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**Sujet**

# The Daugavet property for Lipschitz case

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# Table of contents

# Abstract

We will study the generalization of the Daugavet property of the linear case to the nonlinear case; more precisely the Lipschitz case. Among other, we are interested in a some conditions sufficient for a Lipschitz operator to satisfy the Daugavet's property.

**Key-words**: Daugavet property, Lipschitz operator, the compact linear operator, the weakly-compact linear operator, the operators of finit-rank, $\ell_1$-singular operator, strong Radon-Nikodým operator, copies spaces of $\ell_1$, the slice of the unit ball $B_X$, the Lipschitz slice of $S_X$,slope$(T)$ set and SCD set.

# Résumé

Nous étudierons la généralisation de la propriété de Daugavet du cas linéaire au cas non linéaire; plus précisement le cas lipschitzien. Entre autre, on s'interesse à quelques conditions suffisantes pour qu'un opérateur de Lipschitz satisfait la proriété de Daugavet.

**Mots-clés**: Propriété de Daugavet, opérateur Lipschitz, l'opérateur linéaire compact, l'opérateur linéaire faiblement compact, les opérateurs de range finie,$\ell_1$-singulier opérateur, l'opérateur de strong Radon-Nikodým , copie des espaces de $\ell_1$, la tranche de la boule d'unité $B_X$, la tranche Lipschitz de $S_X$, l'ensemble de Slope $(T)$ et l'ensemble de SCD.

**Notations**

| | |
|---|---|
| $\mathbb{K}$ | The field of real or complex numbers |
| $\mathbb{R}^n$ | The vector space of $n$ dimension |
| $X, Y$ | Banach spaces |
| $M$ | Metric space |
| $X^*$ | The topological dual of $X$ |
| $B_X$ | The unit ball of $X$ |
| $S_X$ | The unit sphere of $X$ |
| $L(X, Y)$ | The set of all linear operators |
| $B(X, Y)$ or $\mathcal{L}(X, Y)$ | The set of all bonded linear operators |
| $X/Y$ | The quotient of a vector space $X$ by a subspace $Y$ |
| $Y^\perp$ | The orthogonal set |
| $\mathrm{Lip}_0\,(X, Y)$ | The set of all Lipschitz operators between $X$ and $Y$ that vanish in 0 |
| $X^\# = \mathrm{Lip}_0\,(X, \mathbb{R})$ | The Lipschitz dual of the pointed metric space $X$ |
| $\mathcal{M}(X)$ | The linear space of all molecules on the metric space $X$ |
| $m_{xx'}$ | The molecule defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$ for $x, x' \in X$ |
| $\chi_A$ | The characteristic function of the set $A$ |
| $\text{Æ}(X)$ | The Arens-Eells space of $X$ |
| $\mathcal{F}(M)$ | The Lipschitz free space of $X$ |
| $T_L$ | The linearization of the Lipschitz operator $T$ |
| $T^*$ | The adjoint linear operator of $T$ |
| $(\Omega, \Sigma, \mu)$ | A measure space |
| $ext(C), exC$ | Stands for the set of extreme points of a set $C$ |
| $\omega$ | The weak topology |
| $\omega^*$ or $\sigma(X^*, X)$ | The weak $^*$ topology |
| $\mathrm{lin}(e_1, e_2, ..., e_n)$ | The linear combination of $(e_i)_{i=1}^n$ |
| $G_\delta$ | Expressible as a countable intersection of open subsets |
| $S(x^*, \varepsilon)$ | The slice of the unit ball $B_X$ |
| $S(S_X, f, \varepsilon)$ | The Lipschitz slice of $S_X$ |

# Introduction

We say that a Banach space $X$ has the Daugavet property (DP, in short), if every rank-1 operator $T : X \to X$ satisfies the Daugavet equation

$$||I + T|| = 1 + ||T|| \tag{1.1}$$

where $I$ is the identity map on $X$. The study of the Daugavet property was inaugurated by I.-K. Daugavet [5] in 1961, he proved that every compact operator on $C([0, 1])$ satisfies (1.1). Consequently, $C([0, 1])$ has the DP, and, in 1966 G.Ya. Lozanoskii proved the same on $L_1[0, 1]$ satisfies (1.1) for every compact operator. Investigation of the Daugavet property continued in the seventies and eighties. In particular, J. Holub proved in [7] and [8] (see also [15]) that for any Hausdorff compact topological space $K$, $C(K)$ has the DP if, and only if, $K$ has no isolated points, and, similarly, $L_1(\mu)$ has the DP if and only if $\mu$ is non-atomic. In the early nineties, the work of Y. Abramovich, C. Aliprantis and O. Burkinshaw (see [1] and [2]) infused new ideas into the field. The state-of-the-art information on the Daugavet property can be found, for instance, in [11] and [17]. The Daugavet property holds for the disk algebra $A(D)$ and the algebra of bounded analytic functions $H^\infty$ (Wojtaszczyk, 1992), the Daugavet equation holds for operators not fixing a copy of $C([0, 1])$ defined on certain "large" subspaces of $C(K)$, where $K$ is a compact space without isolated points, and for operators not fixing a copy of $L_1[0, 1]$ defined on certain "large" subspaces of $L_1[0, 1]$ ([10], [20]) (Kadets–Popov, 1997). A $C^*-$algebra has the Daugavet property if and only if it is nonatomic, and then (1.1) is automatically valid for the completely bounded norm. Consequently, the predual of a nonatomic von Neumann algebra has the Daugavet property (Oikhberg, 2002). In 2007, Ivankhno, Kadets, Werner, they prove that space of Lipschitz functions (Lip$(K)$) has Daugavet property when $K \subseteq \mathbb{R}^n$ is compact and convex and in 2017, Abraham Rueda Zoca (joint with L. García-Lirola and A. Procházka) gave in [6] the metric charactersitation of DP in Lipschitz-free spaces.$(\mathcal{F}(M))$.

The Daugavet equation has proved useful in approximation theory, where it was used to find the best constants in certain inequalities by S. B. Stečkin, and in the geometry of Banach spaces.

In the first chapter, we recall of important mathematical concepts of concerning the Schauder basis and unconditional bases, Atomic and NonAtomic measures, the Radon-Nikodým property, Asplund property, copies spaces of $\ell_1$. We show certain classes of operators like (the compact linear operator and the weakly-compact linear operator, the operator of finit-rank, $\ell_1$-singular operator and strong Radon-Nikodým operator), and we remind some basic definitions and properties concerning the metric spaces, Lipschitz maps, non linear Hahn-Banach theorem, Lipschitz spaces, the Lipshitz dual of metric spaces, the predual of $\mathrm{Lip}_0(X)$, and Arens Eels space.

In the second chapter, we present the definition of Daugavet equation, so we see the definition of the Daugavet property, then we pass to the important point at which thesis is based, it is the Geomtric characterisation of the Daugavet property in terms of slices of the unit ball. It helped to find a lot of isomorphic properties of the Daugavet spaces, and we show some classical examples for which Banach spaces this proposition is true $(L_1[0,1], L_1(\mu)$ if and only if $\mu$ is non-atomic and, similarly $C(K)$ if $K$ has no isolated points.). We see the largest reasonable classes of linears operators that satisfy (1.1), in particular, weakly compact operators (as a matter of fact even for strong Radon-Nikodým operators) and operators not fixing a copy of $\ell_1$, after that we show another characterizations of the Daugavet property. In particular, a space with the Daugavet property contains a copy of $\ell_1$ [11], it does not have an unconditional basis [9] and it does not even embed into a space with an unconditional basis [11]. We also, present some new hereditary properties (Theorem 2.6.3). In particular, a pair $(X, Y)$ has the Daugavet property, provided $Y$ is a Daugavet space and $Y/X$ has the Radon-Nikodým property, we note that the Daugavet property passes from $X^*$ to $X$, but not necessarily from $X$ to $X^*$; indeed $C([0,1])$ has the

Daugavet property, but its dual fails it, even more, every slice of the unit ball has diameter 2. In particular, $X$ is not reflexive [16].

In the last chapter, we study the Daugavet property for the Lipschitz case, we see a notion of slice generated by a Lipschitz functional which will play a fundamental role in our discussion. As a consequence of this result we can show that this new concept shares some fundamental properties of linear slices (see the fundamental Lemma 3.1.1) which allow to show that well-known geometrical characterizations of the Daugavet in terms of slices are also true for Lipschitz slices and it present the same wild behaviour as linear ones (Corollary 3.1.1). In particular, if $X$ has the Daugavet property, then , every Lipschitz slice has diameter 2, and for $T \in \mathrm{Lip}(X)$. is such that is $\{\frac{Tx - Ty}{\|x - y\|} : x \neq y\}$ is relatively weakly compact, then (1.1) is valid. Finally we see some sufficient conditions for a Lipschitz operator to satisfy the Daugavet equation. Befor that we see a new concepts like slope(T) set and SCD set (SCD is an abbreviation for slicely countably determined).

# Chapter 1

# Prelimnaries

## 1.1 The Schauder Basis

In mathematics, a Schauder basis or countable basis is similar to the usual (Hamel) basis of a vector space, the difference is that Hamel bases use linear combinations that are finite sums, while for Schauder bases they may be infinite sums. This makes Schauder bases more suitable for the analysis of infinite-dimensional topological vector spaces including Banach spaces. Schauder bases were described by Juliusz Schauder in 1927, although such bases were discussed earlier. For example, the Haar basis was given in 1909, and Faber (1910) discussed a basis for continuous functions on an interval, sometimes called a Faber–Schauder system.

**Definition 1.1.1 (Schauder Bases)** *Let $E$ be a Banach space and let $(e_n)$ be a sequence of vectors in $E$. Then $(e_n)$ is a Schauder basis (or simply a basis) if every $x \in E$ admits an expansion of the form*

$$x = \sum_{n=1}^{\infty} x_n e_n$$

*for some unique sequence of scalars $(x_n)$.*

**Example 1.1.1**

1) *Let* $E = l_p$ *for* $1 \leq p < \infty$*, or* $E = c_0$*. Note that some authors write* $l_p$*,* $\ell^p$*, or* $\ell_p$ *instead. For* $n \geq 1$*, let* $e_n \in E$ *be the sequence which is* 0 *except with a* 1 *in the nth position. Then* $(e_n)$ *is a Schauder basis for* $E$*, called the "standard unit-vector basis" of* $E$*.*

2) *Let* $H$ *be a separable Hilbert space with an orthonormal basis* $(e_n)$*. Then* $(e_n)$ *is a basis (in fact, an unconditional basis, see later) for* $H$*.*

A basis for a Banach space is not, in many cases, of a huge amount of use, as the convergence properties are rather weak. A more useful notion is that of an uncondtional basis.

A series $(x_n)$ in a Banach space $E$ is said to sum unconditionally if, for each permutation $\sigma$ on $\mathbb{N}$, the sum

$$\sum_{n=1}^{\infty} x_{\sigma(n)}$$

converges, and converges to the same limit (although, see below, this is automatic), independently of $\sigma$. Recall that in a finite-dimensional Banach space (or just in $\mathbb{R}$ or $\mathbb{C}$) this notion is equivalent to $\sum_{n=1}^{\infty} \|x_n\| < \infty$. This is not true in the infinite-dimensional case (indeed, it characterises finite-dimensional Banach spaces).

**Definition 1.1.2 (Unconditional Basis)** *a basis* $(e_n)$ *for a Banach space* $E$ *is an unconditional basis if, for each* $x \in E$*, there exists a unique expansion of the form*

$$x = \sum_{n=1}^{\infty} x_n e_n,$$

*where the sum converges unconditionally.*

**Example 1.1.2**

1) *The standard bases of the sequence spaces* $c_0$ *and* $\ell_p$ *for* $1 \leq p < \infty$*, as well as every orthonormal basis in a Hilbert space, are unconditional.*

2) *The trigonometric system is not an unconditional basis in* $L_p$*, except for* $p = 2$*.*

3) *The Haar system is a unconditional basis in* $L_p$ *for any* $1 < p < \infty$*. Actually, the space* $L_1$ *has no unconditional basis.*

## 1.2 Atomic and NonAtomic Measures

In mathematics, more precisely in measure theory, an atom is a measurable set which has positive measure and contains no set of smaller positive measure. A measure which has no atoms is called non-atomic or atomless.

**Definition 1.2.1** *Suppose $\mu$ is a (nonnegative, countably additive) measure on the $\sigma$-ring Ş (a nonempty collection of sets). A set $E \in$ Ş will be called an atom for $\mu$, if*

(1) $\mu(E) > 0$ *and ,*

(2) *given $F \in$ Ş, either $\mu(E \cap F)$ or $\mu(E - F)$ is $0$.*

**Remark 1.2.1** *If $E$ is an atom for $\mu$. and $\mu(E \cap F) > 0$, then $E \cap F$ is also an atom for $\mu$.*

**Definition 1.2.2** *We shall say that $\mu$ is purely atomic or simply atomic if every measurable set of positive measure contains an atom.*
*We shall say that $\mu$ is nonatomic if there are no atoms for $\mu$.*

**Remark 1.2.2** *The zero measure is the only measure which is both purely atomic and nonatomic.*

**Example 1.2.1** *Let $\mu$ and $\nu$ be counting measure and Lebesgue measure, resp., on the Borel sets of the unit interval. It is easy to see that $\nu \leqq \mu$, even though $\nu$ is nonatomic and $\mu$ is atomic.*

## 1.3 The Radon-Nikodým property

The Radon-Nikodým property (RNP, for short) is one of the most important isomorphic invariants of Banach spaces.

**Definition 1.3.1** *A closed convex and bounded set $C$ in a Banach space $E$ has the Radon-Nikodým property (RNP) if it satisfies the following property.*

Let $(\Omega, B)$ be a measurable space, and let $\tau$ be an $E-$valued measure and $\mu$ a scalar probability measure on $(\Omega, B)$. Assume that $\tau(A)/\mu(A) \in C$ for all $A \in B$ with $\mu(A) \neq 0$. Then there is an $f \in L_1(\mu, E)$ such that

$$\tau(A) = \int_A f(\omega) d\mu(\omega), \ A \in B.$$

The space $E$ is said to have the RNP if its unit ball has the RNP.

**Example 1.3.1** *Every reflexive Banach space has the Radon–Nikodým property, in particular, $L_p$ spaces, $(1 < p < +\infty)$ and $\ell_1$ have RNP.*
*But $L_1([0,1])$ and $C(K)$ spaces whenever $K$ is an infinite compact space fail this property.*

**Theorem 1.3.1** *If $X$ is a Banach space, T.F.A.E.*

*(1) $X$ has the Radon-Nikodým property.*

*(2) For all $f \in S_{X^*}$ and all $\epsilon > 0$, there exists $t : X \longrightarrow S_{X^*} \cap B(f, \epsilon)$ such that for all sequence $(x_n)$ in $X$, if $(f(x_n) - \epsilon\|x_n\|)$ is bounded below and if $< t(x_n), x_{n+1} - x_n > \leq 0$ for all $n$, then $(x_n)$ converges.*

## 1.4 Asplund property

In mathematics — specifically, in functional analysis — an Asplund space or strong differentiability space is a type of well-behaved Banach space. Asplund spaces were introduced in 1968 by the mathematician Edgar Asplund, who was interested in the Fréchet differentiability properties of Lipschitz functions on Banach spaces.

**Definition 1.4.1** *A convex function $f$ defined on a nonempty convex open subset $D$ of $E$ is said to be Frechet differentiable at $x_0 \in D$ provided there exists a continuous linear functional $f'(x_0)$ on $E$ such that for any bounded subset $B$ of $E$ and $\varepsilon > 0$, there exists $\delta > 0$.such that*

$$\sup_{x \in B} |\frac{f(x_0 + tx)}{t} - \langle f'(x_0), x\rangle| < \varepsilon$$

*whenever $0 < |t| < \delta$. We call $f'(x_0)$ the Frechet differential of $f$ at $x_0$.*

**Definition 1.4.2** *A locally convex space $E$ is said to have the Asplund property provided every continuous convex function defined on a nonempty open convex subset $D$ of $E$ is Frechet differentiable at each point of some dense $G_\delta$ subset of $D$.*

## 1.5 Copies spaces of $\ell_1$

**Definition 1.5.1** *Let $X$ and $Y$ be Banach spaces. We say that $X$ has a (complemented) copy of $Y$ if $X$ has a (complemented) subspace which is isomorphic to $Y$. We'll denote these by*

$$X \supset Y$$

*and*

$$X \supset_{(c)} Y.$$

**Definition 1.5.2** *$X$ contains a copy of $\ell_1$ means that there is an isomorphic embedding $T : \ell_1 \longrightarrow X$ so that $\ell_1$ is isomorphic to a subspace of $X$. Recall that $T : X \longrightarrow Y$ is an isomorphism onto its image if $m \cdot \|x\| \leq \|Tx\| \leq M \cdot \|x\|$,*
*for every $x \in X$, where $m, M > 0$ are some positive constants.*

## 1.6 Some classes of operators

### 1.6.1 The compact linear operator

**Definition 1.6.1** *$T \in B(X, Y)$ is compact, if the image of the unit ball of $X$ has compact closure in $Y$ (($\overline{T(B_X)}$ is compact ).*

**Definition 1.6.2 (Equivalent Definition)** *An operator $T \in B(X, Y)$ is compact if and only if $(x_n) \in X$ being a bounded sequence implies that there is a subsequence $(x_{a(n)})$ such that $(Tx_{a(n)})$ converges in $Y$.*

**Remark 1.6.1** *We denote*

$$K(X,Y) = \{T \in B(X,Y), T \text{ compact }\}$$

*the set of compact operators from $X$ into $Y$ Banach spaces.*

**Theorem 1.6.1 (J. Schauder, 1930)** *Suppose that $X$ and $Y$ are two Banach spaces and that $T \in B(X,Y)$. Then $T$ is compact if, and only if, $T^*$ (the adjoint operator of $T$) is compact.*

**Remark 1.6.2** *The proof of the theorem use the so called Arzelà–Ascoli theorem.*

**Proposition 1.6.1** *Let $T : X \longrightarrow Y$, a linear operator between normed linear spaces, if the range space* $\mathrm{Ran}(T)$ *is of finite dimension,* $\dim(\mathrm{Ran}(T)) < \infty$ *($T$ a finite-dimensional operator), then $T$ is compact.*

**Proof.** This is seen by noting that for a bounded set $E \subseteq X$, $T(E)$ is closed and bounded in the finite-dimensional subspace $\mathrm{Ran}(T) \subseteq Y$. Therefore, Heine-Borel Theorem applies, and $T(E)$ is compact in $\mathrm{Ran}(T) \subseteq Y$. ■

**Example 1.6.1**

*- For some fixed $g \in C([0,1],\mathbb{R})$, define by the Proposition linear operator $T$ from $C([0,1],\mathbb{R})$ to $C([0,1],\mathbb{R})$ by*

$$(Tf)(x) = \int_0^x f(t)g(t)dt.$$

*That the operator $T$ is indeed compact follows from the Ascoli theorem.*

*- More generally, if $\Omega$ is any domain in $\mathbb{R}^n$ and the integral kernel $k : \Omega \times \Omega \to \mathbb{R}$ is a Hilbert-Schmidt kernel, then the operator $T$ on $L_2(\Omega,\mathbb{R})$ defined by*

$$(Tf)(x) = \int_\Omega k(x,y)f(y)dy$$

*is a compact operator.*

*- By Riesz's lemma, the identity operator is a compact operator if and only if the space is finite-dimensional.*

### 1.6.2 The weakly-compact linear operator

**Definition 1.6.3** *$T \in B(X,Y)$ is weakly compact, if the image of the unit ball of $X$ has weakly compact closure in $Y$ ($\overline{T(B_X)}$ is compact weakly).*

**Definition 1.6.4 (Equivalent Definition)** *An operator $T \in B(X,Y)$ is weakly compact if and only if $(x_n) \in X$ being a bounded sequence implies that there is a subsequence $(x_{a(n)})$ such that $(Tx_{a(n)})$ converges weakly in $Y$.*

### 1.6.3 The operator of finit-rank

**Definition 1.6.5** *Let $T : X \longrightarrow Y$ be a continuous linear mapping between normed linear spaces If the range space* $\mathrm{Ran}(T)$ *is of finite dimension,* $\dim(\mathrm{Ran}(T)) < \infty$*, we call $T$ a finite-dimensional operator.*

**Proposition 1.6.2** *Let $X$ be a Banach space. $T \in B(X)$ be a finite rank bounded linear operator. Then*

$$T(x) = \sum_{i=1}^{n} f_i(x)x_i$$

*where $x_1, x_2, \dots, x_n$ from $X$ and $f_1, f_2, \dots, f_n$ from $X^*$ are linearly independent.*

### 1.6.4 $\ell_1$-Singular operator

**Definition 1.6.6** *Given a Banach space $Z$, we say that an operator $T : X \to Y$ is $Z$-singular whenever the restriction $T|_M$ is not an isomorphism for any subspace $M \subset X$ isomorphic to $Z$. The notion of $\ell_p$-singular operator are of particular importance, since these have been used recently to study certain properties of strictly singular operators.*

**Definition 1.6.7 ($\ell_1$-Singular Operator)** *We say that an operator $T : X \to Y$ is $\ell_1$-singular (or that $T$ does not fix a copy of $\ell_1$) whenever the restriction $T|_M$ is not an isomorphism for any subspace $M \subset X$ isomorphic to $\ell_1$.*

### 1.6.5 Strong Radon-Nikodým operator

For a probability space $(\Omega, \Sigma, \mu)$ and $X$ is a Banach space, the following Definition are stated.

**Definition 1.6.8** *An operator $T : L_1(\mu) \longrightarrow X$ is a strong Radon-Nikodým operator if the closure of $T(B_X)$ has the Radon-Nikodým property.*

## 1.7 Lipschitz spaces

**Definition 1.7.1** *Let $X$ and $Y$ be metric spaces. A map $f : (X, d_X) \longrightarrow (Y, d_Y)$ is called Lipschitz if there is a positive constant $C$ such that*

$$\forall x, y \in X, \quad d_Y(f(x), f(y)) \leq C d_X(x, y). \tag{*}$$

If $C = 1$, the map is called nonexpansive (and contraction if $C < 1$). For a Lipschitz map $f$, we define its Lipschitz constant by

$$L(f) := \text{Lip}(f) := \sup_{x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} = \inf\{C : C \text{ verifying } (*)\}.$$

When $(X, d_X, e_X), (Y, d_Y, e_Y)$ be pointed metric spaces. We say a map $f : (X, d_X, e_X) \longrightarrow (Y, d_Y, e_Y)$ preserves distinguished point if $f(e_X) = e_Y$.

**Definition 1.7.2** *Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces. A map $f : (X, d_X) \longrightarrow (Y, d_Y)$ is called bi-Lipschitz or quasi-isometry, if $f$ is bijective and both $f$, $f^{-1}$ are Lipschitz.*

In this case $X$ and $Y$ are called Lipschitz isomorphic or Lipschitz homeomorphic (Kalton) and quasi-isometric (Weaver [19]).
A bi-Lipschitz function $f$ is an isometry if

$$\forall x, y \in X, \ d_Y(f(x), f(y)) = d_X(x, y).$$

We shall give some properties concerning the composition and the extention of Lipschitz operators.

**Proposition 1.7.1** (*Weaver* [19]) *Let $X_0$ and $Y_0$ be metric spaces and let $X$ and $Y$ be their completions. Let a Lipschitz map $f_0 : X_0 \longrightarrow Y_0$. Then $f_0$ has a unique Lipschitz extension $f : X \longrightarrow Y$, and furthermore $\mathrm{Lip}(f) = \mathrm{Lip}(f_0)$.*

We now see the classical theorem of Hahn-Banach.

**Theorem 1.7.1** *Let $X$ be a metric space and let $X_0 \subset X$ .*
*For any Lipschitz function $f_0 : X_0 \longrightarrow \mathbb{R}$ there exists $f : X \longrightarrow \mathbb{R}$ be a Lipschitz function such that $f|_{X_0} = f_0, \mathrm{Lip}(f) = \mathrm{Lip}(f_0)$, and $\|f\|_\infty = \|f_0\|_\infty$.*

**Proof.** Dfine the function $f_0 : X \longrightarrow \mathbb{R}$ by the formula

$$f_0(z) = \inf_{x \in E}(f(x) + \mathrm{Lip}(f)d(x, z)), \quad z \in X.$$

To see that this function satisfies the results, fix an arbitrary $x_0 \in E$. Then, for any $x \in E$

$$\begin{aligned} f(x_0) - f(x) &\leq \mathrm{Lip}(f)d(x_0, x), \\ &\leq \mathrm{Lip}(f)(d(x_0, z) + d(z, x)) \ . \end{aligned}$$

This implies (that $f(x) + \mathrm{Lip}(f)d(x, z)$ is bounded below)

$$f(x_0) - \mathrm{Lip}(f)d(x_0, z) \leq f(x) + \mathrm{Lip}(f)d(x, z).$$

So $f_0(z)$ is well-defined. Also, if $z \in E$, the above shows that $f_0(z) = f(z)$. Finally (by definition of the inf), for $z, y \in X$ and $\epsilon > 0$, choose $x_z \in E$ such that

$$\begin{aligned} f_0(z) &\geq f(x_z) + \mathrm{Lip}(f)d(z, x_z) - \epsilon \\ -f_0(z) &\leq -f(x_z) - \mathrm{Lip}(f)d(z, x_z) + \epsilon \end{aligned}$$

Then

$$\begin{aligned} f_0(y) - f_0(z) &\leq f(x_z) + \mathrm{Lip}(f)d(y, x_z) - f(x_z) - \mathrm{Lip}(f)d(z, x_z) + \epsilon \\ &\leq \mathrm{Lip}(f)d(y, z) + \epsilon \ . \end{aligned}$$

Thus, we see that $f_0$ is indeed $\mathrm{Lip}(f)$-Lipschitz. ■

### 1.7.1 Lipschitz spaces

**Definition 1.7.3** *(a) Let $(X, d_X)$ be metric space. Then* $\mathrm{Lip}(X)$ *is the space of all bounded scalar valued Lipschitz functions on $X$ with the norm*

$$\|f\|_L = \max\{\|f\|_\infty, \mathrm{Lip}(f)\}.$$

*(b) Let $(X, d_X)$, $(Y, d_Y)$ be pointed metric spaces. We denote by* $\mathrm{Lip}_0(X, Y)$ *the set of all base-point preserving Lipschitz maps from $X$ to $Y$ with the norm*

$$\mathrm{Lip}(f) := \sup_{x\neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)}.$$

*If $E$ is a Banach space,* $\mathrm{Lip}_0(X, E)$ *is a Banach space under the Lipschitz norm given by*

$$\mathrm{Lip}(f) = \sup\{\frac{\|f(x) - f(y)\|}{d(x, y)}, \quad x \neq y\}.$$

*For $E = \mathbb{K}$, we designate* $\mathrm{Lip}_0(X, \Bbbk) = \mathrm{Lip}_0(X) = X^{\#}$*. The Banach space $X^{\#}$ is called also Lipschitz dual of $X$. It has been used by various mathematicians as a framework to extend results from linear functional analysis to the nonlinear case.*

**Notation 1.7.1** *Designed by $B_{X^{\#}}$ the unit ball of $X^{\#}$. Then $B_{X^{\#}}$ is a compact Hausdorff space in the topology of pointwise convergence on $X$ (see* [19,*Page* 39]*).*

**Example 1.7.1** *Let $X$ be a pointed metric space of finite diameter, i.e., $\sup\limits_{x,y\in X} d(x, y) < 1$. We have $\mathrm{Lip}_0(X) \subset (X)$. Indeed, $\mathrm{Lip}(f) = \sup\limits_{x\neq y}\dfrac{|f(x) - f(y)|}{d(x, y)}$. This implies that by taking $y = 0$, $|f(x)| \leq \mathrm{Lip}(f)d(x, 0)$. Consequently, $f \in \ell_\infty(X)$. And if $X$ is Hausdorff space then $Lip_0(X)$ isometry to $\ell_\infty(X)$.*

### 1.7.2 The predual of $\mathrm{Lip}_0(X)$ and adjoint of Lipschitz mapping

We constract this space.

Let $(X, e, d)$ be a metric space. A molecule on $X$ is a real valued function $m$ on $X$ with finite support (i.e., the set where $m$ has non-zero values) and satisfies

$$\sum_{x\in supp(m)} m(x) = 0.$$

Denote by $M(X)$ the real linear space of molecules on $X$. We can write

$$\begin{aligned} m &= \sum_{x\in supp(m)} m(x)\mathbf{1}_{\{x\}} \\ &= \sum_{i=1}^{n} m(x_i)\mathbf{1}_{\{x_i\}} = 1. \end{aligned}$$

Where $supp(m) = \{x_1, ..., x_n\}$ and $\mathbf{1}_{\{x\}}$ denotes the characteristic function of the set $\{x\}$. For $x,\ y \in X$ we define the basic molecule $m_{xy} = \mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}$. It is easy to see that every molecule $m$ can be written as a (non unique) finite linear combination of basic molecule. We have

$$\begin{aligned} m &= \sum_{j=1}^{n} \lambda_j\ (\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{x_j'\}}) \\ &= \sum_{j=1}^{n} \lambda_j\ m_{x_j x_j'} \end{aligned}$$

The condition $\sum_{i=1}^{n} m(x_i) = 0$ insures that such representations of $m$ exist. Put

$$\|m\|_{M(X)} = \inf\{\sum_{j=1}^{n} |\lambda_j| d(x_j, x_j^{'})\}$$

over all representation of $m = \sum_{j=1}^{n} \lambda_j(\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{x_j'\}})$.
It follows that $\|.\|_{M(X)}$ is a norm on the vector space $M(X)$. Arens Eels space is the completion of the normed space $(M(X), \|.\|_{M(X)})$ and denote by $Æ(X, d_X)$. This space was first introduced by Arens and Eells in 1956. Originally, the basic idea goes back to Kantorovich. The terminology Arens-Eells space $Æ(X, d_X)$ is due to Weaver [19]. A different notation was used by Godefroy and Kalton. It is the Lipschitz-free space denoted by $\mathcal{F}(X, d_X)$ and is defined to be the canonical predual of $\mathrm{Lip}_0(X)$, i.e., the closed linear span of the points evaluations

$$\delta(x)(f) = \langle \delta(x), f\rangle = f(x),\ \ x \in X$$

in $\mathrm{Lip}_0(X)^* \overline{(span\{\delta_X(x)\}_{x\in X}}^{\mathrm{Lip}_0(X)^*} = \mathcal{F}(X, d_X))$.
The application $\delta_X : X \longrightarrow \mathcal{F}(X, d_X)$ is an isometric embedding. We can see $\mathcal{F}(X, d_X))$ as the completion of the set of all measures $\mu$ of finite support under the norm

$$\|\mu\| = \sup\{\int f \; d\mu : Lip(f) \leq 1\}.$$

**Theorem 1.7.2** ([19]) *Let $(X, e, d)$ be a pointed metric space. Then $\textit{Æ}^*(X, d_X) \equiv \mathrm{Lip}_0(X)$.*

**Proof.** The application $i_X : X \longrightarrow \text{Æ}(X, d_X)$ defined by

$$i_X(x) = (\mathbf{1}_{\{x\}} - \mathbf{1}_{\{e\}})$$

is an isometric embedding of $X$ into $\text{Æ}(X, d_X))$. Indeed, we have

$$\|i_X(x) - i_X(y)\|_{\text{Æ}} = \|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\|_{\text{Æ}} = d(x, y).$$

Define

$$S : \text{Æ}^*(X, d_X) \longrightarrow Lip_0(X)$$

by

$$(S\varphi)(x) = \varphi((\mathbf{1}_{\{x\}} - \mathbf{1}_{\{e\}})).$$

Since $\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{x'\}}\|_{\text{Æ}(X,d_X)} \leq d(x, x^{'})$ for all $x, x^{'} \in X$, we have

$$\begin{aligned} |(S\varphi)(x) - (S\varphi)(x^{'})| &= |\varphi((\mathbf{1}_{\{x\}} - \mathbf{1}_{\{e\}})) - \varphi((\mathbf{1}_{\{x'\}} - \mathbf{1}_{\{e\}}))| \\ &= |\varphi((\mathbf{1}_{\{x\}} - \mathbf{1}_{\{x'\}}))| \\ &\leq \|\varphi\| d(x, x'). \end{aligned}$$

Also $(S\varphi)(e) = \varphi(0)$, so indeed $S\varphi \in \mathrm{Lip}_0(X)$. It follows that $S$ is a nonexpansive linear mapping from $\text{Æ}^*(X, d_X)$ to $\mathrm{Lip}_0(X)$.
Define now $R : \mathrm{Lip}_0(X) \longrightarrow \text{Æ}(X, d_X)$ by

$$(Rf)(m) = \sum_x m(x) f(x).$$

For $f \in \mathrm{Lip}_0(X)$ and $m$ a molecule. If $m = \sum_{j=1}^{m} \lambda_j(\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{x_{j'}\}})$, we have

$$\begin{aligned}
|(Rf)(m)| &= |(Rf)\sum_{x} m(x)f(x)| \\
&\leq \sum_{j=1}^{m} |\lambda_j|\; |(\mathbf{f}(x_j) - f(x_j^{'})| \\
&\leq \mathrm{Lip}(f)\sum_{j=1}^{m} |\lambda_j|\; d(x, x^{'})
\end{aligned}$$

Hence $|(Rf)(m)| \leq \mathrm{Lip}(f)\|m\|_{\text{Æ}(X)}$, which uniquely extends to a continuous linear functional on the completion Æ$(X, d_X)$ of $M(X)$, denoted by the same symbol $Rf$. Thus $Rf \in$ Æ$^*(X, d_X)$ and $\|Rf\| \leq \mathrm{Lip}(f)$. Straightforward calculations show that $R$ and $S$ are inverses, so that $\mathrm{Lip}_0(X)$ is isometrically isomorphic to Æ$^*(X, d_X)$. ■

**Corollary 1.7.1** ([19]) *Let $(X, e, d)$ be a pointed metric space.*

*$(a)$ For any molecule $m$ we have $\|m\|_{\text{Æ}(X)} = \sup\limits_{f \in B_{X^\#}} |\langle m, f\rangle|$ and there existes $f \in B_{X^\#}$ with $|\langle m, f\rangle| = \|m\|_{\text{Æ}(X)}$.*

*$(b)$ $\|.\|_{\text{Æ}(X)}$ is a norm on the space of molecules and $\|m_{xy}\|_{\text{Æ}(X)} = d(x, y)$ for all $x$, $y \in X$.*

*$(c)$ $\|.\|_{\text{Æ}(X)}$ is the largest seminorm on the space of molecules which satisfies $\|m_{xy}\|_{\text{Æ}(X)} \leq d(x, y)$ for all $x, y \in X$.*

Let $(X, e, d)$ be a pointed metric space.

**Theorem 1.7.3 (Weaver [19], Theorem2.2.4])** *Let $T : X \longrightarrow E$ be a Lipschitz map which preserves base point $($i.e., $T(e) = 0)$. Then there is a unique bounded linear operator $T_L$ :Æ$(X, d_X) \longrightarrow E$ such that $T = T_L \circ i_X$ and $\|T_L\| = \mathrm{Lip}(T)$.The linear operator $T_L$ is called the linearization of $T$.*

$$\begin{tikzcd}
\text{Æ}(X) \arrow[dr, "T_L"] & \\
X \arrow[u, leftarrow, "i(x)"'] \arrow[r] & E
\end{tikzcd}$$

**Proof.** (a) By Corollary 1.7.1, we have for all $x, y \in X$

$$\|m_{xy}\|_{\text{Æ}(X)} = \|m_{xe} - m_{ye}\|_{\text{Æ}(X)} = \|i_X(x) - i_X(y)\|_{\text{Æ}} = \|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\|_{\text{Æ}} = d(x, y).$$

So $i_X$ is an isometry.

(b) Every molecule $m$ is uniquely expressible in the form

$$\sum_{j=1}^{n} \lambda_j \; m_{x_j e} = \sum_{j=1}^{n} \lambda_j \; (\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{x'_j\}})$$

Where the points $x_j$ are all distinct and none equals $e$. We then define $T_L$ by

$$T_L(m) = \sum_{j=1}^{n} \lambda_j \; T(x_j).$$

Since $T_L$ is essentially an extension of $T$ that is, $T = T_L \circ i_X$ we automatically have $\|T_L\| \geq \mathrm{Lip}(T)$. For the rest it will suffice to show that $\|T_L\| \geq \mathrm{Lip}(T)$ (in particular, this implies that $T$ is bounded, hence it extends to all of $\text{Æ}(X)$). Define now a seminorm $\|.\|_0$ on the space of molecules by setting $\|m\|_0 = \dfrac{\|T_L\|}{Lip(T)}$. Then

$$\|m_{xy}\|_0 = \|\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{y\}}\|_0 = \frac{\|T(x) - T(y)\|}{\mathrm{Lip}(T)} \leq d(x, y)$$

for all $x, y \in X$, so Corollary 1.7.1 implies that $\|.\|_0 \leq \|.\|_{\text{Æ}}$. Thus

$$\|T_L(m)\| \leq \mathrm{Lip}(T).\|m\|_{\text{Æ}},$$

which shows that $\|T_L(m)\| \leq \mathrm{Lip}(T).\|m\|_{\text{Æ}}$, as desired. ■

# Chapter 2

# The Daugavet property for the linear case

We use standard notation such as $B_X$ and $S_X$ for the unit ball and the unit sphere of a Banach space $X$, and we employ the notation

$$S(x^*, \varepsilon) = \{x \in B_X : x^*(x) \geq \|x^*\| - \varepsilon\}$$

for some functional $x^* \in X^*$ of norm 1 and some $\varepsilon > 0$.for the slice of $B_X$ determined by $x \in S_{X^*}$ and $\varepsilon > 0$. $exC$ stands for the set of extreme points of a set $C$. In this study we deal with real Banach spaces although the results extend to the complex case with minor modifications.

## 2.1 Daugavet equation

We start with a formal definition following [11].

**Definition 2.1.1** *Let $X$ be a Banach space, a linear operator $T : X \longrightarrow X$ is said to satisfy the Daugavet equation whenever*

$$\|Id + T\| = 1 + \|T\| \tag{DE}$$

*where $I$ is the identity operator on $X$.*

**Lemma 2.1.1** (*see* [ 2, *page* 217]) *Two vectors $u$ and $v$ in a normed space satisfy $\|u+v\| = \|u\| + \|v\|$ if, and only if, $\|\alpha u + \beta v\| = \alpha\|u\| + \beta\|v\|$ holds for $\alpha, \beta > 0$. In particular, a continuous operator $T : X \longrightarrow X$ on a Banach space satisfies the Daugavet equation if and only if the operator $\alpha T$ likewise satisfies the Daugavet equation for each $\alpha > 0$.*

**Proof.** Let two vectors $u$ and $v$ in a normed vector space satisfy $\|u+v\| = \|u\| + \|v\|$ and let $\alpha, \beta > 0$. By the symmetry of the situation, we can assume that $\alpha \geq \beta \geq 0$. Now note that

$$\begin{aligned} \|\alpha u + \beta v\| &= \|\alpha(u+v) - (\alpha - \beta)v\| \\ &\geq \alpha\|u+v\| - (\alpha - \beta)\|v\| \\ &= \alpha(\|u\| + \|v\|) - (\alpha - \beta)\|v\| \\ &= \alpha\|u\| + \beta\|v\| \end{aligned}$$

and the desired equality $\|\alpha u + \beta v\| = \alpha\|u\| + \beta\|v\|$ follows. ■

## 2.2 Daugavet property

**Definition 2.2.1** *Banach space $X$ has the Daugavet property (DP, in short) if every rank-1 operator $T : X \longrightarrow X$ (for all operators $T : X \longrightarrow X$ of the form $T(x) = x_0^*(x)x_0 \backslash x_0^* \in X^*$, $x_0 \in X$) satisfies the Daugavet equation (DE).*

## 2.3 Daugavet property (Geomtric characterisation)

Geometric characterization of the Daugavet property in terms of slices of the unit ball. It helped to find a lot of isomorphic properties of the Daugavet spaces. Slices of the unit sphere play a crucial role in the geometric approach to the Daugavet and alternative Daugavet equations, thanks to the following two results.

**Definition 2.3.1** *Let $X$ be a Banach space. A slice of $B(X)$ is called the following set*

$$S(x^*, \varepsilon) = \{x \in B_X : x^*(x) \geq 1 - \varepsilon\}$$

*where* $x^* \in X^*$ *and* $\varepsilon > 0$*. We always assume that* $x^* \in S(X^*)$*. If* $X$ *is a dual space and* $x^*$ *is taken from the predual, then* $S(x^*, \varepsilon)$ *is called a weak* slice.*

**Lemma 2.3.1** (*This follows from* [21]) *The following assertions about a Banach space* $X$ *are equivalent.*
(i) $X$ *has the Daugavet property.*
(ii) *For every slice* $S = S(x_0^*, \varepsilon_0)$ *of* $B_X$*, every* $x_0 \in S_X$ *and every* $\varepsilon > 0$ *there exists a point* $x \in S$ *such that* $\|x + x_0\| \geq 2 - \varepsilon$*.*
(iii) *For every slice* $S = S(x_0^*, \varepsilon_0)$ *of* $B_X$*, every* $x_0 \in S_X$ *and every* $\varepsilon > 0$ *there exists a slice* $S'$ *of* $B_X$ *contained in* $S$ *such that* $\|x + x_0\| \geq 2 - \varepsilon$*, for all* $x \in S'$*.*
(iv) *For every relatively weakly open subset* $U$ *of* $B_X$*, every* $x_0 \in S_X$ *and every* $\varepsilon > 0$ *there exists a relatively weakly open subset* $U'$ *of* $B_X$ *contained in* $U$ *such that* $\|x + x_0\| \geq 2 - \varepsilon$*, for all* $x \in U'$*.*

**Proof.** The equivalence of (i) and (ii) is quickly established by looking at the rank-1 operator defined by $T(x) = x_0^*(x)x_0$ when a slice and a point are given; conversely, every such operator determines a slice and a point. For the implication (i) $\implies$ (iii) note that $\|Id + T^*\| = \|Id + T\| = 2$ for the above $T$. Hence there is a functional $x^* \in S_{X^*}$ such that $\|x^* + T^*x^*\| \geq 2 - \varepsilon_0$ and $x^*(x_0) \geq 0$. Put

$$x_1^* = \frac{x^* + T^*x^*}{\|x^* + T^*x^*\|}, \ \varepsilon_1 = 1 - \frac{2 - \varepsilon_0}{\|x^* + T^*x^*\|}$$

Then we have, given $x \in S' := S(x_1^*, \varepsilon_1)$,

$$\langle (Id + T^*)x^*, x \rangle \geq (1 - \varepsilon_1)\|x^* + T^*x^*\| = 2 - \varepsilon_0,$$

therefore

$$x^*(x) + x^*(x_0)x_0^*(x) \geq 2 - \varepsilon_0, \tag{2.3}$$

which implies that $x_0^*(x) \geq 1 - \varepsilon_0$, i.e., $x \in S(x_0^*, \varepsilon_0)$. Moreover, by (2.3) we have $x^*(x) + x^*(x_0) \geq 2 - \varepsilon_0$ and hence $\|x + x_0\| \geq 2 - \varepsilon_0$; note that there is no loss of generality in assuming that $\varepsilon = \varepsilon_0$.

The implication $(i) \implies (iv)$ is more difficult and originally due to R. Shvidkoy [17]. He starts off by rediscovering a lemma that has originated in Bourgain's Paris lecture notes on the Radon-Nikodým property to the effect that a relatively weakly open subset of $B_X$ contains a convex combination of slices, and then one uses (iii). Finally, the implications (iv) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (ii) are clear. ■

By means of the Hahn-Banach theorem the equivalence of (i) and (ii) above can be rephrased as follows. We use the notation

$$\Delta_\varepsilon(x) = \{y \in B_X : \|x - y\| \geq 2 - \varepsilon\}$$

for $x \in S_X$.

**Corollary 2.3.1** *A Banach space $X$ has the Daugavet property if, and only if, $B_X = \overline{co}\Delta_\varepsilon(x)$ for all $x \in S_X$ and $\varepsilon > 0$.*

There is also a weak* version of Lemma 2.3.1 that can be verified by working with the adjoint of the above operator $T$.

**Lemma 2.3.2** *The following assertions about a Banach space $X$ are equivalent.*

(i) *$X$ has the Daugavet property.*

(ii) *For every weak* closed slice $S^*$ of $B_{X^*}$, every $x_0^* \in S_{X^*}$ and every $\varepsilon > 0$ there exists a point $x^* \in S^*$ such that $\|x^* + x_0^*\| \geq 2 - \varepsilon$.*

(iii) *For every weak* closed slice $S^*$ of $B_{X^*}$, every $x_0^* \in S_{X^*}$ and every $\varepsilon > 0$ there exists a slice $S^{*'}$ of $B_{X^*}$ contained in $S^*$ such that $\|x^* + x_0^*\| \geq 2 - \varepsilon$, for all $x^* \in S^{*'}$.*

Lemmas 2.3.1 and 2.3.2 allow us to deduce an immediate necessary condition for the Daugavet property.

Actually, those lemmas characterises Daugavet pairs $(X, Y)$, meaning a Banach space $X$ and a subspace $Y \subset X$ such that

$$\|J + T\| = 1 + \|T\|$$

for every operator from $X$ into $Y$ of rank-1, here $J$ denotes the canonical embedding map. The only modification to be made in the formulation of Lemma 2.3.1 and 2.3.2 is that $S$ and $U$ refer to slices and subsets of $B(Y)$.

## 2.4 Examples (Classical examples)

In this section we see for which Banach spaces is this proposition true?

**Example 2.4.1** *Let us use Lemma 2.3.1 to show that* $L_1[0,1]$ *has the Daugavet property. In fact, let* $f_0 \in S_{L_1}$ *and* $g_0 \in S_{L_\infty}$*. For* $\varepsilon > 0$*, find a measurable subset* $B$ *of* $[0,1]$ *such that* $\|\chi_B f_0\|_{L_1} \leq \frac{\varepsilon}{2}$ *and* $\|\chi_B g_0\|_{L_\infty} \geq 1-\frac{\varepsilon}{2}$*, and pick* $f \in S_{L_1}$ *so that* $\chi_B f = f$ *and* $\langle g_0, f\rangle \geq 1-\varepsilon$*. Since clearly* $\|f + f_0\| \geq 2-\varepsilon$*, condition* $(ii)$ *of Lemma 2.3.1 is fulfilled.*

The same arguments work for $L_1(\mu)$ if $(\Omega, \Sigma, \mu)$ is a nonatomic measure space, and they easily extend to the Bochner space $L_1(\mu, E)$ irrespective of the range space.

**Example 2.4.2** *Similarly one can show that* $C(K)$ *and indeed the space of vector valued functions* $C(K,E)$ *has the Daugavet property if* $K$ *has no isolated points. It is somewhat more convenient to work with Corollary 2.3.1 now. Thus, let* $f \in S_{C(K,E)}$ *and* $\varepsilon > 0$ *be given. Let* $U$ *be the open set* $\{t \in K : \|f(t)\|_E > 1-\frac{\varepsilon}{2}\}$ *and pick, given* $n \in \mathbb{N}$*, open pairwise disjoint nonvoid subsets* $U_1, ..., U_n \subset U$ *and points* $t_j \in U_j$*. Now let* $h \in B_{C(K,E)}$*. Choose continuous* $E$*-valued functions* $g_j$ *such that* $g_j = h$ *off* $U_j$*,* $g_j(t_j) = -f(t_j)$ *and* $\|g_j\|_\infty \leq 1$*; this can be achieved by multiplying* $f$ *and* $h$ *by suitable Urysohn functions. Then* $g_j \in \Delta_\varepsilon(f)$*, and for* $t \in U_i$

$$\begin{aligned} \|h(t) - \frac{1}{n}\sum_{j=1}^{n} g_j(t)\|_\infty &= \|h(t) - \frac{n-1}{n}h(t) - \frac{1}{n}g_i(t)\|_\infty \\ &= \frac{1}{n}\|h(t) - g_i(t)\|_\infty \leq \frac{2}{n} \end{aligned}$$

*whereas for* $t \notin \cup_j Uj$

$$h(t) - \frac{1}{n}\sum_{j=1}^{n} g_j(t) = 0.$$

*This proves that* $h \in \overline{co}\Delta_\varepsilon(f)$ *and, consequently, that* $B_{C(K,E)} = \overline{co}\Delta_\varepsilon(f)$*.*

The same argument implies that the spaces of weakly resp. weak*continuous functions $C_\omega(K,E)$ resp.$C_{\omega^*}(K,E)$ have the Daugavet property if $K$ fails to have isolated points.

## 2.5 Wider classes of operators with D.P

The Daugavet equation holds for much wider classes of operators.

**Theorem 2.5.1** *If $X$ has the Daugavet property and $T : X \longrightarrow X$ is a weakly compact operator, then $T$ satisfies (DE).in fact this is so for all "strong Radon-Nikodým operators" (i.e., $\overline{T(B_X)}$ has the RNP)* ([*see* 11]).

**Proof.** Let $T : X \longrightarrow X$ be a weakly compact operator with $\|T\| = 1$. Then $K = \overline{T(B_X)}$ is weakly compact and therefore coincides with the closed convex hull of its denting points; in fact, $K$ is the closed convex hull of its strongly exposed points. So for every $\varepsilon > 0$ there is a denting point $y_0$ of $K$ with $\|y_0\| > \sup\{\|y\| : y \in K\} - \varepsilon = 1 - \varepsilon$ and for some $0 < \delta < \varepsilon$ there is a slice $S = \{y \in K : y^*(y) \geq 1 - \delta\}$ of $K$ containing $y_0$ and having diameter $< \varepsilon$; here $y^* \in Y^*$ and $\sup\limits_{y \in K} y^*(y) = 1$. Consider $x^* = T^* y^*$. By construction $\|x^*\| = 1$ and

$$\begin{aligned} T(S(x^*, \delta)) &= \{Tx : x \in B_X,\ x^*(x) \geq 1 - \delta\} \\ &= \{Tx : x \in B_X,\ y^*(Tx) \geq 1 - \delta\} \subset S. \end{aligned}$$

So for every $x \in S(x^*, \delta)$ wehave $\|Tx\| \geq 1 - 2\varepsilon$. Now by Lemma 2.2 **[11]** select an element $x_0 \in S(x^*, \delta)$ such that$\| \frac{x_0 + y_0}{\|y_0\|} \| \geq 2 - \delta$ and hence $\| x_0 + y_0 \| \geq 2 - 2\varepsilon$. But $Tx_0 \in S$, so $\| Tx_0 - y_0 \| < \varepsilon$, and we have

$$\|J + T\| \geq \| x_0 + Tx_0 \| \geq \| x_0 + y_0 \| - \varepsilon \geq 2 - 3\varepsilon$$

as desired. ■

**Theorem 2.5.2 (Theorem4 [17])** *If the pair $(X, Y)$ has the Daugavet property, then every operator from $\mathcal{L}(X, Y)$ not fixing a copy of $\ell_1$($\ell_1$-singular operators $T$) satisfies the Daugavet equation.*

**Proof.** Let $T \in \mathcal{L}(X, Y)$, $\|T\| = 1$, be such an operator and $\varepsilon > 0$ be arbitrary.
Our considerations will rely on the following "releasing principle", suppose for some finite set of vectors $\{x_i\}_{i=1}^n \subset B(X)$ and some $\varepsilon > 0$ the inequalities

$$\| \sum_{i=1}^{n} \theta_i x_i \| > n - \varepsilon \qquad (*)$$

and

$$\|\sum_{i\in I_1} a_i x_i + \sum_{i\in I_2} a_i T x_i\| > (\sum_{i\in I_1\cup I_2} a_i)(1-\varepsilon) \tag{$**$}$$

hold for all non-negative reals $a_i$, signs $\theta_i$, and some disjoint sets $I_1, I_2 \subset \{1,2,...,n\}$. Then there is a weak open set $U \subset X$ such that $(*)$ and $(**)$ remain true for all $x_n \in U \cap B(X)$. Let us prove it. By the compactness argument, there is a $\delta > 0$ such that

$$\|\sum_{i\in I_1} a_i x_i + \sum_{i\in I_2} a_i T x_i\| > 1-\varepsilon+\delta \tag{$***$}$$

whenever $\sum\limits_{i\in I_1\cup I_2} a_i = 1$ and $I_1$, $I_2$ as above. Fix a finite $\frac{\delta}{2}$-net $\{(a_{k,1}, a_{k,2}, ..., a_{k,n})\}_{k=1}^{K}$ in the set $\{(a_1, a_2, ..., a_n) : \sum\limits_{i=1}^{n} a_i = 1, a_i \geq 0\}$ equipped with the $\ell_1$-metric. Using the lower weak semicontinuity of a norm and weak continuity of a bounded linear operator we conclude that there is a weak open set $U$ such that both $(*)$ and $(***)$ hold for $a_i = a_{k,i}$, $i = 1,2,...,n, k = 1,2,...,K.$ and all $z_n \in U \cap B(X)$. It is not hard to see that $U$ is desired.

Now we construct a sequence$\{x_i\}_{i=1}^{\infty} \subset B(X)$ which satisfies $(*)$ and $(**)$ for all non-negative reals $a_i$, signs $\theta_i$ and all disjoint finite sets $I_1$, $I_2 \subset \mathbb{N}$.

Assume that we have constructed such a sequence $\{x_i\}_{i=1}^{n}$ of length $n$, We want to prove now that altering only the last term $x_n$ one can find another vector $x_{n+1}$ such that the resulting sequence of length $n+1$ satisfies $(*)$ and $(**)$. Arguing in such a way, we produce the desired infinite sequence if only take $x_1 \in S(X)$ with $\|Tx_1\| > 1-\varepsilon$ on the first step.

Let us put $x'_{n+1} = x_n$ for a moment. Clearly, $(**)$ remains true for the sequence $x_1, x_2, ..., x_n, x'_{n+1}$ and all $I_1$, $I_2$ with additional restriction: if one of them contains $n$, then the other does not contain $n+1$. We get rid of this restriction by alteration of $x_n$ and $x'_{n+1}$. To this end, we use the "releasing principle" for $x'_{n+1}$ and find the corresponding weak open set $U \subset X$. Application of Lemma 3(b)**[17]** several times yields a vector $x_{n+1} \in U \cap B(X)$ such that $(*)$ is valid for the sequence $x_1, x_2, ..., x_n, x_{n+1}$ and $(**)$ holds without the restriction: if $I_1$ contains $n+1$, then $I_2$ does not contain $n$. Then we use the "releasing principle" to release $x_n$ so that both $(*)$ and $(**)$ remain true. Appealing to Lemma 3(b)**[17]** we finally get an $x'_n$ such that $(**)$ holds for the sequence $x_1, x_2, ..., x'_n, x_{n+1}$ without any restrictions on $I_1$ and $I_2$. Inequality $(*)$ is satisfied automatically.

The constructed sequence is $(1-\varepsilon)$-equivalent to the canonical basis of $\ell_1$, for if $\sum_{i=1}^{n}|\lambda_i|$ $=1$, then by $(*)$ we have

$$\begin{aligned}
\|\sum_{i=1}^{n}\lambda_i x_i\| &= \|\sum_{i=1}^{n} sign\lambda_i.x_i+\sum_{i=1}^{n}(\lambda_{i-}sign\lambda_i).x_i\| \\
&> n-\varepsilon-\sum_{i=1}^{n}|\lambda_{i-}sign\lambda_i| = n-\varepsilon-\sum_{i=1}^{n}|1_-|\lambda_i|| \\
&= n-\varepsilon-n+1=1-\varepsilon.
\end{aligned}$$

Since $T$ fixes no copies of $\ell_1$, by Rosenthal's Lemma we may assume that the sequence $(Tx_n)_{n=1}^{\infty}$ is weakly Cauchy. Thus, $(Tx_{2n+1}-Tx_{2n})_{n=1}^{\infty}$ is weakly null. By Mazur's Theorem there are two finite disjoint sets $I_1$, $I_2\subset\mathbb{N}$ such that for some $p\in conv\{x_i : i\in I_1\}$ and $q\in conv\{x_i : i\in I_2\}$ we have $\|Tp-Tq\|<\varepsilon$. From this and $(**)$ we finally obtain

$$\|p+Tp\|>\|p-Tq\|-\varepsilon>2(1-\varepsilon)-\varepsilon=2-3\varepsilon,$$

which implies $\|J+T\|=2$ in view of arbitrariness of $\varepsilon$.
This finishes the proof. ■

## 2.6 Daugavet property (Auther characterisations)

Now we use Lemma 2.3.1 to produce $\ell_1$-copies in spaces with the Daugavet property. First, an extension of that lemma.

**Lemma 2.6.1** *If $(X,Y)$ has the Daugavet property, then for every finite-dimensional subspace $Y_0$ of $Y$, every $\varepsilon>0$ and every slice $S(x_0^*,\varepsilon_0)$ of $B_X$ there is a slice $S(x_1^*,\varepsilon_1)$ of $B_X$ such that*

$$\| y+tx \|\geq(1-\varepsilon_0)(\| y\|+|t|)\quad\forall y\in Y_0, x\in S(x_1^*,\varepsilon_1). \tag{2.4}$$

**Proof.** Let $\delta=\frac{\varepsilon_0}{2}$ and pick a finite $\delta$-net $\{y_1,...,y_n\}$ in $S_{Y_0}$. By a repeated application of Lemma 2.3.1(ii) we obtain a sequence of slices $S(x_0^*,\varepsilon_0)\supset S(x^{*(1)},\varepsilon^{(1)})\supset ...\supset S(x^{*(n)},\varepsilon^{(n)})$ such that one has

$$\| y_k+x \|\geq 2-\delta \tag{2.5}$$

for all $x \in (x^{*(k)}, \varepsilon^{(k)})$. Put $x_1^* = x^{*(n)}$ and $\varepsilon_1 = \varepsilon^{(n)}$, then (2.5) is valid for every $x \in S(x_1^*, \varepsilon_1)$ and $k = 1, ..., n$. This implies that for every $x \in S(x_1^*, \varepsilon_1)$ and every $y \in S_{Y_0}$ the condition

$$\| y + x \| \geq 2 - 2\delta = 2 - \varepsilon_0$$

holds.

Let $0 \leq t_1, t_2 \leq 1$ with $t_1 + t_2 = 1$. If $t_1 \geq t_2$, we have for $x$ and $y$ as above

$$\begin{aligned}
&\| t_1 x + t_2 y \| = \| t_1(x + y) + (t_2 - t_1)y \| \\
&\geq t_1 \| (x + y) - |t_2 - t_1| \| y \| \\
&\geq t_1(2 - \varepsilon_0) + t_2 - t_1 \\
&= t_1 + t_2 - t_1\varepsilon_0 \geq 1 - \varepsilon_0
\end{aligned}$$

and an analogous argument shows this estimate in case $t_1 < t_2$.

This implies (2.4), by the homogeneity of the norm and the symmetry of $S_{Y_0}$. ■

**Lemma 2.6.2** *If $X$ has the Daugavet property, then for every $x \in S_X$, every $\varepsilon > 0$ and every separable subspace $V$ of $X^*$ there is an element $x^* \in S_{X^*}$such that $x^*(x) \geq 1 - \varepsilon$ and*

$$\|x^* + v^*\| = 1 + \|v^*\| \quad \textit{for all } v^* \in V.$$

**Proof.** Take a dense sequence $(v_n^*)$ in $S_V$ and a sequence of weak* compact slices $S(x, \varepsilon) \supset S(x_1, \varepsilon_1) \supset S(x_2, \varepsilon_2) \supset ...$ such that

$$\| x^* + v_k^* \| \geq 2 - \frac{1}{n} \quad \forall x^* \in S(x_n, \varepsilon_n), \; k = 1, 2, ..., n$$

this is possible by a repeated application of Lemma 2.3.1, as in the proof of Lemma 2.6.1. Clearly, any $x^* \in \bigcap_{n \in N} S(x_n, \varepsilon_n)$ works. ■

**Theorem 2.6.1** *If $X$ has the Daugavet property, then $X$ contains a copy of $\ell_1$ (By, $[10, page 861]$).*

**Proof.** Using Lemma 2.6.1 inductively, it is easy to construct a sequence of vectors $e_1, e_2, ...$ and a sequence of slices $S(x_n^*, \varepsilon_n)$, $\varepsilon_n = 4^{-n}$, $n \in \mathbb{N}$, such that $e_{n+1} \in S(x_{n+1}^*, \varepsilon_{n+1})$ and every element of $S(x_{n+1}^*, \varepsilon_{n+1})$ is– "up to $\varepsilon_n$" $-\ell_1$-orthogonal to lin$\{e_1, ..., e_n\}$ which means

$$\| y + x \| \geq (1 - \varepsilon_n)(\| y\| + \|x \|) \quad \forall y \in lin\{e_1, ..., e_n\},\ x \in S(x_{n+1}^*, \varepsilon_{n+1}).$$

The sequence $(e_n)$ is then equivalent to the unit vector basis in $\ell_1$. ■

The proof even shows the stronger result that $X$ contains asymptotically isometric copies of $\ell_1$

**Corollary 2.6.1** ([10]) *If $X$ has the Daugavet property, then $X^*$ contains an isometric copy of $\ell_1$. In particular, $X^*$ is neither strictly convex nor smooth.*

**Proof.** An isometric $\ell_1$-copy can be produced from Lemma 2.6.2 by an obvious inductive procedure. ■

### Unconditional bases

Recall that a Schauder basis $(e_n)$ with coefficient functionals $(e_n^*)$ on a (separable) Banach space $X$ is called an unconditional basis if for
each $x \in X$ the expansion

$$x = \sum_{n=1}^{\infty} e_n^*(x) e_n$$

converges unconditionally. This can be equivalently rephrased by saying that for each $x$ the net of finite-rank projections defined by

$$P_F = \sum_{n=1}^{\infty} e_n^*(x) e_n, \quad F \subset \mathbb{N} \text{ finite}$$

converges to $x$ along the index set **FIN** of finite subsets of $\mathbb{N}$.

It is a well-known fact that neither $C[0,1]$ nor $L_1[0,1]$ have unconditional bases. Indeed, this follows easily from the Daugavet property as shown by V. Kadets [9].

**Proposition 2.6.1** *Any separable Banach space with the Daugavet property fails to have an unconditional basis,* (*this follows from* [21, *page* 85]).

**Proof.** Suppose $X$ has an unconditional basis, and define the operators $P_F$ as above. Then $\sup\{\|P_F(x)\| : F \subset \mathbb{N} \text{ finite}\} < \infty$ for each $x$ so that the uniform boundedness principle implies

$$\alpha := \sup\{\|P_F(x)\| : F \subset \mathbb{N} \text{ finite}\} < \infty$$

If $X$ has the Daugavet property, this leads to a contradiction as follows.
Pick $F_0$ such that $\|P_{F_0}\| > \alpha + \frac{1}{2}$. Then the Daugavet equation implies that

$$\|Id - P_{F_0}\| = 1 + \|P_{F_0}\| > \alpha + \frac{1}{2} \tag{2.6}$$

note that it is applicable by Theorem 2.5.1 (in fact by the special case we have proved above). But

$$\begin{aligned} \|(Id - P_{F_0})(x)\| &= \|\sum_{n \notin F_0} e_n^*(x) e_n\| \\ &\leq \sup\{\|P_G(x)\| : G \subset \mathbb{N} \setminus F_0 \text{ finite}\} < \alpha\|x\| \end{aligned}$$

so that $\|Id - P_{F_0}\| \leq \alpha$, contradicting (2.6). ■

The same argument applies to show that a Banach space with the Daugavet property fails to have an unconditional finite-dimensional Schauder decomposition or indeed an unconditional Schauder decomposition into reflexive spaces or even into spaces not containing $\ell_1$.

The arguments presented in this study so far seem to provide the most easily accessible proof to show that $C[0,1]$ and $L_1[0,1]$ don't have unconditional bases. But for these spaces more is true, they do not even embed into spaces with unconditional bases. In fact, this also results from the Daugavet property.

**Theorem 2.6.2** ([**11**]) *Suppose $X$ has the Daugavet property and $Y \subset X$ is a subspace with a separable annihilator $Y^{\perp}$. Then $Y$ has the Daugavet property.*

**Proof.** Fix $y \in S_Y$ and $\varepsilon > 0$ and consider the slice

$$S = \{y^* \in Y^* = X^*/Y^\perp :\| y^*\| \leq 1, y^*(y) \geq 1 - \varepsilon\}.$$

Also, fix an element $[x_1^*] \in S_{X^*/Y^\perp}$. To prove the theorem it suffices, by Lemma 2.2[**11**](iii), to find some $[x_2^*] \in S$ such that $\|[x_1^* + x_2^*]\| = 2$. To achieve this, apply Lemma 2.6.2 with $x = y$ and $V = \text{lin}(\{x_1^*\} \cup Y^\perp)$. We get a functional $x_2^* \in S_{X^*}$ such that $x_2^*(y) \geq 1 - \varepsilon$ and

$$\| x_2^* + v^* \| \geq 1 + \|v^* \| \quad \forall v^* \in V$$

Then $[x_2^*] \in S$ and

$$\begin{aligned} \|[x_1^* + x_2^*]\| &= \inf\{\|x_1^* + x_2^* + z^*\| : z^* \in Y^\perp\} \\ &= \inf\{1 + \|x_1^* + z^*\| : z^* \in Y^\perp\} \\ (\text{since } x_1^* + z^* &\in V) \\ &= 1 + \|[x_1^*]\| = 2 \end{aligned}$$

This completes the proof of the theorem. ■

**Theorem 2.6.3** ([**17**, *page* 205]) *Let $X$ have the Daugavet property and $Y$ be a subspace of $X$.*

(a) *If $X/Y$ has the Radon-Nikodým property, then the pair $(Y, X)$ has the Daugavet property.*

(b) *If $Y$ is reflexive, then $X/Y$ has the Daugavet property.*

In the particular case when $X = L_1[0, 1]$ part (b) of Theorem 2.6.3 was proved in [11].

**Proof.** Part (a). According to Lemma 2.3.1 (iii) it is sufficient to prove that given any $\delta > 0, S(y^*, \varepsilon)$ and $x \in B_X$ there is a $y \in S(y^*, \varepsilon)$ such that $\|x + y\| > 2 - \delta$.

Denote by $j$ the quotient map $:X \longrightarrow X/Y$. Saving the notation for the functional $y^*$, we extend it to all of $X$ by the Hahn-Banach Theorem. The set $A = j(S(y^*, \varepsilon))$ is convex and contains the origin. Since $X/Y$ has the Radon-Nikodým property, the Phelps Theorem (see for example [17, ref.6]) yields a convex combination $\sum_{i=1}^{n} \lambda_i a_i$ of strongly exposed points $\{a_i\}_{i=1}^n$ of the set $\overline{A}$ for which

$$\sum_{i=1}^{n} \lambda_i a_i < \frac{\delta}{2}. \tag{$*$}$$

Let $\{a_i^*\}_{i=1}^n \subset (X/Y)^*$ be functionals exposing $\{a_i\}_{i=1}^n$ respectively and let positive numbers $\{\varepsilon_i\}_{i=1}^n$ be such that

$$\operatorname{diam}\{S(a_i^*, \varepsilon_i) \cap \overline{A}\} < \frac{\delta}{4},\ i = 1, 2, ..., n. \qquad (**)$$

Since $S(a_i^*, \varepsilon_i) \cap A \neq \varnothing$, we have $S(j^* a_i^*, \varepsilon_i) \cap S(y^*, \varepsilon) \neq \varnothing$. Applying Lemma 3(b)[17] we find $x_i \in S(j^* a_i^*, \varepsilon_i) \cap S(y^*, \varepsilon)$ such that

$$\| \sum_{i=1}^n \lambda_i x_i + x \| > 2 - \frac{\delta}{4}$$

Now taking into account $(*)$ and $(**)$ we obtain the following estimate:

$$\| j(\sum_{i=1}^n \lambda_i x_i) \| < \| \sum_{i=1}^n \lambda_i a_i \| + \frac{\delta}{4} < \frac{\delta}{2}.$$

It means that there is a $y \in B_Y$ for which

$$\| \sum_{i=1}^n \lambda_i x_i - y \| < \delta.$$

Then by $(**)$ we finally get

$$\| x - y \| > 2 - \frac{3}{2}\delta.$$

Clearly, $y \in S(y^*, \varepsilon + \delta)$.

Because of arbitrariness of $\varepsilon$ and $\delta$, part (a) is proved.

The proof of part (b) is analogous (we have only to use the weak* topology and apply Lemma 3(c))**[17]**. ■

**Corollary 2.6.2** *Suppose $X$ is a Daugavet space and $Y$ is a complemented subspace in $X$ such that $X/Y$ contains no copies of $\ell_1$, then the norm of every projection from $X$ onto $Y$ is at least* 2.

**Proof.** Let $P : X \longrightarrow X$ be any projection onto $Y$. Then $-Id + P$ fixes no copies of $\ell_1$ and hence, by Theorem 2.5.2, satisfies the Daugavet equation. So, we have $\|P\| = \|Id + (-Id + P)\| = 1 + \|P - Id\| \geq 2$. ■

**Corollary 2.6.3** *If $X$ has the Daugavet property, then every slice of the unit ball has diameter* 2. *In particular, $X$ is not reflexive* [*see* 21].

**Proof.** This follows from (ii) of Lemma 2.3.1 if we apply it to an $x_0$ with $-x_0 \in S$, respectively from (ii) of Lemma 2.3.2; note that closed convex bounded sets in spaces with the Radon-Nikodým property admit slices of arbitrarily small diameter (cf. Ref.3, Th. 5.8 from [21]). ■

**Proposition 2.6.2** *If $X^*$ has the Daugavet property, so does $X$. The converse is not true, $C[0,1]$ has it but $C[0,1]^*$ not.*

## 2.7 Recent results

In this section we will see new examples of Banach spaces in [6] enjoying the Daugavet property.

**Definition 2.7.1** *Let $M$ be a metric space. $M$ is said to be length if, for every pair of points $x, y \in M$, it follows that $d(x,y)$ is the infimum of the length rectifiable curves joining them.*

It is well known that a complete metric space $M$ is length if, and only if, for every pair of points and every $x, y \in M$ and every $\varepsilon > 0$ then there exists $z \in M$ such that

$$d(x,z) \leq \frac{1+\varepsilon}{2} d(x,y) \text{ and } d(y,z) \leq \frac{1+\varepsilon}{2} d(x,y).$$

**Definition 2.7.2** *A metric space $M$ is said to be local if, for every $\varepsilon > 0$. and every Lipschitz function $f : M \to \mathbb{R}$ there exist $u \neq v \in M$ such that $d(u,v) < \varepsilon$ and $\dfrac{f(u)-f(v)}{d(u,v)} > \|f\|_L - \varepsilon$.*

**Theorem 2.7.1 (L. García-Lirola, A. Procházka and A.R.Z. (preprint [6]))** *Let $M$ be a complete metric space. The following assertions are equivalent.*

1) *$M$ is length.*

2) Lip$(M)$ *has the Daugavet property.*

3) *$\mathcal{F}(M)$ has the Daugavet property.*

4) *$M$ is local.*

# Chapter 3

# The Daugavet property for the Lipschitz case

The results obtained in this chapter are in[14]. We see a new notion of slice generated by a Lipschitz functional which will play a fundamental role in our discussion, note that non-linear slices were also considered in [13].

Let us presenting some common notation. For a set $A$ of a Banach space $X$, $\overline{\text{conv}}(A)$ and $\overline{\text{conv}}(A)$ stand for the convex hull and the closed convex hull of A, respectively, and $\overline{\text{conv}}(TA)$ and $\overline{\text{conv}}(TA)$ are the absolutely convex hull and the absolutely closed convex hull of A, respectively. By Re(.) we denote the real part function, understanding that it is just the identity if we are dealing only with real numbers.

## 3.1 Lipschitz slices

**Definition 3.1.1** *Let $X$ be a Banach space. A Lip-slice of $S_X$ is a non-empty set of the form*

$$\{\frac{x_1 - x_2}{\|x_1 - x_2\|} : x_1 \neq x_2, \frac{f(x_1) - f(x_2)}{\|x_1 - x_2\|} > \alpha\}$$

*where $f \in \text{Lip}(X, \mathbb{R})$ is non-zero and $\alpha \in \mathbb{R}$. The following notation will be useful: for $f \in \text{Lip}(X, \mathbb{R}) \backslash \{0\}$ and $\varepsilon > 0$, we write*

$$S(S_X, f, \varepsilon) := \{\frac{x_1 - x_2}{\|x_1 - x_2\|} : x_1 \neq x_2, \frac{f(x_1) - f(x_2)}{\|x_1 - x_2\|} > \text{Lip}(f) - \varepsilon\}$$

*and observe that this is never empty and so it is a Lip-slice of $S_X$, conversely, every Lip-slice of $S_X$ can be written in this form.*
*Remark that for a real-linear functional $f = \operatorname{Re} x^*$ with $x^* \in X^*$, the above definition gives a usual slice of $S_X$*

$$S(S_X, Re\ x^*, \varepsilon) := \{x \in S_X : \operatorname{Re} x^* > \|x^*\| - \varepsilon\}.$$

The next result shows that slices generated by Lipschitz functionals behave similarly to those generated by linear functionals.

**Lemma 3.1.1 (Fundamental lemma)** *Let $X$ be a Banach space, $f \in \operatorname{Lip}(X, \mathbb{R})$, $\varepsilon > 0$, and $A \subset S_X$. If $\overline{\operatorname{conv}}(A) \cap S(S_X, f, \varepsilon) \neq \emptyset$ then $A \cap S(S_X, f, \varepsilon) \neq \emptyset$.*
*We need a preliminary result which shows that every rescaling of a functional has the same Lipschitz slices.*

**Lemma 3.1.2** *Let $f \in \operatorname{Lip}(X, \mathbb{R})$ and $\varepsilon > 0$. Then, for every $r > 0$ the functional defined by $f_r(x) = \frac{1}{r} f(rx)$ for $x \in X$ satisfies $\|f_r\| = \|f\|$ and*

$$S(S_X, f_r, \varepsilon) = S(S_X, f, \varepsilon).$$

**Proof.** Fix $r > 0$ and $x, y \in X$ with $x \neq y$, and observe that

$$\frac{f_r(x) - f_r(y)}{\|x - y\|} = \frac{f(rx) - f(ry)}{\|rx - ry\|}.$$

Using this it is immediate that $\|f_r\| = \|f\|$. Besides, if $\frac{x - y}{\|x - y\|} \in S(S_X, f_r, \varepsilon)$ then, by the above equality, we have that

$$\frac{x - y}{\|x - y\|} = \frac{rx - ry}{\|rx - ry\|} \in S(S_X, f, \varepsilon)$$

which gives the inclusion $S(S_X, f_r, \varepsilon) \subset S(S_X, f, \varepsilon)$. The converse inclusion is proved analogously, or one just observes that $(f_r)_{1/r} = f$. ■

**Proof.** (Proof of Lemma 3.1.1) Let $y_1, y_2$ be distinct elements in $X$ such that $\frac{y_1 - y_2}{\|y_1 - y_2\|} \in \overline{\text{conv}}(A) \cap S(S_X, f, \varepsilon)$. By rescaling the functional, we can suppose that $\|y_1 - y_2\| = 1$. Indeed, let

$r = \|y_1 - y_2\|$ and observe that

$$f_r(\frac{y_1}{\|y_1 - y_2\|}) - f_r(\frac{y_2}{\|y_1 - y_2\|}) = \frac{f(y_1) - f(y_2)}{\|y_1 - y_2\|}.$$

Taking into account Lemma 3.1.2, it is easy to observe that the functional $f_r$ and the points $\frac{y_1}{\|y_1 - y_2\|}, \frac{y_2}{\|y_1 - y_2\|}$ satisfy the desired conditions.
Now, we have that

$$y_1 - y_2 \in \overline{\text{conv}}(A) \text{ and } f(y_1) - f(y_1) > \text{Lip}(f) - \varepsilon.$$

So we can find $x_1, ..., x_n \in A$ and $\lambda_1, ..., \lambda_n \in [0, 1]$ with $\sum_{k=1}^{n} \lambda_k = 1$ satisfying

$$f(y_1) - f(y_2) - \|y_1 - y_2 - \sum_{k=1}^{n} \lambda_k x_k\| \text{Lip}(f) > Lip(f) - \varepsilon$$

and, therefore

$$f(y_1) - f(y_2) - |f(y_2) - f(y_1 - \sum_{k=1}^{n} \lambda_k x_k)| > \text{Lip}(f) - \varepsilon.$$

We can write

$$\begin{aligned} & \frac{f(y_1) - f(y_1 - \lambda_1 x_1)}{\lambda_1}\lambda_1 + \frac{f(y_1 - \lambda_1 x_1) - f(y_1 - (\lambda_1 x_1 + \lambda_2 x_2))}{\lambda_2}\lambda_2 \\ & +... + \frac{f(y_1 - \sum_{k=1}^{n-1} \lambda_k x_k) - f(y_1 - \sum_{k=1}^{n} \lambda_k x_k)}{\lambda_n}\lambda_n \\ = \; & f(y_1) - f(y_2) + [f(y_2) - f(y_1 - \sum_{k=1}^{n} \lambda_k x_k)] \\ > \; & \text{Lip}(f) - \varepsilon. \end{aligned}$$

Now, an evident convexity argument gives the existence of $\ell \in \{1, ..., n\}$ such that

$$\frac{f(y_1 - \sum_{k=1}^{\ell-1} \lambda_k x_k) - f(y_1 - \sum_{k=1}^{\ell} \lambda_k x_k)}{\lambda_\ell} > Lip(f) - \varepsilon$$

understanding that in case $\ell = 1$ the element $\sum_{k=1}^{\ell-1} \lambda_k x_k$ is zero. Therefore, we get that

$$x_\ell = \frac{(y_1 - \sum_{k=1}^{\ell-1} \lambda_k x_k) - (y_1 - \sum_{k=1}^{\ell} \lambda_k x_k)}{\lambda_\ell} \in S(S_X, f_r, \varepsilon)$$

(recall that $\|x_\ell\| = 1$), which finishes the proof. ■

As a consequence of this result we can show that Lipschitz slices in a space with the Daugavet property present the same wild behaviour as linear ones.

**Corollary 3.1.1** *Let $X$ be a space with the Daugavet property, let $\varepsilon > 0$ and let $S$ be a Lip-slice. Then, for every $x \in S_X$ there is $y \in S$ such that $\|x + y\| > 2 - \varepsilon$.*

**Proof.** Fix $x \in S_X$. Since $X$ has the Daugavet property we have by [11, Lemma 2.2] that

$$\overline{\text{conv}}(\{y \in S_X : \|x + y\| > 2 - \varepsilon\}) = B_X,$$

so Lemma 3.1.1 gives that

$$\{y \in S_X : \|x + y\| > 2 - \varepsilon\} \cap S \neq \emptyset$$

as claimed. ■

**Corollary 3.1.2** *If $X$ has the Daugavet property, then every Lipschitz slice has diameter 2.*

## 3.2 The Daugavet equation for Lipschitz operators

We devote this section to obtain some sufficient conditions for a Lipschitz operator to satisfy the Daugavet equation.

From [3] that a bounded subset $A$ of a Banach space $X$ is an SCD set (SCD is an abbreviation for slicely countably determined) if there is a determining sequence $\{S_n : n \in \mathbb{N}\}$ of slices of $A$, i.e., a sequence $\{S_n : n \in \mathbb{N}\}$ such that $A \subset \overline{\text{conv}}(B)$ whenever $B \subset A$

intersects all the $S_n$'s. This property, which clearly implies separability, is possessed by many classes of separable bounded convex subsets, for example by dentable sets (in particular by Radon-Nikodým sets), by sets with the Asplund property, by strongly regular sets, by CPCP sets, by sets which do not contain $\ell_1$ sequences [3] and by the unit ball of any space with a 1-unconditional basis [11]. Remark that in [3] the property SCD was defined only for convex sets, so for future applications of results from [3] we need the following simple lemma (from [14])

**Lemma 3.2.1** *If $A \subset X$ and $\overline{\text{conv}}A$ is an SCD set, then $A$ is also an SCD set.*

**Proof.** Let $\{S_n :\in \mathbb{N}\}$ be a determining sequence of slices of $\overline{\text{conv}}A$. Then, the sets $S_n^{'} = S_n \cap A$ are not empty and are slices of $A$. It remains to show that $\{S_n^{'} :\in \mathbb{N}\}$ is a determining sequence for $A$. This is evident: if $B \subset A$ intersects all the $S_n^{'}$, then $B$ intersects all the $S_n$, so $A \subset \overline{\text{conv}}A \subset \overline{\text{conv}}(B)$ and we obtain the result . ■

We need some more notation in the spirit of [3]. Given a Banach space $X$, we denote by $K(X^*)$ the intersection of $S_{X^*}$ with the weak*-closure in $X^*$ of $ext(B_{X^*})$. We consider $K(X^*)$ as a topological space equipped with the weak* topology $\sigma(X^*, X)$. For a Lip-slice $S$ of $S_X$ and $\varepsilon > 0$, we consider the set

$$\begin{aligned} D(S,\varepsilon) \quad : & \quad = \{x^* \in K(X^*) : \exists y \in S \text{ with } \mathrm{Re}\ x^*(y) > 1-\varepsilon\} \\ & = \quad \{x^* \in K(X^*) : S \cap S(S_X, \mathrm{Re}\ x^*, \varepsilon) \neq \emptyset\}. \end{aligned}$$

**Lemma 3.2.2** ([14]) *$K(X^*)$ is a Baire space and $D(S,\varepsilon)$ is an open subset of $K(X^*)$ for every $\varepsilon > 0$. If, moreover, $X$ has the Daugavet property, then $D(S,\varepsilon)$ is dense in $K(X^*)$.*

**Proof.** Let $K^{'}(X^*)$ be the weak*-closure in $X^*$ of $ext(B_{X^*})$, which is weak*-compact, and observe that

$$K^{'}(X^*) \backslash K(X^*) = \underset{n\in\mathbb{N}}{\cup}[(1-\frac{1}{n})B(X^*) \cap K^{'}(X^*)]$$

is of the first category in $K^{'}(X^*)$, so $K(X^*)$ is a Baire space.
Since each set $D_y := \{x^* \in K(X) : \text{Re } x^*(y) > 1-\varepsilon\}$ is relatively $\sigma(X^*, X)$-open in $K(X^*)$, $D(S,\varepsilon) = \underset{y\in S}{\cup} D_y$, $D(S,\varepsilon)$ is evidently relatively $\sigma(X^*, X)$-open in $K(X^*)$.
Finally, to show that $D(S,\varepsilon)$ is weak$^*$ dense in $K(X^*)$ for a Banach space with the Daugavet property it is sufficient to demonstrate that the weak$^*$ closure of $D(S,\varepsilon)$ contains every extreme point of $B_{X^*}$. Since weak$^*$-slices form a base of (relative) neighborhoods of any extreme point of $B_{X^*}$ (Choquet's lemma, see [4, Definition 25.3 and Proposition 25.13]), it is sufficient to prove that every weak*-slice

$$S(B_{X^*}, x, \delta) := \{x^* \in B_{X^*} : Re\ x^*(x) > 1-\delta\}$$

with $\delta \in (0,\varepsilon)$ and $x \in S_X$ intersects $D(S,\varepsilon)$. To this end, let us use Corollary 3.1.1: Given $x$ and $\delta$ as above, there is a $y \in S$ such that $\|x+y\| > 2-\delta$. By Krein-Milman theorem, there is $y^* \in ext(B_{X^*})$ such that Re $y^*(x+y) > 2-\delta$. Therefore, both Re $y^*(x) > 1-\delta$ and Re $y^*(y) > 1-\delta$, which implies that $y^* \in S(B_{X^*}, x, \delta) \cap D(S,\varepsilon)$. ∎

An application of the Baire Theorem gives the following result.

**Corollary 3.2.1** *Let $X$ be a Banach space with the Daugavet property. Given any sequence of Lip-slices $\{S_n : n \in \mathbb{N}\}$ and any sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive numbers, we have that $\underset{n\in\mathbb{N}}{\cap} D(S_n, \delta_n)$ is a dense $G_\delta-$subset of $K(X^*)$.*

We are now ready to state the main result of the present section.(Theorem 3.4 of [14])

**Theorem 3.2.1** *Let $X$ be a Banach space with the Daugavet property. Then, every $T \in$* Lip$(X)$ *for which* slope$(T)$ *is an SCD-set (e.g., relatively weakly compact), satisfies (DE).*

For a map $T \in \text{Lip}(X,Y)$. We denote

$$\text{slope}(T) := \{\frac{T(x_1) - T(x_2)}{\|x_1 - x_2\|} : x_1 \neq x_2 \in X\}.$$

Observe that if $T \in L(X)$, then slope$(T) = T(S_X)$.

**Proof.** It is sufficient to consider the case of $\|T\| = 1$. For $\varepsilon > 0$ fixed, let $u, v \in X$ with $u \neq v$ such that

$$\frac{\|T(u) - T(v)\|}{\|u - v\|} > 1 - \varepsilon. \tag{$*$}$$

Since slope$(T)$ is an SCD-set, there is a sequence $\{S_n\}$ of slices of slope$(T)$ which is determining. For every $n \in \mathbb{N}$, we write

$$\begin{aligned} S_n &= \{z \in \text{slope}(T) : \text{Re } x_n^*(z) > 1 - \varepsilon_n\} \\ &= \{\frac{\|T(x) - T(y)\|}{\|x - y\|} : x \neq y, \frac{\|\text{Re } x_n^*(T(x)) - \text{Re } x_n^*(T(y))\|}{\|x - y\|} > 1 - \varepsilon_n\} \end{aligned}$$

where $\varepsilon_n > 0$ and $x_n^* \in S_{X^*}$. Next, for each $n \in \mathbb{N}$ we consider the subset of $S_X$ given by

$$\begin{aligned} \widetilde{S}_n &= \{\frac{x - y}{\|x - y\|} : x \neq y, \frac{\|T(x) - T(y)\|}{\|x - y\|} \in S_n\} \\ &= \{\frac{x - y}{\|x - y\|} : x \neq y, \frac{\|\text{Re } x_n^*(T(x)) - \text{Re } x_n^*(T(y))\|}{\|x - y\|} > 1 - \varepsilon_n\} \end{aligned}$$

and observe that $\widetilde{S}_n$ is not empty (as $S_n$ is a non-empty subset of slope$(T)$) and so, as Re $x_n^* \circ T \in \text{Lip}(X, \mathbb{R})$, $\widetilde{S}_n$ is a Lip-slice of $S_X$. Therefore, we can apply Corollary 3.2.1 for the sequence $\{\widetilde{S}_n :\in \mathbb{N}\}$ to obtain that the set

$$A = \underset{n \in \mathbb{N}}{\cap} D(\widetilde{S}_n, \varepsilon)$$

is dense in $K(X^*)$. Thus, there is $x^* \in A$ such that

$$\text{Re } x^*(\frac{T(u) - T(v)}{\|u - v\|}) > 1 - \varepsilon.$$

Besides, since $x^* \in A$, for every $n \in \mathbb{N}$ we have that $S(S_X, \text{Re } x^*, \varepsilon) \cap \widetilde{S}_n \neq \emptyset$. Hence, for every $n \in \mathbb{N}$ we can find distinct $x_n, y_n \in X$ such that

$$\frac{x_n - y_n}{\|x_n - y_n\|} \in S(S_X, \text{Re } x^*, \varepsilon) \quad \text{and} \quad \frac{T(x_n) - T(y_n)}{\|(x_n) - (y_n)\|} \in S_n.$$

Therefore, using that the sequence $\{S_n\}$ is determining for slope$(T)$, we get that

$$\text{slope}(T) \subset \overline{\text{conv}}\{\frac{T(x_n) - T(y_n)}{\|(x_n) - (y_n)\|} : n \in \mathbb{N}\}.$$

This, together with $(*)$, allows us to find $\lambda_1, ..., \lambda_N > 0$ with $\sum_{k=1}^{N} \lambda_k = 1$ such that

$$\text{Re } x^*(\sum_{k=1}^{N} \lambda_k \frac{T(x_k) - T(y_k)}{\|x_k - y_k\|}) > 1 - \varepsilon.$$

So, by convexity, there is $\ell \in \{1, ..., N\}$ satisfying

$$\text{Re } x^*(\frac{T(x_\ell) - T(y_\ell)}{\|x_\ell - y_\ell\|}) > 1 - \varepsilon.$$

Recalling that $\frac{x_\ell - y_\ell}{\|x_\ell - y_\ell\|} \in S(S_X, Rex^*, \varepsilon)$, we can write

$$\begin{aligned} \|Id + T\| &\geq \|\frac{x_\ell - y_\ell}{\|x_\ell - y_\ell\|} + \frac{T(x_\ell) - T(y_\ell)}{\|x_\ell - y_\ell\|}\| \\ &\geq \text{Re } x^*(\frac{x_\ell - y_\ell}{\|x_\ell - y_\ell\|} + \frac{T(x_\ell) - T(y_\ell)}{\|x_\ell - y_\ell\|}) \\ &> 2 - 2\varepsilon. \end{aligned}$$

Letting $\varepsilon \longrightarrow 0$, we get $\|Id + T\| > 2$, which finishes the proof since the converse inequality always holds. ■

The condition that slope($T$) is an SCD-set of Theorem 3.2.1 means, in particular, that $T$ has separable image. In order to get rid of this separability restriction, one can use the following lemma.in [14].

We need this lemma to prove the last theorem which is due to **??**.

**Lemma 3.2.3** *Let $X$ be a Banach space with the Daugavet property and let $T \in \text{Lip}(X)$. Then there is a sep-arable subspace $E \subset X$ having the Daugavet property, with $T(E) \subset E$ and $\|T|_E\| = \|T\|$.*

**Proof.** Fix a sequence$(x_k) \subset X$ such that

$$\lim_{k \longrightarrow \infty} \|\frac{T(x_{2k}) - T(y_{_{2k-1}})\|}{\|x_{2k} - y_{_{2k-1}}\|} = \|T\| \qquad (**)$$

and define recursively two sequences of separable subspaces $X_n \subset X$ and $E_n \subset X$,

$$X_1 \subset E_1 \subset X_2 \subset E_2...,$$

in the following way. Put $X_1 = \overline{\text{lin}}\{x_k\}_{k \in \mathbb{N}}$. When a separable subspace $X_n$ is defined, we select a separable subspace $E_n \supset X_n$ having the Daugavet property (the possibility of such a choice is guaranteed by [12, Theorem 4.5]) and put $X_{n+1} = \overline{\text{lin}}(E_n \cup T(E_n))$. Under this construction $E = \overline{\underset{n \in \mathbb{N}}{\cup} E_n}$ is the subspace we need (the Daugavet property of $E$ easily follows from the Daugavet property of the $E_n$ and from a particular case of [11, Lemma 2.2]). ■

**Corollary 3.2.2** *Let $X$ be a Banach space with the Daugavet property, $T \in Lip(X)$ and suppose that* $\overline{\text{conv}}(\text{slope}(T))$*has one of the following properties, Radon-Nikodým property, Asplund property, CPCP or absence of $\ell_1$-sequences. Then $T$ satisfies* (*DE*) (*see* [14]).

**Proof.** Let $E \subset X$ be the subspace from Lemma 3.2.3. Denote by $T_E \in \text{Lip}(E)$ the restriction of $T$ to $E$. Then $\overline{\text{conv}}(\text{slope}(T_E)) \subset \overline{\text{conv}}(\text{slope}(T))$, so $\overline{\text{conv}}(\text{slope}(T_E))$ also has one of the properties listed above and hence, as it is also separable, $\overline{\text{conv}}(\text{slope}(T_E))$ is an SCD-set, see [3]. Thanks to Lemma 3.2.1 this means that $\text{slope}(T_E)$ is an SCD-set, so by Theorem 3.2.1 $T_E$ satisfies the Daugavet equation. Consequently,

$$\|Id + T\| > \|Id_E + T_E\| = 1 + \text{Lip}(T_E) = 1 + \text{Lip}(T)$$

as claimed. ■

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## الملخص

سندرس التعميم لخاصية دوغافي (Daugavet) في الحالة الخطية إلى الحالة غير الخطية، للتحديد أكثر الحالة الليبشيتزية، كما سنهتم أكثر في هذه الدراسة برؤية بعض الشروط الضروريه و الكافية التي بدورها تسمح لمؤثرات ليبشيتز بتحقيق خاصية دوغافي.

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## Abstract

We will study the generalization of the Daugavet property of the linear case to the nonlinear case; more precisely the Lipschitz case. Among other, we are interested in a some conditions sufficient for a Lipschitz operator to satisfy the Daugavet's property.

**Key-words.** Daugavet property, Lipschitz operator, $\ell_1$-singular operator, strong Radon-Nikodým operator, copies spaces of $\ell_1$, the slice of the unit ball $B_X$, the Lipschitz slice of $S_X$, slope $(T)$ set and SCD set.

## Résumé

Nous étudierons la généralisation de la propriété de Daugavet du cas linéaire au cas non linéaire; plus précisément le cas lipschitzien. Entre autre, on s'intéresse à quelques conditions suffisantes pour qu'un opérateur de Lipschitz satisfasse la propriété de Daugavet.

**Mots-clés**. Propriété de Daugavet, opérateur Lipschitz, l'opérateur, $\ell_1$-singulier opérateur, l'opérateur de strong Radon-Nikodým, copie des espaces de $\ell_1$, la tranche de la boule d'unité $B_X$, la tranche Lipschitz de $S_X$, l'ensemble de Slope $(T)$ et l'ensemble de SCD.