u_j|^{p-2}u_j v dx - \int_0^1 |u|^{p-2}uvdx| \\ &\le p[\int_0^1 ||u_j|^{p-2}u_n - |u|^{p-2}u|^{\frac{p}{p-1}}dx](\int_0^1 |v|^p dx)^{\frac{1}{p}}. \end{aligned}$$

We pose $u_j = |u_j|^{p-2}\, u_j, u = |u|^{p-2}\, u$, *we know that* $u_j \to u$ *in* $L^p(0,1)$, *there is a sub-sequence* $u_{jk} = u_j$, *such that* $u_j(x) \to u(x)$ *in* $]0,1[$.
moreover there exists $g \in L^P$ *such that*

$$|u_j(x)| \leq g(x) \;\; p.p \; in \;]0,1[.$$

so

$$|u_j(x)|^{p-1} \leq |g(x)|^{p-1} \quad \in L^{\frac{p}{p-1}}$$

according to $u_j \to u$ *in* $\frac{p}{p-1}$ *we recall that* $\|J'_G(u_j) - J'(u)\| \leq \int_\Omega |u_j|^{p-2}u_j - |u|^{p-2}udx \to 0$ *and then* $\lim_{j\to+\infty} \|J'_G(u_j) - J'(u)\|_{(L^p(\Omega))'} = 0$
as J *is G-differentiable and* J'_G *is continuous, so* J *is Frechet differentiable and* $J'_G = J'$.

1.3.3 Critical Points

Definition 1.17 *[15] Let* Ω *an open set of Banach space* E, *suppose that* $I \in \mathcal{C}^1(\Omega.\mathbb{R})$, *we say that* $u \in \Omega$ *is critical point of* I, *if*

$$I'(u) = 0$$

if u *is not critical point, then we say that* u *is regular point of* I.
if $c \in \mathbb{R}$, *then we say that* c *is a critical value of* I, *if there exists* $u \in \Omega$ *such as*

$$I(u) = c \;\; and \;\; I'(u) = 0.$$

if c *is not a critical value, then* c *is said to be a regular value of* I.

1.4 Some Theorems used

Definition 1.18 *(**weakly lower semi-continuous**) Let* $J : E \longrightarrow \mathbb{R}$ *be a functional, we say that* J *is weakly lower semi-continuous if* $\forall u_j \subset E : u_j \rightharpoonup u_0 \;\; in \;\; E$, *we have*

$$J(u_0) \leq \lim_{j\longrightarrow+\infty} \inf J(u_j).$$

Definition 1.19 *[8](**coercive**) Let* $J : E \longrightarrow \mathbb{R}$ *a functional on* E . *We say that* J *is coercive if and only if*

$$\lim_{\|u\|_E \longrightarrow+\infty} \frac{J(u)}{\|u\|_E} = +\infty.$$

Theorem 1.10 *[8] Any functional continuous convex on* E, *then* J *is weakly lower semi-continuous.*

Theorem 1.11 *[8](**Lebesgue's Dominated Convergence Theorem**) Let* $\{f_j\}$ *be a sequence of functions of* L^1. *We suppose that*

1. $f_j(x) \longrightarrow f(x)$ *a.e.on* Ω,
2. *there exists a function* $g \in L^1$ *such that for every* j, $|f_j(x)| \leq g(x)$ *a.e.in* Ω.

We have $f \in L^1(\Omega)$ *and* $\|f_j - f\|_1 \longrightarrow 0$.

Theorem 1.12 *[8](**Lebesgue's Dominated Convergence Inverse Theorem**) Let* $\{f_j\}$ *be a sequence in* L^p *and let* $f \in L^p$ *be such that* $\|f_j - f\|_p \longrightarrow 0$.
Then, there exist a sub-sequence $\{f_{j_k}\}$ *and a function* $h \in L^p$ *such that*

1. $f_{j_k}(x) \longrightarrow f(x)$ *a.e.on* Ω,
2. $|f_{j_k}(x)| \leq h(x) \quad \forall k$, *a.e.in* Ω.

Theorem 1.13 *(**Fubini's Theorem**) [8] Assume that* $F \in L^1(\Omega_1 \times \Omega_2)$. *Then for a.e.* $x \in \Omega_1$,

$$F(x,y) \in L^1_y(\Omega_2) \quad \text{and} \quad \int_{\Omega_2} F(x,y)\ dy \in L^1_x(\Omega_1).$$

Similarly, for a.e. $y \in \Omega_2$,

$$F(x,y) \in L^1_x(\Omega_1) \quad \text{and} \quad \int_{\Omega_1} F(x,y)\ dx \in L^1_y(\Omega_2).$$

Moreover, one has

$$\int_{\Omega_1} dx \int_{\Omega_2} F(x,y)\ dy = \int_{\Omega_2} dy \int_{\Omega_1} F(x,y) dx = \int\int_{\Omega_1 \times \Omega_2} F(x,y)\ dx\ dy.$$

CHAPTER 2

HARDY INEQUALITY AND APPLICATIONS

In this chapter, we study Hardy inequality and we present some applications of this inequality, by variational method, we deal with the elliptic linear problem to prove that it admits a unique weak solution by using Lax Milligram theorem, Then we consider quasilinear elliptic problem involving Hardy potential where we study the existence at least non-trivial weak solution using Mountain-Pass Theorem.

2.1 Hardy Inequality

In this section is to discuss the following classical result, essentially due to Hardy inequality in the space $W^{1,p}(\mathbb{R}^N)$ (See [16]), The first version it was proved of Hardy's inequality in **1915** as **G.H.Hardy**. and the following version it was proved in **1998**.

Lemma 2.1 *[12] Assume* $1 < p < N$, *then if* $u \in W^{1,p}(\mathbb{R}^N)$

1. $\frac{u}{|x|} \in L^p(\mathbb{R}^N)$.
2. ***(Hardy Inequality)***
$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p}\, dx \leq C_{N,p} \int_{\mathbb{R}^N} |\nabla u|^p\, dx.$$
with $C_{N,p} = \left(\frac{p}{N-p}\right)^p$.
3. *The constant* $C_{N,p}$ *is optimal.*

Proof.

`Step 1` . A density argument allows us to consider only smooth functions $u \in C_0^\infty(\mathbb{R}^N)$. Under this hypothesis we have the following identity

$$
\begin{aligned}
|u(x)|^p = -\int_1^{\infty} \frac{d}{d\lambda} |u(\lambda x)|^p \, d\lambda &= -p\int_1^{\infty} |u(\lambda x)|^{p-2} u(\lambda x) \frac{du}{d\lambda} \, d\lambda \\
&= -p\int_1^{\infty} |u(\lambda x)|^{p-2} u(\lambda x) x \cdot \frac{\partial u}{\partial X}(\lambda x) \, d\lambda \\
&= -p\int_1^{\infty} |u(\lambda x)|^{p-2} u(\lambda x) x \cdot \nabla u(\lambda x) \, d\lambda \\
&= -p\int_1^{\infty} u^{p-1}(\lambda x) x \cdot \nabla u(\lambda x) \, d\lambda,
\end{aligned}
$$

where

$$
\frac{du}{d\lambda}(\lambda x) = \frac{\partial u}{\partial X}\frac{\partial X}{\partial \lambda}, \quad \frac{\partial X}{\partial \lambda} = x.
$$

Let $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N)$, the scalar product in $\mathbb{R}^N$ is $x \cdot y = \sum\limits_{i=1}^{N} x_i y_i$

By taking the radial function which is $r = |x| = \left(\sum\limits_{i=1}^{N} x_i^2\right)^{\frac{1}{2}}$. $\frac{du}{dx_i} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x_i}$ such that $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$, by direct calculating we get

$$
\begin{aligned}
\frac{x}{|x|} \cdot \nabla u(x) &= \sum_{i=1}^{N} \frac{x_i}{r}\frac{du}{dx_i} \\
&= \frac{1}{r^2}\frac{\partial u}{\partial r}\sum_{i=1}^{N} x_i^2 \\
&= \frac{\partial u}{\partial r}.
\end{aligned}
\tag{2.1}
$$

Divided by $|x|^p$ and we integrate according to $\mathbb{R}^N$, then we put $y = \lambda x$, so that $dy = \lambda^N dx$, and by using (2.1), we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} dx &= -p\int_1^{\infty}\int_{\mathbb{R}^N} \frac{u^{p-1}(\lambda x)}{|x|^{p-1}} \frac{x}{|x|} \cdot \nabla u(\lambda x) \, dx \, d\lambda \\
&= -p\int_1^{\infty}\int_{\mathbb{R}^N} \frac{u^{p-1(y)}}{\left|\frac{1}{\lambda}y\right|^{p-1}} \frac{y}{|y|} \cdot \nabla u(y) \frac{1}{\lambda^N} \, dy \, d\lambda \\
&= -p\int_1^{\infty}\int_{\mathbb{R}^N} \frac{1}{\lambda^{N-p+1}} \frac{u^{p-1(y)}}{|y|^{p-1}} \frac{\partial u(y)}{\partial r} \, dy.
\end{aligned}
$$

Next, we apply Fubini's theorem, we get

$$
\begin{aligned}
\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} dx &= -p\int_1^{\infty} \frac{d\lambda}{\lambda^{N-p+1}} \int_{\mathbb{R}^N} \frac{u(y)^{p-1}}{|y|^{p-1}} \frac{\partial u(y)}{\partial r} \, dy \\
&= -\frac{p}{N-p}\int_{\mathbb{R}^N} \frac{u(y)^{p-1}}{|y|^{p-1}} \frac{\partial u(y)}{\partial r} \, dy.
\end{aligned}
$$

Now, by using Hölder inequality where $\left(\frac{1}{p} + \frac{p-1}{p} = 1\right)$, it follows that

$$
\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} \, dx \leq \frac{p}{N-p}\left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|y|^p} \, dy\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} \left|\frac{\partial u(y)}{\partial r}\right|^p \, dy\right)^{\frac{1}{p}}.
$$

Then, we conclude that

$$\int_{\mathbb{R}^N} \frac{u(x)^p}{|x|^p}\ dx \leq \left(\frac{p}{N-p}\right)^p \int_{\mathbb{R}^N} |\nabla u(x)|^p\ dx$$

`Step 2.` Optimality of the constant. Following the idea of Hardy for the one dimensional case, we show that the best constant is $C_{N,p} = \left(\frac{p}{N-p}\right)^p$.
Given $\epsilon > 0$, take the radial function

$$u(r) = \begin{cases} A_{N,p,\epsilon} & \text{if } r \in [0,1], \\ A_{N,p,\epsilon} r^{\frac{p-N}{p}-\epsilon} & \text{if } r > 1, \end{cases}$$

where $A_{N,p,\epsilon} = \frac{p}{N-p+p\epsilon}$, whose derivative is

$$u'(r) = \begin{cases} 0, & \text{if } r \in [0,1], \\ -r^{-(\frac{N}{p})-\epsilon} & \text{if } r > 1, \end{cases}$$

By direct integrated, using the definition of radial function and we set $x = rw$, such that $dx = r^{N-1}drdw$, then $|x|^p = r^p$ we get

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{u(x)^p}{|x|^p}dx &= \int_B \frac{u(x)^p}{|x|^p}dx + \int_{\mathbb{R}^N - B} \frac{u(x)^p}{|x|^p}dx \\ &= A^p_{N,p,\epsilon}\omega_N \left(\int_0^1 r^{N-1-p}dr + \int_1^\infty r^{-(1+p\epsilon)}dr\right) \\ &= A_{N,p,\epsilon}\omega_N \int_0^1 r^{N-1-p}dr + A^p_{N,p,\epsilon}\int_{\mathbb{R}^N} |\nabla u(x)|^p\, dx, \end{aligned}$$

where

$$\begin{aligned} \int_B \frac{u(x)^p}{|x|^p}dx &= \int_0^1 \int_{S^{N-1}} \frac{u(r)^p}{r^p} r^{N-1} drdw \\ &= A^p_{N,p,\epsilon} \int_0^1 r^{N-p-1}dr \int_{S^{N-1}} dw \\ &= A^p_{N,p,\epsilon}\omega_N \frac{1}{N-p}. \end{aligned}$$

and, we have

$$\begin{aligned} \int_{\mathbb{R}^N - B} \frac{u(x)^p}{|x|^p}dx &= \int_1^\infty \int_{S^{N-1}} \frac{u(r)^p}{r^p} r^{N-1} drdw \\ &= A^p_{N,p,\epsilon} \int_1^\infty \frac{r^{\frac{p-N}{p}-\epsilon}}{r^p} r^{N-1} dr \int_{S^{N-1}} dw \\ &= A^p_{N,p,\epsilon}\omega_N \int_1^\infty r^{-1-p\epsilon-N+1+N-1}dr \\ &= A^p_{N,p,\epsilon}\omega_N \int_1^\infty r^{\frac{p(-1-p\epsilon-N+1)}{p}} r^{N-1}dr \\ &= A^p_{N,p,\epsilon}\omega_N \int_1^\infty \left(\frac{\partial u}{\partial r}\right)^p r^{N-1}dr \\ &= A^p_{N,p,\epsilon}\omega_N \int_{\mathbb{R}^N} |\nabla u(x)|^p\, dx. \end{aligned}$$

Where ω_N is the measure of the $(N-1)$-dimensional unit sphere$\int_{S^{N-1}} dw = \omega_N$.
The last equality holds because we have

$$|\nabla u(x)| = \frac{\partial u}{\partial r}$$

and,

$$|\nabla u(x)| = \left(\sum_{i=1}^{N}\left(\frac{du}{dx_i}\right)^2\right)^{\frac{1}{2}} |\nabla u(x)|^2 = \sum_{i=1}^{N}\left(\frac{du}{dx_i}\right)^2 = \sum_{i=1}^{N}\left(\frac{\partial u}{\partial r}\frac{x_i}{r}\right)^2 = \frac{1}{r^2}\left(\frac{\partial u}{\partial r}\right)^2\sum_{i=1}^{N} x_i^2 = \left(\frac{\partial u}{\partial r}\right)^2.$$

We conclude by letting $\epsilon \longrightarrow 0$. This completes the proof. ■

2.2 Weighted Lebesgue Spaces

In this section, we present some auxiliary results on the Weighted Lebesgue spaces.(See [6])

Definition 2.1 *($L^p(\Omega, \frac{1}{|x|^p})$ Spaces)*
Let $\frac{1}{|x|^p}$ is a Weighted on $\mathbb{R}^N$, that is a measurable function such that $\frac{1}{|x|^p} > 0$ a.e. in $\mathbb{R}^N$.
We defined $L^p(\Omega, \frac{1}{|x|^p})$ space by:

$$L^p(\Omega, \frac{1}{|x|^p}) = \left\{u : \Omega \to \mathbb{R}; \textit{measurable and} \int_\Omega \frac{|u(x)|^p}{|x|^p} dx < \infty\right\}$$

With the norm:

$$\|u\|_{p(\Omega,\frac{1}{|x|^p})} = \left(\int_\Omega \frac{|u|^p}{|x|^p} dx\right)^{\frac{1}{p}}$$

We notice $\|\cdot\|_{L^p(\Omega,\frac{1}{|x|^p})} = \|\cdot\|_{p,\frac{1}{|x|^p}}$.

Proposition 2.1 *Let $1 < p < \infty$, then Banach space $L^p(\mathbb{R}^N, \frac{1}{|x|^p}, |\cdot|_{p,\frac{1}{|x|^p}})$ is uniformly convex.*

Proposition 2.2 *Assume that $\frac{1}{|x|^p} \in L^1_{loc}(\mathbb{R}^N)$,$1 \le p < \infty$, and let $\{u_j\}_j$,$u \in L^p(\mathbb{R}^N, \frac{1}{|x|^p})$ be such that, $u_j \rightharpoonup u$ in $L^p(\mathbb{R}^N, \frac{1}{|x|^p})$ and $u_j \to v$ a.e. in $\mathbb{R}^N$ as $j \to \infty$ then, $u = v$ a.e. in $\mathbb{R}^N$.*

Proposition 2.3 *Assume $\frac{1}{|x|^p} \in L^1_{loc}(\mathbb{R}^N)$, Let $1 < p < \infty$ and let $\{u_j\}_j$, $u \in L^p(\mathbb{R}^N, \frac{1}{|x|^p})$.*

1. *If $\{u_j\}_j$ bounded in $L^p(\mathbb{R}^N, \frac{1}{|x|^p})$ and $u_j \to u$ a.e. in $\mathbb{R}^N$,then $u_j \rightharpoonup u$ in $L^p(\mathbb{R}^N, \frac{1}{|x|^p})$ and $|u_j|^{p-2} u_j \rightharpoonup |u|^{p-2} u$ in $L^{p'}(\mathbb{R}^N, \frac{1}{|x|^p})$.*
2. *If $\|u_j\|_{p,\frac{1}{|x|^p}} \to \|u\|_{p,\frac{1}{|x|^p}}$ and $u_j \rightharpoonup u$ in $L^p(\mathbb{R}^N, \frac{1}{|x|^p})$.then $u_j \to u$ in $L^p(\mathbb{R}^N, \frac{1}{|x|^p})$ and $|u_j|^{p-2} u_j \rightharpoonup |u|^{p-2} u$ in $L^{p'}(\mathbb{R}^N, \frac{1}{|x|^p})$.*

Proof. We choose a sub-sequence $\{u_{jk}\}_{jk}$ of sequence $\{u_j\}_j$ still denoted by $\{u_j\}_j$. Let $\{u_j\}_j$, u be in $L^p(\mathbb{R}^N, \frac{1}{|x|^p})$.
Case(i). Thanks to Proposition 2.1, and Theorem 1.2 There exists sub-sequence $\{u_{jk}\}_k$ and

$v \in L^p(\mathbb{R}^N, \frac{1}{|x|^p})$. such that $u_{jk} \rightharpoonup v$ in $L^p(\mathbb{R}^N, \frac{1}{|x|^p})$, being $\{u_{jk}\}_k$ bounded in $L^p(\mathbb{R}^N, \frac{1}{|x|^p})$ and $u_j \to u$ *a.e.* in $\mathbb{R}^N$ implied $u_{jk} \to u$ *a.e.* in $\mathbb{R}^N$. Then $u = v$ *a.e.* $\mathbb{R}^N$ by proposition 2.2. we deduce that sequence $u_j \rightharpoonup u$ in $L^p(\mathbb{R}^N, \frac{1}{|x|^p})$. Applying the same argument to the sequence $j \to |u_j|^{p-2} u_j$,we obtain that $|u_j|^{p-2} u_j \rightharpoonup |u|^{p-2} u$ in $L^{p'}(\mathbb{R}^N, \frac{1}{|x|^p})$.

Case(ii). By Corollary 1.1 and Proposition 2.1 imply that $u_j \to u$ in $L^p(\mathbb{R}^N, \frac{1}{|x|^p})$.
Now,fix a sub-sequence $\{v_{jk}\}_k$ of $\{v_j\}_j$,$v_j = |u_j|^{p-2} u_j$. Hence $u_j \to u$ in $L^p(\mathbb{R}^N, \frac{1}{|x|^p})$, and so there exists sub-sequence $\{u_{nj}\}_n$ such that $u_{nj} \to u$ *a.e.* in $\mathbb{R}^N$. By Lebesgue's Dominated Convergence Inverse Theorem of course, $v_j \to v$ *a.e.* in $\mathbb{R}^N$, and we have:

$$\begin{cases} |u_{nj}|^{p-2} u_{nj} \to |u|^{p-2} u \quad a.e.\text{in} \quad \mathbb{R}^N. \\ \left| |u_{nj}|^{p-2} u_{nj} \right| \le |h(x)|^{p-1} \in L^{p'}(\mathbb{R}^N, \frac{1}{|x|^p}) \end{cases}$$

By Lebesgue's Dominated Convergence Theorem, $|u_{nj}|^{p-2} u_{nj} \to |u|^{p-2} u$ in $L^{p'}(\mathbb{R}^N, \frac{1}{|x|^p})$.This completes the proof. ■

The following result is known as Lieb-Brezis Lemma which introduced in [9].

Theorem 2.1 *Suppose $u_j \to u$ a.e. and $\|u_j\|_p \le C < \infty$ for all j and for some $0 < p < \infty$.Then the limit in*

$$\lim_{j\to\infty} \left\{ \|u_j\|_p^p - \|u_j - u\|_p^p \right\} = \|u\|_p^p. \tag{2.2}$$

exists and this equality holds.

2.3 Classical Mountain Pass Theorem

Definition 2.2 [13] ***(Palais-Smale Sequence)***

Let E be a Banach space and let $J : E \to \mathbb{R}$ be a differentiable functional, A sequence $\{u_j\}_j \subset E$, such that $\{J(u_j)\}_j$ is bounded in $\mathbb{R}$ and $J'(u_j) \to 0$ in E^* , as $j \to +\infty$ is called a Palais-Smaile sequence for J.
Let $c \in \mathbb{R}$, if $\{J(u_j)\}_j$ is bounded in $\mathbb{R}$ and $J'(u_j) \to c$ in E^* , as $j \to +\infty$ Then $\{u_j\}$ is called a Palais-Smale sequence for J at level c is called a Palais-Smale level for J.

Definition 2.3 [13] ***(Palais-Smale Condition)***
Let E be Banach spaces and let $J : E \to \mathbb{R}$ be a differentiable functional ,we say that J satisfies the Palais-Smale Condition if every Palais-Smale Sequence for J
i.e.
$\{J(u_j)\}_j$ is bounded in $\mathbb{R}$ and $J'(u_j) \to 0$ in $(E(\Omega))^*$, as $j \to +\infty$.
we say that J satisfies the Palais-Smale Condition at level $c \in \mathbb{R}$ if every Palais-Smale Sequence at level c
i.e.
$\{J(u_j)\}_j$ is bounded in $\mathbb{R}$ and $J'(u_j) \to c$ in $(E(\Omega))^*$, as $j \to +\infty$.
Has a converging sub-sequence in E.

Theorem 2.2** [15]**(Classical Mountain Pass Theorem)
Let E be a Banach space, $J \in \mathcal{C}^1(E, \mathbb{R})$ satisfying the Palais-Smale condition. We assume that $J(0) = 0$ and that :

(*a*) *there are* $\rho > 0$ *and* $\alpha > 0$ *such that* $\|u\|_E = \rho$, *then* $J(u) \geq \alpha$

(*b*) *there exists* $u_0 \in E$, *such that* $\|u_0\|_E > \rho$ *and* $J(u_0) < \alpha$.

Then J *has a critical value* c *such that* $c \geq \alpha$. *More precisely, if we put*

$$\mathscr{B} := \left\{\varphi([0,1]); \varphi \in C([0,1], E), \varphi(0) = 0, \varphi(1) = u_0\right\},$$

And

$$c := \inf_{A \in \mathscr{B}} \max_{v \in A} J(v),$$

Then it is a critical value of J, *and* $c \geq \alpha$.

2.4 Study elliptic problems involving Hardy Potentials

In this section, we apply Hardy's inequality to two types of elliptic problems with Hardy potentials. For the linear problem, we use Lax-Milligram Theorem to show there exist a unique weak solution. However, for the nonlinear problem where Lax-Milligram does not apply because the difficulty in last problem is the nonlinear therm $f(x,u)$, so we use Mountain-Pass Theorem to prove the existence at least non-trivial weak solution. In addition, in last problem we need some auxiliary results on Weighted Lebesgue Space.

2.4.1 An elliptic linear problem involving Hardy potential

In this section, we consider the following problem

$$\begin{cases} -\Delta u = \gamma \frac{u}{x^2} + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where $f \in L^2(\Omega)$, $0 \in \Omega$ and γ is a real parameter

Definition 2.4 *We say that* $u \in H_0^1(\Omega)$ *is a weak solution of Problem (2.3), if and only if*

$$\int_\Omega \nabla u \nabla \varphi dx - \gamma \int_\Omega \frac{u}{x^2} \varphi dx = \int_\Omega f(x) \varphi dx$$

for any $\varphi \in H_0^1(\Omega)$.

we prove that Problem (2.3) admits a unique weak solution in $H_0^1(\Omega)$ using Lax Milligram theorem. we choose $a(\cdot,\cdot)$: $H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$

$$a(u, \varphi) = \int_\Omega \nabla u \nabla \varphi \; dx - \gamma \int_\Omega \frac{u}{x^2} \varphi \; dx.$$

and $L_f(\cdot)$: $H_0^1(\Omega) \to \mathbb{R}$

$$L_f(\varphi) = \int_\Omega f(x) \varphi \; dx.$$

Obviously, $a(\cdot,\cdot)$ is well defined and bi-linear application. also, $L_f(\cdot)$ is a linear and continuous application. Now, we show that $a(\cdot,\cdot)$ is continuous and coercive. Indeed, Let $u, \varphi \in H_0^1(\Omega)$, by Cauchy Schwartz inequality and Hardy inequality and poincaré inequality, we have

$$\begin{aligned}
|a(u,\varphi)| &\leq \int_\Omega |\nabla u|\,|\nabla\varphi|\ dx + |\gamma| \int_\Omega \frac{|u|}{x^2}\,|\varphi|\ dx \\
&\leq \|\nabla u\|_2\,\|\nabla\varphi\|_2 + |\gamma| \left(\int_\Omega \frac{|u|^2}{x^2}\,|\varphi|\ dx \right)^{\frac{1}{2}} \|\varphi\|_2 \\
&\leq \|u\|_{H_0^1}\,\|\varphi\|_{H_0^1} + |\gamma|\, C_{N,2}^{\frac{1}{2}}\,\|u\|_{H_0^1}\, C_\Omega\,\|\varphi\|_{H_0^1} \\
&= \left(1 + |\gamma|\, C_{N,2}^{\frac{1}{2}} C_\Omega\right) \|u\|_{H_0^1}\,\|\varphi\|_{H_0^1} \\
&= M\,\|u\|_{H_0^1}\,\|\varphi\|_{H_0^1}\,.
\end{aligned}$$

where $M = 1 + |\gamma|\, C_{N,2}^{\frac{1}{2}} C_\Omega$, then $a(\cdot,\cdot)$ is continuous in $H_0^1(\Omega)$.
Now, let $\varphi \in H_0^1(\Omega)$, we have

$$\begin{aligned}
a(\varphi,\varphi) &= \int_\Omega |\nabla\varphi|^2\,dx - \gamma \int_\Omega \frac{\varphi^2}{x^2} dx \\
&\geq \|\varphi\|_{H_0^1}^2 + C_{N,2}\gamma\,\|\varphi\|_{H_0^1}^2 \\
&= (1 + C_{N,2}\gamma)\,\|\varphi\|_{H_0^1}^2 \\
&= C\,\|\varphi\|_{H_0^1}^2
\end{aligned}$$

where $C = 1 + C_{N,2}\gamma$. Then $a(\cdot,\cdot)$ is coercive in $H_0^1(\Omega)$.

2.4.2 A quasilinear elliptic problem involving Hardy potential

Throughout this section, we assume that $\Omega \subset \mathbb{R}^N (N > 2)$ is an open bounded set, $0 \in \Omega$, γ is a real parameter, also we denote with

$$t^+ = max\,\{t, 0\} = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

and

$$t^- = max\,\{-t, 0\} = \begin{cases} 0 & \text{if } t \geq 0 \\ -t & \text{if } t < 0 \end{cases}$$

respectively positive and negative parts of $t \in \mathbb{R}$. Let us consider the following quasilinear problem:

$$\begin{cases} -\Delta u = \gamma \frac{u}{|x|^2} + f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

Where $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a caratheodory function, that is
$t \to f(x,t)$ is continuous for *a.e.* $x \in \Omega$,
$x \to f(x,t)$ is Lebesgue measurable for all $t \in \mathbb{R}^N$.

We assume that f satisfies the following hypotheses (f_1) there exists $r \in (2, 2^*)$, with the critical sobolev exponent $2^* = \frac{2N}{N-2}$, such that for any $\epsilon > 0$ there exists $\delta_\epsilon = \delta(\epsilon) > 0$ and

$$|f(x,t)| \leq 2\epsilon\,|t| + r\delta_\epsilon\,|t|^{r-1}$$

holds for a.e $x \in \Omega$ and any $t \in \mathbb{R}$.
(f_2) There exist $\theta \in (2, 2^*)$, $c > 0$, $t_0 \geq 0$ such that

$$c \leq \theta F(x,t) \leq t f(x,t).$$

for a.e. $x \in \Omega$ and any $|t| \geq t_0$, where

$$F(x,t) = \int_0^t f(x,\tau)d\tau$$

This hypothesis is known as Ambrosetti-Rabinowitz condition.
(f_3) there exists $d_1 > 0$ and $d_2 \geq 0$ such that

$$F(x,t) \geq d_1\,|t|^\theta - d_2 \quad for \quad \text{a.e. } \; x \in \Omega$$

We are now ready to introduce the variational setting for problem (2.4).

Definition 2.5 *We say that a function $u \in H_0^1(\Omega)$ is a weak solution of problem (2.4), if and only if*

$$\int_\Omega \nabla u \nabla \varphi dx = \gamma \int_\Omega \frac{u}{|x|^2}\varphi dx + \int_\Omega f(x,u)\varphi dx$$

for any $\varphi \in H_0^1(\Omega)$.

Clearly, the weak solutions of problem (2.4) are exactly the critical points of the functional $J_\gamma : H_0^1(\Omega) \to \mathbb{R}$, given by

$$J_\gamma(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\gamma}{2}\int_\Omega \frac{u^2}{|x|^2}dx - \int_\Omega F(x,u)dx.$$

which is well defined and of class $\mathcal{C}^1$ on $H_0^1(\Omega)$. and:

$$\left\langle J_\gamma'(u), \varphi \right\rangle = \int_\Omega \nabla u \nabla \varphi dx = \gamma \int_\Omega \frac{u}{|x|^2}\varphi dx + \int_\Omega f(x,u)\varphi dx$$

In the following, we present the main result of this section

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set, $0 \in \Omega$, γ is a real parameter and $N > 2$. Let $f(1) - f(2)$ hold true. Then, for any $\gamma < \frac{1}{C_{N,2}}$, Where $C_{N,2} = \left(\frac{2}{N-2}\right)^2$ problem (2.4) admits a non-trivial weak solution.*

The proof of Theorem 2.3 is based on the application of Classical Mountain-Pass Theorem.

Remark 2.1 *The verification of the $(P.S)$ condition for J_γ is fairly delicate, considering the contribution of the Hardy potentials. Indeed. even if the embedding of $H_0^1(\Omega)$ into $L^2(\Omega, \frac{1}{|x|^2})$ by Hardy inequality, (See Lemma 2.1), these embeddings are not compact. For this, we exploit a suitable tricky step analysis combined with the celebrated Brézis and Lieb lemma, (See Theorem 2.1), which can be applied in $H_0^1(\Omega)$ if we first prove the convergence $\nabla u_j(x) \to \nabla u(x)$ a.e. in Ω, as $j \to \infty$.*

Proposition 2.4 *Let (f_1) - (f_2) hold true. Then, for any $\gamma < \frac{1}{C_{N,2}}$ The functional J_γ verifies the $(P.S)$ condition*

Proof. Let us fix $\gamma < \frac{1}{C_{N,2}}$ and let $\{u_j\}_j \subset H_0^1(\Omega)$ be a sequence satisfying $(P.S)$ condition; that is

$$\{J_\gamma(u_j)\}_j \text{ is bounded in } \quad \mathbb{R} \quad \text{and} \quad J_\gamma^{'}(u_j) \to 0 \; in \; \left(H_0^1(\Omega)\right)^*, \text{as} j \to +\infty. \tag{2.5}$$

we first show that $\{u_j\}_j$ is bounded in $H_0^1(\Omega)$, arguing by contradiction. Then, going to a sub-sequence, still denoted by $\{u_j\}_j$, we have

$$\lim_{j\to\infty} \|u_j\|_{H_0^1} = \infty, \tag{2.6}$$

Thus, according to (f_2) and Hardy inequality Lemma 2.1, we get

$$\begin{aligned} J_\gamma(u_j) - \frac{1}{\theta}\left\langle J_\gamma^{'}(u_j), u_j\right\rangle &= \frac{1}{2}\int_\Omega |\nabla u_j|^2\,dx - \frac{\gamma}{2}\int_\Omega \frac{{u_j}^2}{|x|^2}dx - \int_\Omega F(x,u_j)dx - \frac{1}{\theta}\int_\Omega |\nabla u_j|^2\,dx \\ &\quad + \frac{\gamma}{\theta}\int_\Omega \frac{{u_j}^2}{|x|^2}dx + \frac{1}{\theta}\int_\Omega f(x,u_j)u_j dx. \\ &= \left(\frac{1}{2}-\frac{1}{\theta}\right)\int_\Omega |\nabla u_j|^2\,dx - \gamma\left(\frac{1}{2}-\frac{1}{\theta}\right)\int_\Omega \frac{{u_j}^2}{|x|^2}dx \\ &\quad - \int_\Omega \left[F(x,u_j) - \frac{1}{\theta}f(x,u_j)u_j\right]dx. \end{aligned}$$

We introduce some lemmas which we need in this proof.

Lemma 2.2 *We have*

$$\gamma\left(\frac{1}{2}-\frac{1}{\theta}\right)\int_\Omega \frac{|u|^2}{|x|^2}dx \le \gamma^+\left(\frac{1}{2}-\frac{1}{\theta}\right)C_{N,2}\int_\Omega |\nabla u|^2\,dx.$$

Proof.

Since $\left(\frac{1}{2}-\frac{1}{\theta}\right) > 0$, because $\theta > 2$ and $\gamma < \frac{1}{C_{N,2}}$
We can show the inequality according to the two cases of γ using Hardy inequality ; Indeed

If $\gamma > 0$, then $\gamma^+ = \gamma$, and we have

$$\gamma\left(\frac{1}{2}-\frac{1}{\theta}\right)\|u\|^2_{2,\frac{1}{|x|^2}} \le \gamma\left(\frac{1}{2}-\frac{1}{\theta}\right)C_{N,2}\|\nabla u\|_2^2.$$

If $\gamma < 0$, then $\gamma^+ = 0$, and we get

$$\gamma\left(\frac{1}{2}-\frac{1}{\theta}\right)\|u\|^2_{2,\frac{1}{|x|^2}} \le 0.$$

Hence, the inequality is true for all cases of γ, This completes the proof. ■
the second auxiliary result is:

Lemma 2.3 *We have*

$$\int_\Omega \left[F(x,u_j) - \frac{1}{\theta}f(x,u_j)u_j\right]dx \le |\Omega| \sup_{x\in\Omega,|t|\le t_0}\left[F(x,t) - \frac{1}{\theta}f(x,t)t\right]^+.$$

Proof. According to $(f2)$ we conclude that

$$\begin{aligned}\int_\Omega\left[F(x,u_j)-\frac{1}{\theta}f(x,u_j)u_j\right]dx &= \int_{\{|u_j|\le t_0\}}\left[F(x,u_j)+\frac{1}{\theta}f(x,u_j)u_j\right]dx\\ &\quad+\int_{\{|u_j|>t_0\}}\left[F(x,u_j)-\frac{1}{\theta}f(x,u_j)u_j\right]dx\\ &\le \int_{\{|u_j|\le t_0\}}\left[F(x,u_j)-\frac{1}{\theta}f(x,u_j)u_j\right]dx\\ &\le |\Omega|\sup_{x\in\Omega,|t|\le t_0}\left[F(x,t)-\frac{1}{\theta}f(x,t)t\right]^+.\end{aligned}$$

This completes the proof. ■

Now, we complete the Proof of Proposition 2.4

According Lemma 2.2 and Lemma 2.3 , we get

$$\begin{aligned}J_\gamma(u_j)-\frac{1}{\theta}\left\langle J_\gamma'(u_j),u_j\right\rangle &\ge \left(\frac{1}{2}-\frac{1}{\theta}\right)\int_\Omega|\nabla u_j|^2dx-\gamma^+C_{N,2}\left(\frac{1}{2}-\frac{1}{\theta}\right)\int_\Omega|\nabla u_j|^2dx-D\\ &=\left(\frac{1}{2}-\frac{1}{\theta}\right)\left(1-\gamma^+C_{N,2}\right)\|\nabla u_j\|_2^2-D\\ &=\left(\frac{1}{2}-\frac{1}{\theta}\right)\left(1-\gamma^+C_{N,2}\right)\|u_j\|_{H_0^1}^2-D.\end{aligned}\tag{2.7}$$

where

$$D=|\Omega|\sup_{x\in\Omega,|t|\le t_0}\left[F(x,t)-\frac{1}{\theta}f(x,t)t\right]^+<\infty,$$

On the other hand we have, by (2.5) there exist c_1,c_2,$c_3>0$ such that

$$\begin{aligned}J_\gamma(u_j)-\frac{1}{\theta}\left\langle J_\gamma'(u_j),u_j\right\rangle &\le \left|J_\gamma(u_j)-\frac{1}{\theta}\left\langle J_\gamma'(u_j),u_j\right\rangle\right|\\ &\le|J_\gamma(u_j)|-\frac{1}{\theta}\left|\left\langle J_\gamma'(u_j),u_j\right\rangle\right|\\ &\le c_1+\frac{1}{\theta}\|J'(u_j)\|_{(H_0^1)'}\|u_j\|_{H_0^1}\\ &\le c_1+\frac{1}{\theta}c_2\|u_j\|_{H_0^1}+o(1).\end{aligned}\tag{2.8}$$

Consequently, from (2.7) and (2.8), we get

$$c_1+c_3\|u_j\|_{H_0^1}+o(1)\ge\left(\frac{1}{2}-\frac{1}{\theta}\right)\left(1-\gamma^+C_{N,2}\right)\|u_j\|_{H_0^1}^2-D.\tag{2.9}$$

where $c_3=\frac{1}{\theta}c_2$

this implies that

$$c_1+o(1)+D\ge\left(\frac{1}{2}-\frac{1}{\theta}\right)\left(1-\gamma^+C_{N,2}\right)\|u_j\|_{H_0^1}^2-c_3\|u_j\|_{H_0^1}.\tag{2.10}$$

from (2.6), we get

$$\lim_{j\to\infty}\left\{\left(\frac{1}{2}-\frac{1}{\theta}\right)\left(1-\gamma^+C_{N,2}\right)\|u_j\|_{H_0^1}^2-c_3\|u_j\|_{H_0^1}\right\}=+\infty.\tag{2.11}$$

which is contradiction with (2.10). Hence, we deduce that the sequence $\{u_j\}_j$ is bounded in $H_0^1(\Omega)$.

Hence, $\{u_j\}_j$ is bounded in $H_0^1(\Omega)$, Which is reflexive space there exist a sub-sequence denoted by $\{u_j\}_j$, such that

$$u_j \rightharpoonup u \ \ in \ \ H_0^1(\Omega), \tag{2.12}$$

which means that

$$\nabla u_j \rightharpoonup \nabla u \ \ in \ \ \left[L^2(\Omega)\right]^N, \tag{2.13}$$

Then, thanks to the compactly embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, we have

$$u_j \to u \quad in \quad L^2(\Omega), \tag{2.14}$$

by using Lebesgue's dominated convergence inverse (Theorem 1.12), There exist a sub-sequence denoted by $\{u_j\}_j$ and function $h \in L^2$, such that

$$u_j(x) \to u(x) a.e. \ \ in \ \ \Omega, as j \to \infty, \tag{2.15}$$

and,

$$|u_j(x)| \leq h(x) a.e. \ \ in \ \ \Omega, \tag{2.16}$$

Using Hardy inequality Lemma 2.1, we can deduce that the sequence $\{u_j\}_j$ is bounded in $L^2(\Omega, \frac{1}{|x|^2})$. By proposition 2.3 we deduce that there exist a sub-sub sequence denoted by $\{u_j\}_j$, such that

$$u_j \rightharpoonup u \ \ in \ \ L^2(\Omega, \frac{1}{|x|^2}). \tag{2.17}$$

Now, we claim this Lemma

Lemma 2.4 *We have*

$$\nabla u_j(x) \to \nabla u(x) \quad a.e. in \quad \Omega, \quad as j \to \infty. \tag{2.18}$$

Proof. Let $\phi \in C^\infty(\mathbb{R}^N)$ be a cut-off function with $0 \leq \phi \leq 1$, such that

$$\phi(x) = \begin{cases} 1 & \text{if} \quad x \in B(0, \frac{1}{2}) \\ 0 & \text{if} \quad x \in B(0,1). \end{cases}$$

Then, we define $\Psi_R(x) = 1 - \phi(\frac{x}{R})$ for any $R > 0$, so that $\Psi_R \in C^\infty(\mathbb{R}^N)$ with $0 \leq \Psi_R \leq 1$, such that

$$\Psi_R(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{R}^N \backslash B(0,R) \\ 0 & \text{if} \quad x \in B(0, \frac{R}{2}). \end{cases}$$

And the sequence $\{\Psi_R u_j\}_j$ is bounded in $H_0^1(\Omega)$. By simple calculation, for any $j \in \mathbb{N}$ we have

$$\begin{aligned} \left\langle J'_\gamma(u_j), \Psi_R(u_j - u) \right\rangle &= \int_\Omega \nabla u_j \nabla(\Psi_R(u_j - u)) \, dx - \gamma \int_\Omega \Psi_R \frac{u_j}{|x|^2}(u_j - u) \, dx \\ &- \int_\Omega \Psi_R f(x, u_j)(u_j - u) \, dx \\ &= \int_\Omega \nabla u_j \nabla \Psi_R (u_j - u) \, dx + \int_\Omega \nabla u_j \Psi_R \nabla(u_j - u) \, dx \\ &- \gamma \int_\Omega \Psi_R \frac{u_j}{|x|^2}(u_j - u) \, dx - \int_\Omega \Psi_R f(x, u_j)(u_j - u) \, dx. \end{aligned} \tag{2.19}$$

Of course, all integrals are zero whenever $\overline{\Omega} \subset B(0, \frac{R}{2})$, since $\Psi_R \equiv 0$ in $B(0, \frac{R}{2})$. Thus, let us consider $R > 0$ sufficiently small such that

$$\left[\mathbb{R}^N \backslash B(0, \frac{R}{2})\right] \cap \overline{\Omega} \neq \emptyset \tag{2.20}$$

We introduce the following steps which is necessary later.

`Step 1 .` we show that

$$\int_\Omega \nabla u_j \nabla \Psi_R (u_j - u) dx \to 0, \quad as j \to \infty. \tag{2.21}$$

By Cauchy Schwartz inequality, and $\{u_j\}_j$ is bounded in $H_0^1(\Omega)$, we get

$$\begin{aligned}
\int_\Omega \nabla u_j \nabla \Psi_R (u_j - u) dx &\leq \left| \int_\Omega \nabla u_j \nabla \Psi_R (u_j - u) dx \right| \\
&\leq \int_\Omega |\nabla u_j| \left| \nabla \Psi_R (u_j - u) \right| dx \\
&\leq \left(\int_\Omega |\nabla u_j|^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla \Psi_R (u_j - u)|^2 dx \right)^{\frac{1}{2}} \\
&\leq C \|\nabla u_j\|_2 \|u_j - u\|_2 \\
&\leq CC' \|u_j - u\|_2 .
\end{aligned}$$

then, by the compactly embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, we have

$$\int_\Omega \nabla u_j \nabla \Psi_R (u_j - u) dx \leq \widetilde{C} \|u_j - u\|_2 \to 0.$$

where $\widetilde{C} = CC'$, as $j \to \infty$.

`Step 2 .` we show that

$$\int_\Omega \Psi_R f(x, u_j)(u_j - u)\ dx \to 0, \quad as j \to \infty. \tag{2.22}$$

By considering also (f_1) with $\epsilon = 1$, $|\Psi_R| \leq 1$ we obtain

$$\begin{aligned}
\int_\Omega \Psi_R f(x, u_j)(u_j - u) dx &\leq \left| \int_\Omega \Psi_R f(x, u_j)(u_j - u) dx \right| \\
&\leq \int_\Omega |\Psi_R| \left| f(x, u_j)(u_j - u) \right| dx \\
&\leq \int_\Omega \left(2 |u_j| + r\delta_1 |u_j|^{r-1} \right) |u_j - u| dx \\
&\leq \int_\Omega 2 |u_j| |u_j - u| dx + \int_\Omega \left(r\delta_1 |u_j|^{r-1} \right) |u_j - u| dx.
\end{aligned}$$

Next, applying Cauchy Schwartz and Hölder inequality where $\frac{1}{r} + \frac{1}{r'} = 1$, we get

$$\begin{aligned}
\int_\Omega \Psi_R f(x, u_j)(u_j - u)\ dx &\leq 2 \|u_j\|_2 \|u_j - u\|_2 + \left(\int_\Omega \left(r\delta_1 |u_j|^{r-1} \right)^{\frac{r}{r-1}} dx \right)^{\frac{r-1}{r}} \left(\int_\Omega |u_j - u|^r dx \right)^{\frac{1}{r}} \\
&\leq 2 \|u_j\|_2 \|u_j - u\|_2 + (r\delta_1)^{r-1} \|u_j\|_r^{r-1} \|u_j - u\|_r .
\end{aligned}$$

By using (2.14) and $\{u_j\}_j$ is bounded in $H_0^1(\Omega)$, we have

$$\begin{aligned}\int_\Omega \Psi_R f(x,u_j)(u_j-u)dx &\le 2C_1\left\|u_j\right\|_{H_0^1}\left\|u_j-u\right\|_2+(r\delta_1)^{r-1}(C_2)^{r-1}\left\|u_j\right\|_{H_0^1}^{r-1}\left\|u_j-u\right\|_r\\ &\le 2C_1\left\|u_j\right\|_{H_0^1}\left\|u_j-u\right\|_2+(r\delta_1)^{r-1}(C_2)^{r-1}\left\|u_j\right\|_{H_0^1}^{r-1}\left\|u_j-u\right\|_r\\ &\le C_3\left\|u_j-u\right\|_2+C_4\left\|u_j-u\right\|_r\to 0.\end{aligned}$$

as $j\to\infty$, for a constants C_1, C_2, C_3 and $C_4=(r\delta_1)^{r-1}(C_2)^{r-1}\left\|u_j\right\|_{H_0^1}^{r-1}$.

`Step 3 .` we show that

$$\int_\Omega \psi_R\frac{u_j}{|x|^2}(u_j-u)\;dx\to 0,\quad as j\to\infty. \tag{2.23}$$

From relations (2.17) we deduce that

$$\int_\Omega \psi_R\frac{u_j}{|x|^2}u\;dx\to\int_\Omega \psi_R\frac{u^2}{|x|^2}\;dx.\quad \text{as} j\to\infty. \tag{2.24}$$

Now, we try to show that

$$\int_\Omega \psi_R\frac{u_j^2}{|x|^2}\;dx\to\int_\Omega \psi_R\frac{u^2}{|x|^2}\;dx.\quad \text{as} j\to\infty. \tag{2.25}$$

using the dominated convergence theorem, From (2.15) we have

$$\psi_R(x)\frac{u_j(x)^2}{|x|^2}\to\psi_R(x)\frac{u(x)^2}{|x|^2}\quad a.e.\;\; in\;\;\Omega\quad \text{as} j\to\infty. \tag{2.26}$$

From (2.16) we have

$$\Omega\backslash B(0,\frac{R}{2})=\left\{x\in\Omega:|x|>\frac{R}{2}\right\}\quad for\quad any\;\; R>0.$$

Then,

$$\psi_R(x)\frac{|u_j|^2}{|x|^2}\le\left(\frac{2}{R}\right)^2|u_j|^2\le\left(\frac{2}{R}\right)^2h^2(x)\;\; a.e.\;\; in\;\;\Omega\backslash B\Big(0,\frac{R}{2}\Big), \tag{2.27}$$

Since $\psi_R(x)=0$ in $B(0,\frac{R}{2})$, and from (2.26), (2.27), the dominated convergence theorem gives

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega \psi_R\frac{|u_j|^2}{|x|^2}\;dx &=\int_{B(0,\frac{R}{2})}\psi_R\frac{|u_j|^2}{|x|^2}\;dx+\int_{\Omega\backslash B(0,\frac{R}{2})}\psi_R\frac{|u_j|^2}{|x|^2}\;dx\\ &=\int_{\Omega\backslash B(0,\frac{R}{2})}\psi_R\frac{|u_j|^2}{|x|^2}\;dx=\int_{\Omega\backslash B(0,\frac{R}{2})}\psi_R\frac{|u|^2}{|x|^2}\;dx\\ &=\int_\Omega \psi_R\frac{|u|^2}{|x|^2}\;dx.\end{aligned} \tag{2.28}$$

such that

$$\int_{B(0,\frac{R}{2})}\psi_R\frac{|u_j|^2}{|x|^2}\;dx=0.$$

Thus, By (2.5), we have

$$\Big\langle J_\gamma'(u_j),\Psi_R(u_j-u)\Big\rangle\to 0,\quad as j\to\infty. \tag{2.29}$$

So from (2.21), (2.22),(2.23), (2.29) and using (2.19), we can deduce that:

$$\int_\Omega \psi_R \nabla u_j (\nabla u_j - \nabla u)\ dx \to 0. \tag{2.30}$$

`Step 4` . we show that

$$\int_\Omega \psi_R \nabla u (\nabla u_j - \nabla u)\ dx \to 0, \quad as j \to \infty. \tag{2.31}$$

Using (2.13), and the fact that $\psi_R \nabla u \in L^2(\Omega)$. Then, we have

$$\int_\Omega \psi_R \nabla u_j \nabla u\ dx \to \int_\Omega \psi_R \left|\nabla u\right|^2\ dx.$$

which means that,

$$\int_\Omega \psi_R \nabla u (\nabla u_j - \nabla u)\ dx \to 0. \tag{2.32}$$

as $j \to \infty$.
We introduce this Step to prove that $\nabla u_j \to \nabla u$ strongly in $[L^2(\Omega_R)]^N$ as $j \to \infty$.
`Step 5` . we show that

$$\int_{\Omega_R} \left|\nabla u_j - \nabla u\right|^2\ dx \to 0, \quad as j \to \infty. \tag{2.33}$$

denoting $\Omega \backslash B(0,R)$ by Ω_R, Where $\Omega_R = \{x \in \Omega : |x| > R\}$ for any $R > 0$, we get

$$\int_{\Omega_R} \psi_R \nabla u (\nabla u_j - \nabla u)\ dx \leq \int_\Omega \nabla u (\nabla u_j - \nabla u)\ dx \to 0. \tag{2.34}$$

since $\psi_R(x) = 1$ in Ω_R.
By subtracting this relationship (2.30), from (2.32), we get

$$\int_\Omega \psi_R \left|\nabla u_j - \nabla u\right|^2\ dx \to 0, \quad as j \to \infty. \tag{2.35}$$

Thus, combining (2.34), (2.35) we get

$$\int_\Omega \psi_R \left|\nabla u_j - \nabla u\right|^2\ dx \leq \int_{\Omega_R} \left|\nabla u_j - \nabla u\right|^2\ dx \to 0, \quad as j \to \infty. \tag{2.36}$$

we prove that $\nabla u_j \to \nabla u$ strongly in $[L^2(\Omega_R)]^N$ as $j \to \infty$, whenever $R > 0$ satisfies (2.20). However, when $\overline{\Omega} \subset B(0, \frac{R}{2})$ we have $\Omega_R = \emptyset$. Thus, for any $R > 0$ the sequence $\nabla u_j \to \nabla u$ in $[L^2(\Omega_R)]^N$ as $j \to \infty$, and by diagonalization we prove claim Lemma 2.4. This complete the proof of Lemma 2.4. ■

Now, we introduce this Step to prove that $\nabla u_j \to \nabla u$ strongly in $[L^2(\Omega)]^N$ as $j \to \infty$, and we need the Brézis and Lieb lemma in (Theorem 2.1) to prove it.
`Step 6` . Assume that

$$\|u_j - u\|^2_{2,\frac{1}{|x|^2}} \to \ell, \quad \ell > 0. \tag{2.37}$$

Since the sequence $\{u_j\}_j$ is bounded in $[L^2(\Omega)]^N$, By Lemma 2.4 we get

$$\lim_{j\to\infty} \int_\Omega \nabla u_j \nabla u\ dx = \int_\Omega \lim_{j\to\infty} \nabla u_j \nabla u\ dx = \|\nabla u\|_2^2. \tag{2.38}$$

Also, arguing as in (2.22), we can prove

$$\begin{aligned} \lim_{j\to\infty} \int_\Omega f(x,u_j)(u_j-u)\,dx = 0, \\ \lim_{j\to\infty} \int_\Omega \frac{u_j}{|x|^2} u\,dx = \|u\|^2_{2,\frac{1}{|x|^2}}. \end{aligned} \tag{2.39}$$

Furthermore, using (2.4) and the Brézis and Lieb Lemma in (Theorem 2.1), we obtain

$$\begin{aligned} \|\nabla u_j\|_2^2 - \|\nabla u_j - \nabla u\|_2^2 = \|\nabla u\|_2^2 + o(1), \\ \|u_j\|^2_{2,\frac{1}{|x|^2}} - \|u_j-u\|^2_{2,\frac{1}{|x|^2}} = \|u\|^2_{2,\frac{1}{|x|^2}} + o(1). \end{aligned} \tag{2.40}$$

as $j \to \infty$.
Thus, by (2.5), (2.38) and (2.39), we get

$$\begin{aligned} o(1) = \left\langle J'_\gamma(u_j), u_j-u \right\rangle &= \int_\Omega \nabla u_j(\nabla u_j - \nabla u)\,dx - \gamma \int_\Omega \frac{u_j}{|x|^2}(u_j-u)\,dx - \int_\Omega f(x,u_j)(u_j-u)\,dx \\ &= \|\nabla u_j\|_2^2 - \|\nabla u\|_2^2 - \gamma(\|u_j\|^2_{2,\frac{1}{|x|^2}} - \|u\|^2_{2,\frac{1}{|x|^2}}) + o(1). \end{aligned} \tag{2.41}$$

as $j \to \infty$.
Hence, by (2.40) it follows that

$$\begin{aligned} \|\nabla u_j - \nabla u\|_2^2 &= \gamma(\|u_j-u\|^2_{2,\frac{1}{|x|^2}}) + o(1) \\ &= \gamma\ell + o(1). \end{aligned} \tag{2.42}$$

as $j \to \infty$,
Now assume for contradiction that $\ell > 0$. then, from Hardy inequality Lemma 2.1, (2.42) and $\gamma < \frac{1}{C_{N,2}}$, we have

$$\begin{aligned} \lim_{j\to\infty} \|\nabla u_j - \nabla u\|_2^2 &= \gamma(\lim_{j\to\infty} \|u_j-u\|^2_{2,\frac{1}{|x|^2}}) \\ &< \frac{1}{C_{N,2}}(\lim_{j\to\infty} \|u_j-u\|^2_{2,\frac{1}{|x|^2}}) \\ &< \lim_{j\to\infty} \|\nabla u_j - \nabla u\|_2^2. \end{aligned}$$

which is impossible. Therefore $\ell = 0$, so that by (2.42) we have $\nabla u_j \to \nabla u$ strongly in $[L^2(\Omega)]^N$ as $j \to \infty$, implying that $u_j \to u$ strongly in $H_0^1(\Omega)$.
This complete the proof of Proposition 2.4. ■
Now, We prove the geometric conditions of Mountain Pass Theorem, We start by the first geometric condition.

Lemma 2.5 *Let* (f_1) *holds true. Then, for any* $\gamma < \frac{1}{C_{N,2}}$ *there exist* $\rho = \rho(\gamma)$ *such that* $0 < \rho \le 1$ *and* $\alpha = \alpha(\rho) > 0$ *such that* $J_\gamma(u) \ge \alpha$ *for any* $u \in H_0^1(\Omega)$, *with* $\|u\|_{H_0^1} = \rho$.

Proof. Let us fix $\gamma < \frac{1}{C_{N,2}}$. By (f_1), for any $\epsilon > 0$ we have $\delta_\epsilon > 0$ such that

$$|F(x,t)| \le \epsilon |t|^2 + \delta_\epsilon |t|^r, \quad for \;\; a.e. x \in \Omega \;\; and \quad any \quad t \in \mathbb{R}. \tag{2.43}$$

First, we have

$$\begin{aligned}J_\gamma(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\gamma}{2}\int_\Omega \frac{u^2}{|x|^2}dx - \int_\Omega F(x,u)dx\\ &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{\gamma}{2}\|u\|^2_{2,\frac{1}{|x|^2}} - \int_\Omega F(x,u)dx.\end{aligned}$$

By using Hardy inequality, we have

$$J_\gamma(u) \geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{\gamma}{2}C_{N,2}\|\nabla u\|_2^2 - \int_\Omega F(x,u)dx. \tag{2.44}$$

Next, by 2.43, Since the following continuous embeddings of $H_0^1(\Omega)$ into $L^2(\Omega)$ and $L^r(\Omega)$, $r\in]2,2^*]$, So $\|u\|_r^r \leq C_r\|u\|^r_{H_0^1}$.

$$\begin{aligned}J_\gamma(u) &\geq \frac{1}{2}(1-\gamma C_{N,2})\|\nabla u\|_2^2 - \epsilon\|u\|_2^2 - \delta_\epsilon\|u\|_r^r\\ &\geq \frac{1}{2}(1-\gamma C_{N,2})\|u\|^2_{H_0^1} - \epsilon C_2\|u\|^2_{H_0^1} - \delta_\epsilon C_r\|u\|^r_{H_0^1}.\end{aligned}$$

We choose $\epsilon>0$ enough small, Then for $\rho>0$ enough small, we have

$$\frac{1}{2}(1-\gamma C_{N,2})\rho^2 - \epsilon C_2\rho^2 - \delta_\epsilon C_r\rho^r > 0.$$

for any $u\in H_0^1(\Omega)$ with $\|u\|_{H_0^1}=\rho$
Next, we put $\sigma_\epsilon = \frac{1}{2}(1-\gamma C_{N,2}) - \epsilon C_2 > 0$, and $\rho \in \left(0, min\left\{1, (\frac{\sigma_\epsilon}{2\delta_\epsilon}C_r)^{\frac{1}{r-2}}\right\}\right]$

$$\begin{aligned}J_\gamma(u) &\geq \sigma_\epsilon\rho^2 - \delta_\epsilon C_r\rho^r\\ &\geq \left(\sigma_\epsilon - \delta_\epsilon C_r\rho^{r-2}\right)\rho^2 = \alpha > 0.\end{aligned}$$

This completes the proof of Lemma 2.5. ■
Next, We Prove second geometric condition of Mountain Pass Theorem.

Lemma 2.6 *Let* $(f_1)-(f_2)$ *hold true. Then, for any* $\gamma\in\mathbb{R}^N$ *there exists* $e\in H_0^1(\Omega)$ *such that* $J_\gamma(e)<0$ *and* $\|e\|_{H_0^1}>1$.

Proof. Let us fix $\gamma\in\mathbb{R}^N$. By (f_1) and (f_2), there exist $d_1>0$ and $d_2\geq 0$ such that

$$F(x,t)\geq d_1|t|^\theta - d_2 \quad for \quad a.e. \quad x\in\Omega \quad and \quad any \quad t\in\mathbb{R}. \tag{2.45}$$

Thus, if $\varphi\in H_0^1(\Omega)$ with $\|\varphi\|_{H_0^1}=1$, so that by 2.45, for any $t\geq 1$ we have

$$J_\gamma(t\varphi) = \frac{1}{2}\|t\varphi\|^2_{H_0^1} - \frac{\gamma}{2}\|t\varphi\|^2_{2,\frac{1}{|x|^2}} - \int_\Omega F(x,t\varphi)\,dx,$$

Then, by applying (2.45) we have

$$J_\gamma(t\varphi) \leq \frac{1}{2}t^2\|\varphi\|^2_{H_0^1} - \frac{\gamma}{2}t^2\|\varphi\|^2_{2,\frac{1}{|x|^2}} - d_1t^\theta\|\varphi\|^\theta_\theta - d_2|\Omega|,$$

Next, by using Hardy inequality, we have

$$J_\gamma(t\varphi) \leq \frac{1}{2}t^2\|\varphi\|^2_{H_0^1} - \frac{\gamma}{2}C_{N,2}t^2\|\nabla\varphi\|_2^2 - d_1t^\theta\|\varphi\|^\theta_\theta - d_2|\Omega|,$$

Since $\theta > 2$ and $\gamma < \frac{1}{C_{N,2}}$ by (f_2), passing to the limit as $t \to \infty$ with $\|\varphi\|_{H_0^1} = 1$ we get

$$\begin{aligned}\lim_{t\to\infty} J_\gamma(t\varphi) &= \lim_{t\to\infty}\left[\frac{1}{2}t^2 - \frac{\gamma}{2}C_{N,2}t^2 - d_1t^\theta - d_2\left|\Omega\right|\right]\\ &= \lim_{t\to\infty}\left[\frac{1}{2}\left(1-\gamma C_{N,2}\right)t^2 - d_1t^\theta - d_2\left|\Omega\right|\right]\\ &= -\infty.\end{aligned}$$

So, we choose $t_\infty > 0$ enough large such that

$$\frac{1}{2}\left(1-\gamma C_{N,2}\right)t_\infty^2 - d_1t_\infty^\theta - d_2\left|\Omega\right| < 0$$

And, $\|t_\infty\varphi\|_{H_0^1} > 1$ this is equivalent $t_\infty > \frac{1}{\|\varphi\|_{H_0^1}}$ so we take $e = t_\infty\varphi$, with t_∞ sufficiently large. From this we conclude that $J_\gamma(e) < 0$ and $\|e\|_{H_0^1} > 1$. hold true. This completes the proof of Lemma 2.6. ■

proof of Theorem 2.3.

Since $J_\gamma(0) = 0$ and J_γ verifies the $(P.S)$ condition (See Proposition 2.4), and verifies the geometric conditions, (See Lemma 2.5 and Lemma 2.6), Which means the functional J_γ satisfies all conditions of Mountain Pass Theorem, then J_γ admits at least non-trivial critical point, So the existence of a non-trivial weak solution of problem (2.4). ■

CHAPTER 3

A KIRCHHOFF PROBLEM INVOLVING HARDY POTENTIAL

This chapter, By variational method we study the multiplicity of solutions for a class of Kirchhoff type problem with Hardy type potentials.

3.1 Presentation of Problem

We consider the following Kirchhoff type problem

$$\begin{cases} -M(\int_\Omega |\nabla u|^2 \ dx)\Delta u = \mu \ a(x)\frac{u}{|x|^2} + \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is bounded domain with smooth boundary $\partial\Omega$, $0 \in \Omega$, $M : \mathbb{R}_0^+ \to \mathbb{R}$ is continuous and increasing function with $\mathbb{R}_0^+ := [0, +\infty)$, the function $a : \Omega \to \mathbb{R}$ may change sign, λ is positive parameter, $0 \leq \mu < \frac{1}{C_{N,2}}$, where $C_{N,2} = \left(\frac{2}{N-2}\right)^2$ is optimal constant in the Hardy Inequality.

In order to state the main result of this section, let us introduce the following assumptions for problem (3.1) :

(A) The function $a : \overline{\Omega} \to \mathbb{R}$ is continuous, there exists $A_0 > 0$ such that

$$-A_0 \leq a(x) \leq A_0 \ \ for \ \ \text{a.e.}x \in \overline{\Omega}.$$

(M_0) There exist $m_0 > 0$ such that

$$M(t) \geq m_0 \ \ for \ \ all \ \ t \in \mathbb{R}_0^+.$$

(F_1) $f : \mathbb{R} \to \mathbb{R}$ is continuous and sub-linear at infinity, *i.e.*

$$\lim_{|t| \to +\infty} \frac{f(t)}{t} = 0$$

(F_2) f is super-linear at zero, *i.e.*

$$\lim_{t \to 0} \frac{f(t)}{t} = 0$$

(F_3) It holds that

$$\sup_{t\in\mathbb{R}} F(t) > 0,$$

where

$$F(t) = \int_0^t f(\tau)d\tau. \tag{3.2}$$

Definition 3.1 *We say that $u \in H_0^1$ is a weak solution of problem (3.1) if*

$$M\left(\int_\Omega |\nabla u|^2 \ dx\right)\int_\Omega \nabla u \nabla \varphi \ dx - \mu \int_\Omega \frac{a(x)}{|x|^2} u\varphi \ dx - \lambda \int_\Omega f(u)\varphi \ dx = 0$$

for all $\varphi \in H_0^1(\Omega)$.

Let us define the functional $J_{\mu,\lambda} : H_0^1(\Omega) \to \mathbb{R}$ by the following formula

$$\begin{aligned} J_{\mu,\lambda}(u) &= \frac{1}{2}\hat{M}\left(\int_\Omega |\nabla u|^2 \ dx\right) - \frac{\mu}{2}\int_\Omega \frac{a(x)}{|x|^2}|u|^2 \ dx - \lambda\int_\Omega F(u) \ dx, \\ &= \mathcal{A}(u) - \lambda\mathcal{F}(u), \end{aligned}$$

where

$$\mathcal{A}(u) = \frac{1}{2}\hat{M}\left(\int_\Omega |\nabla u|^2 \ dx\right) - \frac{\mu}{2}\int_\Omega \frac{a(x)}{|x|^2}|u|^2 \ dx,$$

$$\mathcal{F}(u) = \int_\Omega F(u) \ dx, \hat{M}(x,t) = \int_0^t M(\tau) \ d\tau, F(t) = \int_0^t f(\tau) \ d\tau. \tag{3.3}$$

3.2 Main result

Now, we present the main result of this section

Theorem 3.1 *[14] Assume that the condition (A), (M_0) and $(F_1) - (F_3)$ are satisfied. Then there exists $\overline{\mu} > 0$ such that for any $\mu \in [0, \overline{\mu})$, there exist an open interval $\Lambda_\mu \subset (0, +\infty)$ and a real number $\delta_\mu > 0$ such that for every $\lambda \in \Lambda_\mu$ problem (3.1) has at least two distinct, nontrivial weak solutions in $H_0^1(\Omega)$ whose $H_0^1(\Omega)$-norms are less than δ_μ.*

Now, we will prove Theorem 3.1 by using a recent result on the existence of at least three critical points which was proved by **G. Bonanno** (see [5]). As we explain it in the following

Proposition 3.1 *[14] Let $(E, \|\cdot\|)$ be a separable and reflexive real Banach space, $\mathcal{A}, \mathcal{F} : E \to \mathbb{R}$ be two continuously Gateaux differentiable functionals. Assume that there exists $x_0 \in E$ such that $\mathcal{A}(x_0) = \mathcal{F}(x_0) = 0$, $\mathcal{A}(x) \geq 0$ for all $x \in E$ and there exist $x_1 \in E, \rho > 0$ such that*

(i) $\rho < \mathcal{A}(x_1)$,

(ii) $\sup_{\{\mathcal{A}(x)<\rho\}} \mathcal{F}(x) < \rho\dfrac{\mathcal{F}(x_1)}{\mathcal{A}(x_1)}$.
Further, put

$$\overline{a} = \frac{\xi\rho}{\rho\frac{\mathcal{F}(x_1)}{\mathcal{A}(x_1)} - \sup\limits_{\{\mathcal{A}(x)<\rho\}} \mathcal{F}(x)} \quad with \quad \xi > 1,$$

and assume that the functional $\mathcal{A} - \lambda\mathcal{F}$ is sequentially weakly lower semi-continuous, satisfies the Palais-Smaile condition and *(iii)* $\lim\limits_{\|x\|_E \to \infty} [\mathcal{A}(x) - \lambda \mathcal{F}(x)] = +\infty$ *for every* $\lambda \in [0, \overline{a}]$.

Then, there exist an open interval $\Lambda \subset [0, \overline{a}]$ *and a positive real number* δ *such that each* $\lambda \in \Lambda$, *the equation*

$$D\mathcal{A}(u) - \lambda D\mathcal{F}(u) = 0$$

has at least three solution in E *whose* $\|\cdot\|$*-norms are less than* δ.

In the rest of this section, we denote by S_q the best constant of the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$, that is, $S_q \|u\|_q \leq \|u\|_{H_0^1}$, $\|u\|_{H_0^1} = \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}$.

3.3 Proof of Theorem 3.1

We introduce the following Lemma to prove this condition

$$|f(t)| \leq C_1(1 + |t|), \quad for \quad all \quad t \in \mathbb{R}.$$

which we will need in the rest of the Proofs of Lemma 3.3, Lemma 3.4 and Lemma 3.5 later.

Lemma 3.1 *Assume that* (F_1) *is holds. Then there exists a constant* $C_1 > 0$, *such that*

$$|f(t)| \leq C_1(1 + |t|), \quad for \quad all \quad t \in \mathbb{R}.$$

Proof. By considering (F_1), we have f is continuous which means

$$\forall \epsilon > 0 \ni \delta_\epsilon > 0 \quad \forall t \in \mathbb{R}, |t| > \delta_\epsilon \Rightarrow \left|\frac{f(t)}{t}\right| \leq \epsilon.$$

Taking $\epsilon = 1$, we obtain

$$\forall \epsilon > 0 \ni \delta_1 > 0 \quad \forall t \in \mathbb{R}, |t| > \delta_1 \Rightarrow \left|\frac{f(t)}{t}\right| \leq 1. \tag{3.4}$$

Let $t \in \mathbb{R}^N$;
if $|t| \leq \delta_1$ which means $t \in [-\delta_1, \delta_1]$,
since f is continuous on a compact $[-\delta_1, \delta_1]$, then f is bounded, that is

$$\ni C > 0 \, |f(t)| \leq C \tag{3.5}$$

if $|t| > \delta_1$, by (3.4) we have

$$|f(t)| \leq (1 + |t|) \tag{3.6}$$

from (3.5) and (3.6), we deduce that

$$|f(t)| \leq C_1(1 + |t|), \quad for \quad all \quad t \in \mathbb{R}.$$

Where $C_1 = max\{1, C\}$.
This complete the proof. ■
We introduce the following Lemma to prove this inclusion $B_\rho^1 \subset B_\rho^2$ which we will need in proof of Lemma 3.5 later.

Lemma 3.2 *Assume that* (A) *and* (M_0) *is holds.*
We have $B_\rho^1 \subset B_\rho^2$.

Proof. For each $\rho > 0$, we define the sets

$$B_\rho^1 = \Big\{ u \in H_0^1(\Omega) : \quad \mathcal{A}(u) < \rho \Big\},$$

$$B_\rho^2 = \Big\{ u \in H_0^1(\Omega) : \quad (m_0 - \mu A_0 C_{N,2}) \|u\|_{H_0^1}^2 < 2\rho \Big\}.$$

we have $u \in B_\rho^1$ which means

$$\mathcal{A}(u) = \frac{1}{2}\hat{M}\left(\int_\Omega |\nabla u|^2 \ dx\right) - \frac{\mu}{2}\int_\Omega \frac{a(x)}{|x|^2}|u|^2 \ dx < \rho,$$

then,

$$\mathcal{A}(u) = \hat{M}\left(\int_\Omega |\nabla u|^2 \ dx\right) - \mu\int_\Omega \frac{a(x)}{|x|^2}|u|^2 \ dx < 2\rho,$$

by using (A), (M_0) and Hardy inequality, we get

$$\begin{aligned}
\mathcal{A}(u) &= \hat{M}\left(\int_\Omega |\nabla u|^2 \ dx\right) - \mu\int_\Omega \frac{a(x)}{|x|^2}|u|^2 \ dx \\
&\geq m_0\left(\int_\Omega |\nabla u|^2 \ dx\right) - \mu A_0 C_{N,2}\int_\Omega |\nabla u|^2 \ dx \\
&= m_0 \|u\|_{H_0^1}^2 - \mu A_0 C_{N,2} \|u\|_{H_0^1}^2 \\
&= (m_0 - \mu A_0 C_{N,2}) \|u\|_{H_0^1}^2 .
\end{aligned}$$

we obtain,

$$(m_0 - \mu A_0 C_{N,2}) \|u\|_{H_0^1}^2 < 2\rho.$$

consequently, $u \in B_\rho^2$.
we conclude this proof. ■

We introduce the following Lemma which represent condition of Proposition 3.1 to verify that functional $J_{\mu,\lambda}$ is sequentially weakly lower semi-continuous on $H_0^1(\Omega)$.

Lemma 3.3 *There exists* $\overline{\mu} > 0$ *such that for any* $\mu \in [0, \overline{\mu})$, *the functional* $J_{\mu,\lambda}$ *is sequentially weakly lower semi-continuous on* $H_0^1(\Omega)$.

Proof. Let $\{u_j\}$ be a sequence that converges weakly to u in $H_0^1(\Omega)$. By the conditions (A) and (M_0), taking $\overline{\mu} = \frac{m_0}{A_0 C_{N,2}}$, then for each $0 \leq \mu < \overline{\mu}$, using the same arguments as in the Proof of [[4],Theorem 3.2], we can obtain

$$\begin{aligned}
\liminf_{j\to\infty}\left\{\frac{1}{2}\hat{M}\left(\int_\Omega |\nabla u_j|^2 \ dx\right) - \frac{\mu}{2}\int_\Omega \frac{a(x)}{|x|^2}|u_j|^2 \ dx\right\} &\geq \frac{1}{2}\hat{M}\left(\int_\Omega |\nabla u|^2 \ dx\right) \\
&\quad - \frac{\mu}{2}\int_\Omega \frac{a(x)}{|x|^2}|u|^2 \ dx.
\end{aligned} \tag{3.7}$$

On the other hand, we shows that

$$\lim_{j\to\infty}\int_\Omega F(u_j) \ dx = \int_\Omega F(u) \ dx. \tag{3.8}$$

From finite increase theorem, we have

$$\begin{aligned}\left|\int_\Omega F(u_j)\ dx - \int_\Omega F(u)\ dx\right| &\leq \int_\Omega |F(u_j)-F(u)|\ dx\\ &\leq \int_\Omega |f(u+\theta_j(u_j-u))|\,|u_j-u|\ dx\end{aligned}$$

then, by using Lemma 3.1 we get

$$\left|\int_\Omega F(u_j)\ dx - \int_\Omega F(u)\ dx\right| \leq C_1\int_\Omega (1+|u+\theta_j(u_j-u)|)\,|u_j-u|\ dx$$

from Lemma 3.1 and Hölder inequality, we get

$$\begin{aligned}\left|\int_\Omega F(u_j)\ dx - \int_\Omega F(u)\ dx\right| &\leq C_1\left(|\Omega|^{\frac{1}{2}} + \int_\Omega |u+\theta_j(u_j-u)|^2\ dx\right)^{\frac{1}{2}}\left(|u_j-u|^2\ dx\right)^{\frac{1}{2}}\\ &= C_1\left[(|\Omega|^{\frac{1}{2}}) + \|u+\theta_j(u_j-u)\|_2\right]\|u_j-u\|_2\,,\end{aligned}$$

where $\theta_j \in (0,1)$. From relations (3.7) and (3.8), we conclude that

$$\liminf_{j\to\infty} J_{\mu,\lambda}(u_j) \geq J_{\mu,\lambda}(u)$$

and thus, the functional $J_{\mu,\lambda}$ is sequentially weakly lower semi-continuous in $H_0^1(\Omega)$.
This complete the proof. ■
We introduce the following Lemma which represent two conditions of Proposition 3.1 to verify that functional $J_{\mu,\lambda}$ is coercive and satisfies the Palais-Smale condition.

Lemma 3.4 *For every $\mu \in [0,\overline{\mu})$ and $\lambda \in \mathbb{R}$, the functional $J_{\mu,\lambda}$ is coercive and satisfies the Palais-Smale condition.*

Proof.
Firstly, we prove the functional $J_{\mu,\lambda}$ is coercive.
Let us fix $\lambda \in \mathbb{R}$, arbitrary. By (F_1), there exists $\delta = \delta(\lambda) > 0$, such that

$$|f(t)| \leq S_2^2\left(m_0 - \mu A_0 C_{N,2}\right)(1+|\lambda|)^{-1}|t|\,, \text{for all}\, |t| > \delta.$$

then,

$$|f(t)| \leq S_2^2\left(m_0 - \mu A_0 C_{N,2}\right)(1+|\lambda|)^{-1}|t| + \max_{\tau\in[-\delta,\delta]}|f(\tau)|\,, \text{for all} \quad t \in \mathbb{R}^N.$$

Integrating the above inequality, by (3.2) we have
If $t > 0$, we obtain

$$\int_0^t f(\tau)\ d\tau \leq S_2^2\left(m_0 - \mu A_0 C_{N,2}\right)(1+|\lambda|)^{-1}\int_0^t \tau\ d\tau + \int_0^t \max_{\tau\in[-\delta,\delta]}|f(\tau)|\ d\tau. \tag{3.9}$$

and,

$$\int_0^t f(\tau)\ d\tau \geq -S_2^2\left(m_0 - \mu A_0 C_{N,2}\right)(1+|\lambda|)^{-1}\int_0^t \tau\ d\tau - \int_0^t \max_{\tau\in[-\delta,\delta]}|f(\tau)|\ d\tau \tag{3.10}$$

Thus, combining (3.9) and (3.10) we have

$$F(t) \leq \frac{S_2^2}{2}\left(m_0 - \mu A_0 C_{N,2}\right)(1+|\lambda|)^{-1}t^2 + \max_{|\tau|\leq\delta}|f(\tau)|\,t. \tag{3.11}$$

and,

$$F(t) \geq -\frac{S_2^2}{2}\left(m_0-\mu A_0 C_{N,2}\right)(1+|\lambda|)^{-1} t^2-\max_{|\tau| \leq \delta}|f(\tau)| t. \tag{3.12}$$

so, combining relationship (3.11) and (3.12) we obtain

$$|F(t)| \leq \frac{S_2^2}{2}\left(m_0-\mu A_0 C_{N,2}\right)(1+|\lambda|)^{-1}|t|^2+\max_{|\tau| \leq \delta}|f(\tau)||t| \quad for \quad any \quad t>0.$$

If $t<0$, we obtain

$$\int_t^0 f(\tau) d\tau \leq S_2^2\left(m_0-\mu A_0 C_{N,2}\right)(1+|\lambda|)^{-1} \int_t^0 \tau d\tau+\int_t^0 \max_{\tau \in[-\delta, \delta]}|f(\tau)| \; d\tau. \tag{3.13}$$

and,

$$\int_t^0 f(\tau) d\tau \geq -S_2^2\left(m_0-\mu A_0 C_{N,2}\right)(1+|\lambda|)^{-1} \int_t^0 \tau d\tau-\int_t^0 \max_{\tau \in[-\delta, \delta]}|f(\tau)| \; d\tau. \tag{3.14}$$

then, combining (3.13) and (3.14) we obtain

$$-F(t) \leq \frac{S_2^2}{2}\left(m_0-\mu A_0 C_{N,2}\right)(1+|\lambda|)^{-1} t^2+\max_{|\tau| \leq \delta}|f(\tau)| t. \tag{3.15}$$

and,

$$-F(t) \geq -\frac{S_2^2}{2}\left(m_0-\mu A_0 C_{N,2}\right)(1+|\lambda|)^{-1} t^2-\max_{|\tau| \leq \delta}|f(\tau)| t. \tag{3.16}$$

So, combining relationship (3.15) and (3.16) we obtain

$$|F(t)| \leq \frac{S_2^2}{2}\left(m_0-\mu A_0 C_{N,2}\right)(1+|\lambda|)^{-1}|t|^2+\max_{|\tau| \leq \delta}|f(\tau)||t| \quad for \quad any \quad t<0.$$

so we conclude that,

$$|F(t)| \leq \frac{S_2^2}{2}\left(m_0-\mu A_0 C_{N,2}\right)(1+|\lambda|)^{-1}|t|^2+\max_{|\tau| \leq \delta}|f(\tau)||t| \quad for \quad all \quad t \in \mathbb{R}. \tag{3.17}$$

Hence, by (A) and (M_0), we obtain

$$\begin{aligned}
J_{\mu, \lambda}(u) &= \frac{1}{2} \hat{M}\left(\int_{\Omega}|\nabla u|^2 \; dx\right)-\frac{\mu}{2} \int_{\Omega} \frac{a(x)}{|x|^2}|u|^2 \; dx-\lambda \int_{\Omega} F(x, u) \; dx, \\
&\geq \frac{m_0}{2} \int_{\Omega}|\nabla u|^2 \; dx-\frac{\mu A_0}{2} \int_{\Omega} \frac{|u|^2}{|x|^2} dx-|\lambda| \int_{\Omega}|F(x, u)| \; dx
\end{aligned}$$

Then, by using Hardy inequality Lemma 2.1 and (3.17) we have

$$\begin{aligned}
J_{\mu, \lambda}(u) \geq & \frac{m_0}{2} \int_{\Omega}|\nabla u|^2 \; dx-\frac{\mu A_0}{2} C_{N,2} \int_{\Omega}|\nabla u|^2 \; dx-|\lambda| \int_{\Omega}|F(x, u)| \; dx \\
\geq & \frac{1}{2}\left(m_0-\mu A_0 C_{N,2}\right) \int_{\Omega}|\nabla u|^2 \; dx-\frac{|\lambda| S_2^2}{2(1+|\lambda|)}\left(m_0-\mu A_0 C_{N,2}\right) \int_{\Omega}|u|^2 \; dx \\
& -|\lambda| \max_{|t| \leq \delta}|f(t)| \int_{\Omega}|u| \; dx.
\end{aligned}$$

From the continuous embeddings and the Hölder inequality

$$J_{\mu, \lambda}(u) \geq \frac{1}{2(1+|\lambda|)}\left(m_0-\mu A_0 C_{N,2}\right)\|u\|_{H_0^1}^2-\max_{|t| \leq \delta}|f(t)| \frac{|\lambda|}{S_1}|\Omega|^{\frac{1}{2}}\|u\|_{H_0^1}.$$

Consequently, we have

$$\lim_{\|u\|_{H_0^1}\to\infty}\left\{\frac{1}{2(1+|\lambda|)}\left(m_0-\mu A_0C_{N,2}\right)\|u\|_{H_0^1}^2-\max_{|t|\leq\delta}|f(t)|\frac{|\lambda|}{S_1}|\Omega|^{\frac{1}{2}}\|u\|_{H_0^1}\right\}=\infty.$$

Since $\overline{\mu}=\frac{m_0}{A_0C_{N,2}}>0$, we deduce that for each $\mu\in[0,\overline{\mu})$ and $\lambda\in\mathbb{R}$, the functional $J_{\mu,\lambda}$ is coercive.
Next, we prove the functional $J_{\mu,\lambda}$ satisfies the Palais-Smale condition.
let $\{u_j\}$ be sequence in $H_0^1(\Omega)$, such that

$$J_{\mu,\lambda}(u_j)\to c<\infty\ \ and\ \ J_{\mu,\lambda}(u)\to 0\ \ in\ \ (H_0^1(\Omega))^*\ \ j\to\infty, \tag{3.18}$$

Since $J_{\mu,\lambda}$ is coercive, the sequence $\{u_j\}$ is bounded in $H_0^1(\Omega)$ which is reflexive space, so we deduce that there exists a sub-sequence of $\{u_j\}$, still denoted by $\{u_j\}$, that converges weakly to some $u\in H_0^1(\Omega)$ and thanks to the compactly embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, we have $\{u_j\}$ converges strongly to u in $L^2(\Omega)$.
We will prove that for any $u,\varphi\in H_0^1(\Omega)$, we have

$$\phi(u,\varphi)\geq m_0\|u-\varphi\|_{H_0^1}^2, \tag{3.19}$$

where m_0 is given by (M_0) and

$$\begin{aligned}\phi(u,\varphi)&=M\left(\int_\Omega|\nabla u|^2\ dx\right)\int_\Omega\nabla u(\nabla u-\nabla\varphi)\\&-M\left(\int_\Omega|\nabla\varphi|^2\ dx\right)\int_\Omega\nabla\varphi(\nabla u-\nabla\varphi).\end{aligned}$$

Indeed, using the following inequality

$$ab\leq\frac{1}{2}\left(a^2+b^2\right)$$

and,

$$a\left(a-b\right)\leq\frac{1}{2}\left(a^2-b^2\right)$$

we have,

$$\nabla u\nabla\varphi\leq|\nabla u|\,|\nabla\varphi|\leq\frac{1}{2}\left(|\nabla u|^2+|\nabla\varphi|^2\right)$$

or

$$\nabla u\left(\nabla u-\nabla\varphi\right)\geq\frac{1}{2}\left(|\nabla u|^2-|\nabla\varphi|^2\right). \tag{3.20}$$

We have

$$\begin{aligned}\phi(u,\varphi)&=M\left(\int_\Omega|\nabla u|^2\ dx\right)\int_\Omega\nabla u(\nabla u-\nabla\varphi)\ dx-M\left(\int_\Omega|\nabla\varphi|^2\ dx\right)\int_\Omega\nabla\varphi(\nabla u-\nabla\varphi)\ dx\\&-M\left(\int_\Omega|\nabla\varphi|^2\ dx\right)\int_\Omega\nabla u(\nabla u-\nabla\varphi)\ dx+M\left(\int_\Omega|\nabla\varphi|^2\ dx\right)\int_\Omega\nabla u(\nabla u-\nabla\varphi)\ dx\\&=\left\{M\left(\int_\Omega|\nabla u|^2\ dx\right)\ dx-M\left(\int_\Omega|\nabla\varphi|^2\ dx\right)\right\}\int_\Omega\nabla u(\nabla u-\nabla\varphi)\ dx\\&+M\left(\int_\Omega|\nabla\varphi|^2\ dx\right)\int_\Omega|\nabla u-\nabla\varphi|^2\ dx\end{aligned}$$

Hence, by using (3.20) and because $M(t)$ is increasing, it implies that

$$\begin{aligned}\phi(u,\varphi) &\geq \frac{1}{2}\left\{M\left(\int_\Omega |\nabla u|^2\ dx\right) - M\left(\int_\Omega |\nabla\varphi|^2\ dx\right)\right\}\left[\int_\Omega |\nabla u|^2\ dx - \int_\Omega |\nabla \varphi|^2\ dx\right]\\ &\quad + M\left(\int_\Omega |\nabla\varphi|^2\ dx\right)\int_\Omega |\nabla u - \nabla\varphi|^2\ dx\\ &\geq m_0 \|u-\varphi\|^2_{H^1_0}.\end{aligned}$$

Now, from (3.18), (3.19) and Hardy inequality Lemma 2.1, we find that

$$\begin{aligned} o(1) &= \langle J'(u_j) - J'(u), u_j - u\rangle\\ &= \frac{1}{2}M\left(\int_\Omega |\nabla u_j|^2\ dx\right)\int_\Omega \nabla u_j(\nabla u_j - \nabla u)\ dx\\ &\quad - \frac{1}{2}M\left(\int_\Omega |\nabla u|^2\ dx\right)\int_\Omega \nabla u(\nabla u_j - \nabla u)\ dx\\ &\quad - \frac{\mu}{2}\int_\Omega \frac{a(x)}{|x|^2}|u_j-u|^2\ dx - \lambda\int_\Omega (f(u_j)-f(u))(u_j-u)\ dx\\ &\geq \frac{m_0}{2}\|u_j-u\|^2_{H^1_0} - \frac{\mu A_0 C_{N,2}}{2}\|u_j-u\|^2_{H^1_0} - \lambda\int_\Omega (f(u_j)-f(u))(u_j-u)\ dx\\ &= \frac{1}{2}(m_0 - \mu A_0 C_{N,2})\|u_j-u\|^2_{H^1_0} - \lambda\int_\Omega (f(u_j)-f(u))(u_j-u)\ dx. \end{aligned} \tag{3.21}$$

On the other hand, by Hölder inequality,

$$\begin{aligned}\left|\int_\Omega (f(u_j)-f(u))(u_j-u)\ dx\right| &\leq C_1\int_\Omega (2+|u_j|+|u|)|u_j-u|\ dx\\ &\leq C_1\int_\Omega \left[2|\Omega|^{\frac{1}{2}} + \|u_j\|_2 + \|u\|_2\right]\|u\|_2,\end{aligned} \tag{3.22}$$

which approaches 0 as $j\to\infty$.
From (3.21), (3.22) and the fact that $0\leq\mu<\overline{\mu}=\frac{m_0}{A_0C_{N,2}}$, we deduce that u_j converges strongly to u in $H^1_0(\Omega)$. This complete the Proof. ■
In order to verify that conditions (**i**) and (**ii**) of Proposition 3.1 are satisfied, we try to prove this condition in the following Lemma.

Lemma 3.5 *For each $\mu\in[0,\overline{\mu})$ we have*

$$\lim_{\rho\to 0^+}\frac{\sup\{\mathcal{F}(u) : \mathcal{A}(u)<\rho\}}{\rho} = 0,$$

where the functionals $\mathcal{A}$ and $\mathcal{F}$ are given by (3.3).

Proof. By (F_2), for an arbitrary small $\epsilon>0$, there exists $\delta>0$, such that

$$|f(t)| \leq \frac{\epsilon S_2^2}{2}(m_0-\mu A_0C_{N,2})|t| \quad for\ \ all \quad |t|<\delta.$$

Combining the above inequality with Lemma 3.1, we get

$$|F(t)| \leq \frac{\epsilon S_2^2}{4}(m_0-\mu A_0C_{N,2})|t|^2 + C_\delta|t|^q \quad for\ \ t\in\mathbb{R}, \tag{3.23}$$

where $q\in\left(2,\frac{2N}{N-2}\right)$, and C_δ is constant that does not depend on t.
By Lemma 3.2. Moreover, using (3.23), it follows that for any $u\in B^2_\rho$,

$$\mathcal{F}(u) \leq \frac{\epsilon}{4}(m_0-\mu A_0C_{N,2})\|u\|^2_{H^1_0} + C_\delta S_q^{-q}\|u\|^q_{H^1_0}. \tag{3.24}$$

Since $0 \in B_\rho^1$ and $J_{\mu,\lambda}(0) = 0$, we have $0 \leq \sup\limits_{u \in B_\rho^1} J_{\mu,\lambda}(u)$. On the other hand, if $u \in B_\rho^2$, then

$$\|u\|_{H_0^1} \leq (m_0 - \mu A_0 C_{N,2})^{\frac{-1}{2}} (2\rho)^{\frac{1}{2}}.$$

Now, using (3.24), we deduce that

$$\begin{aligned} 0 \leq \frac{\sup\limits_{u \in B_\rho^1} \mathcal{F}(u)}{\rho} &\leq \frac{\sup\limits_{u \in B_\rho^2} \mathcal{F}(u)}{\rho} \\ &\leq \frac{\epsilon}{2} + C_\delta S_q^{-q} (m_0 - \mu A_0 C_{N,2})^{\frac{-q}{2}} (2\rho)^{\frac{q}{2}-1}. \end{aligned} \tag{3.25}$$

Since $q > 2$, letting $\rho \to 0^+$, because $\epsilon > 0$ is arbitrary, we get the conclusion. ■

Proof of Theorem 3.1. In order to prove Theorem 3.1, we shall apply Proposition (3.1) as well as $\mathcal{A}$ and $\mathcal{F}$ as in (3.3). Now, we shall check all assumptions of Proposition 3.1. Indeed, we have $\mathcal{A}(0) = \mathcal{F}(0) = 0$ and since $-A_0 \leq a(x) \leq A_0$ for all $x \in \overline{\Omega}$, we deduce from Hardy inequality Lemma 2.1 that for any $0 \leq \mu < \overline{\mu}$, $\mathcal{A} \geq 0$ for any $u \in H_0^1(\Omega)$.

From (F_3), let $t_0 \in \mathbb{R}$ be such that $F(t_0) > 0$. Also choose $R_0 > 0$ such a way that

$$R_0 < dist(0, \partial\Omega). \;\; For \;\; \sigma \in (0,1),$$

we define the function u_σ by

$$u_\sigma(x) = \begin{cases} 0, & \text{for } x \in \mathbb{R}^N \backslash B_{R_0}(0), \\ t_0, & \text{for } x \in B_{\sigma R_0}(0), \\ \frac{t_0}{R_0(1-\sigma)}(R_0 - |x|), & \text{for } x \in B_{R_0}(0) \backslash B_{\sigma R_0}(0), \end{cases}$$

where $B_r(0)$ denotes the N-dimensional open ball with center 0 and radius $r > 0$. It is clear that $u_\sigma \in H_0^1(\Omega)$.

whose derivative is,

$$u'_\sigma(x) = \begin{cases} 0, & \text{for } x \in \mathbb{R}^N \backslash B_{R_0}(0), \\ 0, & \text{for } x \in B_{\sigma R_0}(0), \\ \frac{t_0}{R_0(1-\sigma)}\left(\frac{-x}{|x|}\right), & \text{for } x \in B_{R_0}(0) \backslash B_{\sigma R_0}(0), \end{cases}$$

From the definition of u_σ and we set $x = rw$, such that $dx = r^{N-1} dr dw$, we get

$$\begin{aligned} \|u_\sigma\|_{H_0^1}^2 &= \int_{\mathbb{R}^N} |\nabla u_\sigma|^2 \, dx \\ &= \int_{B_{R_0}(0) \backslash B_{\sigma R_0}(0)} \left(\frac{t_0}{R_0(1-\sigma)}\right)^2 \left(\frac{x}{|x|}\right)^2 dx \\ &= \left(\frac{t_0}{R_0(1-\sigma)}\right)^2 \int_{S^{N-1}} dw \int_{\sigma R_0}^{R_0} r^{N-1} \, dr \\ &= \frac{t_0^2}{R_0^2(1-\sigma)^2} \frac{\omega_N}{N} \left(R_0^N - (\sigma R_0)^N\right) \\ &= t_0^2 R_0^{-2} (1-\sigma)^{-2} \frac{\omega_N}{N} \left(R_0^N - \sigma^N R_0^N\right). \end{aligned}$$

then,

$$\|u_\sigma\|^2_{H^1_0} = t_0^2(1-\sigma)^{-2}(1-\sigma^N)\omega_N R_0^{N-2}$$

and

$$\begin{aligned} \mathcal{F}(u_\sigma) &= \int_{B_{\sigma R_0}(0)} F(u_\sigma)\, dx + \int_{B_{R_0}(0)\setminus B_{\sigma R_0}(0)} F(u_\sigma)\, dx \\ &\geq \left[F(t_0)\sigma^N - \max_{|t|\leq R_0} |F(t)|\,(1-\sigma)^N \right] R_0^N \omega_N, \end{aligned}$$

where ω_N is the volume of the unit ball $B_1(0)$. If we choose $\sigma \in (0,1)$ close enough to 1, says σ_0, then the right-hand side of the last inequality becomes strictly positive. By Lemma 3.5, we can choose $\rho_0 \in (0,1)$ such that

$$\rho_0 < (m_0 - \mu A_0 C_{N,2})\, \|u_{\sigma_0}\|^2_{H^1_0} \leq \mathcal{A}(u_{\sigma_0})$$

and

$$\begin{aligned} \frac{\sup_{\mathcal{A}(u) < \rho_0} \mathcal{F}(u)}{\rho_0} &< \frac{\left[F(t_0)\sigma^N - \max_{|t|\leq R_0} |F(t)|\,(1-\sigma)^N \right] R_0^N \omega_N}{2\mathcal{A}(u_{\sigma_0})} \\ &< \frac{\mathcal{F}(u_{\sigma_0})}{\mathcal{A}(u_{\sigma_0})}. \end{aligned}$$

Now, in Proposition 3.1, we choose $x_0 = 0$, $x_1 = u_{\sigma_0}$, $\xi = 1 + \rho_0$ and For any $\mu \in [0, \overline{\mu})$, taking into account the above Lemmas, all assumptions of Proposition 3.1 are verified. Then there exist an open interval $\Lambda_{\overline{\mu}} \subset [0, \overline{a}]$ and a number $\delta_{\overline{\mu}}$, such that for each $\lambda \in$ the equation $D\mathcal{A}(u) - \lambda D\mathcal{A}(u) = 0$ has at least three solutions in $H^1_0(\Omega)$ whose $H^1_0(\Omega)$-norms are less than $\delta_{\overline{\mu}}$. By (F_2), $f(0) = 0$, one of them may be the trivial one, so Problem (3.1) has at least two non-trivial weak solutions with the required properties.

■

CONCLUSION

In this work, we have studied some techniques to solve elliptic problem involving Hardy type potentials, posed on a bounded open set $\Omega \subset \mathbb{R}^N$. we presented some auxiliary results on the L_p Weighted Lebesgue Spaces. To show some applications of Hardy's inequality we proved the existence of a unique weak solution of elliptic linear problem by using Lax Milligram Theorem, then we proved that the existence at least non-trivial weak solution using Mountain-Pass Theorem. Also we studied the existence at least three critical points in a class of Kirchhoff. There are some cases that we have not studied, so we look forward to be studied by our colleagues in the coming years, such as, the similar result for the p-Laplacian operator.

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Abstract

The objective of our work is to study the Hardy inequality, Then we try to apply it to the following elliptic linear problem

$$\begin{cases} -\Delta u = \gamma \frac{u}{x^2} + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $f \in L^2(\Omega)$, $0 \in \Omega$, γ is a real parameter.

To prove that it admits a unique weak solution by using Lax Milligram theorem.

Then we have the following some elliptic problems involving Hardy potential

$$\begin{cases} -\Delta u = \gamma \frac{u}{|x|^2} + f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N > 2)$ be open and bounded, $0 \in \Omega$,γ is a real parameter.

Where we study the existence at least non-trivial weak solution using Mountain-Pass theorem, that is the associated functional J_γ admits at least a non trivial critical point.

Keywords: Hardy potentials, Variational methods, Critical point, Weak solution, mountain pass theorem.

We present the following a class of Kirchhoff type problem involving Hardy type potentials

$$\begin{cases} -M(\int_\Omega |\nabla u|^2 \; dx)\Delta u = \frac{\mu}{x^2} a(x)u + \lambda f(x,u) & \texttt{in } \Omega \\ u = 0 & \texttt{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is bounded domain with smooth boundary $\partial\Omega$, $0 \in \Omega$, $M : \mathbb{R}_0^+ \to \mathbb{R}$ is continuous and increasing function with $\mathbb{R}_0^+ := [0, +\infty)$, the function $a : \Omega \to \mathbb{R}$ may change sign, λ is positive parameter,$0 \leq \mu < \frac{1}{C_{N,2}}$, where $C_{N,2} = \left(\frac{2}{N-2}\right)^2$ is optimal constant in the Hardy Inequality.

Keywords: Kirchhoff type problem, Hardy type potential, Sub-linear non-linearity, Multiple solutions, Three critical points theorem.

Résumé

L' objective de notre travail est d'étudier L'inégalité de Hardy, puis nous essayons de l' appliquer au problème linéaire elliptique suivant

$$\begin{cases} -\Delta u = \gamma \frac{u}{x^2} + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

où $f \in L^2(\Omega)$, $0 \in \Omega$, γ est un paramètre réel.
prouver qu'il admet une solution faible unique en utilisant le théorème de Lax Milligram.
Nous avouns alors les problèmes elliptiques suivants impliquant le potentiel de Hardy

$$\begin{cases} -\Delta u = \gamma \frac{u}{|x|^2} + f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

où $\Omega \subset \mathbb{R}^N (N > 2)$ est ouvert et borneé $0 \in \Omega$,γ est un paramètre réel .
Nous étudions l'existence d'au moins une solution faible non triviale en utilisant le théorème du col de la montagne, c'est-a-dire la fonctionnelle J associée admet au moins un point critique non trivial.

Mots clés : Potentiels de type Hardy, Méthodes variationnelles, Point critique, Solution faible, théorème du col de la montagne.

Nous présentons ci-dessous une classe de problèmes de type kirchhoff impliquant des potentiels de type Hardy.

$$\begin{cases} -M(\int_\Omega |\nabla u|^2 \ dx)\Delta u = \frac{\mu}{x^2} a(x)u + \lambda f(x,u) & \text{dans } \Omega \\ u = 0 & \text{sur } \partial\Omega \end{cases}$$

où $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ est un domaine borné de frontière $\partial\Omega$, $0 \in \Omega$, $M : \mathbb{R}_0^+ \to \mathbb{R}$ est une fonction continue et croissante avec $\mathbb{R}_0^+ := [0, +\infty)$, la fonction $a : \Omega \to \mathbb{R}$ peut changer de signe, λ est un paramètre positif, $0 \leq \mu < \frac{1}{C_{N,2}}$, où $C_{N,2} = \left(\frac{2}{N-2}\right)^2$ est une constante optimale dans l'inégalité de Hardy.

Mots clés : Problèmes de type kirchhoff, Potentiels de type Hardy, Non-linéarité subliéaire, Solution multiples, Théorème des trois point critiques.

الملخص

في هذا العمل نقوم بدراسة بعض تطبيقات التي تظهر فيها أهمية متراجحة هاردي على بعض المسائل الخطية والغير خطية. في المسألة الأولى نثبت وجود حل وحيد بإستعمال نظرية لاكس ميليغرام

$$\begin{cases} -\Delta u = \gamma \frac{u}{x^2} + f(x) & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases}$$

بحيث $f \in L^2(\Omega)$ ، $0 \in \Omega$ ، γ متغير حقيقي.
ناقشنا كذلك في هذه المذكرة إشكالية وجود الحلول بإستعمال نظرية المر الجبلي. نقوم بإثبات أن المسألة التالية تقبل على الأقل حلين غير تافهين.

$$\begin{cases} -\Delta u = \gamma \frac{u}{|x|^2} + f(x,u) & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases}$$

بحيث $\Omega \subset \mathbb{R}^N (N > 2)$ مفتوح محدود، $0 \in \Omega$ ، γ متغير حقيقي.

الكلمات المفتاحية : مسائل متظمنة لهاردي، طريقة التغاير، النقطة الحرجة، الحل ضعيف، نظرية المر الجبلي.

تم تدعيم هذه المذكرة بمسألة أخرى من نوع كيرشوف متظمنة لهاردي. لدراسة وجود الحلول لهذه المسألة نثبت نظرية وجود على الأقل ثلاث نقاط حرجة.

$$\begin{cases} -M(\int_\Omega |\nabla u|^2 \ dx)\Delta u = \frac{\mu}{x^2} a(x) u + \lambda f(x,u) & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases}$$

بحيث $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ مفتوح محدود على الحافة $\partial\Omega$ ، $0 \in \Omega$ ، $M : \mathbb{R}_0^+ \to \mathbb{R}$ دالة مستمرة و متزايدة على المجال $\mathbb{R}_0^+ := [0, +\infty)$. $a : \Omega \to \mathbb{R}$ قد تغير من اشارتها. λ متغير حقيقي موجب. $0 \leq \mu < \frac{1}{C_{N,2}}$ ، حيث $C_{N,2} = \left(\frac{2}{N-2}\right)^2$ الثابت المثالي لمتراجحة هاردي .

الكلمات المفتاحية : نظرية ثلاث نقاط حرجة، مسائل من نوع كيرشوف، مسائل متظمنة لهاردي، الحل المضاعف.

قائمة المصطلحات

FRANÇAIS	ENGLISH	عربية
Points Critique	Critical points	النقاط الحرجة
Faiblement semi-continue inférieurement	Weakly lower semi-continous	نصف مستمر من الاسفل بضعف
Minimisation	Minimization	تصغير
Majoré	Bounded above	محدود من الاعلى
Minoré	Bounded below	محدود من الادنى
Borné	Bounded	محدود
Convexe	Convex	محدب
Frontière	Boundary	حافة
Boule	Ball	كرة
Boule -Unité	Unit-Ball	كرة الوحدة
Compact	Compact	متراص
Injection continue	Injection continous	تباين مستمر
Injection compact	Injection compact	تباين متراص
Convergence	Convergence	التقارب
Convergence fort	Strong convergence	التقارب بقوة
Convergence faible	Weak convergence	التقارب بضعف
Densité	Density	كثافة
Problèmes non linéaires	Non linear proplems	مسائل غير خطية
Point singulier	Singular point	نقطة شاذة

Dérivable	Differentiable	قابل للاشتقاق اوللمفاضلة
Fonctionnel	Functional	دالي
Dérivée partielle	Partial derivative	مشتق جزئي
Gradient	Gradient	تدرج
Continu	Continous	مستمر
Linéaire	Linear	خطي
Conjugué	Conjugate	المرافق
Fonctions régulière	Smooth functions	الدوال المنتظمة
Solution non-trivial	Non-trivial solution	حل غير بديهي او غير تافه
Norme	Norm	نظيم
Partie	Part	جزء
Ouvert	Open	مفتوح
Optimal	Optimal	امثل او مثلى
Limite supérieure	Limit superior upper	نهاية عليا « Lim sup »
Limite inférieure	Limit inferior lower	نهاية سفلى « Lim inf »
Contradiction	Contradiction	تناقض
Contant	Contant	ثابت
Définition	Definition	تعريف
Thérème	Theorem	نظرية
Lemme	Lemma	توطئة

Corollaire	Corollary	لازمة او نتيجة
Démonstration	Proof,Demonstration	برهان او اثبات
Exemple	Example	مثال
Notation	Notation	ترميز
Cas	Case	حالة
Calcul	Calculation, Calculus	حساب

بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

﴿ وَلقَدّ مَنَنَّا عَلَيْكَ مَرةً أُخري ﴾

وفي الختام ... أقف اليوم أمامكم ، قلبي يملؤه مزيج من المشاعر المتناقضة، فرحة الإنجاز وحزن الفراق، مشاعر ممزوجة بذكريات رحلة تعليمية طويلة مليئة بالتحديات والإنجازات.

لم تكن الرحلة قصيرة، ولا ينبغي لها أن تكون، لم يكن الحلم قريباً، ولا طريق كان محفوفاً بالتسهيلات، لــكني فعلتها.

إلَى من كللَه الله بالَهُيبُة والوقارْ من كان يُراهن على نجاحي ويؤمن بشجاعتي ... يا من اودعتني لله ها أنا أكمل المسير الذي بدأناه معا.

" أبــي" _ حفظه الله _

إلَى اليد الخفية التي أزالت الأشواك عن طريقي إلى من لا يضاهيها أحد في الكون ملاكي الحارس قرة عيني نعمتي وجنة قلبي.

"أمــي "_ حفظها الله _

إلى شخصي المفضل شكرا لك دائما لأنك كنت قدوتي وعلى ما بذلته من أجلي منذ صغري .

"جدي"_ رحمه الله _

إلَى نفسي الطموحة التي تدفعني للأمام ولا تقبل الاستسلام

إلَى كل من علمني حرفا طيلة مسيرتي التعليمية لكم مني كل الاحترام والتقدير.

الحمد لله الذي ما تم جهد ولا ختم سعي إلا بفضله ، وما تخطيت هذه العقبات والصعوبات إلا بتوفيقه، تخرجت ليس بجهدي وإنما بتوفيق من ربي، فاللهم إجعل هذا العلم شفيعا لي يوم تسألني عن شبابي فيما أفنيته، وزدني علما ونفعا به.

و أخيرا ،أود أن أؤكد أن هذه ليست النهاية، بل هي بداية جديدة لرحلة مليئة بالتحديات والفرص ،سنواجه صعوبات جديدة ، لكننا سنستمر في السعي لتحقيق طموحاتنا.

﴿ وَءَاخِرُ دَعۡوَىٰهُمۡ أَنِ ٱلۡحَمۡدُ لِلَّهِ رَبِّ ٱلۡعَٰلَمِينَ ﴾

أشواق